Hedging options with transaction costs

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March 2025

Abstract

This paper investigates an optimal investment problem in which an investor hedges a short position in a European call option. Under a utility maximization framework, we first derive the optimal portfolio allocation without transaction costs by formulating and solving the corresponding Hamilton–Jacobi–Bellman equation. In the presence of a short call position, we extend the analysis to reveal how the indirect utility function adjusts and how the optimal strategy is consequently modified. To facilitate practical implementation, a numerical scheme based on a binomial tree is introduced, enabling the approximation of the value function and recovery of the optimal strategy in a discretized setting. We then incorporate proportional transaction costs into the model and extend the numerical method to account for the resulting variational inequality. Numerical results highlight the approach's effectiveness in capturing both the frictionless and cost-adjusted optimal strategies.

Key Words : Stochastic control, Option Hedging, Transaction costs

1 Optimal investment problem without transaction costs

1.1 Problem setting

We consider a portfolio with two assets : one risky and one non risky asset. The first one follows a geometric brownian motion :

$$dS = \alpha S dt + \sigma S dZ_t \tag{1}$$

The second is such that :

$$B_t = Be^{rt} \tag{2}$$

with B the initial amount invested in the asset. Lastly, the portfolio has a short position in an European call option on the risky asset, with strike K and maturity T, with payoff :

$$C_T = (S_T - K)^+ \tag{3}$$

The investor has a utility function u twice differentiable and concave, such that his objective is to maximise his expected utility of the terminal wealth W_T, with :

$$W_T = B_T + y_T S_T - C_T \tag{4}$$

$y_T S_T$ being the cash value of his position on the risky asset at maturity.

1.2 Optimal investment without options

There is no consumption before maturity so the portfolio is self-financing, so when there is no option position in the portfolio, we have :

$$\begin{aligned} dW_t &= dB_t + y_t dS_t && (5) \\ &= rB_t dt + y_t(\alpha S_t dt + \sigma S_t dZ_t) && (6) \end{aligned}$$

By setting $\pi_t W_t = y_t S_t$ (the proportion of wealth allocated in the risky asset at time t), and therefore $(1-\pi_t)W_t = B_t$, we have :

$$dW_t = (\alpha - r)\pi_t W_t dt + rW_t dt + \sigma \pi_t W_t dZ_t \tag{7}$$

As we mentioned, the investor's goal is to maximize his expected utility at maturity, which translates as :

$$\max_\pi E(u(W_T)) \tag{8}$$

and we define the indirect function utility function at time t for a wealth W (the value function of our stochastic control problem) as :

$$U(t, W) = \max_\pi E_t(u(W_T)) \tag{9}$$

By Itô's formula, we have :

$$\begin{aligned} dU(t, W) &= \frac{\partial U}{\partial t} dt + \frac{\partial U}{\partial w} dW_t + \frac{1}{2}\frac{\partial^2 U}{\partial w^2}(dW_t)^2 && (10) \\ &= \frac{\partial U}{\partial t} dt + \frac{\partial U}{\partial w}((\alpha - r)\pi_t W_t dt + rW_t dt + \sigma \pi_t W_t dZ_t) && (11) \\ &+ \frac{1}{2}\frac{\partial^2 U}{\partial w^2}\sigma^2 \pi^2 W_t^2 dt && (12) \end{aligned}$$

Therefore :

$$E(dU(t, W)) = \frac{\partial U}{\partial t} dt + \frac{\partial U}{\partial w}((\alpha - r)\pi_t W_t + rW_t)dt + \frac{1}{2}\frac{\partial^2 U}{\partial w^2}\sigma^2 \pi^2 W_t^2 dt \tag{13}$$

And :

$$\max_\pi E(dU(t, W)) = \frac{\partial U}{\partial t} dt + \max_\pi (\frac{\partial U}{\partial w}((\alpha - r)\pi_t W_t + rW_t)dt + \frac{1}{2}\frac{\partial^2 U}{\partial w^2}\sigma^2 \pi_t^2 W_t^2 dt) \tag{14}$$

By Bellman's principle of optimality ("If I'm already following the optimal strategy, then doing an optimal action in the next dt should give no advantage — I'm already on the best possible path."), if U is optimal for t onward, at time t the best decision we can make regarding the control π leads to $E(\frac{dU}{dt}) = 0$, which leads to the Hamilton-Jacobi-Bellman equation, dividing by dt and setting the derivative to 0 for optimality :

$$\boxed{\frac{\partial U}{\partial t} + \max_\pi L^\pi U = 0} \tag{15}$$

with the following differential operator :

$$L^\pi U = \frac{\partial U}{\partial w}((\alpha - r)\pi_t W_t + rW_t) + \frac{1}{2}\frac{\partial^2 U}{\partial w^2}\sigma^2 \pi_t^2 W_t^2 \tag{16}$$

and the boundary condition :

$$U(T, W) = u(W), \forall W \geq 0 \tag{17}$$

Once we know the indirect utility function U, we can find the optimal allocation π^* using first order conditions to solve for $\max_\pi L^\pi U$:

$$\frac{\partial L^\pi U}{\partial \pi} = \pi_t W^2 \sigma^2 \frac{\partial^2 U}{\partial w^2} + (\alpha - r)W\frac{\partial U}{\partial w} \tag{18}$$

which leads to :

$$\pi^* = -\frac{(\alpha - r)U_w}{W\sigma^2 U_{ww}} \tag{19}$$

with $U_w = \frac{\partial U}{\partial w}$.

In this paper, we will consider two utility function :

1. $u(W) = 1 - e^{-\gamma W}$, of the class CARA of utilities
2. $u(W) = \frac{W^\gamma}{\gamma}$, of the class CRRA of utilities.

- **CRRA Utility** : When there is no option position in the portfolio and for a utility function of the shape $u(W) = \frac{W^\gamma}{\gamma}, 0 < \gamma < 1$, the boundary condition gives $U(T, W) = \frac{W^\gamma}{\gamma}$ and then the indirect utility function can be found as (Proof in the annex) :

$$U(t, W) = g(t)\frac{W^\gamma}{\gamma} \tag{20}$$

with :

$$g(t) = e^{vr(T-t)} \tag{21}$$

and

$$v = \frac{1}{2}(\frac{\alpha - r}{\sigma})^2 \frac{1}{1-\gamma} + r \tag{22}$$

This leads to :

$$\boxed{\pi^* = -\frac{(\alpha - r)U_w}{W\sigma^2 U_{ww}} = \frac{\alpha - r}{\sigma^2(1-\gamma)}} \tag{23}$$

- **CARA Utility** : When there is no option position in the portfolio and for a utility function of the shape $u(W) = 1 - e^{-\gamma W}, 0 < \gamma < 1$, the boundary condition gives $U(T, W) = 1 - e^{-\gamma W}$ and then the indirect utility function can be found as (Proof in the annex) :

$$U(t, W) = 1 - e^{-\gamma\phi(t)W + \psi(t)} \tag{24}$$

with :

