Modern Classical Mechanics

Solution Manual

T. HELLIWELL

V. SAHAKIAN

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Preamble

Our book “Modern Classical Mechanics” includes more than 500 problems, attached as usual in a few pages at the end of every chapter. Some problems are easy – labeled by (*), some are challenging tagged with (***), many others are somewhere in between and are denoted by (**). Truth be told, it took us about as long to collect or invent these problems, solve them, and typeset them, as to write and typeset the main body of the textbook itself. Some we have assigned already many times in our courses, while others are newly minted.

It is of course through solving problems that most students learn, really learn, the material. And for that to succeed it is obviously important to keep the solutions manual confidential. As every physicist knows, it is only through grappling with problems oneself, making mistakes, following blind alleys, and otherwise challenging oneself, is the material truly learned. Having solutions with which students can compare their work only at the end then becomes an invaluable resource for learning from their mistakes.

In this problem solutions manual we have attempted to solve all of the problems at the end of the chapters. Surprisingly, we have been known to make mistakes. There may even be typos here and there. So we hope that people who use the book and this manual will let us know when they find errors so that we can correct them. We will keep an errata webpage updated at:

sahakian.physics.hmc.edu/mcm

You can submit corrections and suggestions by contacting **sahakian@hmc.edu** via email; please include the words “MCM errata” in the subject of the email. We will periodically update the content of the booklet at the Cambridge University Press website.

Finally, we are grateful to Ivy Yuan for help in typesetting this manual.

Part I

1 Newtonian Particle Mechanics

1.1 Problems and Solutions

★ **Problem 1.1** A meterstick is at rest in a primed frame of reference, with one end at the origin and the other at $x' = 1.0$ m. (a) Using the Galilean transformation find the location of each end of the stick in the unprimed frame at a particular time t, and then find the length of the meter stick in the unprimed frame. (b) Repeat for the case that the stick is laid out along the positive y' axis, with one end at the origin and the other at $y' = 1.0$ m. What is the length of the stick in the unprimed frame?

Solution

(a) $x = x' + vt' = x' + vt$, so the left end has $x_l = 0 + vt = vt$, and the right end has $x_r = 1.0 \text{ m} + vt$. Therefore the length $= x_r - x_l = 1.0$ m.

(b) $x' = 0$ for both ends in this case, and $y' = 0$ and $y' = 1.0$ m always. Therefore $x = x' + vt = vt$ for both ends, and $y = y' = 0$ and 1.0 m for the two ends, so the length in the unprimed frame is $\Delta y = \Delta y' = 1.0$ m. ■

★ **Problem 1.2** A river of width D flows uniformly at speed V relative to the shore. A swimmer swims always at speed $2V$ relative to the water. (a) If the swimmer dives in from one shore and swims in a direction perpendicular to the shoreline in the reference frame of the flowing river, how long does it take her to reach the opposite shore, and how far downstream has she been swept relative to the shore? (b) If instead she wants to swim to a point on the opposite shore directly across from her starting point, at what angle should she swim relative to the direction of the river flow, and how long would it take her to swim across?

Solution

(a) Her velocity perpendicular to the shoreline is $2V$, so the time to reach the opposite shore is $t = \frac{D}{2V}$. During this time, she is also swept downstream a distance $d = Vt = \frac{D}{2}$.

(b) She must have an upstream component of velocity V to make up for the river flow. From the Pythagorean theorem, her velocity component across the river is $\sqrt{(2V)^2 - V^2} = \sqrt{3}V$ and the angle

$$\theta = \tan^{-1}\frac{V}{\sqrt{3}V} = \sin^{-1}\frac{V}{2V} = \sin^{-1}\frac{1}{2} = 30^\circ$$

Therefore her angle relative to the flow direction is $30^\circ + 90^\circ = 120^\circ$. Her time to swim across is $t = \frac{D}{\sqrt{3}V}$. ■

★ **Problem 1.3** The crews of two eight-man sculls decide to race one another on a river of width D that flows at uniform velocity V_0. The crew of scull A rows downstream a distance D and then back upstream, while the crew of scull B rows to a point on the opposite shore directly across from the starting point, and then back to the starting point. They begin simultaneously, and each crew rows at the same speed V relative to the water, with $V > V_0$. Who wins the race, and by how much time?

Solution

A: Relative to the shore, A has velocity $V_0 + V$ downstream and $V - V_0$ upstream. The time spent downstream is $\frac{D}{V_0+V}$ and upstream $\frac{D}{V-V_0}$, so the total time for A is

$$\frac{D}{V_0 + V} + \frac{D}{V - V_0} = \frac{D(V - V_0 + V + V_0)}{V^2 - V_0^2} = \frac{2DV}{V^2 - V_0^2} = t_A$$

B: The velocity of B relative to the shore is $\sqrt{V^2 - V_0^2}$, so the total time across the stream and back for B is $t_B = \frac{2D}{\sqrt{V^2-V_0^2}}$. Therefore

$$t_A - t_B = 2D\left[\frac{V}{V^2 - V_0^2} - \frac{1}{\sqrt{V^2 - V_0^2}}\right] = \frac{2D}{V^2 - V_0^2}\left[V - \sqrt{V^2 - V_0^2}\right] > 0.$$

So B wins the race by $\Delta t = \frac{2D(V-\sqrt{V^2-V_0^2})}{V^2-V_0^2}$. ■

★ **Problem 1.4** Passengers standing in a coasting spaceship observe a distant star at the zenith, *i.e.*, directly overhead. If the spaceship then accelerates to speed $c/100$ where c is the speed of light, at what angle to the zenith (to three significant figures) do the passengers now see the star?

Solution

Note that $\sin\theta = \frac{c/100}{c} = \frac{1}{100} \simeq \theta$ ($\sin\theta \simeq \theta$ for $\theta \ll 1$). Alternatively, perhaps the hypotenuse *should* be $\sqrt{c^2 + \frac{c^2}{10^4}}$, so $\tan\theta = \frac{c/100}{c} \simeq \theta$, so $\theta \simeq \frac{1}{100}$, the same either way to three significant figures. ■

★★ **Problem 1.5** (a) Snow is falling vertically toward the ground at speed v. (a) A bus driver is driving through the snowstorm on a horizontal road at speed $v/3$. At what angle to the vertical are the snowflakes falling as seen by the driver? (b) Suppose that the large windshield in the flat, vertical front of the bus has been knocked out, leaving a hole of area A in the vertical plane. Given that N is the number of falling snowflakes per unit horizontal area per unit time, if the bus moves at constant speed $v/3$ to reach a destination at distance d, how many snowflakes fall into the bus before the destination is reached? (c) To minimize the total number of snowflakes that fall in, the driver considers driving faster or slower. What would be the best speed to take?

Solution

(a) From the point of view of the ground, snowflakes fall straight down at speed v, so from the point of view of the bus the snowflakes fall at an angle of $\theta = \tan^{-1} \frac{V/3}{V} = \tan^{-1}(\frac{1}{3})$.

(b) In a time t the volume swept into the bus is $V = A(\frac{V}{3}t) = Ad$, so the number of snowflakes entering is NAd, regardless of speed.

(c) The speed of the bus doesn't matter. If the bus has higher velocity, more snowflakes come in per unit time, but the time to travel the distance d is less. The number of snowflakes entering the bus is the same whether the bus moves fast or slow. ■

★★ **Problem 1.6** The jet stream is flowing due east at velocity v_J relative to the ground. An aircraft is traveling at velocity v_C in the northeast direction relative to the air. (a) Relative to the ground, find the speed of the aircraft and the angle of its motion relative to the east. (b) Keeping the same speed v_C relative to the air, at what angle would the plane have to move through the air relative to the east so that it would travel northeast relative to the ground?

Solution

(a) Note that

$$v_{\text{net,horizontal}} = v_J + v_C \cos 45^\circ \quad \text{and} \quad v_{\text{net,vertical}} = v_C \sin 45^\circ$$

$$\text{Therefore, since } \cos 45^\circ = \sin 45^\circ = 1/\sqrt{2},$$

it follows that

$$\vec{v}_{net} = \left(v_J + v_C/\sqrt{2}\right)\hat{x} + (v_C/\sqrt{2})\hat{y}$$

($\hat{x} + \hat{y}$ are unit vectors).

$$\theta = \tan^{-1} \frac{v_C/\sqrt{2}}{v_J + v_C/\sqrt{2}}$$

$$v_{net} = \sqrt{(v_J + v_C/\sqrt{2})^2 + (v_0/\sqrt{2})^2}$$

(b) Note that

$$v_{\text{net}} \sin 45^\circ = v_C \sin(\pi - \theta) = v_C \sin\theta.$$

Also

$$v_J = v_{net} \cos 45 + v_C \cos(\pi - \theta) = v_{\text{net}} \cos 45 - v_C \cos\theta$$

$$v_J = v_c \sin\theta - v_C \cos\theta = v_C(\sin\theta - \cos\theta) = v_C\sqrt{2}\sin(\theta - (\pi/4))$$

since

$$\sin(\theta - (\pi/4)) = \sin\theta\cos(\pi/4) - \cos\theta\sin(\pi/4) = \frac{1}{\sqrt{2}}(\sin\theta - \cos\theta).$$

Thus

$$\sin(\theta - (\pi/4)) = \frac{v_J}{\sqrt{2}v_C}$$

$$\theta - (\pi/4) = \sin^{-1}\frac{v_J}{\sqrt{2}v_C}$$

$$\theta = \frac{\pi}{4} + \sin^{-1}\frac{v_J}{\sqrt{2}v_C}.$$

For example, suppose $v_J = v_C \cos 45 = \sqrt{2}v_C$; then

$$\theta = \frac{\pi}{4} + \sin^{-1} 1 = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4},$$

which is correct. ■

⋆ **Problem 1.7** The earth orbits the sun once/year in a nearly circular orbit of radius 150×10^6 km. The speed of light is $c = 3 \times 10^5$ km/s. Looking through a telescope, we observe that a particular star is directly overhead. If the earth were quickly stopped and made to move in the opposite direction at the same speed, at what angle to the vertical would the same star now be observed?

Solution

The speed of the earth's orbit is found from

$$F = ma : \quad \frac{-GM_{\text{sun}}m}{r_e^2} = -\frac{mv_e^2}{r_e}$$

$$\Rightarrow v_e = \sqrt{\frac{GM_{\text{sun}}}{r_e}}$$

Here $G = 6.67 \times 10^{-11}\frac{m^3}{\text{kg s}^2}$, $M_{\text{sun}} = 2.0 \times 10^{30}$ kg, $r_e = 1.5 \times 10^{11}$ m. Therefore

$$v_e = \sqrt{\frac{(2/3)\cdot 10^{-10}\cdot 2\cdot 10^{30}}{1.5\cdot 10^{11}}} = \sqrt{8.89 \times 10^8}\ \text{m/s} = 2.98 \times 10^4\ \text{m/s}$$

The speed of light is $c = 3 \times 10^8$ m/s, so since $v_e/c \ll 1$, $\theta \simeq \frac{v_e}{c} = \frac{2.98\times 10^4}{3.0\times 10^8} \approx 10^{-4}$ radians.

Moving in the original direction the star (apparently overhead) is actually 10^{-4} radians in the forward direction. So if the earth were moving in the opposite direction the star would appear to be $\theta \simeq 2 \times 10^{-4}$ radians from the vertical. ■

⋆ **Problem 1.8** A long chain is tied tightly between two trees and a horizontal force F_0 is applied at right angles to the chain at its midpoint. The chain comes to equilibrium so that each half of the chain is at angle θ from the straight line between the chain endpoints. Neglecting gravity, what is the tension in the chain?

Solution

Balancing forces perpendicular to the chain, $F_0 = 2T \sin\theta \Rightarrow T = F_0/2\sin\theta$, where T is the tension. ■

⋆⋆ **Problem 1.9** An object of mass m is subject to a drag force $F = -kv^n$, where v is its velocity in the medium, and k and n are constants. If the object begins with velocity v_0 at time $t = 0$, find its subsequent velocity as a function of time.

Solution

$F = -kv^n = mdv/dt$ by Newton's Second Law. Therefore

$$\int_0^t dt = -\frac{m}{k}\int_{v_0}^{v} v^{-n}dv \Rightarrow t = -\left(\frac{m}{k}\right)\frac{v^{-n+1}}{-n+1}\Big|_{v_0}^{v}$$

$$-\frac{kt}{m} = \frac{v^{-n+1} - v_0^{-n+1}}{-n+1} \Rightarrow v^{-n+1} = v_0^{-n+1} - \frac{kt}{m}(-n+1)$$

$$\Rightarrow v \equiv v^{\frac{-n+1}{-n+1}} = \left[v_0^{-n+1} - (-n+1)\frac{kt}{m}\right]^{\frac{1}{-n+1}}.$$

■

⋆⋆ **Problem 1.10** A small spherical ball of mass m and radius R is dropped from rest into a liquid of high viscosity η, such as honey, tar, or molasses. The only appreciable forces on it are gravity mg and a linear drag force given by Stokes's law, $F_{\text{Stokes}} = -6\pi\eta Rv$, where v is the ball's velocity, and the minus sign indicates that the drag force is opposite to the direction of v. (a) Find the velocity of the ball as a function of time. Then show that your answer makes sense for (b) small times; (c) large times.

Solution

Let $\alpha = 6\pi\eta R$, so $F_{\text{Stokes}} = -\alpha v$. Then $F_{net} = mg - \alpha v = ma = mdv/dt$.

(a) It follows from $F = m\frac{dv}{dt}$ that $dt = \frac{m\,dv}{mg-\alpha v}$, where v is positive downward. Then

$$t = \int dt = \int_0^v \frac{m\,dv}{mg - \alpha v}.$$

Let $u \equiv mg - \alpha v$, so $du = -\alpha dv$. Therefore

$$t = \int_{mg}^{mg-\alpha v} \frac{m(1/-\alpha)du}{u} = -\frac{n}{\alpha}\ln u\Big|_{mg}^{mg-\alpha v} = \frac{m}{\alpha}\ln\left(\frac{mg}{mg-\alpha v}\right).$$

Therefore $e^{\frac{\alpha t}{m}} = \frac{mg}{mg-\alpha v}$, so $(mg - \alpha v) = mge^{-\alpha t/m}$. Then

$$\alpha v = mg\left[1 - e^{-\alpha t/m}\right] \quad \text{so} \quad v(t) = \left(\frac{mg}{\alpha}\right)(1 - e^{-\alpha t/m}).$$

(b) For small times $e^{-\alpha t/m} \cong 1 - \frac{\alpha t}{m} = 1 - (\frac{6\pi\eta R}{m})t$. (series expansion $e^x = 1 + x + x^2/2! + \cdots$). Therefore

$$v(t) \cong \frac{mg}{\alpha}\left(1 - \left(1 - \frac{6\pi\eta R}{m}t\right)\right) + \cdots \cong \frac{mg}{6\pi\eta R}\left(\frac{6\pi\eta R}{m}t\right) = gt,$$

which is correct, because for very small times the drag force is negligible.

(c) For large times $e^{-\alpha t/m} \to 0$, so $v(t) \cong mg/6\pi\eta R$, $mg = 6\pi\eta R v$. In this case the forces balance, with no additional acceleration. The ball is approaching its terminal velocity. ∎

⋆⋆⋆ **Problem 1.11** We showed in Example 1.2 that the distance a ball falls as a function of time, starting from rest and subject to both gravity g downward and a quadratic drag force upward, is

$$y = (v_T^2/g)\ln(\cosh(gt/v_T)),$$

where v_T is its terminal velocity. (a) Invert this equation to find how long it takes the ball to reach the ground in terms of its initial height h. (b) Check your result in the limits of *small* h and *large* h. (For part (b) it is useful to know the infinite series expansions of the functions e^x, $(1+x)^n$, and $\ln(1+x)$ for small x.)

Solution

(a) From the given equation, it follows that $\frac{gy}{v_T^2} = \ln(\cosh(gt/v_T))$, so

$$\cosh\frac{gt}{v_T} = e^{gy/v_T^2} \equiv \frac{e^{gt/v_T} + e^{-gt/v_T}}{2}$$

Multiply by e^{gt/v_T}: $(e^{gt/v_T})^2 - 2e^{gy/v_T}(e^{gt/v_T}) + 1 = 0$, which is a quadratic equation in e^{gt/v_T}, with solutions

$$e^{gt/v_T} = e^{\frac{gy}{v_T^2}}\left[1 \pm \sqrt{1 - e^{-2gh/v_T^2}}\right]$$

using the quadratic equation, and where now h is the initial height and t is the time to reach the ground. Which sign is correct?

Note that as $h \to \infty$, $e^{-2gh/v_T^2} \to 0$ and $(1 - e^{-2gh/v_T^2})^{1/2} \to 1 - \frac{1}{2}e^{-2gh/v_T^2}$ by the binomial approximation. So with the lower sign,

$$e^{gt/v_T} \simeq e^{gh/v_T^2}\left[\frac{1}{2}e^{-2gh/v_T^2}\right] = \frac{1}{2}e^{-gh\sqrt{v_T^2}}$$

which is incorrect, because it implies that t decreases as h increases. So using the upper sign,

$$e^{gt/v_T} = e^{gh/v_T^2}\left[1 + \sqrt{1 - e^{-2gh/v_T^2}}\right].$$

Take the natural log of both sides, giving

$$\frac{gt}{v_T} = \frac{gh}{v_T^2} + \ln\left[1 + \sqrt{1 - e^{-2gh/v_T^2}}\right]$$

so

$$t = \frac{h}{v_T} + \frac{v_T}{g}\ln\left[1 + \sqrt{1 - e^{-2gh/v_T^2}}\right].$$

(b) Check the result:
For small h, $e^{-2gh/v_T^2} \simeq 1 - (2gh/v_T^2)$, since $e^x = 1 + x + x^2/2! + \cdots$ for small x, and so

$$\sqrt{1 - e^{-2gh/v_T^2}} \simeq \sqrt{2gh/v_T^2}$$

Therefore

$$\ln\left[1 + \sqrt{2gh/v_T^2}\right] \simeq \sqrt{2gh/v_T^2}$$

since $\ln(1 + x) \simeq x$ for $x \ll 1$. Thus

$$t \simeq \frac{h}{v_T} + \frac{v_T}{g}\sqrt{2gh/v_T^2} \simeq \sqrt{\frac{2h}{g}}$$

for small h. Therefore $t = \sqrt{\frac{2h}{g}}$, uniformly accelerated motion for small times, valid before the drag force becomes appreciable. For large h,

$$(1 - e^{-2gh/v_T^2})^{1/2} \simeq 1 - \frac{1}{2}e^{-2gh/v_T^2}$$

so

$$t \simeq \frac{h}{v_T} + \frac{v_T}{g}\ln 2 \simeq \frac{h}{v_T}$$

which is also correct, since then most of the trip is essentially at the terminal velocity v_T. ■

⋆ **Problem 1.12** For objects with linear size between a few millimeters and a few meters moving through air near the ground, and with speed less than a few hundred meters per second, the drag force is close to a quadratic function of velocity, $F_D = (1/2)C_D A\rho v^2$, where ρ is the mass density of air near the ground, A is the cross-sectional area of the object, and C_D is the drag coefficient, which depends upon the shape of the object. A rule of thumb is that in air near the ground (where $\rho = 1.2$ kg/m^3), then $F_D \simeq \frac{1}{4}Av^2$.
(a) Estimate the terminal velocity v_T of a skydiver of mass m and cross-sectional area A.
(b) Find v_T for a skydiver with $A = 0.75$ m^2 and mass 75 kg. (The result is large, but a few lucky people have survived a fall without a parachute. An example is 21-year old Nicholas Alkemade, a British Royal Air Force tail gunner during World War II. On March 24, 1944 his plane caught fire over Germany and his parachute was destroyed. He had the choice of burning to death or jumping out. He jumped and fell about 6 km, slowed at the end by

falling though pine trees and landing in soft snow, ending up with nothing but a sprained leg. He was captured by the Gestapo, who at first did not believe his story, but when they found his plane they changed their minds. He was imprisoned, and at the end of the war set free, with a certificate signed by the Germans corroborating his story.)

Solution
At terminal velocity, $F_D \simeq \frac{1}{4}Av^2 \simeq mg$ in SI units, so $v_T \simeq \sqrt{\frac{4mg}{A}}$. Then $v_T \cong \sqrt{\frac{4mg}{A}} = 2\sqrt{\frac{(75)(9.8)}{0.75}} = 62.6$ m/s $\cong 225$ km/hr $\cong 140$ mi/hr. ■

⋆ **Problem 1.13** A damped oscillator consists of a mass m attached to a spring k, with frictional damping forces. If the mass is released from rest with amplitude A, and after 100 oscillations the amplitude is $A/2$, what is the total work done by friction during the 100 oscillations?

Solution
We can simply see how much energy is lost. The initial amplitude is A, so the initial energy is all potential energy $\frac{1}{2}kA^2$. After 100 oscillations the amplitude is $A/2$, so the energy is $\frac{1}{2}k(A/2)^2 = \frac{1}{8}kA^2$. The energy lost is $\frac{1}{2}kA^2 - \frac{1}{8}kA^2 = \frac{3}{8}kA^2$, so the work done by friction according to the work-energy theorem is $-\frac{3}{8}kA^2$. ■

⋆ **Problem 1.14** The solution of the underdamped harmonic oscillator is $x(t) = Ae^{-\beta t}\cos(\omega_1 t + \varphi)$, where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$. Find the arbitrary constants A and φ in terms of the initial position x_0 and initial velocity v_0.

Solution
Given $x(t) = Ae^{-\beta t}\cos(\omega_1 t + \varphi)$, where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$ and A and ϕ are arbitrary constants that can be found in terms of the initial conditions $x(0) = x_0$ and $v(0) = v_0$. So $x_0 = A\cos\phi$ and $v_0 = A[-\beta\cos\phi - \omega_1\sin\phi]$. Note $\sin\phi = \sqrt{1 - \cos\phi^2}$ (using the plus sign), $\sin\phi = \sqrt{1 - (x_0/A)^2}$. Therefore

$$v_0 = A\left[-\beta x_0/A - \omega_1\sqrt{1 - (x_0/A)^2}\right] = -\beta x_0 - \omega_1\sqrt{A^2 - x_0^2}$$

It follows that

$$A^2 - x_0^2 = \frac{(v_0 + \beta x_0)^2}{\omega_1^2} \quad \text{so} \quad A = \frac{\sqrt{v_0^2 + 2\beta x_0 v_0 + x_0^2(\beta^2 + \omega_1^2)}}{\omega_1}$$

and

$$\phi = \sin^{-1}\left[1 - (\frac{x_0}{A})^2\right] = \sin^{-1}\left[1 - \frac{(x_0\omega_1)^2}{v_0^2 + 2\beta x_0 v_0 + x_0^2(\beta^2 + \omega_1^2)}\right]$$

$$\phi = \sin^{-1}\left[\frac{v_0^2 + 2\beta x_0 v_0 + x_0^2\beta^2}{v_0^2 + 2\beta x_0 v_0 + x_0^2(\beta^2 + \omega_1^2)}\right].$$

■

⋆⋆ **Problem 1.15** An overdamped oscillator is released at location $x = x_0$ with initial velocity v_0. What is the maximum number of times the oscillator can subsequently pass through $x = 0$?

Solution

The overdamped solution is

$$x(t) = A_1 e^{\gamma_1 t} + A_2 e^{\gamma_2 t} \,,$$

where

$$\gamma_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

with $\beta > \omega_0$.

$$v(t) = x(t) = A_1\gamma_1 e^{\gamma_1 t} + A_2\gamma_2 e^{\gamma_2 t} \,.$$

At $t = 0$, $x = x_0$ and $x(0) = v(0) = v_0$. Therefore $x_0 = A_1 + A_2$ and

$$v_0 = A_1(-\beta + \sqrt{\beta^2 - \omega_0^2}) + A_2(-\beta - \sqrt{\beta^2 - \omega_0^2}) \,.$$

Eliminate A_2, using $A_2 = x_0 - A_1$, so

$$v_0 = A_1(-\beta + \sqrt{\beta^2 - \omega_0^2}) + (x_0 - A_1)(-\beta - \sqrt{\beta^2 - \omega_0^2}) = x_0 \left[-\beta - \sqrt{\beta^2 - \omega_0^2}\right]$$
$$+ A_1 \left[-\beta + \sqrt{\beta^2 - \omega_0^2} + \beta + \sqrt{\beta^2 - \omega_0^2}\right] = -x_0(\beta + \sqrt{\beta^2 - \omega_0^2}) + A_1 2\sqrt{\beta^2 - \omega_0^2} \,.$$

Therefore

$$A_1 = \frac{v_0 + x_0(\beta + \sqrt{\beta^2 - \omega_0^2})}{2\sqrt{\beta^2 - \omega_0^2}}$$

$$A_2 = x_0 - A_1 = \frac{x_0 2\sqrt{\beta^2 - \omega_0^2} - v_0 - x_0(\beta + \sqrt{\beta^2 - \omega_0^2})}{2\sqrt{\beta^2 - \omega_0^2}}.$$

$$A_2 = \frac{x_0\sqrt{\beta^2 - \omega_0^2} - v_0 - \beta x_0}{2\sqrt{\beta^2 - \omega_0^2}} = \frac{-v_0 + x_0(\sqrt{\beta^2 - \omega_0^2} - \beta)}{2\sqrt{\beta^2 - \omega_0^2}}$$

$$x(t) = \frac{v_0 + x_0(\beta + \sqrt{\beta^2 - \omega_0^2})}{2\sqrt{\beta^2 - \omega_0^2}} e^{\gamma_1 t} + \frac{-v_0 - x_0(\beta - \sqrt{\beta^2 - \omega_0^2})}{2\sqrt{\beta^2 - \omega_0^2}} e^{\gamma_2 t}.$$

$$x(t) = \frac{\left[v_0 + x_0(\beta + \sqrt{\beta^2 - \omega_0^2})\right] e^{\gamma_1 t} - \left[v_0 + x_0(\beta - \sqrt{\beta^2 - \omega_0^2})\right] e^{\gamma_2 t}}{2\sqrt{\beta^2 - \omega_0^2}} \,.$$

Without loss of generality, we can assume $x_0 > 0$. Then if the mass reaches $x = 0$ we must have

$$\left[v_0 + x_0(\beta + \sqrt{\beta^2 - \omega_0^2})\right] e^{\gamma_1 t} = \left[v_0 + x_0(\beta - \sqrt{\beta^2 - \omega_0^2})\right] e^{\gamma_2 t}$$

or

$$e^{(\gamma_2-\gamma_1)t} = e^{-2\sqrt{\beta^2-\omega_0^2}t} = \frac{v_0 + x_0(\beta + \sqrt{\beta^2 - \omega_0^2})}{v_0 + x_0(\beta - \sqrt{\beta^2 - \omega_0^2})} .$$

Now for $t > 0$,

$$e^{-2\sqrt{\beta^2-\omega_0^2}t} < 1$$

But for $x_0 > 0, \beta > 0$, this is only possible if $v_0 < 0$, in fact, $v_0 + x_0(\beta + \sqrt{\beta^2 - \omega_0^2}) < 0$. So also $v_0 + x_0(\beta - \sqrt{\beta^2 - \omega_0^2}) < 0$ as well.

$$e^{-2\sqrt{\beta^2-\omega_0^2}t} = \frac{-|v_0| + x_0(\beta + \sqrt{\beta^2 - \omega_0^2})}{-|v_0| + x_0(\beta - \sqrt{\beta^2 - \omega_0^2})} = \frac{|v_0| - x_0(\beta + \sqrt{\beta^2 - \omega_0^2})}{|v_0| - x_0(\beta - \sqrt{\beta^2 - \omega_0^2})} < 1 .$$

(Note $v_0 < 0$). There can be only a single time t_0 when the mass passes through $x = 0$. Plotting $x(t)$ for a strongly negative v_0 shows that $x(t)$ can pass from positive to negative values one time, but then approaches $x = 0$ asymptotically from below. ■

⋆ **Problem 1.16** There are thought to be three types of the particles called *neutrinos*: electron-type (ν_e), muon type (ν_μ), and tau-type (ν_τ). If they were all massless they could not spontaneously convert from one type into a different type. But if there is a mass difference between two types, call them types ν_1 and ν_2, the probability that a neutrino starting out as a ν_1 becomes a ν_2 is given by the oscillating probability $P = S_{12} \sin^2(L/\lambda)$, where S_{12} is called the *mixing strength parameter*, which we take to be constant, L is the distance traveled by the neutrino, and λ is a characteristic length, given in kilometers by

$$\lambda = \frac{E}{1.27\Delta(m)^2}$$

where E is the energy of the neutrino in units of GeV (1 GeV = 10^9 eV) and $\Delta(m)^2$ is the difference in the *squares* of the two masses in units (eV)^2). Neutrinos are formed in earth's atmosphere by the collision of cosmic-ray protons from outer space with atomic nuclei in the atmosphere. The giant detector *Super Kamiokande*, located deep underground in a mine west of Tokyo, saw equal numbers of electron-type neutrinos coming (1) from the atmosphere above the detector (2) from the atmosphere on the other side of the earth, which pass through our planet on their way to the detector. However, Super K saw more muon-type neutrinos coming down from above than those coming up from above. This was strong evidence that muon-type neutrinos oscillated into tau-type neutrinos (which Super

K could not detect) as they penetrated the earth, since it requires more time to go 13,000 km through the earth than 20 km through the atmosphere above the mine. (a) Suppose $(\Delta m)^2 = 0.01$ eV^2 between ν_μ and ν_τ type neutrinos, and that the neutrino energy is $E = 5$ GeV. What is λ? How would this explain the fewer number of muon neutrinos seen from below than from above? (b) The best experimental fit is $(\Delta m)^2 = 0.0022$ eV^2. Again assuming $E = 5$ GeV, what is λ? Make a crude estimate of the ratio one might expect for the number of muon neutrinos from below and from above.

Solution

(a) $\lambda = \frac{E}{1.27\Delta(m)^2} = \frac{5}{1.27(0.01)} = 394$ km. Therefore since the atmosphere has a thickness of only about 20 km, few of the muon-type neutrinos would have had time to convert to τ-type neutrinos, but there would have been several oscillations coming through the 13, 000 km of the earth.

(b) $\lambda = 5/[1.27(0.0022)] = 1790$ km, so very few of the muon neutrinos coming through the atmosphere only will convert. In penetrating the earth the probability of conversion is approximately

$$P = S_{12}\sin^2(L/\lambda) = S_{12}\sin^2\frac{13,000}{1790} = S_{12}\sin^2 7.26 \simeq 0.687S_{12}$$

so the probability of remaining a muon-type neutriino is $S_{12} \cdot 100\%$ if penetrating the atmosphere only, and $S_{12} \cdot 31.3\%$ if penetrating the earth. So a crude estimate of the number of muon neutrinos from below compared with the number of muon neutrinos from above is roughly 0.31. This is very rough, because some neutrinos will pass through only a portion of the earth. ∎

★★ **Problem 1.17** The "quality factor" Q of an underdamped oscillator can be defined as

$$Q = 2\pi\frac{E}{|\Delta E|}$$

where at some time E is the total energy of the oscillator and $|\Delta E|$ is the energy loss in one cycle. (a) Show that $Q \simeq \pi/\beta P$, where β is the damping constant and P is the period of oscillation. Therefore if the damping increases, Q decreases. (b) What is Q for a simple pendulum that loses 1% of its energy during each cycle? (c) The quality factor also describes the sharpness of the resonance curve of a driven, lightly-damped oscillator. Show that to a good approximation $Q \simeq \omega/(\Delta\omega)$, where $\Delta\omega$ is the angular frequency difference between the two locations on the amplitude resonance curve for which the amplitude is $1/\sqrt{2}$ that at peak resonance.

Solution

(a) The oscillator follows the solution $x(t) = Ae^{-\beta t}\cos(\omega t + \varphi)$ where the energy of the oscillator is proportional to x^2, so $E \propto A^2e^{-2\beta t}$. One cycle corresponds to a period of $P = 2\pi/\omega$, so

$$\begin{aligned}\Delta E &= A^2e^{-2\beta t} - A^2e^{-2\beta(t+P)} = A^2e^{-2\beta t}(1-e^{-2\beta P})\\ &\simeq A^2e^{-2\beta t}\left[1-(1-2\beta P)\right] = A^2e^{-2\beta t}(2\beta P)\end{aligned}$$

(using $e^x = 1 + x + x^2/2! + \cdots$). Thus

$$Q = 2\pi\frac{E}{|\Delta E|} = \frac{2\pi A^2e^{-2\beta t}}{A^2e^{-2\beta t}(2\beta P)} = \frac{\pi}{\beta P}$$

(b) $Q = 2\pi E/0.01E \cong 628$

(c) The resonance curve is (for the amplitude of oscillation):

$$C(\omega) = \frac{f_0}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2}}\,.$$

The resonance peak is at

$$C_R = \frac{f_0}{2\beta\omega_1} \simeq \frac{f_0}{2\beta\omega_0}$$

for light damping. Suppose

$$C(\omega) = \frac{1}{\sqrt{2}}C_R = \frac{1}{\sqrt{2}}\left(\frac{f_0}{2\beta\omega_0}\right) = \frac{f_0}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2}}\,.$$

Again, $4\beta^2\omega^2 = 4\beta^2\omega_0^2$ for a narrow resonance curve, so

$$\sqrt{2}(2\beta\omega_0) \simeq \sqrt{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2} \quad \text{so} \quad (\omega_0^2-\omega^2)^2 \simeq 4\beta^2\omega_0^2.$$

Therefore

$$\omega = \omega_0\left[1 \pm \frac{2\beta}{\omega_0}\right]^{1/2} \Rightarrow \omega_+ = \omega_0\left(1+\frac{\beta}{\omega_0}\right) \text{ and } \omega_- = \omega_0\left(1-\frac{\beta}{\omega_0}\right)$$

by the binomial approximation. Therefore, the difference is

$$\Delta\omega \equiv \omega_+ - \omega_- \equiv 2\beta.$$

Thus

$$\frac{\omega}{\Delta\omega} \cong \frac{\omega_0}{2\beta} = \frac{(2\pi/P)}{2\beta}\,.$$

$$\frac{\omega}{\Delta\omega} = \frac{\pi}{\beta P} = Q\,.$$

■

⋆ **Problem 1.18** Consider the unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{r}}$, and $\hat{\boldsymbol{\theta}}$ in a plane. (a) Find $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ in terms of any or all of $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, x, and y.(b) Find $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ in terms of any or all of $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, r, and θ.

Solution

By drawing a picture in the x, y plane, it is easy to show that (a)

$$\hat{\mathbf{r}} = \cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}} \tag{1.1}$$
$$\hat{\boldsymbol{\theta}} = -\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}} \tag{1.2}$$

where $\sin\theta = y/r = y/\sqrt{x^2+y^2}$ and $\cos\theta = x/\sqrt{x^2+y^2}$. (b) Multiply 6.1 by $\cos\theta$ and 6.1 by $\sin\theta$;

$$\hat{\mathbf{r}}\cos\theta = \cos^2\theta\hat{\mathbf{x}} + \sin\theta\cos\theta\hat{\mathbf{y}}$$
$$\hat{\boldsymbol{\theta}}\sin\theta = -\sin^2\theta\hat{\mathbf{x}} + \sin\theta\cos\theta\hat{\mathbf{y}}$$

Subtract these equations: $\hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta = \hat{\mathbf{x}}$, $\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\theta - \hat{\theta}\sin\theta$.

Then multiply 6.1 by $\sin\theta$, 6.1 by $\cos\theta$.

$$\hat{\mathbf{r}}\sin\theta = \sin\theta\cos\theta\hat{\mathbf{x}} + \sin^2\theta\hat{\mathbf{y}}$$
$$\hat{\boldsymbol{\theta}}\cos\theta = -\sin\theta\cos\theta\hat{\mathbf{x}} + \cos^2\theta\hat{\mathbf{y}}$$

add these to find $\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta$. ∎

⋆ **Problem 1.19** The mass and mean radius of the moon are $m = 7.35 \times 10^{22}$ kg and $R = 1.74 \times 10^6$ m. (a) From these parameters, along with Newton's constant of gravity $G = 6.674 \times 10^{-11}$ m^{3}kg^{-1}s^{-2}, find the moon's escape velocity in m/s. (b) For a slingshot boom of length 50 m, what must be the minimum rotation frequency ω to sling material off the moon, as described in Example 1.3? Take into account both the radial and tangential components of the payload velocity when it comes off the end of the boom. Assume payloads are initially set upon the boom at radius $r = 3$ meters and with $\dot{r} = 0$.

Solution

(a) At escape velocity $E = \frac{1}{2}mv_{esc}^2 - GMm/r = 0$, so

$$v_{esc} = \sqrt{\frac{2GM}{r}} = 2.37 \times 10^3\text{m/s} = 2.37 \text{ km/s}$$

(b) $r = r_0\cosh\omega t$ as shown in the chapter, so $\cosh\omega t = r/r_0 = 50\text{ m}/3\text{ m} = 16.7$. The velocity of the payload is

$$\begin{aligned}\mathbf{v} &= \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} = r_0\omega(\sinh\omega t\hat{\mathbf{r}} + \cosh\omega t\hat{\boldsymbol{\theta}}) \Rightarrow v^2 = \mathbf{v}\cdot\mathbf{v}\\ &= r_0^2\omega^2(\sinh^2\omega t + \cosh^2\omega t) = r_0^2\omega^2(2\cosh^2\omega t - 1)\end{aligned}$$

(since $\cosh^2 - \sinh^2 = 1$). Therefore,

$$\omega = \frac{v_{esc}/r_0}{\sqrt{2\cosh^2\omega t - 1}} = \frac{(2.37\times 10^3\text{ m/s})/3\text{ m}}{\sqrt{2(16.7)^2 - 1}} = \frac{0.79\text{ s}^{-1}\times 10^3}{23.6} = 33.5\text{ s}^{-1},$$

so is swinging around very very fast for a 50 m boom. ∎

⋆ **Problem 1.20** Ninety percent of the initial mass of a rocket is in the form of fuel. If the rocket starts from rest and then moves in gravity-free empty space, find its final velocity v if the speed u of its exhaust is (a) 3.0 km/s (typical chemical burning), (b) 1000 km/s, (c) $c/10$, where c is the speed of light. (d) If the exhaust velocity is 3.0 km/s, for how long can the rocket maintain the acceleration $a = 10$ m/s^2?

Solution

The rocket equation is $v = v_0 + u\ln\frac{m_0}{m} = 0 + u\ln\frac{m_0}{0.1m_0} = u\ln 10 = 2.30u.$

(a) $v = 2.30(3.0 \text{ km/s}) = 6.9$ km/s.

(b) $v = 2.30(1000 \text{ km/s}) = 2300$ km/s.

(c) $v = 2.30(3 \times 10^7 \text{ m/s}) = 6.9 \times 10^4$ km/s.

(d) $a = \frac{dv}{dt} = u\frac{d}{dt}(\ln m_0 - \ln m) = -um^{-1}\frac{dm}{dt}$. $10 \text{ m/s}^2 = -(3.0\text{km/s})\frac{1}{m}(\frac{dm}{dt})$. $\frac{1}{m}\frac{dm}{dt} = -\frac{10 \text{ m/s}^2}{3000 \text{ m/s}} = -\frac{1}{3}10^{-2} \text{ s}^{-1}$ =constant. so

$$\int \frac{dm}{m} = -\frac{1}{300}\text{s}^{-1}t \Rightarrow \int_{m_0}^{m} \frac{dm}{m} = \ln\frac{m}{m_0} = -\frac{t}{300}\text{ s}^{-1} \Rightarrow$$
$$t = 300\ln\frac{m_0}{m}s = 300\ln\frac{m_0}{0.1m_0}\text{s} = 300\ln 10 \text{ s} = 690 \text{ s} = 11.5 \text{ minutes.}$$

■

⋆ **Problem 1.21** A space traveler pushes off from his coasting spaceship with relative speed v_0; he and his spacesuit together have mass M, and he is carrying a wrench of mass m. Twenty minutes later he decides to return, but his thruster doesn't work. In another forty minutes his oxygen supply will run out, so he immediately throws the wrench away from the ship direction at speed v_w relative to himself prior to the throw. (a) What then is his speed relative to the ship? (b) In terms of given parameters, what is the minimum value of v_w required so he will return in time?

Solution

(a) Conserving momentum of the traveler and wrench,

$$(M+m)v_0 = Mv_f + m(v_0+v_w) \Rightarrow Mv_f = (M+m)v_0 - m(v_0+v_w)$$
$$= Mv_0 - mv_w \Rightarrow v_f = \frac{Mv_0 - mv_w}{M}$$

(> 0 if he is moving away from the ship). His velocity must be $-\frac{v_0}{2}$ to make it back in 40 minutes, since he has twice as long.

(b) $v_f = -\frac{v_0}{2} = v_0 - \frac{m}{M}v_w \Rightarrow v_w = (\frac{M}{m})\frac{3}{2}v_0.$ ■

⋆⋆ **Problem 1.22** An astronaut of mass M, initially at rest in some inertial frame in gravity-free empty space, holds n wrenches, each of mass $M/2n$. (a) Calculate her recoil velocity v_1 if she throws all the wrenches at once in the same direction with speed u relative to her original inertial frame. (b) Find her final velocity v_2 if she first throws half of the wrenches with speed u relative to her original inertial frame, and then the other half with speed u relative to the frame she reached after the first throw. Compare v_2 with v_1 from part (a).

(c) Then find her total recoil velocity v_n if she throws all n wrenches, one at a time and in the same direction, and each with speed u relative to her instantaneous inertial frame just before she throws it. (d) Find her total recoil velocity in the limit $n \to \infty$, and compare with the rocket equation.

Solution

(a) $v_{recoil} = u/2$

(b) Throw half the wrenches: $v_{\text{recoil}} = u/5$. Throw the second half: $v_{\text{recoil}} = u/4$. Total recoil velocity $\frac{u}{5} + \frac{u}{4} = \frac{9}{20}u$.

(c) Throw 1/3 at a time: first throw gives $v_{\text{recoil}} = u/8$; second throw $v_{\text{recoil}} = u/7$; third throw $v_{\text{recoil}} = u/6$. So throwing $1/3$ at a time gives a total recoil

$$u\left[\frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right] = u(\frac{78}{168}).$$

One throw: $\frac{u}{2}$; two: $\frac{u}{4}+\frac{u}{5}$; three: $\frac{u}{6}+\frac{u}{7}+\frac{u}{8}$; four: $\frac{u}{8}+\frac{u}{9}+\frac{u}{10}+\frac{u}{11}$; five: $\frac{u}{10}+\frac{u}{11}+\frac{u}{12}+\frac{u}{13}+\frac{u}{14}$.

(c) In general,

$$(\frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2} + \cdots \frac{1}{2n+n-1})u = \sum_{n=0}^{n-1} \frac{dk}{2n+k} \to$$

let $x = 2n + k, dx = dk$.

$$u\int_{2n}^{3n-1} \frac{dx}{x} = u \ln \frac{3n-1}{2n} \to \left(\ln \frac{3}{2}\right) u$$

as $n \to \infty$. The rocket equation gives

$$v = u \ln \frac{m_0}{m} = u \ln \frac{3/2\ m}{m} = u \ln \frac{3}{2}$$

which agrees in the limit $n \to \infty$. ■

★★ **Problem 1.23** We are planning to travel in a rocket for 6 months with acceleration 10 m/s^2, and with a final payload mass 1000 tonnes (1 tonne = 1000 kg). (a) Using a chemically-fueled rocket with exhaust speed 3160 m/s, what must be the original ship mass m_0? Compare m_0 with the mass of the observed universe. (Including so-called "dark matter", the mass density is approximately 6×10^{-30} g/cm^3 and the observed radius is of order 10^{10} light years.) (b) Redo part (a) if instead we use a fuel that can be ejected at 3.16×10^7 m/s, about 10 percent the speed of light. (c) How fast would this ship be moving at the end of 6 months? (d) How far will the ship have gone by this time? Compare this distance with the distance to the star Alpha Centauri, about 4 light-years away.

Solution

The rocket equation is $v = u\ln(m_0/m)$, so the acceleration of the rocket is

$$a = dv/dt = u\frac{d/dt(m_0/m)}{m_0/m} = \frac{um_0(-\frac{dm/dt}{m^2})}{m_0/m} = -\frac{u}{m}\frac{dm}{dt} = 10 \text{ m/s}^2.$$

Therefore

$$\left|\frac{dm}{dt}\right| = \frac{a\,dt}{u} \text{ resulting in } m = m_0 e^{-at/u}, \text{ where } = 10 \text{ m/s}^2.$$

(a) If $u = 3160$ m/s and $t = \frac{1}{2}$ year $= \frac{1}{2}(3.16 \times 10^7\text{s})$, then $m/m_0 = e^{-10\cdot\frac{1}{2}\cdot 3/16\times 10^7/3.16\times 10^3} = e^{-6\times 10^4} \cdot \log_{10}(m/m_0) = -5 \times 10^4 \log_{10} e$.

Then $m = 10^6$ kg so $6 = \log m_0 - 5 \times 10^4$, and so $\log m_0 = 6 + 5 \times 10^4(0.434) = 6 + 21700 = 21706$. This gives $m_0 \simeq 10^{21,700}$ kg.

The mass of the observed universe is

$$\begin{aligned}
&\sim \frac{4}{3}\pi R^3 \rho \sim \frac{4}{3}\pi(10^{10} c \text{ yrs})^3 (6 \times 10^{-30} \text{ g/cm}^3) \\
&\sim \frac{4\pi}{3}(10^{10}\ 3 \times 10^8 \text{ m/s } 3.16 \times 10^7 \text{ s})^3\ 6 \times 10^{-30} \text{ g/cm}^3 \frac{\text{kg}}{1000\text{g}} \left(\frac{100 \text{ cm}}{1 \text{ m}}\right)^3 \\
&\sim 4(9.5 \times 10^{25} \text{ m})^3\ 6 \times 10^{-30} \text{ kg/m}^3 \times 10^3 \\
&\sim 20,000 \times 10^{48} \text{ kg} \sim 2 \times 10^{52} \text{ kg}
\end{aligned}$$

The mass of the ship would be hypothetically much much larger.

(b) If instead $u = 3.16 \times 10^7$ m/s, then

$$m/m_0 = e^{-(10/2)\frac{3.16\times 10^7}{3.16\times 10^7}} = e^{-5}$$

$\log_{10}(m/m_0) = -5 \log_{10} e$.

$$m = 10^6\text{kg} \Rightarrow 6 = \log m_0 - 5(.484), \quad \log m_0 = 8.17, \quad m_0 = 10^{8.17}\text{kg},$$

which is more reasonable.

(c) At the end of six months

$$\begin{aligned}
v &= u \ln(m_0/m) = 3.16 \times 10^7 \text{ m/s} \ln\left[\frac{10^{8.17}}{10^6}\right] = 3.16 \times 10^7 \text{ m/s} \ln 10^{2.17} \\
&= (3.16)(2.17)\ln(10)10^7 \text{ m/s} = (6.86)(2.30) \times 10^7 \text{ m/s} = 15.8 \times 10^7 \text{ m/s} \\
&= 1.58 \times 10^8 \text{ m/s}
\end{aligned}$$

about half the speed of light. (This is a relativistic speed, so it would be prudent to redo the problem using equations for relativistic rockets. See Chapter 2 problems.)

(d) At uniform acceleration $d = \frac{1}{2}at^2 = \frac{1}{2}(10 \text{ m/s}^2)(\frac{3.16\times 10^7 \text{ s}}{2})^2 = 12.5 \times 10^{14}$ m $= 1.25 \times 10^{15}$ m. One light-year $= 3 \times 10^8$ m/s $\cdot\ 3.16 \times 10^7$ s $\cong 9.5 \times 10^{15}$ m, so

$$1.25 \times 10^{15} \text{ m} = 1.25 \times 10^{15} \text{ m}\left(\frac{1c \cdot yr}{9.5 \times 10^{15} \text{ m}}\right) \cong 0.13 \text{ light-year}$$

So in 6 months, the ship would get only a small fraction of the distance to α Centauri. ■

⋆⋆⋆ **Problem 1.24** A single-stage rocket rises vertically from its launchpad by burning liquid fuel in its combustion chamber; the gases escape with a net momentum downward, while the rocket, in reaction, accelerates upward. The gravitational field is g. (a) Pretending that air resistance is negligible, show that the rocket's equation of motion is

$$m\frac{dv}{dt} = -u\frac{dm}{dt} - mg$$

where m is the instantaneous mass of the rocket at time t, v is its upward velocity, and u is the speed of the exhaust relative to the rocket. (b) Assume that g and u remain constant while the fuel is burning, and that fuel is burned at a constant rate $|dm/dt| = \alpha$. Integrate the rocket equation to find $v(m)$. (c) Suppose that $u = 4.4$ km/s and that all the fuel is burned up in one minute. If the rocket achieves the escape velocity from earth of 11.2 km/s, what percentage of the original launchpad mass was fuel?

Solution

(a) At time t the rocket has mass m and is moving vertically upward at velocity v. At time $t + \Delta t$ the rocket has mass $m + \Delta m$ (with $\Delta m < 0$), and is moving upward at velocity $v + \Delta v$. There is also a bit of exhaust $-\Delta m \equiv |\Delta m|$ moving downward with velocity $u - v$. From Newton's second law, the change in total momentum is $\Delta p = p(t + \Delta t) - p(t) = F\Delta t = -mg\Delta t$, where the positive direction is upward. Here

$$\Delta p = (m + \Delta m)(v + \Delta v) - |\Delta m|(u - v) - mv.$$

Cancelling some terms and neglecting the second-order product $\Delta m\ \Delta v$, we find $m\Delta v = -\Delta m u - mg\Delta t$. Dividing by Δt and taking the limit $\Delta t \to 0$, we find the differential equation given in the problem statement.

(b) Given that $dm/dt = -\alpha$, where α is a positive constant, it follows that $m = m_0 - \alpha t$. Also, using the chain rule,

$$\frac{dv}{dt} = \frac{u}{m}\alpha - g = \frac{dv}{dm}\frac{dm}{dt} = -\alpha\frac{dv}{dm}$$

Dividing by $-\alpha$ and integrating over m, we find the velocity as a function of mass during fuel burning,

$$v = v_0 + u\ln\frac{m_0}{m} - \frac{g}{\alpha}(m_0 - m).$$

(c) Alternatively, we can write the velocity as a function of time during fuel burning,

$$v = v_0 + u\ln\frac{m_0}{m_0 - \alpha t} - gt.$$

Here $\alpha = m_{fuel}/60$ seconds. We find $\ln(1 - m_{fuel}/m_0) = -(v + gt)/u = -2.68$, from which we find that the initial percentage in fuel is 93.2%. ■

★★ **Problem 1.25** A rocket in gravity-free empty space has fueled mass M_0 and exhaust velocity u equal to that of a first-stage Saturn V rocket (as used in sending men to the moon): $M_0 = 3100$ tons $= 28 \times 10^6$ kg and $u = 2500$ m/s. The ship's acceleration is kept constant at 10 m/s^2. (a) Find the initial rate of fuel ejection $|dM/dt|_{t=0}$. (b) After how many minutes will the ship mass be reduced to $1/e$ of its initial value? (c) Suppose the ship accelerates as described for 20 minutes. What percent of its initial mass is left? How many kilograms is this? What is the ship's velocity at this time?

