SGN – Assignment #1

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1 Periodic orbit

Exercise 1

Consider the 3D Sun–Earth Circular Restricted Three-Body Problem with $\mu = 3.0359 \times 10^{-6}$.

1) Find the x-coordinate of the Lagrange point L_2 in the rotating, adimensional reference frame with at least 10-digit accuracy.

Solutions to the 3D CRTBP satisfy the symmetry

$$\mathcal{S} : (x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \rightarrow (x, -y, z, -\dot{x}, \dot{y}, -\dot{z}, -t).$$

Thus, a trajectory that crosses perpendicularly the $y = 0$ plane twice is a periodic orbit.

2) Given the initial guess $\mathbf{x}_0 = (x_0, y_0, z_0, v_{x0}, v_{y0}, v_{z0})$, with

$$\begin{aligned} x_0 &= 1.008296144180133 \\ y_0 &= 0 \\ z_0 &= 0.001214294450297 \\ v_{x0} &= 0 \\ v_{y0} &= 0.010020975499502 \\ v_{z0} &= 0 \end{aligned}$$

Find the periodic halo orbit that passes through z_0; that is, develop the theoretical framework and implement a differential correction scheme that uses the STM either approximated through finite differences or achieved by integrating the variational equation.

The periodic orbits in the CRTBP exist in families. These can be computed by continuing the orbits along one coordinate, e.g., z_0. This is an iterative process in which one component of the state is varied, while the other components are taken from the solution of the previous iteration.

3) By gradually increasing z_0 and using numerical continuation, compute the families of halo orbits until $z_0 = 0.0046$.

(8 points)

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1)Lagrangian points with $y = 0, z = 0$ in the rotating, adimensional reference frame of the CRTBP (L_1, L_2 and L_3) are solutions of the following equation :

$$x - (1 - \mu)\frac{x + \mu}{|x + \mu|^3} - \mu\frac{x + \mu - 1}{|x + \mu - 1|^3} = 0$$

To compute L_2, we should use $x_0 >> 1 - \mu$. Doing that, I compute x_{L_2} =1.0100701875.

2)The objective here is to extend the method used in the homework to find the periodic that passes by the initial guess. **Figure 1** shows the 3D-plot of the orbit, in the rotating frame. Thus, I integrated the righ-hand side of 3D CRTBP using events options to stop when $y = 0$.

After this, the solution is differentialy corrected to get an orthogonal crossing of the $y = 0$ plane, which means $v_{x0} = v_{z0} = 0$. I implemented a while loop which will loop until v_{x0}, v_{z0} are superior to 1e-13, which is near zero. In this loop, I correct x_0 and v_{y0} using the STM previously computed. We have the correction to make with the following equation, omitting the zero terms, with Φ_{ij} the i-th line, j-th column element of the STM, $\delta X = (\delta x_0, 0, \delta z_0, 0, 0, 0$ the correction to make and t_e the time when the event happens :

$$\begin{pmatrix}\delta v_x(t_e)\\ \delta v_z(t_e)\end{pmatrix} = [\begin{pmatrix}\Phi_{41}(t_e) & \Phi_{45}(t_e)\\ \Phi_{61}(t_e) & \Phi_{65}(t_e)\end{pmatrix} - \frac{1}{v_y(t_e)}\begin{pmatrix}\delta a_x(t_e)\\ \delta a_z(t_e)\end{pmatrix}(\Phi_{21}(t_e)\Phi_{25}(t_e))]\begin{pmatrix}\delta x_0(t_e)\\ \delta v_{y0}(t_e)\end{pmatrix}$$

I implemented the inversion of this equation to find the correction and the orbit found is the following and then I computed the symetric of the half-orbit found. It seems useful to plot $_2$ because this halo orbit is around this point.

