Reverse-time (and Kirchhoff) migration

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1 Preliminaries.

We consider the two-dimensional euclidian space as the medium, purported to be homogeneous ($c(x) \equiv 1$). Let us derive an analytical formula for the time-harmonic Green's function $\widehat{G}_0(\omega, x, y)$, that is the Fourier transform of the Green's function $G_0(t, x, y)$ w.r.t. the time variable t, which we recall to satisfy the *Helmholtz equation*, *i.e.*

$$\Delta_x \widehat{G}_0(\omega, x, y) + \omega^2 \widehat{G}_0(\omega, x, y) = -\delta(x - y). \tag{1.1}$$

By taking the Fourier transform $\mathscr{F}_x$ w.r.t. the space variable x of (1.1), we obtain

$$-|k_x|^2 \mathscr{F}_x \widehat{G}_0(\omega, k, y) + \omega^2 \mathscr{F}_x \widehat{G}_0(\omega, k, y) = -\exp(ik \cdot y)/(2\pi). \tag{1.2}$$

from which it follows that $\mathscr{F}_x \widehat{G}_0(\omega, k, y) = -\exp(ik \cdot y)/(2\pi[\omega^2 - |k|^2])$. Therefore, one has

$$\widehat{G}_0(\omega, x, y) = \mathscr{F}_x^{-1}[\mathscr{F}_x \widehat{G}_0](\omega, x, y) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{\exp(ik \cdot (x - y))}{|k_x|^2 - \omega^2} \mathrm{d}k. \tag{1.3}$$

Remark 1.1. *Given a function $\mathbb{R} \ni r \mapsto f(r)$, one defines the ν-th* Hankel transform *of f as*

$$\mathscr{H}_\nu[f](k) := \int_0^\infty f(r) J_\nu(kr) r dr,$$

where J_ν is the ν-th Bessel function of the first kind*, that is*

$$J_\nu(z) = \frac{(-i)^n}{2\pi} \int_0^{2\pi} \exp(iz\cos(\varphi) + in\varphi) d\varphi.$$

The ν-th Hankel transform expresses any given function f as a weighted-sum of an infinite number of rescaled *Bessel functions of the first kind J_ν. That is, in comparison with the Fourier transform, the dual variable is no longer the* frequency *as it is some* scaling factor.

Wisely rotating the frame of coordinates and carrying the computations into the polar domain yields that

$$\widehat{G}_0(\omega,x,y) = \frac{1}{(2\pi)^2}\int_0^\infty \frac{r\mathrm{d}r}{r^2-\omega^2}\left[\int_0^{2\pi} \exp\left(ir\cos(\varphi)|x-y|\right)\mathrm{d}\varphi\right] = \frac{1}{2\pi}\mathscr{H}_0\left[\frac{1}{r^2-\omega^2}\right](|x-y|).$$

It remains to compute the 0-th Hankel transform of $f : r \mapsto 1/(r^2-\omega^2)$. The neat trick here is to write

$$\mathscr{H}_0[f](k) = \int_0^\infty \mathscr{L}^{-1}[rf(r)](s)\mathscr{L}[J_0(kr)](s)\mathrm{d}s,$$

where $\mathscr{L}$ (resp. $\mathscr{L}^{-1}$) denotes the (resp. inverse) Laplace transform, and we have

$$\mathscr{L}[J_0(kr)](s) = \frac{1}{\sqrt{k^2+s^2}}, \qquad \mathscr{L}^{-1}[rf(r)] = \cos(-i\omega s),$$

so that we obtain (after change of variable $s \to s/|x-y|$)

$$\widehat{G}_0(\omega,x,y) = \frac{1}{2\pi}\int_0^\infty \frac{\cos(-i|x-y|\omega s)}{\sqrt{s^2+1}}\mathrm{d}s = \frac{1}{2\pi}K_0(-i|x-y|\omega) = \frac{i}{4}H_0(\omega|x-y|),$$

with K_0 the 0*-th modified Bessel function of the second kind*, and H_0 the 0*-th Hankel function*, where the last two equalities are "common knowledge" in the field.

2 Time-harmonic localization – full aperture.

The embedded reflector x_{ref} is modeled by a local variation $V(x)$ of the propagation speed $c(x)$ in the vicinity of the reflector, where we suppose that $V(x) = \sigma_{\mathrm{ref}} \cdot \mathbb{1}_{\Omega_{\mathrm{ref}}}(x-x_{\mathrm{ref}})$ where σ_{ref} is the target reflectivity and Ω_{ref} is the small scattering region of area ℓ_{ref}^2 which corresponds to the reflector apparatus. We assume that ℓ_{ref} is negligible w.r.t. the typical wavelength $\lambda = 2\pi/\omega$, such that we can suppose that $V(x) \simeq \sigma_{\mathrm{ref}}\ell_{\mathrm{ref}}^2 \cdot \delta(x-x_{\mathrm{ref}})$. Using Born approximation, the Green's function reads

$$\widehat{G}_{\mathrm{ref}}(\omega,x,y) = \widehat{G}_0(\omega,x,y) + \omega^2\sigma_{\mathrm{ref}}\ell_{\mathrm{ref}}^2 \cdot \widehat{G}_0(\omega,x,x_{\mathrm{ref}})\widehat{G}_0(\omega,x_{\mathrm{ref}},y).$$