  - $\phi(t) = e^{r(T-t)}$
  - $\psi(t) = \frac{(\alpha - r)^2}{2\sigma^2}(t - T)$

This leads to :

$$\boxed{\pi^* = \frac{(\alpha - r)}{W\sigma^2\gamma e^{r(T-t)}}} \tag{25}$$

And if we define y_t as the optimal holdings in the risky asset at time t, we have :

$$y^* = \frac{(\alpha - r)}{S\sigma^2\gamma e^{r(T-t)}} \tag{26}$$

1.3 When there is a short position in option

In the case where the investor has a short position in an European call option , the terminal wealth becomes :

$$\overline{W_T} = W_T - (S_T - K)^+ \tag{27}$$

With this new formulation, the indirect utility function (value function of the stochastic control problem) becomes :

$$U(t, S, W) = \max_\pi E_t(u(\overline{W_T})) \tag{28}$$

In this case, the stock price becomes a variable in the indirect utility function, which changes the computations leading to the HJB equation :

$$dU(t, S, W) = \frac{\partial U}{\partial t}dt + \frac{\partial U}{\partial w}dW_t + \frac{1}{2}\frac{\partial^2 U}{\partial w^2}(dW_t)^2 + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}(dS_t)^2 + \frac{\partial^2 U}{\partial S \partial W}dS_t dW_t \tag{29}$$

We have $dS_t = \alpha S_t dt + \sigma S_t dZ_t$ and $dW_t = ((\alpha - r)\pi_t W_t + rW_t)dt + \sigma\pi_t W_t dZ_t$.
These two are correlated (same brownian motion), therefore $dS_t dW_t = \sigma^2\pi_t S_t W_t dt$. This leads to :

$$E(dU(t, S, W)) = (\frac{\partial U}{\partial t} + \frac{\partial U}{\partial w}((\alpha - r)\pi_t W_t + rW_t) + \frac{\partial U}{\partial S}\alpha S_t + \frac{1}{2}\frac{\partial^2 U}{\partial w^2}\sigma^2\pi_t^2 W_t^2 \tag{30}$$

$$+ \frac{1}{2}\frac{\partial^2 U}{\partial S^2}\sigma^2 S_t^2 + \sigma^2\pi_t S_t W_t \frac{\partial^2 U}{\partial S \partial W})dt \tag{31}$$

This time, we define :

$$\overline{L}^\pi U = \frac{\partial U}{\partial w}((\alpha - r)\pi_t W_t + rW_t) + \frac{\partial U}{\partial S}\alpha S_t + \frac{1}{2}\frac{\partial^2 U}{\partial w^2}\sigma^2\pi_t^2 W_t^2 + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}\sigma^2 S_t^2 + \sigma^2\pi_t S_t W_t \frac{\partial^2 U}{\partial S \partial W} \tag{32}$$

and by the same argument, this leads to the HJB equation :

$$\boxed{\frac{\partial U}{\partial t} + \max_\pi \overline{L}^\pi U = 0} \tag{33}$$

with the boundary condition :

$$U(T, S, W) = u(W - (S - K)^+) \tag{34}$$

The same way as before, we can derive the optimal strategy which will depend on the form of the indirect utility function :

$$\boxed{\pi^* = -\frac{\alpha - r}{\sigma^2 W}\frac{U_w}{U_{ww}} - \frac{S}{W}\frac{U_{Sw}}{U_{ww}}} \tag{35}$$

We can prove that in this case, the indirect utility function is linked to the one when there is no position in the option, by the following relation :

$$\overline{U}(t, W) = U(t, W - f(t, S)) \tag{36}$$

where $f(t, S)$ is the price of the call option under the Black-Scholes framework.
Since we know the explicit form of both indirect utility functions, it allows us to get :

- **CRRA Utility** :

$$\boxed{\pi^* = \frac{\alpha - r}{\sigma^2(1-\gamma)} \frac{W - f}{W} + \frac{S}{W} f_S} \tag{37}$$

And :

$$\pi^* W = \frac{\alpha - r}{\sigma^2(1-\gamma)}(W - f) + S f_S \tag{38}$$

- **CARA Utility** :

$$\boxed{\pi^* = \frac{S}{W} f_S - \frac{(\alpha - r)}{W\sigma^2 \gamma e^{r(T-t)}}} \tag{39}$$

And :

$$y^* = f_S - \frac{(\alpha - r)}{S\sigma^2 \gamma e^{r(T-t)}} \tag{40}$$

Interpretation for the CRRA Utility : The proportion of wealth allocated to the risky asset can be decomposed into two components:

- **Classic Merton Component (adjusted):**

$$\frac{\alpha - r}{\sigma^2(1-\gamma)}(W - f)$$

This term reflects the investment proportion based on *wealth at risk*, i.e., $W - f$. Since we are effectively short an option worth f, we cannot treat this value as available wealth. It must be treated as if it is already lost. Thus, the classic Merton proportion still appears, but scaled to reflect the wealth actually at risk.

- **Delta-Hedge Component:**

$$S f_S$$

This is the delta of the option, scaled by the spot price S. It represents the portion of the portfolio used to replicate the option's payoff, thereby hedging our exposure.

Asymptotic Behavior:

- **When W is large:**

$$\pi \to \frac{\alpha - r}{\sigma^2(1-\gamma)}$$

The Merton line re-emerges as the dominant term, and the impact of the option becomes negligible relative to total wealth.

- **When $W \to f$:**

$$\pi \to \frac{S f_S}{W}$$

In this case, we are highly exposed and have no surplus to speculate with. The portfolio is primarily governed by the necessity to hedge the option.

2 Numerical Scheme Without Transaction Costs Using a Binomial Tree

In practical applications, it is often difficult to derive an explicit form for the indirect utility function

$$U(t, S, W) = \max_{\pi} E_t\Big[u\Big(W_T - (S_T - K)^+\Big)\Big], \tag{41}$$

especially when additional complications such as transaction costs are present. In this section, we present a numerical scheme based on a binomial tree method that approximates the value function $U(t, S, W)$ and yields the optimal strategy by backward induction.

2.1 Overview of the Scheme

The aim is to compute, for each node on a discrete grid in time, stock price, and wealth, the optimal expected utility when starting from that node and acting optimally thereafter. In particular, the scheme will:

1. **Model the stock process:** We assume the stock follows a recombining binomial tree. Over a time step Δt, the stock price evolves as

$$S_{t+\Delta t} = \begin{cases} S_t\, u, & \text{with probability } p, \\ S_t\, d, & \text{with probability } 1-p, \end{cases} \tag{42}$$

   where

$$u = \exp\Big(\sigma\sqrt{\Delta t}\Big), \quad d = \exp\Big(-\sigma\sqrt{\Delta t}\Big), \tag{43}$$

   and the probability p is given by

$$p = \frac{e^{\alpha\,\Delta t} - d}{u - d}. \tag{44}$$

2. **Model the wealth dynamics:** The investor allocates a fraction π_t of wealth W_t to the risky asset. For a small time step Δt, the wealth update is approximated by

$$W_{t+\Delta t} = W_t\Big(1 + r\,\Delta t + \pi_t\Big[(\alpha - r)\Delta t \pm \sigma\sqrt{\Delta t}\Big]\Big), \tag{45}$$

   where the "+" corresponds to the up move (i.e., $S_{t+\Delta t} = S_t\, u$) and the "−" to the down move (i.e., $S_{t+\Delta t} = S_t\, d$).