Solution

(a) The rocket equation is $v = u\ln\frac{m_0}{m}$, so for constant acceleration we have

$$a = \frac{dv}{dt} = u\frac{d}{dt}(\ln m_0 - \ln m) = -u\frac{dm/dt}{m} = u\frac{|dm/dt|}{m}$$

Therefore

$$|dm/dt|_0 = \frac{am_0}{u} = \frac{10\ \text{m/s}^2 \times 28 \times 10^6\ \text{kg}}{2500\ \text{m/s}} = 1.1 \times 10^5\ \text{kg/s}.$$

(b)

$$a = u\frac{|dm/dt|}{m} = -u\frac{dm/dt}{m}, \quad \text{so} \quad -\int \frac{adt}{u} = \int \frac{dm}{m}.$$

Therefore

$$m = m_0 e^{-\frac{a}{u}t}, \quad \frac{gt}{u} = 1 \Rightarrow t = \frac{u}{g} = \frac{2500\ \text{m/s}}{90\ \text{m/s}^2} = 250\ \text{s} = 4.17\ \text{minutes}.$$

(c)

$$m = m_0 e^{-\frac{g}{u}(20\ \text{min})}, (\frac{m}{m_0}) = e^{-\frac{20\ \text{min}}{4.17\ \text{min}}} = e^{-4.80} = 8.23 \times 10^{-3},$$

so 0.832% of the mass is left.

$$m = .00823\left[23 \times 10^8\text{kg}\right] = 0.23 \times 10^6\text{kg} = 230,000\ \text{kg} = 230\ \text{tons}$$

$$v = 2500\ \text{m/s}\ \ln(\frac{3100}{230}) = 2500\ \text{m/s} \cdot \ln(13.5) = 6500\ \text{m/s} = 6.50\ \text{km/s}$$

■

⋆⋆ **Problem 1.26** Beginning at time $t = 0$, astronauts in a landing module are descending toward the surface of an airless moon with a downward initial velocity $-|v_0|$ and altitude $y = h$ above the surface. The gravitational field g is essentially constant throughout this descent. An onboard retrorocket can provide a fixed downward exhaust velocity u. The astronauts need to select a fixed exhaust rate $\lambda = |dm/dt|$ in order to provide a soft landing with velocity $v = 0$ when they reach the surface at $y = 0$. (a) Explain briefly why Newton's second law for the module during its descent has the form

$$m(t)\frac{dv}{dt} = u\left|\frac{dm}{dt}\right| - m(t)g$$

(b) Find the velocity v of the module as a function of time, in terms of $|v_0|, u, m_0, \lambda$, and g. (c) During the descent its velocity is $v = dy/dt$, negative because it is downward. Find an expression for $y(t)$ in terms of $|v_0|, g, u, \lambda, m_0$, and h.

Solution

(a) The thrust $u|dm/dt|$ behaves like an upward force, while the gravitational force mg is downward. So Newton's second law becomes $ma = mdv/dt = u|dm/dt| - mg$ where a is positive upward.

(b) Given that $\frac{dm}{dt} = -\lambda$, where λ is a positive constant, we have $\frac{dv}{dt} = \frac{u}{m}\lambda - g = \frac{dv}{dm}\frac{dm}{d4t}$ by the chain rule, so $-\lambda\frac{dv}{dm} = \lambda\frac{u}{m} - g$. Divide by $(-\lambda)$ and integrate over m:

$$\int_{v_0}^{v} dv = -u\int_{m_0}^{m}\frac{dm}{m} + \frac{g}{\lambda}\int_{m_0}^{m} dm$$

so

$$v = v_0 - gt + u\ln\left(\frac{m_0}{m_0 - \lambda t}\right)$$

where $v_0 = -|v_0|$.

(c) Integrating once again, starting at $y = h$,

$$y - h = \int_0^t v\,dt = -|v_0|t - \frac{1}{2}gt^2 - u\int_0^t dt\,\ln\left(\frac{m_0 - \lambda t}{m_0}\right)$$

Let $q \equiv \frac{m_0 - \lambda t}{m_0}$, so then

$$\int dt\,\ln q = -(m_0/\lambda)\int dq\,\ln q = -\frac{m_0}{\lambda}[q(\ln q - 1)]$$

so

$$\begin{aligned} y &= h - |v_0|t - \frac{1}{2}gt^2 + \frac{u}{\lambda}\left[(m_0 - \lambda t)\left(\ln\frac{m_0 - \lambda t}{m_0} - 1\right) + m_0\right] \\ &= h - (|v_0| - u)t - \frac{1}{2}gt^2 + \frac{u}{\lambda}(m_0 - \lambda t)\ln\left(\frac{m_0 - \lambda t}{m_0}\right). \end{aligned}$$

■

★★★ **Problem 1.27** A spaceprobe of mass M is propelled by light fired continuously from a bank of lasers on the moon. A mirror covers the rear of the probe; light from the lasers strikes the mirrors and bounces directly back. In the rest-frame of the lasers, n_γ photons are fired per second, each with momentum $p_\gamma = h\nu_\gamma/c$, where h is Planck's constant, c is the speed of light, and ν is the photon's frequency. (a) Show that in a short time interval Δt the change in the probe's momentum is $2n'_\gamma p'_\gamma \Delta t$, where n'_γ is the number of photons striking the mirror per second, and p'_γ is the momentum of each photon, both in the *probe's* frame of reference. (b) The photons are Doppler-shifted in the probe's frame, so their frequency is only $\nu' \approx \nu(1-v/c)$, where v is the velocity of the probe. Show also that $n'_\gamma = n_\gamma(1-v/c)$, and then show that the ship's acceleration has the form $a = \alpha(1 - v/c)^2$ where α is a constant. Express α in terms of M, n_γ, and p_γ. (c) Find an expression for the probe's velocity as a function of time. Briefly discuss the nature of this result as the probe travels faster and faster.

Solution

(a) The change in momentum of one photon in the instantaneous rest-frame of the probe is $2p'_\gamma$, so that is also the change in the probe's momentum for each photon. During a short time interval Δt the number of photons striking the probe is $n'_\gamma\Delta t$, so the overall change of momentum of the probe is $2p'_\gamma(n'_\gamma\Delta t)$.

(b) The frequency of a photon in the probe's frame is $\nu' \approx \nu(1 - v/c)$ where ν is the frequency in the moon's frame. Therefore $p'_\gamma = p_\gamma(1 - v/c)$ since for each photon there is a Doppler shift. Also $n'_\gamma = n_\gamma(1 - v/c)$, where n_γ is the number per second in the moon's frame, and n'_γ is the number per second in the probe's frame. This can be seen by picturing a tube of radiation which, in the frame of the moon, has a length of one light-second. This radiation is directed towards the right, aimed at the probe. Then all the photons in this tube will pass through the right end of the tube within a time of one second. The probe is also moving toward the right, so in one second it will move a distance $v\times$ 1 second. Therefore there are some photons in the tube that will not be able to reach the probe in 1 second: namely, those within a length $v\times$ 1 second at the left end of the tube, which comprise a fraction v/c of all the photons in the tube. Those reaching the probe in 1 second are therefore a fraction $(1 - v/c)$ of the total.

Now the overall change of momentum of the probe in time Δt is $\frac{\Delta P}{\Delta t} = M\frac{\Delta v}{\Delta t} = Ma$, so its acceleration is

$$\frac{\Delta P/\Delta t}{M} = \frac{2n'_\gamma p'_\gamma}{M} = \frac{2n_\gamma p_\gamma}{M}(1 - v/c)^2 \equiv \alpha(1 - v/c)^2$$

where $\alpha = 2n_\gamma p_\gamma/M$.

(c)

$$\frac{dv}{dt} = \alpha(1 - v/c)^2 \Rightarrow \int \frac{dv}{(1 - v/c)^2} = \alpha \int dt\,.$$

Let

$$u \equiv 1 - v/c \Rightarrow du = -dv/c, -\int \frac{du\, c}{u^2} = \alpha t\,,$$

$$\begin{aligned} t &= -\frac{c}{\alpha}\int \frac{du}{u^2} = \frac{c}{\alpha u}\bigg|_{u_0}^{u}, \frac{\alpha t}{c} = \frac{1}{u} - \frac{1}{u_0} \\ &= \frac{1}{1 - v/c} - 1 \Rightarrow 1 - v/c = (1 + \frac{\alpha t}{c})^{-1} \Rightarrow v/c = 1 - \frac{1}{1 + \alpha t/c}\,. \end{aligned}$$

$$\frac{v}{c} = \frac{\alpha t/c}{1 + \alpha t/c}\,.$$

At first the probe accelerates quickly, with acceleration α. The acceleration falls off with time, because each photon has been Doppler-shifted to the red, and also fewer photons per second strike the probe as the probe moves faster and faster. ■

⋆⋆ **Problem 1.28** A proposed interstellar ram-jet would sweep up deuterons in space, burn them in an onboard fusion reactor, and expel the reaction products out the tail of the ship. In a reference frame instantaneously at rest relative to the ship, deuterons, each of mass m, approach the ship at relative velocity v. They are burned, and the burn products, with essentially the same total mass, are ejected from the rear of the ship at velocity $v + u$. The ship mass M stays constant, the cross-sectional area of the ship is A, and the number of deuterons per unit volume is n. (a) Find dN/dt, the number of deuterons swept up per unit time, in terms of n, A, and v. (b) Find dP/dt, the change in total momentum of the ship per unit time. (c) Show that the velocity of the ship increases exponentially, with $v = v_0 e^{\alpha t}$ where v_0 is the ship's initial velocity and where α is a constant, which can be expressed in terms of given parameters. Assume that u is constant.

Solution
(a) In time Δt, the ship picks up $\Delta N = n(\Delta \mathrm{Vol}) = nAv\Delta t$ neutrons. So $dN/dt = nAv$.

(b) $\frac{dP}{dt} = M\frac{dv}{dt} = unAmv$.

That is, conserving overall momentum between times t and $t + \Delta t$, we have $-\Delta mv = M\Delta v - \Delta m(v+u) \Rightarrow M\Delta v = \Delta mu \Rightarrow$
$M\frac{dv}{dt} = \frac{dm}{dt}u = m\frac{dN}{dt}u = nAvmu$.

(c) Separating variables and integrating,

$$dv/v = \frac{nAmu}{M}dt \Rightarrow \ln(v/v_0) = \left(\frac{nAmu}{M}t\right)$$

Therefore $v = v_0 e^{\frac{nAmu}{M}t} = v_0 e^{\alpha t}$ where $\alpha = nAmu/M$. ■

★ **Problem 1.29** (a) An open railroad coal car of mass M is rolling along a horizontal track at velocity v_0 when a coal chute suddenly dumps a load of coal of mass m into the coal car, vertically in the frame of the ground. When the load of coal has come to rest relative to the coal car, how fast is the coal car moving? (b) A similar coal car, of mass M and velocity v_0, has a covered hole in its bottom; when the cover is suddenly opened a mass m of coal falls onto the tracks. How fast is the coal car then moving?

Solution
(a) Conserve momentum in the horizontal direction. Therefore

$$Mv_0 = (M+m)v_f \Rightarrow v_f = (M/(M+m_0))v_0$$

(b) In this case the coal "remembers" its initial horizontal velocity v_0 as it falls out, by the law of inertia. That is, it has the same horizontal velocity afterwards as it did before. Therefore the coal car keeps moving at speed v_0 in order to conserve overall momentum. ■

★ **Problem 1.30** Half of a chain of total mass M and length L is placed on a frictionless table top, while the other half hangs over the edge.

If the chain is released from rest, what is the speed of the last link just as it leaves the table top?

Solution
Conserve energy. Initially the chain's energy is entirely potential energy U. Let $U = 0$ for any link at the level of the table top. Then the initial total energy is $U = -(\frac{M}{2})\frac{L}{4}g = -\frac{ML}{8}g$, since a mass $M/2$ of the chain hangs over the edge, and the center of mass of the hanging part of the chain is a distance $L/4$ below the tabletop. As the last link leaves the tabletop, the potential energy of the chain is $-MgL/2$. (Note that now $L/2$ is the distance of the overall center of mass below the table). Therefore

$$T_0 + U_0 = T_f + U_f : 0 - \frac{MLg}{8} = T_f - \frac{MLg}{2},$$

so

$$T_f = MgL(\frac{1}{2} - \frac{1}{8}) = \frac{3}{8}MgL = \frac{1}{2}Mv^2.$$

Finally, $v^2 = \frac{3}{4}gL$, so $v = \frac{\sqrt{3gL}}{2}$. ■

★ **Problem 1.31** A particle of mass m is free to move in one dimension between the coordinates $x = 0$ and $x = 2\pi/k$, where k is a positive constant. Within this range the particle is subject to the force $F = \alpha \sin(kx)$, where α is a constant. (a) If the maximum value of the corresponding potential energy is α/k, what are the turning points of the particle if its energy is $E = \alpha/2k$? (b) Find the speed of the particle as a function of x.

Solution

(a) The potential energy is

$$U = -\int F dx = -\alpha \int^x \sin kx\, dx = \frac{\alpha}{k} \cos kx + con.$$

The constant is zero here, since $U_{max} = U_{in}$. The turning points occur where $T = 0$, i.e. where

$$U(x) = \frac{\alpha}{k} \cos kx = E = \frac{\alpha}{2k},$$

so $\cos kx = \frac{1}{2}$ or $kx = \frac{\pi}{3}$ and $\frac{5\pi}{3}$. Therefore the turning points are at $x = \frac{\pi}{3k}$ and $x = \frac{5\pi}{3k}$.

(b) Therefore

$$\frac{1}{2}mv^2 = E - U = \frac{\alpha}{2k} - \frac{\alpha}{k} \cos kx$$

$$v(x) = \sqrt{\frac{2}{m}\left(\frac{\alpha}{k}\right)\left(\frac{1}{2} - \cos kx\right)} = \sqrt{\frac{\alpha}{km}}(1 - 2\cos kx).$$

■

★★ **Problem 1.32** One end of a string of length ℓ is attached to a small ball, and the other end is tied to a hook in the ceiling. A nail juts out from the wall, a distance d ($d < \ell$) below the hook. With the string straight and horizontal, the ball is released. When the string becomes vertical it meets the nail, and then the ball swings upward until it is directly above the nail. (a) What speed does the ball have when it reaches this highest point? (b) Find the minimum value of ℓ such that the ball can reach this point at all.

Solution

Energy is conserved. The initial energy is $T_0 + U_0 = 0 + 0$ where we measure gravitational potential energy from the ball's initial position. The final energy is $T_f + U_f = \frac{1}{2}mv^2 - mg(2d - l)$. Therefore

$$\frac{1}{2}mv^2 - mg(2d - l) = 0 \Rightarrow v_f^2 = 2g(2d - l)$$

Now apply $F = ma$ at the final point.

$$F = \text{tension} + mg(\text{positive downward}) = ma = mv^2/R = mv_f^2/(l - d)$$

(Here v^2/R is the acceleration at the top, because the ball is moving in a circle.) Note that the tension in a string cannot be negative. Therefore,

$$\text{Tension}+mg = \frac{mv_f^2}{l-d} = \frac{m(2g)(2d-l)}{l-d} \geq mg \Rightarrow 2(2d-l) \geq l-d \Rightarrow 5d \geq 3l, d \geq (3/5)l$$

for the ball to reach the upper point. ■

★ **Problem 1.33** A rope of mass/length λ is in the shape of a circular loop of radius R. If it is made to rotate about its center with angular velocity ω, find the tension in the rope. *Hint*: Consider a small slice of the rope to be a "particle."

Solution

Here $\Delta m = \lambda\Delta s = \lambda R\Delta\theta$ is the mass of the small piece of rope subtending angle $\Delta\theta$, measured from the center of the loop. There are tension forces at each end of this small piece, so that the net inward force due to the tension is $2T\sin\frac{\Delta\theta}{2} \simeq 2T\cdot\frac{\Delta\theta}{2}$ (for $\Delta\theta \ll 1$) $= \Delta m R\omega^2$ (using $F = ma$) $= \lambda R\Delta\theta \cdot R\omega^2$. So $T = \lambda R^2\omega^2$. ■

★ **Problem 1.34** A particle is attached to one end of an unstretched Hooke's-law spring of force-constant k. The other end of the spring is fixed in place. If now the particle is pulled so the spring is stretched by a distance x, the potential energy of the particle is $U = (1/2)kx^2$. (a) Now suppose there are *two* springs with the same force constant that are laid end-to-end in the y direction, with a particle attached between them. The other ends of the springs are fixed in place. Now the particle is pulled in the *transverse* direction a distance x. Find its potential energy $U(x)$. (b) $U(x)$ is proportional to what power of x in the limit of small x, and to what power of x in the limit of large x?

Solution

(a) $U = 2 \times \frac{1}{2}k\cdot(\text{stretch})^2$. Let the unstretched length of each spring be l_0. Then (by Pythagoras), the stretch is $\sqrt{l_0^2 + x^2} - l_0$, so

$$U = k\left[\sqrt{l_0^2 + x^2} - l_0\right]^2 = kl_0^2\left[\sqrt{1 + (x^2/l_0^2)} - 1\right]^2$$

or

$$U = k\left[2l_0^2 + x^2 - 2l_0\sqrt{l_0^2 + x^2}\right].$$

(b) For small x, $(1 + x^2/l_0^2)^{1/2} = 1 + \frac{x^2}{2l_0^2}$ using the binomial expansion, so

$$U \simeq kl_0^2\left(\frac{x^2}{2l_0^2}\right)^2 = \frac{1}{4}k\frac{x^4}{l_0^2}$$

for $x \ll l_0$. For large x instead, $\sqrt{1 + x^2/l_0^2} \simeq x/l_0$ valid for $x/l_0 \gg 1$, so then

$$U \simeq kl_0^2(x/l_0)^2 = kx^2.$$

■

⋆⋆ **Problem 1.35** A spherical pendulum consists of a bob of mass m on the end of a light string of length R hung from a point on the ceiling, and with a uniform gravitational field g downward. The position of the bob can be specified by the polar angle θ of the string (the angle of the string and bob from the vertical) and the azimuthal angle φ (the angle of the string and bob from, say, the north as projected down onto a horizontal base plane.) (a) Show that the square of the velocity of the bob at any moment is $v^2 = R^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2)$. Then in terms of any or all of m, R, g, and the two coordinates θ and φ and their first time derivatives (b) find an expression for the energy E of the bob and explain why it is conserved; (c) find an expression for the angular momentum ℓ of the bob about the vertical axis passing through the point of support, and explain why it is conserved.

Solution

(a) The velocity in the θ direction is $R\dot{\theta}$, and the velocity in the φ direction is $(R\sin\theta)\dot{\varphi}$. So the square of the velocity is

$$v^2 = R^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2)$$

(b) The energy is

$$T + U = \frac{1}{2}mv^2 - mgR\cos\theta$$

$$E = \frac{1}{2}mR^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) - mgR\cos\theta$$

(The potential energy has been taken as zero at the point of support). Energy is conserved because no work is being done on the bob apart from gravity, and that is a conservative force with a potential energy that has been included in E.

(c) The angular momentum about the vertical axis is $mR^2\sin^2\theta\dot{\phi}$, which is conserved because there is no torque in the $\hat{\phi}$ direction. ■

⋆ **Problem 1.36** Consider an arbitrary power-law central force $\mathbf{F}(\mathbf{r}) = -kr^n\hat{\mathbf{r}}$, where k and n are constants and r is the radius in spherical coordinates. Prove that such a force is conservative, and find the associated potential energy of a particle subject to this force.

Solution

We can use the $\nabla \times \mathbf{F} = 0$ test. In spherical coordinates,

$$\nabla \times \mathbf{F} = \frac{1}{r^2\sin\theta}\begin{bmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\phi} \\ \partial_r & \partial_\theta & \partial_\phi \\ F_r & rF_\theta & r\sin\theta F_\phi \end{bmatrix}$$

$$= \frac{1}{r^2 \sin\theta} \begin{bmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\phi} \\ \partial_r & \partial_\theta & \partial_\phi \\ kr^n & 0 & 0 \end{bmatrix}$$

$$= \frac{1}{r^2 \sin\theta} \left(-r\hat{\boldsymbol{\theta}} \begin{bmatrix} \partial_r & \partial_\phi \\ kr^n & 0 \end{bmatrix} + r\sin\theta\hat{\boldsymbol{\phi}} \begin{bmatrix} \partial_r & \partial_\theta \\ kr^n & 0 \end{bmatrix} \right) \tag{1.3}$$

$$= \frac{1}{r^2 \sin\theta} [0+0] = 0, \text{so } \mathbf{F} = kr^n \hat{\mathbf{r}}$$

is conservative. (In fact, any central force is conservative.)

$$U(r) = -\int \mathbf{F}(r) \cdot d\mathbf{s} = -k \int r^n \hat{\mathbf{r}} \cdot d\mathbf{r} = -k \int r^n dr = \frac{-kr^{n+1}}{n+1} + \text{constant}$$

$$\Rightarrow U(r) = -\frac{k}{n+1} r^{n+1} + \text{constant}$$

The integration constant is arbitrary. We can set it equal to zero, usually the simplest choice. ■

⋆ **Problem 1.37** The potential energy of a mass m on the end of a Hooke's-law spring of force constant k is $(1/2)kx^2$. if the maximum speed of the mass with this potential energy is v_0, what are the turning points of the motion?

Solution

The maximum speed is where the potential energy $U = 0$ here, which is at $x = 0$. So conserving energy, $E = T + U =$constant.

$$\frac{1}{2}mv_0^2 + 0 = 0 + \frac{1}{2}kx_t^2 \Rightarrow x_{tp} = \pm\sqrt{\frac{m}{k}}v_0$$

There are two turning points, at $x_{tp} = \pm\sqrt{\frac{m}{k}}v_0$. ■

⋆ **Problem 1.38** Planets have roughly circular orbits around the sun. Using the table below of the orbital radii and periods of the inner planets, how does the centripetal acceleration of the planets depend upon their orbital radii? That is, find the exponent n in $a = \text{con} \times r^n$. (Note that 1 A. U. = 1 astronomical unit, the mean sun-earth distance.)

Planet	Mean orbital radius (A. U.)	Period (Years)
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.000	1.000
Mars	1.523	1.881

Solution

The acceleration for circular orbits is $a = v^2/r = r\omega^2 = r(\frac{2\pi}{T})^2 = 4\pi^2 r/T^2 = \text{con} \times r^n$. Therefore, $\text{con} \times T^2 = 4\pi^2 r^{1-n}$, so $r = \text{con}T^{2/(1-n)}$, where T is the period. Take the $\log_{10}$ of this equation, $\log r = \log(\text{con}) + \frac{2}{1-n}\log T$. We can then construct a table:

Planet	r	$\log r$	T	$\log T$
Mercury	.387	–.412	.241	–.618
Venus	.723	–.141	.615	–.211
Earth	1.000	0	1.000	0
Mars	1.523	.183	1.881	.274

From earth's numbers we find $\log(\text{con}) = 0$.

Therefore for Mercury: $-.412 = \frac{2}{1-n}(-.618) \Rightarrow = .667$.

For Venus: $-.141 = \frac{2}{1-n}(-.211) \Rightarrow \frac{2}{1-n} = .668$.

For Mars: $0.183 = \frac{2}{1-n}(.274) \Rightarrow \frac{2}{1-n} = .667$.

All are equal to $2/3$ within the significant figures given. Therefore $n = -2$. This is correct, since $a = F/m = GM/r^2$. ■

⋆ **Problem 1.39** Four mathematically equivalent conditions for a force to be conservative are given in the chapter. One condition is that a conservative force can always be written as $\mathbf{F} = -\nabla U$. Show then that each of the other three conditions is a necessary consequence.

Solution

Given that $\mathbf{F} = -\nabla U$, it follows that $\nabla \times \mathbf{F} = 0$, because $\nabla \times \nabla U = 0$ for any scalar field U. *Thus condition 3 is satisfied*. Then also $\oint \mathbf{F} \cdot d\mathbf{s} = \oint -\nabla U \cdot d\mathbf{s} = -\oint U = -\int_a^a dU = -[U(a) - U(a)] = 0$ where a is any point on the closed path. *Thus condition 2 is satisfied.* Then

$$\int_a^b \mathbf{F} \cdot d\mathbf{s} = -\int_a^b \nabla U \cdot ds = -\int_a^b dU = -(U(a) - U(a))$$

which depends on the end points only, and not on the path. *So condition (1) is also satisfied.* Therefore all three of the conditions are satisfied. ■

⋆⋆ **Problem 1.40** A rock of mass m is thrown radially outward from the surface of a spherical, airless moon of radius R. From Newton's second law its acceleration is $\ddot{r} = -GM/r^2$, where M is the moon's mass and r is the distance from the moon's center to the rock. The energy of the rock is conserved, so $(1/2)m\dot{r}^2 - GMm/r = E = \text{constant}$. (a) Show by differentiating this equation that energy conservation is a first integral of $F = m\ddot{r}$ in this case. (b) What is the minimum value of E, in terms of given parameters, for which the rock will escape from the moon? (c) For this case what is $\dot{r}(t)$, the velocity of the rock as a function of time since it was thrown?

Solution

(a) Given that $\frac{1}{2}m\dot{r}^2 - GMm/r = E$, differentiation gives

$$m\dot{r}\ddot{r} + \frac{GMm}{r^2}\dot{r} = 0 \Rightarrow -\frac{GMm}{r^2} = m\ddot{r}$$

or $F = ma$ in the radial direction.

(b) As $r \to \infty$, energy conservation gives $\frac{1}{2}m\dot{r}^2 = E$. But $\dot{r}^2 \geq 0$, so the minimum energy for the rock to go out to infinity is $E = 0$.

(c) $E = 0 = \frac{1}{2}m\dot{r}^2 - GMm/r$ so

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2GM}{r}} = \frac{dr}{dt}$$

Therefore $\int_R^r dr\sqrt{r} = \int_0^t \sqrt{2GM}dt$, so

$$\frac{r^{3/2}}{3/2}\int_R^r = \sqrt{2GM}t, \quad r^{3/2} - R^{3/2} = \frac{3}{2}\sqrt{2GM}t$$

Solving for $r(t)$,

$$r(t) = \left[R^{3/2} + \frac{3}{2}\sqrt{2GM}t\right]^{2/3}$$

so

$$\dot{r}(t) = \frac{\sqrt{2GM}}{\left[R^{3/2} + \frac{3}{2}\sqrt{2GM}t\right]^{1/3}}$$

Check at $t = 0$: $\dot{r}(0) = \sqrt{\frac{2GM}{r}}$. Check as $t \to \infty$: $\dot{r}(t) \to \frac{\sqrt{2GM}}{\left[3/2\sqrt{2GM}t\right]^{1/3}} \to \left(\frac{4GM}{3}\right)^{1/3} t^{-1/3}$.

■

⋆⋆⋆ **Problem 1.41** Consider a point mass m located a distance R from the origin, and a spherical shell of mass ΔM, radius a, and thickness Δa, centered on the origin. The shell has uniform mass density ρ. (a) Find ΔM in terms of the other parameters given, assuming $\Delta a \ll a$. Show that the gravitational potential energy of the point mass m due to the shell's gravity is (b) $-G\Delta Mm/R$ for $R > a$; (c) a constant for $R < a$. (d) Then show that if a mass distribution is spherically symmetric the gravitational field inside it is directed radially inward, and its magnitude at radius R from the center is simply $GM(R)/R^2$, where $M(R)$ is the mass *within* the sphere whose radius is R. That is, a shell whose radius is greater than R exerts no net gravitational force on m.

Solution

(a) The mass

$$\Delta M = \rho(\text{Vol}) = \rho\text{Area} \cdot \text{thickness} = \rho(4\pi a^2)\Delta a.$$

(b) Picture the shell sitting on a table top. Slice the shell horizontally into thin rings, each ring identified by its angle from the shell's uppermost point (the "north pole" of the shell), with angles measured using radial lines from the center of the shell. The radius of the ring is $a\sin\theta$. Its thickness in the radial direction is Δa and its thickness in the tangential direction is $a\Delta\theta$. Therefore the volume of the ring is $(2\pi a\sin\theta)\Delta a\Delta\theta$, and so its mass is

$$\Delta M_{\text{ring}} = \rho 2\pi a^2 \sin\theta\Delta a\Delta\theta.$$

Therefore the potential energy between m and the ring is

$$U_{\text{ring}} = -\frac{G\Delta M_{ring}m}{r} = -\frac{G\rho_m 2\pi a^2 \sin\theta\Delta a\Delta\theta}{r}$$

where $r^2 = R^2 + a^2 - 2Ra\cos\theta$ by the law of cosines. (Note that all points of the ring are equidistant from m).

Now let us find the potential energy due to an entire shell, by summing over the potentials due to the rings. The differential

$$\Delta(r^2) = 2r\Delta r = \Delta(R^2 + a^2 - 2aR\cos\theta) = 2aR\sin\theta\Delta\theta$$

since R is fixed. Therefore using

$$2aR\sin\theta\Delta\theta = 2r\Delta r, U_{ring} = -\frac{Gm\rho\pi a\Delta a(2r\Delta r)}{rR} = \frac{-Gm\rho 2\pi a\Delta a\Delta r}{R}$$

(b) Now we can sum over all the rings to get the potential energy of the entire shell. Note that if $R > a$, so the point mass m is outside the shell, then the quantity r varies from $R - a$ for the ring closest to M (i.e, with $\theta = 0$) up to $R + a$ for the ring farthest away (with $\theta = \pi$). Therefore

$$U_{\text{shell}} = \Sigma U_{\text{rings}} = \frac{-Gm\rho 2\pi a\Delta a\Delta r}{R}\int_{R-a}^{R+a} dr = \frac{-4\pi a^2 GM\rho\Delta a}{R}$$

The volume of the shell is $\Delta V_{\text{shell}} = 4\pi a^2\Delta a$ so the mass of the shell is $\Delta M_{\text{shell}} = \rho\Delta V_{\text{shell}}$, so $U_{\text{shell}} = -\frac{Gm\Delta M_{\text{shell}}}{R}$ if $R > a$. That is, the potential due to such a shell is the same as it would be if the entire shell were located at the origin.

(c) If $R < a$ (i.e, if the point mass m is inside the shell) then the distance r varies form $a - R$ upto $a + R$. So the potential due to such a shell is

$$U_{\text{shell}} = -\frac{2Gm\rho\pi a\Delta a}{R}[(a + R) - (a - R)] = -4Gm\rho\pi a\Delta a$$

which is a constant, independent of the position of m. The force due to such a shell is $F = -dU_{\text{shell}}/dr = 0$, so the force exerted by a shell of radius a such that $a > r$ is zero.

(d) Only shells of radius a ($a < r$) exert a force on m and the force they exert is

$$F = -\frac{dU_{\text{shell}}}{dr} = -\frac{Gm\Sigma\Delta M_{\text{shell}}}{R^2}$$

$F = -GmM_{\text{inside}}/R^2$ where M_{inside} = mass of all shells of radius $a < R$. ■

★★ **Problem 1.42** A tunnel is drilled straight through a uniform-density nonrotating spherically-symmetric airless asteroid of radius R. The tunnel is oriented along the x axis, with $x = 0$ at the center of the asteroid and of the tunnel. Using the results of the preceding problem, (a) show that if an astronaut of mass m steps into one side of the tunnel she will experience a spring-like force $F = -kx$ as she falls through the tunnel. (b) Find k in terms of any or all of G and the mass M and radius R of the asteroid. (c) Find the time it would take for her to oscillate from one end of the tunnel to the other and back again, in terms of the same parameters.

Solution

(a) In the preceding problem we learned that the force on an astronaut at a distance r from the center of the asteroid is $F = -\frac{GM(r)m}{r^2}$ where $M(r)$ is the mass within a distance r, which is $M(r) = \rho \cdot \frac{4}{3}\pi r^3$ where here $r = x$. Therefore $F(r) = -(G\rho \cdot \frac{4}{3}\pi m)x = -kx$ where $k = \frac{4}{3}\pi G\rho m$.

(b) $\rho = \frac{M}{vol} = \frac{M}{\frac{4}{3}\pi R^3} \Rightarrow k = \frac{4}{3}\pi Gm\frac{M}{\frac{4}{3}\pi R^3} = \frac{GMm}{R^3}$.

(c) For simple harmonic oscillators the angular frequency is $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{GM}{R^3}} = \frac{2\pi}{T}$, where T =period of oscillation. Therefore $T = 2\pi\sqrt{\frac{R^3}{GM}}$ is the time to fall through the tunnel and back. ■

★★ **Problem 1.43** Referring to the preceding problem, if a different straight tunnel is drilled through the same asteroid, where this time the tunnel misses the asteroid's center by a distance $R/2$, (a) how long would it take the astronaut to fall from one end of the tunnel to the other and back, assuming no friction between the sides of the tunnel and the astronaut? (b) Suppose that instead of falling through the tunnel, she is given an initial tangential velocity of just the right magnitude to insert her into a circular orbit just above the surface. How long will it take her to return to the starting point in this case?

Solution

(a) Picture the asteroid set on a table top, with a tunnel drilled through it horizontally. Any point in the tunnel can be identified by the angle θ of the point measured from the center of the asteroid, or by the distance x of the point measured from the center of the tunnel. The gravitational force toward the center of the asteroid is $\frac{GM(r)m}{r^2} = \frac{GMmr}{R^3}$, so the component of force toward the center of the tunnel is

$$-\frac{GMmr}{R^3}\sin\theta = -\frac{GMmr}{R^3}\frac{x}{r} = -\left(\frac{GMm}{R^3}\right)x = -kx$$

where k is the same as it is for a tunnel through the center of the asteroid. Therefore since the period of a simple harmonic oscillator is $T = 2\pi\sqrt{\frac{m}{k}}$, we again get $T = 2\pi\sqrt{R^3/GM}$, the same as for a tunnel through the center of the asteroid.

(b) In circular orbit $F = ma = -\frac{mv^2}{r} = -mr\omega^2$, so using $\omega = 2\pi/T$, we have

$$-\frac{GMm}{R^2} = -mR\left(\frac{2\pi}{T}\right)^2.$$

Solving for T we find $T = 2\pi\sqrt{R^3/GM}$, the same time as to oscillate once through the tunnel and back. ■

★ **Problem 1.44** Estimate the radius of the largest spherical asteroid an astronaut could escape from by jumping.

Solution

Suppose the astronaut can raise her center of mass a distance h when jumping up from the earth's surface. Then $\frac{1}{2}mv_{\text{jump}}^2 = mg_E h$ or $v_{\text{jump}}^2 = 2g_E h$. To escape the spherical asteroid,

$$\frac{1}{2}mv_{\text{jump}}^2 - \frac{GM_{\text{ast}}m}{R_{\text{ast}}} = 0, \text{ so } v_{\text{jump}}^2 = \frac{2GM_{\text{ast}}}{R_{\text{ast}}} = \frac{2G\frac{4}{3}\pi R^3\rho_{\text{ast}}}{R}$$

$$v_{\text{jump}}^2 = \frac{8\pi}{3}G\rho_{\text{ast}}R_{\text{ast}}^2 = 2g_E h = \frac{2GM_E}{R_E^2}h = 2G\frac{\frac{4}{3}\pi R_E^3\rho_E h}{R_E^2} = \frac{8\pi}{3}GR_E\rho_E h$$

$$\Rightarrow R_{\text{ast}}^2 = \frac{\rho_E}{\rho_{\text{ast}}}hR_E$$

$$\text{so } R_{\text{ast}} \sim \sqrt{\frac{\rho_E}{\rho_{\text{ast}}}}\sqrt{hR_E}$$

Now $R_E \simeq 6400$ km, and estimate $h \cong 1$ m. Then

$$\rho_E/\rho_{\text{ast}} \simeq 2 \Rightarrow R_{\text{ast}} \sim 1.4 \times (6.4 \times 10^6 \text{m})^{1/2} \simeq 1.4 \times 2.5 \text{ km} \sim 4 \text{ km},$$

which is a very rough estimate. (Reasonable answers might vary from 1 km to 10 km). ■

⋆⋆⋆ **Problem 1.45** A particle of mass m is subject to the central attractive force $\mathbf{F} = -k\mathbf{r}$, like that of a Hooke's-law spring of zero unstretched length, whose other end is fixed to the origin. The particle is placed at a position $\mathbf{r_0}$ and given an initial velocity $\mathbf{v_0}$ that is not colinear with $\mathbf{r_0}$. (a) Explain why the subsequent motion of the particle is confined to a plane containing the two vectors $\mathbf{r_0}$ and $\mathbf{v_0}$. (b) Find the potential energy of the particle as a function of r. (c) Explain why the particle's angular momentum is conserved about the origin, and use this fact to obtain a first-order differential equation of motion involving r and dr/dt. (d) Show that the particle has both an inner and an outer turning point, and solve the equation for $t(r)$, where the particle is located at an outer turning point at time $t = 0$. (e) Invert the result to find $r(t)$ in this case.

Solution

(a) The initial velocity is $\mathbf{v}_0$ and the initial acceleration is

$$\mathbf{a}_0 = \mathbf{F}_0/m = -(k/m)\mathbf{r}_0.$$

A bit later the velocity is

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{a}_0\Delta t = \mathbf{v}_0 - (k/m)\mathbf{r}_0\Delta t,$$

still in the plane defined by the vectors $\mathbf{v}_0$ and $\mathbf{r}_0$. There is no acceleration perpendicular to this plane, at any later time, so the motion is confined to this plane indefinitely.

(b) The potential energy is

$$U = -\int \mathbf{F}\cdot d\mathbf{r} = k\int \mathbf{r}\cdot d\mathbf{r} = k\int r dr = \frac{1}{2}kr^2$$

(c) The torque is

$$\mathbf{N} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times (-k\mathbf{r}) = 0$$

since $\mathbf{r} \times \mathbf{r} = 0$. Therefore the angular momentum is conserved, since

$$d\boldsymbol{\ell}/dt = \mathbf{N} = 0.$$

That is,

$$\boldsymbol{\ell} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\dot{\mathbf{r}} = \ell_0 = \text{constant}$$

In polar coordinates

$$\mathbf{r} = r\hat{\mathbf{r}}, \qquad \dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta}$$

so

$$\mathbf{r} \times m\dot{\mathbf{r}} = r\hat{\mathbf{r}} \times m\left[\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta}\right] = 0 + mr^2\dot{\theta}\hat{\mathbf{z}}\ ,$$

using $\hat{\mathbf{r}} \times \hat{\theta} = \hat{\mathbf{z}}$, a unit vector perpendicular to the plane. Conserving energy,

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}kr^2 = E, \text{or } \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2(\frac{l^2}{m^2r^4}) + \frac{1}{2}kr^2 = E = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + \frac{1}{2}kr^2.$$

The equation of motion is

$$E = \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} + \frac{1}{2}kr^2,$$

a first-order differential equation.

(d) The turning points occur where $\dot{r} = 0$, so then

$$\frac{1}{2}kr^2 + \frac{l^2}{2mr^2} - E = 0 \text{ or } r^4 + (\frac{l^2}{mk}) - \frac{2}{k}Er^2 = 0.$$

Using the quadratic equation, we have

$$r^2 = \frac{2E/k \pm \sqrt{4E^2/k^2 - 4l^2/mk}}{2},\ r_\pm^2 = E/k \pm \sqrt{E^2/k^2 - l^2/mk}.$$

These are the two turning points. Solving the equation of motion for $\dot{r}$, we have

$$\frac{1}{2}m\dot{r}^2 = E - \frac{l^2}{2mr^2} - \frac{1}{2}kr^2 \text{ or } \dot{r} = \pm\left[\frac{2}{m}(E - \frac{l^2}{2mr^2} - \frac{1}{2}kr^2)\right]^{1/2}$$

$$\int_0^t dt = \pm\int_{r_0}^r dr\left[\frac{2}{m}(E - \frac{l^2}{2mr^2} - \frac{1}{2}kr^2)\right]^{-1/2}$$

$$t(r) = \int_{r_0}^r \frac{dr}{\left[\frac{2}{m}(E - \frac{l^2}{2mr^2} - \frac{1}{2}kr^2)\right]^{1/2}} = \sqrt{\frac{m}{2}}\int_{r_0}^r \frac{dr\ r}{\sqrt{-\frac{l^2}{2m} + Er^2 - \frac{1}{2}kr^4}}$$

Let $x = r^2, dx = 2rdr$. Then

$$t(x) = \frac{1}{2}\sqrt{\frac{m}{2}} \int_{r_0^2}^{r^2} \frac{dx}{\sqrt{-\frac{l^2}{2m} + Ex - \frac{1}{2}kx^2}} = -\frac{1}{2}\sqrt{\frac{m}{2}} \frac{1}{\sqrt{k/2}} \sin^{-1}\left(\frac{-kx+E}{\sqrt{E^2 - l^2k/m}}\right)_{r_0^2}^{r^2}$$

$$= -\frac{1}{2}\sqrt{\frac{m}{k}}\left[\sin^{-1}\left(\frac{E-kr^2}{\sqrt{E^2-\ell^2k/m}}\right) - \sin^{-1}\left(\frac{E-kr_0^2}{\sqrt{E^2-\ell^2k/m}}\right)\right]$$

where

$$r_0^2 = E/k + \sqrt{E^2/k^2 - \ell^2/mk}, kr_0^2 = E + \sqrt{E^2 - \ell^2k/m}$$

$$t(r) = -\frac{1}{2}\sqrt{\frac{m}{k}}\left[\sin^{-1}(\frac{E-kr^2}{\sqrt{E^2-\ell^2k/m}}) - \sin^{-1}(-1)\right]$$
$$= -\frac{1}{2}\sqrt{\frac{m}{k}}\left[\sin^{-1}(\frac{E-kr^2}{\sqrt{E^2-\ell^2k/m}}) + \frac{\pi}{2}\right]$$

(e) Inverting,

$$-2\sqrt{\frac{k}{m}} - \frac{\pi}{2} = \sin^{-1}\frac{E-kr^2}{\sqrt{E^2-\ell^2k/m}}$$

$$\frac{E-kr^2}{\sqrt{E^2-\ell^2k/m}} = \sin\left[-2\sqrt{k/m}t + -\frac{\pi}{2}\right] = -\sin 2\sqrt{k/m}t + \pi/2 = -\left[\cos 2\sqrt{\frac{k}{m}}t\right]$$

Therefore

$$E - kr^2 = -\sqrt{E^2 - k\ell^2/m}\cos(2\sqrt{\frac{k}{m}}t), r^2 = \frac{1}{k}\left[E + \sqrt{E^2 - k\ell^2/m}\cos(2\sqrt{\frac{k}{m}}t)\right]$$

Check at $t = 0$:

$$r^2 = \frac{1}{k}\left[E + \sqrt{E^2 - (k\ell^2/m)}\right]$$

which is indeed the outer turning point. So in general

$$r(t) = \left[\frac{E}{k} + \sqrt{(\frac{E}{k})^2 - (\ell^2/mk)}\cos(2\sqrt{k/m}t)\right]^{1/2}.$$

■

⋆ **Problem 1.46** A water molecule consists of an oxygen atom with a hydrogen atom on each side. The smaller of the two angles between the two *OH* bonds is 108°. Find the distance of the center of mass of a water molecule from the oxygen atom in terms of the distance *d* between the oxygen atom and either hydrogen atom. The O has 16 times the mass of each *H*.

Solution

The center of mass (CM) is on the line of symmetry of the molecule, extending from the O atom staying halfway between the H atoms. If y_{CM} is the distance of the CM from the O atom, then

$$M_o \cdot y_{CM} = 2M_o(d\cos 54^{\circ} - y_{CM})$$

$$16y_{CM} = 2(d\cos 54^{\circ} - y_{CM})$$

$$\Rightarrow 18y_{CM} = 2d\cos 54^{\circ} \Rightarrow y_{CM} = \frac{1}{9}d\cos 54^{\circ} = \frac{0.5878}{9}d = 0.065d.$$

■

★ **Problem 1.47** A solid semicircle of radius R and mass M is cut from sheet aluminum. Find the position of its center of mass, measured from the midpoint of the straight side of the semicircle.

Solution

Set the straight side of the solid semicircle flat on a table top; then the CM is on the vertical line extending up from the straight side of the semicircle some distance y_{CM}. Now slice the sheet into thin horizontal strips whose centers are a distance y above the straight side, and whose half-length is x. Then

$$R^2 = x^2 + y^2 \Rightarrow x = \sqrt{R^2 - y^2}$$

so

$$dm = \sigma(2x)dy = 2\sigma\sqrt{R^2 - y^2}dy \quad (\sigma = \text{mass/area}).$$

Therefore

$$y_{\text{CM}} = \frac{2\sigma\int_0^R \sqrt{R^2 - y^2}\,y\,dy}{2\sigma\int_0^R \sqrt{R^2 - y^2}\,dy}, \quad y = R\sin\theta \; dy = R\cos\theta d\theta$$

$$y_{\text{CM}} = \frac{\int R\sqrt{1 - \sin^2\theta}\, R\sin\theta R\cos\theta d\theta}{R\sqrt{1 - \sin^2\theta}\, R\cos\theta d\theta} = R\frac{\int_0^{\pi/2}\cos^2\theta\sin\theta d\theta}{\int_0^{\pi/2}\cos^2\theta d\theta}$$

$$= R\frac{-(\cos^3\theta)/3|_0^{\pi/2}}{\frac{1}{2}(\theta + \frac{1}{2}\sin 2\theta)|_0^{\pi/2}} = R\frac{1/3}{\frac{1}{2}(\frac{\pi}{2} + 0)} = \frac{4}{3\pi}R.$$

■

★ **Problem 1.48** Star α, of mass m, is headed directly towards Star β, of mass $3m$, with velocity v_0 as measured in $\beta's$ rest frame. (a) What is the velocity of their mutual center of mass, measured in $\beta's$ frame? (b) How fast is each star moving in the CM frame? (c) If the two stars merge upon colliding, how fast is the new star moving in the CM frame?

Solution

(a) By definition

$$v_{\mathrm{CM}} = \frac{m_\alpha v_\alpha + m_\beta v_\beta}{m_\alpha + m_\beta} = \frac{m_\alpha v_\alpha + 0}{m_\alpha + m_\beta} = (\frac{m_\alpha}{m_\alpha + m_\beta})v_0$$

(b) Now

$$v_\alpha = v_0 - v_{\mathrm{CM}} = \left[\frac{(m_\alpha + m_\beta) - m_\alpha}{m_\alpha + m_\beta}\right] v_0 = (\frac{m_\beta}{m_\alpha + m_\beta})v_0$$

$$v_\beta = -v_{\mathrm{CM}} = -(\frac{m_\alpha}{m_\alpha + m_\beta})v_0$$

(c) Zero. The combined star must be at rest in the CM frame, since by momentum conservation its momentum must be zero. ■

⋆ **Problem 1.49** (a) A neutron in a nuclear reactor makes a head-on elastic collision with a carbon nucleus, which is initially at rest and has 12 times the mass of a neutron. What fraction of the neutron's initial speed is lost in the collision? (b) Repeat part (a) if instead the neutron collides head-on with a deuteron (mass twice that of the neutron) within a heavy-water ($D_2 0$) molecule? (Carbon nuclei and deuterons can both be used as moderators in a reactor, whose purpose is to *moderate* their speeds, *i.e.* slow down neutrons, as slower neutrons are more likely to cause nuclear fission.)

Solution

(a) In the center of mass frame let the initial velocity of the neutron be v_0', to the *right*. Then in that frame the initial velocity of the carbon atom is $-\frac{1}{12}v_0'$, so that the total momentum is zero, as required in the CM frame. After the neutron and carbon atom collide, in the CM frame both initial velocities are simply reversed, so the total momentum is still zero, and both momentum and kinetic energy are conserved in the collision. Now the lab frame clearly moves to the left at velocity v_0' relative to the CM frame, because in the lab frame the carbon atom is initially at rest. Therefore in the lab frame the initial velocity of the neutron is $\frac{13}{12}v_0'$, and its final velocity is $-\frac{11}{12}v_0'$. Therefore the fraction of its initial speed lost is

$$\frac{\frac{13}{12}v_0' - \frac{11}{12}v_0'}{\frac{13}{12}v_0'} = (\frac{2}{12})/(13/12) = \frac{2}{13} = 15.4\%$$

That is, 15.4% of the neutron's initial speed is lost in the collision.

(b) The neutron's speed has fallen from $\frac{3}{2}v_0'$ to $\frac{1}{2}v_0'$, losing v_0'. The fraction lost is therefore

$$\frac{v_0'}{\frac{3}{2}v_0'} = \frac{2}{3} = 66.7\%$$

Therefore deuterons are more effective moderators; they slow the neutrons more effectively than carbon atoms. ■

⋆ **Problem 1.50** A neutron of mass m and velocity v_0 collides head-on with a ^{235}U nucleus of mass M at rest in a nuclear reactor, and the neutron is absorbed to form a ^{236}U nucleus. (a) Find the velocity v_A of the ^{236}U nucleus in terms of m, M, and v_0. (b) The ^{236}U nucleus subsequently fissions into two nuclei of equal mass, each emerging at angle θ to the forward direction. Find the speed v_B of each final nucleus in terms of given parameters. Use classical Newtonian physics to solve the problem.

Solution

(a) Conserve momentum:

$$mv_0 = (M+m)v_A$$

so

$$v_A = \left(\frac{m}{M+m}\right) v_0$$

where we have assumed mass conservation: the final mass is $M+m$.

(b) Again conserving momentum and mass,

$$(M+m)v_A = 2\left(\frac{M+m}{2}\right) v_B \cos\theta \quad \text{so} \quad v_B = v_A/\cos\theta.$$

Note that kinetic energy is *not* conserved, since

$$2 \times \frac{1}{2}\left(\frac{M+m}{2}\right) v_B^2 = \left(\frac{M+m}{2}\right)\frac{v_B^2}{\cos^2\theta} \neq \frac{1}{2}(M+m)v_A^2$$

In fact the K.E. has increased, indicating that some potential energy was stored in the ^{236}U nucleus. ■

⋆ **Problem 1.51** Two balls, with masses m_1 and m_2, both moving along the same straight line, strike one another head-on in a one-dimensional elastic collision. (a) Show that the magnitude of the relative velocity between the two balls is the same before and after the collision. (b) Also show that if a video were made of such a collision, and then shown to an audience, the viewers could not be sure from the motion of the balls whether the video were being run forward or backward in time. That is, such collisions are said to be time-reversal invariant. (c) Would this time-reversal invariance still be true if the collision were inelastic? Give an example.

Solution

(a) Let the initial velocities of the balls be v_{10} and v_{20}, and their final velocities be v_{1f} and v_{2f}. Conservation of momentum then gives

$$m_1 v_{10} + m_2 v_{20} = m_1 v_{1f} + m_2 v_{2f}.$$

The collision is elastic, so kinetic energy is also conserved:

$$\frac{1}{2}m_1 v_{10}^2 + \frac{1}{2}m_2 v_{20}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2.$$

We can rewrite these equations in the form

$$(1)\ m_1(v_{10} - v_{1f}) = m_2(v_{2f} - v_{20})$$

$$(2)\ m_1(v_{10}^2 - v_{1f}^2) = m_2(v_{2f}^2 - v_{20}^2).$$

Now divide Eq. 2 by Eq. 1, and use the identity $A^2 - B^2 = (A+B)(A-B)$. This gives $v_{10} + v_{1f} = v_{2f} + v_{20}$ or $v_{10} - v_{20} = v_{2f} - v_{1f}$. Therefore $|v_{10} - v_{20}| = |v_{2f} - v_{1f}|$, meaning that the relative velocities of the two balls are the same before and after.

(b) If time were reversed, $v_{10} \leftrightarrow -v_{1f}$ and $v_{20} \leftrightarrow -v_{2f}$, so both momentum and kinetic energy would still be conserved. Therefore time-reversed collisions obey the laws of physics, so can happen.

(c) No. If the collision were inelastic, in the extreme case the balls might stick together after the collision, creating heat in the process. Backwards in time, the combined balls would cool down as the two balls separate. This violates the law of entropy, i.e, the second law of thermodynamics. ■

⋆⋆ **Problem 1.52** Three perfectly elastic superballs are dropped simultaneously from rest at height h_0 above a hard floor. They are arranged vertically, in order of mass, with $M_1 >> M_2 >> M_3$, where M_1 is at the bottom. There are small separations between the balls. When M_1 strikes the floor it bounces back up elastically, striking M_2 on its way down. M_2 then bounces back up, striking M_3 on its way down. What is the subsequent maximum altitude achieved by each ball? *Hint*: Analyze each collision in the CM frame of that collision, which is essentially the rest-frame of the heavier ball because the other ball is so much lighter. Neglect the balls' radii and the small separations between them.

