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Existence of the Dynamic Symmetries O_4 and SU_3 for All Classical Central Potential Problems

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It is found that all classical dynamic problems (relativistic as well as non-relativistic) involving central potentials, inherently possess both O_4 and SU_3 symmetry. This leads to a generalization of both the Runge-Lenz vector in the Kepler problem and the conserved symmetric tensor in the harmonic oscillator problem. For a general central potential, an explicit construction of the elements of the Lie algebra of O_4 and SU_3 in terms of canonical variables is given. The question of a possible quantum-mechanical analog is discussed. Also, a constructive technique is given for imbedding the Lorentz group and SU_3 in an infinite-dimensional Lie algebra.

§ 1. Introduction

After the initial success of SU_3 in particle physics, various attempts were made to embed both the internal and external symmetry groups within a larger structure in order to relate different SU_3 multiplets. How the internal and external symmetry groups interlock, and whether useful results can be obtained from a finite larger algebra has been the subject of much discussion.[1] In related investigations, the whole question of internal invariance has come under close scrutiny. The internal invariances that are most clearly understood are those which arise in classical problems; for these, the invariances derive from a dynamical origin. Recently, a number of classical problems have been extensively re-investigated from the group-structure point of view. Our paper will also be primarily concerned with internal invariances of classical dynamical problems. We shall show that all such central potential problems—relativistic as well as non-relativistic—inherently have *both* O_4 and SU_3 symmetries.

An internal symmetry associated with the non-relativistic Kepler problem (which is responsible for its accidental degeneracy) has been known for a long time.[2] As a consequence of this symmetry there exists a conserved quantity—the so-called Runge-Lenz vector. The invariance group itself has been shown to be isomorphic to O_4, the four-dimensional rotation group. [A short review of the Runge-Lenz vector and the Lie algebra of O_4 in terms of Poisson brackets, rather than commutators, is given in § 2.] Recent work[3] on the Kepler problem, has been directed towards imbedding O_4 within a larger finite non-compact group, for example, the De Sitter $O(4, 1)$ group. This latter group is not an invariance group of the Hamiltonian, but rather is an invariance group for all the bound states.

Another classical problem that has a well known internal symmetry is the three dimensional isotropic harmonic oscillator.[4] In addition to the angular momentum (which is conserved for all central potential problems), there is a conserved symmetric tensor, which has some properties analogous to those of the Runge-Lenz vector.[5] The invariance group in this case is isomorphic to SU_3. [See § 3 for a brief review of the pertinent results—again, the Lie algebra will be given in terms of Poisson brackets rather than commutators.]

In the literature, the presence of O_4 symmetry for the Kepler problem, and SU_3 symmetry for the harmonic oscillator is often connected with the observation that for bound states, both of these problems have orbits that are closed. Thus, it is reasoned, the geometric property of closed orbits implies the existence of one of these invariance groups*) (*which* one does not follow unambiguously from this general argument). If this is indeed the reason for an invariance group larger than SU_2 (rotational invariance), then the existence of such a larger symmetry is restricted to those classical problems involving potentials which produce closed bound orbits. However, if the existence of such larger invariance groups is merely a reflection of the fact that there exists a constant plane of orbit in the problem, then one would expect both SU_3 *and* O_4 symmetries to exist for all central potentials. In § 4, we shall show that this is indeed the case, and we shall give an explicit method of construction for the generators in terms of canonical variables $\boldsymbol{r}$ and $\boldsymbol{p}$.

The problem of finding constants of the motion and the corresponding symmetries for classical problems having bilinear Hamiltonians has been investigated by Dulock.[7] By means of a Hopf mapping, he is able to directly transform such problems into a form recognizable as an isotropic harmonic oscillator, thus demonstrating SU_3 symmetry. He also explicitly shows that the Kepler problem has this additional SU_3 symmetry, and he displays the transcendental connection among the generators. Our results, which overlap in part with Dulock's, refer to a different class of potentials (central), are valid for relativistic mechanics, and derive from different considerations. Although purported to be a generalization of the Kepler and harmonic oscillator symmetries, the recent involved technique of Fivel[8] for constructing "approximate dynamical symmetries" for isotropic potential problems also appears to have little contact with the present work.

§ 2. The Runge-Lenz vector in the non-relativistic Kepler problem

In the Kepler problem

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*) For a recent discussion concerning closed trajectories and degeneracy, see reference 6).

$$dp/dt = -\lambda r^{-2}\hat{r} \tag{2·1}$$

where $\hat{r} = \boldsymbol{r}/r$ and $\boldsymbol{p} = md\boldsymbol{r}/dt$, the familiar integrals of the motion are the energy E and the angular momentum $\boldsymbol{L}$:*)

$$E = p^2/(2m) - \lambda/r\,, \tag{2·2}$$

$$\boldsymbol{L} = \boldsymbol{r}\wedge\boldsymbol{p}\,. \tag{2·3}$$

Also, for this problem, the vector

$$\boldsymbol{R} = \boldsymbol{p}\wedge\boldsymbol{L} - \lambda m\hat{r} \tag{2·4}$$

is conserved. Its magnitude is connected with the other two constants,

$$R^2 = 2mEL^2 + \lambda^2 m^2, \tag{2·5}$$

and its direction is in the plane of the orbit since

$$\boldsymbol{R}\cdot\boldsymbol{L} = 0\,. \tag{2·6}$$

Taking $\boldsymbol{R}$ to be the fixed direction from which the azimuthal angle is measured, we get the orbit equation directly by contracting $\boldsymbol{R}$ with the position vector $\boldsymbol{r}$:

$$\boldsymbol{R}\cdot\boldsymbol{r} = Rr\cos\theta = L^2 - \lambda mr\,. \tag{2·7}$$