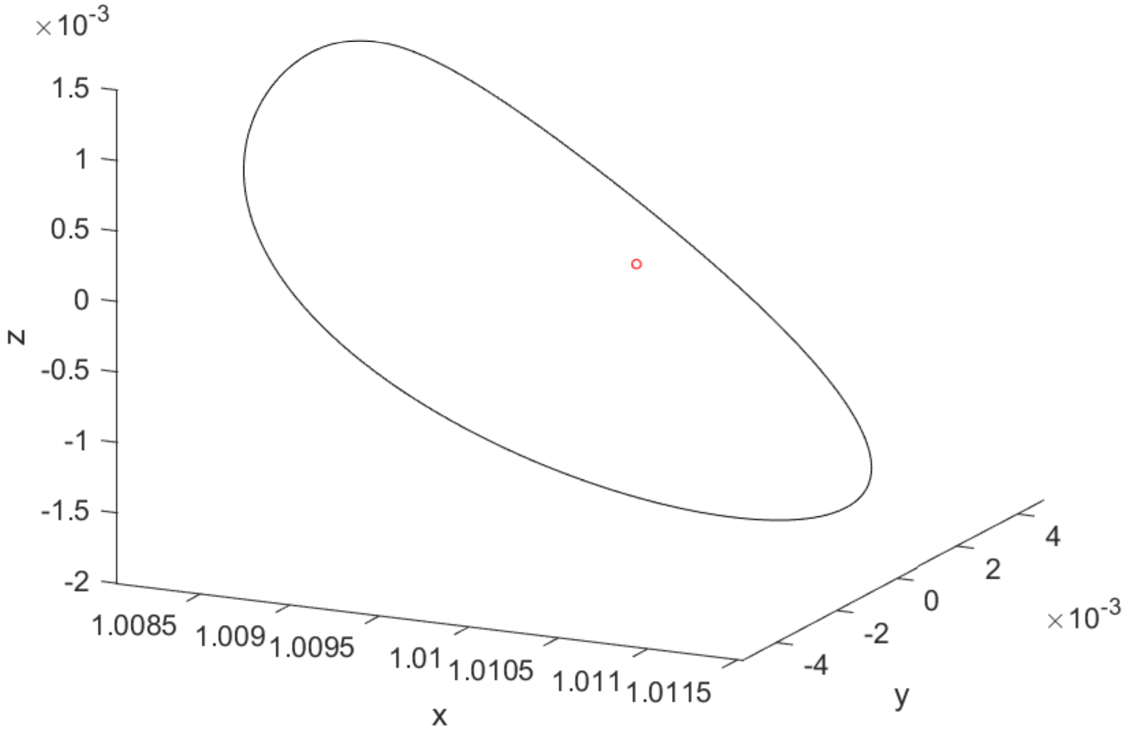

Figure 1: Orbit passing through $z_0 = 1.214294450297.10^{-3}$ with L_2 in red in the rotating frame

3)Using a for loop from $z_0 = 1.214294450297.10^{-3}$ to $z_0 = 0.0046$ with a time-step of 1e-4, I implemented the previous scheme to get the periodic orbit. **Figure 2, Figure 3, Figure 4** shows the projection projected on xy,xz and yz planes with L2 in red, in the rotating frame.

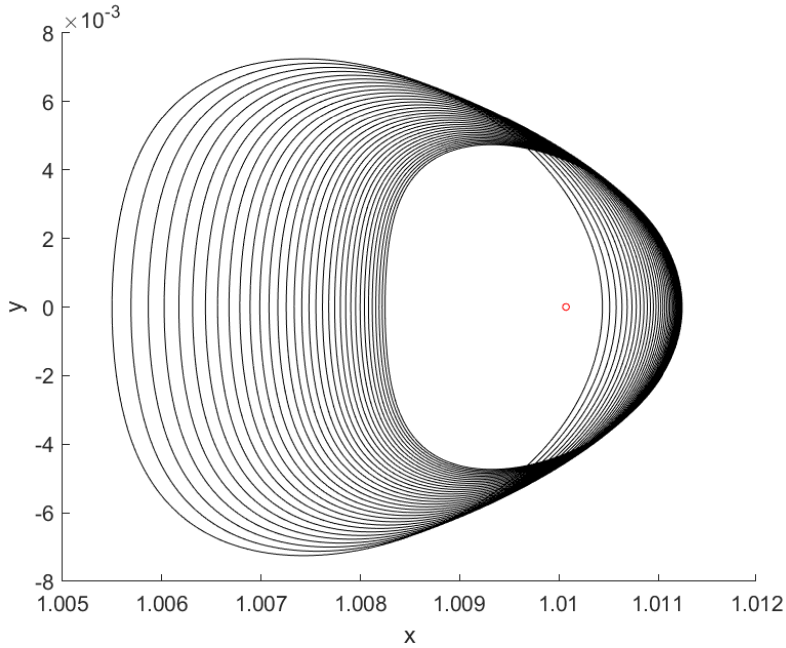

Figure 2: Orbits passing through $z_0 = 1.214294450297.10^{-3}$ to $z_0 = 4.6.10^{-3}$ projected on xy-plane with L_2 in red in the rotating frame