Solution

On the initial descent the three balls achieve a velocity v_0 downward, where

$$\frac{1}{2}mv_0^2 = mgh_0 \Rightarrow v_0 = \sqrt{2gh_0}$$

So M_1 bounces back up with velocity v_0, striking M_2 descending with v_0. Then M_2 bounces back with velocity $v_0 + v_0 = 2v_0$ in M_1's frame. (i.e, in M_1's frame M_2 has the same speed $2v_0$ before and after the collision of M_1 & M_2)

In the lab frame, M_2's speed changes from v_0 downward to $3v_0$ upward. Then comes the $M_2\,M_3$ collision. In the lab we have M_2 moving upward with $3v_0$ and M_2 moving downward at speed v_0. In M_2's frame M_3 is moving downward with speed $3v_0 + v_0 = 4v_0$, so after the collision M_3 is moving upward with speed $4v_0$ in M_2's frame. In the lab frame, M_2 is moving upward at $3v_0$, so in the lab frame M_3 moves upward with speed $4v_0 + 3v_0 = 7v_0$. So ball M_1 achieves maximum altitude h_0, ball M_2 achieves maximum altitude

$$\frac{v^2}{2g} = \frac{(3v_0)^2}{2g} = 9\frac{v_0^2}{2g} = 9h_0$$

Ball M_3 achieves max altitude

$$\frac{v^2}{2g} = \frac{(7v_0)^2}{2g} = 49h_0$$

■

⋆⋆⋆ **Problem 1.53** *Classical big-bang cosmological models.* Consider a very large sphere of uniform-density dust of mass density $\rho(t)$. That is, at any given time the density is the same everywhere within the sphere, but the density decreases with time if the sphere expands, or increases with time if the sphere contracts, so that the total mass of the sphere remains fixed. At time $t = 0$ the sphere is all gathered at the origin, with infinite density and infinite outward velocity, so it is undergoing a "big bang" explosion. At some instant t_0 after the big bang the density everywhere within the sphere is ρ_0 and the outward speed of a particular dust particle at radius r_0 is v_0. Use Newton's gravitational constant G and also the result found in problem 1.41, that in spherical symmetry only those mass-shells whose radius r is *less* than the radius of the particular dust particle exert a net force on the particle. Let M_r be the (time-independent) total mass within radius r. (a) Find an expression for $r(t)$, the radius of the particle as a function of time, *supposing that the particle has the escape velocity*. That is, the particle, in common with all particles in the sphere, keeps moving outward but slows down, approaching zero velocity as $r \to \infty$. (b) Then consider the same model of spherically-symmetric dust, except that instead of having the escape velocity, each dust particle has at any moment a velocity *less* than the escape velocity for that particle. This means that the energy of a particle of mass m is negative, where

$$E = \frac{1}{2}m\dot{r}^2 - \frac{GM_r m}{r}.$$

and where $r(t)$ is its distance from the center of the sphere. Show that in this case the time $t(r)$ expressed in terms of r can be written

$$t(r) = \sqrt{\frac{m}{2|E|}} \int^r dr \sqrt{\frac{r}{\alpha - r}}$$

and find the constant α in terms of G, M_r, m, and $|E|$. (c) To perform the integration, substitute $r = \alpha \sin^2(\eta/2) \equiv (\alpha/2)(1 - \cos\eta)$, where η is a new variable, and show that

$$t = \left(\frac{\alpha^3}{8GM_r}\right)^{1/2} (\eta - \sin\eta).$$

(d) Make a table of t and r for a few values of η between 0 and 2π, and plot $r(t)$ for these values. The resulting shape is a *cycloid*, and the equations for $t(\eta)$ and $r(\eta)$ are in fact the parametric equations for a cycloid. Note that this negative-energy cosmological model begins with a "big bang" and ends with a "big crunch." (e) Finally, consider the same model of spherically-symmetric dust, except that instead of having the escape velocity or less, dust particles have at any moment a velocity *greater* than the escape velocity. This means that the energy of a dust particle of mass m is positive. Using an approach analogous to that just used for negative-energy cosmologies, show that in this case

$$t(r) = \sqrt{\frac{m}{2E}} \int^r dr \sqrt{\frac{r}{\alpha + r}}$$

and find the constant α in terms of G, M_r, m, and E. (f) Perform the integration by substituting $r = \alpha \sinh^2(\eta/2) \equiv (\alpha/2)(\cosh\eta - 1)$, where η is a new variable and sinh and cosh are hyperbolic sine and hyperbolic cosine functions, respectively. (g) Then write the solution in parametric form, analogous to that just carried out for negative-energy cosmologies. That is, give formulas for both $t(\eta)$ and $r(\eta)$ for positive-energy cosmologies. (h) Make a table of t and r for several values of η, and plot $r(t)$ for these values. What is the ultimate fate of such a classical model universe?

Solution

(a) Conservation of energy gives

$$\frac{1}{2}m\dot{r}^2 - \frac{GMm}{r} = E = 0$$

for the escape velocity, since then clearly $\dot{r} \to 0$ as $r \to \infty$. Solving for $\dot{r}$, $\dot{r} = \sqrt{\frac{2GM_r}{r}}$ so

$$dr\sqrt{r} = \sqrt{2GM_r}dt$$

Integrating,

$$\frac{r^{3/2}}{3/2} = \sqrt{2GM_r}t$$

so

$$r(t) = \left[\frac{3}{2}(2GM_r)^{1/2}\right]^{2/3} t^{2/3} = \left(\frac{9GM_r}{2}\right)^{1/3} t^{2/3}$$

(b) The energy is negative, so

$$\frac{1}{2}m\dot{r}^2 - \frac{GMm}{r} = E = -|E|$$

so

$$\dot{r}^2 = \frac{2GM_r}{r} - \frac{2|E|}{m} \text{ or } \dot{r} = \sqrt{\frac{2GM_r}{r} - \frac{2|E|}{m}}$$

$$\frac{dr\sqrt{r}}{\sqrt{2GM_r - \frac{2|E|r}{m}}} = dt.$$

Therefore

$$t(r) = \int^r \frac{dr\sqrt{r}}{\sqrt{\frac{2GM_r}{2|E|/m} - \frac{2|E|}{m}}}\sqrt{\frac{m}{2|E|}} = \sqrt{\frac{m}{2|E|}}\int^r dr\sqrt{\frac{r}{\alpha - r}}$$

where $\alpha = \frac{GM_r m}{|E|}$.

(c) Substitute $r = \alpha \sin^2 \eta/2$, so that

$$dr = \alpha 2 \sin(\eta/2)\cos(\eta/2)\frac{d\eta}{2} = \alpha \sin(\eta/2)\cos(\eta/2)d\eta.$$

Then

$$t(r) = \sqrt{\frac{m}{2|E|}}\alpha \int d\eta \sin\eta/2 \cos\eta/2 \sqrt{\frac{\alpha \sin^2\eta/2}{\alpha - \alpha\sin^2\eta/2}}$$
$$= \sqrt{\frac{m}{2|E|}}\alpha \int d\eta \sin\eta/2 \cos\eta/2 \sqrt{\frac{\sin\eta/2}{\cos\eta/2}}$$

$$t(r) = \sqrt{\frac{m}{2|E|}}\alpha \int^{\eta} d\eta \sin^2\eta/2 = \sqrt{\frac{m}{2|E|}}\alpha\frac{1}{2}(\eta - \sin\eta) = (\frac{\alpha^3}{8GM_r})^{1/2}(\eta - \sin\eta)$$

and

$$r = \alpha\sin^2\eta/2 = \frac{\alpha}{2}(1 - \cos\eta).$$

The equations for $t(\eta)$ and $r(\eta)$ are the parametric equations for a cycloid.

(d) In tabular form, we have

η	$r/(\alpha/2)$	$t/(\frac{\alpha^3}{8GM_r})^{1/2}$
0	0	0
$\pi/2$	1	$\pi/2 - 1$
π	2	π
$3\pi/2$	1	$\frac{3\pi}{2} + 1$
2π	0	2π

(e) Now we have

$$\frac{1}{2}m\dot{r}^2 - \frac{GM_r m}{r} = E = |E| \text{ so } \dot{r} = \sqrt{\frac{2GM_r m}{r} + \frac{2|E|}{m}}$$

$$t(r) = \frac{dr\sqrt{r}}{2GM_r + \frac{2|E|r}{m}} = \sqrt{\frac{m}{2|E|}}\int^{r} dr\sqrt{\frac{r}{\alpha + r}}$$

where $\alpha = \frac{GM_r m}{|E|}$

(f) Substitute

$$r = \alpha\sinh^2\eta/2 \equiv \frac{\alpha}{2}(\cosh\eta - 1)$$

so

$$dr = \alpha\sinh\eta/2\cosh\eta/2 d\eta.$$

Then

$$t(r) = \sqrt{\frac{m}{2E}}\alpha \int d\eta \sinh\eta/2 \cosh\eta/2 \sqrt{\frac{\alpha\sinh^2\eta/2}{\alpha + \alpha\sinh^2\eta/2}}$$
$$= 2\alpha\sqrt{\frac{m}{2E}}\int d\frac{\eta}{2}\sinh^2\eta/2 = (\frac{\alpha^3}{8GM_r}^{1/2}(\sinh\eta - \eta))$$

$$r(\eta) = \frac{\alpha}{2}(\cosh\eta - 1)$$

η	$r/(\alpha/2)$	$t/(\frac{\alpha^3}{8GM_r})^{1/2}$
0	0	0
1	.543	.175
2	2.762	1.627
3	9.068	7.018

This model grows, slows down, but keeps expanding as $t \to \infty$ ■

⋆⋆⋆ **Problem 1.54** The Friedmann equations have played an important role in relativistic big-bang cosmologies. They feature a "scale factor" $a(t)$, proportional to the distance between any two points (such as the positions of two galaxies) that are sufficiently remote from one another that local random motions can be ignored. If a increases with time, the distance between galaxies increases proportionally, corresponding to an expanding universe. If we model for simplicity the universe as filled with pressure-free dust of uniform density ρ, the Friedmann equations for $a(t)$ are

$$\ddot{a} = -\frac{4\pi G\rho}{3}a \quad \text{and} \quad \dot{a}^2 = \frac{8\pi G\rho}{3}a^2 - \frac{kc^2}{R_0^2}$$

where G is Newton's gravitational constant, c is the speed of light, R_0 is the distance between two dust particles at some particular time t_0, and k = +1, –1, or 0. The density of the dust is inversely proportional to the cube of the scale factor $a(t)$, *i.e.*, $\rho = \rho_0(a_0/a)^3$, where ρ_0 is the density when $a = a_0$. Therefore

$$\ddot{a} = -\frac{4\pi G\rho_0 a_0^3}{3a^2} \quad \text{and} \quad \dot{a}^2 = \frac{8\pi G\rho_0 a_0^3}{3a} - \frac{kc^2}{R_0^2}.$$

(a) Show that if we set the origin to be at one of the two chosen dust particles, then if M is the total mass of dust within a sphere surrounding this origin out to the radius of the other chosen particle, and if at arbitrary time t_0 we set $a_0 = 1$, then the equations can be written

$$\ddot{a} = -\frac{(GM/R_0^3)}{a^2} \quad \text{and} \quad \frac{1}{2}\dot{a}^2 - \frac{(GM/R_0^3)}{a} = -\frac{kc^2}{2R_0^2} \equiv \epsilon$$

where ϵ and M are constants.

(b) Show that the second equation is a first integral of the first equation. (c) Compare these equations to the $F = ma$ and energy conservation equations of a particle moving radially under the influence of the gravity of a spherical moon of mass M. (d) Einstein hoped that his general-relativistic equations would lead to a static solution for the universe, since he (like just about everyone before him) believed that the universe was basically at rest. The Friedmann equations resulting from his theory show that the universe is generally expanding or contracting, however, just as a rock far from the Earth is not going to stay there, but will generally be either falling inward or on its way out. So Einstein modified his theory with the addition of a "cosmological constant" Λ, which changed the Friedmann equations for pressure-free dust to

$$\ddot{a} = -\frac{(GM/R_0^3)}{a^2} + \frac{\Lambda}{3}a \quad \text{and} \quad \frac{1}{2}\dot{a}^2 - \frac{(GM/R_0^3)}{a} - \frac{\Lambda}{6}a^2 = \epsilon.$$

Show that these equations *do* have a static solution, and find the value of Λ for which the solution is static. (e) Show however (by sketching the effective potential energy function in the second equation) that the static solution is *unstable*, so that if the universe is kicked even slightly outward it will accelerate outward, or if it is kicked even slightly inward it will collapse. A static solution is therefore physically unrealistic. (Einstein failed to realize that his static solution was unstable, and later, when Edwin Hubble showed from his observations at the Mount Wilson Observatory that the universe is in fact expanding, Einstein declared that introducing the cosmological constant was "my biggest blunder".) (f) Suppose the cosmological constant is retained in the equations, but that the dust is removed so that $M = 0$. Solve the equations for $a(t)$ in this case. The solution is the **de Sitter model**, an "inflationary" model of the expanding universe. What is the constant ϵ for the de Sitter model? (g) Make a qualitative sketch of $a(t)$ if both M and Λ are positive constants. Of the terms containing M and Λ, which dominates for small times? For large times?

Solution

(a) The mass $M = \rho_0(\text{Vol})_0 = \rho_0[(4/3)\pi R_0^3]$, so $\rho_0 = M/(4/3\pi R_0^3$. Substitute this expression into the first Friedmann equation, which gives $\ddot{a} = -GM/R_0^3/a^2$, since we have set $a_0 = 1$. The second equation can then be written in the form

$$\frac{1}{2}\dot{a}^2 - \frac{GM/R_0^3}{a} = -\frac{kc^2}{2R_0^2} \equiv \epsilon$$

(b) Differentiating the second equation gives us the first. This shows that the second equation is a first integral of the first equation.

(c) Newton's law $F = ma$ gives $\ddot{r} = -GM/r^2$, so letting $r = R_0 a$, we find the first Friedmann equation. Here R_0 is a constant. Similarly, energy conservation gives

$$\frac{1}{2}\dot{a}^2 - \frac{GM/R_0^3}{a} = E/mR_0^2 = \text{constant},$$

a negative constant if $E < 0$, corresponding to a velocity less than the escape velocity.

(d) The solution is stable if both $\dot{a} = 0$ and $\ddot{a} = 0$. Note that both are true if $\Lambda = 3GM/R_0^2 a^3$ (where now a is a constant) and ϵ is chosen appropriately.

(e) The second equation has the mathematical form of a conservation-of-energy equation, with the first term on the left proportional to the kinetic energy, and the other two terms on the left proportional to the potential energy. To show that the solution is unstable, simply demonstrate that the potential energy has a maximum at the equilibrium point.

(f) For the first question, note that if $M = 0$, the solution of the first equation is $a(t) = a_0 \exp(\sqrt{\Lambda/3}\, t)$, an exponential expansion analogous to the increasing price of goods in an inflationary economy.

(g) Including both the M term and the Λ term, we have

$$\dot{a}^2 = 2\epsilon + \frac{2GM/R_0^3}{a} + \frac{\Lambda}{3}a^2.$$

If we assume $\epsilon = 0$ (and this appears to be close to the truth in our universe right now), then for small t the M term dominates and for large t the Λ term dominates. That is, for small times a grows, but at a decreasing rate; for large times a grows exponentially, i.e., becomes inflationary. ■

2 Relativity

2.1 Problems and Solutions

★ **Problem 2.1** *Time dilation and length contraction.* Clock A is placed at the origin of the primed frame; it reads time $t' = 0$ just as the origins of the primed and unprimed frames coincide. At a later time t to observers in unprimed frame, find from the Lorentz transformation of Eqs. 2.15 (a) how far A has moved, and (b) what time A reads. This is an example of the fact that *moving clocks run slow.* A stick of length L_0 is placed at rest along the x' axis of the primed frame. Observers in the unprimed frame measure the position of both ends of the stick at the same time t to them as the stick is moving along at speed V. (c) Using the Lorentz transformation, find the length $L \equiv (x_2 - x_1)$ of the stick in the unprimed frame, in terms of L_0 and the relative frame velocity V. Here x_2 and x_1 are the locations of each end of the stick, as measured in the unprimed frame. The fact that $L < L_0$ is an example of the fact that *moving lengths are contracted in their direction of motion.* This phenomenon is called the Lorentz contraction or the Lorentz-Fitzgerald contraction.

Solution

The Lorentz transformation includes

$$x = \gamma(x' + vt') \text{ and } t = \gamma(t' + vx'/c^2)$$

also

$$x' = \gamma(x - vt) \text{ and } t' = \gamma(t - vx/c^2)$$

(a) For clock A, $x' = 0$. Therefore, $x = vt$ is how far A has moved in the unprimed frame.

(b) A reads

$$t' = \gamma(t - vx/c^2) = \gamma\left(t - \frac{v}{c^2}vt\right) = \sqrt{1 - v^2/c^2}t$$

since $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$.

(c) $L_0 = x_2' - x_1'$, the difference in position of the two ends of the stick as measured in the primed frame, the rest-frame of the stick. Therefore

$$L_0 = x_2' - x_1' = \gamma(x_2 - x_1 - v(t_2 - t_1)).$$

But in the unprimed frame we measure the two ends simultaneously in that frame, with $t_2 = t_1$, So

$$L \equiv x_2 - x_1 = L_0/\gamma = L_0\sqrt{1 - v^2/c^2} \leq L_0.$$

■

⋆ **Problem 2.2** *The invariance of transverse lengths.* A stick of length L_0 is placed at rest along the y' axis of the primed frame, extending from $y' = y'_1$ to $y' = y'_2$. Observers in the unprimed frame measure the position of both ends of the stick at the same time t to them as the stick is moving along at speed V. Using the Lorentz transformation of Eqs. 2.15, find the length $L \equiv (y_2 - y_1)$ of the stick in the unprimed frame, in terms of L_0 and the relative frame velocity V. Here y_2 and y_1 are the locations of each end of the stick, as measured in the unprimed frame. The fact that $L = L_0$ is an example of the fact that *moving transverse lengths are invariant under Lorentz transformations.*

Solution

The Lorentz transformation includes the equation $y = y'$ for transverse coordinates. Therefore $y'_1 = y_1$ and $y'_2 = y_2$, so

$$L_0 \equiv y'_2 - y'_1 = y_2 - y_1 = L$$

so

$$L = y_2 - y_1 = L_0.$$

Transverse lengths are invariant. ■

⋆ **Problem 2.3** *The relativity of simultaneity.* Two clocks are placed at rest on the x' axis of the primed frame, clock A at $x' = 0$ and clock B at $x' = L_0$. They are therefore a distance L_0 apart in their mutual (primed) rest frame. Observers in the unprimed frame see both clocks moving at velocity V, B leading the way and A following it. Then at some particular time t, unprimed observers measure the readings of t'_A and t'_B of the two clocks. Show from the Lorentz transformation of Eqs. 2.15 that $t'_B < t'_A$, and that in fact $t'_B = t'_A - VL_0/c^2$. This is an example of the fact that *leading clocks lag*, *i.e.*, that the clock leading the way reads a lesser time than the chasing clock. It also shows that simultaneity is not universal but relative. In non-relativistic physics if two events are simultaneous according to observers in one frame or reference, they are simultaneous in all frames. That is not true in relativity.

Solution

The Lorentz transformation equations include

$$t' = \gamma(t - Vx/c^2), \quad x' = \gamma(x - Vt)$$
$$t = \gamma(t' + Vx'/c^2), \quad x = \gamma(x' + Vt')$$

Clock A is always at $x' = 0$, so $x_A = Vt_A$. Clock B is always at $x' = L_0$, so $L_0 = \gamma(x_B - Vt_B)$.

Now we measure both at $t_A = t_B$, so $L_0 = \gamma(x_B - x_A)$.

$$t'_B - t'_A = \gamma\left[(t_B - t_A) - V/c^2(x_B - x_A)\right] = \gamma\left[0 - V/c^2(L_0/\gamma)\right] = -VL_0/c^2.$$

Therefore the leading clock lags by an amount VL_0/c^2, where L_0 is the rest distance between the two clocks. ■

★ **Problem 2.4** A primed frame moves at $V = (3/5)c$ relative to an unprimed frame. Just as their origins pass, clocks at the origins of both frames read zero, and a flashbulb explodes at that point. Later, the flash is seen by observer A at rest in the primed frame, whose position is x', y', z' = (3 m, 0, 0). (a) What does A's clock read when A sees the flash? (b) When A sees the flash, where is she located according to unprimed observers? (c) To unprimed observers, what do their own clocks read when A sees the flash? Use the Lorentz transformation of Eqs. 2.15.

Solution

(a) Observer A is a distance 3 m from the flash ($x' = 3$ m), so observes the flash at

$$t'_A = \frac{x'}{c} = \frac{3\text{ m}}{c}$$

(b) Her position is

$$x = \gamma(x' + Vt') = \frac{5}{4}\left(3\text{ m} + \frac{3}{5}c \cdot \frac{3\text{ m}}{c}\right) = \frac{5}{4}\left(3\text{ m} + \frac{9}{5}\text{ m}\right) = \frac{24}{4}\text{ m} = 6\text{ m}.$$

(c) Their clocks read

$$t = \gamma\left(t' + \frac{V}{c^2}x'\right) = \frac{5}{4}\left(\frac{3\text{ m}}{c} + \frac{3}{5c}\cdot 3\text{ m}\right) = \frac{5}{4}\left(\frac{24}{5}\frac{\text{m}}{c}\right) = 6\text{ m}/c$$

as expected. ■

★ **Problem 2.5** Synchronized clocks A and B are at rest in our frame of reference, a distance five light-minutes apart. Clock C passes A at speed (12/13) c bound for B, when C, and also both A and B, read $t = 0$ in our frame. (a) What time does C read when it reaches B? (b) How far apart are A and B in C's frame? (c) In C's frame, when A passes C, what time does B read?

Solution

(a) The time it takes in the frame of A and B is

$$t = D/v = \frac{5c\text{ min}}{12/13\ c} = \frac{5(13)}{12}\text{min}.$$

However, C's clock runs slow by the factor $\sqrt{1-(12/13)^2} = 5/13$, so C's clock reads

$$\frac{5(13)}{12}\left(\frac{5}{13}\right) = \frac{25}{12}\text{ min}.$$

(b) In C's frame the distance is contracted by $\sqrt{1-(12/13)^2}$, so the distance between A and B is

$$5\ c\text{ min} \times \frac{5}{13} = \frac{25}{13}\ c\text{ minutes}$$

(c) Leading clocks lag from C's point of view. Here A is the leading clock because B is chasing A to the left in C's frame. A reads $t = 0$, so B reads the later time

$$VD/c^2 = \frac{12}{13}c \cdot \frac{5\ c\ \text{min}}{c^2} = \frac{60}{13}\ \text{min}.$$

■

★ **Problem 2.6** Two spaceships are approaching one another. According to observers in our frame, (a) the left-hand ship moves to the right at $(4/5)c$ and the right-hand ship moves to the left at $(3/5)c$. How fast is the right-hand ship moving in the frame of the left-hand ship? (b) the left-hand ship moves to the right at speed $(1-\epsilon)c$ and the right-hand ship moves to the left at $(1-\epsilon)c$, where $0 < \epsilon < 1$. How fast is the right-hand ship moving in the frame of the left-hand ship? Show that this speed is less than c, no matter the value of ϵ within the range allowed.

Solution

(a) The primed frame is the rest frame of the left-hand ship, while we are at rest in the unprimed frame. The relative velocity is $V = \frac{4}{5}c$. The right-hand ship has $v_x = -\frac{3}{5}c$, so

$$v'_x = \frac{v_x - V}{1 - v_x V/c^2} = \frac{-\frac{3}{5}c - \frac{4}{5}c}{1 - (-\frac{3}{5})\frac{4}{5}c/c^2} = \frac{-\frac{7}{5}c}{1 + \frac{12}{25}} = \frac{-\frac{7}{5}c}{\frac{37}{25}} = -\frac{35}{37}c$$

(b) In this case

$$V = (1-\epsilon)c, v_x = -(1-\epsilon)c$$

$$\Rightarrow v'_x = \frac{-(1-\epsilon)c - (1-\epsilon)c}{1 - (-)(1-\epsilon)(1-\epsilon)} = \frac{(-2+2\epsilon)c}{1 + (1 - 2\epsilon + \epsilon^2)}$$

$$= -2\left[\frac{1-\epsilon}{2 - 2\epsilon + \epsilon^2}\right]c = -(\frac{1-\epsilon}{1 - \epsilon + (\epsilon^2/2)})c$$

The magnitude of this velocity is $(\frac{1-\epsilon}{1-\epsilon+(\epsilon^2/2)})c < c$ for any $\epsilon > 0$. ■

★ **Problem 2.7** Astronaut A boards a spaceship leaving earth for the star Alpha Centauri, 4 light-years from earth, while her friend B stays at home. The ship travels at speed 4/5 c, and upon arrival immediately turns around and travels back to earth at the same speed 4/5 c. (a) How much has A aged during the entire trip? (b) How much has B aged during the time A has been gone?

Solution

In B's rest frame (a single inertial frame throughout) the time it takes A to reach α-Centauri is

$$t = D/v = \frac{4c\ \text{yrs}}{\frac{4}{5}c} = 5\ \text{yrs}$$

(a) A's clock runs slow, so A ages $5\ \text{yrs}\sqrt{1-(4/5)^2} = 3$ yrs on the way out, and another 3 yrs coming home. So overall A has aged 6 yrs for the round trip.

(b) In B's inertial frame B has aged 2×5 yrs $= 10$ yrs. ■

⋆⋆ **Problem 2.8** Al and Bert are identical twins. When Bert is 24 years old he travels to a distant planet at speed 12/13 c, turns around and heads back at the same speed, arriving home at age 44. Al stays at home. (a) How old is Al when Bert returns? (b) How far away was the planet in Al's frame? (c) Why can't Bert reasonably claim that from his point of view it was Al who was moving, so that Al's clocks should be time dilated, making Al younger than Bert when they reunite?

Solution

Let the planet be a distance D from the earth. So in Al's frame, which is the same inertial frame throughout, the time it takes Bert to reach the planet is

$$t = D/v = \frac{D}{(12/13)c} = \frac{13}{12}D/c.$$

Bert's clock is time-dilated, however, so Bert will have aged

$$\frac{13}{12}D/c\sqrt{1-\left(\frac{12}{13}\right)^2} = \frac{5}{12}D/c,$$

and for the round trip Bert will age by

$$\frac{5}{12}D/c \times 2 = \frac{5}{6}D/c.$$

We are told he has aged by 44 yrs – 24 yrs = 20 yrs, so $\frac{5}{6}D/c = 20$ yrs. Solving for D, $D = \frac{6}{5}20c$ yrs = 24 c yrs. So the time in A's frame is

$$\frac{D}{(12/13)c} = \frac{24c \text{ yrs}}{(12/13)c} = 26 \text{ yrs}$$

for a total of 52 yrs for the round trip.

(a) Al's age upon Bert's return is (24 + 52) yrs = 76 yrs.

(b) The distance to the planet is 24 c-yrs.

(c) The flaw in Bert's potential argument is that the story is not symmetrical. Only Al stays at rest in the same inertial frame throughout. ■

⋆ **Problem 2.9** The Global Positioning System (GPS) features 24 earth satellites orbiting at altitude 20,200 km above earth's surface. Each satellite carries 4 highly precise atomic clocks; this precision is essential in allowing us to know our positions on the ground within a few meters or less. Special relativistic time dilation effects, although tiny, must be taken into account. They are due to the speed v of the satellites relative to a clock at rest in some appropriate inertial frame. Let us take this reference clock to be a hypothetical clock at rest at the center of the earth. (To call such a clock inertial is only an approximation, because the earth has a small acceleration toward the sun and moon, which are themselves accelerating toward the center of our galaxy, etc., etc.) (a) Find the special-relativistic time dilation factor $\sqrt{1-v^2/c^2}$ for clocks in a GPS satellite, expressed in the form $1-\epsilon$, where ϵ is a very small number. (b) How much time would they lose in one year due to this effect? (There is a *second* relativistic effect on GPS clocks, as described in Chapter 10, due not to their velocity but to their altitude in earth's gravity. Given information: Mass and mean radius of the Earth: 5.98 $\times 10^{24}$ kg and 6370 km; Newton's gravitational constant $G = 6.674 \times 10^{-11} \mathrm{m}^3/(\mathrm{kg\ s}^2)$.)

Solution

An object in circular orbit around the earth obeys Newton's second law

$$\dot{F} = ma, \text{ or } \frac{GMm}{r^2} = \frac{mv^2}{r},$$

so the object's speed is $v = \sqrt{\frac{GM}{r}}$ where M is earth's mass and r is the radius of the orbit measured from the center of the earth. That is,

$$v = \sqrt{\frac{(6.674 \times 10^{-11}\ \mathrm{m^3/kg\,s^2})(5.98 \times 10^{24}\mathrm{kg})}{(6.37 \times 10^6\mathrm{m}) + (2.02 \times 10^7\mathrm{m})}} = 3.88 \times 10^3 \mathrm{m/s}$$

(a) The time-dilation factor is

$$\sqrt{1 - v^2/c^2} = \sqrt{1 - \left(\frac{3.88}{3.00}\right)^2 10^{-10}} = 1 - \frac{1}{2}(1.67) \times 10^{-10} = 1 - 8.36 \times 10^{-11},$$

using the binomial approximation $(1 + x)^n \simeq 1 + nx$ for $x \ll 1$.

(b) The number of seconds in one year is 3.16×10^7. So the orbiting clocks would lose

$$(3.16 \times 10^7\ \mathrm{s})(8.36 \times 10^{-11}) = 2.64 \times 10^{-3}\ \mathrm{s} = 2.64\ \mathrm{ms}$$

in one year, due to special relativistic time dilation. ■

⋆ **Problem 2.10** Incoming high-energy cosmic-ray protons strike Earth's upper atmosphere and collide with the nuclei of atmospheric atoms, producing a downward-directed shower of particles, including (among much else) the pions π^+, π^-, and π^0. The charged pions decay quickly into muons and neutrinos,

$$\pi^+ \to \mu^+ + \nu \quad \text{and} \quad \pi^- \to \mu^- + \nu.$$

The muons are themselves unstable, with a half-life of 1.52 microseconds in their rest frame, decaying into electrons or positrons and additional neutrinos. Nearly all muons are created at altitudes of about 15 km and more, and then those that have not yet decayed rain down upon the earth's surface. Consider muons with speeds $(0.995 \pm 0.001)c$, with their numbers measured on the ground and in a balloon-lofted experiment at altitude 12 km. (a) How far would such muons descend toward the ground in one half-life if there were no time dilation? (b) What fraction of these muons observed at 12 km would reach the ground? (c) Now take into account time dilation, in which the muon clocks run slow, extending their half-lives in the frame of the earth. Then what fraction of those observed at 12 km would make it to the ground? (Such experiments supported the fact of time dilation.)

Solution

(a) The distance moved in one half-life would be

$$d = vt = 0.995\ c(1.52 \times 10^{-6}\mathrm{s}) = 454\ \mathrm{m}.$$

(b) The number of half-life distances required to reach the ground would be

$$(12\ \text{kg}/454\ \text{m}) = 26.4,$$

so the fraction remaining at the ground would be $f = (1/2)^{26.4}$. That is,

$$f = e^{-18.3} = 1.13 \times 10^{-8},$$

so approximately one in 100 million reach the ground.

(c) Including time dilation,

$$\sqrt{1 - v^2/c^2} = \sqrt{1 - (.995)^2} = 0.1,$$

so muons would travel 10 times as far in one half-life, 4540 m. Now the fraction reaching the ground is

$$f = (1/2)^{2.64} = 0.160$$

That is, in this case 16% make it to the ground. Many muons are in fact observed at ground level, providing evidence that time dilation is real. ■

★ **Problem 2.11** Suppose that in the distant future astronomers build a telescope so powerful they can see aliens on a planet that is 10 light-years from earth. One day they observe the aliens board a spaceship and blast off toward earth. According to earth clocks, the ship and its alien crew arrive at the earth exactly one week later. Assuming the velocity of the ship was constant during almost the entire trip, find its velocity ratio $\beta = v/c$ in earth's frame, valid to three significant figures. (Note that $v/c < 1$.)

Solution

By the time earth astronomers see the ship leave it is 10 yrs after the ship actually left. By this time the ship has traveled a distance $d = vt = v(10\text{ yrs})$. The ship arrives at earth one week $= \frac{1}{52}$ yr later, traveling at speed v. Note that

$$v(10\text{ yrs} + \frac{1}{52}\text{yrs}) = 10c\text{ yrs},$$

so the ship speed was

$$v/c = \frac{10\text{ yrs}}{(10 + \frac{1}{52})\text{yrs}} = \frac{520}{521} = 0.998,$$

0.2% less than the speed of light. ■

★★★ **Problem 2.12** A distant galactic nucleus ejects a jet of material at right angles (90°) to our line of sight. We know the distance of the galaxy from the redshift of its spectral lines, so we can calculate how far the jet has traveled in a given time using the very small but growing angle between the galactic nucleus and jet as observed through our telescope. From this information we can find the velocity of the jet. Note that for such transverse motion it takes essentially the same time for light from the jet to reach us from the end of its journey as it does from its beginning, because it stays essentially the same distance from us throughout. But now suppose the jet is ejected at some angle θ relative to our line of sight, so the jet's transverse velocity component is $v_\perp = v\sin\theta$ and its velocity component towards us is $v_\| = v\cos\theta$. And because the jet is getting closer to us, the time it takes light to reach us from it becomes smaller and smaller. (a) In this case find an equation for the jet's *apparent* transverse velocity in the sky, defined as the transverse distance moved divided by the time interval as observed on earth, and show that this apparent velocity v_{app} can exceed the speed of light. (b) For a given actual velocity v, find the angle θ that *maximizes* v_{app}, and then find the magnitude of v_{app} in this case. (c) Evaluate such a maximal v_{app} for the case $v = 0.99c$. Such apparent superluminal velocities have often been observed by astronomers, even though no matter actually travels faster than light.

Solution

(a) Suppose that at $t = 0$ the jet is located at point A, a distance d from our telescope. At that time the jet starts to move with velocity v at an angle θ to our line of sight, so that it moves a distance $v\tau\sin\theta$ perpendicular to our view, where τ is the time during which it moves, and a distance $v\tau\cos\theta$ *towards* us. So then at time $t = \tau$ the jet is located at point B, a distance $v\tau\sin\theta$ perpendicular to our line of sight, and $d - v\tau\cos\theta$ from us, to an excellent approximation since it is so far away. We do not *observe* the jet to be at point A until time d/c, since the signal approaches us at speed c. We then observe the jet to be at point B at time $\tau + (d - v\tau\cos\theta)/c$, the time at which it actually reaches B, plus the time it takes light from that event to reach us. That is, the time *interval* between observing the jet at A and at B is

$$\Delta t = \tau + \frac{d - v\tau\cos\theta}{c} - \frac{d}{c} = \tau(1 - (v/c)\cos\theta).$$

During this time, the jet's sideways displacement is $\Delta x = v\tau\sin\theta$, so its *apparent* sideways velocity is

$$v_{\text{app}} \equiv \frac{\Delta x}{\Delta t} = \frac{v\tau\sin\theta}{\tau(1 - v\cos\theta/c)} = \frac{v\sin\theta}{1 - (v/c)\cos\theta}.$$

Suppose θ is small, (so $\sin\theta \approx \theta$ and $\cos\theta \cong 1 - \theta^2/2$). So then

$$v_{\text{app}} \cong \frac{v\theta}{1 - (v/c)}$$

to first order in θ. So if for example $\theta = 0.1$ and $v/c = 0.99$, we find that

$$v_{\text{app}} = \frac{0.99c(0.1)}{0.01} = 9.9c.$$

The jet in this case *appears* to be moving sideways at almost ten times the speed of light, whereas in fact it is moving mostly towards us, with a smaller sideways component, at a net speed $v < c$.

(b) We can maximize v_{app} by setting $dv_{\text{app}}/d\theta = 0$. That is,

$$\frac{dv_{\text{app}}/c}{d\theta} = \frac{(v/c)\cos\theta}{1 - (v/c)\cos\theta} - \frac{(v/c)\sin\theta}{(1 - (v/c)\cos\theta)^2}((v/c)\sin\theta) = 0$$

The result is that the angle which maximizes $v_{\rm app}$ is $\theta = \cos^{-1}(v/c)$. Substituting this result into the expression for $v_{\rm app}/c$ gives

$$v_{\rm app}/c = \frac{v/c}{\sqrt{1 - v^2/c^2}}$$

(c) For $v/c = 0.99$,

$$v_{\rm app}/c = \frac{0.99}{\sqrt{1 - (0.99)^2}} = 7.00.$$

■

★ **Problem 2.13** A bullet train of rest-length 500 m is chugging along a straight track at speed 4/5 c when it enters a tunnel of length 400 m. Due to length contraction in the frame of the tunnel, the train apparently briefly fits inside the tunnel all at once. From the point of view of train passengers, however, it is the tunnel that is contracted, with a length of only 400 m × 3/5 = 240 m, so the 500 m train seemingly *cannot* fit inside all at once. The question is: Does the train fit inside the tunnel all at once, or not? Explain.

Solution

The answer is that the entire train fits inside the tunnel at once in the reference frame of the tunnel, but it does not do so in the frame of the train. There is no inconsistency here, due to the fact that if two events are simultaneous in one frame they are not necessarily simultaneous in another frame. From the point of view of the tunnel, although the train is inside the tunnel all at once, the train clocks are not synchronized with one another: the clock on the last train car reads a later time than the clock on the engine. Therefore observers in the tunnel frame will understand that train passengers will think the last car is inside the tunnel at a later time than the engine is inside the tunnel, which is exactly what passenger do think, as the onrushing tunnel first encloses their engine, and by the time it encloses their last car, the engine has already left the tunnel. ■

★★ **Problem 2.14** A carrot slicing machine consists of 8 parallel blades spaced 5 cm apart, held together in a framework that allows all the blades to descend at once upon an unsuspecting carrot laid out horizontally in the machine. The result is several carrot pieces of length 5 cm, plus random bits left over at each end. Now suppose that a carrot is made to move lengthwise at speed 4/5 c into the machine just as the blades descend. The Lorentz contraction ensures that the carrot will be shorter in the machine frame than in its rest frame, so there will be fewer carrot pieces. Each of these non-end pieces will still have length 5 cm in the machine frame because that is the spacing of the blades, so it appears they must be *longer* than 5 cm when finally brought to rest: In fact, each should have rest-length 5 cm/(3/5 c) = 8 1/3 cm. Now view the exact same procedure in the rest-frame of the carrot. Then it is the slicing machine that moves at 4/5 c, so it contracts as a whole, and the distance between blades is Lorentz-contracted to 5 cm$\sqrt{1 - (4/5)^2}$ = 3 cm. That is, it seems that it produces carrot pieces 3 cm long in their rest-frame. These conclusions (8 1/3 cm and 3 cm) cannot both be correct, since it is the same carrot that was involved in both sets of reasoning. Which is the correct answer (if either) and *why* is the other answer or answers wrong?

Solution

The important feature to note is that if all the blades descend simultaneously in the slicing machine frame, they cannot descend simultaneously in the carrot frame. Therefore since all are cut simultaneously in the machine frame, the correct answer is that each of the pieces will have length $8\frac{1}{3}$ cm in their rest-frame.

How does this work in the carrot frame? We know that leading clock lag, so if a clock is attached to each blade of the machine, and if the carrot moves to the right relative to the machine, so the machine moves to the left relative to the carrot, the clocks on the left-most blade will lag clocks attached to the right-most blade, as observed in the carrot's frame. Therefore to the carrot, the right-most blade slices first, and the left-most blade slices last. So the left end of a carrot piece is sliced after the right end has already been sliced, so the carrot piece will be longer than 3 cm in the carrot's frame. ■

⋆⋆ **Problem 2.15** By differentiating the velocity transformation equations one can obtain transformation laws for acceleration. Find the acceleration transformations for the x component a_x, in terms of a_x, v_x, and V, the relative frame velocity.

Solution

The velocity transformation is

$$v'_x = \frac{v_x - V}{1 - v_x V/c^2}, \quad v'_y = \frac{v_y\sqrt{1 - V^2/c^2}}{1 - v_x V/c^2} \text{ (similar for } v'_z)$$

$$\Rightarrow a'_x = \frac{dv'_x}{dt'} = \frac{d}{dt'}\left(\frac{v_x - V}{1 - v_x V/c^2}\right) = \left(\frac{dv_x/dt'}{1 - v_x V/c^2}\right) - \frac{(v_x - V)}{(1 - v_x V/c^2)^2}\left(\frac{-V}{c^2}\right)\frac{dv_x}{dt'}$$

$$= \frac{1 - v_x(V/c^2) + v_x(V/c^2) - (V^2/c^2)}{(1 - v_x(V/c^2))^2}\frac{dv_x}{dt}\frac{dt}{dt'} = \frac{1 - (V^2/c^2)}{(1 - v_x(V/c^2))^2} a_x \frac{1}{dt'/dt},$$

using the chain rule and

$$dv_x/dt = a_x, \quad dt/dt' = 1/(dt'/dt).$$

Then from

$$t' = \gamma(t - Vx/c^2) \text{ it follows that } dt'/dt = \gamma(1 - V dx/dt/c^2) \equiv \gamma(1 - Vv_x/c^2)$$

$$\Rightarrow a'_x = \frac{a_x(1 - (V/c^2))^{3/2}}{(1 - v_x(V/c^2))^3} = \frac{a_x}{\gamma^3(1 - v_x(V/c^2))^3}$$

Similarly,

$$a'_y = \frac{dv'_y}{dt'} = \frac{d}{dt'}\left(\frac{v_y\sqrt{1 - (V^2/c^2)}}{1 - v_x(V/c^2)}\right) \equiv \frac{1}{\gamma}\frac{d}{dt'}\left(\frac{v_y}{1 - v_x(V/c^2)}\right)$$

$$a'_y = \frac{1}{\gamma}\left[\frac{dv_y/dt'}{1 - v_x(V/c^2)} - \frac{v_y}{(1 - v_x(V/c^2))^2}\left(-(V/c^2)\frac{dv_x}{dt'}\right)\right]$$

$$= \frac{1}{\gamma(1 - v_x(V/c^2))^2}\left[(1 - \frac{v_x V}{c^2})\frac{dv_y}{dt'} + v_y \frac{dv_x}{dt'}(V/c^2)\right]$$

Again,

$$dt'/dt = \gamma(1 - Vv_x/c^2),$$

so

$$a'_y = \frac{1}{\gamma(1 - v_x(V/c^2))^2}\left[(1 - \frac{v_x v}{c^2})a_y \frac{1}{\gamma(1 - V(v_x/c^2))} + \frac{v_y V}{c^2} a_x \frac{1}{\gamma(1 - V(v_x/c^2))}\right]$$

$$= \frac{1}{\gamma^2(1 - v_x(V/c^2))^3}\left[(1 - \frac{v_x V}{c^2})a_y + \frac{v_y V}{c^2} a_x\right]$$

$$a'_y = \frac{a_y - (V/c^2)(v_x a_y - v_y a_x)}{\gamma^2(1 - v_x V/c^2)^3} \text{and so also}$$

$$a'_z = \frac{a_z - (V/c^2)(v_x a_z - v_z a_x)}{\gamma^2(1 - v_x(V/c^2))^3}.$$

■

⋆ **Problem 2.16** An electron moves at velocity 0.9 c. How fast must it move to double its momentum?

Solution

The initial momentum is

$$p_0 = \frac{mv_0}{\sqrt{1 - v_0^2/c^2}}, \text{so also } p_f = 2p_0 = \frac{mv_f}{\sqrt{1 - v_f^2/c^2}}, \text{which can be inverted to give}$$

$$(v_f/c)^2 = \frac{p_f^2}{p_f^2 + m^2c^2} = \frac{(2p_0)^2}{(2p_0)^2 + m^2c^2} = \frac{(\frac{2mv_0}{\sqrt{1-v_0^2/c^2}})^2}{(\frac{2mv_0}{\sqrt{1-v_0^2/c^2}})^2 + m^2c^2}$$

$$= \frac{4m^2v_0^2}{4m^2v_0^2 + m^2c^2(1 - v_0^2/c^2)} = \frac{1}{1 + \frac{1}{4}(\frac{c^2}{v_0^2} - 1)} = \frac{1}{1.05864} = 0.945.$$

Therefore $v_f/c = 0.972$. ■

⋆ **Problem 2.17** An atomic nucleus starts at rest in the lab, and is then struck by two photons, one after the other, each with momentum p_γ in the same direction. The photons are absorbed in the nucleus. If the mass of the final (excited) nucleus is M^*, calculate its velocity.

Solution

The total incoming momentum is $2p_\gamma$, so momentum conservation gives

$$2p_\gamma = \frac{M^* v}{\sqrt{1 - v^2/c^2}}.$$

Solve for v:

$$4p_\gamma^2(1 - v^2/c^2) = M^{*2}v^2 \Rightarrow \frac{v^2}{c^2}(M^{*2}c^2 + 4p_\gamma^2) = 4p_\gamma^2$$

$$\frac{v}{c} = \frac{2p_\gamma}{\sqrt{4p_\gamma^2 + M^{*2}c^2}}$$

■

⋆ **Problem 2.18** Two particles make a head-on collision, stick together, and stop dead. The first particle has mass m and speed (24/25)c, and the second has mass M and speed (5/13) c. Find M in terms of m.

Solution

Their momenta must be equal but opposite, because the final momentum is zero. That is

$$\frac{m \cdot \frac{24}{25}c}{\sqrt{1 - (24/25)^2}} = \frac{M\frac{5}{13}c}{\sqrt{1 - (5/13)^2}}$$

$$\frac{24mc}{\sqrt{49}} = \frac{5Mc}{\sqrt{144}} \Rightarrow M/m = \frac{24}{5}(\frac{12}{7}) = 8.2286.$$

■

⋆ **Problem 2.19** Spaceship A, moving away from the earth at velocity $3/5c$, is sending messages to spaceship B, which left the earth earlier at speed $4/5c$ in the same direction. The messages sent by A are contained in pulses sent by a laser on A, with the pulses separated by 100 femtoseconds in A's frame of reference. As B receives the pulses, what is the pulse separation according to the crew on B?

Solution

We will find B's velocity in A's reference frame.

$$v'_B = \frac{v_B - V}{1 - v_B V/c^2} = \frac{\frac{4}{5}c - \frac{3}{5}c}{1 - \frac{4}{5} \cdot \frac{3}{5}} = \frac{\frac{c}{5}}{1 - \frac{12}{25}} = \frac{5c}{37}$$

Now in A's (primed) frame, the time between pulses is 100 fs. B will experience a "red shift" toward lower frequencies (and a large period between pulses). By the Doppler formula,

$$\frac{\Delta t_{toB}}{\Delta t_{toA}} = \sqrt{\frac{1 + v_{rel}/c}{1 - v_{rel}/c}} = \sqrt{\frac{1 + (5/37)}{1 - (5/37)}} = \sqrt{\frac{42}{32}} = 1.1456$$

Therefore the pulses will be separated by $114.56\cdots$ fs in B's frame. ■

⋆ **Problem 2.20** An alien vessel is detected approaching earth at 3/5 c. An intercepting probe is sent from earth at speed 4/5 c toward the vessel. As they approach one another the probe uses a pulsed laser to send a message to the oncoming aliens, where the time interval between pulses is 12 picoseconds in the frame of the probe. What is the time interval between pulses as observed by the aliens?

Solution

What is the vessel's velocity in the probe's frame?

$$v' = \frac{v - V}{1 - V/c^2} = \frac{-\frac{3}{5}c - \frac{4}{5}c}{1 - (-\frac{3}{5}c)(\frac{4}{5}c)} = \frac{-\frac{7}{5}c}{1 + \frac{12}{25}} = -\frac{35}{37}c$$

where the minus sign means leftward motion. In the probe's frame the time interval between pulses is 12 ps. The time interval will be shorter in the vessel's frame by the Doppler factor

$$\sqrt{\frac{1 - \frac{35}{37}}{1 + \frac{35}{37}}} = \sqrt{\frac{\frac{2}{37}}{\frac{72}{37}}} = \sqrt{\frac{1}{36}} = \frac{1}{6}.$$

So the time interval between pulses as detected by the vessel will be $\frac{12\text{ ps}}{6} = 2$ ps. ■

⋆ **Problem 2.21** An organist on earth is playing Bach's Toccata and Fugue in D Minor, which is being broadcast by a powerful radio antenna. Travelers in a spaceship moving at speed $v = 3/5c$ away from the earth are listening in. In what key do they hear the music?

Solution

According to the Doppler effect, the frequencies heard will all be reduced by the factor

$$\frac{\nu_{\text{heard}}}{\nu_{\text{transmitted}}} = \sqrt{\frac{1 - v/c}{1 + v/c}} = \sqrt{\frac{1 - 3/5}{1 + 3/5}} = \frac{1}{2}.$$

If a frequency is cut in half, in music that corresponds to one octave lower. All pitches are one octave lower in this case, so the key is still D minor.

■

⋆ **Problem 2.22** The Andromeda galaxy, also known to astronomers by the catalog number M31, is in our local group of galaxies, about 2.5 million light-years from our own Milky Way (MW) galaxy. When using spectrometers to measure the wavelengths of light emitted by stars in M31, astronomers find the redshift to be $\Delta\lambda/\lambda = -0.001001$, where λ is the wavelength of the spectral line in the laboratory, $\Delta\lambda$ is the shift in wavelength, and the minus sign indicates that $\Delta\lambda < 0$, corresponding to a *blue* shift. (a) If one assumes this change in wavelength is due to the Doppler effect, how fast (in km/s) is M31 approaching us? (b) If this velocity were also M31's velocity toward our galactic nucleus, and it did not change with time, how long would it take M31 to collide with the MW? [In fact, M31's velocity toward the MW nucleus is less than the result calculated in part (a), because the

Fig. 2.15 The M31 (Andromeda galaxy) referred to in a problem. Image credit: NASA/JPL-Caltech, from the GALEX mission.

solar system is orbiting around our MW nucleus, with a component of velocity directed towards M31, so in fact M31 is moving only about 110 km/s toward our MW nucleus. We would also expect the M31/MW relative velocity of approach to increase with time due to their mutual gravitational attraction. Taking all this into account predicts they will collide in about 4 billion years.]

Solution

$$\frac{\lambda_{observed}}{\lambda_{natural}} = \frac{\lambda_{natural} + \Delta\lambda}{\lambda_{natural}} = 1 + \frac{\Delta\lambda}{\lambda_{natural}} = 1 - 0.001001$$

$$= \sqrt{\frac{1-(v/c)}{1+(v/c)}} = (1-(v/c))^{1/2}(1+(v/c))^{-1/2} \simeq 1 - \frac{v}{2c} - \frac{v}{2c} = 1-(v/c)$$

to first order in v/c (using a binomial expansion). Then $v/c \simeq 0.001$ so

$$v = 10^{-3}c \cong 3 \times 10^5 \text{m/s} = 300 \text{ km/s}.$$

(b)

$$t = D/v = \frac{2.5 \times 10^6 c - \text{yrs}}{10^{-3}c} = 2.5 \times 10^9 \text{yrs} = 2.5 \text{ billion years}.$$

■

★★ **Problem 2.23** A proton moves in the x, y plane with velocity $v = (3/5)c$, at an angle of 45° to both the x and y axes. (a) Find all four components of the proton's four-vector velocity v^μ and evaluate the invariant square of its components $\eta_{\mu\nu}v^\mu v^\nu$. (b) Find all four components of the proton's four-vector velocity in a frame moving in the positive x direction at velocity $V = (4/5)c$. (c) Evaluate explicitly the invariant square of its components in this frame.