3. **Set the terminal condition:** At maturity T, the investor's terminal wealth, is

$$\overline{W_T} = W_T - (S_T - K)^+. \tag{46}$$

   Therefore, for the utility function $u(x) = \frac{x^\gamma}{\gamma}$ (with $0 < \gamma < 1$), the terminal condition is

$$U(T, S, W) = \frac{\Big(W - (S - K)^+\Big)^\gamma}{\gamma}. \tag{47}$$

4. **Dynamic Programming Recursion:** At an arbitrary node (t, S, W), the value function is computed by optimizing over the control π:

$$U(t, S, W) = \max_{\pi \in \Pi}\Big\{p\, U\Big(t + \Delta t, S\, u,\, W_{\text{up}}(\pi)\Big) + (1-p)\, U\Big(t + \Delta t, S\, d,\, W_{\text{down}}(\pi)\Big)\Big\}, \tag{48}$$

   where the wealth updates for the up and down moves are:

$$W_{\text{up}}(\pi) = W\Big(1 + r\,\Delta t + \pi\Big[(\alpha - r)\Delta t + \sigma\sqrt{\Delta t}\Big]\Big), \tag{49}$$

$$W_{\text{down}}(\pi) = W\Big(1 + r\,\Delta t + \pi\Big[(\alpha - r)\Delta t - \sigma\sqrt{\Delta t}\Big]\Big). \tag{50}$$

5. **Backward Induction:** Starting from the terminal condition at $t = T$, we recursively compute the value function at each earlier time step. At every node, a discrete set of control values $\pi \in \Pi$ is tested. For each candidate π, the next-step wealth levels are computed and the corresponding U-values are interpolated (if necessary) on the wealth grid, since W_{up} and W_{down} might not belong in the wealth grid used to compute the U-values at the next step. The control that maximizes the expected utility is chosen and the value function is updated accordingly.

2.2 Detailed Algorithm

We now summarize the steps of the algorithm:

1. **Grid Setup:** Discretize the time interval $[0, T]$ into N_t steps with $\Delta t = T/N_t$, the stock price over a range $[S_{\min}, S_{\max}]$, and the wealth over a grid $[W_{\min}, W_{\max}]$.
2. **Terminal Condition:** For each terminal node at time T, where the stock price is

$$S_T = S_0\, u^i\, d^{N_t - i} \quad (i = 0, 1, \ldots, N_t), \tag{51}$$

and for each wealth value W, set

$$U(T, S_T, W) = \frac{\Big(W - (S_T - K)^+\Big)^\gamma}{\gamma}. \tag{52}$$

3. **Backward Induction:** For $n = N_t - 1, N_t - 2, \ldots, 0$ and for each node (characterized by the number of up moves i, with stock price $S = S_0\, u^i\, d^{n-i}$):

   (a) For each wealth grid point W, loop over the discrete set of candidate controls $\pi \in \Pi$.

   (b) Compute the next-period wealth in both the up and down scenarios:

$$W_{\text{up}}(\pi) = W\Big(1 + r\,\Delta t + \pi\Big[(\alpha - r)\Delta t + \sigma\sqrt{\Delta t}\Big]\Big), \tag{53}$$

$$W_{\text{down}}(\pi) = W\Big(1 + r\,\Delta t + \pi\Big[(\alpha - r)\Delta t - \sigma\sqrt{\Delta t}\Big]\Big). \tag{54}$$

   (c) For the up move, the next node is $(t + \Delta t, S\,u)$; for the down move, it is $(t + \Delta t, S\,d)$. Use linear interpolation on the wealth grid to approximate the value function at these nodes, i.e.,

$$U\Big(t + \Delta t, S\,u, W_{\text{up}}(\pi)\Big) \quad \text{and} \quad U\Big(t + \Delta t, S\,d, W_{\text{down}}(\pi)\Big). \tag{55}$$

   (d) Compute the expected utility for the control π:

$$V(\pi) = p\,U\Big(t + \Delta t, S\,u, W_{\text{up}}(\pi)\Big) + (1 - p)\,U\Big(t + \Delta t, S\,d, W_{\text{down}}(\pi)\Big). \tag{56}$$

   (e) Choose the control π^* that maximizes $V(\pi)$, and set

$$U(t, S, W) = \max_{\pi \in \Pi} V(\pi). \tag{57}$$

4. **Optimal Control Recovery:** At the end of the backward induction, the value function $U(0, S_0, W)$ at the root node (time 0, stock price S_0) is obtained. Also, by recording the control π^* that maximizes $V(\pi)$ at each node, we can recover the optimal investment strategy.

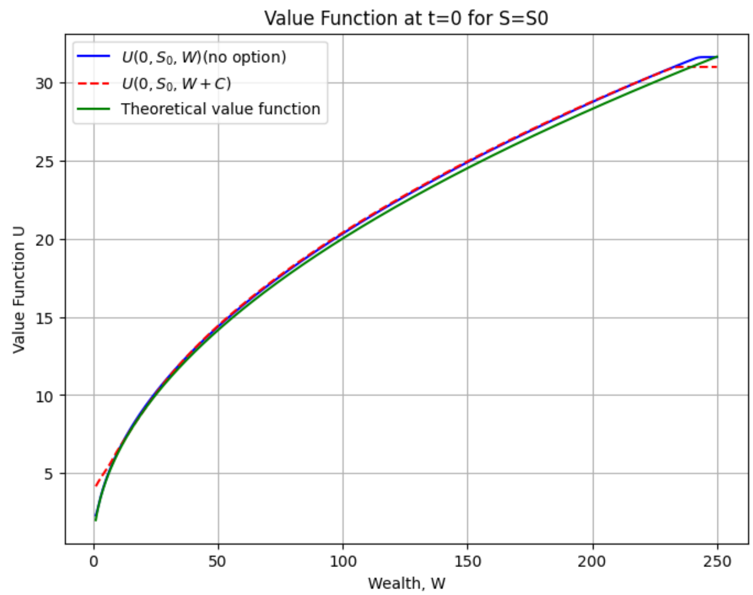
2.3 CARA Utility

The numerical scheme for the CARA Utility is essentially the same, but thanks to a trick explained in the next section related to the shape of this utility function, the price and optimal strategy are independent of the wealth, which allow us to greatly simplify the numerical scheme by eliminating the need for a grid of W values, the indirect utility function depending on W only by a multiplicationg factor.

2.4 Results

2.4.1 CRRA Utility

For the following parameters : $T = 1, S_0 = 100, \sigma = 0.2, \alpha = 0.05, r = 0.03, K = 100, \gamma = 0.5$, we observe that when adding the Black-Scholes price (C) to the initial wealth for the portfolio with a position in option, the curve of the indirect utility function at $t = 0$, with regards to the intial wealth, corresponds with the one with no option position, and the theoretical one.