So,

$$\frac{1}{r} = \frac{\lambda m}{L^2}\left[1 + \left(\frac{R}{\lambda m}\right)\cos\theta\right]. \tag{2·8}$$

Thus, we find the eccentricity is simply $R/(\lambda m)$, and that not only does $\boldsymbol{R}$ lie in the plane of the orbit, it points in the direction of the major axis.

The existence of the conserved quantities $\boldsymbol{R}$ and $\boldsymbol{L}$ is, of course, related to the symmetries inherent in the Kepler problem. The particular invariance group implied by those symmetries can be inferred from the structure of the Lie algebra of the conserved quantities. For classical quantities, the Lie algebra itself can be developed in terms of Poisson brackets**) (instead of commutators):

$$\{u, v\} \equiv \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial x_i}\,. \tag{2·9}$$

Conserved quantities are just those expressions (not explicitly functions of time) whose Poisson brackets with the energy are zero.

In order to get a closed algebra, it is convenient to consider the Runge-Lenz vector $\boldsymbol{A}$

$$\boldsymbol{A} = (-2mE)^{-1/2}\boldsymbol{R} \tag{2·10}$$

instead of $\boldsymbol{R}$. For negative energies, $\boldsymbol{A}$ is a real vector. Performing the nec-

*) Here, the vector cross product is indicated by the notation $\boldsymbol{a}\wedge\boldsymbol{b}$. Also, a quantity under a caret will uniformly indicate a unit vector.

**) We employ summation convention. Latin indices range from 1 to 3, Greek indices from 1 to 4.

essary differentiations, we establish that the Lie algebra is:

$$\{\boldsymbol{A}, E\} = \{\boldsymbol{L}, E\} = 0\,, \tag{2·11}$$

$$\{L_i, L_j\} = \varepsilon_{ijk} L_k\,, \tag{2·12}$$

$$\{L_i, A_j\} = \varepsilon_{ijk} A_k\,, \tag{2·13}$$

$$\{A_i, A_j\} = \varepsilon_{ijk} L_k\,. \tag{2·14}$$

We see that the six quantities $\boldsymbol{A}$ and $\boldsymbol{L}$ have a closed algebra. Moreover, this algebra is identical to that for the generators of the group O_4—the orthogonal group of rotations in four dimensions.

In order to demonstrate this, consider a real 4-dimensional space with variables x_1, x_2, x_3, x_4 and p_1, p_2, p_3, p_4, which has as rotation generators

$$W_{\alpha\beta} = x_\alpha p_\beta - x_\beta p_\alpha\,. \qquad (\alpha, \beta = 1, \cdots 4) \tag{2·15}$$

Then in this space the Poisson brackets are

$$\{u, v\}_4 \equiv \frac{\partial u}{\partial x_\lambda}\frac{\partial v}{\partial p_\lambda} - \frac{\partial u}{\partial p_\lambda}\frac{\partial v}{\partial x_\lambda}\,. \tag{2·16}$$

Setting

$$\mathcal{L}_i = \tfrac{1}{2}\varepsilon_{ijk} W_{jk}\,, \qquad \mathcal{A}_i = W_{i4}\,, \tag{1·17}$$

one finds that

$$\{\mathcal{L}_i, \mathcal{L}_j\}_4 = \varepsilon_{ijk}\mathcal{L}_k\,, \tag{2·18}$$

$$\{\mathcal{L}_i, \mathcal{A}_j\}_4 = \varepsilon_{ijk}\mathcal{A}_k\,,$$

$$\{\mathcal{A}_i, \mathcal{A}_j\}_4 = \varepsilon_{ijk}\mathcal{L}_k\,.$$

Thus, the algebra for O_4 is isomorphic to that we had found for $\boldsymbol{A}$ and $\boldsymbol{L}$. This discussion also shows, that by taking as elements, $\boldsymbol{L}$, $i\boldsymbol{A}$ (instead of simply $\boldsymbol{A}$), one in effect makes the 4-th dimension imaginary. Consequently, those elements describe rotation in such a space, and hence are the generators of the Lorentz transformation. This is called the "unitary trick" for changing O_4 into L_4. In the Kepler prolem for positive energies, $i\boldsymbol{A} = (2mE)^{-1/2}\boldsymbol{R}$ yields a real vector.

