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Introduction to linear systems

$\begin{cases} x+y=2 \\ x-y=0 \end{cases}$ (2 unknowns) $\rightarrow \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 0 \end{array}\right)$

linear system   augmented matrix

MATRICES

$\left(\begin{array}{ccc|c} * & \cdots & & * \\ * & \cdots & & * \\ * & \cdots & & * \end{array}\right)$ augmented matrix

$\left(\begin{array}{cccc|c} * & \cdots & * & & * \\ 0 & \cdots & * & & * \\ 0 & \cdots & 0 & 0 & 0 \end{array}\right)$ reduced row echelon form.

inconsistent if RHS column is pivot column in these forms.

$\left(\begin{array}{cccc|c} 0 & \cdots & 1 & 0 \cdots & * \\ 0 & \cdots & 0 & 1 \cdots & * \\ 0 & \cdots & \cdots & 0 & * \end{array}\right)$ reduced row echelon form

Solutions:
- no solution - inconsistent
- infinitely many solutions - general sol^n
- unique solution - a set of constants

Special matrices

$\begin{pmatrix} a \\ b \\ c \end{pmatrix}_{3\times 1}$ $[a\ b\ c]_{1\times 3}$ vectors

zero matrices $O_{2\times 3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

square matrix $A_{3\times 3} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

diagonal matrix, $D = \begin{pmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & d_n \end{pmatrix} = \mathrm{diag}(d_1\ d_2 \dots d_n)$ → scalar matrix $c = \mathrm{diag}(c, \dots, c) = \begin{pmatrix} c & & 0 \\ & c & \\ 0 & & c \end{pmatrix}$ → identity matrix, $I_n = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$

triangular matrix

upper triangular, strictly upper triangular, lower triangular, strictly lower triangular.

symmetrical matrices
$A = A^T$

$Ax = 0$ homogeneous
a homogeneous system is always consistent.
$A0 = 0$. trivial solution is a solution. therefore

if the system has a nontrivial solution, it has infinitely many solutions.

Scalar multiplication

$cA = \begin{pmatrix} ca_{11} & \cdots & ca_{1n} \\ \vdots & \ddots & \end{pmatrix}$

addition

$a+b = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots \\ \vdots & \ddots & \end{pmatrix}$

properties of Matrix addition and scalar multiplication apply

Matrix multiplication

$a \times b = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}$ → multiplication

$a_{m\times n} b_{n\times p} = c_{m\times p}$ properties of matrix multiplication apply

$A\times B \neq B\times A$
premultiplying A to B
$I_m A = A = A I_n$
$A O_{n\times p} = O_{m\times p}$
$O_{p\times m} = O_{p\times n} A$

power of square matrices

$A^0 = I_n$
$A^n = A A^{n-1}$, $n \geq 1$

transpose

$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, $A^T = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$

dimensions of matrix are swapped.

$(cA)^T = cA^T$
$(A+B)^T = A^T + B^T$
$(AB)^T = B^T A^T$

Matrix manipulation

elementary row operations
- ↳ row swap $A([1\ 2],:) = A([2\ 1],:)$
- ↳ add multiple $A(1,:) = A(1,:) + 3*A(2,:)$
- ↳ scale by constant $A(1,:) = 3*A(1,:)$

Gaussian elimination
- bring rightmost entry to top row
- make entries under it zero (reducing to REF)
- cover top row and repeat

Gauss Jordan elimination
- make leading entries 1
- make all entries above leading entry 0.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases} \iff \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & \cdots & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

coefficient matrix   variable vector   constant vector

$a\underset{\sim}{x} = \underset{\sim}{b}$

$$a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n = b$$

Elementary matrices

$I_n \xrightarrow{r} E$, result of a single ERO.