Solution

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{5}{4}, \quad v^x = v\cos 45^\circ = v^y$$

$$v^\mu = (\gamma c, \gamma v^x, \gamma v^y, \gamma v^z) = (\frac{5}{4}c, \frac{3\sqrt{2}}{8}c, \frac{3\sqrt{2}}{8}c, 0)$$

$$\eta_{\mu\nu} v^\mu v^\nu = -(v^t)^2 + (v^k)^2 + (v^y)^2 + (v^z)^2 = -\frac{25}{16}c^2 + \frac{18}{64}c^2 + \frac{18}{64}c^2 + 0^2$$

$$-\frac{50c^2}{32} + \frac{18}{32}c^2 = -c^2$$

(b)

$$V = \frac{4}{5}c \Rightarrow \gamma_V = \frac{1}{\sqrt{1 - V^2/c^2}} = \frac{5}{3}$$

$$v^{t'} = \gamma_V(v^t - \frac{V}{c}v^x) \qquad v^{y'} = v^y$$

$$v^{x'} = \gamma_V(v^x - \frac{V}{c}v^t) \qquad v^{z'} = v^z$$

$$v^{t'} = \frac{5}{3}(\frac{5}{4}c - \frac{4}{5}\frac{3\sqrt{2}}{8}c) = \frac{1}{12}c(25 - 6\sqrt{2})$$

$$v^{x'} = \frac{5}{3}(\frac{3\sqrt{2}}{8}c - \frac{4}{5}\frac{5}{4}c) = \frac{5}{24}c(3\sqrt{2} - 8)$$

$$v^{y'} = \frac{3\sqrt{2}}{8}c \qquad v^{z'} = 0$$

(c)

$$\eta_{\mu'\nu'} v^{\mu'} v^{\nu'} = -(v^{t'})^2 + (v^{x'})^2 + (v^{y'})^2 + (v^{z'})^2$$

$$= -\frac{c^2}{144}(25 - 6\sqrt{2})^2 + \frac{25}{576}c^2(-8 + 3\sqrt{2})^2 + \frac{18c^2}{64} = -c^2$$

■

★★ **Problem 2.24** A particular pion π^+ decays in 26 nanoseconds in its own rest-frame. Suppose a particle accelerator produces the pion with total energy $E = 100\ mc^2$, where m is its mass. (a) How far (in meters) will it travel before decaying? (b) A different pion has a kinetic energy equal to its mass energy. If it travels a distance D before decaying, find how long it lived in its own rest-frame.

Solution

(a) The distance it moves is

$$D = vt = v\frac{t_{\text{rest frame}}}{\sqrt{1 - v^2/c^2}}$$

since it lasts longer in the lab than in its rest frame. Now

$$\frac{E}{mc^2} = 100 = \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \quad \text{so}$$

$$(1 - v^2/c^2) = 10^{-4} \Rightarrow v/c = \sqrt{1 - 10^{-4}}. \quad \text{Therefore}$$

$$D = (\sqrt{1 - 10^{-4}}c)\ 26 \times 10^{-9}\ \text{s} \times 100 \simeq (3 \times 10^8 \text{m/s})26 \times 10^{-7}\ \text{s} = 780\ \text{m}$$

(b)

$$K.E. = (\gamma - 1)mc^2 = mc^2 \Rightarrow \gamma = 2 = \frac{1}{\sqrt{1 - v^2/c^2}} \Rightarrow \qquad v/c = \frac{\sqrt{3}}{2}.$$

$$D = vt_{\text{lab}} = \frac{vt_{\text{rest frame}}}{\sqrt{1 - v^2/c^2}} \qquad \text{so} \qquad t_{\text{rest frame}} = \frac{D\sqrt{1 - v^2/c^2}}{v} = \frac{D\sqrt{1 - 3/4}}{\frac{\sqrt{3}}{2}c} = \frac{D}{\sqrt{3}c}$$

■

⋆ **Problem 2.25** A photon of total energy $E = 12,000$ MeV is absorbed by a nucleus of mass M_0, originally at rest. Afterwards, the excited nucleus has mass M and is moving at speed (12/13) c. Find its momentum in the units MeV/c, and both M and M_0 in the units MeV/c^2.

Solution

Conserve momentum:

$$12,000\ \text{MeV/c} = \frac{M \cdot \frac{12}{13}c}{\sqrt{1 - (\frac{12}{13})^2}} = \frac{M \cdot \frac{12}{12}c}{5/13} = \frac{12}{5}Mc \Rightarrow M = 5000\ \text{MeV/c}^2$$

Conserve energy:

$$12,000\ \text{MeV/c} + M_0c^2 = \frac{Mc^2}{\sqrt{1 - (12/13)^2}} = \frac{13}{5}Mc^2$$

Combining,

$$12,000\ \text{MeV} + M_0c^2 = 13,000\ \text{MeV} \Rightarrow M_0 = \frac{1000\ \text{MeV}}{c^2}.$$

So altogether,

$$p = 12,000\ \text{MeV}/c, \quad M = 5000\ \text{MeV}/c^2, \quad M_0 = 1000\ \text{MeV}/c^2.$$

■

★ **Problem 2.26** A team plans to accelerate a probe of mass 2.0 kg away from the far side of the Moon by a bank of lasers that push the probe with the constant force F in the rest frame of the Moon. What F would be required to accelerate the probe to velocity $0.9c$ in one week?

Solution

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(\gamma m\mathbf{v}) \quad \text{so since } \mathbf{F} \text{ is constant,}$$

$$\int F dt = Ft = \gamma mv\Big|_0^v = \gamma mv = \frac{mv}{\sqrt{1 - v^2/c^2}}.$$

$$\text{Therefore} \quad F = \frac{m}{t}\frac{v}{\sqrt{1 - v^2/c^2}} = \frac{2.0 \text{ kg}}{7 \times 24 \times 2400 \text{ s}}\frac{0.9c}{\sqrt{1 - (0.9)^2}}$$

$$F = \frac{(2.0)(0.9)(3 \times 10^8 \text{m/s})}{(7)(2.4)(5.6) \text{ s}} \text{kg}\frac{10^{-1-3}}{\sqrt{1 - 0.81}} = (\frac{(2.0)(0.9)(3)}{(7)(2.4)(5.6)(.436)})10^4 \text{ kg m/s}$$

$$= 2050 \text{ kg m/s}^2 = 2050 \text{ Newtons.}$$

■

★ **Problem 2.27** Show that the momentum and velocity four-vectors are both timelike, and that the force four-vector is spacelike .

Solution

$$P^\mu = (\frac{E}{c}, p^x, p^y, p^z) =\equiv (\frac{E}{c}, \mathbf{p}) \Rightarrow p^\mu p^\nu \eta_{\mu\nu} = -\frac{E^2}{c^2} + p^2 = -M^2c^2 \ (< 0, \text{so timelike}).$$

$$u^\mu = (\gamma_c, \gamma\mathbf{v}) \Rightarrow u^\mu u^\nu \eta_{\mu\nu} = -\gamma^2 c^2 + \gamma^2 v^2$$

$$u^\mu u^\nu \eta_{\mu\nu} = -\gamma^2 c^2 (1 - v^2/c^2) \ (< 0, \text{so timelike}).$$

$$f^\mu = \left(\gamma\frac{\mathbf{v}}{c} \cdot \mathbf{F}, \gamma\mathbf{F}\right) = \gamma\left(\frac{v}{c}F\cos\theta, \mathbf{F}\right)$$

where θ is the angle between $\mathbf{v}$ & $\mathbf{F}$.

$$\to f^\mu f^\nu \eta_{\mu\nu} = \gamma^2\left(-\frac{v^2}{c^2}F^2\cos^2\theta + F^2\right) = \gamma^2 F^2\left(1 - \frac{v^2}{c^2}\cos^2\theta\right) > 0 \text{ (so spacelike).}$$

■

★★ **Problem 2.28** A wave equation for light is

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0.$$

where ϕ is a scalar potential . Show that the set of all linear transformations of the spacetime coordinates that permit this wave equation to be written as we did, correspond to (i) four possible translations in space and time, (ii) three constant rotations of space, and (iii) three

Lorentz transformations. Collectively, these are called the **Poincaré transformations** of spacetime.

Solution

The wave equation can be written in the form

$$\eta^{\mu\nu}\gamma_\mu\gamma_\nu\phi = 0 \qquad \partial_\mu = \partial_\mu x^{\mu'}\partial_{\mu'} \equiv A_\mu^{\mu'}\partial_{\mu'} \Rightarrow \eta^{\mu\nu}A_\mu^{\mu'}A_\nu^{\nu'}\partial_{\mu'}\partial_{\nu'}\varphi = 0$$

$$\Rightarrow A_\mu^{\mu'}A_\nu^{\nu'}\eta^{\mu\nu} = \eta^{\mu'\nu'}$$

In matrix notation

$$\hat{\mathbf{A}}\hat{\eta}\hat{\mathbf{A}}^T = \hat{\eta}$$

Note that $\eta^{\mu\nu}$ and $\eta^{\mu'\nu'}$ have the same diagonal matrices.

Rotations:

$$\hat{\mathbf{A}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \hat{\mathbf{R}} & \\ 0 & & & \end{pmatrix} \Rightarrow \hat{\mathbf{R}}^T\hat{\mathbf{R}} = 1$$

Translations:

$$x^{\mu'} = x^\mu + a^\mu \Rightarrow \partial_{\mu'} = \partial_\mu$$

Lorentz transformations: If

$$r^\mu r^\nu \eta_{\mu\nu} = 0 \Rightarrow r^{\mu'}r^{\nu'}\eta_{\mu'\nu'} = 0$$

where r^μ is the position vectors. But

$$r^\mu r^\nu \eta_{\mu\nu} = 0 \Rightarrow -c^2t^2 + x^2 + y^2 + z^2 = 0 \ \text{ light trajectory}$$

$$\Rightarrow -c^2t'^2 + x'^2 + y'^2 + z'^2 = 0 \Rightarrow \hat{\mathbf{A}} \ \text{ preserves speed of light} \rightarrow \text{Lorentz boosts allowed.}$$

■

★★ **Problem 2.29** *Two spaceships with string "paradox"*. Consider two spaceships, both at rest in our inertial frame, a distance D apart, one behind the other. There is a light string of rest-length D tied between them. Now the ships, both at the same time in our frame, begin to accelerate uniformly to the right, with the string still tied between them. The ships start at the same time and have the same acceleration, so the distance between them, and therefore the length of the string, must be constant in our frame. However, we know that a moving string *should* be Lorentz-contracted in its direction of motion, by the usual factor $\sqrt{1-\beta^2}$. Therefore, does the "need" of the string to become shorter in our frame cause it to break eventually, or does the fact that its length remains the same in our frame mean that it will not break? Explain which is correct. *Hint*: The "proper length" of an accelerating object can be measured by observers in an inertial frame instantaneously comoving with the object, that is, in an inertial frame that at some moment is at rest relative to the object. This "paradox" was originally posed by E. Dewan and M. Beran in 1959 and later modified by J. S. Bell in 1987. As Bell describes in Chapter 9 of his book "Speakable and unspeakable in quantum mechanics," Cambridge University Press, 1987, some very good physicists have gotten the wrong answer, at least initially.

Solution

Many solutions to this "paradox" can be found in the literature. The correct explanations all show that the string will break, and why it will break. Here is one way to understand the situation. (1) A string in which there is no tension will not break, while a string under sufficient tension will break. (2) Initially, before the spaceships start to accelerate, they are at rest in some inertial frame S, the string between them has its natural length (say) D, is under no tension, and so does not break. (3) Now view this same situation from a different inertial frame S' moving with velocity v to the right relative to frame S, the ships, and the string. In this frame S' the string and ships are moving to the *left* at constant velocity v. The string has length $D\sqrt{1-v^2/c^2}$ in S' due to the Lorentz contraction. It does not break (since it didn't break when seen in S and whether a string breaks or not is a relativistic invariant) and so it is under no stress, *i.e.*, no tension forces or compressional forces. (For example, we could untie the string from both ships, and the string would not change length.) Its Lorentz-contracted length is its *natural* length when it is moving at uniform speed. If it has some other length than its Lorentz-contracted length it must be under stress. (4) So now return to the view in frame S, but after the ships have begun to accelerate. Now the ships and string all have the same velocity v to the right. However, since the ships began to accelerate at the same time, and with the same acceleration, their distance apart (and the length of the string) must stay constant, equal to D. That is, the string has length D in this frame, and not the Lorentz-contracted length $D\sqrt{1-v^2/c^2}$, which would be the length for which it has no tension. Therefore the string, keeping length D, must be under tension, and therefore will break if v is large enough.

It is also interesting to view events from the point of view of observers at rest in frame S'. Whereas in frame S the two ships began to accelerate simultaneously, this is not true in frame S'. From the Lorentz transformation we learn that if there are two moving clocks, each moving with speed v, and which have been synchronized in their mutual rest frame, the leading clock will lag the trailing clock in time by the amount vD/c^2, where D is the rest distance (the proper distance) between the two clocks. Now in frame S' the two ships are moving to the left, so the left-hand ship leads the right-hand ship from the point of view of observers in S'. Therefore the clock on the left-hand ship lags the clock on the right-hand ship. Now suppose each ship begins to accelerate when its clock reads (say) $t = 0$, simultaneously with one another in frame S. But because the left-hand ship clock lags the right-hand ship clock as seen in S', it is the right-hand ship clock that reads $t = 0$ first, and therefore the right-hand ship will begin to accelerate first as seen in S'. This will begin to stretch the string from its natural length $D\sqrt{1-v^2/c^2}$ in S', since the left-hand ship has not yet begun to accelerate, so the string will break if v is large enough.

■

⋆⋆⋆ **Problem 2.30** (a) Prove that the time order of two events is the same in all inertial frames if and only if they can be connected by a signal traveling at or below speed c. (b) Suppose that in an unprimed inertial frame a particular signal from A to B can travel at velocity

$v = 2c$. Then find a relative velocity V with a primed frame (where $|V| < c$) such that in the primed frame the same signal reaches B before it was sent by A.

Solution

(a) Say event A causes event B such that in frame $\mathcal{O}$, $\Delta t = t_B - t_A > 0$. We want to show that $\Delta t' > 0$ for any frame $\mathcal{O}'$ whose relative speed $V < c$. Without loss of generality, we can align the x and x' axes such that $\mathcal{O}'$ moves with V in the positive x direction. We then have:

$$ct'_{A,B} = \gamma \left(ct_{A,B} - \frac{V}{c} x_{A,B} \right)$$

$$\Rightarrow c\Delta t' = c(t'_B - t'_A) = \gamma(c(t_B - t_A) - \frac{V}{c}(x_B - x_A)) = \gamma \left(c\Delta t - \frac{V}{c}\Delta x \right)$$

$$\text{if } \Delta t' > 0 \Rightarrow c\Delta t - \frac{V}{c}\Delta x > 0 \Rightarrow c > V > \frac{\Delta x}{\Delta t}$$

where $\Delta x = x_B - x_A$ is the spatial separation between the two events as seen by $\mathcal{O}$. Hence, we must have $\frac{\Delta x}{\Delta t} < c$ as desired.

(b) Now we have

$$\frac{\Delta x}{\Delta t} = 2c \text{ with } \Delta x > 0, \Delta t > 0$$

From (a), we have

$$c\Delta t' = \gamma \left(c\Delta t - \frac{V}{c}\Delta x \right) = \gamma c\Delta t \left(1 - \frac{V}{c}\frac{\Delta x}{c\Delta t} \right) = \gamma c\Delta t \left(1 - \frac{V}{c} 2 \right)$$

We want $\Delta t' < 0$ so that B is before A in $\mathcal{O}'$

$$\Rightarrow 1 - \frac{2V}{c} < 0 \Rightarrow \frac{c}{2} < V$$

Any $V > c/2$ will do. ■

★★ **Problem 2.31** In the text we derived the Doppler formulae for light. Using the same strategy, find the relativistic Doppler formulae for waves traveling at a speed $v < c$. For example, the waves may be sound waves in some very stiff material whose sound speed is a few percent that of c.

Solution

In frame O, we have the 4-momentum

$$p^\mu = (\frac{E}{c}, p^x, p^y, p^z) = (\frac{E}{c}, p, 0, 0) \quad \text{as in the text.}$$

$$Here E = h\nu \quad \text{and} \quad p = \frac{h}{\lambda}.$$

but now $\lambda\nu = v \neq c$. In O', we will have $p^{\mu'} = (\frac{E'}{c}, p', 0, 0)$. We also know (where V is the relative speed of $\mathcal{O}$ and $\mathcal{O}'$),

$$\frac{E'}{c} = \gamma_V(\frac{E}{c} - \frac{V}{c}p) \Rightarrow \frac{h\nu'}{c} = \gamma_V(\frac{h\nu}{c} - \frac{V}{c}\frac{h}{\lambda}) \Rightarrow \nu' = \gamma_V(\nu - \frac{V}{c}\frac{c}{v}\nu) = \gamma_V(1 - \frac{V}{v})\nu$$

where ν is the frequency seen in the rest frame of source $\mathcal{O}$. ■

⋆ **Problem 2.32** An algebraic expression is said to be **Lorentz covariant** if its form is the same in all inertial frames: the expression differs in two inertial frames $\mathcal{O}$ and $\mathcal{O}'$ only by putting prime marks on the coordinate labels. For example, $A^\mu\eta_{\mu\nu}B^\nu = K$ is a Lorentz covariant expression, where A^μ and B^ν are four-vectors and K a constant. Under the Lorentz transformation, $A^\mu\eta_{\mu\nu}B^\nu = A^{\mu'}\Lambda^\mu_{\mu'}\eta_{\mu\nu}B^{\nu'}\Lambda^\nu\nu' = A^{\mu'}\eta_{\mu'\nu'}B^{\nu'} = K$, where we used $\eta_{\mu'\nu'} = \Lambda^\mu_{\mu'}\eta_{\mu\nu}\Lambda^\nu_{\nu'}$. Because the indices come matched in pairs across a metric factor $\eta_{\mu\nu}$, the expression preserves its structural form. The quantity is also a **Lorentz scalar**: its *value* is unchanged under a Lorentz transformation. Which of the following quantities are Lorentz scalars, given that K is a constant and any quantity with a single superscript is a four-vector? (a) $KA^\mu\eta_{\mu\nu}$ (b) $C^\mu = D^\mu(A^\lambda\eta_{\lambda\nu}B^\nu)$. (c) $KA^\mu\eta_{\mu\nu}B^\lambda\eta_{\lambda\sigma}D^\nu F^\sigma$

Solution

(a) Is a Lorentz vector, *not* a Lorentz scalar.

(b) C^μ is a Lorentz vector, *not* a Lorentz scalar.

(c) *Is* a Lorentz scalar. ■

⋆ **Problem 2.33** Consider a Lorentz covariant expression that is *not* a Lorentz scalar, $C^\lambda = K^\lambda h(A^\mu\eta_{\mu\nu}B^\nu)$, where h is any function of the quantity in parentheses. Here quantities with a single superscript are four-vectors. Under a Lorentz transformation, $A^\mu\eta_{\mu\nu}B^\nu$ is Lorentz covariant and is also a Lorentz scalar. Hence, its form and value are unchanged, which means that the function $h(A^\mu\eta_{\mu\nu}B^\nu)$ is unchanged in form or value as well. The quantity K^μ on the other hand is a four-vector; this means that it transforms as $K^\mu = \Lambda^\mu_{\mu'}K^{\mu'}$. The right-hand side of the equation for C^λ transforms as a four-vector as whole, which implies that C^λ also transforms as a four-vector and observer $\mathcal{O}'$ can write $C^{\lambda'} = K^{\lambda'}h(A^{\mu'}\eta_{\mu'\nu'}B^{\nu'})$. This quantity is said to be a **Lorentz vector** (instead of a scalar), since it transforms as a four-vector: That is, its components change, but through the well-defined prescription for a four-vector. Which of the following quantities are Lorentz vectors, given that K is a Lorentz scalar and any quantity with a single subscript or superscript is a Lorentz vector? (a) $K\eta_{\mu\nu}$ (b) $C^\lambda = D^\mu A^\lambda\eta_{\mu\nu}B_\nu$ (c) $KA^\mu\eta_{\mu\nu}B^\lambda\eta_{\lambda\sigma}D^\nu F^\sigma$

Solution

(a) No. (b) Yes. (c) No. ■

⋆ **Problem 2.34** The concept of Lorentz covariance is important because it allows us to quickly determine the transformation properties of expressions under changes of inertial reference frames. The principle of relativity requires that all laws of physics are unchanged as seen by different inertial observers. Hence, we need to ensure that expressions reflecting statements of a law of physics are Lorentz covariant, *i.e.*, that they retain their structural form under Lorentz transformations. A useful application of this comes from the modified second law of dynamics,

$$f_\mu = \frac{dp^\mu}{d\tau} .$$

Forces that we insert on the left hand side of this equation must be Lorentz covariant expressions that transform as four-vectors. This ensures that observer $\mathcal{O}'$ can write simply

$$f_{\mu'} = \frac{dp^{\mu'}}{d\tau} .$$

For example, we could write $f_\mu = K_\mu$ with a constant four-vector K_μ. (a) Is a "relativistic spring law" $f_\mu = -(0, k\mathbf{r})$ for some constant k, a Lorentz covariant expression? (b) What about a modified spring law $f_\mu = -Kr^\mu = -k(c\,t, \mathbf{r})$? (c) What about Newtonian gravity $\mathbf{F} = -(k/r^3)\mathbf{r}$? Is such a force covariant?

Solution

(a) No, not covariant. In a primed frame it would not have the form $(0, k\mathbf{r}')$.

(b) Yes, covariant. In a primed frame $f_\mu = -k(ct', \mathbf{r}')$ because $(ct, \mathbf{r})$ is a four-vector.

(c) No, not covariant. In a primed frame $\mathbf{F}' \neq -(k/r'^3)\mathbf{r}'$ using a Lorentz transformation. ■

★★ **Problem 2.35** Show that the most general Lorentz transformation can be written as a 4×4 matrix $\hat{\mathbf{\Lambda}}$ satisfying

$$\hat{\mathbf{\Lambda}}^{\mathrm{T}} \cdot \hat{\eta} \cdot \hat{\mathbf{\Lambda}} = \hat{\eta} \quad \text{and} \quad |\hat{\mathbf{\Lambda}}| = 1 .$$

Since a Lorentz transformation is by definition a linear transformation of time and space that preserves the speed of light, you simply need to show that these two properties as necessary and sufficient for this. Note also that reflections get ruled out by the second condition by choice.

Solution

The Lorentz transformation matrix is

$$\hat{\mathbf{\Lambda}} = \begin{bmatrix} \gamma & \pm\gamma\beta & 0 & 0 \\ \pm\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = \frac{V}{c}.$$

The position 4-vector is (ct, x, y, z), so that

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \hat{\mathbf{\Lambda}} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \text{Check} \quad \hat{\mathbf{\Lambda}}^{\mathbf{T}}\hat{\eta}\hat{\mathbf{\Lambda}} = \hat{\eta} \qquad \det \hat{\mathbf{\Lambda}} = 1$$

In general, for a light ray with position

$$r^{\mu} = (ct, x, y, z) \qquad \hat{\mathbf{r}}^{\mathbf{T}}\hat{\eta}\hat{\mathbf{r}} = 0 \Rightarrow \hat{\mathbf{r}}'^{\mathbf{T}}\hat{\eta}\hat{\mathbf{r}}' = 0$$

$$\text{But } \hat{\mathbf{r}}^{\mathbf{T}}\hat{\eta}\hat{\mathbf{r}} = \hat{\mathbf{r}}'^{\mathbf{T}}\hat{\mathbf{\Lambda}}^{\mathbf{T}}\hat{\eta}\hat{\mathbf{\Lambda}}\hat{\mathbf{r}}' \Rightarrow \hat{\mathbf{\Lambda}}^{\mathbf{T}}\hat{\eta}\hat{\mathbf{\Lambda}} = \hat{\eta} \quad \text{as needed.}$$

■

★★ **Problem 2.36** A π^0 meson with mass $m_\pi = 135.0\ \text{MeV}/c^2$ is created in the upper atmosphere when a cosmic-ray proton collides with a nitrogen nucleus. The mean lifetime of π^0s is 8.4×10^{-17} s; they almost always decay into two photons. Suppose this particular pion has total energy $E = 500$ MeV and moves vertically downward toward the ground, and also that it decays in three mean lifetimes into two photons, one moving up and one moving down. (a) How far does the π^0 move relative to the ground from its creation until it decays? (b) Find the frequency of each final photon measured in the frame of the ground.

Solution

(a) The pion's energy is

$$E = \gamma mc^2 \Rightarrow \gamma = E/mc^2 = \frac{500\ \text{MeV}}{135\ \text{MeV}} = 3.70$$

$$\tau = 3 \times 8.4 \times 10^{-17}\ \text{s} = 2.52 \times 10^{-16}\ \text{s} = \text{lifetime in its rest frame.}$$

The time it takes to decay in the frame of the earth is longer,

$$t = \gamma\tau = 3.70 \times 2.52 \times 10^{-16}\ \text{s} = 9.32 \times 10^{-16}\ \text{s}$$

Now we need the pion's velocity:

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = 3.70 \quad \Rightarrow v/c = \sqrt{.927} = 0.963,$$

so the distance moved is

$9.32 \times 10^{-16}\ \text{s} \times 0.963 \times 3 \times 10^8\ \text{m/s} = 26.9 \times 10^{-8}\text{m} = 2.69 \times 10^{-7}\ \text{m}$, less than one micron.

(b) In the pion rest frame, each photon has energy $\frac{500\ \text{MeV}}{2} = 250\ \text{MeV} = h\nu_0$, so the frequency of each photon is

$$\nu_0 = \frac{250\ \text{MeV}}{6.63 \times 10^{-34}\ \text{Js}} = \frac{250 \times 10^6\ \text{eV}}{6.63 \times 10^{-34}\ \text{Js}} = \frac{250 \times 10^6 \times 1.60 \times 10^{-14}\ \text{J}}{6.63 \times 10^{-34}\ \text{Js}}$$

$$= \frac{(2.50)(1.60)}{6.63} 10^{34-19}\ \text{s}^{-1} = 0.603 \times 10^{15}\ \text{s}^{-1} = 6.03 \times 10^{14}\ \text{s}^{-1} \qquad \text{(in the pion rest-frame.)}$$

In the frame of the ground the frequency of the downward-moving photon will be higher,

$$\nu_{\text{down}} = \nu_0 \sqrt{\frac{1 + (v/c)}{1 - (v/c)}} = 6.03 \times 10^{14}\ \text{s}^{-1} \sqrt{\frac{1 + 0.96\ \text{s}}{1 - 0.96\ \text{s}}}$$

$$= (6.03)\sqrt{\frac{1.963}{.037}} \times 10^{14}\ \text{s}^{-1} = 43.9 \times 10^{14}\ \text{s}^{-1} = 4.39 \times 10^{15}\ \text{s}^{-1}$$

and the frequency of the upward-moving photon will be lower,

$$\nu_{v_p} = \nu_0 \sqrt{\frac{1 - v/c}{1 + v/c}} = 6.03\sqrt{\frac{.037}{1.963}} \times 10^{14}\ \text{s}^{-1} = (6.05)(.137) \times 10^{14}\ \text{s}^{-1}$$

$$= 0.831 \times 10^{14}\ \text{s}^{-1} = 8.31 \times 10^{13}\ \text{s}^{-1}.$$

■

⋆ **Problem 2.37** A π^- meson with mass $m_\pi = 140.0$ MeV/c^2 is produced in a (p, p) collision in an accelerator. The pion subsequently decays into a muon and a muon-type antineutrino, in the reaction $\pi^- \to \mu^- + \bar{\nu}_\mu$. The antineutrino has a non-zero but very small mass, so in this calculation you can ignore it. The muon has a mass-energy of 105.7 MeV/c^2. In the rest-frame of the original pion, find (a) The total energy and three-momentum of the muon, (b) The total energy and three-momentum of the antineutrino.

Solution

(a) Conservation of energy:

$$E_\pi = m_\pi c^2 = E_\mu + E_\nu$$

Conservation of momentum:

$$\mathbf{p}_\pi = 0 = \mathbf{p}_\mu + \mathbf{p}_\nu$$

so in magnitude

$$p_\mu = p_\nu \Rightarrow p_\mu^2 c^2 = p_\nu^2 c^2 \Rightarrow E_\mu^2 - m_\mu^2 c^4 \simeq E_\nu^2$$

$$\text{But } E_\mu = m_\pi c^2 - E_\nu \text{ or } E_\nu = m_\pi c^2 - E_\mu.$$

Let us eliminate E_ν,

$$-m_\mu^2 c^4 = (m_\pi c^2 - E_\mu)^2 = m_\pi^2 c^4 - 2 m_\pi c^2 E_\mu$$

(a) So

$$E_\mu = (\frac{m_\pi^2 + m_\mu^2}{2m_\pi})c^2 = (\frac{(140.0)^2 + (105.7)^2}{2(140.0)})\ \text{MeV} = 109.9\ \text{MeV}$$

$$p_\mu c = \sqrt{E_\mu^2 - m_\mu^2 c^4} = \sqrt{(109.9)^2 - (105.7)^2} = 30.0\ \text{MeV} \Rightarrow p_\mu = 30.0\ \text{MeV/c}$$

(b)

$$E_{\bar{\nu}} = m_\pi c^2 - E_\mu = (140.0 - 109.9)\ \text{MeV} = 30.1\ \text{MeV} \qquad p_\nu = 30.0\ \text{MeV/c}$$

■

★★ **Problem 2.38** The Higgs particle has a mass-energy of 125 GeV/c^2. Once created it decays very quickly into various sets of particles: for example, about 60% of the time it decays into a $(b\bar{b})$ quark- antiquark pair. Such b quarks have a mass energy of about 4.2 GeV each. (The b quarks are also called "bottom" quarks, and the reaction is written $H \to b\bar{b}$ or simply $H \to b\bar{b}$ for short.) Suppose a particular Higgs particle is moving at $v = 4/5\ c$ in the lab, that it decays into a $(b\bar{b})$ pair with the b quark moving in the forward direction and the $\bar{b}$ quark moving in the backward direction. Find the energy and momentum in the lab frame of (a) the b quark; (b) the $\bar{b}$ quark.

Solution

We will first find the energy and momentum for each final particle in the CM frame, and then transform to the lab frame. In the CM frame the Higgs is at rest, with $E_H = 125$ GeV and $p_H = 0$. After the decay,

$$E_{\text{b quark}} = E_H/2 = 62.5\ \text{GeV and}\ p_b c = \sqrt{E_b^2 - M_b^2 c^4}$$

$$= \sqrt{(62.5)^2 - (4.2)^2}\ \text{GeV} = 62.36\ \text{GeV}$$

$$\text{and}\ E_{\bar{b}} = 62.5\ \text{GeV and}\ p_{\bar{b}}c = -62.36\ \text{GeV in the CM frame.}$$

The lab frame moves to the left at $\frac{4}{5}c$ relative to the CM frame, (since the Higgs moves to the right at $\frac{4}{5}c$ in the lab.) Then using the Lorentz transformation for the energy-momentum four-vector,

(a) for the b quark

$$E_{\text{lab}} = \gamma(E_{CM} + Vp_{CM}) = \frac{5}{3}\left[62.5 + \frac{4}{5}(62.4)\right]\ \text{GeV} = 187.5\ \text{GeV}$$

$$p_{\text{lab}} = \gamma(p_{CM} + \frac{V}{c^2}E_{CM}) = \frac{5}{3}\left[62.4 + \frac{4}{3}(62.5)\right]\ \text{GeV} = 187.5\ \text{GeV/c}$$

(b) For the $\bar{b}$ quark,

$$E_{\text{lab}} = \gamma(E_{CM} + Vp_{CM}) = \frac{5}{3}\left[62.5 - \frac{4}{5}(62.4)\right]\ \text{GeV} = 21.0\ \text{GeV}$$

$$p_{\text{lab}} = \gamma(p_{CM} + \frac{V}{c^2}E_{CM}) = \frac{5}{3}\left[-62.4 + \frac{4}{3}(62.5)\right]\ \text{GeV} = -20.7\ \text{GeV/c}$$

■

★ **Problem 2.39** *Tachyons* are hypothetical (and so-far undetected) particles that always travel faster than light. (a) Show that all components of a tachyon's momentum four-vector are real if we assign the tachyon an imaginary mass, say $m = im_0$, where m_0 is real. (b) Then show that the invariant square of the momentum four-vector $\eta_{\mu\nu}p^\mu p^\nu$ is necessarily positive. Thus the world of particles could be separated into three regimes: (i) Ordinary massive particles, with $\eta_{\mu\nu}p^\mu p^\nu < 0$; photons or other possible massless particles, with $\eta_{\mu\nu}p^\mu p^\nu = 0$; and tachyons, with $\eta_{\mu\nu}p^\mu p^\nu > 0$.

Solution

(a) The four-vector momentum is

$$p^\mu = (-\frac{E}{c}, p^x, p^y, p^z) \text{ where } E = \frac{mc^2}{\sqrt{1 - v^2/c^2}}, p^x = \frac{mv^x}{\sqrt{1 - v^2/c^2}}$$

etc. for a normal particle of mass m. For a tachyon $v/c > 1$, so

$$E = \frac{mc^2}{i\sqrt{v^2/c^2 - 1}} \quad , \quad p^x = \frac{mv^x}{i\sqrt{v^2/c^2 - 1}}$$

etc, since $i = \sqrt{-1}$. So if $m = im_0$, where m_0 is real, then E and p are real, with

$$E = \frac{m_0 c^2}{\sqrt{v^2/c^2 - 1}} \qquad p^x = \frac{m_0 v^x}{\sqrt{v^2/c^2 - 1}} \qquad \text{etc. for tachyons.}$$

These expressions are real, since $(v/c)^2 > 1$.

(b) For ordinary massive particles,

$$\eta_{\mu\nu}p^\mu p^\nu = -(E/c)^2 + p_x^2 + p_y^2 + p_z^2 = -\frac{m_0^2 c^2}{v^2/c^2 - 1} + \frac{m_0^2(v_x^2 + v_y^2 + v_z^2)}{v^2/c^2 - 1}$$

$$= \frac{m_0^2}{\frac{v^2}{c^2} - 1}\left[-c^2 + v^2\right] = \frac{m_0^2 c^2}{v^2/c^2 - 1}\left[-1 + (v^2/c^2)\right] = m_0^2 c^2 > 0.$$

■

★ **Problem 2.40** *Pion photoproduction.* Positive pi mesons can be created in the reaction $\gamma + p \to n + \pi^+$, in which an incoming photon strikes a proton at rest, forming a neutron and a π^+. (a) Find the threshold photon energy for this reaction given the masses (in units MeV/c^2), $m_p = 938$; $m_n = 939.6$; and $m_{\pi+} = 139.6$. (b) For this photon energy, how fast is the CM frame moving relative to the lab frame, expressed in the form $V_{\rm CM}/c$? (c) What is the momentum of the initial photon in the CM frame, expressed in units MeV/c?

Solution

(a) The threshold energy is

$$E_{\text{thresh}} = \left[(2m_f)^2 - m_0^2 - m_1^2\right] c^2/2m_1 = \frac{(m_n + m_\pi)^2 - m_p^2}{2m_p} c^2$$

$$= \frac{(939.6 + 139.6)^2 - (938)^2}{2(938)} \text{MeV} = 152 \text{ MeV}$$

(b) Then

$$\frac{V_{CM}}{c} = \frac{p_f c}{E_f} = \frac{p_\gamma c}{p_\gamma c + m_p c^2} = \frac{152\text{MeV}}{(152 + 938)\text{MeV}} = 0.14$$

(c) 152 MeV/c, since for photons $E = pc$. ■

⋆⋆ **Problem 2.41** Lambda (Λ^0) baryons can be created in high-energy $(p, \bar{p})$ collisions of protons and antiprotons in the reaction $p + \bar{p} \rightarrow \Lambda^0 + k^+ + \bar{p}$, where k^+ is a positive k meson. (a) Find the minimum (*i.e.*, threshold) energy required for the incident antiproton if the target proton is at rest in the lab. The masses of the particles (in MeV/c^2) are p or $\bar{p}$: 938.3; Λ^0 : 1115.7; and k^+ : 493.7. (b) Find the minimum energy of each initial particle in a collider experiment, in which the total momentum is zero. (c) Suppose that in the collider experiment the energy of each initial particle is twice the minimum energy required. Find then how far the subsequent Λ^0 will travel in the collider detector before it decays, assuming the Λ^0 lasts for a time 2.63 $\times 10^{-10}$ s (the mean lifetime of a Λ^0) in its own rest frame, and also assuming that the final antiproton is at rest in the lab.

Solution

(a) The threshold energy is

$$E_{\text{thresh}} = \left[\left(\sum m_f\right)^2 - m_0^2 - m_1^2 \right] c^2/(2m),$$

$$\text{where } m_0^2 = m_1^2 = m_p^2 \quad \text{and} \quad \sum m_f = m_\wedge + m_k + m_p.$$

Therefore

$$\left(\sum m_f\right)^2 = (m_\Lambda + m_k + m_p)^2 = (1115.7 + 493.7 + 938.3)^2 = 6.50 \times 10^6 (\text{MeV/c}^2)^2$$

$$m_0^2 + m_1^2 = (.9383)^2 \times 2 = (1.76)(\text{MeV/c}^2)^2 \times 10^6$$

$$\text{so } E_{\text{thresh}} = \frac{[6.50 - 1.76]\,\text{MeV}^2 \times 10^6/c^2}{2(.938) \times 10^3\,\text{MeV/c}^2} = 2.53 \times 10^3\ \text{MeV} = 2,530\text{MeV} = 2.53\ \text{GeV}$$

(b) The total final energy in the CM frame at threshold is

$$(m_{\text{or}} + m_k + m_p)c^2 = 2.548 \times 10^3\ \text{MeV}$$

so the initial energy of each particle is

$$\frac{2.548 \times 10^3}{2} = 1.274 \times 10^3\ \text{MeV}$$

(c) The energy of each initial particle in the colliding beam experiment is $2 \times 2.548 \times 10^3$MeV, so the initial kinetic energy of each particle is 2.548×10^3 MeV, so the total energy of each initial particle is 5.10×10^3 MeV. After the collision the final antiproton is at rest, with an energy of $m_p c^2 = 938.3$ MeV. So the remaining particles $\wedge^0$ and k^+ have a combined energy of

$$2 \times 5.10 \times 10^3\ \text{MeV} - 938.3\ \text{MeV} = (10,200 - 938)\ \text{MeV} = 9,260\ \text{MeV}.$$

Also to conserve momentum

$$|p_\wedge| = |p_k|,\ \text{so } p_\wedge^2 c^2 = p_k^2 c^2 \Rightarrow E_\wedge^2 - m_\wedge^2 c^4 = E_k^2 - m_k^2 c^4$$

and to conserve energy

$$E_\wedge + E_k = 9,260\ \text{MeV},\ \text{or } E_k = 9,260\ \text{MeV} - E_\wedge.$$

$$\text{Thus, } E_\wedge^2 - m_\wedge^2 c^4 = (9{,}260\text{ MeV} - E_\wedge)^2 - m_k^2 c^4$$
$$\text{or } \; -m_\wedge^2 c^4 = (9{,}260\text{ MeV})^2 - 2(9260)E_\wedge - m_k^2 c^4$$

$$\text{so} \quad E_\wedge = \frac{(9{,}260\text{ MeV})^2 + m_\wedge^2 c^4 - m_k^2 c^4}{2(9260\text{ MeV})} = \frac{85{,}750{,}000 + (1115.7)^2 - (494)^2}{2(9260)}\text{ MeV}$$

$$= \frac{\left[85.75 \times 10^6 + 1.245 \times 10^6 - .244 \times 10^6\right]}{18{,}520} = 4.83 \times 10^3\text{ MeV}$$

$$\text{Thus, } \gamma_\wedge = \frac{E_\wedge}{M_\wedge c^2} = \frac{4.83 \times 10^3}{1.116 \times 10^3} = 4.33$$

The proper time is 2.63×10^3 s, so

$$t_{\text{lab}} = \gamma\tau = 1.14 \times 10^{-9}\text{ s}$$

The velocity of the $\wedge$ is

$$v/c = \sqrt{\frac{v^2}{c^2}} = \sqrt{1 - \frac{1}{\gamma^2}} = \sqrt{1 - \frac{1}{(4.33)^2}} = 0.97$$

so the distance traveled is

$$d = vt_{\text{lab}} = 0.97c(1.14 \times 10^{-9}\text{ s}) = (0.97)(1.14)3 \times 10^{-1}\text{ m} = 0.33\text{ m} = 33\text{ cm}.$$

■

⋆ **Problem 2.42** Positive Sigma (Σ^+) baryons can be created along with positive k mesons k^+ in high-energy collisions of protons with protons, in the reactions $p + p \to \Sigma^+ + k^+ + n$, where n is a neutron. (a) Find the minimum (*i.e.*, threshold) energy required for the incident proton if the target proton is at rest in the lab. The masses of the particles (in units MeV/c^2) are p : 938.3; Σ : 1189.4; k^+ : 493.7, and n : 939.6. (b) Find the minimum energy of each proton required in a (p,p) collider experiment, in which the total momentum is zero. (c) Find the velocity of the CM (*i.e.*, zero-momentum) frame in this experiment, relative to the lab frame in which one initial proton is at rest, expressed as V_{CM}/c.

Solution

(a) The threshold energy is

$$E_{\text{thresh}} = \frac{\left[(\sum m_f)^2 - m_0^2 - m_1^2\right] c^2}{2m_1}$$

where

$$m_0^2 = m_1^2 = m_p^2 \quad \text{and} \quad \sum m_f = m_\Sigma + m_k + m_n.$$

That is, in MeV units,

$$E_{\text{thresh}} = \frac{\left[(1189.4 + 493.7 + 939.6)^2 - 2(938.3)^2\right]}{2(938.3)} = \frac{(2.62)^2 - 1.76}{1.88} \times 10^3$$
$$= 2.72 \times 10^3\text{ MeV}$$

(b) The minimum corresponds to creating the mass only,

$$(m_\tau + m_k + m_N)c^2 = 2.62 \times 10^3 \text{ MeV, due to both initial protons.}$$

Therefore the energy required for each proton is

$$\frac{2.62}{2} \times 10^3 \text{ MeV} = 1.31 \times 10^3 \text{ MeV.}$$

(c) The velocity of the center of mass relative to the speed of light is

$$v_{CM}/c = \frac{p_f c}{E_f} = \frac{p_1 c}{E_p + m_0 c^2} = \frac{\sqrt{E_1^2 - m_1^2 c^4}}{E_1 + m_p c^2}$$

$$E_1 = 2.72 \times 10^3 \text{ MeV} \quad m_1 c^2 = m_p c^2 = 938.3 \text{ MeV}$$

so

$$V_{CM}/c = \frac{\sqrt{(2.72)^2 \times 10^6 - (.9383)^2 \times 10^6}}{(2.72 \times 10^3 + 0.9383 \times 10^3)} \frac{\text{MeV}}{\text{MeV}} = \left[\sqrt{7.40 - 0.88}/(3.66)\right] = 0.70.$$

■

⋆ **Problem 2.43** Electrons (e^-) and antielectrons (e^+) (called *positrons*) each have mass-energy 0.511 MeV. A positron can be created in an (e^-, e^-) collision as long as an electron is created along with it, thus conserving both electric charge and lepton number. (Electrons have lepton number +1 and positrons have lepton number –1.) (a) In a linear accelerator in which high-energy electrons are incident upon other electrons at rest in the lab, what is the minimum required energy of each of the incident electrons, in MeV? (b) If two beams of electrons are instead fired at one another in a collider with equal but opposite momenta, what now is the minimum energy of each electron required to create a positron, in MeV?

Solution

(a) This problem is similar to the antiproton production problem discussed in Example 2.6 in the text. The threshold energy is

$$E_{\text{thresh}} = \frac{\left[(\sum m_f)^2 - m_0^2 - m_1^2\right] c^2}{2m_1} = \frac{\left[(4m_e)^2 - 2m_e^2\right] c^2}{2m_e}$$

$$= 7m_e c^2 = 7(0.511\text{MeV}) = 3.58 \text{ MeV.}$$

This is the minimum total energy of each incident electron. The minimum kinetic energy of each incident electron is

$$E - m_e c^2 = 3.58 \text{ MeV} - 0.51 \text{ MeV} = 3.07 \text{ MeV.}$$

(b) Each initial electron must have

$$E = 2m_e c^2 = 2 \times (0.511 \text{ MeV}).$$

$E = 1.022$ MeV must be the minimum total energy of each initial electron in the CM frame. ■

⋆ **Problem 2.44** (a) A photon of energy E_0 strikes a free electron at rest in the lab. ("Free" here means the electron is not bound inside an atom.) Is it possible for the photon to be absorbed by the electron? If so, find the energy and momentum of the final electron. If not, explain why not. (b) A free electron of energy E_0 is moving in the lab. Is it possible for the electron to emit a photon, so that after the emission there is a photon and an electron moving more slowly than before? If so, find the final energy and momentum of both the photon and the electron. If not, explain why not.

Solution

(a) It is easiest to view the collision in the CM (center-of-momentum) frame. Although the collision does conserve momentum, it obviously does not conserve energy, because at the end $E = m_e c^2$ and before $E > m_e c^2$. So this reaction cannot happen.

(b) This would conserve momentum, but it would not conserve energy, because

$$E_{\text{before}} = m_e c^2 \qquad E_{\text{after}} > m_e c^2.$$

■

⋆ **Problem 2.45** The quantity $\lambda_C \equiv h/m_e c$ is called the "Compton wavelength" of the electron. (a) If a photon scatters off an electron at rest with scattering angle $\theta = 45^\circ$, what is the photon's change of wavelength in terms of λ_C? (b) For what scattering angle is the change of wavelength a maximum, and what is the change of wavelength in that case?

Solution

(a) In Compton scattering

$$\lambda' - \lambda = \lambda_C(1 - \cos\theta) \quad \text{where} \quad \lambda_C = h/(m_e c).$$

$$\text{If } \theta = 45^\circ, \Delta\lambda = \lambda' - \lambda = \lambda_C\left(1 - \frac{1}{\sqrt{2}}\right) = \lambda_C\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)$$

(b) Obviously, $\Delta\lambda$ is a maximum at $\theta = 180^\circ$, because then $\cos\theta = -1$, so

$$\Delta\lambda = \lambda' - \lambda = 2\lambda_C.$$

■

⋆⋆ **Problem 2.46** Consider a relativistic elastic collision between two particles of equal mass, such as two protons. In the lab frame the target proton is at rest, and the incident proton has three-vector velocity v. For nonrelativistic equal-mass collisions the two protons emerge at right angles to one another, except for the special case where one of the protons moves straight ahead while the other is at rest. Is that also true for relativistic collisions? Prove that it is or that it is not. *Hint*: Draw before-and-after pictures of the three-vector velocities for each proton in the CM frame, and then transform to pictures in the lab frame.

Solution

We need to compute $\mathbf{v}_{(1)f} \cdot \mathbf{v}_{(2)f}$. In the CM frame $\mathbf{P}'_{tot} = 0$, so the two particles must move away from each other back-to-back. Therefore

$$\Rightarrow \mathrm{v}^{x'}{}_{(1)f} = v'_f \cos\theta' \qquad \mathrm{v}^{y'}{}_{(1)f} = v'_f \sin\theta'$$

$$\mathrm{v}^{x'}{}_{(2)f} = -v'_f \cos\theta' \qquad \mathrm{v}^{y'}{}_{(2)f} = -v'_f \sin\theta'$$

We need to transform these to the lab frame using

$$\mathrm{v}^{x} = \frac{\mathrm{v}^{x'} + V}{1 + \frac{\mathrm{v}^{x'} V}{c^2}} \qquad \mathrm{v}^{y} = \frac{\mathrm{v}^{y'}}{\gamma_V (1 + \frac{\mathrm{v}^{x'} V}{c^2})}$$

for (1) and (2), where V is the relative speed of the frames. We can find V from the transformation of initial total momentum between lab and CM frames

$$P'_{(tot)i} = \gamma_V (P_{(tot)i} - \frac{V}{c}\frac{E_{(tot)i}}{c}) \qquad 0 = \gamma_V(\gamma_v m v - \frac{V}{c}(\gamma_v + 1) m c)$$

$$\Rightarrow \frac{V}{c} = \frac{\gamma_v}{\gamma_v + 1}\frac{v}{c} \quad \text{where } v \text{ is the speed of the incoming particle.}$$

$$\Rightarrow \gamma_V = \frac{1}{\sqrt{1 - V^2/c^2}} = \frac{\sqrt{1 + \gamma_v}}{\sqrt{2}}$$

We also need v'_f in term of v. From 4-momentum conservation the in the lab frame, we have

$$2\gamma_{v'_{2'}} mc^2 = 2\gamma_{v'_f} mc^2 \Rightarrow v'_{2'} = v'_f$$

But since the target is at rest in the lab frame, we know that $v'_{2'} = V$. So we have

$$\mathrm{v}^{x}{}_{(1,2)f} = \frac{\pm V\cos\theta' + V}{1 \pm \frac{V^2}{c^2}\cos\theta'}, \qquad \mathrm{v}^{y}{}_{(1,2)f} = \frac{\pm V \sin\theta'}{\gamma_V (1 \pm \frac{V^2}{c^2}\cos\theta')}$$

where $(\pm)$ correspond to particles (1) and (2). We then have

$$\mathbf{v}_{(1)\mathbf{f}} \cdot \mathbf{v}_{(2)\mathbf{f}} = + \frac{V^2}{\gamma_V^2} \frac{(\gamma_V^2 - 1)\sin^2\theta'}{(1 - \frac{V^2}{c^2}\cos^2\theta')}$$

This will vanish only if

$$\gamma_V = 1 \text{ (or } \gamma_v = 1) \rightarrow \text{ incident particle not moving}$$

$$\text{OR } \sin\theta' = 0 \rightarrow \mathbf{V}_{(2)f} = 0.$$

So for relativistic collisions, the angle between the emerging particles is *not* $\frac{\pi}{2}$ if all particles involved are identical. In the nonrelativistic limit $\gamma_v \simeq 1$, however, so then the two final particles do emerge at right angles. ■

⋆⋆⋆ **Problem 2.47** *A relativistic rocket.* In Chapter 1 we derived the differential equation of motion of a nonrelativistic rocket, by conserving both momentum and total mass over a short time interval Δt. That is, the momentum of the rocket at time t was set equal to the sum of the momenta of the rocket and bit of exhaust at time $t + \Delta t$, and similarly the mass of the rocket at t was set equal to the sum of the masses of the rocket and bit of exhaust at time $t + \Delta t$. We can find the equation of motion of a relativistic rocket in a similar way, except that the total mass is not conserved in this case; it is now the total momentum and the total energy that are conserved. At time t, let the rocket have mass m and velocity v; and at time $t + \Delta t$ let the rocket have mass $m + \Delta m$ (where $\Delta m < 0$) and velocity $v + \Delta v$, and let the bit of exhaust have mass ΔM and backward velocity $\bar{u}$. Note that $\Delta M \neq -\Delta m$ in relativistic physics. (a) Show from the velocity transformation that the velocity u of the bit of exhaust in the instantaneous rest frame of the rocket is given by

$$u = \frac{\bar{u} + v}{1 + \bar{u}v/c^2}.$$

(b) By conserving momentum show that, to first order in changes in m, v, and M,

$$\frac{\Delta m v}{\sqrt{1 - v^2/c^2}} + \frac{m\Delta v}{(1 - v^2/c^2)^{3/2}} = \frac{\Delta M\bar{u}}{\sqrt{1 - \bar{u}^2/c^2}}$$

(c) Then conserve energy, again keeping no terms beyond those with first order changes. Using both the conservation of momentum and conservation of energy expressions, show that the terms with ΔM can be eliminated. Then the results of problem (a) can be used to eliminate $\bar{u}$ in favor of u, and by dividing through by Δm and taking the limit $\Delta m \to 0$ show that

$$m\frac{dv}{dm} + u(1 - (v^2/c^2)) = 0,$$

which is the relativistic rocket differential equation of motion. Show also that this reduces to the equation for a classical rocket in the limit of small velocities.