In general, $\boldsymbol{A}$ (like $\boldsymbol{R}$) has the physical significance of a constant vector in the plane of the orbit pointing along the major axis (for bound states). It follows that $\boldsymbol{L}\wedge\boldsymbol{A}$ is also a constant vector in the plane of the orbit, which points in the direction of the minor axis. In fact, the quantity

$$c_1\boldsymbol{A} + c_2\boldsymbol{L}\wedge\boldsymbol{A}$$

for c_1, c_2 being arbitrary numbers, is itself a constant vector in the plane of the orbit which points in an arbitrary direction (determined by choice of c_1, c_2). If the implication of the Runge-Lenz vector is only that there is a constant plane of the orbit, then since this is the case for all central potentials, one should be able to find an analogous Runge vector for all such potentials.

§ 3. SU_3 for the non-relativistic isotropic three-dimensional harmonic oscillator

In the oscillator problem,

$$d\boldsymbol{p}/dt = -(\lambda^2/m)\boldsymbol{r}\,, \tag{3·1}$$

where again $\boldsymbol{p}=m d\boldsymbol{r}/dt$, the familiar integrals of the motion are the energy E and the angular momentum $\boldsymbol{L}$:

$$E=(2m)^{-1}(p^2+\lambda^2 r^2)\,, \tag{3·2}$$

$$\boldsymbol{L}=\boldsymbol{r}\wedge\boldsymbol{p}\,. \tag{3·3}$$

Also, in this problem, the symmetric tensor (six components)

$$\mathbf{A}_{ij}=\frac{1}{\lambda}(p_i p_j+\lambda^2 r_i r_j) \tag{3·4}$$

is conserved. Its trace is just proportional to the energy,

$$\text{trace}\,\mathbf{A}=(2m/\lambda)E\,, \tag{3·5}$$

so the other five components represent the independent conserved quantities.

The tensor $\mathbf{A}$ is in the plane of the orbit (i.e., perpendicular to the angular momentum), in the sense that

$$\mathbf{A}_{ij}L_j=0\,. \tag{3·6}$$

Also, the orbit equation itself is obtained from $\mathbf{A}$ by means of complete contraction with the position vector r:

$$r_i\mathbf{A}_{ij}r_j=\lambda^{-1}(2mEr^2-L^2)\,. \tag{3·7}$$

For the bound solution

$$\boldsymbol{r}=D_1\cos(\lambda t/m)\,\hat{x}_1+D_2\sin(\lambda t/m)\,\hat{x}_2\,, \tag{3·8}$$

where

$$2mE=\lambda^2(D_1^2+D_2^2)\,,\qquad L^2=(\lambda D_1D_2)^2, \tag{3·9}$$

the tensor $\mathbf{A}$ is diagonal,

$$\mathbf{A}=\begin{pmatrix}\lambda D_1^2 & 0 & 0\\ 0 & \lambda D_2^2 & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{3·10}$$

and for this special coordinate system, the orbit equation is simply

$$(r_1/D_1)^2+(r_2/D_2)^2=1\,. \tag{3·11}$$

The Lie algebra involving $\mathbf{A}$ is found using the Poisson brackets. The results are:

$$\{\mathbf{A}, E\} = \{\boldsymbol{L}, E\} = 0\,, \tag{3·12}$$

$$\{L_i, L_j\} = \varepsilon_{ijk} L_k\,, \tag{3·13}$$

$$\{L_i, \mathbf{A}_{jk}\} = \varepsilon_{ijn}\mathbf{A}_{nk} + \varepsilon_{ikn}\mathbf{A}_{jn}\,, \tag{3·14}$$

$$\begin{aligned}\{\mathbf{A}_{ij}, \mathbf{A}_{kl}\} = (&\delta_{ij}\varepsilon_{jln} + \delta_{il}\varepsilon_{jkn} \\ &+ \delta_{jk}\varepsilon_{iln} + \delta_{jl}\varepsilon_{ikn}) L_n\,.\end{aligned} \tag{3·15}$$

The five (traceless) components of $\mathbf{A}$ and the 3 components of $\boldsymbol{L}$ yield a closed algebra—the algebra of SU_3. In standard form, the algebra may be displayed by setting

$$\begin{aligned}
\mathbf{H}_1 &= (2\sqrt{3})^{-1} L_3\,, \\
\mathbf{H}_2 &= (12)^{-1}(\mathbf{A}_{11} + \mathbf{A}_{22} - 2\mathbf{A}_{33})\,, \\
\mathbf{E}_\varepsilon{}^\sigma &= (4\sqrt{3})^{-1}(L_1 - \sigma\mathbf{A}_{13} + i\varepsilon L_2 - i\varepsilon\sigma\mathbf{A}_{23})\,, \\
\mathbf{E}_{2\varepsilon} &= (4\sqrt{6})^{-1}(\mathbf{A}_{11} - \mathbf{A}_{22} + 2i\varepsilon\mathbf{A}_{12})\,,
\end{aligned} \tag{3·16}$$

where ε and σ are independently ± 1. Then, the bracket algebra of SU_3 is the following:

$$\begin{aligned}
&\{\mathbf{H}_1, \mathbf{H}_2\} = 0\,, \\
&\{\mathbf{H}_1, \mathbf{E}_\varepsilon{}^\sigma\} = -i\varepsilon(2\sqrt{3})^{-1}\mathbf{E}_\varepsilon{}^\sigma, \\
&\{\mathbf{H}_2, \mathbf{E}_\varepsilon{}^\sigma\} = i\varepsilon\sigma(1/2)\mathbf{E}_\varepsilon{}^\sigma, \\
&\{\mathbf{H}_1, \mathbf{E}_{2\varepsilon}\} = -i\varepsilon(\sqrt{3})^{-1}\mathbf{E}_{2\varepsilon}\,, \\
&\{\mathbf{H}_2, \mathbf{E}_{2\varepsilon}\} = 0\,, \\
&\{\mathbf{E}_\varepsilon{}^\sigma, \mathbf{E}_{-\varepsilon}^{-\sigma}\} = \{\mathbf{E}_\varepsilon{}^\sigma, \mathbf{E}_{2\varepsilon}\} = 0\,, \\
&\{\mathbf{E}_\varepsilon{}^\sigma, \mathbf{E}_{-\varepsilon}^{\sigma}\} = -i\varepsilon(2\sqrt{3})^{-1}\mathbf{H}_1 - \tfrac{1}{2}i\varepsilon\sigma\mathbf{H}_2\,, \\
&\{\mathbf{E}_{2\varepsilon}, \mathbf{E}_{-2\varepsilon}\} = -i\varepsilon(\sqrt{3})^{-1}\mathbf{H}_1\,, \\
&\{\mathbf{E}_\varepsilon{}^\sigma, \mathbf{E}_\varepsilon{}^{-\sigma}\} = i\varepsilon\sigma(\sqrt{6})^{-1}\mathbf{E}_{2\varepsilon}\,, \\
&\{\mathbf{E}_\varepsilon{}^\sigma, \mathbf{E}_{-2\varepsilon}\} = -i\varepsilon\sigma(\sqrt{6})^{-1}\mathbf{E}_{-\varepsilon}^{-\sigma}\,.
\end{aligned} \tag{3·17}$$

Since $\mathbf{A}$ is "perpendicular" to $\boldsymbol{L}$, one can find another conserved symmetric tensor, say

$$\mathbf{T}_{sj} = \varepsilon_{iks}\mathbf{A}_{ij} L_k + \varepsilon_{ijk}\mathbf{A}_{is} L_k\,,$$

that is also "perpendicular" to $\boldsymbol{L}$. In general, the quantity

$$c_1\mathbf{A} + c_2\mathbf{T}$$

for c_1, c_2 being arbitrary numbers, is itself a constant symmetric tensor "perpendicular" to the fixed momentum direction $\boldsymbol{L}$. If the implication of the SU_3 symmetry for the harmonic oscillator is only that there is a symmetric tensor, diagonal in the constant plane of the orbit, then since this plane exists for all central potentials, one should be able to find an analogous symmetry for all such potentials.

§ 4. Symmetries inherent in all central potentials

A feature of central potential problems is that there exists a fixed plane of orbit. How can we describe a constant vector in the plane of the orbit in terms of canonical variables $\boldsymbol{r}$ and $\boldsymbol{p}$. In this section, we shall develop an almost trivial technique for constructing such a vector. This vector will then be employed in the construction of O_4 and SU_3 algebras.

Let $\hat{k}$ (hereafter referred to as a "unit Runge vector") be a unit vector in the plane of the orbit. Then by resolution along three orthogonal directions $\hat{r}$, $\hat{L}$, and $\hat{r}\wedge\hat{L}$,

$$\hat{k}=(\hat{k}\cdot\hat{r})\hat{r}+(\hat{k}\cdot\hat{L})\hat{L}+(\hat{k}\cdot\hat{r}\wedge\hat{L})\hat{r}\wedge\hat{L}\,. \tag{4·1}$$

However,

$$\hat{k}\cdot\hat{L}=0\,, \tag{4·2}$$

by hypothesis. Also, if one chooses $\hat{k}$ as the direction from which to measure the azimuthal angle θ (the positive sense given by a right-handed rotation about $\hat{L}$), then

$$(\hat{r}\cdot\hat{k})=\cos\theta\,;\;\;(\hat{k}\cdot\hat{r}\wedge\hat{L})=\sin\theta\,. \tag{4·3}$$

Consequently,

$$\hat{k}=(\cos\theta)\hat{r}+(\sin\theta)\hat{r}\wedge\hat{L}\,. \tag{4·4}$$

Now, if we can write $\cos\theta$, $\sin\theta$ in terms of $\boldsymbol{r}$ and $\boldsymbol{p}$, then we will have expressed a constant unit vector in the plane of the orbit in terms of canonical variables. In order to do this, we turn to the equations of motion for a general central potential.*)

For a central potential $V(r)$,**) the equations of motion for the relativistic classical problem are:

$$d\boldsymbol{p}/dt=-(dV/dr)\hat{r}\,, \tag{4·5}$$

where

*) An analogous technique of obtaining non-trivial results from a mathematical identity by substituting the equations of motion at a critical point, has been used by F. Calogero (unpublished) to establish bilinear and non-bilinear conservation laws in field theory. The author gratefully acknowledges numerous conversations with Professor Calogero relative to the present problem.