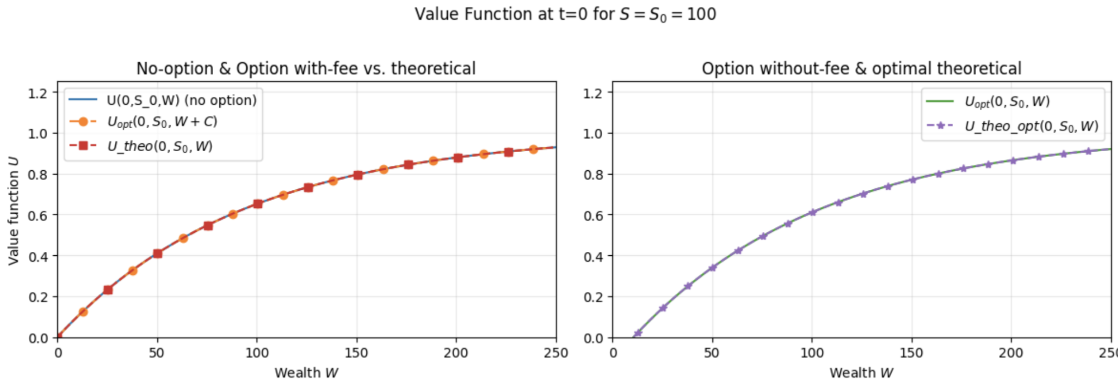


2.4.2 CARA Utility

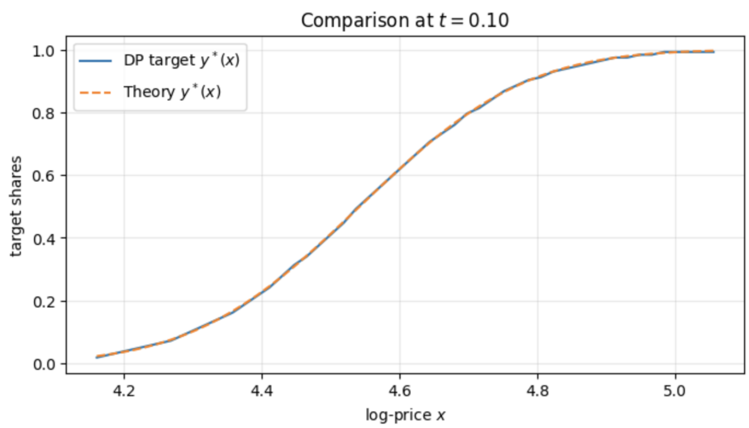
For the following parameters : T = 1, S0 = 100, $\sigma = 0.2$, $\alpha = 0.05$, r = 0.05, K = 100, $\gamma = 0.001$, the price obtained is :

$$C = 10,45$$

corresponding exactly with the Black Scholes price.
We observe that by adding to the initial wealth for the portfolio with a position in option C, the error between the value function when there is no option in the portfolio and the one with an option, is minimised (RMSE = 8,3e-16).
These two are also aligned with the theoretical value function in absence of option, with a RMSE of 0.0 between the theoretical value function and the one found with DP.
Finally, the value function found by DP when there is an option aligns perfectly (RMSE of 0.0001) with the theoretical one ($\overline{U}(t, W, S) = U(t, W - f)$) when the risk aversion coefficient γ is close to 0 (making the investor risk-neutral). This comes from the fact that the price without transaction costs diverges slightly from the Black-Scholes price as the investor becomes more risk averse. This is explained by the fact that Black-Scholes always assumes the investor as risk-neutral.



We can also observe that the optimal policy found by DP corresponds to the theoretical one, with a RMSE of approximately 0.001 across prices.



3 Optimal investment with transaction costs

We now extend the problem to account for proportional transaction costs. The investor can trade a risky asset S_t and a non-risky one B_t, and is charged a proportional transaction cost $\lambda > 0$ on both buying and selling. His holdings are represented by:

- y_t: quantity of risky asset held
- W_t: wealth held in the cash account

We model the system as follows:

$$\begin{aligned}
dS_t &= \alpha S_t dt + \sigma S_t dZ_t, &(58)\\
dy_t &= l_t dt - m_t dt, &(59)\\
dW_t &= rW_t dt - (1+\lambda)S_t l_t dt + (1-\lambda)S_t m_t dt, &(60)
\end{aligned}$$

where $l_t, m_t \geq 0$ are the purchases and sales of the risky asset, and λ is the transaction cost (assumed symmetric in our study).

Remark. When setting the transaction costs to 0, we observe that the portfolio has the following dynamics (total wealth being $X_t = W_t + y_t S_t$) :

$$\begin{aligned}
dX_t &= dW_t + dS_t y_t + dy_t S_t &(61)\\
&= rW_t dt - S_t dy_t + y_t(\alpha S_t dt + \sigma S_t dZ_t) + S_t dy_t &(62)\\
&= r(X_t - y_t S_t)dt + \alpha y_t S_t dt + \sigma y_t S_t dZ_t &(63)\\
&= rX_t dt + y_t S_t(\alpha - r)dt + y_t S_t \sigma dZ_t. &(64)
\end{aligned}$$

By setting $\pi_t = \frac{y_t S_t}{X_t}$ (proportion of wealth in risky asset), we find the exact same expression as in Part 1, which confirms coherence between the formulations of both problems.

3.1 Optimal investment without options

The investor's objective is now to maximize expected utility of his terminal wealth including the liquidation value of his risky holdings:

$$\sup_{(l,m)} \mathbb{E}\left[u\left(W_T + y_T S_T - \lambda S_T |y_T|\right)\right]. \tag{65}$$

This leads us to define the indirect utility function (value function):

$$U(t,S,y,W) = \sup_{(l,m)} \mathbb{E}_{t,S,y,W}\left[u\left(W_T + y_T S_T - \lambda S_T |y_T|\right)\right]. \tag{66}$$

We apply Itô's lemma to the function $U(t, S_t, y_t, W_t)$, assuming regularity:

$$\begin{aligned}
dU &= \frac{\partial U}{\partial t}dt + \frac{\partial U}{\partial S}dS_t + \frac{\partial U}{\partial y}dy_t + \frac{\partial U}{\partial W}dW_t + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}(dS_t)^2 &(67)\\
&= \left[\frac{\partial U}{\partial t} + \alpha S\frac{\partial U}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 U}{\partial S^2} + rW\frac{\partial U}{\partial W}\right.\\
&\quad \left. + l\left(\frac{\partial U}{\partial y} - (1+\lambda)S\frac{\partial U}{\partial W}\right) + m\left(-\frac{\partial U}{\partial y} + (1-\lambda)S\frac{\partial U}{\partial W}\right)\right]dt + \sigma S\frac{\partial U}{\partial S}dZ_t. &(68)
\end{aligned}$$