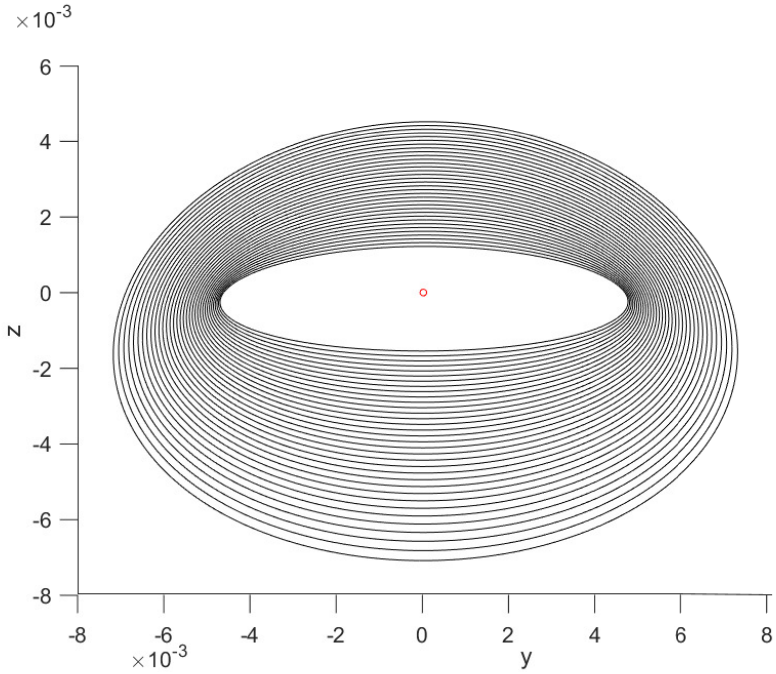

Figure 3: Orbits passing through $z_0 = 1.214294450297.10^{-3}$ to $z_0 = 4.6.10^{-3}$ projected on yz-plane with L_2 in red in the rotating frame

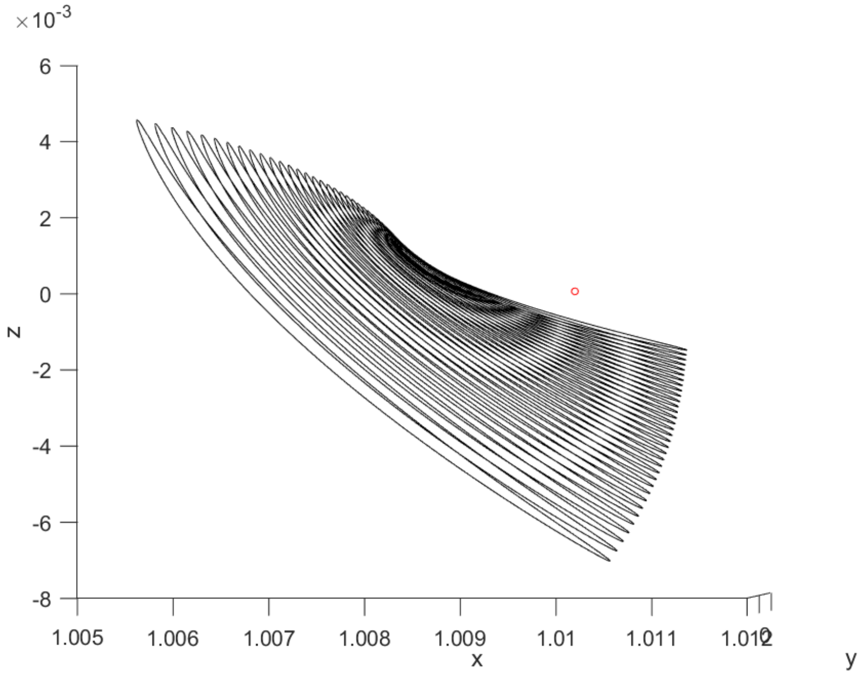

Figure 4: Orbits passing through $z_0 = 1.214294450297.10^{-3}$ to $z_0 = 4.6.10^{-3}$ projected on xz-plane with L_2 in red in the rotating frame

2 Impulsive guidance

Exercise 2

Consider the two-impulse transfer problem stated in Section 3.1 (Topputo, 2013)*.

1) Using the procedure in Section 3.2, produce a first guess solution using $\alpha = 1.5\pi$, $\beta = 1.41$, $\delta = 7$, and $t_i = 0$. Plot the solution in both the rotating frame and Earth-centered inertial frame (see Appendix 1 in (Topputo, 2013)).

2) Considering the first guess in 1) and using $\{\mathbf{x}_i, t_i, t_f\}$ as variables, solve the problem in Section 3.1 with simple shooting in the following cases

   a) without providing any derivative to the solver, and

   b) by providing the derivatives and by estimating the state transition matrix with variational equations.

3) Considering the first guess solution in 1) and the procedure in Section 3.3, solve the problem with multiple shooting taking $N = 4$ and using the variational equation to compute the Jacobian of the nonlinear equality constraints.

(11 points)

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1) I apply the following formulas which are mentionned in the indicated section of the article. I then integrate the right-hand side of the PBRFBP as mentionned in the article (Equations (5) and (6), which can be found in the section 2.2 of Topputo, 2013) with initial point equal to my first guess to obtain Figures 5 and 6 which show the solution respectively in the rotating frame and in the Earth-centered frame.

I used all the constants of the Table 3 in Topputo (2013) and $(\alpha, \beta, t_i, \delta)$ as asked, and I saved them in a constants function, which is a struct with fields equal to the constants. All the formulas are adimensionnal in this exercise. I used ode113 which is recommended by MatLab for this kind of non-stiff problem.

$$r_0 = R_{Earth} + h_i, v_0 = \beta\sqrt{(1-\mu)/r_0}$$

$$x_0 = r_0 \cos\alpha, y_0 = r_0 \sin\alpha, \dot{x}_0 = -(v_0 - r_0)\sin\alpha, \dot{y}_0 = (v_0 - r_0)\cos\alpha,$$

In our case, the vector is the following :

$$[x_0, y_0, \dot{x}_0, \dot{y}_0] = [-0.0121506683, -0.0170263133, 10.7229705956, -1.9697777426e-15]$$

*F. Topputo, "On optimal two-impulse Earth–Moon transfers in a four-body model", Celestial Mechanics and Dynamical Astronomy, Vol. 117, pp. 279–313, 2013, DOI: 10.1007/s10569-013-9513-8.