**) Hereafter, we shall supress the functional dependence and simply write $V(r)\equiv V$.

$$\boldsymbol{p} = m\gamma(d\boldsymbol{r}/dt) \tag{4·6}$$

and

$$\gamma = [1 - c^{-2}(d\boldsymbol{r}/dt)\cdot(d\boldsymbol{r}/dt)]^{-1/2}. \tag{4·7}$$

The familiar integrals of the motion are the energy $\mathcal{E}$ and the angular momentum $\boldsymbol{L}$:

$$\mathcal{E} = mc^2\gamma + V, \tag{4·8}$$

$$\boldsymbol{L} = \boldsymbol{r}\wedge\boldsymbol{p}. \tag{4·9}$$

With the azimuthal angle θ defined to be positive for right-handed rotations about $\hat{L}$ [as in Eq. (4·3)], the magnitude $L = (\boldsymbol{L}\cdot\boldsymbol{L})^{1/2}$ is given by

$$L = m\gamma r^2(d\theta/dt). \tag{4·10}$$

Also, from Eqs. (4·6) and (4·7) it follows that

$$(mc)^2(\gamma^2 - 1) = \boldsymbol{p}\cdot\boldsymbol{p} = (m\gamma)^2[(dr/dt)^2 + r^2(d\theta/dt)^2], \tag{4·11}$$

$$(\boldsymbol{r}\cdot\boldsymbol{p}) = m\gamma r(dr/dt). \tag{4·12}$$

From these equations and Eq. (4·5) follow the differential equation for the orbit,

$$\left(\frac{du}{d\theta}\right)^2 = \left\{\left(\frac{mc}{L}\right)^2\left[\left(\frac{\mathcal{E}-V}{mc^2}\right)^2 - 1\right] - u^2\right\} \tag{4·13}$$

and the relation

$$(du/d\theta) = -(\hat{r}\cdot\boldsymbol{p})/L. \tag{4·14}$$

Here and in the following we change from the variable r to its reciprocal

$$u = 1/r. \tag{4·15}$$

From Eq. (4·13), it follows that*)

$$\theta = L(mc)^{-1}\int^u h^{-1/2}(u)\,du, \tag{4·16}$$

where

$$h(u) = [(\mathcal{E}-V)/(mc^2)]^2 - 1 - [(Lu)/(mc)]^2, \tag{4·17}$$

and the integration constant is contained in the arbitrary choice of reference line for measurement of the angle θ. The ambiguity in the sign in front of the square root in Eq. (4·16) may also be subsumed into the choice of the arbitrary reference line.**)

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*) See, for example reference 9) for the analogous development of $(du/d\theta)^2$ and $\theta = \theta(u, L^2, \mathcal{E})$ for the non-relativistic central force problem. A relativistic treatment for the Coulomb potential, which is easily generalized to any central potential, is given in reference 10).

**) The *identical* argument to this is given by Landau and Lifshitz in reference 11).

The function $\theta=\theta(u, L^2, \mathcal{E})$ given by the integral displayed in Eq. (4·16) may or may not (the last possibility most likely!) represent some well tabulated function; however, the availability of values for this function is unimportant for the following discussion. Also, in general, the function θ will not be periodic with a periodicity of 2π.*) But, such periodicity is not essential for the following construction of the elements of the desired algebras, and we shall not assume such "single-valuedness" here.

From Eq. (4·16), it is obvious that we may obtain the function

$$\cos\theta = f(u, L^2, \mathcal{E}) \equiv f. \tag{4·18}$$

In other words, $\cos\theta$ is expressed in terms of canonical variables $\boldsymbol{r}$ and $\boldsymbol{p}$, since the arguments u, L^2 and $\mathcal{E}$ are themselves functions of these canonical variables.

Also, from Eq. (4·14), it follows that

$$\sin\theta = (\partial f/\partial u)\,(\hat{r}\cdot\boldsymbol{p})/L. \tag{4·19}$$

Substituting these expressions into the form of the unit Runge vector [Eq. (4·4)], we obtain

$$\hat{k} = [f - u(\partial f/\partial u)]\hat{r} + L^{-2}(\partial f/\partial u)\,\boldsymbol{p}\wedge\boldsymbol{L}, \tag{4·20}$$

where f is that function given in Eq. (4·18). This expression is then the desired unit vector expressed in terms of canonical variables.

It can be shown that the function f satisfies the partial differential equation,

$$\left(\frac{\mathcal{E}-V}{c^2}\right)\left(\frac{dV}{du}\right)\frac{\partial f}{\partial u} - \left[\left(\frac{\mathcal{E}-V}{c}\right)^2 - (mc)^2 - L^2u^2\right]\frac{\partial^2 f}{\partial u^2}$$
$$- L^2\left(f - u\frac{\partial f}{\partial u}\right) = 0, \tag{4·21}$$

and also since $\sin^2+\cos^2=1$, it follows that

$$f^2 + \left(\frac{\partial f}{\partial u}\right)^2 \frac{(\hat{r}\cdot\boldsymbol{p})^2}{L^2} = 1, \tag{4·22}$$

$$2f\left(\frac{\partial f}{\partial L^2}\right) + \frac{2(\hat{r}\cdot\boldsymbol{p})^2}{L^2}\left(\frac{\partial f}{\partial u}\right)\left(\frac{\partial^2 f}{\partial L^2\partial u}\right) - \frac{p^2}{L^4}\left(\frac{\partial f}{\partial u}\right)^2 = 0. \tag{4·23}$$