Then, we use the same reasoning as in Section 1.2, taking the expectation and using the dynamic programming principle (and optimality), we obtain the HJB equation:

$$\boxed{\frac{\partial U}{\partial t} + \alpha S\frac{\partial U}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 U}{\partial S^2} + rW\frac{\partial U}{\partial W} + \max_{l\geq 0, m\geq 0}\left\{l\left(\frac{\partial U}{\partial y} - (1+\lambda)S\frac{\partial U}{\partial W}\right) + m\left(-\frac{\partial U}{\partial y} + (1-\lambda)S\frac{\partial U}{\partial W}\right)\right\} = 0.} \tag{69}$$

From this equation, we can isolate the terms that drive the control l and m:

$$F(U) := \max\left\{\frac{\partial U}{\partial y} - (1+\lambda)S\frac{\partial U}{\partial W},\ -\frac{\partial U}{\partial y} + (1-\lambda)S\frac{\partial U}{\partial W}\right\}. \tag{70}$$

Then the problem can be reformulated as a variational inequality:

$$\max\left\{F(U),\ -\frac{\partial U}{\partial t} - \alpha S\frac{\partial U}{\partial S} - \frac{1}{2}\sigma^2 S^2\frac{\partial^2 U}{\partial S^2} - rW\frac{\partial U}{\partial W}\right\} = 0. \tag{71}$$

By setting :

$$A = \frac{\partial U}{\partial y} - (1+\lambda)S\frac{\partial U}{\partial W}\text{(Selling term)} \tag{72}$$

$$B = -\frac{\partial U}{\partial y} + (1-\lambda)S\frac{\partial U}{\partial W}\text{(Buying term)} \tag{73}$$

$$C = -\frac{\partial U}{\partial t} - \alpha S\frac{\partial U}{\partial S} - \frac{1}{2}\sigma^2 S^2\frac{\partial^2 U}{\partial S^2} - rW\frac{\partial U}{\partial W}\text{(No Transaction term)} \tag{74}$$

the Variational Inequality becomes :

$$\max\{A, B, C\} = 0 \tag{75}$$

This divides the (S, y, W) space in three regions, each characterized by one of the terms being active (=0).

No Transaction region : Defined by

$$(1-\lambda)S \leq \frac{\partial U/\partial y}{\partial U/\partial w} \leq (1+\lambda)S \tag{76}$$

$$-\frac{\partial U}{\partial t} - \alpha S\frac{\partial U}{\partial S} - \frac{1}{2}\sigma^2 S^2\frac{\partial^2 U}{\partial S^2} - rW\frac{\partial U}{\partial W} = 0 \tag{77}$$

Selling region : Defined by

$$A = 0 \tag{78}$$

$$B < 0 \tag{79}$$

$$C \leq 0 \tag{80}$$

Buying region : Defined by

$$A < 0 \tag{81}$$

$$B = 0 \tag{82}$$

$$C \leq 0 \tag{83}$$

The terminal condition reflects the full liquidated value of the portfolio at maturity:

$$U(T, S, y, W) = u\left(W + yS - \lambda S|y|\right). \tag{84}$$

3.2 When there is a short position in option

We now consider the case where the investor holds a short position in a contingent claim (e.g., an option) with a payoff function $\phi(S_T)$. This means that, in addition to managing their liquid and risky positions, the investor must deliver a terminal payoff equal to $\phi(S_T)$ at time T.

Modified objective. The optimization problem becomes:

$$\sup_{(l,m)} \mathbb{E}\left[u\left(W_T + y_T S_T - \lambda S_T |y_T| - \phi(S_T)\right)\right]. \tag{85}$$

This corresponds to an investor aiming to maximize expected utility after liquidating all risky holdings and delivering the payoff of the contingent liability.

Value function. The associated value function is:

$$U(t,S,y,W) = \sup_{(l,m)} \mathbb{E}_{t,S,y,W}\left[u\left(W_T + y_T S_T - \lambda S_T |y_T| - \phi(S_T)\right)\right]. \tag{86}$$

HJB and variational inequality. As before, the HJB equation and the variational inequality remain unchanged in structure since the dynamics of the controls and state variables are unaffected by the presence of the claim. The only modification lies in the terminal condition.

Terminal condition. The new terminal condition becomes:

$$U(T,S,y,W) = u\left(W + yS - \lambda S|y| - \phi(S)\right). \tag{87}$$

Remark. If the investor receives a premium p at time t for selling the claim, one may define an indifference price by comparing the value functions with and without the claim, and solving:

$$U^{\text{with claim}}(t,S,y,W+p) = U^{\text{without claim}}(t,S,y,W). \tag{88}$$

3.3 Dimension Reduction via CARA Utility

We will now study the problem when using the exponential utility, which will help us reduce the dimension of the problem, by factoring out the wealth of the maximization problem. This comes from the following reformulation (with $\delta(T,t) = e^{-r(T-t)}$:

$$W_T = \frac{B_s}{\delta(T,s)} + \int_s^T \frac{1}{\delta(T,t)}((1-\lambda)S_t dL_t - (1+\lambda)S_t dM_t) \tag{89}$$

Therefore, we can write the maximization problem as :

$$V(t,B,y,S) = \max_{\pi} \mathbb{E}_{t,B,y,S}(u(W_T + y_T S_T - \lambda S_T|y_T| - \phi(S_T))) \tag{90}$$

$$= 1 - e^{-\gamma \frac{B}{\delta(T,t)}} Q(t,y,S) \tag{91}$$

with :

$$Q(t,y,S) = \min_{\pi} \mathbb{E}_{t,y,S}(e^{-\gamma \int_t^T \frac{1}{\delta(T,s)}((1-\lambda)S_s dL_s - (1+\lambda)S_s dM_s)} e^{-\gamma(y_T S_T - \lambda S_T|y_T| - \phi(S_T))}) \tag{92}$$

$$Q(t,y,S) = 1 - V(t,0,y,S) \tag{93}$$

Therefore, we can transform the variational inequality we obtained for U, into the following p.d.e for $Q(t,y,S)$:

$$\min\{Q_y + \frac{\gamma(1+\lambda)S}{\delta(T,t)}Q, -(Q_y + \frac{\gamma(1-\lambda)S}{\delta(T,t)}Q), Q_t + \alpha S Q_S + \frac{1}{2}\sigma^2 S^2 Q_{SS}\} = 0 \tag{94}$$

We write this using the previous formula and noticing that :

- $V_B = 0$ ($Q_B = 0$)
- $V_y = -e^{-\gamma B/\delta} Q_y$
- $V_S = -e^{-\gamma B/\delta} Q_S$
- $V_{SS} = -e^{-\gamma B/\delta} Q_{SS}$
- $V_t = -e^{-\gamma B/\delta} (\frac{\gamma r B}{\delta} Q - Q_t)$

The boundary condition is :

$$Q(T, y, S) = \exp(-\gamma(y_T S_T - \lambda S_T |y_T| + \phi(S_T))) \tag{95}$$

(For the case without the option, only the boundary condition changes and becomes $Q(T, y, S) = \exp(-\gamma((1-\lambda) y_T S_T)$.

This transformation allows us to solve for Q, which doesn't depend on the wealth level, and then simply multiply by a quantity depending only on the initial wealth B to get V, instead of solving for V directly, allowing for a much simpler numerical scheme.