Solution

(a) Imagine two pictures in an unprimed frame: The first, at time t, shows a rocket of mass m moving to the right at speed v. The second, at time $t + \Delta t$, shows the rocket with mass $m + \Delta m$ (where $\Delta m < 0$), moving at speed $v + \Delta v$, and also a bit of exhaust moving to the left at speed $\bar{u}$, which has mass ΔM. (Note that in relativistic physics $\Delta M \neq -\Delta m$, since mass is not conserved.)

Now let the instantaneous rest-frame of the rocket be the *primed* frame. Then as in Example 2.2, the velocities of an object as measured in the two frames are related by the velocity transformation

$$v = \frac{v' + v}{1 + vv'/c^2}$$

where v is the velocity of the rocket as seen in the unprimed frame. Applying this to the bit of exhaust, we have

$$u = \frac{\bar{u} + v}{1 + \bar{u}v/c^2}$$

where u is the exhaust speed relative to the rocket and $\bar{u}$ is its speed in the unprimed frame, as in the first picture described above.

(b) Conserving momentum between times t and $t + \Delta t$,

$$\frac{mv}{\sqrt{1 - v^2/c^2}} = \frac{(m + \Delta m)(v + \Delta v)}{\sqrt{1 - (v + \Delta v)^2/c^2}} - \frac{\Delta M\bar{u}}{\sqrt{1 - \bar{u}^2/c^2}}$$

where $\bar{u}$ is positive to the left. Now the quantity

$$\left(1-(v+\Delta v)^2/c^2\right)^{-1/2} \simeq \left(1-(v^2+2v\Delta v)/c^2\right)^{-1/2} \simeq (1-v^2/c^2)^{-1/2}\left(1+\frac{v\Delta v/c^2}{1-v^2/c^2}\right)$$

using the binomial approximation $(1+x)^n \simeq 1+nx$ to keep only terms through first order in Δv.

Therefore momentum conservation between the two pictures gives, to first order in small quantities,

$$\frac{mv}{\sqrt{1-v^2/c^2}} = \frac{(m+\Delta m+m\Delta v)}{\sqrt{1-v^2/c^2}} + \frac{mv^2/c^2\,\Delta v}{(1-v^2/c^2)^{3/2}} - \frac{\Delta M\bar{u}}{\sqrt{1-\bar{u}^2/c^2}}.$$

Cancelling the *zero*th order terms and rearranging gives

$$\frac{\Delta mv}{\sqrt{1-v^2/c^2}} + \frac{m\Delta v}{(1-v^2/c^2)^{3/2}} = \frac{\Delta M\bar{u}}{\sqrt{1-\bar{u}^2/c^2}}$$

as claimed in the problem statement.

(c) Now conserving *energy* between the two pictures,

$$\frac{mc^2}{\sqrt{1-v^2/c^2}} = \frac{(m+\Delta m)c^2}{\sqrt{1-(v+\Delta v)^2/c^2}} + \frac{\Delta Mc^2}{\sqrt{1-\bar{u}^2/c^2}}.$$

Using the binomial approximation and cancelling zeroth order terms,

$$\frac{\Delta mc^2}{\sqrt{1-v^2/c^2}} + \frac{mv\Delta vc^2}{(1-v^2/c^2)^{3/2}} = \frac{-\Delta Mc^2}{\sqrt{1-\bar{u}^2/c^2}}.$$

Now multiply this last equation by $\bar{u}/c^2$ and add the result to the momentum conservation equation. This eliminates the ΔM terms. This gives

$$\Delta m(\bar{u}+v)(1-v^2/c^2) + m\Delta v(1+\bar{u}v/c^2) = 0.$$

Then using the result from part (a) we find

$$\Delta mu(1-v^2/c^2) + m\Delta v = 0.$$

Finally, divide by Δm and take the limit $\Delta m \to 0$:

$$m\frac{dv}{dm} + u(1-v^2/c^2) = 0,$$

which is the relativistic rocket equation. This reduces to $mdv/dm + u = 0$ as $v/c \to 0$, which when integrated gives the well-known nonrelativistic result

$$v = v_0 + u\ln(m_0/m).$$

■

★★ **Problem 2.48** By integrating the relativistic rocket differential equation of motion from the preceding problem, show that in terms of the ratio m/m_0, the relative rocket velocity v/c is given by

$$\frac{v}{c} = \frac{1-(m/m_0))^{2u/c}}{1+(m/m_0))^{2u/c}}$$

where m is the rocket mass at any time and m_0 is its mass at time $t = 0$ when the rocket starts from rest. We assume that the exhaust velocity u = constant.

Solution

A relativistic rocket obeys

$$m\,dv/dm + u(1 - v^2/c^2) = 0 \quad \text{so } u\frac{dm}{m} = -\frac{dv}{1-v^2/c^2}.$$

Assuming u =constant, we can integrate

$$u\ln m\Big|_{m_0}^{m} = -\frac{c}{2}\ln\left(\frac{1+(v/c)}{1-(v/c)}\right)_{v=0}^{v=v} \quad \text{so} \quad (\frac{m}{m_0})^{2u/c} = (\frac{1-(v/c)}{1+(v/c)}).$$

Finally solve for v/c:

$$v/c = \frac{1-(m/m_0)^{2u/c}}{1+(m/m_0)^{2u/c}} \quad \text{as claimed.}$$

■

★★ **Problem 2.49** Consider the special case of a relativistic "photon" rocket in which the exhaust consists of photons only. The photons might be produced by an onboard laser or from the annihilation of particles and antiparticles carried with the rocket, for example. (a) From the result given in the preceding problem, how fast would the rocket be moving by the time it burned all its fuel, which was initially 90% of the rocket's mass? (b) Prove that for any given ratio of final to initial rocket mass, photon rockets are more efficient than rockets whose exhausts consist of massive particles, in that the final rocket velocity is greatest for photon rockets.

Solution

(a) If $u = c$ we have

$$v/c = \frac{1-(m/m_0)^2}{1+(m/m_0)^2}$$

Here we have $m = 0.1m_0$, in which case

$$v/c = \frac{1-0.01}{1+0.01} = 0.9802\cdots \quad \text{(moving about 98\% the speed of light)}$$

(b) Let $R = m/m_0$ and $x = 2u/c$, so

$$v/c = \frac{1-R^x}{1+R^x}.$$

If $\frac{d(v/c)}{dx} > 0$, then as x increases we get larger velocities. Now if R is fixed,

$$\frac{d}{dx}R^x = R^x \ln R,$$

$$\text{so,} \quad \frac{d}{dx}(v/c) = \frac{-R^x \ln R}{(1+R^x)} - \frac{(1-R^x)R^x \ln R}{(1+R^x)^2}$$

$$= \frac{-R^x \ln R\,[(1+R^x)+(1-R^x)]}{(1+R^x)^2} = \frac{-2R^x \ln R}{(1+R^x)^2}$$

Note that $\ln R = \ln(\frac{m}{m_0}) < 0$ because $m/m_0 < 1$. Therefore $\frac{d(v/c)}{dx} > 0$, meaning that as $x = 2u/c$ increases, v/c increases as well. So $u = c$ gives the best velocity. ■

★★ **Problem 2.50** The Captain of an interstellar photon-rocket spaceship wishes to maintain a constant acceleration a in the instantaneous rest-frame of the ship, since that would provide a constant effective gravity for passengers. In that case, at what rate $|dm/dt|$ (as a function of time) should the ship's mass decrease with time?

Solution

From previous problems the ship velocity is, for $u = c$,

$$v/c = \frac{1-(m/m_0)^2}{1+(m/m_0)^2},$$

But from example 2.7,

$$v/c = \frac{at/c}{\sqrt{1+(at/c)^2}}, \qquad \text{where } a = \text{acceleration in the ship frame.}$$

Squaring each and setting the right-hand sides equal,

$$\frac{(at/c)^2}{1+(at/c)^2} = \left[\frac{1-(m/m_0)^2}{1+(m/m_0)^2}\right]^2$$

so

$$(at/c)^2 = \left[\frac{1-(m/m_0)^2}{1+(m/m_0)^2}\right]^2 \left[1+(at/c)^2\right] \Rightarrow (at/c)^2 \left[1 - \left[\frac{1-(m/m_0)^2}{1+(m/m_0)^2}\right]\right]$$

$$= \left[\frac{1-(m/m_0)^2}{1+(m/m_0)^2}\right]^2$$

$$\Rightarrow \left(\frac{at}{c}\right)^2 = \frac{\left[\frac{1-(m/m_0)^2}{1+(m/m_0)^2}\right]^2}{1-\left[\frac{1-(m/m_0)^2}{1+(m/m_0)^2}\right]^2} = \frac{\left[1-(m/m_0)^2\right]^2}{\left[1+(m/m_0)^2\right]^2 - \left[1-(m/m_0)^2\right]^2}$$

$$= \frac{\left[1-(m/m_0)^2\right]^2}{1+2(m/m_0)^2+(m/m_0)^4-[1-2(m/m_0)^2+(m/m_0)^4]}$$

$$= \frac{\left[1-(m/m_0)^2\right]^2}{4(m/m_0)^2}$$

Therefore

$$\frac{at}{c} = \frac{1-(m/m_0)^2}{2(m/m_0)} \quad \text{so} \quad 1-(m/m_0)^2 = 2m/m_0(\frac{at}{c})$$

$$1 - \frac{2at}{c}(m/m_0) - (m/m_0)^2 = 0$$

Check at $m = m_0$, $t = 0$. As $m/m_0 \to 0$, $t \to \infty$,

$$\text{rewrite} \quad (m/m_0)^2 + (\frac{2at}{c})(m/m_0) - 1 = 0$$

$$m/m_0 = \frac{-2at/c \pm \sqrt{(2at/c)^2+4}}{2} = \frac{\sqrt{(at/c)^2+1} - at/c}{2}$$

$$m = m_0\left[\sqrt{(at/c)^2+1} - at/c\right]$$

$$dm/dt = m_0\left[\frac{1}{2}\left[\left(\frac{at}{c}\right)^2+1\right]^{-1/2}\frac{a^2}{c^2}2t - (a/c)\right] = \frac{m_0 a}{c}\left[\frac{(at/c)}{\sqrt{(at/c)^2+1-1}}\right] \quad < 0$$

which is correct. Finally, the rate of fuel burning is

$$|\frac{dm}{dt}| = \frac{m_0 a}{c}\left[1 - \frac{at/c}{\sqrt{(at/c)^2+1}}\right]$$

■

⋆ **Problem 2.51** *The transverse Doppler effect.* In Example 2.8 we derived the relativistic Doppler formulas for light sources that move either directly toward or away from the observer. Another possibility is that the source moves in a perpendicular direction, transverse to the observer's line of sight. In nonrelativistic physics there is no Doppler effect in this case. Show that if a light source is at rest at the origin of the primed frame, while moving at speed V in the x direction as seen by an unprimed observer, then if the momentum of the photons is purely in the y direction according to the observer, it follows that $\nu = \nu'\sqrt{1 - V^2/c^2}$. There is therefore a relativistic red-shift in the case of transverse motion, related to the fact that the source's time is dilated in the observer's frame.

Solution

The photon's 4-momentum is given by

$$\mathrm{p}^\mu = (\frac{E}{c}, 0, \frac{E}{c}, 0)$$

Then

$$\mathrm{p}^{t'} = \gamma_v(\mathrm{p}^t - \frac{V}{c}\mathrm{p}^x)$$

$$\mathrm{p}^{x'} = \gamma_v(\mathrm{p}^x - \frac{V}{c}\mathrm{p}^t)$$

$$\mathrm{p}^{y'} = \mathrm{p}^{y} \quad \mathrm{p}^{z'} = \mathrm{p}^{z} \Rightarrow \mathrm{p}^{t'} = \frac{E'}{c} = \gamma_V \frac{E}{c}$$

$$\Rightarrow E' = h\nu' = \gamma_V E = \gamma_V h\nu \Rightarrow \nu = \sqrt{1 - V^2/c^2}\nu'$$

One way to understand this result is to note that although there is no traditional Doppler effect for purely transverse motion, for relativistic motion the source-clock runs slow from the observer's point of view, so that the source frequency will be reddened from the stationary observer's viewpoint. ■

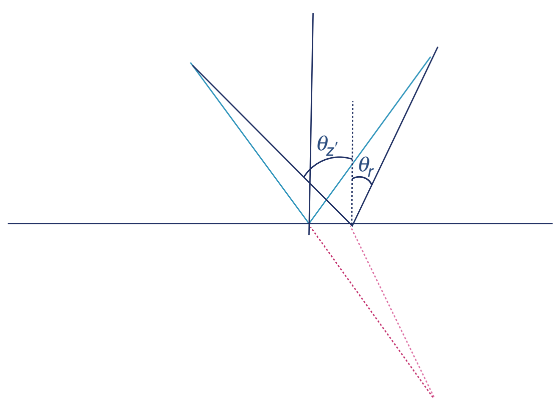
3 The Variational Principle

3.1 Problems and Solutions

⋆ **Problem 3.1** Prove from Fermat's Principle that the angles of incidence and reflection are equal for light bouncing off a mirror. Use neither algebra nor calculus in your proof! (*Hint*: The result was proven by Hero of Alexandria 2000 years ago.)

Solution

Consider a path in which the incident angle θ_i of the light ray is not necessarily the same as its angle of reflection θ_r, as shown at the left below.



Now draw also the "reflection" of the bouncing ray, as shown above. It strikes a fictitious point below the mirror. Note that the distance between the source at the left and the fictitious point is minimized if the path from the source to the fictitious point is a straight line. But for a straight line the angles $\theta_r = \theta_i$. Clearly if $\theta_r = \theta_i$ is the minimum distance, it is also the shortest-time path. This proof uses geometry only, which of course was known by the ancient Greeks. ■

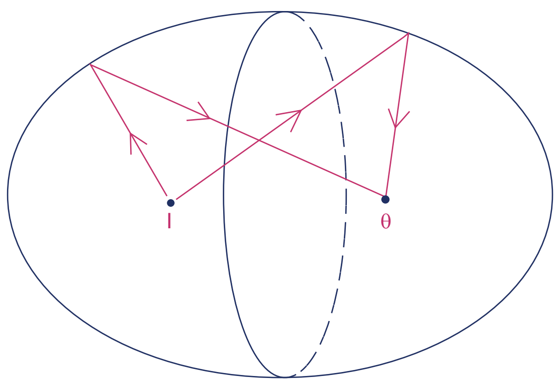
⋆ **Problem 3.2** An ideal converging lens focusses light from a point object onto a point image. Consider only rays that are straight lines except when crossing an air-glass boundary, such as those shown in the figure. Relative to the ray that passes straight through the center of the lens, do the other rays require more time, less time, or the same time to go from O to I? That is, in terms of Fermat's Principle, is the central path a local minimum, maximum, or a stationary path that is neither a minimum nor a maximum?

Solution

They all require the same time. Note that the longer paths, toward the edges of the lens, where the lens is thinner, spend less time in the lens than the shorter paths through the center of the lens. The central point is a stationary path which is neither a minimum nor a maximum. ■

★ **Problem 3.3** Light focusses onto a point I from a point O after reflecting off a surface that completely surrounds the two points, as shown in cross section below. The shape of the surface is such that all rays leaving O (excepting the single ray which returns to O) reflect to I. (a) What is the shape of the surface? (b) Pick any one of the paths. Is it a path of minimum time, maximum time, or is it stationary but of neither minimum nor maximum time for all nearby paths?

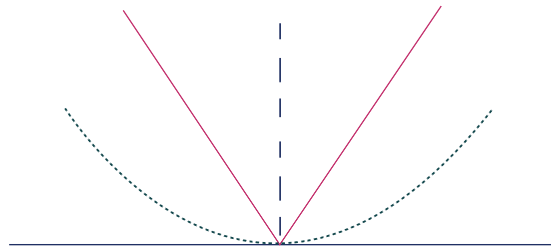
Solution



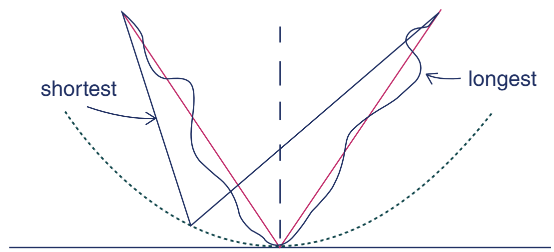
(a) The drawing shows an ellipse, which is a slice through an ellipsoid of revolution. The foci of the ellipse are at O and I. Light travels from O to I by bouncing off the ellipsoid. The travel-time for light rays is constant for an ellipse. (b) The straight-line bouncing paths in this plane all have the same length, so they form a set of stationary paths which have neither a minimum nor maximum length. ■

★ **Problem 3.4** Consider the ray shown bouncing off the bottom of the surface in the preceding problem. Replace the surface at this point by the more highly-curved surface shown in dotted lines. The ray still bounces from O to I. Is the ray now a path of minimum time, maximum time, or is it stationary but of neither minimum nor maximum time? Compare with nearby paths that bounce once but are otherwise straight. Suppose the paths must bounce once but need not be segments of straight lines. What then?

Solution



Nearby paths that still consist of two straight line segments would be slightly *shorter* than the line path shown. So the ray shown is a *maximum* time path among such straight line segments. If the paths are no longer straight-line segments, such as the wavy curves shown below, these other paths are mostly longer. So for such paths the straight path may actually be the shortest.

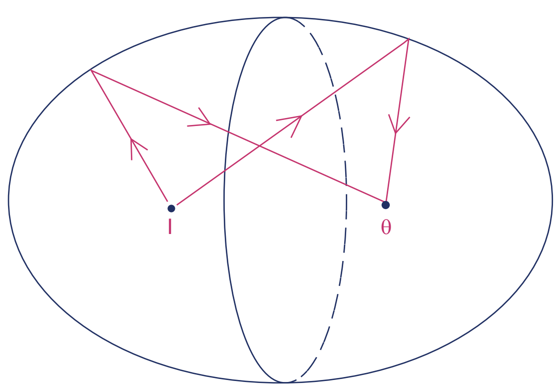


■

★★ **Problem 3.5** When bouncing off a flat mirror, a light ray travels by a minimum time path. (a) For what shape mirror would the paths of all bouncing light-rays take equal times? (b) Is there a shape for which a bouncing ray would take a path of greatest time, relative to nearby paths?

Solution

(a) An *ellipsoidal* mirror. The ellipsoid has foci at a and b, the initial and final points of the rays. The points on an ellipsoid (with ellipses as cross sections) are all such that the sum of the distances from a to the surface, and then bouncing to b, is the same wherever the bounce point is located, as shown at the left below;

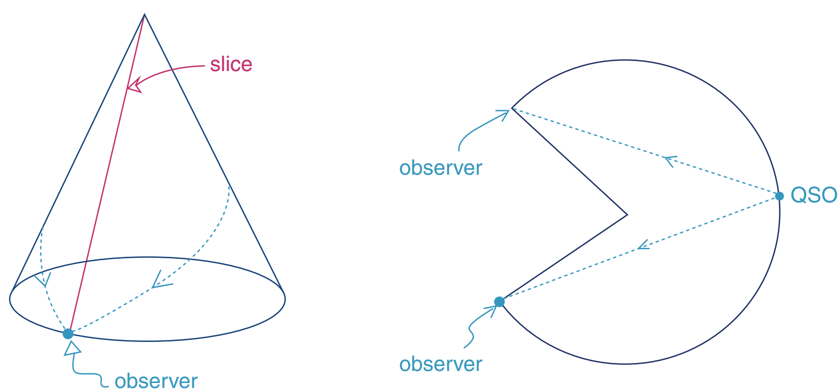


(b) Consider a path, originating at point O, which strikes the bottom of a paraboloid rather than an ellipsoid, and then bounces back to point I. The path consists of two straight line segments. This path takes a *greater* time than similar paths that strike the paraboloid on either side of the bounce. More generally, the bottom of the paraboloid corresponds to a *stationary*-path location. Considering the set of all possible paths that bounce off this point, including curvy paths, it is neither an absolute minimum nor an absolute maximum time path. ■

⋆⋆ **Problem 3.6** A hypothetical object called a straight **cosmic string** (which may have been formed in the early universe and may persist today) makes the r, θ space around it conical. That is, set an infinite straight cosmic string along the z axis; the two-dimensional space perpendicular to this, measured by the polar coordinates r and θ, then has the geometry of a cone rather than a plane. Suppose there is a cosmic string between Earth and a distant quasi-stellar object (QSO). What might we see when we look at this QSO? [Assume light travels in least-time paths (here also least-distance paths) relative to nearby paths.]

Solution

To visualize the light paths, one can "develop" the cone. That is, slice it along some straight line originating at the vertex and ending at the observer, and then lay it flat, as shown in the figure.

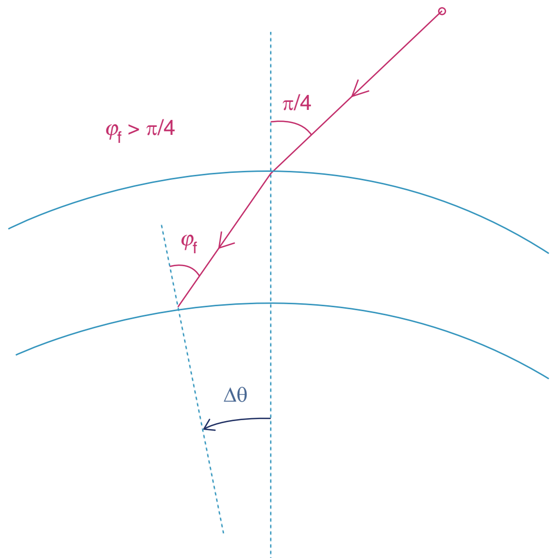


Note that there now appears to be "two" positions for the observer, which are in reality the same point when the slice is glued back together. Light travels in straight lines on the developed cone; shown are two different light rays (the dashed lines) originating at the QSO and traveling to the observer. That is, the observer will see two images of the QSO, one to the left and one to the right. (Astronomers have looked for such double images. Although they do sometimes see multiple images of a QSO or other object, those seen have all been caused by the gravitational bending of light past galaxies between us and the source. None so far have the particular signature of passing by a cosmic string.) ■

⋆⋆⋆ **Problem 3.7** Model earth's atmosphere as a spherical shell 100 km thick, with index of refraction $n_t = 1.00000$ at the top and $n_b = 1.00027$ at the bottom. Is a light ray's final angle $\varphi_{\rm f}$ relative to the normal at the ground greater or less than its initial angle $\varphi_{\rm i}$ relative to the normal at the top of the atmosphere? (Earth's radius is R = 6400 km.) Assume the ray strikes the upper atmosphere at a 45° angle.

Solution

A sketch of the ray's trajectory is shown below.



For a *flat* atmosphere, $n \sin\varphi$ = con; that is, $n_t \sin\varphi_i = n_b \sin\varphi_f$. Therefore if $\varphi_b = \varphi_t - \Delta\varphi$, we have

$$\sin\varphi_t = n_b \sin(\varphi_t - \Delta\varphi) = n_b(\sin\varphi_t \cos\Delta\varphi - \cos\varphi_t \sin\Delta\varphi).$$

Now $\Delta\varphi$ is very small, so $\sin\Delta\varphi << \cos\Delta\varphi$. Therefore for a ray near 45° we can neglect the second term. Also for small $\Delta\varphi$, we can expand $\cos\Delta\varphi$, so

$$1 \simeq n_b \cos\Delta\varphi \simeq n_b(1 - (\Delta\varphi)^2/2!\,).$$

Solving for $\Delta\varphi$, we have $\Delta\varphi \simeq \sqrt{2(0.00027)} \simeq 0.023$ radians. That is, for a flat atmosphere, the ray would strike the ground at a steeper angle than it entered the upper atmosphere, by an amount 0.023 radians. However, the ground curves around, as shown in the diagram, making the ray *less* steep relative to the local normal than for a flat atmosphere. Let us estimate how large this effect is. In the sketch above, the ray enters the upper atmosphere at 45°, and strikes the ground after it has traveled through an angle $\Delta\theta$ as measured from the center of the earth.

To a pretty good approximation this angle is $\Delta\theta \simeq$ 100 km / 6400 km $\simeq$ 0.016. Now 0.016 is less than 0.023, so refraction of the ray in earth's atmosphere is somewhat more important than earth's curvature in this case. So even though the earth's surface curves around, the ray still strikes the ground at a somewhat steeper angle than when it entered the upper atmosphere. ■

★ **Problem 3.8** We seek to find the path $y(x)$ that minimizes the integral $I = \int f(x,y,y')dx$. Find Euler's equation for $y(x)$ for each of the following integrands f, and then find the solutions $y(x)$ of each of the resulting differential equations if the two endpoints are (x,y) = (0, 1) and (1, 3) in each case. (a) $f = ax + by + cy'^2$ (b) $f = ax^2 + by^2 + cy'^2$.

Solution

(a) The first derivatives are

$$\frac{\partial f}{\partial x} = a \qquad \frac{\partial f}{\partial y} = b \qquad \frac{\partial f}{\partial y'} = 2cy', \quad \text{so Euler's equation}$$

$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0 \text{ gives} \qquad 2cy'' - b = 0 \qquad y'' = \frac{b}{2c},$$

$$\text{Integrating,} \quad y' = \frac{b}{2c}x + \alpha, \qquad y(x) = \frac{b}{4c}x^2 + \alpha x + \beta.$$

The endpoints are $(x,y) = 0,1$ and $(1,3)$, so from the 1st endpoint we find $1 = \beta$, and from the 2nd endpoint

$$3 = \frac{b}{4c} + \alpha + 1 \Rightarrow \alpha = 2 - b/4c, \qquad \text{Therefore}$$

So finally

$$y(x) = \frac{b}{4c}x^2 + (2 - \frac{b}{4c})x + 1.$$

(b) Now

$$f = ax^2 + by^2 + cy'^2 \qquad \frac{\partial f}{\partial x} = 2ax \qquad \frac{\partial f}{\partial y} = 2by \qquad \frac{\partial f}{\partial y'} = 2cy'$$

$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0 \Rightarrow 2cy'' - 2by = 0 \qquad y'' - (\frac{b}{c})y = 0$$

$$\text{Solution } e^{\alpha x} \Rightarrow \alpha^2 - b/c = 0 \qquad \alpha \pm \sqrt{b/c}.$$

$$\text{so, general solution is} \qquad y(x) = Ae^{\sqrt{b/c}x} + Be^{-\sqrt{b/c}x}.$$

End point condition 1: $(x,y) = (0,1)$ leads to $1 = A + B$. End point 2:

$$3 = Ae^{\sqrt{b/c}} + Be^{-\sqrt{b/c}}$$

$$\text{combine:} \qquad 3 = Ae^{\sqrt{b/c}} + (1 - A)e^{-\sqrt{b/c}}$$

$$A(e^{\sqrt{b/c}} - e^{-\sqrt{b/c}}) = 3 - e^{-\sqrt{b/c}} \Rightarrow A = \frac{3 - e^{-\sqrt{b/c}}}{e^{\sqrt{b/c}} - e^{-\sqrt{b/c}}}$$

$$B = 1 - A = \frac{e^{\sqrt{b/c}} - e^{-\sqrt{b/c}} - (3 - e^{-\sqrt{b/c}})}{e^{\sqrt{b/c}} - e^{-\sqrt{b/c}}} = \frac{e^{\sqrt{b/c}} - 3}{e^{\sqrt{b/c}} - e^{-\sqrt{b/c}}}$$

So the final solution matching the end points can be written

$$y(x) = \left(\frac{3 - e^{-\sqrt{b/c}}}{e^{\sqrt{b/c}} - e^{-\sqrt{b/c}}} \right) e^{\sqrt{b/c}x} - \left(\frac{3 - e^{\sqrt{b/c}}}{e^{\sqrt{b/c}} - e^{-\sqrt{b/c}}} \right) e^{-\sqrt{b/c}x}$$

■

★★ **Problem 3.9** Find a differential equation obeyed by geodesics in a plane using polar coordinates r, θ. Integrate the equation and show that the solutions are straight lines.

Solution

The distance ds between two points separated infinitesimally obeys $ds^2 = dr^2 + r^2 d\theta^2$ in polar coordinates. Therefore

$$s = \int \sqrt{1 + r^2\theta'^2}\, dr = \int \sqrt{r'^2 + r^2}\, d\theta.$$

depending upon whether we use r or θ as the independent variable. Here $\theta' \equiv d\theta/dr$ and $r' \equiv dr/d\theta$. It is easier to use r as the independent variable here: Then $f = \sqrt{1 + r^2\theta'^2}$ in Euler's equation. In that case $\partial f/\partial\theta = 0$, and so

$$\frac{\partial f}{\partial \theta'} = \frac{r^2\theta'}{\sqrt{1 + r^2\theta'^2}} = k,$$

a constant. Solving for θ',

$$\theta' = \frac{d\theta}{dr} = \frac{\pm k}{r\sqrt{r^2 - k^2}}, \quad \text{so} \quad \theta(r) = \pm k \int \frac{dr/r}{\sqrt{r^2 - k^2}}.$$

The integral can be solved using the substitution $r = k \sec q$ in terms of a new variable q. We finally get

$$r\cos(\theta - C) = x\cos C + y\sin C = \pm k.$$

where C is a constant of integration. This is the equation of a straight line. (We have used the identity $\cos(\theta - C) = \cos\theta\cos C + \sin\theta\sin C$ and the relationships $x = r\cos\theta$, $y = r\sin\theta$ between Cartesian and polar coordinates.) ■

★ **Problem 3.10** Find two first-order differential equations obeyed by geodesics in three-dimensional Euclidean space, using spherical coordinates r, θ, φ.

Solution

Beginning with the metric on a sphere

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2,$$

we can write the distance between any two points as

$$s = \int \sqrt{\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2 dt}.$$

or alternatively, and more simply, we can leave off the square root, so

$$s = \int I dt \ \text{ where } \ I = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2.$$

We can then take all the partial derivatives, $\partial I/\partial r$, $\partial I/\partial \dot{r}, \partial I \partial \theta$, etc., and assemble them into three coupled Euler equations involving the three coordinates r, θ, ϕ. This results in the three second-order ordinary differential equations

$$\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\theta}^2 = 0,$$

$$\frac{d}{dt}(r^2\dot{\theta}) - r^2\sin\theta\cos\theta\dot{\phi}^2 = 0, \quad \text{and}$$

$$\frac{d}{dt}(r^2\sin^2\theta\dot{\phi}^2) = 0.$$

Two first integrals are

$$r^2\sin^2\theta\dot{\phi} = \ell, \ \text{ a constant,} \quad \text{and}$$

$$\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2 = \epsilon, \quad \text{another constant.}$$

These two first integrals are all we need if we orient the spherical coordinate system so that both endpoints of the geodesics are in the equatorial plane $\theta = \pi/2$. This we can always do without loss of generality. Then the two equations reduce to $r^2\dot{\phi} = \ell$ and $\dot{r}^2 + r^2\dot{\phi}^2 = \epsilon$, and the entire geodesic lies in the equatorial plane. ■

⋆⋆⋆ **Problem 3.11** Two-dimensional surfaces that can be made by rolling up a sheet of paper are called *developable* surfaces. Find the geodesic equations on the following developable surfaces and solve the equations. (a) A circular cylinder of radius R, using coordinates θ and z. (b) A circular cone of half-angle α (which is the angle between the cone and the axis of symmetry) using coordinates θ and ℓ, where ℓ is the distance of a point on the cone from the apex. *Hint*: Find the distance ds between nearby points on the surface in terms of $\ell, \alpha, d\theta$, and $d\ell$.

Solution

(a) A tiny triangle on the cylinder has hypothenuse $ds = \sqrt{dz^2 + R^2 d\theta^2}$, where z is measured along the length of the cylinder and R is its radius. So an arbitrary path between points a and b has length $s = \int_a^b ds = \int_a^b \sqrt{(dz/d\theta)^2 + R^2}d\theta$. Geodesics can then be found from Euler's equation

$$\frac{\partial f}{\partial z} - \frac{d}{d\theta}\frac{\partial f}{\partial z'}.$$

with $f = \sqrt{z'^2 + R^2}$ and where $z' \equiv dz/d\theta$. The derivative $\partial f/\partial z = 0$, so

$$\frac{\partial f}{\partial z'} = \frac{z'}{\sqrt{z'^2 + R^2}} = k.$$

where k is a constant. This gives

$$z' = \frac{kR}{\sqrt{1 - k^2}}.$$

which is clearly a constant, which we will name "C", The solution is $z(\theta) = C\theta + C'$, where C' is a constant of integration. This is the equation of a helix, which makes sense, because a straight line drawn on the unrolled sheet of paper becomes a helix when the sheet is rolled up to form a circular cylinder.

(b) A tiny triangle on the cone has hypothenuse $ds = \sqrt{d\ell^2 + \ell^2 \sin^2\alpha d\theta^2}$. (Note that a line perpendicular to the symmetry axis of the cone has length $\ell \sin\theta$.) The distance between two points on the cone is therefore

$$s = \int \sqrt{1 + \ell^2 \sin^2\alpha (d\theta/d\ell^2)} d\ell,$$

so from Euler's equation we find that (since $\partial f/\partial\theta = 0$)

$$\frac{\partial f}{\partial \theta'} = \frac{\ell^2 \sin^2 \alpha \theta'}{\sqrt{1 + \ell^2 \sin^2 \alpha \theta'^2}} = k,$$

a constant. Solving for θ',

$$\theta' = \frac{\pm k}{\ell \sin\alpha \sqrt{\ell^2 \sin^2\alpha - k^2}},$$

so

$$\theta = \frac{\pm k}{\sin\alpha} \int \frac{dy}{y\sqrt{y^2 - k^2}} + \theta_0 = \pm\frac{1}{\sin\alpha} \cos^{-1}\left(\frac{k}{\ell \sin\alpha}\right) + \theta_0.$$

where $y \equiv \ell \sin\alpha$ and θ_0 is a constant of integration. Solving for ℓ, we find

$$\ell = \frac{k}{\sin\alpha} \sec(\sin\alpha(\theta - \theta_0)$$

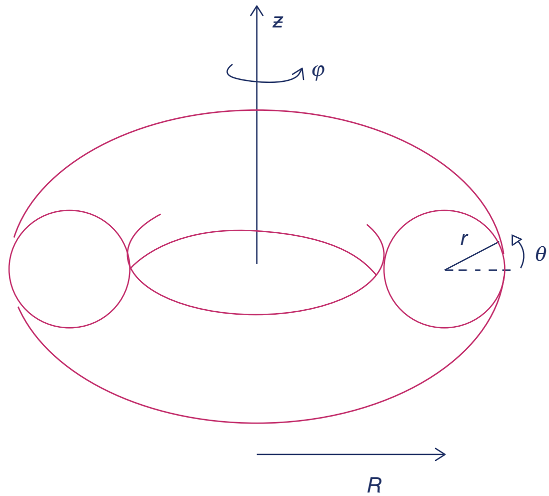
where the constants k and θ_0 can be found from the locations of a and b. ■

⋆⋆ **Problem 3.12** A torus can be defined by two radii: A large radius R running around the center of the torus, and a small radius r corresponding to a cross-sectional slice. Let R live in the x, y plane. Then if φ is an angle relative to the x axis and lying in the x, y plane, and θ is an angle within a cross-sectional slice, with $\theta = 0$ corresponding to the outermost radius of the torus $R + r$, then the Cartesian coordinates of points on the torus are

$$x = (R + r\cos\theta)\sin\phi \quad y = (R + r\cos\theta)\cos\phi \quad z = r\sin\theta.$$

(a) Find an expression for the distance ds between nearby points on the torus, using the angles φ and θ as coordinates. (b) Find a second-order differential equation for geodesics on the torus in terms of θ, θ' and θ'', where $\theta' = d\theta/d\varphi$, etc. (c) Show that paths with constant $\theta = 0$ or with constant $\theta = \pi$ are geodesics, but that a path with constant $\theta = \pi/2$ is *not* a geodesic.

Solution



(a) A tiny distance along the surface is ds, where

$$ds^2 = dx^2 + dy^2 + dz^2 = (R + r\cos\theta)^2 d\phi^2 + r^2 d\theta^2,$$

using the formulas for x, y, z given in the problem statement. Geodesics satisfy $\delta \int ds = \delta \int f(\theta, \theta')d\phi$, where $f = [(R + r\cos\theta)^2 + r^2\theta'^2]^{1/2}$. Euler's equation then becomes

$$\frac{\partial f}{\partial \theta} - \frac{d}{d\phi}\frac{\partial f}{\partial \theta'} = -\frac{(R + r\cos\theta)r\sin\theta}{f} - \frac{d}{d\phi}\frac{r^2\theta'}{f} = 0.$$

This gives the second-order differential equation

$$r\theta'' + \frac{2r^2\sin\theta\theta'^2}{(R + r\cos\theta} + (R + r\cos\theta)\sin\theta = 0$$

where $\theta' = d\theta/d\phi$, $\theta'' = d^2\theta/d\phi^2$. We can guess some solutions:

Running around the entire torus at maximum radius $R + r$, corresponding to $\theta = 0$. With initial conditions $\theta = 0$ and $\theta' = 0$, note from the equation that also $\theta'' = 0$. Therefore the path remains at $\theta = 0$. This path is a geodesic.

Running around the torus at $\theta = \pi$, the reasoning is the same. The geodesic remains at radius $R - r$. This path is a geodesic.

Starting at $\theta = \pi/2$ and $\theta' = 0$, note that $\theta'' \neq 0$, so there is no geodesic that remains at $\theta = \pi/2$. ∎

⋆⋆ **Problem 3.13** Using Euler's equation for $y(x)$, prove that

$$\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0.$$

This equation provides an alternative method for solving problems in which the integrand f is not an explicit function of x, because in that case the quantity $f - y'\partial f/\partial y'$ is constant, which is only a first-order differential equation.

Solution

Euler's equation is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0.$$

Also from multivariable calculus we know that

$$\frac{df(x,y,y')}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx}.$$

However,

$$\frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

from Euler's equation. Therefore

$$\frac{df(x,y,y')}{dx} = \frac{\partial f}{\partial x} + \frac{d}{dx}\left[\frac{\partial f}{\partial y'}y'\right].$$

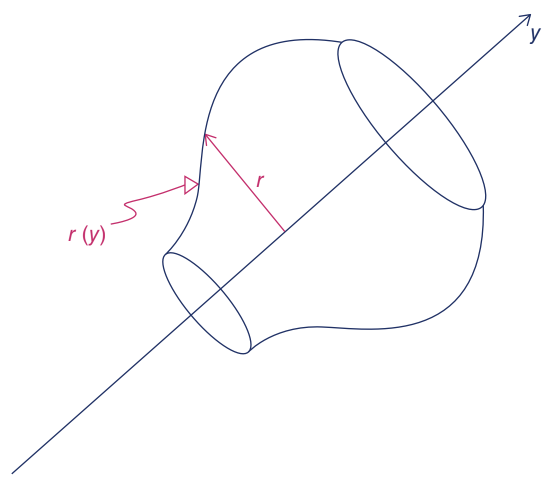
Rearranging,

$$\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0.$$

as required. ■

★★ **Problem 3.14** A line and two points not on the line are drawn in a plane. A smooth curve is drawn between the two points and then rotated about the given line. Find the shape of the curve that minimizes the area generated by the rotated curve. A lampshade manufacturer might use this result to minimize the material used to produce a lampshade of given upper and lower radii.

Solution



We want to find the shape $r(y)$ for the curve between a and b. Here y is the vertical distance along the line, and r is perpendicular to the line. The area generated by the rotated curve is summed over the small areas $\Delta A = 2\pi r \Delta s$, where Δs is a small length along the curve. That is, the total area is

$$A = \int dA = 2\pi \int_{r_1}^{r_2} r\sqrt{dy^2 + dr^2} = 2\pi \int_{r_1}^{r_2} r\sqrt{1+y'^2}dr = 2\pi \int_{y_1}^{y_2} r\sqrt{1+r'^2}dy$$

where $r' \equiv dr/dy$ and $y' \equiv dy/dr$. Therefore we can use Euler's equation in either of two ways:

$$\frac{d}{dr}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0 \quad \text{where} \quad f = r\sqrt{1+y'^2},$$

or

$$\frac{d}{dy}\frac{\partial f}{\partial r'} - \frac{\partial f}{\partial r} = 0 \quad \text{where} \quad f = r\sqrt{1+r'^2}.$$

The first form is easier to use because $\partial f/\partial y = 0$. Therefore we find that

$$\frac{\partial f}{\partial y'} = r\frac{y'}{\sqrt{1+y'^2}} = k,$$

where k is a constant. This gives $y' = \pm k/\sqrt{r^2 - k^2}$. We can perform the integration by substituting $r = k\cosh x$, so that finally

$$r = k\cosh\left(\frac{y - y_0}{k}\right).$$

where y_0 is a constant of integration. The constants k and y_0 can be found from the locations of a and b. (Note that the hyperbolic cosine function is sometimes called a *catenary*, which is the shape of a hanging chain in uniform gravity.) ■

⋆⋆⋆ **Problem 3.15** The time required for a particle to slide from the cusp of a cycloid to the bottom was shown in Section 3.4 to be $t = \pi\sqrt{a/2g}$. Show that if the particle starts from rest at any point *other* than the cusp, it will take this same length of time to reach the bottom. The cycloid is therefore also the solution of the *tautochrone*, or "equal-time" problem. *Hint*: The energy equation for the particle speed in terms of y written in Section 3.4 must be modified to take into account the new starting condition. [The tautochrone result was known to the author Herman Melville. In the chapter called "The Try-Works" in *Moby-Dick*, the narrator Ishmael, on board the whaling ship Pequod, describes the great try-pots used for boiling whale blubber: "Sometimes they are polished with soapstone and sand, till they shine within like silver punchbowls. ... It was in the lefthand try-pot of the Pequod, with the soapstone diligently circling around me, that I was first indirectly struck by the remarkable fact, that in geometry all bodies gliding along the cycloid, my soapstone for example, will descend from any point in precisely the same time."]

Solution

A cycloid is defined by the parametric equations $x = (a/2)(\theta - \sin\theta), y = (a/2)(1-\cos\theta)$. In this problem x is positive to the right, and y is positive downwards. If the particle starts from an arbitrary θ_0, and slides to the bottom at $\theta = \pi$, which is located at $x = a\pi/2$, $y = a$, then the particle's energy is $E = (1/2)mv^2 + mg(-y) = -mgy_0$, the minus signs needed because y is positive downward. So the speed of the particle at any point is $v = \sqrt{2g(y - y_0)}$ the time to reach the bottom is

$$t = \int \frac{ds}{v} = \int_{\theta=\theta_0}^{\pi} \frac{\sqrt{dx^2 + dy^2}}{2g(y - y_0)}.$$

In terms of θ, after taking differentials and a bit of algebra,

$$dx^2 + dy^2 = \frac{a^2}{2}(1 - \cos\theta)d\theta^2.$$

Therefore

$$t = \sqrt{\frac{a}{2g}} \int_{\theta_0}^{\pi} \frac{\sqrt{1 - \cos\theta}}{\cos\theta_0 - \cos\theta}.$$

Now use the trig identity $1 - \cos\theta = 2\sin^2\theta/2$, and also let $\varphi = \theta/2$ and use $\sin^2\varphi = 1 - \cos^2\varphi$ to give

$$t = \sqrt{\frac{a}{2g}} \int_{\varphi_0}^{\pi/2} \frac{2\sin\varphi\, d\varphi}{\sqrt{\cos^2\varphi_0 - \cos^2\varphi}}.$$

Now let $\cos\varphi = z$, so that

$$t = 2\sqrt{\frac{a}{2g}} \int_0^{z_0} \frac{dz}{\sqrt{z_0^2 - z^2}} = 2\sqrt{\frac{a}{2g}} \int_0^{\pi/2} \frac{\cos\alpha d\alpha}{\sqrt{1 - \sin^2\alpha}} = \pi\sqrt{\frac{a}{2g}}.$$

where we have set $z = z_0 \sin\alpha$. Note that the final result is independent of z_0. That is, the time to slide to the bottom of the cycloid is independent of θ_0, or y_0, the starting position.

■

★★ **Problem 3.16** Derive Snell's law from Fermat's Principle.

Solution

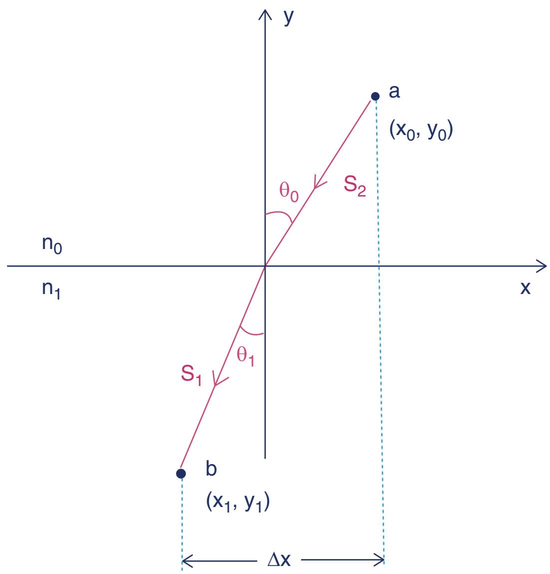
A light ray starts at point a in a medium with index of refraction n_0 and ends at point b in a medium with index of refraction n_1, with a flat interface in between, as shown in the diagram below. Point a is a vertical distance y_0 above the interface, and a horizontal distance x_0 to the right of the point O where the ray strikes the interface, which we take to be the origin of coordinates. Point b is a vertical distance y_1 below the interface and a horizontal distance x_1 to the left of point O. Thus the ray moves a distance $s_0 = \sqrt{x_0^2 + y_0^2}$ in the upper medium and a distance $s_1 = \sqrt{x_1^2 + y_1^2}$ in the lower medium. The light-ray paths are at angles θ_0 and θ_1 to the interface normals, as shown in the diagram.The total time for light to travel from a to b is $(s_0/v_0 + s_1/v_1)$, where $v_0 = c/n_0$ and $v_1 = c/n_1$. That is, the total time is

$$t = \frac{1}{c}\left[n_0\sqrt{x_0^2 + y_0^2} + n_2\sqrt{(\Delta x - x_0)^2 + y_1^2)}\right]$$

where $\Delta x = x_0 + x_1$ is the total horizontal distance between A and B, which is a fixed distance, as are y_0 and y_1. The only variable then is x_0. So we vary x_0 so as to minimize the total travel time of the ray. That is,

$$\frac{dt}{dx_0} = \frac{1}{c}\left[n_0\frac{x_0}{\sqrt{x_0^2+y_0^2}} - \frac{n_1(\Delta x - x_0)}{\sqrt{(\Delta x - x_0)^2 + y_1^2}}\right] = \frac{1}{c}\left[n_0\frac{x_0}{\sqrt{x_0^2+y_0^2}} - n_1\frac{x_1}{\sqrt{x_1^2+y_1^2}}\right] = 0.$$

for a minimum time. That is, $n_0 \sin\theta_0 = n_1 \sin\theta_1$, which is Snell's law.



■

⋆ **Problem 3.17** A lifeguard is standing on the beach some distance from the shoreline, when he hears a swimmer calling for help. The swimmer is some distance offshore and also some lateral distance from the lifeguard. The lifeguard knows he can run twice as fast as he can swim. To minimize the time it takes to reach the swimmer, show that his path should consist of two line segments: relative to the shoreline, his running path should be at angle θ_1 and his swimming path should be at angle θ_2, where $\cos\theta_1 = 2\cos\theta_2$.

Solution

The problem is an exact analog to the refraction of light at the boundary between two sheets of glass whose indicies of refraction n_1, n_2 differ by a factor of two. By Fermat's principle of least time, we can derive Snell's law $n_1 \sin\varphi_1 = n_2 \sin\varphi_2$, where $\varphi = \pi/2 - \theta$ in each case, since φ is the angle relative to the interface normal. The claimed result follows. ■

⋆ **Problem 3.18** Describe the geodesics on a right circular cylinder. That is, given two arbitrary points on the surface of a cylinder, what is the shape of the path of minimum length between them, where the path is confined to the surface? *Hint*: A cylinder can be made by rolling up a sheet of paper.

Solution

If we slit the cylinder lengthwise, and then lay it flat (that is, "develop" the cylinder) we can draw a straight line between the two arbitrary points using a ruler. If the cylinder is then rolled back up, it is seen that the straight line becomes a *helix*, which is then the shape of minimum length. ■

★★ **Problem 3.19** A particle falls along a cycloidal path from the origin to the final point $(x, y) = (\pi a/2, a)$; the time required is $\pi\sqrt{a/2g}$, as shown in Section 3.4. How long would it take the particle to slide along a straight-line path between the same points? Express the time for the straight-line path in the form $t_{\text{straight}} = kt_{\text{cycloid}}$, and find the numerical factor k.

Solution

The straight line between the given endpoints is $y = (2/\pi)x$, where x is positive to the right and y is positive downwards. Sliding down the straight-line slope, the particle obeys $F = ma$, or $mg\sin\theta = m\ddot{s}$, where s is the distance measured along the slope. Here θ is the angle of the slope relative to the horizontal. Therefore we have

$$\ddot{s} = g\sin\theta = g\frac{a}{\sqrt{a^2 + \pi^2 a^2/4}} = \frac{g}{\sqrt{1 + \pi^2/4}}.$$

This acceleration is constant, so the distance traveled is $s = (1/2)\ddot{s}t^2$. Inverting, the time to travel a distance s, starting from rest, is

$$t = \sqrt{\frac{2s}{\ddot{s}}} = \sqrt{\frac{2a}{g}}\sqrt{1 + \pi^2/4}.$$

This is larger than the time to travel by the cycloidal path (which is $t_C = (\pi/\sqrt{2})\sqrt{a/g}$) by the factor $k = \sqrt{1 + 4/\pi^2}$. ■

★ **Problem 3.20** A unique transport system is built between two stations 1 km apart on the surface of the moon. A tunnel in the shape of a full cycloid cycle is dug, and the tunnel is lined with a frictionless material. If mail is dropped into the tube at one station, how much later (in seconds) does it appear at the other station? How deep is the lowest point of the tunnel? (Gravity on the moon is about 1/6th that on earth.)