After much tedious algebra, involving the actual differentiation required by the Poisson bracket operation, it has been explicitly verified that the unit vector $\hat{k}$ exhibited in Eq. (4·20) satisfies:

$$\{\mathcal{E}, \hat{k}\} = 0, \tag{4·24}$$

*) Periodicity of 2π corresponds to the case of a closed orbit. Szymacha and Werle[12] have shown that if one requires such periodicity, then only the Kepler and oscillator problems lead to O_4 and SU_3 algebras, respectively. Such a restriction, though appealing, is not necessary *a priori* in the construction of the relevant elements of the algebras.

$$\{\hat{k}_i,\ \hat{k}_j\}=0\,. \tag{4·25}$$

Of course, these relations are to be expected from the fact that $\hat{k}$, in the form given before appropriate substitution in terms of canonical variables, is indeed a constant unit vector [see Eq. (4·1)]. Further tedious algebra, also involving the actual differentiation required by the Poisson bracket operation, explicitly shows that the unit vector $\hat{k}$ exhibited in Eq. (4·20), which is expressed in terms of canonical variables, also satisfies

$$\{L_i,\ \hat{k}_j\}=\varepsilon_{ijk}\hat{k}_k\,. \tag{4·26}$$

Again, this relation is to be expected since $\hat{k}$ as expressed by Eq. (4·20), obviously transforms as a vector.

Now, we may multiply the unit Runge vector by any function of L^2 and $\mathcal{E}$, and the result will still be a conserved vector. By a judicious choice of multiplying function, one may obtain a Lie algebra isomorphic to O_4 (and by taking a tensor combination of the $\hat{k}$'s, a Lie algebra isomorphic to SU_3). Accordingly, from the construction

$$\begin{aligned}&\boldsymbol{A}=A\hat{k}\,, \qquad (4·27)\\ &\boldsymbol{B}=B\boldsymbol{L}\wedge\hat{k}\,,\\ &\mathbf{A}_{ij}=F\hat{k}_i\hat{k}_j+G(\hat{L}\wedge\hat{k})_i(\hat{L}\wedge\hat{k})_j\,,\end{aligned}$$

where A, B, F and G are as yet undetermined functions of $\mathcal{E}$ and L^2, the vector and tensor character is displayed by the bracket relations

$$\begin{aligned}&\{L_i,\ A_j\}=\varepsilon_{ijk}A_k\,, \qquad (4·28)\\ &\{L_i,\ B_j\}=\varepsilon_{ijk}B_k\,,\\ &\{L_i,\ \mathbf{A}_{jk}\}=\varepsilon_{ijk}\mathbf{A}_{nk}+\varepsilon_{ikn}\mathbf{A}_{jn}.\end{aligned}$$

Our constructed quantities have internal bracket relations:

$$\{A_i,\ A_j\}=-\varepsilon_{ijk}(\partial A^2/\partial L^2)L_k\,, \tag{4·29}$$

$$\{B_i,\ B_j\}=-\varepsilon_{ijk}(\partial B^2/\partial L^2)L_k\,, \tag{4·30}$$

$$\begin{aligned}\{\mathbf{A}_{ij},\ \mathbf{A}_{kl}\}=&(\varepsilon_{ikn}\delta_{jb}\delta_{lc}+\varepsilon_{jln}\delta_{ib}\delta_{kc} \qquad (4·31)\\ &+\varepsilon_{iln}\delta_{jb}\delta_{kc}+\varepsilon_{jkn}\delta_{ib}\delta_{lc})L_n\ \text{times}\\ &\{(FG/L^2)\delta_{bc}+[(G-F)(\partial F/\partial L^2)-(FG/L^2)]\hat{k}_b\hat{k}_c\\ &-L^{-2}[(G-F)(\partial G/\partial L^2)+(FG/L^2)](\hat{L}\wedge\hat{k})_b(\hat{L}\wedge\hat{k})_c\}\,.\end{aligned}$$

For the choice of the undetermined functions which makes

$$\boldsymbol{A}=[P^2(\mathcal{E})-L^2]^{1/2}\hat{k}\,, \tag{4·32}$$

$$\boldsymbol{B}=[P^2(\mathcal{E})-L^2]^{1/2}\hat{L}\wedge\hat{k}\,, \tag{4·33}$$

$$\mathbf{A}_{ij}=\{P(\mathcal{E})\mp[P^2(\mathcal{E})-L^2]^{1/2}\hat{k}_i\hat{k}_j$$
$$+\{P(\mathcal{E})\pm[P^2(\mathcal{E})-L^2]^{1/2}\}(\hat{L}\wedge\hat{k})_i(\hat{L}\wedge\hat{k})_j\,, \tag{4·34}$$

where $P(\mathcal{E})$ is an arbitrary function of $\mathcal{E}$ (but *not* L^2),*) the bracket relations of Eqs. (4·29) to (4·31) become simply:

$$\{A_i,\ A_j\}=\varepsilon_{ijk}L_k\,, \tag{4·35}$$

$$\{B_i,\ B_j\}=\varepsilon_{ijk}L_k\,, \tag{4·36}$$

$$\{\mathbf{A}_{ij},\ \mathbf{A}_{kl}\}=(\varepsilon_{ikn}\delta_{jl}+\varepsilon_{jln}\delta_{ik}$$
$$+\varepsilon_{iln}\delta_{jk}+\varepsilon_{jkn}\delta_{il})L_n\,. \tag{4·37}$$