4 Numerical Scheme for the Problem with Transaction Costs and Option Position

In this section, we present a numerical scheme to compute the indirect utility function $U(t, S, y, W)$ in the case where the investor is subject to proportional transaction costs and holds a short position in a European option. We recall that the effective terminal wealth accounts for the liquidation value of the remaining position in the risky asset:

$$\bar{W}_T = W_T + S_T y_T - \lambda S_T |y_T|.$$

As a result, the terminal condition becomes:

$$U(T, S, y, W) = u(W + Sy - \lambda S|y|).$$

4.1 Discretization of the State Space

We discretize the continuous variables t, S, y, and W as follows:

- The time interval $[0, T]$ is divided into N steps of size $\Delta t = T/N$.
- The stock price S_t is approximated by a binomial process:

$$S(i+1) = \begin{cases} S(i) \cdot e^{\alpha \Delta t + \sigma \sqrt{\Delta t}} & \text{with probability } 1/2, \\ S(i) \cdot e^{\alpha \Delta t - \sigma \sqrt{\Delta t}} & \text{with probability } 1/2. \end{cases}$$

- The number of shares held y is discretized in steps of Δy:

$$y_k = k\Delta y, \quad k \in \{-M, ..., M\}.$$

- The cash wealth W is discretized in steps of ΔW:

$$W_l = l\Delta W, \quad l \in \{0, ..., K\}.$$

We denote the discretized utility value by:

$$u(i, j, k, l) \approx U(i\Delta t, S_j, y_k, W_l).$$

4.2 Terminal Condition

At maturity $t = T$, we initialize:

$$u(N, j, k, l) = u\left(W_l + S_j y_k - \lambda S_j |y_k|\right).$$

4.3 Backward Recursion over Time Steps

For $i = N - 1$ down to 0, we compute $u(i, j, k, l)$ at each grid point by considering the three possible actions: do nothing, buy the minimal allowed number of shares, or sell the minimal allowed number of shares.

1. **No Transaction (NT):**

   If no transaction occurs, wealth grows at the risk-free rate:

$$W' = W_l \cdot e^{r\Delta t}.$$

   Let l^+ be the index closest to W'. The continuation value is:

$$u^{\mathrm{NT}}(i, j, k, l) = \frac{1}{2}\left[u(i+1, j+1, k, l^+) + u(i+1, j-1, k, l^+)\right].$$

2. **Buy Action:**
   The investor buys Δy shares:
$$y' = y_k + \Delta y, \quad W' = W_l - (1+\lambda) S_j \Delta y.$$
   Let l_b be the nearest index to W', then:
$$u^{\text{Buy}}(i,j,k,l) = u(i,j,k+1,l_b).$$

3. **Sell Action:**
   The investor sells Δy shares:
$$y' = y_k - \Delta y, \quad W' = W_l + (1-\lambda) S_j \Delta y.$$
   Let l_s be the nearest index to W', then:
$$u^{\text{Sell}}(i,j,k,l) = u(i,j,k-1,l_s).$$

4. **Value and Optimal Action:**
   The optimal decision is to take the maximum value among the three possibilities:
$$u(i,j,k,l) = \max\left\{u^{\text{NT}}(i,j,k,l), u^{\text{Buy}}(i,j,k,l), u^{\text{Sell}}(i,j,k,l)\right\}.$$
   The action that maximizes $u(i,j,k,l)$ determines whether the point lies in region $\mathcal{NT}$, $\mathcal{B}$ (buy), or $\mathcal{S}$ (sell).

4.4 Implementation Notes

- In practice, we use interpolation when W' does not exactly fall on a mesh point.
- The scheme tests only minimal trade increments ($\pm\Delta y$), rather than computing the exact trade required to re-enter $\mathcal{NT}$.
- Nevertheless, this recursive maximization allows us to capture the optimal control logic implied by the variational inequality.

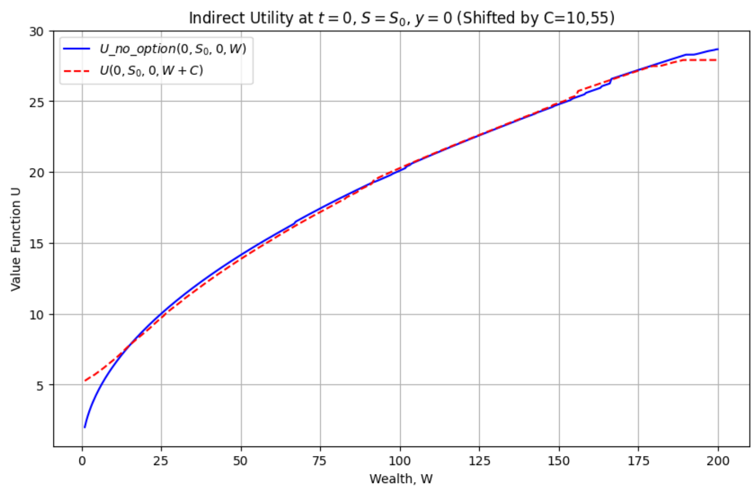
4.5 CARA Utility

The numerical scheme for the CARA Utility is essentially the same, but thanks to the trick we used in the previous section, the price and optimal strategy are independent of the wealth. The Dynamic Programming scheme minimizes $Q(t,y,S)$ for the three possible actions, at each step, which doesn't depend on W, therefore there is no need to create a grid for W.
The indirect utility function does depend on W, but only by a multiplication factor once we have found Q, we have :
$$U(0,W,y,S) = 1 - e^{-\frac{\gamma W}{\delta(T,0)}} Q(0,y,S) \tag{96}$$

4.6 Results

4.6.1 CRRA Utility

For the following parameters : $T = 1, S_0 = 100, \sigma = 0.2, \alpha = 0.05, r = 0.03, K = 100, \gamma = 0.5, \lambda = 0.01$, we observe that by adding to the initial wealth for the portfolio with a position in option $C = 11$, the error between the value function when there is no option in the portfolio and the one with an option, is minimised. This relates to the indifference pricing method.



4.6.2 CARA Utility

For the following parameters : $T = 1, S_0 = 100, \sigma = 0.2, \alpha = 0.05, r = 0.05, K = 100, \gamma = 0.5$ and $\lambda = 0.01$ (transaction costs), the price of the option is :