Solution

One end of the tunnel is at (x, y) = (0, 0) and the other end is at $(x, y) = (\pi a, 0)$ where the value of "a" is $a = 1\ \text{km}/\pi = 1000\ \text{m}/\pi$. Here x is the horizontal coordinate and y is the vertical coordinate, with y positive downward. The trip time is $t = 2\pi\sqrt{a/2g}$, twice the time it takes to slide to the bottom of the tunnel. So the total travel time is

$$T \simeq 2\pi\sqrt{\frac{1000\ \text{m}}{2\pi(9.8\ \text{m/s}^2)/6}} \simeq 63\ \text{s}$$

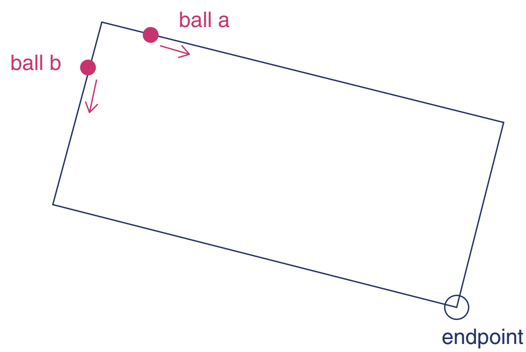
or about one minute. The tunnel depth can be found from the cycloid equations

$$x = (a/2)(\theta - \sin\theta) \quad y = (a/2)(1 - \cos\theta).$$

At (x, y) = (0, 0) we have $\theta = 0$, at the left end of the tunnel. At $\theta = \pi$, y reaches a maximum, which is the lowest point. So at $\theta = \pi$ we have $x = \pi a/2$ and $y = a$. The tunnel depth is therefore $y = a = (1000/\pi\ \text{m}) \simeq 320$ m. ■

⋆ **Problem 3.21** A hollow glass tube is bent into the form of a slightly tilted rectangle, as shown in the figure. Two small ball bearings can be introduced into the tubes at one corner; one rolls clockwise and the other counterclockwise down to the opposite corner at the bottom. The balls are started out simultaneously from rest, and note that each ball must roll the same distance to reach the destination. The question is: which ball reaches the lower corner first, or do they arrive simultaneously? Why?

Solution



The ball bearings roll the same distance, but ball "a", which goes by the upper route, gets off to a slow start, since its initial tilt angle is small; ball "b" gets off to a fast start, because it falls steeply initially. So "b" wins the race, since it retains its initial large speed throughout the trip. Note that by energy conservation both balls end up with the same speed at the bottom, but "b's" average speed is greater. ∎

⋆ **Problem 3.22** Assume earth's atmosphere is essentially flat, with index of refraction $n = 1$ at the top and $n = n(y)$ below, with y measured from the top, and the positive y direction downward. Suppose also that $n^2(y) = 1 + \alpha y$, where α is a positive constant. Find the light-ray trajectory $x(y)$ in this case.

Solution

Fermat's principle holds that the light travel time is a minimum, where here $t = \int dt = \int ds/v = (1/c)\int n(y)\sqrt{1+x'^2}dy$ where $x' = dx/dy$. Euler's equation is

$$\frac{\partial f}{\partial x} - \frac{d}{dy}\frac{\partial f}{\partial x'} = 0$$

in this case, and since $\partial f/\partial x = 0$, we find that

$$\frac{\partial f}{\partial x'} = \frac{n(y)x'}{\sqrt{1+x'^2}} = k,$$

where k is a constant. Solving for x' and using $n^2 = 1 + \alpha y$, it follows that

$$\frac{dx}{dy} = \pm\frac{k}{\sqrt{1-k^2+\alpha y}}.$$

Integrating, we find

$$x(y) = \pm\frac{2k}{\alpha}\sqrt{1-k^2+\alpha y} + C.$$

where C is a constant of integration. The two constants k and C can be determined from the initial values of x and x'. ■

★★ **Problem 3.23** Suppose that earth's atmosphere is as described in the preceding problem, except that $n^2(y) = 1 + \alpha y + \beta y^2$, where α and β are positive constants. Find the light-ray trajectory $x(y)$ in this case.

Solution

Fermat's principle holds that the light travel time is a minimum, where here $t = \int dt = \int ds/v = (1/c)\int n(y)\sqrt{1+x'^2}dy$ where $x' = dx/dy$. Euler's equation is

$$\frac{\partial f}{\partial x} - \frac{d}{dy}\frac{\partial f}{\partial x'} = 0 \quad \text{where} \quad f = n(y)\sqrt{1 = x'^2}$$

in this case, and since $\partial f/\partial x = 0$, we find that

$$\frac{\partial f}{\partial x'} = \frac{n(y)x'}{\sqrt{1+x'^2}} = k,$$

where k is a constant. Solving for x' and using $n^2 = 1 + \alpha y + \beta y^2$, it follows that

$$\frac{dx}{dy} = \pm\frac{k}{\sqrt{1-k^2+\alpha y+\beta y^2}}.$$

Integrating, we find

$$x(y) = \pm\frac{k}{\sqrt{\beta}}\ln(2\sqrt{\beta(1-k^2+\alpha y+\beta y^2)} + 2\beta y + \alpha) + C.$$

where C is a constant of integration, and where we have assumed $\beta > 0$. Using the $+$ sign, and considering the path to travel from y_0 to y,

$$x(y) = \frac{k}{\sqrt{\beta}}\ln(2\sqrt{\beta(1-k^2+\alpha y+\beta y^2)} + 2\beta y + \alpha)|_{y_0}^{y}.$$

Suppose we set $y_0 = 0$ at the top of the atmosphere. Then

$$\begin{aligned} x(y) &= \frac{k}{\sqrt{\beta}}[\ln(2\sqrt{\beta(1-k^2)+\alpha y+\beta y^2)} + 2\beta y + \alpha) - \ln(2\sqrt{\beta(1-k^2)} + \alpha) \\ &= \frac{k}{\sqrt{\beta}}\ln\left[\frac{2\sqrt{\beta(1-k^2)+\alpha y+\beta y^2)} + 2\beta y + \alpha}{2\sqrt{\beta(1-k^2)} + \alpha}\right]. \end{aligned}$$

resulting in $x = 0$ at $y = 0$. As x increases, so does y. We can choose k to fix the slope dx/dy at the top of the atmosphere. ■

★★ **Problem 3.24** Consider earth's atmosphere to be spherically symmetric above the surface, with index of refraction $n = n(r)$, where r is measured from the center of the earth. Using polar coordinates r, θ to describe the trajectory of a light ray entering the atmosphere from high altitudes, (a) find a first-order differential equation in the variables r and θ that governs the ray trajectory; (b) show that $n(r) r \sin\varphi =$ constant along the ray, where φ is the angle between the ray and a radial line extending outward from the center of the earth. This is the analog of the equation $n(y)\sin\theta =$ constant for a flat atmosphere.

Solution

Fermat's Principle states that

$$\delta \int n(r) ds = \delta \int n(r)\sqrt{dr^2 + r^2 d\theta^2} = \delta \int n(r)\sqrt{1 + r^2\theta'^2}\, dr = 0.$$

where $\theta' \equiv d\theta/dr$. Euler's equation is

$$\frac{\partial f}{\partial \theta} - \frac{d}{dr}\frac{\partial f}{\partial \theta'} = 0.$$

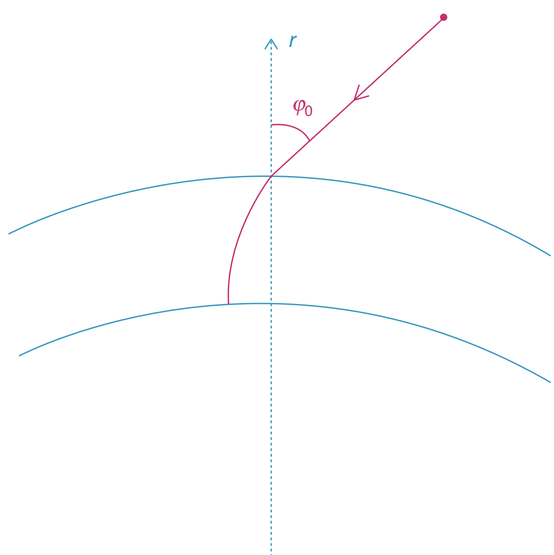
In this problem $\partial f/\partial\theta = 0$, so

(a)

$$\frac{\partial f}{\partial \theta'} = \frac{n(r) r^2 \theta'}{\sqrt{1 + r^2\theta'^2}} = k,$$

where k is a constant. This is a first-order differential equation.

(b) The picture shows a light ray entering the atmosphere at angle φ_0 to a radial line from earth's center, and descending to the ground.



Note that at all points along the ray, $\tan\varphi = r d\theta/dr = r\theta'$, from which one can find that $\sin\varphi = r\theta'/\sqrt{1 + r^2\theta'^2}$. Thus from part (a), we have $n(r) r \sin\varphi = k$, a constant. ■

⋆ **Problem 3.25** Using the result found in the preceding problem, and supposing that $n^2(r) = 1 + \alpha/r^2$ (where α is a constant), find the light-ray trajectory expressed either as $r(\theta)$ or $\theta(r)$.

Solution

From the solution to the preceding problem we have

$$\frac{\partial f}{\partial \theta'} = \frac{n(r)r^2\theta'}{\sqrt{1+r^2\theta'^2}} = k,$$

a constant. Substituting in the given expression for $n(r)$ and solving for θ' gives

$$\theta' = \frac{\pm k/r}{\sqrt{r^2+(\alpha-k^2)}}.$$

Integrating,

$$\theta = \theta_0 \pm k\int \frac{dr/r}{\sqrt{r^2+(\alpha-k^2)}}.$$

Now substitute $r = \sqrt{\alpha-k^2}\ \tan q$, so that

$$\begin{aligned}\theta &= \theta_0 \pm k\int \frac{\sec^2 q dq/\tan q}{\sqrt{\alpha-k^2}\sqrt{\tan^2 q+1}} = \theta_0 \pm \frac{k}{\sqrt{\alpha-k^2}}\int \frac{\sec q dq}{\tan q}\\ &= \theta_0 \pm \frac{k}{\sqrt{\alpha-k^2}}\int \csc q dq = \theta_0 \mp \frac{k}{\sqrt{\alpha-k^2}}\ln|\csc q+\cot q|.\end{aligned}$$

Now $\tan q = r/\sqrt{\alpha-k^2}$ so $\cot q = \sqrt{\alpha-k^2}/r$ and $\csc q = \sqrt{1+\cot^2 q}$, so we have found that

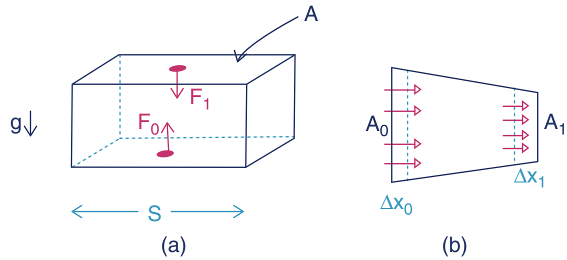
$$\theta = \theta_0 \mp \frac{k}{\sqrt{\alpha-k^2}}\ln\left|\sqrt{1+\frac{\alpha-k^2}{r^2}}+\frac{\sqrt{\alpha-k^2}}{r}\right|,$$

an expression for $\theta(r)$. It is not difficult to invert the equation to find $r(\theta)$. ■

⋆⋆ **Problem 3.26** (a) Show that the pressure difference between two points in an incompressible liquid of density ρ in static equilibrium is $\Delta P = \rho g s$, where s is the vertical separation between the two points and g is the local gravitational field. (b) The liquid is caused to flow through a horizontal pipe of varying cross-sectional area, so that its velocity depends upon position. In a particular section of pipe of length s, the pipe is narrowing, so that the fluid's acceleration has the constant value a. Find the pressure difference ΔP between one end of the section and the other, in terms of ρ and the change in the velocity squared (v^2) between the two ends of the section. Is the pressure larger or smaller at the narrower end of the section? (The result is an example of the *Bernoulli effect*).

Solution

(a) Consider the rectangular slab of fluid shown below. It has mass ρAs, where A is the area of the top or bottom faces of the slab, and s is the length of the slab. The force down on the upper face due to the surrounding fluid is $F_1 = P_1A$ and the force up on the lower face is $F_0 = P_0A$, where P_1 and P_0 are the fluid pressures on the top and bottom faces, respectively. There is also a gravitational force ρAsg downward on the slab, so if the slab is not accelerating these forces balance: That is, $P_0A = P_1A+\rho Asg$, or $\Delta P \equiv P_0-P_1 = \rho gs$.



(b) As the fluid flows through the pipe shown below, the work done at the large end, by fluid at the far left, is $F\Delta x = P_0 A_0 \Delta x_0$. The work done at the right by fluid in the pipe is $P_1 A_1 \Delta x_1$. Therefore the net work done on fluid within the pipe is $\Delta W = P_0 A_0 \Delta x_0 - P_1 A_1 \Delta x_1$ The fluid is incompressible, so the volume of fluid is unchanged as it moves to the right; that is, $A_0 \Delta x_0 = A_1 \Delta x_1$. Therefore the net work done on the fluid is $\Delta W = (P_0 - P_1)(A\Delta x)$ which (by the work-energy theorem) is the change in kinetic energy of the fluid, which is $\Delta KE = \Delta[(1/2)\rho A \Delta x v^2] = (1/2)\rho A \Delta x \Delta v^2$. Therefore $\Delta P(A\Delta x) = (1/20\rho(A\Delta x)\Delta v^2$, so finally $\Delta P = (1/20\rho \Delta v^2$. The pressure $P_0 > P_1$, so the pressure is smaller at the narrower end of the pipe. ■

⋆ **Problem 3.27** The surface of a paraboloid of revolution is defined by $z = a(x^2 + y^2)$ where a is a constant. Find the differential equation for a geodesic originating at a point $(x, y) = (x_0, 0)$ with slope $(dy/dx)_0 = 0$. Does the geodesic return to the same point?

Solution

We will switch to cylindrical coordinates (ρ, φ, z) where $x^2 + y^2 = \rho^2$. Then points on the paraboloid can be described by the two coordinates (ρ, φ) without using z, since $z = a\rho^2$ and $dz = 2a\rho d\rho$. An infinitesimal distance along the paraboloid is then ds, where

$$ds^2 = dz^2 + d\rho^2 + \rho^2 d\varphi^2 = (1 + 4a^2\rho^2)d\rho^2 + \rho^2 d\varphi^2.$$

Thus the length of a path along the paraboloid is

$$s = \int ds = \int \sqrt{(1 + 4a^2\rho^2)d\rho^2 + \rho^2 d\varphi^2}$$
$$= \int d\rho \sqrt{(1 + 4a^2\rho^2) + \rho^2 \left(\frac{d\varphi}{d\rho}\right)^2} \equiv \int d\rho I(\rho, \varphi').$$

Note that the coordinate φ is cyclic, so the Euler equation reduces simply to

$$\frac{\partial I}{\partial \varphi'} = 2\rho^2 \frac{d\varphi}{d\rho}/I = C,$$

where C is a constant. Then after a bit of algebra we can solve for $d\varphi/d\rho$, to find

$$\frac{d\varphi}{d\rho}^2 = \frac{C^2(1 + 4a^2\rho^2)}{\rho^2(\rho^2 - C^2)},$$

which is a first-order differential equation obeyed by all geodesics on the paraboloid. For the particular geodesic described in the problem statement, note that if $dy/dx = 0$ then also $d\varphi/d\rho = 0$, which is only possible if the constant $C = 0$ according to the differential equation. Therefore this describes a geodesic which runs up along the paraboloid with constant $y = 0$ and so it cannot return to its starting point. ■

⋆ **Problem 3.28** According to Einstein's general theory of relativity, light rays are deflected as they pass by a massive object like the sun. The trajectory of a ray influenced by a central, spherically symmetric object of mass M lies in a plane with coordinates r and θ (so-called *Schwarzschild coordinates*); the trajectory must be a solution of the differential equation

$$\frac{d^2u}{d\theta^2} + u = \frac{3GM}{c^2}u^2$$

where $u = 1/r$, G is Newton's gravitational constant, and c is the constant speed of light. (a) The right-hand side of this equation is ordinarily small. In fact, the ratio of the right-hand side to the second term on the left is $3GM/rc^2$. Find the numerical value of this ratio at the surface of the sun. The sun's mass is 2.0×10^{30} kg and its radius is 7×10^5 km. (b) If the right-hand side of the equation is neglected, show that the trajectory is a straight line. (c) The effects of the term on the right-hand side have been observed. It is known that light bends slightly as it passes by the Sun and that the observed deflection agrees with the value calculated from the equation. Near a black hole, which may have a mass comparable to that of the sun but a much smaller radius, the right-hand side becomes very important, and there can be large deflections. In fact, show that there is a single radius at which the trajectory of light is a circle orbiting the black hole, and find the radius r of this circle.

Solution

(a) For the sun, $M = 2 \times 10^{30}$ kg and $R = 7 \times 10^8$ m, so $3GM/Rc^2 \simeq 6 \times 10^{-6}$. (b) Neglecting the right-hand side of the equation given in the problem statement, we have

$$\frac{d^2u}{d\theta^2} + u = 0,$$

which is the simple harmonic oscillator equation, with solutions $u = A\cos\theta + B\sin\theta$. Using $u = 1/r$, we have $Ar\cos\theta + Br\sin\theta = Ax + By = 1$ using the Cartesian coordinates $(x, y) = r(\cos\theta . \sin\theta)$. This is the equation of a straight line. (c) For a circular trajectory we must have $u = 1/r =$ constant, so $d^2u/d\theta^2 = 0$. Then $u = (3GM/c^2)u^2$, so that

$$r = \frac{1}{u} = \frac{3GM}{c^2}.$$

The event horizon radius of a black hole is $2GM/c^2$, so it is possible for a light ray to orbit at the larger radius $3GM/c^2$. (One can show that this circular orbit is unstable, however, so the ray would ultimately either spiral in toward the center or spiral outward.) ■

⋆⋆ **Problem 3.29** A clock is thrown straight upward on an airless planet with uniform gravity g, and it falls back to the surface at a time t_f after it was thrown, according to clocks at rest on the ground. (a) Using the clock's motion as derived in Section 3.7, how much more time than t_f will have elapsed according to this moving clock, in terms of g, t_f, and c, the speed of light? (b) Now suppose that instead of the freely-falling motion used in part (a), the moving clock has constant speed v_0 straight up for time $t_f/2$ according to ground clocks, and then moves straight down again at the same constant speed v_0 for another time interval $t_f/2$, according to ground clocks. How much more time than t_f will have elapsed according to this moving clock, in terms of v_0, g, c, and t_f? (c) Now find the value of v_0, keeping g and t_f fixed, which maximizes the final reading of the moving clock described in part (b). Then evaluate the final reading of this moving clock in terms of g, t_f, and c, and show that it is *less than* the final reading of the freely-falling clock described in part (a). (This is a particular illustration of the fact that the path which *maximizes* the proper time is that of a *freely-falling* clock, *i.e.*, a clock that moves according to Newton's laws. The reader could choose some alternative motion for a clock, and show again that as long as it returns to the beginning point at t_f according to ground clocks, its time will be less than that of the freely-falling clock of part (a).)

Solution

(a) Let dt be a small time interval on the stationary clock, and $d\tau$ be a small interval on the moving clock. Then

$$d\tau = dt\left(1 + \frac{gy}{c^2} - \frac{v^2}{2c^2}\right).$$

Now for freely-falling motion $v = v_0 - gt$ and $y = v_0 g - (1/2)gt^2$. Substituting these into the equation above and using $v_0 = gt_f/2$, we have

$$d\tau = dt\left(1 - \frac{g^2 t_f^2}{8c^2} + \frac{g^2 t_f}{c^2}t - \frac{g^2 t^2}{c^2}\right).$$

Integrating over both sides, $d\tau$ from 0 to τ and dt from 0 to t_f, we find that

$$\tau = t_f + \frac{g^2 t_f^3}{24c^2}.$$

So the clock thrown upward will read a later time than the clock that stayed at rest on the ground.

(b) For a clock going up at constant speed v_0 for time $t_f/2$, and then returning to the ground at the same speed for an additional time $t_f/2$, we can write

$$\tau = 2\int_0^{t_f/2} dt\left(1 + \frac{gv_0 t}{c^2} - \frac{v_0^2}{2c^2}\right) = t_f + \Delta\tau = t_f + \frac{gv_0 t_f^2}{4c^2} - \frac{v_0^2 t_f}{2c^2}.$$

Now we want to maximize $\Delta\tau$ by varying v_0, keeping t_f constant. We find

$$\frac{\partial \Delta\tau}{\partial v_0} = \frac{gt_f^2}{4c^2} - \frac{v_0 t_f}{c^2} = 0.$$

if $v_0 = gt_f/4$. Using this value for v_0 in $\Delta\tau$, we find

$$\Delta\tau = \frac{g^2 t_f^3}{32c^2}.$$

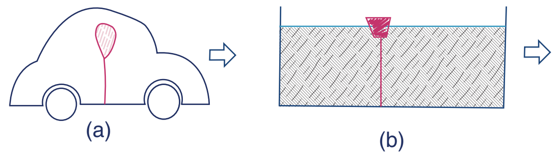


Fig. 3.1 problem.

for the largest possible value of $\Delta\tau$ for constant speed motion up and down. We can compare this with the value of $\Delta\tau$ for freely-falling motion from part (a), which was

$$\Delta\tau = \frac{g^2 t_f^3}{24c^2}.$$

The $\Delta\tau$ for free fall is obviously larger than the $\Delta\tau$ for constant speed. ■

⋆ **Problem 3.30** (a) An automobile driver, stopped at an intersection, ties a helium-filled balloon on a string attached to the floor of her car, so the balloon floats up. When the light turns green she accelerates the car forward. Relative to the car, does the balloon move forward, backward, or remain vertically above the place it is tied? (b) A rectangular fish tank is half filled with water. One end of a rubber band is attached to the bottom of the tank, and the other end is attached to a cork that floats on the water's surface, as shown in the diagram. Now the tank is pushed to the right, giving it a constant rightward acceleration. Eventually, the water surface, cork, and rubber band settle into a new equilibrium configuration. Sketch this new configuration.

Solution

(a) The balloon tips *forward*. The Principle of Equivalence states that a forward acceleration is equivalent to a gravity exerted toward the *rear*. The balloon floats up, in a direction opposite to the effective gravity. (b) An acceleration to the right is equivalent to an artificial gravity backwards. So the total effective gravity is the vector sum of the real gravity downwards and an artificial gravity backwards. The water surface will therefore be slanted, higher toward the rear of the fish tank. The rubber band is at an angle perpendicular to the water surface, and the cork will float at the upper end of the rubber band. ■

⋆ **Problem 3.31** A skyscraper elevator comes equipped with two weighing scales: The first is a typical bathroom scale containing springs that compress when someone stands on it, and the second is the type often used in doctor's offices, where weights are adjusted to balance that of the patient. (a) A rider enters the elevator at the ground floor and stands on the first scale; it reads 150 lbs. Use the principle of equivalence to answer the following questions. (i) As the elevator accelerates upward, will the scale read less than, more than, or equal to 150 lbs? (ii) When the elevator reaches its maximum speed and continues rising at this speed, will the scale read less than, more than, or equal to 150 lbs? (iii) And as the elevator comes to rest at the top floor, what will it read? (b) The rider repeats the experiment, standing this time on the second scale. What will it read during each portion of the trip?

Solution

(a) (i) More than. Accelerating upward generates an artificial gravity downward, compressing the scale springs more than before. (ii) Equal to. Once the acceleration ceases, there is no more artificial gravity. (iii) Less than. During the deceleration there is an acceleration downward, and so an artificial gravity upward. This will decrease the scale reading. (b) The second scale balances the weight of the rider and that of the weights on the balance arms of the scale. Any artificial gravity due to accelerations will act on both rider and weights, so will have no effect on the scale readings. (i), (ii), (iii) all "Equal to". ■

⋆ **Problem 3.32** A laser is aimed horizontally near earth's surface, a distance y_0 above the ground; a pulse of light is then emitted. (a) How far will the pulse fall by the time it has travelled a distance L? (b) What is the value of L if the pulse falls by 0.1 nm, roughly the diameter of a hydrogen atom?

Solution

(a) The pulse "falls" a distance $\Delta y = (1/2)gt^2 = (1/2)g(L/c)^2 = gL^2/2c^2$. (b) $L = c\sqrt{2\Delta y/g} \simeq 1.4$ km. ■

⋆ **Problem 3.33** Note that in the Hafele-Keating experiments the total error in the eastward and westward flights was comparable, ± 23 and ± 21 nanoseconds, respectively, but that the *percentage* error was much greater for the eastward flights. (a) What is the reason for that? What is the lesson one might draw for other experiments or theoretical calculations? (b) Note also that in the calculated differences between the traveling and stay-at-home clocks, the special-relativistic (velocity) effect is negative for the eastward flights and positive for the westward flights. Why was that?

Solution

(a) For the eastern flights the altitude and velocity effects had opposite signs, so tend to cancel one another, leaving a relatively small combined result. Errors therefore can have a large percentage effect. For western flights the altitude and velocity effects have the same sign, leading to a large combined result, so errors have a smaller percentage effect. (b) For eastern flights the planes are flying in the direction of the earth's rotation, so the total speed in an inertial frame (in which the earth is seen to rotate) is large, causing a large time dilation, which is a negative effect on clock rates, more negative than that of a clock at rest on the ground. For western flights the plane's velocity subtracts from that of the earth's rotation, leaving a smaller total speed in an inertial frame, meaning a smaller time dilation effect. When flying west the plane is still moving east in an inertial frame, because the plane's speed relative to the ground is smaller than the earth's rotation relative to the inertial frame. Therefore although running slow in the inertial frame, a clock on the ground runs slower still in the inertial frame, so the time dilation of plane clocks is less than that of ground clocks, leading to a relative positive time dilation effect for clocks carried westward. ■

⋆ **Problem 3.34** A hypothetical planet has an equatorial circumference of 40,000 km, a gravity $g = 10$ m/s^2, and completes one revolution every 24 hours. Aircraft A circles eastward around the equator at constant altitude 10 km, while Aircraft B circles westward around the equator, both at altitude 10 km, except for the brief takeoffs and landings, each requiring 40 hours to make the trip from home base back to home base. Atomic clocks are carried on both planes and others are left at home. What are the calculated differences between the traveling and stay-at-home clocks due to (a) altitude effects (b) velocity (special-relativistic) effects, and (c) the net predicted effect, both for clocks carried on A and on B.

Solution

(a) Altitude effects are the same for each,

$$\Delta t_{\text{high}} = \Delta t_{\text{low}}(1 + \frac{gy}{c^2})$$

$$\Delta t_{\text{airplane}} - \Delta t_{\text{ground}} = +\frac{gy\Delta t_{\text{low}}}{c^2} = -\frac{10 \times 1000}{9 \times 10^{16}}\Delta t_{\text{low}} \simeq +1.11 \times 10^{-12}\Delta t_{\text{low}} = +160 \text{ ns}$$

since $\Delta t_{\text{low}} = 40$ hours $\times$ 3600 seconds/hour. (b) The planet's surface speed is

$$v_E = \frac{40000 \times 10^3}{24 \times 60 \times 60} \simeq 463 \text{ m/s}$$

Airplane A moves eastward parallel to the planet's motion in time $\Delta\tau = 40$ hrs. So its speed relative to the ground is

$$v = 40,000 \text{ km}/40\text{hrs} = 10^3 \text{ km/hr} = 278 \text{ m/s}$$

Therefore the speed of the plane in an inertial frame in which the planet is seen to rotate is (278 + 463) m/s = 741 m/s, so the special-relativistic time dilation factor for the eastward-moving plane is

$$\sqrt{1 - v^2/c^2} \simeq 1 - v^2/2c^2 = 1 - 3.05 \times 10^{-12}.$$

Thus due to time dilation over the 40 hour trip, the slowing of the onboard clocks is $3.04 \times 10^{-12} \times 40$ hours $= 439$ ns relative to an inertial clock. However, the ground clocks are also running slow because of the planet's rotation. In fact, as shown above, the ground is moving at speed 463 m/s, so the ground clocks run slow by the factor

$$1 - v^2/2c^2 = 1 - 1.19 \times 10^{-12}.$$

which amounts to 171 ns by the end of the trip.

Therefore relative to ground clocks, the eastward moving clocks have run slow by (439 – 171) ns = 268 ns due to time dilation, by the end of the 40 hour trip.

Airplane B moves westward antiparallel to the planet's motion in time $\Delta\tau = 40$ hrs. So its speed relative to the ground is still

$$v = 40,000 \text{ km}/40 \text{ hrs} = 10^3 \text{ km/hr} = 278 \text{ m/s}.$$

However, this westward motion is exceeded by the eastward motion of the planet surface, which is 463 m/s, and found above. So the net speed of the plane in an inertial frame in which the planet is seen to rotate is (463 – 278) m/s = 185 m/s eastward, so the special-relativistic time dilation factor for the plane is

$$\sqrt{1 - v^2/c^2} \simeq 1 - v^2/2c^2 = 1 - 1.90 \times 10^{-13}.$$

Multiplying by 40 hours, we get the slowing of clocks on B to be $1.90 \times 10^{-13} \times 40$ hrs = 27 ns relative to inertial clocks. We found above that clocks attached to the surface of the planet run slow by 171 ns relative to inertial clocks, a larger amount than that of clocks on B. So relative to surface clocks, the clocks on B run *fast* by (171 – 27) ns = 144 ns.

In summary, the net time-dilation effects relative to the stay-at-home clocks are 268 ns running slow for clocks on A (eastbound), and 144 ns of running fast for clocks on B (westbound).

(c) Finally, we have to include the altitude effect, which we showed above is 160 ns of running fast for both planes A and B relative to ground clocks. Therefore by the end of the trip A's clocks will have run slow by (268−160) ns = 108 ns relative to stay-at-home clocks, and B's clocks will have run fast by (144 + 160) ns = 304 ns. ■

4 Lagrangian Mechanics

4.1 Problems and Solutions

⋆ **Problem 4.1** In Example 4.3 we found the equation of motion of a block on an inclined plane, using the generalized coordinate X, the distance of the block from the bottom of the incline. Solve the equation for $X(t)$ in terms of an arbitrary initial position $X(0)$ and velocity $\dot{X}(0)$.

Solution

In Example 4.3 we derived the equation of motion $-mg\sin\alpha = m\ddot{X}$, where α is the tilt-angle. Integrating once over time, $-mg(\sin\alpha)t = m\dot{X}$ + con. Now if $\dot{X}(0)$ is the initial velocity, then at $t = 0$ we have $0 = m\dot{X}(0)$ + con, so con = $-m\dot{X}(0)$. Therefore $m\dot{X} = m\dot{X}(0) - mg(\sin\alpha)t$. Integrating once more,

$$mX = m\dot{X}(0)t - \frac{1}{2}mg(\sin\alpha)t^2 + \text{con}.$$

The initial condition is $mX(0) = 0 - 0 + \text{con}$, so the constant of integration is $mX(0)$. Therefore, after cancelling $m's$, we find that

$$X = X(0) + \dot{X}(0)t - \frac{1}{2}g(\sin\alpha)t^2.$$

■

⋆ **Problem 4.2** A particle of mass m slides inside a smooth hemispherical bowl of radius R. Beginning with spherical coordinates r, θ and φ to describe the dynamics, select generalized coordinates, write the Lagrangian, and find the differential equations of motion of the particle.

Solution

The generalized coordinates here are θ, the polar angle, and φ, the azimuthal angle. The kinetic energy is $T = (1/2)mR^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2)$, and the potential energy is $U = mgR\cos\theta$. The Lagrangian is therefore

$$L = T - U = \frac{1}{2}mR^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) - mgR\cos\theta.$$

We could use the Lagrange equations for θ and for φ to get two second-order differential equations of motion. More simply, we note that there are two first-integrals in this problem;

one corresponding to conservation of energy, and one corresponding to conservation of angular momentum. These equations are

$$E = T + U = \frac{1}{2}mR^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2) + mgR\cos\theta.$$

$$\ell = mR^2\sin^2\theta\,\dot{\varphi}.$$

where both E and ℓ are constants. ■

Either these two first-order equations *or* the two Lagrange equations are sufficient.

⋆⋆ **Problem 4.3** Example 4.2 featured a bead sliding on a vertically-oriented helix of radius R. The angle θ about the symmetry axis was related to its vertical coordinate z on the wire by $\theta = \alpha z$. There is a uniform gravitational field g vertically downward. (a) Rewrite the Lagrangian and find the Lagrange equation, using θ as the generalized coordinate. (b) Are there any conserved dynamical quantities? (c) Write the simplest differential equation of motion of the bead, and go as far as you can to solve analytically for θ as a function of time.

Solution

(a) The kinetic energy is

$$\begin{aligned} T = (1/2)mv^2 &= (1/2)m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) + (1/2)m(0 + R^2\dot{\theta}^2 + (\dot{\theta}/\alpha)^2) \\ &= (m/2)(R^2 + \alpha^{-2})\dot{\theta}^2. \end{aligned}$$

and the potential energy is $U = mgz = mg\theta/\alpha$. Therefore the Lagrangian is

$$L = T - U = (m/2)(R^2 + \alpha^{-2})\dot{\theta}^2 - mg\theta/\alpha.$$

The partial derivatives are $\partial L/\partial\theta = -mg/\alpha$ and $\partial L/\partial\dot{\theta} = m(R^2 + \alpha^{-2})\dot{\theta}$, so the resulting Lagrange equation is

$$m(R^2 + \alpha^{-2})\ddot{\theta} + mg/\alpha = 0.$$

(b) Yes, energy is conserved, given by

$$E = T + U = (m/2)(R^2 + \alpha^{-2})\dot{\theta}^2 + mg\theta/\alpha = k,$$

where k is a constant. (c) The equation for E is already a first-order differential equation. Solve it for $\dot{\theta}$: We find

$$\frac{d\theta}{dt} = \pm\sqrt{\frac{k - mg\theta/\alpha}{(m/2)(R^2 + \alpha^{-2})}}.$$

Separating variables and integrating,

$$t = \pm\sqrt{(m/2)(R^2 + \alpha^{-2})}\int\frac{d\theta}{\sqrt{k - mg\theta/\alpha}}.$$

We can integrate using the substitution $q = k - mg\theta/\alpha$. Then we find

$$t = \pm\frac{2}{g}\sqrt{\frac{R^2\alpha^2 + 1}{2m}}\sqrt{k - mg\theta/\alpha} + C,$$

where C is a constant of integration. If we define $k = mg\theta_0/\alpha$ and $C = t_0$, we find that

$$\theta(t) = \theta_0 - \frac{\alpha g}{2(1+R^2\alpha^2)}(t-t_0)^2.$$

■

★★ **Problem 4.4** One end of a wire is tied to a point A on the ceiling and the other end is tied to a point on a ring of radius R and negligible mass. The ring therefore hangs from the wire in a vertical plane and in a gravitational field g. A bead of mass m is threaded onto the ring so it can slide around the ring without friction. The lowest point on the ring is then tied to a second wire whose opposite end is attached to point B on the floor, where point B is directly beneath point A. The wires are then drawn taut. If the ring and attached wires are made to twist sideways through an angle φ away from equilibrium, a potential energy $(1/2)\kappa\varphi^2$ is set up in the wire. (a) Using angles θ and φ as generalized coordinates, where θ is the angle of the bead down from the top of the ring, find the kinetic and potential energies of the bead. (b) Find the equations of motion using Lagrange's equations. Assume that during the motion of the bead it remains entirely on one side of the ring, so it does not meet the wires at $\theta = 0$ and $\theta = \pi$.

Solution

(a) Using spherical coordinates, the kinetic energy is

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2) \quad \text{with} \quad \dot{r} = 0, r = R$$

The potential energy is

$$U = mgR\cos\theta + \frac{1}{2}\kappa\varphi^2.$$

Therefore the Lagrangian is

$$L = \frac{1}{2}m(R^2\dot{\theta}^2 + R^2\sin^2\theta\dot{\varphi}^2) - mgR\cos\theta - \frac{1}{2}\kappa\varphi^2$$

(b) We have 2 degrees of freedom, θ and φ.

$$\frac{d}{dt}(\frac{\partial L}{\partial\dot{\theta}}) = \frac{\partial L}{\partial\theta} \Rightarrow \frac{d}{dt}(mR^2\dot{\theta}) = mR\sin\theta\cos\theta\dot{\varphi}^2 + mgR\sin\theta$$

$$\Rightarrow \left[mR^2\ddot{\theta} - mR\sin\theta\cos\theta\dot{\varphi}^2 - mgR\sin\theta = 0\right]$$

$$\frac{d}{dt}(\frac{\partial L}{\partial\dot{\phi}}) = \frac{\partial L}{\partial\varphi} \Rightarrow \frac{d}{dt}(mR^2\sin^2\theta\dot{\varphi}) = -\kappa\varphi$$

$$\Rightarrow \left[2mR^2\sin\theta\cos\theta\dot{\theta}\dot{\varphi} + mR^2\sin^2\theta\ddot{\varphi} + \kappa\varphi = 0\right]$$

■

★★ **Problem 4.5** A particle moves in a cylindrically symmetric potential $U(\rho, z)$. Use cylindrical coordinates ρ, φ, and z to parameterize the space.

(a) Write the Lagrangian for an unconstrained particle of mass m (using cylindrical coordinates) in the presence of this potential.

(b) Write the Lagrange equations of motion for ρ, φ and z.

(c) Identify any cyclic coordinates, and write a first integral corresponding to each.

Solution

(a) The Lagrangian is

$$L = T - U = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2) - U(\rho, z).$$

(b) Beginning with the partial derivatives $\partial L/\partial \rho = m\rho\dot{\varphi}^2 - \partial U/\partial \rho$ and $\partial L/\partial \dot{\rho} = m\dot{\rho}$, there is the Lagrange equation

$$m\ddot{\rho} - m\rho\dot{\theta}^2 + \frac{\partial U}{\partial \rho} = 0.$$

Similarly for the φ and z coordinates,

$$m\rho^2\dot{\varphi} = \ell = \text{constant}.$$

and

$$m\ddot{z} + \frac{\partial U}{\partial z} = 0.$$

Note that the φ equation is easily written as a first integral, because $\partial L/\partial \varphi = 0$. (c) The angle φ is cyclic, so $\ell = mr^2\dot{\varphi}$ is a constant, corresponding to conservation of angular momentum. This is the only cyclic coordinate here for general $U(\rho, z)$. ■

⋆ **Problem 4.6** A particle of mass m slides inside a smooth paraboloid of revolution whose axis of symmetry z is vertical. The surface is defined by the equation $z = \alpha\rho^2$, where z and ρ are cylindrical coordinates, and α is a constant. There is a uniform gravitational field g. (a) Select two generalized coordinates for m. (b) Find T, U, and L. (c) Identify any ignorable coordinates, and any conserved quantities. (d) Show that there are two first integrals of motion, and find the corresponding equations.

Solution

(a) We can choose as coordinates z and φ. (We could alternatively use ρ, φ.) (b) The kinetic energy is $T = (1/2)mv^2 = (1/2)m[(z/\alpha)\dot{\varphi}^2 + \dot{z}^2]$ and the potential energy is $U = mgz$. Therefore the Lagrangian is

$$L = T - U = \frac{m}{2}(\dot{z}^2 + (z/\alpha)\dot{\varphi}^2) - mgz.$$

(c) E is conserved, because no non-conservative forces act, and angular momentum ℓ about the vertical axis of symmetry is conserved, because there is no torque. So

$$E = T + U = \frac{1}{2}m(\dot{z}^2 + (z/\alpha)\dot{\varphi}^2) + mgz = \text{constant}.$$

and

$$\ell = (mz/\alpha)\dot{\varphi} = \text{constant}.$$

where we have used $\rho^2 = z/\alpha$. Note that φ is ignorable, *i.e.* cyclic. (d) We have already written down the two first integrals in part (c), namely energy conservation and angular momentum conservation. ■

⋆ **Problem 4.7** Repeat the preceding problem for a particle sliding inside a smooth cone defined by $z = \alpha r$.

Solution

(a) Note that the symmetry axis of the cone is vertical, along the z axis. We can use r, φ as generalized coordinates, where r is the distance of any point on the cone away from the z axis, and r is perpendicular to the z axis. (Alternatively, one might use z, φ as generalized coordinates.) (b) The kinetic energy of the particle is $T = (1/2)m(\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2) = (1/2)m[(1+\alpha^2)\dot{r}^2 + r^2\dot{\varphi}^2]$ and $U = mg\alpha r$. So the Lagrangian may be written

$$L = \frac{1}{2}m[(1+\alpha^2)\dot{r}^2 + r^2\dot{\varphi}^2] - mg\alpha r.$$

(c) Note that φ is cyclic, so that angular momentum about the z axis is conserved: That is, $\ell = mr^2\dot{\varphi}$ = constant. (d) The first integrals are (i) angular momentum conservation $\ell = mr^2\dot{\varphi}$ = constant, and (ii) the energy $E = (m/2)[(1+\alpha^2)\dot{r}^2 + r^2\dot{\varphi}^2] + mg\alpha r$ = constant. ■

⋆ **Problem 4.8** A spring pendulum features a pendulum bob of mass m attached to one end of a spring of force-constant k and unstretched length R. The other end of the spring is attached to a fixed point on the ceiling. The pendulum is allowed to swing in a plane. Use r, the distance of the bob from the fixed point, and θ, the angle of the spring relative to the vertical, as generalized coordinates. (a) Find the kinetic and potential energies of the bob in terms of the generalized coordinates and velocities. (b) Find the Lagrangian. Are there any ignorable coordinates? (c) Are there any conserved quantities? (d) Find a complete set of equations of motion, including as many first integrals as possible.

Solution

(a) The kinetic energy is $T = (1/2)mv^2 = (1/2)m(\dot{r}^2 + r^2\dot{\theta}^2)$ and the total potential energy is $U = -mgr\cos\theta + (1/2)k(r-R)^2$. Here the angle θ is measured from the angle of the spring up from the lowest point of the bob, i.e., from the vertical line beneath the point of support. Also note that the spring stretch is $r - R$.
(b) The Lagrangian is

$$L = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta - \frac{1}{2}k(r-R)^2.$$

There are no ignorable coordinates. (c) Energy is conserved; the Hamiltonian is also conserved, because L is not an explicit function of time. (d) There are two generalized coordinates (r and θ) so we need two equations of motion. One of them is energy conservation, a first integral, and the other is either one of the second-order Lagrange equations, either for r or for θ. We will use the Lagrange equation for θ, which is

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0,$$

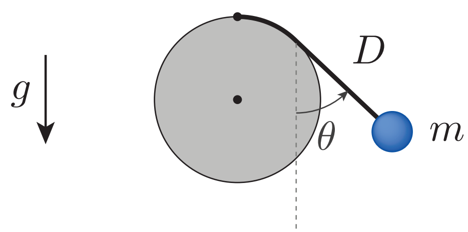
which gives

$$\ddot{\theta} + 2\frac{\dot{r}}{r}\dot{\theta} + \frac{g}{r}\sin\theta = 0.$$

The conservation of energy equation is

$$E = T + U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta + \frac{k}{2}(r - R^2).$$

That is, a complete set of equations of motion consists of energy conservation, which is a first-order differential equation, along with the second-order differential equation for θ. ■



Prob. 4.9 A pendulum hanging from a cylinder (see Problem 4.9).

⋆⋆⋆ **Problem 4.9** A pendulum is constructed from a bob of mass m on one end of a light string of length D. The other end of string is attached to the top of a circular cylinder of radius R ($R < 2D/\pi$). The string makes an angle θ with the vertical, as shown in the figure, as the pendulum swings in the plane. There is a uniform gravity g directed downward. (a) Find the Lagrangian of the bob, using θ as the generalized coordinate. (b) Identify any first integrals of motion. (c) Find the frequency of small oscillations about the stable equilibrium point.

Solution

(a) Let the origin be located at the center of the cylinder. The free-hanging portion of the string at some instant has length $r = D - R(\pi/2 - \theta)$, which is the entire length of the string minus the part of the string still wrapped around the cylinder at that instant.

The Cartesian components of m are, using the figure,

$$\begin{aligned} x &= R\cos\theta + r\sin\theta = R\cos\theta + [D - R(\pi/2 - \theta)]\sin\theta \\ y &= R\sin\theta - r\cos\theta = R\sin\theta - [D - R(\pi/2 - \theta)]\cos\theta. \end{aligned}$$

The potential energy of m is therefore

$$U = mgy = mg[R\sin\theta - [D - R(\pi/2 - \theta)]\cos\theta].$$

We can also find the Cartesian velocity components of m. After cancelling some terms, these are

$$\begin{aligned} \dot{x} &= [D - R(\pi/2 - \theta)]\cos\theta\dot{\theta} \\ \dot{y} &= [D - R(\pi/2 - \theta)]\sin\theta\dot{\theta} \end{aligned}$$

Therefore the kinetic energy of m is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m[D - R(\pi/2 - \theta)]^2\dot{\theta}^2,$$

so the Lagrangian is

$$L = T - U = \frac{1}{2}m[D - R(\pi/2 - \theta)]^2\dot{\theta}^2 - mg[R\sin\theta - [D - R(\pi/2 - \theta)]\cos\theta].$$

(b) Energy is conserved, because no work is being done on the bob apart from gravity, and gravity is accounted for by the potential energy. (The Hamiltonian is also conserved because L is not an explicit function of time. In this problem $E = H$.)

The energy is

$$E = T + U = \frac{1}{2}m[D - R(\pi/2 - \theta)]^2\dot{\theta}^2 + mg[R\sin\theta - [D - R(\pi/2 - \theta)]\cos\theta] = \text{constant},$$

a first-integral of motion.

(c) We can find the frequency of small oscillations by first approximating $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \theta^2/2$ in the equation for E, valid through terms of second order in θ. This gives

$$E \approx \frac{1}{2}m[D - R(\pi/2]^2\dot{\theta}^2 - mg(D - R\pi/2)(1 - \theta^2/2).$$

where we have also neglected the product $\theta\dot{\theta}^2$. Now differentiate the equation with respect to time:

$$m[D - R(\pi/2]^2\dot{\theta}\ddot{\theta} + mg(D - R\pi/2)\theta\dot{\theta} = 0.$$

Factoring out $\dot{\theta}$, we are left with

$$\ddot{\theta} + [g/(D - R\pi/2)]\theta = 0.$$

This is a simple harmonic oscillator equation in θ, with solutions $\sin\omega t$ and $\cos\omega t$, where the frequency of oscillation is

$$\omega = \sqrt{\frac{g}{D - R\pi/2}}$$

about the stable equilibrium point $\theta = 0$. ■

⋆ **Problem 4.10** A particle moves with a cylindrically symmetric potential energy $U = U(\rho, z)$ where ρ, φ, z are cylindrical coordinates. (a) Write the Lagrangian for an unconstrained particle of mass m in this case. (b) Are there any cyclic coordinates? If so, what symmetries do they correspond to, and what are the resulting constants of the motion? (c) Write the Lagrange equation for each cyclic coordinate. (d) Find the Hamiltonian H. Is it conserved? (e) Find the total energy E. Is $E = H$? is E conserved? (f) Write the simplest (*i.e.*, lowest-order) complete set of differential equations of motion of the particle.

Solution

(a) $L = T - U = (1/2)m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2) - U(\rho, z)$ (b) The angle φ is cyclic, indicating symmetry under rotations about the z axis. Angular momentum is conserved about that axis. (c) $\partial L/\partial\varphi = 0$, so from the Lagrange equation is follows that $\partial L/\partial\dot{\varphi} = \ell$, a constant. (d) The Hamiltonian is $H = \Sigma\dot{q}_i p_i - L$ where $p_\rho = \partial L/\partial\dot{\rho} = m\dot{\rho}$ and $p_z = \partial L/\partial\dot{z} = m\dot{z}$. So

$$\begin{aligned} H &= m\rho^2\dot{\varphi}^2 + m\dot{\rho}^2 + m\dot{z}^2 - \frac{1}{2}m(\rho^2\dot{\varphi}^2 + \dot{\rho}^2 + \dot{z}^2) + U(\rho, z) \\ &= \frac{1}{2}m(\rho^2\dot{\varphi}^2 + \dot{\rho}^2 + \dot{z}^2) + U(\rho, z). \end{aligned}$$

Yes, H is conserved, because L is not an explicit function of time. (e) $E = T + U = (1/2)m(\rho^2\dot{\varphi}^2 + \dot{\rho}^2 + \dot{z}^2) + U(\rho, z)$, which is equal to H. So *both* E and H are conserved. (f) (1) $m\rho^2\dot{\varphi} = \ell$, a first order differential equation. Here ℓ is a constant. (2) $E = (1/2)m(\rho^2\dot{\varphi}^2 + \dot{\rho}^2 + \dot{z}^2) + U(\rho, z)$, also a first order equation, where E is a constant. (3) We need one additional differential equation, which must be second order, because nothing else is conserved. This could be

$$m\ddot{z} + \partial U(\rho, z)/\partial z = 0 \quad \text{OR} \quad m\ddot{\rho} - m\rho\dot{\varphi}^2 + \partial U(\rho, z)/\partial\rho = 0.$$ ■

★★ **Problem 4.11** A plane pendulum is made with a plumb bob of mass m hanging on a Hooke's-law spring of negligible mass, force constant k, and unstretched length ℓ_0. The spring can stretch but is not allowed to bend. There is a uniform downward gravitational field g. (a) Select generalized coordinates for the bob, and find the Lagrangian in terms of them. (b) Write out the Lagrange equations of motion (c) Are there any conserved quantities? If so, write down the corresponding conservation law(s). (d) If the bob is swinging back and forth, find the frequency of small oscillations in the general case where the spring can change its length while the bob is swinging back and forth.

Solution

(a) Using polar coordinates

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2}k(r - \ell_0) + mgr\cos\theta$$

(b)

$$\frac{\partial L}{\partial\dot{r}} = m\dot{r} \qquad \frac{\partial L}{\partial\dot{\theta}} = mr^2\dot{\theta}$$

$$\Rightarrow m\ddot{r} = mr\dot{\theta}^2 - k(r - \ell_0) + mg\cos\theta \qquad \frac{d}{dt}(mr^2\dot{\theta}) = -mgr\sin\theta$$

(c) Energy is conserved

$$E = H = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}k(r - \ell_0)^2 - mgr\cos\theta$$

(d) Let r_0 denote equilibrium length. Write $r = r_0 + \Delta r$ where $\Delta r \ll r_0$. Expand energy to quadratic order in Δr

$$E = \frac{1}{2}m\Delta\dot{r}^2 + \frac{1}{2}m(r_0 + \Delta r)^2\dot{\theta}^2 + \frac{1}{2}k(r_0 - \ell_0 + \Delta r)^2 - mg(r_0 + \Delta r)\left(1 - \frac{\theta^2}{2}\right)$$

where we also expand $\cos\theta$ for small angles.

$$\Rightarrow E \cong \frac{1}{2}m\Delta\dot{r}^2 + \frac{1}{2}mr_0^2\dot{\theta}^2 + \frac{1}{2}\frac{(mg)^2}{k}\left(1 + \frac{2\Delta rk}{mg} + \left(\frac{\Delta rk}{mg}\right)^2\right) - mgr_0 - mg\Delta r + \frac{mg}{2}\theta^2$$

where we dropped anything more than quadratic in the small variables Δr and θ and we used equilibrium condition

$$k(r_0 - \ell_0) = mg \Rightarrow E = \frac{1}{2}mr_0^2\dot{\theta}^2 + \frac{1}{2}mg\theta^2 + \frac{1}{2}m\Delta\dot{r}^2 + \frac{1}{2}k\Delta r^2$$

$$\omega_r^2 = \frac{k}{m} \qquad \omega_\theta^2 = \frac{g}{r_0^2 g}$$

■

★ **Problem 4.12** Motion in a slowly-changing uniform electric field

A particle of mass m and charge q moves within a parallel-plate capacitor whose charge Q decays exponentially with time, $Q = Q_0 e^{-t/\tau}$, where τ is the time constant of the decay. Find the equations of motion of the particle. Ignore the effect of any magnetic field that may be generated from the changing electric field.