$A \xrightarrow{r} EA$ performing ERO is just premultiplying E_k to A

$A \xrightarrow{r_1} \xrightarrow{r_2} \xrightarrow{r_3} B \iff B = E_3E_2E_1A \iff A = E_1^{-1}E_2^{-1}E_3^{-1}B$

LU factorisation

WTF: $\begin{cases} Ax = b \\ LUx = b, \text{ where } A = LU \\ Ly = b, \text{ solve for } y \\ Ux = y, \text{ solve for } x. \end{cases}$

both triangular matrices

$A = L * U$

L: Elementary matrices, 1 in diagonal entries
U: REF of A

for $A \xrightarrow{E_1} \xrightarrow{E_2} \xrightarrow{E_3} U$,

$E_3E_2E_1A = U$
$A = \underline{E_1^{-1}E_2^{-1}E_3^{-1}}U$
$= LU$

find U through only $R_i + cR_j$ ERO
cannot use row swap!!
just put $-c_i$ in the (i,j) entry

Submatrices

for matrix $A = \begin{pmatrix} a_{11} & \cdots & \cdots \\ \vdots & \ddots & \\ \vdots & \cdots & a_{mn} \end{pmatrix} \begin{matrix} \rightarrow r_1 \\ \rightarrow r_2 \\ \rightarrow r_n \end{matrix}$

$\downarrow\ \downarrow\ \downarrow\ \downarrow$
$c_1\ c_2\ c_3\ c_n$
are all submatrices.

For an A, B, submatrix A_1 and B_1
$A_1B_1 =$ rows of A_1, columns of B_1 in AB.

of matrix B, columns are $(b_1\ b_2\ b_3 \dots b_n)$
Then $AB = (Ab_1\ Ab_2\ Ab_3 \dots Ab_n)$
and
$= \begin{pmatrix} a_1B \\ a_2B \\ a_3B \\ \vdots \\ a_nB \end{pmatrix}$

For $AX = A\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \end{pmatrix}}^{B}$

★ $A\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = C_1$ $A\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = C_2$ $A\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = C_3$

since A is the same coefficient matrix for all 3 eq^{ns},
construct and solve

$(A \mid C_1 \mid C_2 \mid C_3)$

C_1 gives x_1 $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$; C_2 gives x_2 $\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$; C_3 gives x_3 $\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}$

Inverse of matrices

properties of inverse apply.
equivalent statements of invertibility apply.

the "reciprocal" of a matrix
only square matrices are invertible

Invertible if B exists such that
$AB = I_n = BA$ $AA^{-1} = I_n = A^{-1}A$ (otherwise it is a singular matrix)
the inverse is unique.

the square matrix is invertible <u>if and only if</u> $\det(A) \neq 0$

★ If A is invertible, $Ax = b$ has a unique solution. $u = A^{-1}b$
★ If A is invertible, $Ax = 0$ only has the trivial solution. if it only has trivial solution, it is invertible
★ A is invertible if it a product of elementary matrices. if RREF of A is I, then it is invertible

to find the inverse A^{-1},

- $Ax = b \rightarrow (A \mid b) \xrightarrow{RREF} (I \mid u) \rightarrow u = A^{-1}b$ unique solution for a given vector $\underset{\sim}{b}$.
- $AX = I \rightarrow (A \mid I) \xrightarrow{RREF} (I \mid A^{-1})$ if it is invertible

for $A \xrightarrow{r_1} \xrightarrow{r_2} \xrightarrow{r_3} \xrightarrow{r_4} I$
$\underbrace{E_4E_3E_2E_1}_{A^{-1}}A = I$
$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$

determinant

equivalent statements for invertibility

if $\det(A) \neq 0$, we know there is a unique solution.
if $\det(A) \neq 0$, there is an inverse
we can use $\det(A)$ in cofactor expansion to find A^{-1}

Matrix minor, $M_{11} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$
Cofactor, $A_{ij} = (-1)^{i+j}|M_{ij}|$ $\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ - & + & - & + \cdots \end{pmatrix}$

⇓

$\det(A) = \sum_{k=1}^{n} a_{kj}A_{kj}$ along a row or column.

$\det(A) = \det(A^T)$ A^T is invertible if A is invertible

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
$|A| = ad - bc$

$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ = sum of red − sum of blue

MATLAB: "det(A)"

$\mathrm{adj}(A) = \begin{pmatrix} A_{11} & A_{12} & \cdots \\ \vdots & \ddots & A_{mn} \end{pmatrix}^T$ cofactor matrix transpose

$A(\mathrm{adj}(A)) = \det(A)\,I$

$A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$

properties of determinant

$A \xrightarrow{R_i + aR_j} B$ $\det(B) = \det(A)$
$A \xrightarrow{cR_i} B$ $\det(B) = c\det(A)$
$A \xrightarrow{R_i \leftrightarrow R_j} B$ $\det(B) = -\det(A)$

$E_{(R_i + aR_j)}$, $\det(E) = 1$
$E_{(cR_j)}$, $\det(E) = c$
$E_{(R_i \leftrightarrow R_j)}$, $\det(E) = -1$