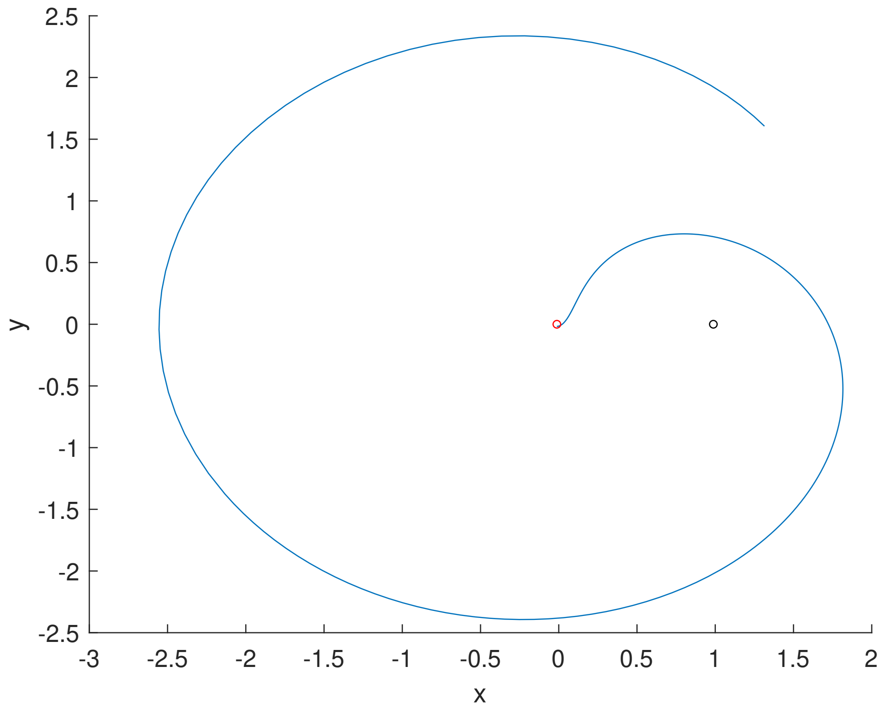

Figure 5: Solution with the provided first guess, in the rotating frame, Earth is the red circle, Moon the black one.

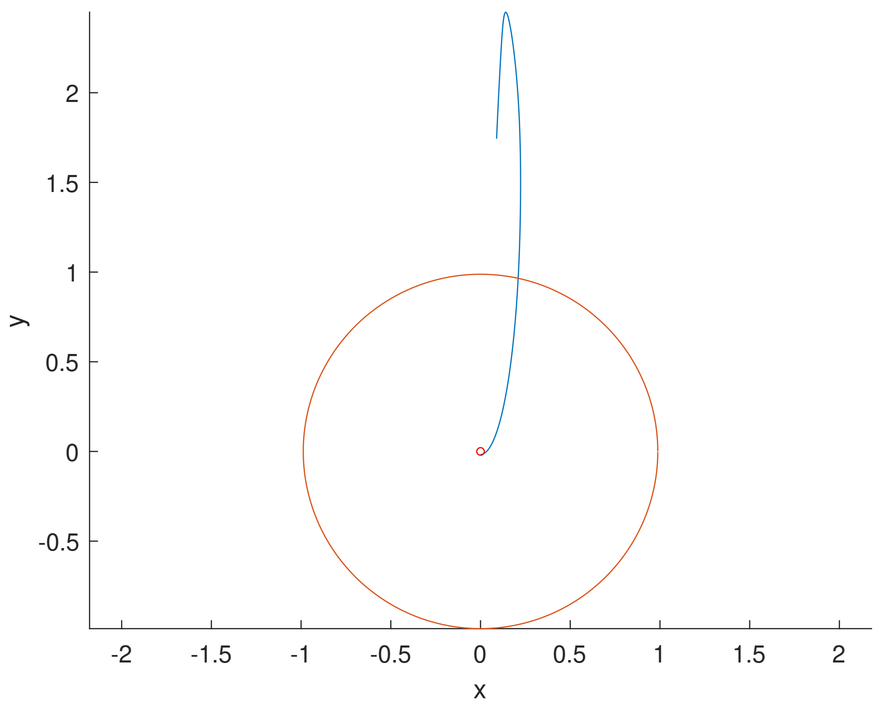

Figure 6: Solution with the provided first guess in the Earth-centered frame, Earth is the red point and the orbit of the moon is the orange circle.