Solution

The electric field between the plates is $E = Q/A = (Q_0/A)e^{-t/\tau}$ where A is the area of the capacitor. If the motion is nonrelativistic we can write

$$m\ddot{x} = qE = (Q_0 q/A)e^{-t/\tau} \equiv m\alpha e^{-t/\tau} \;\Rightarrow\; \ddot{x} = \alpha e^{-t/\tau}.$$

There would be no force in perpendicular directions, so the charge would at most simply drift in the y and z directions. If the motion were relativistic we would have instead

$$\frac{d}{dt}(\gamma v^x) = \alpha e^{-t/\tau}, \text{ where } \gamma = [1 - ((v^x)^2 + (v^y)^2 + (v^z)^2)/c^2]^{-1/2}.$$

■

★ **Problem 4.13** A particle of mass m travels between two points $x = 0$ and $x = x_1$ on Earth's surface, leaving at time $t = 0$ and arriving at $t = t_1$. The gravitational field g is uniform. (a) Suppose m moves along the ground (keeping altitude $z = 0$) at steady speed. Find the total action S to go by this path. (b) Suppose instead that m moves along the least-action parabolic path. Show that the action in this case is

$$S = \frac{mx_1^2}{2t_1} - \frac{mg^2 t_1^3}{24}$$

and verify that it is less than the action for the straight-line path of part (a).

Solution

In general, we can take the Lagrangian to be $L = (1/2)m(\dot{x}^2 + \dot{z}^2) - mgz$.
(a) If the particle moves along the ground only, then $L = (1/2)m\dot{x}^2$, so $S = \int L\,dt = (1/2)m\dot{x}^2 t$ where at constant speed, $\dot{x} = x_1/t_1$. Therefore

$$S = \frac{1}{2}m\frac{x_1^2}{t_1}.$$

for a particle traveling at constant speed at ground level.

(b) Along the physical, parabolic path, we know that $\dot{x} = \dot{x}_0$ and $\dot{z} = \dot{z}_0 - gt, z = \dot{z}_0 t - (1/2)gt^2$. Therefore

$$S = \int_0^{t_1} [(1/2)m(\dot{x}_0^2 + (\dot{z}_0 - gt)^2) - mg(\dot{z}_0 t - (1/2)gt^2)]\, dt = \frac{m}{2}\frac{x_1^2}{t_1} - \frac{mg^2 t_1^3}{24}.$$

Obviously this is less than the straight-line path S found in part (a). ■

★★ **Problem 4.14** Suppose the particle of the preceding problem moves instead at constant speed along an isoceles triangular path between the beginning point and the end point, with the high point at height z_1 above the ground, at $x = x_1/2$ and $t = t_1/2$. (a) Find the action for this path. (b) Find the altitude z_1 corresponding to the least-action path among this class of constant-speed triangular paths. (c) Verify that the total action for this path is greater than that of the parabolic path of the preceding problem.

Solution

(a) From symmetry one can see that the action for the first half of the path, moving from the ground up to height z_1, is the same as the action for the second half, moving from height z_1 back to the ground. Therefore we can calculate the action for the first half and double the result. The Lagrangian is $L = (1/2)m(\dot{x}^2 + \dot{z}^2) - mgz$. While ascending, $\dot{x} = (x_1/2)/(t_1/2) = x_1/t_1$, and $\dot{z} = z_1/(t_1/2) = 2z_1/t_1$. Also $z = \dot{z}t = 2(z_1/t_1)t$. Therefore

$$L = \frac{1}{2}m\left[\left(\frac{x_1}{t_1}\right)^2 + \left(\frac{2z_1}{t_1}\right)^2\right] - mg\left(\frac{2z_1}{t_1}\right)t.$$

The action while ascending is therefore

$$\begin{aligned} S = \int_0^{t_1/2} L\, dt &= \frac{1}{2}m\int_0^{t_1/2}\left[\left(\frac{x_1}{t_1}\right)^2 + \left(\frac{2z_1}{t_1}\right)^2\right] dt - mg\frac{2z_1}{t_1}\int_0^{t_1/2} t\, dt. \\ &= \frac{m}{4}\frac{x_1^2}{t_1} + \frac{mz_1^2}{t_1} - \frac{mgz_1 t_1}{4}. \end{aligned}$$

The total action for the ascent and descent is twice this much,

$$S_{\text{Total}} = \frac{mx_1^2}{2t_1} + \frac{2mz_1^2}{t_1} - \frac{mgz_1 t_1}{2}.$$

(b) Now to find the optimal value of z_1, the value of z_1 that minimizes S for these triangular paths, we require $\partial S/\partial z_1 = 0$. That is, $(4mz_1/t_1) - (mgt_1/2) = 0$, so this best value of z_1 is. $z_1 = gt_1^2/8$.

(c) Now substitute this value of z_1 into the action found in part (a). The result is

$$S_{\text{triangle}} = \frac{mx_1^2}{2t_1} - \frac{mg^2 t_1^3}{32}.$$

The result for a parabolic, free-fall path, from the preceding problem, is

$$S_{\text{free-fall}} = \frac{mx_1^2}{2t_1} - \frac{mg^2t_1^3}{24}.$$

Obviously $S_{\text{triangle}} > S_{\text{free-fall}}$, consistent with the fact that the action for the actual physical motion is less than that for any other path. ■

⋆⋆ **Problem 4.15** A plane pendulum consists of a light rod of length R supporting a plumb bob of mass m in a uniform gravitational field g. The point of support of the top end of the rod is forced to oscillate back and forth in the horizontal direction with $x = A\cos\omega t$. Using the angle θ of the bob from the vertical as the generalized coordinate, (a) find the Lagrangian of the plumb bob. (b) Are there any conserved dynamical quantities? (c) Find the simplest differential equation of motion of the bob.

Solution

(a) The Cartesian components of the plumb-bob's position are $X = x+R\sin\theta = A\cos\omega t + R\sin\theta$ and $Y = R\cos\theta$. Here Y is measured positive *downward* from the point of support. The Lagrangian is therefore

$$\begin{aligned} L = T - U &= \frac{1}{2}m(\dot{X}^2 + \dot{Y}^2) - (-mgR\cos\theta) \\ &= \frac{m}{2}[(-\omega A\sin\omega t + R\cos\theta\,\dot{\theta})^2 + (-R\sin\theta\,\dot{\theta})^2] + mgR\cos\theta. \\ &= \frac{m}{2}[\omega^2A^2\sin^2\omega t - 2\omega AR\sin\omega t\cos\theta\,\dot{\theta} + R^2\dot{\theta}^2]. + mgR\cos\theta. \end{aligned}$$

(b) No. H is not conserved because L is an explicit function of time. E is not conserved because work is being done on the bob by the motor causing the oscillating point of support.
(c) There are no first integrals of motion, so the simplest equation is the second-order equation found from Lagrange's equation. Note that

$$\begin{aligned} \frac{\partial L}{\partial\theta} &= \frac{m}{2}(2\omega AR\sin\omega t\sin\theta\,\dot{\theta}) - mgR\sin\theta. \\ \frac{\partial L}{\partial\dot{\theta}} &= -\frac{m}{2}(2\omega AR\sin\omega t\cos\theta) + mR^2\dot{\theta}. \end{aligned}$$

So the Lagrange equation is

$$\frac{d}{dt}[mR^2\dot{\theta} - m\omega AR\sin\omega t\cos\theta] - m\omega AR\sin\omega t\sin\theta\,\dot{\theta} + mgR\sin\theta = 0.$$

Of course the total time derivative can be taken, resulting in a second order differential equation in the generalized coordinate θ. ■

⋆⋆ **Problem 4.16** Solve the preceding problem if instead of being forced to oscillate in the horizontal direction, the upper end of the rod is forced to oscillate in the vertical direction with $y = A\cos\omega t$.

Solution

(a) The Cartesian components of the plumb-bob's position are $X = R\sin\theta$ and $Y = A\cos\omega t - R\cos\theta$, where Y is measured positive *upward*. The Lagrangian is therefore

$$\begin{aligned} L = T - U &= \frac{1}{2}m(\dot{X}^2 + \dot{Y}^2) - (-mgR\cos\theta) \\ &= \frac{m}{2}[(R\cos\theta\dot{\theta})^2 + (-\omega A\sin\omega t + R\sin\theta\dot{\theta})^2] + mgR\cos\theta. \\ &= \frac{m}{2}[\omega^2A^2\sin^2\omega t - 2\omega AR\sin\omega t\sin\theta\,\dot{\theta} + R^2\dot{\theta}^2] + mgR\cos\theta. \end{aligned}$$

(b) No. H is not conserved because L is an explicit function of time. E is not conserved because work is being done on the bob by the motor causing the oscillating point of support.
(c) There are no first integrals of motion, so the simplest equation is the second-order equation found from the Lagrange equation. Note that

$$\begin{aligned} \frac{\partial L}{\partial\theta} &= -m\omega AR\sin\omega t\cos\theta\,\dot{\theta} - mgR\sin\theta \\ \frac{\partial L}{\partial\dot{\theta}} &= -m\omega AR\sin\omega t\sin\theta + mR^2\dot{\theta}. \end{aligned}$$

So the Lagrange equation is

$$\frac{d}{dt}[-m\omega AR\sin\omega t\sin\theta + mR^2\dot{\theta}] + m\omega AR\sin\omega t\cos\theta\,\dot{\theta} + mgR\sin\theta = 0.$$

Of course the total time derivative can be taken, resulting in a second order differential equation in the generalized coordinate θ. ■

⋆⋆ **Problem 4.17** A particle of mass m on a frictionless table top is attached to one end of a light string. The other end of the string is threaded through a small hole in the table top, and held by a person under the table. If given a sideways velocity v_0, the particle circles the hole with radius r_0. At time $t = 0$ the mass reaches an angle defined to be $\theta = 0$ on the table top, and the person under the table pulls on the string so that the length of the string above the table becomes $r(t) = r_0 - \alpha t$ for a period of time thereafter, where α is a constant. Using θ as the generalized coordinate of the particle, find its Lagrangian, identify any conserved quantities, finds its simplest differential equation of motion, and get as far as you can using analytic means alone toward finding the solution $\theta(t)$ (or $t(\theta)$).

Solution

The kinetic and potential energies of the particle are

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) = \frac{1}{2}m[\alpha^2 + (r_0 - \alpha t)^2\dot{\theta}^2] \quad \text{and} \quad U = 0,$$

so

$$L = T - U = \frac{1}{2}m[\alpha^2 + (r_0 - \alpha t)^2\dot{\theta}^2].$$

The energy E is not conserved because work is being done on the particle by the person pulling on the rope. H is not conserved because L is an explicit function of time. The angular momentum $\ell = mr^2\dot{\theta}$ *is* conserved, because no torque is exerted on the particle. Therefore the simplest equation of motion is

$$\dot{\theta} = \frac{\ell}{mr^2} = \frac{\ell}{m[r_0 - \alpha t]^2}.$$

Integrating,

$$\theta(t) = \theta_0 + \int_0^t \dot{\theta}\, dt = 0 + \frac{\ell}{m}\int_0^t \frac{dt}{(r_0 - \alpha t)^2} = \frac{\ell t}{mr_0(r_0 - \alpha t)}.$$

Note that $\theta = 0$ at $t = 0$, and that θ increases rapidly with time as the denominator factor $\to 0$. ■

★ **Problem 4.18** A rod is bent in the middle by angle α. The bottom portion is kept vertical and the top portion is therefore oriented at angle α to the vertical. A bead of mass m is slipped onto the top portion and the bottom portion is forced by a motor to rotate at constant angular speed ω about the vertical axis. (a) Define a generalized coordinate for the bead and write down the Lagrangian. (b) Identify any conserved quantity or quantities and explain why it (or they) are conserved. (c) Find the generalized momentum of the bead and the Hamiltonian. (d) Are there any equilibrium points of the bead? If so, are they stable or unstable?

Solution

(a) We can use s, the distance up from the bend, as a generalized coordinate. In that case the Lagrangian is

$$L = T - U = \frac{1}{2}m(\dot{s}^2 + (s\sin\alpha)^2\omega^2) - mgs\cos\alpha.$$

(b) The Hamiltonian H is conserved, because L is not an explicit function of time. Neither energy nor angular momentum is conserved (work is begin done on the bead by the rod, and the rod also exerts a torque on the bead.)

(c) $p_s = \partial L/\partial\dot{s} = m\dot{s}$. Therefore

$$\begin{aligned} H = \dot{s}p_s - L &= m\dot{s}^2 - \frac{1}{2}m(\dot{s}^2 + (s\sin\alpha)^2\omega^2) + mgs\cos\alpha \\ &= \frac{1}{2}m(\dot{s}^2 - (s\sin\alpha)^2\omega^2) + mgs\cos\alpha \end{aligned}$$

(d) We can write $H = (1/2)m\dot{s}^2 + U_{\text{eff}}$ where

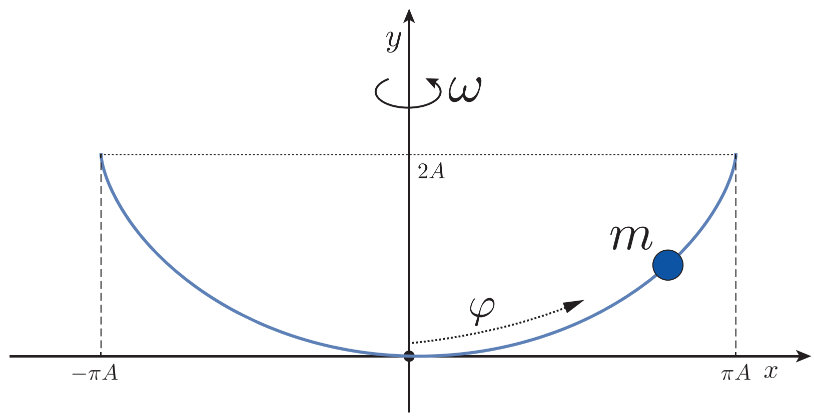
$$U_{\text{eff}} = -\frac{1}{2}m\omega^2\sin^2\alpha s^2 + mgs\cos\alpha.$$

There is an equilibrium point at the maximum of U_{eff}, which is at the point where $dU_{\text{eff}}/ds = 0$, located at $s = (g\cos\alpha)/(\omega^2\sin^2\alpha)$. This equilibrium point is *unstable,* however, because U_{eff} has a local maximum there. ■

★★ **Problem 4.19** A wire is bent into the shape of a cycloid, defined by the parametric equations $x = A(\varphi + \sin\varphi)$ and $y = A(1 - \cos\varphi)$, where φ is the parameter ($-\pi < \varphi < \pi$), and A is a constant. The wire is in a vertical plane, and is spun at constant angular velocity ω about a vertical axis through its center. A bead of mass m is slipped onto the wire. (a) Find the Lagrangian of the bead, using the parameter φ as the generalized coordinate. (b) Identify any first integral of motion of the bead. (c) Reduce the problem to quadrature: That is, show that the time t can be expressed as an integral over φ.

Solution

The parameters φ, x, y are illustrated in the figure below.



(a) In the plane of the cycloid, the velocity squared of the bead is

$$\dot{x}^2 + \dot{y}^2 = A^2 \left[\frac{d}{dt}(\varphi + \sin\varphi) \right]^2 + A^2 \left[(1 - \cos\varphi) \right]^2 = 2A^2\dot{\varphi}^2(1 + \cos\varphi)$$

where we have used the identity $\sin^2\varphi + \cos^2\varphi = 1$. There is also motion perpendicular to the plane of the cycloid, which has $v^2 = x^2\omega^2$. So the total kinetic energy of the bead is proportional to the sum of all the squares of velocity components,

$$T = \frac{1}{2} mA^2[(\varphi + \sin\varphi)^2\omega^2 + 2(1 + \cos\varphi)\dot{\varphi}^2].$$

The potential energy of the bead is $U = mgy = mgA(1 - \cos\varphi)$. Therefore the Lagrangian is

$$L = T - U = \frac{1}{2} mA^2[(\varphi + \sin\varphi)^2\omega^2 + 2(1 + \cos\varphi)\dot{\varphi}^2] - mgA(1 - \cos\varphi).$$

(b) Neither E nor ℓ is conserved, but the Hamiltonian H is conserved, since L is not an explicit function of time. The canonical momentum of the bead is $p = \partial L/\partial\dot{\varphi} = 2mA^2(1 + \cos\varphi)$, so the Hamiltonian is

$$H = \dot{q}p - L = mA^2 \left[-\frac{1}{2}(\varphi + \sin\varphi)^2\omega^2 + (1 + \cos\varphi)\dot{\varphi}^2] + \frac{g}{A}(1 - \cos\varphi) \right].$$

Solving for $\dot{\varphi}^2$,

$$\dot{\varphi}^2 \equiv \left(\frac{d\varphi}{dt} \right)^2 = \frac{H}{mA^2(1 + \cos\varphi)} + \frac{(1/2)(\varphi + \sin\varphi)^2\omega^2 - g/A(1 - \cos\varphi)}{(1 + \cos\varphi)}$$

Taking the square root and inverting, it follows that

$$t(\varphi) = \int_{\varphi_0}^{\varphi} \frac{d\varphi\sqrt{1+\cos\varphi}}{\left[(H/mA^2) + (\omega^2/2)(\varphi + \sin\varphi)^2 - (g/A)(1-\cos\varphi)\right]^{1/2}}$$

so the problem has been "reduced to quadrature." The integral can be solved numerically, given values of the constants H/mA^2, ω^2, and g/A. ■

⋆⋆ **Problem 4.20** *Center of mass and relative coordinates.* Show that for two particles moving in one dimension, with coordinates x_1 and x_2, with a potential that depends only upon their separation $x_2 - x_1$, then the Lagrangian

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - U(x_2 - x_1)$$

can be rewritten in the form

$$L = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}\mu\dot{x}'^2 - U(x'),$$

where $M = m_1 + m_2$ is the total mass and $\mu = m_1 m_2/M$ is the "reduced mass" of the system, and $X = (m_1x_1 + m_2x_2)/M$ is the center of mass coordinate and $x' = x_2 - x_1$ is the relative coordinate.

Solution

Beginning with the second form,

$$L = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}\mu\dot{x}'^2 - U(x')$$

$$= \frac{1}{2}(m_1 + m_2)\frac{(m_1\dot{x}_1 + m_2\dot{x}_2)^2}{(m_1+m_2)^2} + \frac{1}{2}\left(\frac{m_1m_2}{m_1+m_2}\right)(\dot{x}_2 - \dot{x}_1)^2 - U(x_2 - x_1)$$

$$= \frac{(m_1\dot{x}_1^2 + 2m_1m_2\dot{x}_1\dot{x}_2 + m_2^2\dot{x}_2^2)}{2(m_1+m_2)} + \frac{m_1m_2}{2(m_1+m_2)}(\dot{x}_2^2 - 2\dot{x}_1\dot{x}_2 + \dot{x}_1^2) - U(x_2 - x_1).$$

The $\dot{x}_1\dot{x}_2$ terms cancel, and after rearranging, we find the result is

$$L = \frac{m_1}{2}\dot{x}_1^2 + \frac{m_2}{2}\dot{x}_2^2 - U(x_2 - x_1).$$

which is what we were trying to show. ■

⋆ **Problem 4.21** Two blocks of equal mass m, connected by a Hooke's-law spring of unstretched length ℓ, are free to move in one dimension. Find the equations of motion of the system, using the relative and center of mass coordinates introduced in the preceding problem.

Solution

Lay out the system along the horizontal, x direction. Let there be some fixed point at the left, which we take as the origin. Then the left-hand mass has position x_1 and the right-hand mass has position x_2. The total kinetic energy is $T = (1/2)m(\dot{x}_1^2 + \dot{x}_2^2)$ and the potential energy is $U = (1/2)k(x_2 - x_1 - \ell)^2$. (Note that $U = 0$ if $x_2 = x_1 + \ell$, as expected.

The total mass is $M = 2m$ and the reduced mass is $\mu = \frac{mm}{m+m} = m^2/M = m/2$. The center of mass coordinate is $X = (1/2)(x_1 + x_2)$ and the relative coordinate is $x' = x_2 - x_1$. So the Lagrangian can be written

$$L = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}\mu\dot{x}'^2 - \frac{1}{2}k(x' - \ell)^2$$

in terms of the generalized coordinates X and x'. Note that X is ignorable, so the corresponding momentum $P = \partial L/\partial \dot{X} = M\dot{X} =$ constant, so $\dot{X}$ = constant, meaning that the CM moves at constant velocity. The other partial derivatives of L are $\partial L/\partial x' = -k(x' - \ell)$ and $\partial L/\partial \dot{x}' = \mu\dot{x}'$ so that Lagrange's equation gives

$$\ddot{x}' + \frac{k}{\ell}(x' - \ell) = 0,$$

which is a simple harmonic oscillator equation about the equilibrium value $x' = \ell$. ■

⋆⋆⋆ **Problem 4.22** A small block of mass m and a weight of mass M are connected by a string of length D. The string has been threaded through a small hole in a tabletop, so the block can slide without friction on the tabletop, while the weight hangs vertically beneath the tabletop. We can let the hole be the origin of coordinates, and use polar coordinates r, θ for the block, where r is the block's distance from the hole, and z for the distance of the weight below the tabletop. (a) Using generalized coordinates r and θ, write down the Lagrangian of the system of block plus weight. (b) Write down a complete set of first integrals of the motion, explaining the physical meaning of each. (c) Show that the first integrals can be combined to give an equation of the form

$$E = \frac{1}{2}(M + m)\dot{r}^2 + U_{\text{eff}}(r)$$

and write out an expression for $U_{\text{eff}}(r)$. (d) Find the radius of a circular orbit of the block in terms of constants of the motion. (e) Now suppose the block executes small oscillations about a circular orbit. What is the frequency of these oscillations? Is the resulting orbit of the block open or closed? That is, does the perturbed orbit of the block continually return to its former position or not?

Solution

(a) The Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}M\dot{r}^2 + Mg(D - r)$$

using the fact that $\dot{z}^2 = \dot{r}^2$. Rearranging,

$$L = \frac{1}{2}(m + M)\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - Mgr,$$

dropping the constant term MgD, since additive constants make no difference to the equations of motion.

(b) Energy and angular momentum are both conserved. That is, $\ell = mr^2\dot{\theta} =$ con and $E = (1/2)m(\dot{r}^2 + r^2\dot{\theta}^2) + (1/2)M\dot{r}^2 + Mgr$ = con, measuring the potential energy of M from its lowest possible position.

(c) Note that $\dot\theta = \ell/mr^2$, so

$$E = \frac{1}{2}(M+m)\dot r^2 + U_{\text{eff}}(r) \quad \text{where} \quad U_{\text{eff}} = \frac{\ell^2}{2mr^2} + Mgr.$$

(d) The derivative $U_{\text{eff}} = 0$ at the equilibrium value of r. That is, $r_{\text{circle}} = (\ell^3)/Mmg)^{1/3}$.

(e) The effective force constant at the circular orbit is

$$k_{\text{eff}} = U''_{\text{eff}} = \frac{3\ell^2}{mr_0^4} = 3\left(\frac{mM^4g^4}{\ell^2}\right)^{1/3} \quad \text{using } r_0 = \left(\frac{\ell^2}{Mmg}\right)^{1/3}.$$

Therefore the oscillation frequency for small oscillations is, from the effective one-dimensional equation of part (c),

$$\omega = \sqrt{\frac{k_{\text{eff}}}{M+m}} = \left(\frac{3}{M+m}\right)^{1/2}\left(\frac{mM^4g^4}{\ell^2}\right)^{1/6}.$$

and so the period of small oscillations in and out with respect to the circular orbit is

$$T_{\text{osc}} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{M+m}{3}}\left(\frac{\ell^2}{mM^4g^4}\right)^{1/6}.$$

The period of the circular orbit itself is

$$T_{\text{orbit}} = \frac{\text{circumference}}{\text{speed}} = \frac{2\pi r}{r\dot\theta} = \frac{2\pi}{\ell/mr^2} = \frac{2\pi mr^2}{\ell} = \frac{2\pi m}{\ell}\left(\frac{\ell^2}{Mmg}\right)^{2/3}.$$

The ratio of the oscillation period to the orbital period of rotation of the (nearly) circular orbit is therefore

$$\frac{T_{\text{osc}}}{T_{\text{orbit}}} = 2\pi\sqrt{\frac{M+m}{3}}\left(\frac{\ell^2}{mM^4g^4}\right)^{1/6}\frac{\ell}{2\pi m}\left(\frac{Mmg}{\ell^2}\right)^{2/3} = \sqrt{\frac{M+m}{3m}}.$$

Note that if this ratio is an integer, then the orbit is closed. For example, if $M = 2m$ the two periods are equal, so the orbit is closed in this case. Similarly, if $M = 11m$, then the orbit oscillates in and out twice in every orbit, so the orbit is closed. Except for a discrete infinity of such very special cases, the ratio of periods is irrational, so the orbits do not close for a continuous infinity of ratios M/m. ■

★★ **Problem 4.23** Example 1.3 of Chapter 1 proposed that mined material on the moon might be projected off the moon's surface by a rotating boom that *slings* the material into space. Assume the boom rotates in a horizontal plane with constant angular velocity ω, and let r, the distance of the payload from the rotation axis at one end of the boom, be the single generalized coordinate. (a) Find the Lagrangian for a bucket of material of mass m that moves along the boom. (b) Find its equation of motion. (c) Solve it for $r(t)$, subject to the initial conditions $r = r_0$ and $\dot r = 0$ at $t = 0$. (d) If the boom has length R, find the radial and tangential components of the bucket's velocity, and its total speed, as it emerges from the end of the boom. (e) Find the power input $P = dE/dt$ into a bucket of mass m as a function of time. Is the power input larger at the beginning or end of the bucket's journey along the boom? (f) Find the torque exerted by the boom on the bucket, as a function of the position r of the bucket on the boom. There would be an equal but opposite torque back on the boom, caused by the bucket, which might break the boom. At what part of the bucket's journey would this torque most likely break the boom? (g) If $R = 100$ meters and $r_0 = 1$ meter, what must be the rotational period of the boom so that buckets will reach the moon's escape speed as they fly off the boom?

Solution

(a) The kinetic energy is $T = (1/2)m(r^2\omega^2 + \dot{r}^2)$ and the potential energy can be taken to be $U = 0$. Therefore $L = T = (1/2)m(r^2\omega^2 + \dot{r}^2)$.
(b) The Lagrange equation reduces to $\ddot{r} - \omega^2 r = 0$. This is the equation of motion.
(c) The general solution of the equation of motion is $r = Ae^{\omega t} + Be^{-\omega t}$ where A and B are arbitrary constants. Using the initial conditions that $r = r_0$ and $\dot{r}_0 = 0$, we find $A = B = r_0/2$. Therefore

$$r = \frac{r_0}{2}(e^{\omega t} + e^{-\omega t}) = r_0 \cosh \omega t.$$

(d) At $r = R$, $R = r_0 \cosh \omega t_{\text{final}}$. Its tangential velocity is $v_{\tan} = R\omega$, and its radial velocity is $v_{\text{rad}} = \dot{r} = r_0\omega \sin \omega t$. However, $\cosh^2 - \sinh^2 = 1$, so $\sinh = \sqrt{\cosh^2 - 1}$. Therefore $v_{\text{rad}} = r_0\omega\sqrt{(R/r_0)^2 - 1}$.

Its total speed is

$$v = \sqrt{v_{\tan}^2 + v_{\text{rad}}^2} = \sqrt{R^2\omega^2 + r_0^2\omega^2((R/r_0)^2 - 1)} = \omega\sqrt{2R^2 - r_0^2},$$

which is the bucket's speed just as it leaves the boom. (e). The kinetic energy of the bucket at any time as it moves along the boom is

$$\begin{aligned} E = \frac{1}{2}mv^2 &= \frac{1}{2}m[v_{\tan}^2 + v_{\text{rad}}^2] = \frac{1}{2}m[(r\omega)^2 + \dot{r}^2] \\ &= \frac{m}{2}[\omega^2 r_0^2 \cosh^2 \omega t + \omega^2 r_0^2 \sinh^2 \omega t] = \frac{m\omega^2 r_0^2}{2}[1 + 2\sinh^2 \omega t], \end{aligned}$$

so the power input from the boom to the bucket is

$$P = \frac{dE}{dt} = 2m\omega^2 r_0^2 \sinh \omega t \cosh \omega t.$$

(f) The angular momentum of the bucket is $\ell = mr^2\omega = m\omega r_0^2 \cosh^2 \omega t$, so the torque exerted on the bucket by the boom is

$$\tau = \frac{d\ell}{dt} = 2m\omega^2 r_0^2 \cosh \omega t \sinh \omega t = m\omega^2 r_0^2 \sinh 2\omega t.$$

This torque keeps growing with time, so the boom is most likely to break just before the bucket leaves the boom.
(g) The moon's escape velocity is ≈ 2.4 km/s. The speed of the bucket coming off the boom is $v = \omega\sqrt{2R^2 - r_0^2}$ where the angular velocity $\omega = 2\pi/T$, in terms of the period of rotation T. Therefore the period must be $T = (2\pi/v)\sqrt{R^2 - r_0^2}$. Substituting in the numbers given for R, r_0, and v, we find $T \approx 0.37$s. ■

⋆⋆⋆ **Problem 4.24** Consider a vertical circular hoop of radius R rotating about its vertical symmetry axis with constant angular velocity Ω. A bead of mass m is threaded onto the hoop, so is free to move along the hoop. Let the angle θ of the bead be measured up from the bottom of the hoop.

(a) Write the Lagrangian in terms of the generalized coordinate θ. Are there any first-integrals of motion?

(b) Show that there are *two* equilibrium angles of the bead for sufficiently small angular velocities Ω, but that there are *four* equilibrium angles if Ω is sufficiently large.

(c) For each of these equilibrium angles, find out whether that position of the bead is *stable* or *unstable*. That is, if the bead is displaced slightly from equilibrium, does it tend to move back toward the equilibrium angle, or does it depart farther and farther from it?

Solution

(a) The kinetic energy is related to both the rotation of the hoop about the vertical axis and to the motion of the bead along the hoop. The result is $T = (1/2)mv^2 = (m/2)(R^2 \sin^2\theta\Omega^2 + R^2\dot{\theta}^2)$. The potential energy can be taken to be $U = -mgR\cos\theta$, which has its minimum value at $\theta = 0$, the bottom of the hoop, and its maximum value at $\theta = \pi$, the top of the hoop, as expected. Therefore the Lagrangian is

$$L = T - U = \frac{m}{2}R^2(\sin^2\theta\,\Omega^2 + \dot{\theta}^2) + mgR\cos\theta.$$

Note that L is not an explicit function of time, so the Hamiltonian is constant, which gives us a first-integral of motion.

(b) The canonical momentum of the bead is $p_\theta = \partial L/\partial\dot{\theta} = mR^2\dot{\theta}$. Therefore the Hamiltonian is

$$H = \dot{\theta}p_\theta - L = mR^2\dot{\theta}^2 - \frac{m}{2}R^2(\sin^2\theta\,\Omega^2 + \dot{\theta}^2) - mgR\cos\theta.$$

Eliminating $\dot{\theta}$ in favor of p_θ, we can write

$$H = \frac{p_\theta^2}{2mR^2} - \frac{m}{2}R^2(\sin^2\theta\,\Omega^2) - mgR\cos\theta.$$

Note that even though energy is not conserved in this problem, the expression for H has the same *form* as a conservation of energy equation $E = T + U$, where the first term is an effective kinetic energy and the other terms, which depend on position θ only, can be taken to be an effective potential. That is,

$$U_{\text{eff}} = -\frac{m}{2}R^2(\sin^2\theta\,\Omega^2) - mgR\cos\theta.$$

Equilibrium points will be located where the derivative $dU_{\text{eff}}/d\theta = 0$, i.e., where

$$U'(\theta) = mR^2(-\Omega^2\cos\theta + g/R)\sin\theta = 0.$$

The factor $\sin\theta = 0$ at $\theta = 0$ and at $\theta = \pi$, the bottom and top of the hoop. Also $U' = 0$ if $\cos\theta = g/(R\Omega^2)$. However, $\cos\theta \le 1$, so this can only be a distinct solution if $g/(R\Omega)^2 < 1$, that is, if the hoop is spinning with angular velocity $\Omega > \sqrt{g/R}$. In summary, there are two equilibrium points if $\Omega \le \sqrt{g/R}$, namely $\theta = 0, \pi$. There is an additional equilibrium point on each side of the bottom of the hoop if $\Omega > \sqrt{g/R}$; they are located where $\cos\theta = g/(R\Omega^2)$, i.e., where $\theta = \pm\cos^{-1}[g/(R\Omega^2)]$.

(c) Now are these equilibrium points stable or unstable? That depends upon whether the second derivative of the effective potential is negative, zero, or positive. If $U'' \leq 0$ at the equilibrium point, the solution is unstable, but if there is a local minimum in U, with $U'' > 0$, the solution is stable, because then the bead oscillates back and forth, remaining near the equilibrium point. The second derivative of U is

$$\begin{aligned} U''(\theta) &= mR^2(-\Omega^2\cos\theta + g/R)\cos\theta + mR^2\Omega^2\sin^2\theta \\ &= mR^2[\,\Omega^2(1-2\cos^2\theta) + (g/R)\cos\theta] \end{aligned}$$

Now by substituting into this equation the angle $\theta = \pi$, we find $U'' < 0$, so the angle $\theta = \pi$ at the top of the hoop is always an unstable equilibrium point. If $\theta = 0$, at the bottom of the hoop, we find that $U'' \leq 0$ if $\Omega \geq \sqrt{g/R}$, corresponding to *instability*, and that $U'' > 0$ if $\Omega < \sqrt{g/R}$, corresponding to *stability.* Finally the two special angles $\theta = \pm\cos^{-1}[g/R\Omega^2]$, which can only be distinct equilibrium points if $\Omega > \sqrt{g/R}$, are stable, because for them $U'' = mR^2\Omega^2[1-(g/R\Omega^2)] > 0$.

In summary, the equilibrium point $\theta = \pi$ is always unstable; the equilibrium point $\theta = 0$ is stable if and only if $\Omega < \sqrt{g/R}$, and the points $\theta = \pm\cos^{-1}[g/(R\Omega^2)]$, one on either side of the bottom of the hoop, and which only exist as discrete equilibrium points if $\Omega > \sqrt{g/R}$, are stable. ■

⋆ **Problem 4.25** In certain situations, it is possible to incorporate frictional effects in a simple way into a Lagrangian problem. As an example, consider the Lagrangian

$$L = e^{\gamma t}\left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2\right) .$$

(a) Find the equation of motion for the system.
(b) Do a coordinate change $s = e^{\gamma t/2}q$. Rewrite the dynamics in terms of s.
(c) How would you describe the system?

Solution

(a) The partial derivatives of L are $\partial L/\partial q = -e^{\gamma t}kq$ and $\partial L/\partial\dot{q} = e^{\gamma t}m\dot{q}$.. Then the Lagrange equation gives $m\ddot{q} + \gamma m\dot{q} + kq = 0$. (b) Let $s = e^{\gamma t/2}q$ (that is, $q = e^{-\gamma t/2}s$.) Then $\dot{q} = e^{-\gamma t/2}(\dot{s} - \frac{\gamma}{2}s)$, etc., so in terms of s, the equation of motion is

$$\ddot{s} + \left(\frac{k}{m} - \frac{\gamma^2}{4}\right)s = 0.$$

which is the simple harmonic oscillator equation if $k/m > \gamma^2/4$.

(c) If the inequality holds, the frequency of oscillation is $\omega = \sqrt{(k/m) - (\gamma^2/4)}$, reduced due to friction γ, and $q = e^{-\gamma t/2}\sin(\omega t + \varphi)$, the behavior expected for a damped harmonic oscillator. ■

⋆⋆ **Problem 4.26** Consider a particle moving in three dimensions with Lagrangian $L = (1/2)m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + a\dot{x} + b$, where a and b are constants. (a) Find the equations of motion and show that the particle moves in a straight line at constant speed, so that it must be a free particle. (b) The result of (a) shows that there must be another reference frame (x', y', z') such that the Lagrangian is just the usual free-particle Lagrangian $L' = (1/2)m(\dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2)$. However, L' may also be allowed an additive constant, which cannot show up in Lagrange's equations. Find the Galilean transformation between (x, y, z) and (x', y', z') and find the velocity of the new primed frame in terms of a and b.

Solution

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + a\dot{x} + b$$

(a)

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} + a \Rightarrow m\ddot{x} = 0, \quad m\ddot{y} = 0 = m\ddot{z}$$

(b) Write

$$\dot{x}' = \dot{x} + V$$

Then we have

$$\frac{1}{2}m(\dot{x}' - V)^2 + a(\dot{x}' - V) = \frac{1}{2}m\dot{x}'^2 - m\dot{x}'V + \frac{1}{2}mV^2 + a\dot{x}' - aV$$

(Note that $-m\dot{x}'V$ and $+a\dot{x}'$ cancel for $a = mV$, i.e. for $V = \frac{a}{m}$.) ■

⋆ **Problem 4.27** Consider a Lagrangian $L' = L + df/dt$, where the Lagrangian is $L = L(q_k, \dot{q}_k, t)$, and the function $f = f(q_k, t)$. (a) Show that $L' = L'(q_k, \dot{q}_k, t)$, so that it depends upon the proper variables. Show that this would not generally be true if f were allowed to depend upon the $\dot{q}_k'$s. (b) Show that L' obeys Lagrange's equations if L does, by substituting L' into Lagrange's equations. Therefore the equations of motion are the same using L' as using L, so the Lagrangian of a particle is not unique. (This problem requires care in taking total and partial derivatives!)

Solution

(a) First, note that

$$\frac{df}{dt}(q_k, t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_k}\dot{q}_k = F[q_k, \dot{q}_k, t]$$

(b) Then

$$L' = L + \frac{df}{dt}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k} \Rightarrow \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k}\left(L' - \frac{df}{dt}\right) = \frac{\partial L'}{\partial \dot{q}_k} - \frac{\partial f}{\partial q_k}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_k}\right) - \frac{\partial f}{\partial t\,\partial q_k} - \frac{\partial^2 f}{\partial q_k^2}\dot{q}_k$$

$$\frac{\partial L}{\partial q_k} = \frac{\partial L'}{\partial q_k} - \frac{\partial f}{\partial q_k\,\partial t} - \frac{\partial^2 f}{\partial q_k^2}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k} \Rightarrow \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_k}\right) = \frac{\partial L'}{\partial q_k}$$

■

⋆ **Problem 4.28** Show that the function L' given in the preceding problem must obey Lagrange's equations if L does, directly from the principle of stationary action. Lagrange's equations do not have to be written down for this proof!

Solution

$$S = \int dtL = \int dt(L' - \frac{df}{dt}) = \int dtL' - \int dt\frac{df}{dt} = \int dtL'$$

■

⋆⋆ **Problem 4.29** Consider the Lagrangian $L' = m\dot{x}\dot{y} - kxy$ for a particle free to move in two dimensions, where x and y are Cartesian coordinates, and m and k are constants. (a) Show that his Lagrangian gives the equations of motion appropriate for a two-dimensional simple harmonic oscillator. Therefore as far as the motion of the particle is concerned, L' is equivalent to $L = (1/2)m(\dot{x}^2 + \dot{x}^2) - (1/2)k(x^2 + y^2)$. (b) Show that L' and L do *not* differ by the total time derivative of any function $f(x,y)$. Therefore L' is not a member of the class of Lagrangians mentioned in the preceding problems, so there are even more Lagrangians describing a particle than suggested before.

Solution

$$L' = m\dot{x}\dot{y} - kxy$$

(a)

$$m\ddot{x} = -kx \qquad m\ddot{y} = -ky$$

$$\text{since} \quad \frac{\partial L}{\partial \dot{x}} = m\dot{y} \quad \text{and} \quad \frac{\partial L}{\partial \dot{y}} = m\dot{x}$$

$$\frac{\partial L}{\partial x} = -ky \quad \text{and} \quad \frac{\partial L}{\partial y} = -kx$$

(b)

$$L - L' = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}k(x^2 + y^2) - m\dot{x}\dot{y} + kxy = \frac{1}{2}m(\dot{x} - \dot{y})^2 - \frac{1}{2}k(x - y)^2$$

$$= \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2 \quad \text{with} \quad q = x - y$$

This is a simple harmonic oscillator equation in the q-variable, and hence is not a total derivative (which would imply no equilibrium of motion). ■

⋆⋆ **Problem 4.30** Consider a Lagrangian that depends on second derivatives of the coordinates

$$L = L(q_k, \dot{q}_k, \ddot{q}_k, t)\ .$$

Through the variational principle, find the resulting differential equations of motion.

Solution

In this case

$$\delta L = \frac{\partial L}{\partial q_k}\delta q_k + \frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k + \frac{\partial L}{\partial \ddot{q}_k}\delta \ddot{q}_k.$$

The differential equation is therefore

$$\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_k}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) + \frac{\partial L}{\partial q_k} = 0.$$

■

★★ **Problem 4.31** A pendulum consists of a plumb bob of mass m on the end of a string that swings back and forth in a plane. The upper end of the string passes through a small hole in the ceiling, and the angle θ of the bob relative to the vertical changes with time as it swings back and forth. The string is pulled upward at constant rate through the hole, so the length R of the pendulum decreases at a constant rate, with $dR/dt = -\alpha$. (a) Find the Lagrangian of the bob, using θ as the generalized coordinate. (b) Find the Hamiltonian H. Is H equal to the energy E? Why or why not? (c) Is either H or E conserved? Why or why not?

Solution

(a) The bob's kinetic energy is $T = (1/2)m(\dot{R}^2 + R^2\dot{\theta}^2)$ and its potential energy is $U = -mgR\cos\theta$. Therefore, using $R = R_0 - \alpha t$, the Lagrangian is

$$L = T - U = \frac{m}{2}(\alpha^2 + (R_0 - \alpha t)^2\dot{\theta}^2) + mg(R_0 - \alpha t)\cos\theta.$$

(b) The Hamiltonian is $H = \dot{\theta}p_\theta - L$ where the canonical momentum is $p_\theta \equiv \partial L/\partial\dot{\theta} = m(R_0 - \alpha t)^2\dot{\theta}$. This gives

$$H = \frac{m}{2}[-\alpha^2 + (R_0 - \alpha t)^2\dot{\theta}^2] - mg(R_0 - \alpha t)\cos\theta$$

$$= -\frac{m\alpha^2}{2} + \frac{p_\theta^2}{2m(R_0 - \alpha t)^2} - mg(R_0 - \alpha t)\cos\theta.$$

where in the second form we have eliminated $\dot{\theta}$ in favor of p_θ. On the other hand, the energy is

$$E = T + U = \frac{m}{2}[\alpha^2 + (R_0 - \alpha t)^2\dot{\theta}^2] - mg(R_0 - \alpha t)\cos\theta.$$

which is *not* the same as H. That is, $E \neq H$, since the signs of the first terms are different. (c). H is *not* conserved, because L is an explicit function of time. E is *not* conserved because work is being done on the plumb bob as it is pulled upward. ■

★ **Problem 4.32** A spherical pendulum consists of a particle of mass m on the end of a string of length R. The position of the particle can be described by a polar angle θ and an azimuthal angle φ. The length of the string decreases at the rate $dR/dt = -f(t)$, where $f(t)$ is a positive function of time. (a) Find the Lagrangian of the particle, using θ and φ as generalized coordinates. (b) Find the Hamiltonian H. Is H equal to the energy? Why of why not? (c) Is either E or H conserved? Why or why not?

Solution

(a) The angle θ is the polar angle, measured up from a vertical line through the point of support to the string, and φ is the azimuthal angle around the vertical line. The Lagrangian is

$$L = T - U = \frac{m}{2}(\dot{R}^2 + R^2\dot{\theta}^2 + R^2 \sin^2\theta\dot{\varphi}^2) + mgR\cos\theta,$$

where $\dot{R} = -f(t)$, so that $R(t) = -\int^t f(t)\,dt$.

(b) The canonical momenta are $p_\theta = \partial L/\partial\dot{\theta} = mR(t)^2\dot{\theta}$ and $p_\varphi = \partial L/\partial\dot{\varphi} = mR(t)^2 \sin^2\theta\dot{\varphi}$. Therefore

$$H = \Sigma q_i \dot{p}_i - L = \frac{m}{2}[-f(t)^2 + R(t)^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2)] + mgR(t)\cos\theta.$$

The energy $E = T + U$ is different from H, since the sign of the very first term is different in each case. That is, H includes the term $-(m/2)f^2(t)$ whereas there is no minus sign in this term for E.

(c) E is not conserved because work is being done on the particle as the string is pulled up. H is not conserved because L is an explicit function of time. ■

⋆⋆ **Problem 4.33** The Hamiltonian of a bead on a parabolic wire turning with constant angular velocity ω is

$$H = \frac{1}{2}m[(1 + 4\alpha^2 r^2)\dot{r}^2 - r^2\omega^2] + mg\alpha r^2,$$

where H is a constant. Reduce the problem to quadrature: That is, find an equation for the time t is terms of an integral over r.

Solution

$$H = \frac{1}{2}m((1 + 4a^2 r^2)\dot{r}^2 - r^2\omega^2) + mg\alpha r^2 \Rightarrow t = \int dt = \int \frac{\sqrt{1 + 4a^2 r^2}}{\frac{2H}{m} + r^2\omega^2 - 2g\alpha r^2}dr$$

■

⋆⋆ **Problem 4.34** A bead of mass m is placed on a vertically-oriented circular hoop of radius R that is forced to rotate with constant angular velocity ω about a vertical axis through its center. (a) Using the polar angle θ measured up from the bottom as the single generalized coordinate, find the kinetic and potential energies of the bead. (Remember that the bead has motion due to the forced rotation of the hoop as well as motion due to changing θ.) (b) Find the bead's equation of motion using Lagrange's equation. (c) Is its energy conserved? Why or why not? (d) Find its Hamiltonian. Is H conserved? Why or why not? Is $E = H$? Why or why not? (e) Find the equilibrium angle θ_0 for the bead as a function of the hoop's angular velocity ω. Sketch a graph of θ_0 versus ω. Notice that there is a "phase transition" at a certain critical velocity ω_{crit}. (f) Find the frequency of small oscillations of the bead about the equilibrium angle θ_0, as a function of ω.

Solution

(a) The kinetic and potential energies are

$$T = \frac{1}{2}m(R^2\dot{\theta}^2 + R^2\sin^2\theta\omega^2) \qquad U = -mgR\cos\theta \quad L = T - U$$

(b) Lagrange's equation of motion is therefore

$$mR^2\ddot{\theta} = mR^2\sin\theta\cos\theta\omega^2 - mgR\sin\theta$$

(c) The energy is

$$E = \frac{1}{2}m(R^2\dot{\theta}^2 + R^2\sin^2\theta\omega^2) - mgR\cos\theta$$

which is not conserved, because the hoop does work on the bead.

(d) Note that

$$E \neq H$$

Here

$$H = \frac{1}{2}mR^2\dot{\theta}^2 - \frac{1}{2}mR^2\sin^2\theta\omega^2 - mgR\cos\theta$$

Note the sign difference with E. The Hamiltonian is conserved, because L is not an explicit function of time.

(e) We have

$$mR^2\sin\theta\cos\theta\omega^2 = mgR\sin\theta \qquad (\ddot{\theta} = 0) \Rightarrow \cos\theta_0 = \frac{g}{\omega^2 R}$$

$\sin\theta = 0$ or $\theta_0 = 0$ is also a solution. Write potential

$$U(\theta) = -\frac{1}{2}mR^2\sin^2\theta\omega^2 - mgR\cos\theta$$

$$U'(\theta) = mR^2\sin\theta\cos\theta\omega^2 + mgR\sin\theta \quad U''(\theta) = mR^2\cos^2\theta\omega^2 + mgR^2\sin^2\theta\omega^2 + mgR\cos\theta$$

$$U''(\theta) = mR^2\omega^2(1 - 2\cos^2\theta) + mgR\cos\theta \qquad U''(\theta_0) = mR^2\omega^2 - \frac{mg^2}{\omega^2}$$

$$U''(0) = mR^2\omega^2(-1) + mgR = mgR(1 - \frac{R\omega^2}{g})$$

$U''(\theta_0) > 0$ is stable for $\omega^2 > \frac{g}{R}$,

$$U''(0) > 0 \quad \sqrt{\frac{g}{R}} > \omega \Rightarrow \omega_{\text{crit}} = \sqrt{\frac{g}{R}}$$

For $\omega > \omega_{\text{crit}}$, stable at $\cos\theta_0 = \frac{g}{\omega^2 R}$, unstable at $\theta_0 = 0$. For $\omega < \omega_{\text{crit}}$, unstable at $\cos\theta_0 = \frac{g}{\omega^2 R}$, stable at $\theta_0 = 0$. ■

★★ **Problem 4.35** A wire is bent into the shape of a quartic function $y = ax^4$ and oriented in a vertical plane, with x horizontal, y vertical, and a a positive constant. A bead of mass m is threaded onto the wire, and the wire is then forced to rotate with constant angular velocity Ω about the y axis. (a) Let x be the generalized coordinate for the bead and find its Lagrangian. (b) Is the bead's energy conserved? Why or why not? (c) Is the bead's angular momentum conserved about the vertical axis? Why or why not? (d) Find the bead's Hamiltonian. Is H conserved? Why or why not? (e) Is $E = H$? Why or why not? (f) Given $\Omega > 0$, are there any equilibrium positions of the bead? (g) If so, Is each stable or unstable? For any stable equilibrium position, find the frequency ω of small oscillations about the equilibrium point, expressed as a multiple of Ω.

Solution

(a) Using x as the generalized coordinate, the bead's potential energy is

$$U = mgy = mgax^4$$

and its kinetic energy is

$$T = \frac{1}{2}m\left[\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right] = \frac{1}{2}m\left[\dot{x}^2 + (4ax^3\dot{x})^2 + \Omega^2 x^2\right],$$

so the Lagrangian is

$$L = T - U = \frac{m}{2}\left[(1 + 16a^2x^6)\dot{x}^2 + \Omega^2 x^2\right] - mgax^4.$$

(b) The bead's energy is not conserved, because work is being done on the bead by the wire.

(c) The bead's angular momentum is not conserved because the forces on the bead due to the wire cause torques to be exerted as well.

(d) The generalized momentum of the bead is

$$p = \frac{\partial L}{\partial \dot{x}} = m(1 + 16a^2x^6)\dot{x}, \text{ so the Hamiltonian is}$$

$$H = p\dot{x} - L = \frac{m}{2}(1+16a^2x^6)\dot{x}^2 - \frac{m\Omega^2}{2}x^2 + mgax^4 = \frac{p^2}{2m(1+16a^2x^6)} + (mgax^4 - \frac{m\Omega^2}{2}x^2)$$

H is conserved because L is not an explicit function of time.

(e) The energy is

$$E = T + U = \frac{m}{2}\left[(1 + 16a^2x^6)\dot{x}^2 + \Omega^2 x^2\right] + mgax^4$$

$$H = \frac{m}{2}(1 + 16a^2x^6)\dot{x}^2 - \frac{1}{2}m\Omega^2x^2 + mgax^4$$

so $E \neq H$. Also, H is conserved but E is not, so they cannot be the same.

(f) We know

$$\frac{dH}{dt} = 0 \Rightarrow (1 + 16a^2x^6)\dot{x}\ddot{x} + 3m(16a^2x^5\dot{x}^3) - m\Omega^2 x\dot{x} + 4mgax^3\dot{x} = 0, \quad \text{so}$$

$$\text{“A”} \quad (1 + 16a^2x^6)\ddot{x} + 48a^2x^5\dot{x}^2 - \Omega^2 x + 4gax^3 = 0.$$

At equilibrium points both $\dot{x}$ and $\ddot{x}$ are zero, so

$$x\Omega^2 = 4gax^3 \Rightarrow x = 0 \text{ and } x = \frac{\Omega}{2\sqrt{ga}}$$

are equilibrium points. Is each *stable* or *unstable*?