$R = E_k \cdots E_2E_1A$
$\det(R) = \det(E_k) \dots \det(E_2)\det(E_1)\det(A)$
if $R = \mathrm{diag}(d_1 d_2 \dots d_n)$,
$\det(A) = \frac{d_1 d_2 \cdots d_n}{\det(E_k) \cdots \det(E_2)(E_1)}$

for square matrices, $\det(AB) = \det(A)\det(B)$
$A = L \cdot U$
$\det(A) = \det(L)\det(U)$

$\det(A^{-1}) = \det(A)^{-1}$
$\det(cA) = c^n \det(A)$

Euclidean space, $\mathbb{R}^n = \left\{ \underset{\sim}{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \middle| v_i \in \mathbb{R} \text{ for } i=1,\dots n \right\}$

vectors algebra applies

dot product (inner product)
$u \cdot v = u_1v_1 + u_2v_2 + \dots + u_nv_n$
Matlab: dot(A,B)

norm (distance) (magnitude of vector)
for $u = \binom{x}{y}$, $\|u\| = \sqrt{x^2+y^2}$
$= u \cdot u$

for any u, $\frac{u}{\|u\|}$ is the unit vector normalising u

$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|}$

for $u_1, u_2, u_3 \dots u_k$, $c_1u_1 + c_2u_2 + \dots + c_ku_k$
is a linear combination of the vectors.
$c_1, c_2, c_3 \dots c_k$ are coefficients

SPAN

Span of $u_1, u_2, u_3 \dots u_k$ is the subset of $\mathbb{R}^n$ containing all the combinations of $u_1, u_2 \dots u_k$.

$\text{span}\{u_1, u_2 \dots u_k\} = \{c_1u_1 + c_2u_2 + c_3u_3 + \dots + c_ku_k \mid c_1, c_2, c_3 \dots c_k \in \mathbb{R}\}$

$A = (u_1 u_2 \dots u_k)$, $Ax = v$

if $Ax=v$ is consistent, $v \in \text{span}(A)$ $v = c_1u_1 + c_2u_2 + \dots + c_ku_k$

To check if $\text{span}(S) = \mathbb{R}^n$, check every $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \text{span}(S)$

$\left(S \middle| \begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right) \xrightarrow{RREF} \underbrace{(A \mid B)}_{\text{check if all } xyz \text{ will be consistent}}$

if RREF of S has no zero row, it will be consistent

$\text{span}\{e_1 e_2 e_3 \dots e_n\} = \mathbb{R}^n$, standard basis

- linear span must include $\underset{\sim}{0}_n$
- span closed under scalar multiplication.
- span closed under addition.

} definition of subspace

for $v_1 \in \text{span}(S)$
$v_2 \in \text{span}(S)$
$\vdots$
$v_m \in \text{span}(S)$ $\Big\|$ $\text{span}\{v_1 v_2 \dots v_n\} \subseteq \text{span}(S)$

for $T = \{v_1 v_2 \dots v_m\}$
if $v_i \in \text{span}(S)$, then $\text{span}(T) \subseteq \text{span}(S)$
for all $i, 1 \sim m$
if $\text{span}(T) \subseteq \text{span}(S)$ and $\text{span}(S) \subseteq \text{span}(T)$, $\text{span}(T) = \text{span}(S)$

Set of solutions of $Ax=b$,

$V = \{u \in \mathbb{R}^n \mid Au = b\}$

$V = \{\underbrace{u + s_1v_1 + s_2v_2 + \dots + s_kv_k}_{\text{general solution}} \mid s_1, s_2 \dots s_k \in \mathbb{R}\}$

solution set to homogenous system is a vector space (subspace) $\subseteq \mathbb{R}^n$,

$\text{span}\{v_1 v_2 \dots v_k\} = V = \{s_1v_1 + s_2v_2 + \dots s_kv_k \mid s_1, s_2 \dots s_k \in \mathbb{R}\}$

solution set $V = \{u \mid Au = b\}$ for $Ax=b$ is only a subspace (solution space) if system is homogenous.

$W = \{w \mid Aw = b\}$ of $Ax=b$, $b \neq 0$ is given by

$u + V = \{u + v \mid v \in V\}$
$V = \{v \mid Av = 0\}$, soln for homogenous $Au=b$