$$C = 14.65$$

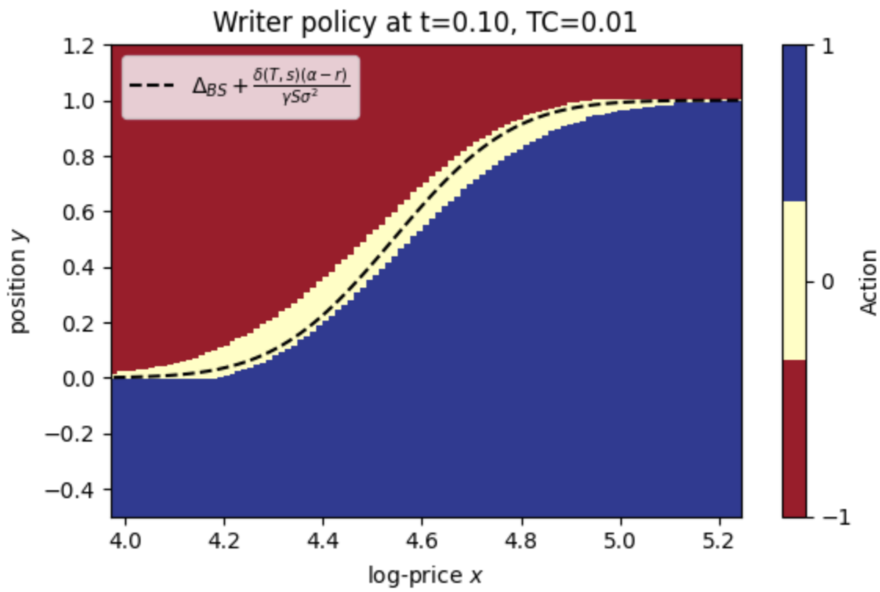
which gives a 40% increase compared to the price without transaction costs.
To compare the way the optimal strategies differ with and without transaction costs, we introduce the following metrics :

- Average shares moved by node (AS) $= \frac{2}{(N+1)(N+2)} \sum_{t=1}^{N} \sum_{i=0}^{t-1} \sum_{j=i,i+1} |y(t,j) - y(t-1,i)|$
- Trade frequency (TF) $= \frac{2}{(N+1)(N+2)} \sum_{t=1}^{N} \sum_{i=0}^{t-1} \sum_{j=i,i+1} \mathbb{1}_{|y(t,j)-y(t-1,i)|>0}$

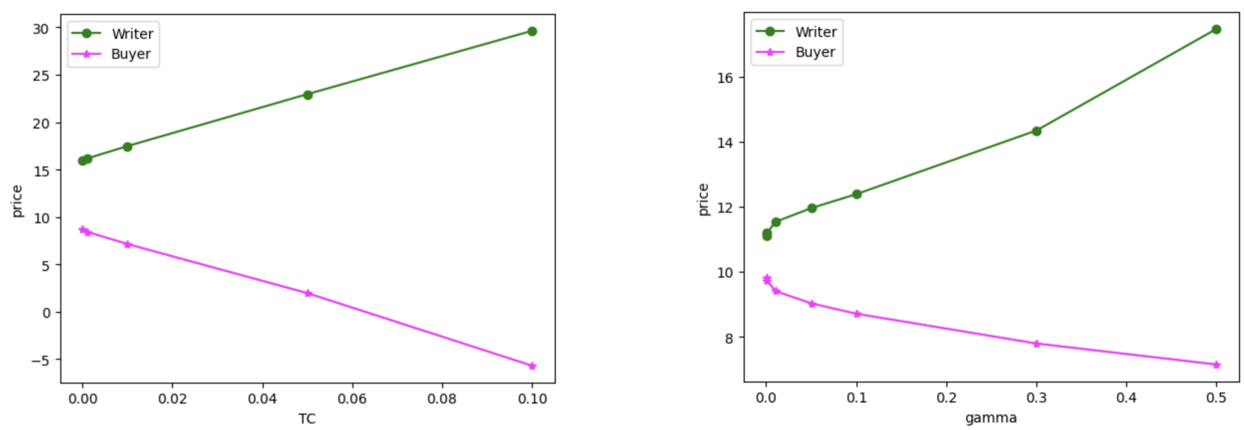
We observe that without transaction costs, TF = 99.7% and AS = 0.009 (we only allow moves of $\pm\Delta_y$, which are fairly small), while for transaction costs of 1%, TF = 75% and AS = 0.0047.
Below is a table comparing price, TF, AS for different strikes, risk aversion, TC levels.

	Gamma	Strike	TC (%)	TC Price	BS Price	RelDiff (%)	Intensity	AvgTrades
0	0.5000	80	0.0	33.231924	24.588835	35.150460	0.007630	0.853091
1	0.5000	80	1.0	34.458102	24.588835	40.137184	0.007573	0.846693
2	0.5000	80	5.0	39.353441	24.588835	60.045974	0.007507	0.839296
3	0.5000	100	0.0	14.973899	10.450584	43.282895	0.007639	0.854102
4	0.5000	100	1.0	16.507851	10.450584	57.961043	0.007585	0.848068
5	0.5000	100	5.0	21.996888	10.450584	110.484785	0.007521	0.840913
6	0.5000	120	0.0	3.801460	3.247477	17.058870	0.007648	0.855107
7	0.5000	120	1.0	5.371964	3.247477	65.419593	0.007598	0.849462
8	0.5000	120	5.0	10.458447	3.247477	222.048332	0.007536	0.842557
9	0.0001	80	0.0	24.589463	24.588835	0.002552	0.008921	0.997434
10	0.0001	80	1.0	25.534968	24.588835	3.847814	0.008496	0.949914
11	0.0001	80	5.0	29.321968	24.588835	19.249112	0.008443	0.943917
12	0.0001	100	0.0	10.453841	10.450584	0.031169	0.008921	0.997435
13	0.0001	100	1.0	11.108797	10.450584	6.298341	0.008494	0.949704
14	0.0001	100	5.0	13.874412	10.450584	32.762082	0.008438	0.943349
15	0.0001	120	0.0	3.250088	3.247477	0.080381	0.008921	0.997435
16	0.0001	120	1.0	3.544063	3.247477	9.132810	0.008501	0.950470
17	0.0001	120	5.0	4.908421	3.247477	51.145655	0.008442	0.943795

Now, one of the most interesting results of this scheme, was the clear apparition of the three regions mentioned in the previous section. Indeed, a 'No-Trade Region' appears in the optimal trading strategy around the theoretical optimal trading strategy without transaction costs, with a 'Buy Region' below and a 'Sell Region' above.



Finally, another interesting observation is the increase of the 'Writer-Buyer' window with regards to transaction costs and risk aversion of the investor, as exposed below.



(a) Sensitivity to Transaction Costs

(b) Sensitivity to risk aversion

Figure 1: Writer-Buyer price sensitivity

5 Bull Callspread Pricing and Hedging

In this section, we study the application of our model to a bull call spread to highlight the fact that linearity in price and hedging strategies disappear for products like call spreads once transaction costs are introduced.

5.1 Without transaction costs

Reminder : A Bull Call Spread is a strategy which consists of buying a call option at a certain strike, while simultaneously selling a call option from the same underlying with the same terminal date, with a higher strike. This allows to benefit from a moderate rise in the underlying asset, while paying a lower price than for the single call.