The quantities ($\boldsymbol{L}$, $\boldsymbol{A}$), ($\boldsymbol{L}$, $\boldsymbol{B}$) and ($\boldsymbol{L}$, $\mathbf{A}$) separately give closed Lie algebras. Reference to Eqs. (2·12) to (2·14) shows that ($\boldsymbol{L}$, $\boldsymbol{A}$) and ($\boldsymbol{L}$, $\boldsymbol{B}$) are separately generators of O_4, and reference to Eq. (3·15) shows that $\boldsymbol{L}$ and the five (traceless) components of $\mathbf{A}$ yield an algebra isomorphic to SU_3. Thus, we have found a generalization of the Runge vector for all (relativistic) central potential problems, and demonstrated that these problems have dynamical symmetries given by O_4 and SU_3.

§ 5. Reduction to the non-relativistic limit and some special cases

The non-relativistic limit of the preceding expressions may be obtained by setting

$$E=\mathcal{E}-mc^2 \tag{5·1}$$

then letting

$$p/(mc)\to 0\,,\qquad (E-V)/(mc^2)\to 0\,. \tag{5·2}$$

Then instead of Eq. (4·8), one obtains

$$E=\frac{p^2}{2m}+V\,, \tag{5·3}$$

and the differential equation for the orbit [Eq. (4·13)] becomes

$$(du/d\theta)^2=[(2m/L^2)(E-V)-u^2]. \tag{5·4}$$

For the *non-relativistic Kepler problem,*

$$V=-\lambda u \tag{5·5}$$

(when λ is positive, the force is attractive), the orbit equation may be integrated in terms of familiar functions. The result is

$$f\equiv\cos\theta=[2mEL^2+(\lambda m)^2]^{-1/2}(L^2u-\lambda m)\,. \tag{5·6}$$

*) For what follows, it is not necessary that the symbol $P(\mathcal{E})$ used in $\boldsymbol{A}$, $\boldsymbol{B}$ and $\mathbf{A}$ represents the same function for the three cases $\boldsymbol{A}$, $\boldsymbol{B}$ and $\mathbf{A}$.

For this case, choosing

$$P(\mathcal{E})=[-\lambda^2 m/(2E)]^{1/2}, \tag{5·7}$$

we obtain with use of Eqs. (4·20) and (4·32) the usual Runge-Lenz vector

$$\boldsymbol{A}=(-2mE)^{-1/2}(\boldsymbol{p}\wedge\boldsymbol{L}-\lambda m\hat{r}). \tag{5·8}$$

Also, from Eq. (4·34) we obtain a rather complicated expression for the symmetric tensor*)

$$\begin{aligned}\mathbf{A}_{ij}=&[P(\mathcal{E})\pm(P^2(\mathcal{E})-L^2)^{1/2}](\delta_{ij}-\hat{L}_i\hat{L}_j)\\ &\mp(-2mE)^{-1}[P^2(\mathcal{E})-L^2]^{-1/2}\{-(\boldsymbol{r}\cdot\boldsymbol{p})[(\boldsymbol{p}\wedge\boldsymbol{L})_i p_j+(\boldsymbol{p}\wedge\boldsymbol{L})_j p_i]\\ &+2mEu^2(\boldsymbol{r}\cdot\boldsymbol{p})[(\boldsymbol{r}\wedge\boldsymbol{L})_i r_j+(\boldsymbol{r}\wedge\boldsymbol{L})_j r_i]\\ &-4mEu^2[P^2(\mathcal{E})-L^2]r_i r_j\}.\end{aligned} \tag{5·9}$$

For the *non-relativistic three-dimensional isotropic harmonic oscillator*

$$V=\lambda^2/(2mu^2), \tag{5·10}$$

here also the orbit equation may be integrated in terms of familiar functions. The result is:

$$\begin{aligned}f\equiv\cos\theta=&[4(m^2E^2-\lambda^2L^2)]^{-1/4}\text{ times}\\ &\{L^2u^2-mE+[m^2E^2-\lambda^2L^2]^{1/2}\}^{1/2}.\end{aligned} \tag{5·11}$$