2)a) The objective function to minimize is as follows, with $\underline{x}_i, r_i, \underline{x}_f, r_f$ respectively the initial and the final states and radii. The constraints are described in Appendix 1.

$$f(\underline{x}_i, t_i, t_f) = |\Delta v_i| + |\Delta v_f|$$

with $\Delta v_i = \sqrt{(\dot{x}_i - y_i)^2 + (\dot{y}_i + x_i + \mu)^2} - \sqrt{\dfrac{1-\mu}{r_i}}$;
$\Delta v_f = \sqrt{(\dot{x}_f - y_f)^2 + (\dot{y}_f + x_f + \mu - 1)^2} - \sqrt{\dfrac{\mu}{r_f}}$

Figures 7 and 8 show the two plots of the solution, respectively in the rotating frame and in the Earth-centered frame, obtained through the optimization of the first guess without providing any derivative.

$$(\underline{x}_{i,sol}, t_{i,sol}, t_{fsol}) = [-0.011755221, -0.017021720, 10.720670376, 0.249061512, 0.378414192, 8.402801128]$$

$$f(\underline{x}_{i,sol}, t_{i,sol}), t_{fsol} = 4.15781280$$

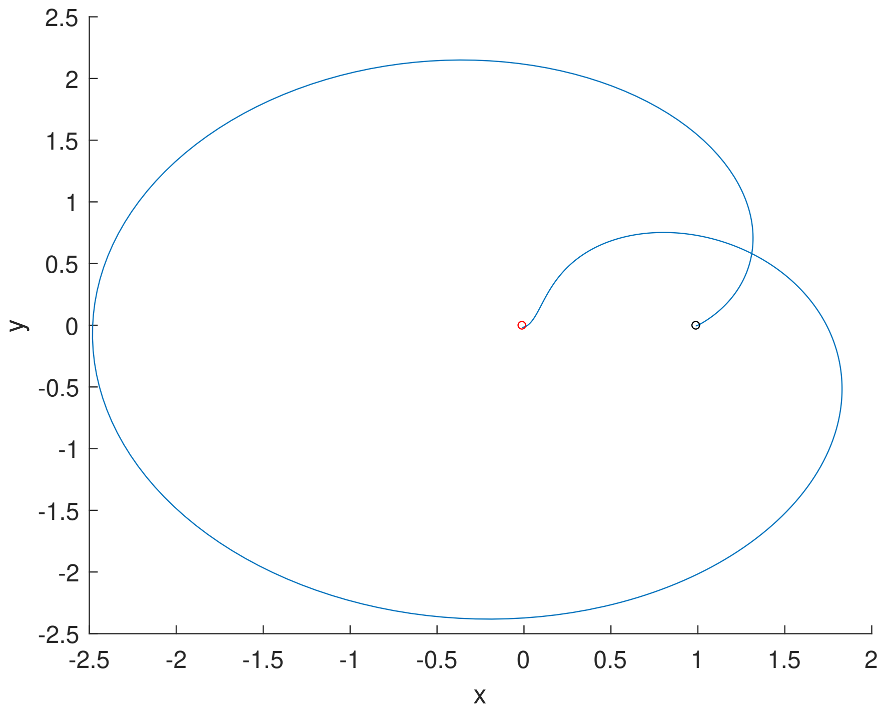

Figure 7: Optimized solution (without derivative) in the rotating frame, Earth is the red circle, Moon the black one.

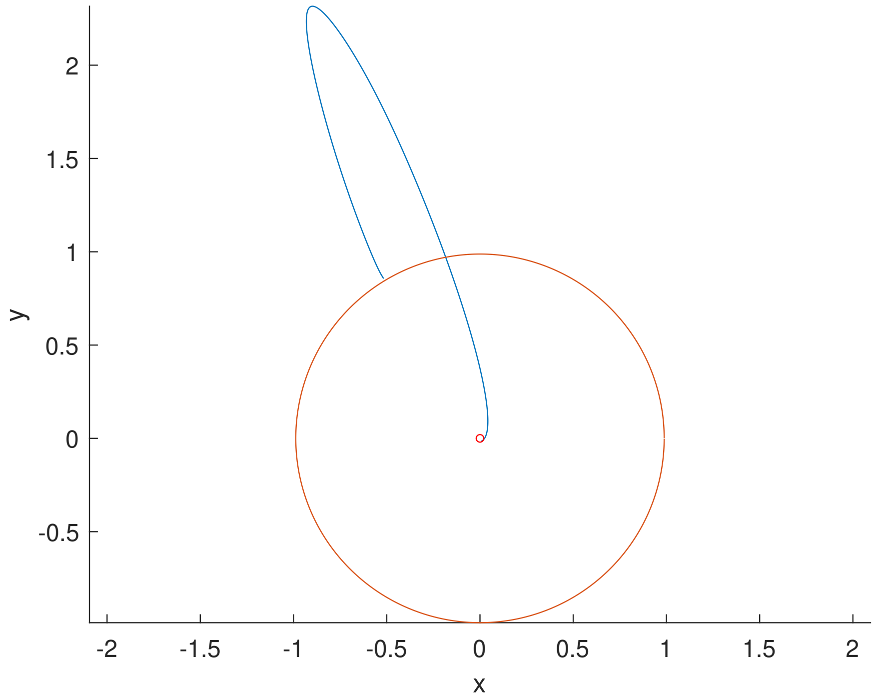

Figure 8: Optimized solution (without derivative) in the Earth-centered frame, Earth is the red circle.