We record the impulse response matrix $(\widehat{u}_{rs}(\omega))_{r,s}$, where $\widehat{u}_{rs}(\omega)$ is the time-harmonic amplitude recorded by the r-th receiver when the s-th source emits a time-harmonic signal with unit amplitude and frequency ω. Equalizing the data, that is removing the incident field (*i.e.* not scattered), we have that

$$\widehat{u}_{rs}(\omega) = \omega^2\sigma_{\mathrm{ref}}\ell_{\mathrm{ref}}^2 \cdot \widehat{G}_0(\omega,x_r,x_{\mathrm{ref}})\widehat{G}_0(\omega,x_{\mathrm{ref}},x_s), \quad \forall r,s. \tag{2.1}$$

2.1 Reverse-time imaging

Recall that the Reverse-time (RT) imaging function for the search point x^S is defined as

$$\mathscr{I}_{\mathrm{RT}}(\omega, x^S) := \frac{1}{N^2}\sum_{r,s}\widehat{G}_0(\omega, x^S, x_r)\widehat{G}_0(\omega, x_s, x^S)\overline{\widehat{u}_{rs}(\omega)} \tag{2.2}$$

If the number of transducers is high enough, *i.e.* $N \gg 1$, we can assume that we have a continuum of transducers on $\partial B(0, R_0)$, so that plugging the impulse responses (2.1) into (2.2) yields the approximation (using *reciprocity* of the time-harmonic Green's function)

$$\mathscr{I}_{\mathrm{RT}}(\omega, x^S) \simeq \omega^2 \sigma_{\mathrm{ref}} \ell_{\mathrm{ref}}^2 \cdot \left[\int_{x\in\partial B(0,R_0)} \widehat{G}_0(\omega, x^S, x)\overline{\widehat{G}_0(\omega, x_{\mathrm{ref}}, x)}\mathrm{d}\sigma(x)\right]^2.$$

Moreover, if $R_0 \gg 1$, we can invoke *Helmholtz-Kirchhoff identity*, so that we eventually have that the theoretical focal spot is given by

$$\mathscr{I}_{\mathrm{RT}}(\omega, x^S) \simeq \sigma_{\mathrm{ref}} \ell_{\mathrm{ref}}^2 \cdot \Im \widehat{G}_0(\omega, x^S, x_{\mathrm{ref}})^2 = \sigma_0 \cdot J_0^2(\omega |x^S - x_{\mathrm{ref}}|)$$

with $\sigma_0 := \sigma_{\mathrm{ref}} \ell_{\mathrm{ref}}^2/16$.

2.2 Kirchhoff migration imaging

Let us now see how to extend the Kirchhoff migration (KM) imaging to the two-dimensional setting. For large $|z|$, we have the following simple asymptotic $H_0(z) \sim \sqrt{\frac{2}{\pi z}} \cdot e^{i(z-\pi/4)}$, so that if we neglect the variations of the amplitude term in the time-harmonic Green's function $\widehat{G}_0$, one has

$$\mathscr{I}_{\mathrm{RT}}(\omega, x^S) \simeq \frac{1}{N^2}\sum_{r,s} e^{i\omega(|x^S - x_r| - |x_{\mathrm{ref}} - x_r|)} e^{i\omega(|x^S - x_s| - |x_{\mathrm{ref}} - x_s|)} = \left[\frac{1}{N}\sum_{r} e^{i\omega(|x^S - x_r| - |x_{\mathrm{ref}} - x_r|)}\right]^2.$$

Once again, if $N \gg 1$, we can assume that we have a continuum of transducers on $\partial B(0, R_0)$, so that

$$\mathscr{I}_{\mathrm{RT}}(\omega, x^S) \simeq \left(\int_{x\in\partial B(0,R_0)} \exp\left[i\omega(|x^S - x| - |x_{\mathrm{ref}} - x|)\right]\mathrm{d}\sigma(x)\right)^2.$$

If $R_0 \gg \max\{|x^S|, |x_{\mathrm{ref}}|\}$, we get from simple geometrical arguments that

$$|x^S - x| - |x_{\mathrm{ref}} - x| \simeq \langle x_{\mathrm{ref}} - x^S, R_0^{-1} x\rangle$$

for every $x \in \partial B(0,R_0)$. Carrying the computations of the contour integral into the polar domain, *i.e.* $x = (R_0\cos(\theta), R_0\sin(\theta))$, and using the fact that there exists $\phi_{S,\mathrm{ref}} \in [0,2\pi)$ such that

$$\langle x_{\mathrm{ref}} - x^S, R_0^{-1}x\rangle = |x^S - x_{\mathrm{ref}}|\cos(\theta + \phi_{S,\mathrm{ref}}), \quad \forall\theta,$$