Choosing

$$P(\mathcal{E})=(mE/\lambda) \tag{5·12}$$

and the top sign in Eq. (4·34), we retrieve [using Eqs. (4·20) and (4·34)] the symmetric tensor

$$\mathbf{A}_{ij}=\frac{1}{\lambda}(p_i p_j+\lambda^2 r_i r_j). \tag{5·13}$$

Also, from Eqs. (4·20) and (4·32) the Runge-Lenz vector turns out to be

$$\boldsymbol{A}=(A/2\lambda)^{1/2}[L^2u^2-mE+\lambda A]^{-1/2}\{(-mE+\lambda A)\hat{r}+u(\boldsymbol{p}\wedge\boldsymbol{L})\}, \tag{5·14}$$

where the magnitude A is

$$A=[(mE/\lambda)^2-L^2]^{1/2}. \tag{5·15}$$

For the *relativistic Kepler problem*

$$V=-\lambda u, \tag{5·16}$$

the orbit equation [Eq. (4·16)] is integrable:

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*) Here we have used the fact that

$$\hat{k}_i\hat{k}_j+(\hat{L}\wedge\hat{k})_i\,(\hat{L}\wedge\hat{k})_j=\delta_{ij}-\hat{L}_i\hat{L}_j.$$

$$\cos[(1-\eta^2)^{1/2}\theta] = [(\mathcal{E}/c^2)^2 - m^2(1-\eta^2)]^{-1/2} \text{ times} \qquad (5\cdot 17)$$
$$\{(Lu/c)(1-\eta^2) - (\mathcal{E}/c^2)\eta\},$$

where $\eta = \lambda/(Lc)$. However, the function of θ that enters into the unit Runge-vector, namely $f \equiv \cos\theta$, is unwieldy, so the results will not be displayed here.

§ 6. Discussion

We have seen that O_4 and SU_3 are dynamical symmetries*) for a central potential problem essentially because motion takes place in a fixed plane. The unit vectors $\hat{k}$, $\hat{L}$ (constrained by $\hat{k}\cdot\hat{L}=0$) along with the scalars L, $\mathcal{E}$ represent five of the six functionally independent constants of motion specifying the initial conditions of a three dimensional problem**) (the time t_0 at which the particle is at $\hat{k}$ would be the sixth constant). Linear combination of the constants in the constructed algebras do not introduce additional constants classically. One would expect that perhaps the same Lie algebras would follow in a quantum-mechanical problem so long as the orbital angular momentum is a constant of the motion. On the other hand, in a quantum-mechanical problem (central potential) involving non-zero spin particles, while the total angular momentum (orbital plus intrinsic) is conserved, the orbital contribution is not separately a constant. Thus, in this case, there is no constant plane and one would not expect that these symmetries would be preserved.

However, even for the spin zero case, a translation of our results into their quantum-mechanical equivalent poses formidable problems. One would have to express the classical quantities as operators,***) and the Lie algebra would involve commutators rather than Poisson brackets. But in general, the classical expression for the SU_3 and O_4 generators involves transcendental functions of the canonical variables $\boldsymbol{r}$ and $\boldsymbol{p}$, so any simple prescription such as $\boldsymbol{p}\to -ih\boldsymbol{\nabla}$ coupled with overall hermiticity does not leave these functions well defined as operators. Indeed one would *not* expect that in general these could be made into operators with the same algebra.****)

First of all, for the Schrödinger equation, *if* the corresponding operators with the same Lie algebras (SU_3 and O_4) could be found, this would imply a degeneracy of energy levels for a general central potential which would be in conflict with degeneracies already known from standard quantum mechanical treat-

*) By *symmetries*, we mean here the existence of Lie algebras isomorphic to the Lie algebras of the groups in question. Whether such constructed Lie algebras could lead to a realization of the associated group in terms of finite canonical transformations remains an open question.

**) An analysis of classical problems on this basis has contemporaneously been made by Bacry, Ruegg and Souriau.[13]

***) The orbit equation could be developed by the technique of Nishiyama.[14]

****) A discussion of the transition from classical integrals of the motion to corresponding quantum mechanical operators, as well as the restrictions on the possibility for such a transition, has been given by Narumi.[15]

ments; consequently we conclude that such operators cannot be found. Secondly, if for a general central potential, such additional operators commuting with the Hamiltonian were found, this would imply the existence of a coordinate system (other than spherical coordinates) in which the Schrödinger equation is separable (in much the same way that the Runge vector for the Kepler problem is associated with separability in parabolic coordinates, and the SU_3 operators for the harmonic oscillator is associated with separability in cartesian coordinates). But it is known[16] that except for a small subset of potentials, the Schrödinger equation for a general central potential is separable only in spherical coordinates.

From the standpoint of embedding the Lorentz group (obtained from the generators of O_4 by the "unitary" trick) and SU_3 in a larger non-compact group, the representation of the generators (in terms of the same canonical variables $\boldsymbol{r}$ and $\boldsymbol{p}$) that we have found in our classical analysis may prove of interest. For instance, one could inquire what algebra contains $\boldsymbol{L}$, $\boldsymbol{A}$ and $\mathbb{A}$. It is found, by performing the differentiation prescribed by the Poisson bracket operation, that from $\boldsymbol{A}$ and $\mathbb{A}$ one does not get a closed algebra, but new elements which in turn lead, by a similar process, to the generation of still other elements, etc. Thus, the imbedding non-compact group involves an infinite-dimensional Lie algebra.*) It remains to be seen whether the infinite-dimensional Lie algebra so obtained has any relevance to elementary particle symmetries.

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