2)b) To compute the gradient of the objective function, I used the definition of the gradient and derivated the value with respect to each variable making ∇f a 6-by-1 vector, as the gradient of inequality constraints. The jacobian of the equality constraints is a 6-by-4 matrix. The state transition matrix is computed with variational equations as asked. To compute the derivatives, I used the chain rule to find known derivatives, like the derivative of the flow with respect to initial time etc, which explain why we find the STM in the equations.

The expression of the derivative is given in Appendix 1. The optimized solution in the rotating frame is the same as before. I found the following results after optimization :

$$(\underline{x}_{i,sol}, t_{i,sol}, t_{f,sol}) = [-0.011604505, -0.017017551, 10.717216701, 0.343959590, -1.736977473e{-}21, 7.84385624$$

$$f(\underline{x}_{i,sol}, t_{i,sol}, t_{f,sol}) = 4.21615420$$

We can observe that the value of the function we have to minimize is higher than without derivatives. However, the algorithm converge very faster.

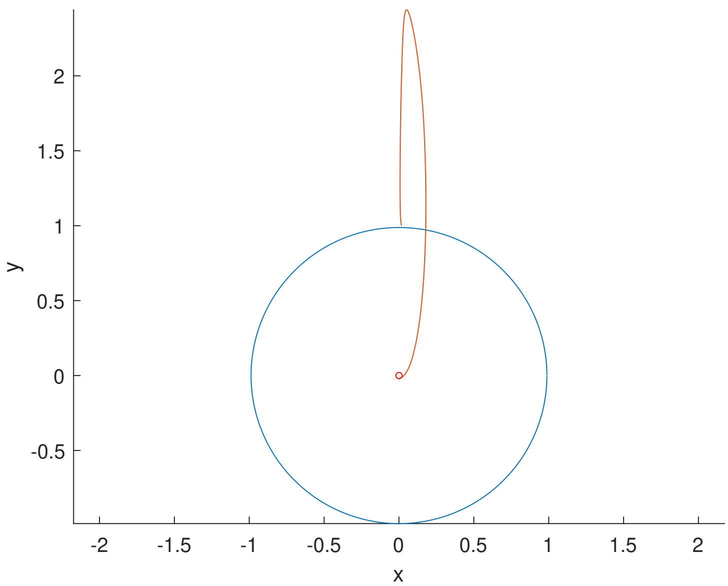

Figure 9: Optimized solution (without derivative) in the Earth-centered frame, Earth is the red circle.

3) As asked, to apply this method, I take the same first-guess generated in question 1). I will use the same notation as in (Topputo,2013) and I apply the formula of (Topputo,2019) to compute the gradient and the jacobian of the objective function and the constraints. The formulas are detailed in Appendix 2 of (Topputo, 2019). Using both gradients of objective function and constraints takes a very long time, so I got my solution using only the one of the objective function. I put 4 colors for the four pieces of the trajectory. The result is coherent with what appears on Figure 5 of Topputo,2013.

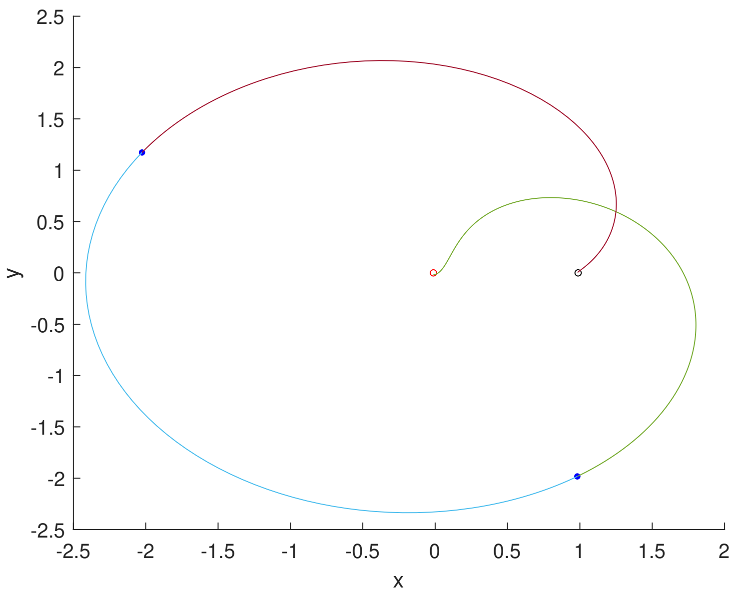

Figure 10: Optimized solution with the multiple shooting method in the rotating frame, Earth is the red circle, Moon the black one. The blue points represent the intermediate shooting point in the multiple shooting method.