we have

$$\int_{x\in\partial B(0,R_0)} \dots \mathrm{d}\sigma(x) \simeq \int_0^{2\pi} \exp\left[i\omega|x^S - x_{\mathrm{ref}}|\cos(\theta + \phi_{S,\mathrm{ref}})\right]\mathrm{d}\theta,$$

so that we recognize the 0-*th modified Bessel function* I_0 *of the first kind*, which we recall to have the integral representation $(2\pi)I_0(z) = \int_0^{2\pi} e^{z\cos(\theta)}\mathrm{d}\theta$, therefore yielding we can take the *theoretical* Kirchhoff migration imaging function at the search point x^S to be

$$\mathscr{I}_{\mathrm{KM}}^{(th)}(\omega, x^S) := I_0^2(i\omega|x^S - x_{\mathrm{ref}}|).$$

In practice, we don't have access to x_{ref}, since we are precisely looking for this quantity ! Therefore, a possible rough ansatz would be to simply define

$$\mathscr{I}_{\mathrm{KM}}^{(an)}(\omega, x^S) := \frac{1}{N^2}\sum_{r,s} e^{i\omega(|x^S - x_s| + |x^S - x_r|)}\overline{\widehat{u}_{rs}(\omega)}.$$

3 Time-harmonic localization – partial aperture.

Let us give an *rough approximate* proof for the exact focal spot formula in the cross-range direction (x-direction) in the partial-aperture setting. First, we assume that we have a continuum of uniformly distributed transducers on $[-R_0/2, R_0/2]$. If the height L of the reflector x_{ref} (*i.e.* $x_{\mathrm{ref}} = (0,L)$) is high enough, so that we can use Hankel's function expansion as before and neglect the variations of the amplitude term, we have the approximation

$$\mathscr{I}_{RT}(\omega, (x^S,0)) \simeq \omega^2\sigma_{\mathrm{ref}}\ell_{\mathrm{ref}}^2 \cdot \left[\int \mathbb{1}_{[-R_0/2,R_0/2]}(x_r)e^{i\omega(|x_{\mathrm{ref}} - (x_r,0)| - |x^S - x_r|)}\mathrm{d}x_r\right]^2.$$

Under the regime where $L \gg R_0$, we have that

$$\begin{aligned}|x_{\mathrm{ref}} - (x_r,0)| - |x^S - x_r| &\simeq \left\langle (x^S,0) - x_{\mathrm{ref}}, \frac{(x_r,0) - x_{\mathrm{ref}}}{|(x_r,0) - x_{\mathrm{ref}}|}\right\rangle \\ &= |(x^S,0) - x_{\mathrm{ref}}| \cdot \underbrace{\mathrm{Angle}\left((x^S,0) - x_{\mathrm{ref}}, (x_r,0) - x_{\mathrm{ref}}\right)}_{\simeq(x^S - x_r)/L}.\end{aligned}$$

Therefore, getting rid of the phase term, we obtain that

$$\begin{aligned}\mathscr{I}_{RT}(\omega,(x^S,0)) &\simeq \omega^2\sigma_{\mathrm{ref}}\ell_{\mathrm{ref}}^2\cdot\left[\int \mathbb{1}_{[-R_0/2,R_0/2]}(x_r)e^{-i\omega x_r\frac{|(x^S,0)-x_r|}{L}}\mathrm{d}x_r\right]^2\\ &= (2\pi)^2\omega^2\sigma_{\mathrm{ref}}\ell_{\mathrm{ref}}^2\cdot\left[\widehat{\mathbb{1}}_{[-R_0/2,R_0/2]}\left(\frac{|(x^S,0)-x_r|}{L}\right)\right]^2,\end{aligned}$$

and using the well-known fact that $\widehat{\mathbb{1}}_{[-R_0/2,R_0/2]}(\xi) = (\pi R_0/2)\cdot\mathrm{sinc}(\xi R_0/2)$, we roughly find the mentioned result. Note that the oscillating behavior as $|x^S|\to\infty$ is coherent with the phenomenon of *edge diffraction*.

4 Time-dependent localization – partial aperture.

5 Stability with respect to measurement noise.

We add measurement noises to our previous settings, that is, we consider that the recorded signals are of the form $\widehat{u}_{rs}(\omega)+W_{rs}^{(1)}(\omega)+iW_{rs}^{(2)}(\omega)$, where $W_{rs}^{(1)}(\omega)$ and $W_{rs}^{(2)}(\omega)$ are i.i.d. Gaussian random variables with mean zero and variance $\sigma^2/2$. Let us write $\mathscr{I}_0(\omega,x^S)$ for the unperturbated imaging function (2.2), such that the Reverse-time imaging function at the search point x^S reads

$$\mathscr{I}_{RT}(\omega,x^S) = \mathscr{I}_0(\omega,x^S)+\mathscr{I}_{\mathrm{noise}}(\omega,x^S),$$

where $\mathscr{I}_{\mathrm{noise}}(\omega,x^S)$ is the complex Gaussian random field generated by the $W_{rs}^{(1/2)}$'s.