In the absence of transaction costs and for a risk neutral investor, the price of the callspread is equal to buying and selling the two calls separately, we have :

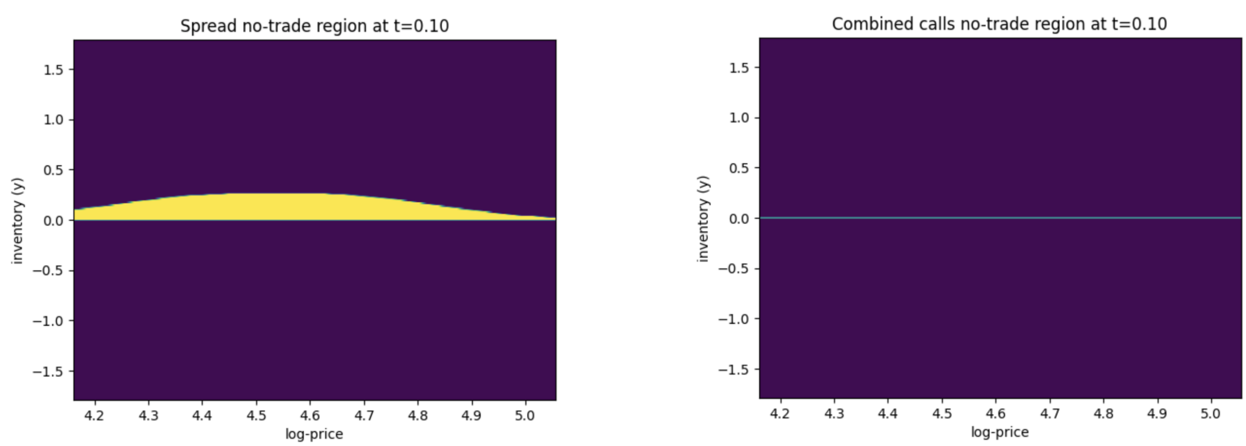
$$\boxed{p_{CS} = p_{K_1} - p_{K_2}} \tag{97}$$

Indeed, using our model with the following parameters : $S_0 = 100, r = \mu = 0.05, \sigma = 0.2, T = 1, K_1 = 95, K_2 = 105, \gamma = 0.0001, c = 0$, we have :

- $p_{CS} = 5.33$
- $p_{K_1} - p_{K_2} = 5.325$

5.2 With transaction costs

Let us now introduce proportional transaction costs and observe how the price and strategy evolve.
Indeed, once transaction costs are introduced, hedging the two calls separately becomes really expensive as it will sometimes require to buy and sell shares of the underlying at the same time, involving twice the necessary transaction costs (the no-trade region will be reduced to the intersection of both no-trade region, which is almost inexistent), which can be avoided by treating the call spread as a single product which we will hedge using our dynammic programming model.

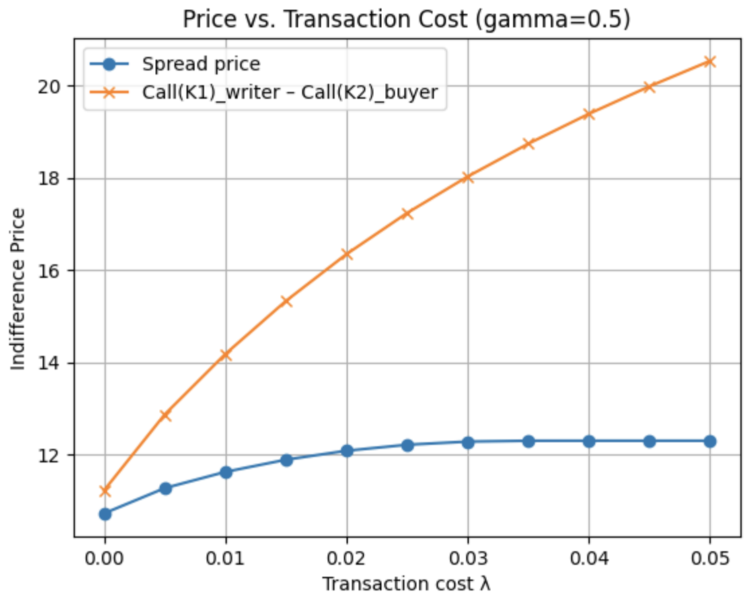


(a) No-trade region for the Spread

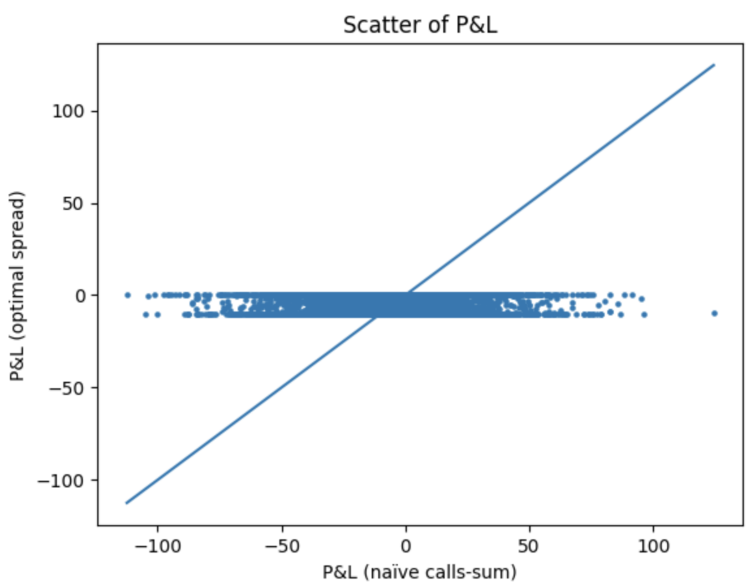
(b) Combined No-trade region

Figure 2: Difference in Hedging strategies

As the price is determined as the minimal amount necessary to hedge the product, this non linearity in hedging strategies as transaction costs will impact the price : the price of the call spread will no longer be equal to the difference of the two call prices for non-zero transaction costs.



All this goes to show the importance treating carefully the questions arising from the introduction of transaction costs in model, as assumptions made before may not hold once some hypothesis are made, which can lead to inefficient hedging and pricing of instruments, like exposed in the figure below.



6 Annex

6.1 Proof of the solution of the HJB (No transaction costs and no option)

Let us consider the CARA utility first.
We make a initial guess that the indirect utility function is of the following form :

$$U(t,W) = 1 - e^{-\gamma\phi(t)W+\psi(t)} \tag{98}$$

Then :

$$U_t = (\gamma\phi'(t)W - \psi'(t))e^{-\gamma\phi(t)W+\psi(t)} \tag{99}$$

$$U_w = \gamma\phi(t)e^{-\gamma\phi(t)W+\psi(t)} \tag{100}$$

$$U_{ww} = -\gamma^2\phi^2(t)e^{-\gamma\phi(t)W+\psi(t)} \tag{101}$$

and :

$$\pi^* = \frac{(\alpha - r)}{W\sigma^2\gamma\phi(t)} \tag{102}$$

Then, the HJB equation becomes :

$$(\gamma\phi'(t) + \gamma\phi(t)r)W - \psi'(t) + \frac{(\alpha - r)^2}{2\sigma^2} = 0 \tag{103}$$

which leads to :

- $\phi(t) = e^{r(T-t)}$
- $\psi(t) = \frac{(\alpha-r)^2}{2\sigma^2}(t - T)$

and :

$$\pi^* = \frac{(\alpha - r)}{W\sigma^2\gamma e^{r(T-t)}} \tag{104}$$

The reasoning behind the proof for the CRRA Utility is exactly the same.