3 Continuous guidance

Exercise 3

A low-thrust option is being considered for an Earth-Mars transfer†. Provide a *time-optimal* solution under the following assumptions: the spacecraft moves in the heliocentric two-body problem, Mars instantaneous acceleration is determined only by the Sun's gravitational attraction, the departure date is fixed, and the spacecraft initial and final states are coincident with those of the Earth and Mars, respectively.

1) Write down the spacecraft equations of motion, the costate dynamics, and the zero-finding problem for the unknowns $\{\boldsymbol{\lambda}_0, t_f\}$ with the appropriate transversality condition.

2) Solve the problem considering the following data:

   - Launch date: `2022-08-03-12:45:20.000 UTC`
   - Spacecraft mass: $m_0 = 1500$ kg
   - Electric properties: $T_{\max} = 150$ mN, $I_{sp} = 3000$ s
   - Number of thrusters: 4

   Report the obtained solution in terms of $\{\boldsymbol{\lambda}_0, t_f\}$ and the error with respect to the target. Validate your results exploiting the properties of time-optimal solutions.

3) Solve the problem for a degraded configuration with only 3 thrusters available, assuming that the failure occurs immediately after launch. Plot the thrust angles and compare them to the nominal case in 2).

(11 points)

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1) Taking into account the mentioned assumptions, the equations of motion of the spacecraft are basically those of the two body problem. The costate dynamics are given by the definition of the Hamiltonian. As stated in class, u is always equal to 1 in a time-optimal problem, so it does not appear in the equations. Vectors are underlined and their norms are just theirs letters not underline. It can be summed as :

$$\begin{cases} \dot{\underline{r}} = \dot{\underline{v}} \\ \dot{\underline{v}} = -\mu\dfrac{\underline{r}}{r^3} - \dfrac{T_{\max}}{m\lambda_v}\underline{\lambda_v} \\ \dot{m} = -\dfrac{T_{max}}{g_0 I_{sp}} \\ \dot{\underline{\lambda_r}} = -3\mu\dfrac{(\underline{r}\cdot\underline{\lambda_v})\underline{r}}{r^5} + \mu\dfrac{\underline{\lambda_v}}{r^3} \\ \dot{\underline{\lambda_v}} = -\dot{\underline{\lambda_r}} \\ \dot{\lambda}_m = -\dfrac{\lambda_v T_{\max}}{m^2} \end{cases}$$

With the following initial and final conditions imposed by the problem :

$$\begin{cases} m_0, t_0 \text{ fixed} \\ \underline{r}(t_0) = \underline{r}_{\text{Earth}}(t_0) \\ \underline{v}(t_0) = \underline{v}_{\text{Earth}}(t_0) \\ \underline{r}(t_f) = \underline{r}_{\text{Mars}}(t_f) \\ \underline{v}(t_f) = \underline{v}_{\text{Mars}}(t_f) \\ \lambda_m(t_f) = 0 \text{ because } m(t_f) \text{ is not fixed} \\ \text{Transversality condition because } t_f \text{ is not fixed} \end{cases}$$

†Read the necessary gravitational constants and planets positions from SPICE. Use the kernels provided on WeBeep for this assignment.

Let's detail the transversality condition which is:

$$H(t_f) - \underline{\lambda}(t_f).\dot{\underline{\psi}}(t_f) = 0$$

with, noting $(\underline{r}, \underline{v}, m) = \underline{x}$, the derivated state $\dot{\underline{x}}$ as $\underline{f}(\underline{x})$:

$$\begin{cases} \underline{\psi}(t) = \underline{x}_{\text{Mars}}(t) \\ H(t) = 1 + \underline{\lambda}(t_f).\underline{f}(t_f). \end{cases}$$

The computation of the derivative of the state of Mars is obtained with two-body problem equations, with only the acceleration of the Sun.

The zero-finding problem is the following :

$$\text{"Find } (\lambda_0, t_f) \text{ such that } \begin{cases} \underline{x}(t_f) - \underline{x}_{\text{Mars}}(t_f) = 0 \\ \lambda_m(t_f) = 0 \\ H(t_f) - \underline{\lambda}(t_f).\dot{\underline{\psi}}(t_f) = 0 \\ \text{Initial conditions} \\ \text{Equations of motions and costate dynamics} \end{cases} \text{"}$$

We have 8 unknowns and 8 conditions, which is what we wanted.

2) To solve this problem I used the following set of units : [AU, year, kg]. I implemented the previous equations and constraints mentioned before. Here are the results :

$\boldsymbol{\lambda}_{0,r}$	-4.8498349	6.8143451	-4.9143564
$\boldsymbol{\lambda}_{0,v}$	0.62856965	-0.51295006	0.85852725
$\lambda_{0,m}$	0.34998376		
t_f	`2023-07-30-07:39:55.6008 UTC`		
TOF [days]	360.78791		

Table 1: Time-optimal Earth-Mars transfer solution.

$\|\|\mathbf{r}_f(t_f) - \mathbf{r}_M(t_f)\|\|$	[km]	0.13968690
$\|\|\mathbf{v}_f(t_f) - \mathbf{v}_M(t_f)\|\|$	[m/s]	$1.4155629e-5$

Table 2: Final state error with respect to Mars' center.

4 Appendix

Appendix 1 : Derivatives of objective functions and constraints

$$\nabla f = \left(\frac{\partial f}{\partial \underline{x}_i}, \frac{\partial f}{\partial t_i}, \frac{\partial f}{\partial t_f}\right)^{\mathrm{T}}$$

with

$$\frac{\partial f}{\partial \underline{x}_i} = \frac{1}{\sqrt{((\dot{x}_i - y_i)^2 + (\dot{y}_i + x_i + \mu)^2)}}[\dot{y}_i + x_i + \mu, y_i - \dot{x}_i, -y_i + \dot{x}_i, \dot{y}_i + x_i + \mu]$$

$$\frac{\partial f}{\partial t_i} = \frac{-1}{\sqrt{((\dot{x}_f - y_f)^2 + (\dot{y}_f + x_f + \mu)^2)}}[\dot{y}_f + x_f + \mu, y_f - \dot{x}_f, -y_f + \dot{x}_f, \dot{y}_f + x_f + \mu]\mathbf{\Phi}(t_i, t_f)\underline{f}(\underline{x}_i, t_i)$$

with $\underline{x}_f$ the final state integrated with initial value $\underline{x}_i$ between t_i and t_f,$\mathbf{\Phi}$ the state transition matrix, and $\underline{f}$ the derivative of the state using the equations of motion.

$$\frac{\partial \underline{f}}{\partial t_f} = \frac{-1}{\sqrt{((\dot{x}_f - y_f)^2 + (\dot{y}_f + x_f + \mu)^2)}} [\dot{y}_f + x_f + \mu, y_f - \dot{x}_f, -y_f + \dot{x}_f, \dot{y}_f + x_f + \mu] \underline{f}(\underline{x}_f, t_f)$$

$$\nabla f = \begin{pmatrix} \frac{\partial \psi_{i1}}{\partial \underline{x}_i} & \frac{\partial \psi_{i1}}{\partial t_i} & \frac{\partial \psi_{i1}}{\partial t_f} \\ \frac{\partial \psi_{i2}}{\partial \underline{x}_i} & \frac{\partial \psi_{i1}}{\partial t_i} & \frac{\partial \psi_{i1}}{\partial t_f} \\ \frac{\partial \psi_{f1}}{\partial \underline{x}_i} & \frac{\partial \psi_{f1}}{\partial t_i} & \frac{\partial \psi_{f1}}{\partial t_f} \\ \frac{\partial \psi_{f2}}{\partial \underline{x}_i} & \frac{\partial \psi_{f1}}{\partial t_i} & \frac{\partial \psi_{f1}}{\partial t_f} \end{pmatrix}$$

with ψ_i, ψ_f as described in Topputo 2013 and 2019 and

$$\frac{\partial \psi_{i1}}{\partial \underline{x}_i} = [2(x_i + \mu), 2y_i, 0, 0]$$

$$\frac{\partial \psi_{i1}}{\partial t_i} = 0$$

$$\frac{\partial \psi_{i1}}{\partial t_f} = 0$$

$$\frac{\partial \psi_{i2}}{\partial \underline{x}_i} = \dot{x}_i, \dot{y}_i, x_i + \mu, y_i];$$

$$\frac{\partial \psi_{i2}}{\partial t_i} = 0$$

$$\frac{\partial \psi_{i2}}{\partial t_f} = 0$$

$$\frac{\partial \psi_{f1}}{\partial \underline{x}_i} = [2(x_f + \mu - 1), 2y_f, 0, 0] \mathbf{\Phi}(t_i, t_f)$$

$$\frac{\partial \psi_{f1}}{\partial t_i} = [2(x_f + \mu - 1), 2y_f, 0, 0] \mathbf{\Phi}(t_i, t_f) \underline{f}(\underline{x}_i, t_i)$$

$$\frac{\partial \psi_{f1}}{\partial t_f} = -[2(x_f + \mu - 1), 2y_f, 0, 0] \underline{f}(\underline{x}_f, t_f)$$

$$\frac{\partial \psi_{f1}}{\partial \underline{x}_i} = [\dot{x}_f, \dot{y}_f, x_f + \mu - 1, y_f] \mathbf{\Phi}(t_i, t_f)$$

$$\frac{\partial \psi_{f1}}{\partial t_i} = [\dot{x}_f, \dot{y}_f, x_f + \mu - 1, y_f] \mathbf{\Phi}(t_i, t_f) \underline{f}(\underline{x}_i, t_i)$$

$$\frac{\partial \psi_{f1}}{\partial t_f} = -[\dot{x}_f, \dot{y}_f, x_f + \mu - 1, y_f] \underline{f}(\underline{x}_i, t_i, t_f)$$