First, near $x = 0$, let $x = \delta$ and $\dot{x} = 0$, where δ is small and positive. Then from Equation “A” above, and neglecting $16a^2x^6$ compared with 1, we have

$$\ddot{x} = \Omega^2\delta - 4ga\delta^3 \simeq \Omega^2\delta$$

neglecting the δ^3 term. So if $\delta > 0$, $\ddot{x}$ is also positive, meaning that x will continue to grow. So $x = 0$ is an *unstable* equilibrium point.

Next, near $x = \frac{\Omega}{2\sqrt{ga}}$, let initially $\dot{x} = 0$ and $x = \frac{\Omega}{2\sqrt{ga}}(1+\delta)$ where $|\delta| \ll 1$. Then from Equation "A" above, we have

$$(1 + 16a^2\left[\frac{\Omega^6}{2^6(ga)^3}\right])\frac{\Omega}{2\sqrt{ga}}\ddot{\delta} - \Omega^2(\frac{\Omega}{2\sqrt{ga}})(1+\delta) + 4ga\frac{\Omega^3}{8(ga)^{3/2}}(1+3\delta) \simeq 0,$$

using the binomial expansion

$$(1+\delta)^3 \simeq 1 + 3\delta \text{ for } |\delta| \ll 1.$$

Simplifying we get

$$\left(1 + \frac{\Omega^6}{2g^3a}\right)\ddot{\delta} - \Omega^2\delta + 3\Omega^2\delta = 0$$

$$\text{or} \quad \ddot{\delta} + \frac{2\Omega^2}{1+\frac{\Omega^6}{2g^3a}}\delta = 0.$$

This is a single harmonic oscillator equation, so $x = \frac{\Omega}{2\sqrt{ga}}$ is a stable equilibrium point. The frequency of small oscillations is

$$\omega = \frac{\sqrt{2}\Omega}{\sqrt{1 + \Omega^6/2g^3a}}$$

■

⋆ **Problem 4.36** In Example 4.8 we analyzed the case of a bead on a rotating parabolic wire. The energy of the bead was not conserved, but the Hamiltonian was:

$$H = \frac{1}{2}m(1 + 4\alpha^2r^2)\dot{r}^2 + U_{\text{eff}} = \text{constant},$$

where

$$U_{\text{eff}} = \frac{1}{2}mr^2(2g\alpha^2 - \omega^2).$$

There is an equilibrium point at $r = 0$ which is unstable if $\omega > \omega_0 \equiv \sqrt{2g}\,\alpha$, neutrally stable if $\omega = \omega_0$, and stable if $\omega < \omega_0$. Find the frequency of small oscillations about $r = 0$ if $\omega < \omega_0$.

Solution

$$H = \frac{1}{2}m(1 + 4\alpha^2r^2)\dot{r}^2 + \frac{1}{2}mr^2(2g\alpha^2 - \omega^2)$$

near $r = 0$, to quadratic order in r,

$$H \simeq \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2(2g\alpha^2 - \omega^2)$$

Therefore

$$\Omega^2 = \frac{\frac{1}{2}m(\omega_0^2 - \omega^2)}{\frac{1}{2}m} = \omega_0^2 - \omega^2 \quad \Rightarrow \quad \Omega = \sqrt{\omega_0^2 - \omega^2}.$$

■

★★ **Problem 4.37** A wire bent in the shape of a hyperbolic cosine function $y = a\cosh(x/x_0)$ is supported in a vertical plane, where x and y are the horizontal and vertical coordinates, respectively. and a and x_0 are positive constants. A bead of mass m is threaded onto the wire and is free to slide without friction along it, and is subject to uniform gravity g directed downward. (a) Find Lagrange's equations for the bead using x as the generalized coordinate, and (b) find the frequency of small oscillations of the bead about the lowest point of the wire.

Solution

(a) This is a 2 dimensional motion. We begin with x and y as coordinates,

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2).$$

However, eliminating y, $\dot{y} = -\frac{a}{x_0}\sinh(x/x_0)\dot{x}$ and $U = mgy = mga\cosh(x/x_0)$

The Lagrangian is then

$$L = \frac{1}{2}m\dot{x}^2(1 + \frac{a^2}{x_0^2}\sinh^2(x/x_0)) - mga\cosh(x/x_0)$$

The equation of motion becomes

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) = \frac{\partial L}{\partial x} \Rightarrow \frac{d}{dt}(m\dot{x}(1 + \frac{a^2}{x_0^2}\sinh^2(x/x_0)))$$
$$= m\dot{x}^2\frac{a^2}{x_0^2}\sinh(x/x_0)\cosh(x/x_0) - \frac{mga}{x_0}\sinh(x/x_0)$$

$$\Rightarrow m\ddot{x}(1 + \frac{a^2}{x_0^2}\sinh^2(x/x_0)) + m\dot{x}^2\frac{a^2}{x_0^2}\sinh(x/x_0)\cosh(x/x_0) = -\frac{mga}{x_0}\sinh x/x_0$$

(b) The easiest approach is to return to the Lagrangian and expand it to quadratic order in small x,

$$L = \frac{1}{2}m\dot{x}^2(1 + \frac{a^2}{x_0^2}\sinh^2(x/x_0)) - mga\cosh(x/x_0) \simeq \frac{1}{2}m\dot{x}^2(1 + \frac{a^2}{x_0^2}\frac{x^2}{x_0^2}) - mga(1 + \frac{1}{2}\frac{x^2}{x_0^2})$$

$$\text{where } \sinh(x/x_0) \simeq \frac{x}{x_0} + \theta((\frac{x}{x_0})^3)$$

But the term $\dot{x}^2x^2$ is quartic in small x and must be dropped,

$$L \simeq \frac{1}{2}m\dot{x}^2 - \frac{1}{2}\frac{mga}{x_0^2}x^2 - mga.$$

The angular frequency of small oscillations ω is then given by

$$\omega^2 = \frac{mga/x_0^2}{m} = \frac{ga}{x_0^2} \quad \text{or} \quad \nu = \frac{\omega}{2\pi} = \frac{1}{2\pi}\frac{\sqrt{ga}}{x_0}$$

■

★★ **Problem 4.38** The wire described in the preceding problem is now forced to rotate about its vertical axis of symmetry with constant angular velocity Ω. (a) Find Ω_c, the critical value of Ω for which the equilibrium point at $x = 0$ is no longer a *stable* equilibrium point, and find the values of x for which there is then a stable equilibrium point for the bead. (b) Find the frequency of small oscillations of the bead about each of any new equilibrium points.

Solution

(a) This is three-dimensional motion with cylindrical symmetry. Using cylindrical coordinates, we have

$$T = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2) \quad \text{with} \quad z = a\cosh(\rho/x_0) \quad \text{and} \quad \dot{\varphi} = \Omega = \text{constant.}$$

We also have

$$\dot{z} = -\frac{a}{x_0}\sinh(\rho/x_0)\dot{\rho} \quad \text{and} \quad U = mgz = mga\cosh(\rho/x_0)$$

The Lagrangian becomes

$$L = \frac{1}{2}m\left(1 + \frac{a^2}{x_0^2}\sinh^2(\rho/x_0)\right)\dot{\rho}^2 + \frac{1}{2}m\rho^2\Omega^2 - mga\cosh(\rho/x_0)$$

for the single degree of freedom ρ. Here $x = 0$ corresponds to $\rho = 0$, so we expand the Lagrangian about $\rho = 0$ to quadratic order:

$$\sinh\frac{\rho}{x_0} \simeq \frac{\rho}{x_0} + \mathcal{O}\left(\left(\frac{\rho}{x_0}\right)^3\right) \quad \text{and} \quad \cosh\frac{\rho}{x_0} \simeq 1 + \frac{1}{2}\left(\frac{\rho}{x_0}\right)^2 + \mathcal{O}\left(\left(\frac{\rho}{x_0}\right)^4\right)$$

But the kinetic term is already quadratic in $\dot{\rho}^2$

$$\Rightarrow L \simeq \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\Omega^2 - \frac{mga}{2x_0^2}\rho^2 - mga = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\left(\Omega^2 - \frac{ga}{x_0^2}\right)\rho^2 - mga$$

The instability develops when

$$\Omega^2 > \frac{ga}{x_0^2} \equiv \Omega_c^2 \Rightarrow \Omega_c = \frac{\sqrt{ga}}{x_0}$$

Say there is another $\rho = \rho_0$ for which we have a stable equilibrium. We would then expand L in $(\rho - \rho_0) = \tilde{\rho}$ small. The kinetic term involves $\dot{\rho} = \frac{d}{dt}(\tilde{\rho} + \rho_0) = \dot{\tilde{\rho}}$ squared, so that the $\sinh^2$ factor would not contribute to quadratic order in $\tilde{\rho}$. The frequency of $\tilde{\rho}$ would then depend only on the rest of L. Hence, we can focus on the "effective" potential

$$U_{\text{eff}(\rho)} = -\frac{1}{2}m\rho^2\Omega^2 + mga\cosh(\rho/x_0)$$

to analyze stability. We then write

$$\frac{\partial U_{\text{eff}}}{\partial\rho} = 0 = -m\rho\Omega^2 + \frac{mga}{x_0}\sinh(\rho/x_0) \Rightarrow \frac{\rho_0}{x_0} = \frac{\Omega_c^2}{\Omega^2}\sinh(\rho_c/x_0)$$

(intersections of a line and the sinh function) where ρ_c is the critical point. The only solution is $\rho_c = 0$ as before which is stable for $\Omega < \Omega_c$. But for $\Omega > \Omega_c$, we have another critical ρ_c that solves the equation, in addition to $\rho_c = 0$ which becomes an unstable point. We can check the stability as follows:

$$\alpha \equiv \frac{\partial^2 U_{\text{eff}}}{\partial \rho^2}\bigg|_{\rho=\rho_c} > 0 \quad \text{for stability}$$

$$\Rightarrow \alpha = -m\Omega^2 + \frac{mga}{x_0^2}\cosh(\frac{\rho_c}{x_0}) = m(-\Omega^2 + \Omega^2\cosh(\frac{\rho_c}{x_0}))$$

$$\alpha > 0 \quad \text{if} \quad \Omega^2 < \Omega_c^2\cosh(\frac{\rho_c}{x_0}), \quad \text{but} \quad \Omega^2 = \Omega_c^2\frac{\sinh(\rho_c/x_0)}{(\rho_c/x_0)}$$

We then need

$$\Omega_c^2\frac{\sinh(\rho_c/x_0)}{(\rho_c/x_0)} < \Omega_c^2\cosh(\frac{\rho_c}{x_0}) \quad \text{or} \quad \frac{\sinh(\rho_c/x_0)}{(\rho_c/x_0)} < \cosh(\rho_c/x_0)$$

which is always satisfied. Hence, for $\Omega > \Omega_c$, the new $\rho_c \neq 0$ is stable.

(b) We now need to expand $u_{\text{eff}}(\rho)$ near $\rho = \rho_c$ to quadratic order:

$$U_{\text{eff}}(\rho) \simeq U_{\text{eff}}(\rho_c) + \frac{1}{2}\frac{\partial^2 U_{\text{eff}}(\rho)}{\partial \rho^2}\bigg|_{\rho=\rho_c}(\rho-\rho_c)^2$$

In (a), we computed

$$\alpha = \frac{\partial^2 U_{\text{eff}}(\rho)}{\partial \rho^2}\bigg|_{\rho=\rho_c}$$

We have

$$\alpha = m(\Omega_c^2\cosh(\frac{\rho_c}{x_0}) - \Omega^2)$$

The Lagrangian then becomes

$$L \simeq \frac{1}{2}m\dot{\tilde{\rho}}^2 - \frac{1}{2}m(\Omega_c^2\cosh(\frac{\rho_c}{x_0}) - \Omega^2)\tilde{\rho}^2 \quad \text{where} \quad \tilde{\rho} = \rho - \rho_c$$

The frequency of small oscillations about ρ_c is then

$$\nu = \frac{\Omega}{2\pi} = \frac{1}{2\pi}\sqrt{\Omega_c^2\cosh(\frac{\rho_c}{x_0}) - \Omega^2}$$

■

⋆⋆⋆ **Problem 4.39** One point on a horizontal circular wire C of radius R is attached to a thin, vertical axle which turns at constant angular velocity Ω about the vertical axis, causing C to turn around with it. A bead of mass m moves without friction on C. (a) Show that relative to C the bead oscillates like a pendulum. (b) Find the frequency of small-amplitude oscillations of the bead in terms of any or all of R, Ω, and m.

Solution

(a) Let $\mathbf{a}$ be the vector from the vertical axle to the center of the ring and $\mathbf{R}$ be the vector from the center of the ring to the bead. Therefore $\mathbf{r} = \mathbf{a} + \mathbf{R}$ is the vector from the axle to the bead, so the velocity of the bead is $\dot{\mathbf{r}} = \dot{\mathbf{a}} + \dot{\mathbf{R}}$. The kinetic energy of the bead is therefore

$$T = \frac{1}{2}m\dot{\mathbf{r}}^2$$

Now we can write

$$\text{but} \quad \dot{\mathbf{a}} = R\Omega\hat{\theta} \quad \text{and} \quad \dot{\mathbf{R}} = R\dot{\varphi}\hat{\varphi}$$

where $\hat{\theta}$ and $\hat{\varphi}$ are the angular unit vectors for polar coordinates centered at the origin and at the center of C, respectively. So

$$\Rightarrow T = \frac{1}{2}m(R\Omega\hat{\theta} + R\dot{\varphi}\hat{\varphi})^2 = L \quad \text{since} \quad U = 0$$

$$\Rightarrow L = \frac{1}{2}m(R^2\Omega^2 + R^2\dot{\varphi}^2 + R^2\Omega\dot{\varphi}\hat{\theta}\cdot\hat{\varphi})$$

$$\text{but} \quad \hat{\theta}\cdot\hat{\varphi} = \cos(\varphi - \theta) = \cos(\varphi - \Omega t)$$

$$\Rightarrow L = \frac{1}{2}m(R^2\Omega^2 + R^2\dot{\varphi}^2 + R^2\Omega\dot{\varphi}\cos(\varphi - \Omega t))$$

The equation of motion for one degree of freedom φ is

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\varphi}}) = \frac{\partial L}{\partial \varphi} \Rightarrow \frac{d}{dt}(mR^2\dot{\varphi} + \frac{1}{2}mR^2\Omega\cos(\varphi - \Omega t)) = -\frac{1}{2}mR^2\Omega\dot{\varphi}\sin(\varphi - \Omega t)$$

$$mR^2\ddot{\varphi} - \frac{1}{2}mR^2\Omega\sin(\varphi - \Omega t)(\dot{\varphi} - \Omega) = -\frac{1}{2}mR^2\Omega\dot{\varphi}\sin(\varphi - \Omega t)$$

$$\Rightarrow \left[mR^2\ddot{\varphi} + \frac{1}{2}mR^2\Omega^2\sin(\varphi - \Omega t) = 0\right]$$

$$\text{write} \quad \alpha \equiv \varphi - \Omega t \Rightarrow \dot{\alpha} = \dot{\varphi} - \Omega \Rightarrow \ddot{\alpha} = \ddot{\varphi}$$

$$\Rightarrow \left[mR^2\ddot{\alpha} + \frac{1}{2}mR^2\Omega^2\sin\alpha = 0\right]$$

which looks like a pendulum of length $\ell = R$

$$m\ell^2\ddot{\alpha} = \frac{\ell}{2}mg\sin\alpha \quad \text{and} \quad g = \ell\Omega^2$$

(b) For small angles α, $\sin\alpha \sim \alpha$

$$\Rightarrow mR^2\ddot{\alpha} \cong -\frac{1}{2}mR^2\Omega^2\alpha$$

The angular frequency of oscillations is

$$\omega^2 = \frac{\frac{1}{2}mR^2\Omega^2}{mR^2} = \frac{1}{2}\Omega^2 \quad \text{or} \quad \left[\omega = \frac{\Omega}{\sqrt{2}}\right].$$

■

⋆⋆ **Problem 4.40** A frictionless slide is constructed in the shape of a cycloid. The horizontal coordinate x and vertical coordinate y of the slide are given in parametric form by

$$x = A(\varphi + \sin\varphi) \quad , \quad y = A(1 - \cos\varphi)$$

where A is a constant. Here the y coordinate is positive *upward*. The slide is the portion of the cycloid with $= -\pi \leq \varphi \leq \pi$, with the bottom of the slide corresponding to $\varphi = 0$. There is a uniform gravitational field g in the negative y direction. (a) Find the Lagrangian of a small block of mass m moving along the slide, using φ as the generalized coordinate. (b) The block will oscillate back and forth near the bottom of the slide. Is its motion simple harmonic in the limit of small amplitudes? If not, explain why not; if so, find the angular frequency of oscillation ω in terms of any or all of A, m, and g.

Solution

The parametric equations are

$$x = A(\varphi + \sin\varphi) \qquad y = A(1 - \cos\varphi)$$

for this two-dimensional problem with kinetic and potential energies

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \quad \text{and} \quad U = mgy.$$

We have

$$\dot{x} = A(\dot{\varphi} + \dot{\varphi}\cos\varphi) \quad \text{and} \quad \dot{y} = A\dot{\varphi}\sin\varphi$$

$$\Rightarrow L = T - U = \frac{1}{2}m(A^2\dot{\varphi}^2(1 + \cos\varphi)^2 + A^2\dot{\varphi}^2\sin^2\varphi) - mgA(1 - \cos\varphi)$$

$$= \frac{1}{2}mA^2\dot{\varphi}^2(1 + \cos^2\varphi + 2\cos\varphi + \sin^2\varphi) - mgA(1 - \cos\varphi),$$

so

$$L = mA^2\dot{\varphi}^2(1 + \cos\varphi) - mgA(1 - \cos\varphi)$$

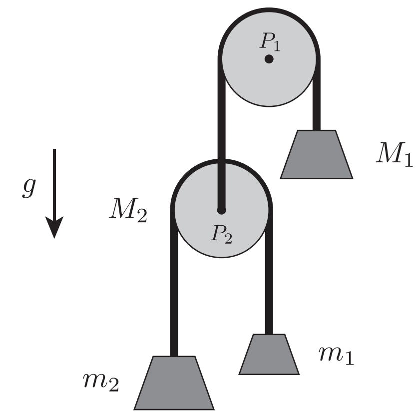
(b) For $\varphi \sim 0$, to quadratic order in φ, L takes the form

$$L \simeq 2mA^2\dot{\varphi}^2 - mg\frac{A\varphi^2}{2}$$

where we have used $\cos\varphi \simeq 1 - \frac{\varphi^2}{2}$ and $(1 + \cos\varphi) \sim 1$ in the kinetic term since $\dot{\varphi}^2$ is already quadratic in small φ. This is indeed a simple harmonic oscillation with angular frequency

$$\omega^2 = \frac{mgA/2}{2mA^2} = \frac{g}{4A} \quad \text{or} \quad \boxed{\omega = \frac{1}{2}\sqrt{\frac{g}{A}}}.$$

■



Prob. 4.41 Three masses and two pulleys (see Problem 4.41).

⋆⋆⋆ **Problem 4.41** A mass M_1 is hung on an unstretchable string A, and the other end of string A is passed over a fixed, frictionless, non-rotating pulley P_1, as shown. This other end of string A is then attached to the center of a second frictionless, non-rotating pulley P_2 of mass M_2, over which is passed a second nonstretchable string B, one end of which is attached to a hanging mass m_1 while the other end is attached to a hanging mass m_2. Let $X_1(t)$ be the length of string A beneath the center of pulley P_1; $X_2(t)$ be the length of string A beneath the center of P_1; $x_1(t)$ be the length of string B beneath the center of pulley P_2; and $x_2(t)$ be the length of string B beneath the center of pulley P_2. There is a uniform gravity g downward. (a) Find the total kinetic energy of the system, in terms of X_1, x_1, and the various masses. (b) Find the total potential energy of the system, measured from the center of fixed pulley P_1. (c) Find the Lagrangian of the system. (d) Find the acceleration of mass M_1 in terms of given quantities.

Solution

(a) Let Y_1, Y_2, y_1, y_2 denote the center of mass locations of M_1, P_2, m_1, m_2, respectively, with the origin at the center of mass of P_1. There are only 2 degrees of freedom given that the lengths of the strings ℓ_A and ℓ_B are constants. The kinetic energy is

$$T = \frac{1}{2}m_1\dot{Y}_1^2 + \frac{1}{2}m_2\dot{Y}_2^2 + \frac{1}{2}M_2\dot{Y}_2^2 + \frac{1}{2}M_1\dot{Y}_1^2$$

But there are 2 constraints that relate these 4 coordinates,

$$\ell_B = -y_2 + Y_2 - y_1 + Y_2 + \frac{C_2}{2} \quad \text{where } C_2 \text{ is the circumference of } P_2$$

$$\ell_A = -Y_2 - Y_1 + \frac{C_1}{2} \quad \text{where } C_1 \text{ is the circumference of } P_1.$$

Taking the time derivative

$$\Rightarrow O = -\dot{y}_2 - \dot{y}_1 + 2\dot{Y}_2 \quad \text{and} \quad 0 = -\dot{Y}_2 - \dot{Y}_1$$

These are 2 constraints that reduce the degrees of freedom from 4 variables to 2. We want to use them to eliminate y_2 and Y_2:

$$\Rightarrow \dot{y}_2 = -\dot{y}_1 - 2\dot{Y}_1 \quad \text{and} \quad \dot{Y}_2 = -\dot{Y}_1$$

We then have

$$\begin{aligned} T &= \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}M_1\dot{Y}_1 + \frac{1}{2}m_2(\dot{y}_1 + 2\dot{Y}_1)^2 + \frac{1}{2}M_2\dot{Y}_1^2 \\ &= \frac{1}{2}(m_1 + m_2)\dot{y}_1^2 + \frac{1}{2}(M_1 + M_2 + 4m_2)\dot{Y}_1^2 + 2m_2\dot{y}_1\dot{Y}_1 \end{aligned}$$

(b) The potential energy is

$$U = m_1gy_1 + m_2gy_2 + M_1gY_1 + M_2gY_2$$

From the constraints in (a), we can solve for y_2 and Y_2,

$$\Rightarrow y_2 = -y_1 - 2Y_1 + \text{constants} \qquad Y_2 = -Y_1 + \text{constants.}$$

The constants are irrelevant as they merely shift the zero of the potential. Setting the origin $y's = 0$ as the zero of the potential, we can drop all these constraints. So

$$U = m_1gy_1 + M_1gY_1 + m_2g(-y_1 - 2Y_1) + M_2g(-Y_1) = (m_1 - m_2)gy_1 + (M_1 - 2m_2 - M_2)gY_1$$

(c) The Lagrangian is then

$$\begin{aligned} L = T - U =& \frac{1}{2}(m_1 + m_2)\dot{y}_1^2 + \frac{1}{2}(M_1 + M_2 + 4m_2)\dot{Y}_1^2 + 2m_2\dot{y}_1\dot{Y}_1 \\ & - (m_1 - m_2)gy_1 - (M_1 - 2m_2 - M_2)gY_1 \end{aligned}$$

(d) The equations of motion for y_1 and Y_1 are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_1}\right) = \frac{\partial L}{\partial y_1} \Rightarrow (m_1 + m_2)\ddot{y}_1 + 2m_2\ddot{Y}_1 = -(m_1 - m_2)g$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Y}_1}\right) = \frac{\partial L}{\partial Y_1} \Rightarrow (M_1 + M_2 + 4m_2)\ddot{Y}_1 + 2m_2\ddot{y}_1 = -(M_1 - 2m_2 - M_2)g$$

We have a system of 2 equations with two unknowns $\ddot{y}_1$ and $\ddot{Y}_1$. Solving for $\ddot{y}_1$, we find

$$\ddot{y}_1 = -\frac{g}{2}\left(\frac{1 + \frac{m_2(M_2 - 3M_1)}{m_1(M_1 + 4m_2 + M_2)}}{1 - \frac{2m_2^2}{m_1(M_1 + 4m_2 + M_2)}}\right)$$

■

⋆ **Problem 4.42** Consider a Lagrangian of the form

$$L = mc^2(1 - \sqrt{1 - v^2/c^2}) - U(x, y, z).$$

Show that the resulting Lagrange equations give Newton's second law $\mathbf{F} = d\mathbf{p}/dt$ for a relativistic particle, if $F^i = -\partial U/\partial x^i$.

Solution

We are given

$$L = mc^2(1 - \sqrt{1 - v^2/c^2} - U(x, y, z))$$

so

$$\frac{\partial L}{\partial \dot{x}} = \frac{-mc^2}{\sqrt{1 - v^2/c^2}}(-\frac{\dot{x}}{c^2}) = \gamma m \dot{x}$$

Similarly,

$$\frac{\partial L}{\partial \dot{y}} = \gamma m \dot{y} \quad \text{and} \quad \frac{\partial L}{\partial \dot{z}} = \gamma m \dot{z}$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) = \frac{d}{dt}(\gamma m \dot{x}) = \frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x}$$

$$\Rightarrow \frac{d}{dt}p_x = -\frac{\partial U}{\partial x} \equiv F^x$$

$$\Rightarrow \frac{d}{dt}p_y = -\frac{\partial U}{\partial y} \equiv F^y$$

$$\Rightarrow \frac{d}{dt}p_z = -\frac{\partial U}{\partial z} \equiv F^z$$

These are indeed Newton's second law for a relativistic particle. ■

5 From Classical to Quantum and Back

5.1 Problems and Solutions

★★ **Problem 5.1** Maxwell's equations for the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ are

$$\nabla \cdot \mathbf{E} = 4\pi\,\rho, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \frac{4\,\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t},$$

where ρ is the charge density, $\mathbf{J}$ is the current density, and ϵ_0 and μ_0 are (respectively) the permittivity and permeability of the vacuum, both constants. Derive the vacuum wave equations for $\mathbf{E}$ and for $\mathbf{B}$. *Hint*: You can use the vector identity $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$ where $\mathbf{A}$ is any vector.

Solution

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \mathbf{B})$$

$$\nabla(\nabla\mathbf{E}) - \nabla^2\mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}(\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}) \Rightarrow \nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \Rightarrow v^2 = c^2$$

$$\nabla \times (\nabla \times \mathbf{B}) = \frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \mathbf{E}) = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{B}$$

$$\Rightarrow \nabla^2\mathbf{B} - \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = 0 \Rightarrow v^2 = c^2$$

■

★ **Problem 5.2** Photons of wavelength 580 nm pass through a double-slit system, where the distance between the slits is $d = 0.16$ nm and the slit width is $a = 0.02$ nm. If the detecting screen is a distance $D = 60$ cm from the slits, what is the linear distance from the central maximum to the first minimum in the diffraction envelope?

Solution

$$\lambda = 580 \text{ nm} \quad d = 0.16 \text{ mm} \quad a = 0.02 \text{ mm} \quad D = 60 \text{ cm}$$

$a < d$, so the first minimum from d

$$d \sin\theta = \frac{\lambda}{2} \quad \text{for min} \Rightarrow \sin\theta \sim \theta \sim 0.0018 \;\Rightarrow \Delta x \cong D\theta \simeq 0.1 \text{ cm}$$

■

⋆⋆ **Problem 5.3** Photons are projected through a double-slit system. (a) What must be the ratio d/a of the slit separation to slit width, so that there will be exactly nine interference maxima within the central diffraction envelope? (b) Is any change observed on the detecting screen if the photon wavelength is changed from λ_0 to $2\lambda_0$? If so, what? (c) If 10^4 photons are counted within the central interference maximum, about how many do you expect will be counted within the last interference maximum that fits within the central diffraction envelope?

Solution

(a) From the information given, we must have the fifth maximum from d must match the first minimum from a. We therefore have

$$d \sin\theta = 5\lambda_0 \qquad a \sin\theta = \lambda_0 \Rightarrow \frac{d}{a} = 5$$

(b)

$$\lambda_0 \to 2\lambda_0$$

There are still nine maxima, but $\Delta\theta$ of the central spot is wider.

(c)

$$d \sin\theta = 4\lambda_0, \text{ so}$$

$$\alpha = \frac{\pi a}{\lambda}\sin\theta = \frac{\pi a}{\lambda_0}\frac{4\lambda_0}{d} = \frac{4\pi}{5}$$

so

$$10^4 \times \frac{\sin^2\alpha}{\alpha^2} \simeq 547.$$

■

⋆⋆ **Problem 5.4** A beam of monoenergetic photons is directed at a triple-slit system, where the distance between adjacent slits is d, and the photon wavelength is $\lambda = d/2$. Find the angles θ from the forward direction for which there are (a) interference maxima (b) interference minima. (c) Then show that some maxima have the same maximum probability as the central peak, but that others have a smaller maximum. Find the ratio of the larger to the smaller maximum probabilities.

Solution

$$\lambda = \frac{d}{2} \Rightarrow k = \frac{2\pi}{\lambda} = \frac{4\pi}{d}$$

$$\text{Amplitude} \quad A \sim 1 + e^{ikd\sin\theta} + e^{2ikd\sin\theta} = 1 + e^{i4\pi\sin\theta} + e^{i8\pi\sin\theta}$$

$$\Rightarrow A * A = (1 + 2\cos(4\pi\sin\theta))^2 \qquad \text{using trig identities for doubled angles}$$

The extrema are located where $\frac{d}{d\theta}|\theta|^2 = 0$

$$\begin{pmatrix} \theta & 0 & \pm\frac{\pi}{2} & \pm\frac{\pi}{6} & \pm\sin^{-1}\frac{1}{6} & \pm\sin^{-1}\frac{1}{4} & \pm\sin^{-1}\frac{1}{3} & \pm\sin^{-1}\frac{2}{3} \\ |A|^2 & 9 & 9 & 9 & 0 & 0 & 1 & 1 \end{pmatrix}$$

The minima are located where $|A| = 0$, and the ratio of maxima is $1:9$

$$\begin{pmatrix} \theta & \pm\sin^{-1}\frac{3}{4} & \pm\sin^{-1}\frac{5}{6} \\ |A|^2 & 0 & 0 \end{pmatrix}$$

■

⋆ **Problem 5.5** A beam of 10 keV photons is directed at a double-slit system and the interference pattern is measured on the detecting plane. The wavelength of these photons is less than the slit separation. Then electrons are accelerated so their (nonrelativistic) kinetic energies are also 10 keV; these electrons are then directed at the same double-slit system, and their interference pattern is measured on the same detecting plane. If the distance between two adjacent photon interference maxima on the detecting screen is y_0, what is the distance between two adjacent *electron* interference maxima? (Note that the mass energy of an electron is 0.5 MeV.)

Solution

Maxima are located at

$$d\sin\theta = m\lambda \Rightarrow d\theta \simeq \lambda \quad \text{for small } \theta$$

$$\Rightarrow \Delta y \sim D\Delta\theta \sim D\theta \sim D\frac{\lambda}{d}$$

$$\text{For photons} \quad E_p = \frac{hc}{\lambda_p}$$

$$\text{For electrons} \quad E_e = \frac{p^2}{2m_e} \quad \text{with} \quad p \sim \frac{h}{\lambda_e} \Rightarrow E_e = \frac{h^2}{2m_e\lambda_e^2}$$

We know

$$E_e = E_p \Rightarrow \lambda_e = \sqrt{\frac{h^2}{2E_pm_e}} \Rightarrow \Delta y_e \simeq \frac{D}{d}\lambda_e = \frac{D}{d}\frac{h}{\sqrt{2E_pm_e}}$$

$$\text{whereas} \quad \Delta y_p \simeq \frac{D}{d}\lambda_p = \frac{D}{d}\frac{hc}{E_p} \Rightarrow \frac{D}{d} = \frac{\Delta y_pE_p}{hc}$$

$$\text{Substitute in} \quad \Delta y_e \Rightarrow \Delta y_c = \Delta y_p\sqrt{\frac{E_p}{2m_ec^2}} = \frac{\Delta y_p}{10}$$

■

⋆ **Problem 5.6** Consider a grating composed of four very narrow slits each separated by a distance d. (a) What is the probability that a photon strikes a detector centered at the central maximum if the probability that a photon is counted by this detector with a single slit open is r? (b) What is the probability that a photon is counted at the first minimum of this four-slit grating if the bottom two slits are closed? (From *Quantum Physics* by John S. Townsend.)

Solution

(a)

$$\text{Amplitude} \quad A = \sqrt{r}(1 + e^{ikd\sin\theta} + e^{2ikd\sin\theta} + e^{3ikd\sin\theta})$$

$$\text{At center} \quad d\sin\theta \to 0 \to |A|^2 = 16r.$$

(b) First min. is when $kd\sin\theta \to \pi/2$ so that the four phases add up to zero (they would form a square with side $\sqrt{r}$). Cover the two slits, we drop two consecutive phase factors, and we get

$$A = \sqrt{r}(1 + e^{ikd\sin\theta}) \to \sqrt{r}(1+i)$$

which implies the probability

$$|A|^2 = 2\,r\,.$$

■

★★ **Problem 5.7** Example 5.2 considered a set of kinked paths about a straight-line path. (a) Using the same set of alternative paths, suppose one considered the sum of phasors about the path with $n = 50$ instead of the sum about the $n = 0$ straight-line path. In particular, if one summed from $n = 25$ to $n = 75$, ± 25 about $n = 50$, how would the sum of phasors differ from the sum for paths about $n = 0$? What physical conclusion can you draw from this? (b) Now returning to the set of kinked paths about the straight-line $n = 0$ path, draw the phasor diagram if the wave number k of the particle were doubled (*i.e.*, if the de Broglie wavelength λ were halved.) What can be concluded about the physical difference between this case and that used in Example 5.2?

Solution

We still have

$$A = e^{ik} e^{i\theta_n s_0} \qquad \theta_n = \frac{2kD_0^2}{s_0} n^2 \quad \text{but now} \quad n = 25 \text{ to } 75.$$

To see the shape of the phasors, write

$$\frac{dy}{dx} = \tan\theta_n = \frac{2kD_0^2}{s_0} n^2 \equiv \alpha n^2$$

$$\text{or} \quad x(t) = \sum_n^t \cos\theta_n \to \int_0^t dn\cos(\alpha n^2)$$

$$y(t) = \sum_n^t \sin\theta_n \to \int_0^t dn\sin(\alpha n^2)$$

This gives a tight spiral in x. The contribution to the amplitude of these modes is ~ 1, or about 10% only. Furthermore, the angles of adjacent phasors monotonically increase, and is significant throughout $n = 25$ to $75 \Rightarrow$, so there is no “extremum” in the sequence.

(b) Note that

$$k \to 2k \Rightarrow \theta_n \to 2\theta_n$$

Each angle doubles, so the interference increases and fewer phasors contribute significantly. Therefore we find a sharper extremum, and we are approaching the classical regime. ■

★★ **Problem 5.8** Example 5.2 considers a particular class of paths near a straight-line path. A different class of paths consists of a set of *parabolas* of the form $y = n\alpha(1 - x/x_0)^2$ fit to the endpoints of the straight line at $(x, y) = (0, 0)$ and $(x, y) = (x_0, 0)$. Here α is a (small) constant, and $n = 0, \pm1, \pm2, \ldots$. Let $\alpha = 0.1x_0$, and draw a careful phasor diagram including enough integers n to see the Cornu spiral behavior and obtain a good estimate of the sum of all these phasors.

Solution

Our apologies! Somewhere along the way the correct parabolas, $y = n\alpha(1-x/x_0)x$, which are zero at both $x = 0$ and $x = x_0$, were transformed into $y = n\alpha(1 - x/x_0)^2$, parabolas which are zero only at $x = x_0$. So now using the correct set of parabolas, we have

$$y = n\alpha(1 - \frac{x}{x_0})x.$$

The distance along the parabolas from $x = 0$ to $x = x_0$ is then

$$s = \int \sqrt{dx^2 + dy^2} = \int_0^{x_0} dx\sqrt{1 + y'^2} \qquad y' = n\alpha(1 - \frac{2x}{x_0})$$

$$s = \int_0^{x_0} dx\sqrt{1 + \alpha^2 n^2 (1 - \frac{2x}{x_0})^2} \simeq \int_0^{x_0} dx(1 + \frac{\alpha^2 n^2}{2}(1 - \frac{2x}{x_0})^2)$$

(for small α & n not too large)

In that case,

$$s \simeq x_0(1 + \frac{n^2\alpha^2}{6}).$$

Therefore

$$e^{iks} = e^{ikx_0} e^{\frac{ikn^2\alpha^2}{6}} \qquad \theta_n = \frac{k\alpha^2}{6} n^2$$

So take $\frac{ka^2}{6} = \frac{\pi}{200} \Rightarrow$, leading to the same spiral pattern as the kinked paths. ■

★ **Problem 5.9** Judge whether or not the following situations are consistent with classical paths. (a) A nitrogen molecule moving with average kinetic energy $\langle 3/2 \rangle\, kT$ at room temperature $T = 300$ K (where k is Boltzmann's constant.) (b) A typical hydrogen atom caught in a trap at temperature $T = 0.1$ K. (c) A typical electron in the center of the sun, at temperature $T = 15 \times 10^6$ K.

Solution

(a) The momentum is

$$p = \sqrt{2ME} \text{ where } E \sim kT \sim 4.2 \times 10^{-21} \text{ J and } M \sim 28 \times 1.7 \times 10^{-27} \text{ kg}$$

$$\Rightarrow p \sim 2 \times 10^{-23} \text{ kg m/s} \qquad \Delta x \sim 1 \text{ m}$$

$$\Rightarrow \Delta x \Delta p \sim 10^{-23} \text{ kg m}^2/\text{s} \ll h \to \text{Classical}$$

(b)

$$p = \sqrt{2ME} \qquad E \sim 1.4 \times 10^{-24} \text{ J} \qquad M \sim 1.7 \times 10^{-27} \text{ kg}$$

$$\Rightarrow p \sim 7.0 \times 10^{26} \text{kg m/s}$$

$$\Delta x \sim \text{trap size} \sim \text{laser wavelength} \sim 10^{-6} \text{m}$$

$$\Rightarrow p\Delta x \sim 10^{-32} > 10^{-34} \text{ kg m}^2/\text{s} \sim h$$

so is approaching the quantum realm.

(c)

$$E \sim 2.1 \times 10^{-26} \text{ J} \qquad M \sim 9.1 \times 10^{-32} \text{kg} \qquad p \sim 2.0 \times 10^{-23} \text{ kg m/s}$$

$$\Delta x \sim \text{radius of the sun} \sim 7 \times 10^{8} \text{ m}$$

$$\Delta p \Delta x \sim 10^{-14} >> h \to \text{Classical}$$

■

★ **Problem 5.10** (a) What condition would have to be met so that the motion of a 135 g baseball would be inconsistent with a classical path? Is this a potentially feasible condition? (b) If we could adjust the value of Planck's constant, how large would it have to be so that the ball in a baseball game would fail to follow classical paths?

Solution

(a)

$$m \simeq 0.135 \text{ kg} \qquad \Delta x \simeq 0.08 \text{ m} \Rightarrow \Delta x \Delta p \sim \Delta x m v \sim h \sim 10^{-34} \text{ J s}$$

$$\Rightarrow v \sim 90 \times 10^{-34} \text{ m/s}$$

Not feasible. For example, this would correspond to random fluctuations in the ball's kinetic energy of order $(1/2)mv^2 \simeq 10^{-66}$ J, while at room temperature the kinetic energy of any air molecules nearby would be of order $kT \simeq 10^{-21}$ J. Therefore any quantum fluctuations would be overwhelmed by thermal fluctuations and so would be undetectable.

(b) Roughly speaking, we would need

$$v \sim 1 \text{ m/s} \Rightarrow h \sim \frac{1}{100} \text{ J s}$$

■

⋆ **Problem 5.11** According to the Heisenberg indeterminacy principle $\Delta x\ \Delta p\ \geq\ \hbar$, the uncertainty in position of a particle multiplied by the uncertainty in its momentum must be greater than Planck's constant divided by 2π. The neutrons in a particular atomic nucleus are confined to be within a nucleus of diameter 2.0 fm (1 fm = 10^{-15} m). Can these neutrons be properly thought of as traveling along classical paths? Explain.

Solution

$$\Delta x \sim 2 \times 10^{-15}\text{m} \qquad \Delta p \sim m_n c \qquad \Delta x \Delta p \sim 4 \times 10^{-34}\text{J s} \sim h \rightarrow \text{Not Classical}$$

■

⋆ **Problem 5.12** Show from the Newtonian equations $x = v_{0x}t$ and $y = y_0 - (1/2)gt^2$ for a particle moving in a uniform gravitational field g, that the shape of its path is a parabola, given by

$$y = y_0 - \frac{mgx^2}{4(E - mgy_0)},$$

the same result we found using the Jacobi principle of least action.

Solution

$$t = \frac{x}{v_0 x} \Rightarrow y = y_0 - \frac{1}{2} g \frac{x^2}{v_{0x}^2}$$

$$E = \frac{1}{2} m v_{0x}^2 + mgy_0 \Rightarrow v_{0x}^2 = \frac{2E}{m} - 2gy_0$$

Substitute this result into the equation above it,

$$y = y_0 - \frac{mgx^2}{4(E - mgy_0)}$$

■

⋆⋆ **Problem 5.13** A particle of mass m can move in two dimensions under the influence of a repulsive spring-like force in the x direction, $F = +kx$. Find the shape of its classical path in the x, y plane using the Jacobi action.

Solution

$$F = +kx = -\frac{\partial U}{\partial x} \Rightarrow U = -\frac{kx^2}{2}$$

$$J = \int_a^b \sqrt{E - U(x)} ds \qquad ds = dx\sqrt{1 + y'^2} \Rightarrow J = \int_a^b \sqrt{E + \frac{kx^2}{2}} \sqrt{1 + y'^2} dx$$

Denote $\sqrt{E + \frac{kx^2}{2}} \sqrt{1 + y'^2}$ as L

$$\frac{\partial L}{\partial y'} = y' \frac{\sqrt{E + kx^2/2}}{\sqrt{1 + y'^2}} \qquad \frac{\partial L}{\partial y} = 0$$

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = \frac{\partial L}{\partial y} = 0 \Rightarrow \frac{y'^2}{1+y'^2} = \frac{C^2}{E+\frac{kx^2}{2}} \qquad C^2 \text{ a constant of integration}$$

$$\Rightarrow \int dy = \pm \int \frac{C}{\sqrt{E - C^2 + \frac{kx^2}{2}}} dx$$

Choose C so that

$$2(C^2 - E)k = 1 \quad \text{or} \quad C = \sqrt{E + \frac{1}{2k}}$$

We find that

$$x = \frac{1}{k} \cosh \left[\sqrt{\frac{k}{2}} \frac{(y - y_0)}{\sqrt{E + \frac{1}{2k}}} \right]$$

with two constants of integration, y_0 and E. This is the shape of a catenary, or hanging chain. ■

⋆⋆ **Problem 5.14** An object of mass m can move in two dimensions in response to the simple harmonic oscillator potential $U = (1/2)kr^2$, where k is the force constant and r is the distance from the origin. Using the Jacobi action, find the shape of the orbits using polar coordinates r and θ; that is, find $r(\theta)$ for the orbit. Show that the shapes are ellipses and circles centered at the origin $r = 0$.

Solution

The Jacobi action is

$$J = \int \sqrt{E - \frac{1}{2}kr^2} \sqrt{1 + r^2\varphi'^2} dr \equiv \int f \, dr.$$

Here $\varphi' = d\varphi/dr$. So

$$\frac{\partial f}{\partial \varphi'} = r^2 \varphi' \frac{\sqrt{E - \frac{1}{2}kr^2}}{\sqrt{1 + r^2\varphi'^2}} \qquad \frac{\partial f}{\partial \varphi} = 0$$

where we write $\varphi(r)$ and $ds = \sqrt{dr^2 + r^2 d\varphi^2}$.

$$\Rightarrow \frac{d}{dr}\left(\frac{\partial f}{\partial \varphi'}\right) = \frac{\partial f}{\partial \varphi} = 0$$

$$\Rightarrow \frac{d\varphi}{dr} = \pm \frac{C}{r^2} \frac{1}{\sqrt{E - \frac{C^2}{r^2} - \frac{1}{2}kr^2}}$$

where C is an integration constant (related to angular momentum). To integrate, use

$$\int \frac{dz/z}{\sqrt{a + bz + cz^2}} = \frac{1}{\sqrt{-a}} \sin^{-1} \left(\frac{bz + 2a}{z\sqrt{b^2 - 4ac}} \right)$$

So

$$\Rightarrow r^2(\varphi) = \frac{2C^2}{E \mp \sqrt{E^2 - 2C^2 k}} \sin(2(\varphi - \varphi_0))$$

This is the equation of an ellipse centered at the origin of the coordinates $r = 0$. A complete analysis of this equation can be found in 7.4.1. ■

⋆ ⋆ ⋆ **Problem 5.15** A comet of mass m moves in two dimensions in response to the central gravitational potential $U = -k/r$, where k is a constant and r is the distance from the Sun. Using the Jacobi action and polar coordinates (r, θ), find the possible shapes of the comet's orbit. Show that these are (a) a parabola, if the energy of the comet is $E = 0$; (b) a hyperbola if $E > 0$; (c) an ellipse or a circle if $E < 0$, where in each case $r = 0$ at one of the foci.

Solution

$$J = \int \sqrt{E + \frac{k}{\rho}} \sqrt{1 + \rho^2 \varphi'^2}\, d\rho \qquad \text{where } ds = \sqrt{d\rho^2 + \rho^2 d\varphi^2} \text{ and } \varphi(\rho)$$

$$\frac{\partial L}{\partial \varphi'} = \rho^2 \varphi' \frac{\sqrt{E + \frac{k}{\rho}}}{\sqrt{1 + \rho^2 \varphi'^2}} \qquad \frac{\partial L}{\partial \varphi} = 0$$

$$\frac{d}{d\rho}\left(\frac{\partial L}{\partial \varphi'}\right) = \frac{\partial L}{\partial \varphi} = 0 \Rightarrow \varphi' = \pm \frac{C}{\rho^2} \frac{1}{\sqrt{E + \frac{k}{\rho} - \frac{C^2}{\rho^2}}}$$

where C is an integration constant related to angular momentum. We can write this as

$$\int d\varphi = \pm C \int \frac{d\rho/\rho}{\sqrt{E\rho^2 + k\rho - C^2}}$$

and using

$$\int \frac{dz/z}{\sqrt{a + bz + cz^2}} = \frac{1}{\sqrt{-a}} \sin^{-1}\left(\frac{bz + 2a}{z\sqrt{b^2 - 4ac}}\right)$$

we find

$$\rho = \frac{2C^2/k}{1 \pm \epsilon \sin(\varphi - \varphi_0)} \quad \text{where} \quad \epsilon = \sqrt{1 + \frac{4EC^2}{k^2}}$$

This describes hyperbolas, parabolas, or ellipses depending on whether $E > 0, E = 0$, or $E < 0$ respectively. A complete analysis of this equation can be found in section 7.4.2. ■

Part II

6 Constraints and Symmetries

6.1 Problems and Solutions

⋆⋆ **Problem 6.1** A particle of mass m slides inside a smooth hemispherical bowl of radius R. Use spherical coordinates r, θ and ϕ to describe the dynamics. (a) Write the Lagrangian in terms of generalized coordinates and solve the dynamics. (b) Repeat the exercise using a Lagrange multiplier. What does the multiplier measure in this case?

Solution

(a) The Lagrangian is

$$L = \frac{1}{2}m(R^2\dot{\theta}^2 + R^2\sin^2\theta\dot{\varphi}^2) - mgR\cos\theta$$

$$\Rightarrow mR\ddot{\theta} = mR^2\sin\theta\cos\theta\dot{\varphi}^2 + mgR\sin\theta \qquad (*)$$

$$\frac{d}{dt}(mR^2\sin^2\theta\dot{\varphi}) = 0 \qquad (**)$$

$$\Rightarrow mR^2\sin^2\theta\dot{\varphi} = \rho = \text{constant} \Rightarrow \dot{\varphi} = \frac{\rho}{mR^2\sin^2\theta}$$

The Hamiltonian H is conserved.

$$\text{constant} = H = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}mR^2\sin^2\theta\dot{\varphi}^2 + mgR\cos\theta$$

Eliminate $\dot{\varphi}$ and integrate,

$$\Rightarrow \int dt = \int \frac{d\theta}{\sqrt{\frac{2H}{mR^2} - \frac{\ell^2}{(mR^2\sin\theta)^2} - \frac{2g}{R}\cos\theta}}$$

(b)

$$dr = 0 \Rightarrow a_{1r} = 1$$

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2) - mgr\cos\theta$$

New r-equation

$$m\ddot{r} = mr\dot{\theta}^2 + mr\sin^2\theta\dot{\varphi}^2 - mg\cos\theta + \lambda_1$$

Equations for θ and φ as in (*) and (**) in (a) with $R \to r$. To find the normal force, we set $r = R$ in the equation above, finding

$$\lambda_1 = \text{normal force} = mR\dot{\theta}^2 + mR\sin^2\theta\dot{\varphi}^2 - mg\cos\theta \ ;$$

we then substitute for the solutions to the equations for θ and φ from part (a). ■

★★ **Problem 6.2** A pendulum consisting of a ball at the end of a rope swings back and forth in a two dimensional vertical plane, with the angle θ between the rope and the vertical evolving in time. The rope is pulled upward at a constant rate so that the length l of the pendulum's arm is decreasing according to $dl/dt = -\alpha \equiv$ constant. (a) Find the Lagrangian for the system with respect to the angle θ. (b) Write the corresponding equations of motion. (c) Repeat parts (a) and (b) using Lagrange multipliers .

Solution

The Lagrangian can be written (a)

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta = \frac{1}{2}m(\alpha^2 + (r_0 - \alpha t)^2\dot{\theta}^2) - mg\alpha t\cos\theta$$

where $r(t) = r_0 - \alpha t$ with r_0 being the initial length. (b)

$$\frac{d}{dt}(m\alpha^2 t^2\dot{\theta}) = mg\alpha t\sin\theta \Rightarrow m\alpha^2 t^2\ddot{\theta} + 2m\alpha^2 t\dot{\theta} - mg\alpha t\sin\theta = 0$$

(c)

$$\frac{dr}{dt} = -\alpha \Rightarrow dr + \alpha dt = 0 \Rightarrow a_{1r} = 1 \quad a_{1t} = \alpha$$

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta \Rightarrow m\ddot{r} = mr\dot{\theta}^2 + mg\cos\theta + \lambda_1$$

Setting $r = r_0 - \alpha t$.

$$\Rightarrow \lambda_1 = +m\alpha t\dot{\theta}^2 - mr_0\dot{\theta}^2 - mg\cos\theta = \text{tension force}$$

$$\frac{d}{dt}(mr^2\dot{\theta}) = -mgr\sin\theta$$

Set $r = r_0 - \alpha t \to$ same as in (a). ■

★★ **Problem 6.3** A particle of mass m slides inside a smooth paraboloid of revolution whose surface is defined by $z = \alpha\rho^2$, where z and ρ are cylindrical coordinates. (a) Write the Lagrangian for the three-dimensional system using the method of Lagrange multipliers. (b) Find the equations of motion.

Solution

(a) Given that $z = \alpha\rho^2$, the Lagrangian is

$$L = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2) - mgz$$

$$dz = 2\alpha\rho d\rho \Rightarrow dz - 2\alpha\rho d\rho = 0$$

$$a_{1z} = 1 \qquad a_{1\rho} = -2\alpha\rho \qquad a_{1\theta} = 0$$

(b)

$$m\ddot{\rho} = m\rho\dot{\theta}^2 - 2\alpha\rho\lambda_1$$

$$\frac{d}{dt}(m\rho^2\dot{\theta}) = 0$$

$$m\ddot{z} = -mg + \lambda_1$$

■

★★ **Problem 6.4** In certain situations, it is possible to incorporate frictional effects without introducing the dissipation function. As an example, consider the Lagrangian

$$L = e^{\gamma t}\left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2\right) .$$

(a) Find the equation of motion for the system. (b) Make the coordinate change $s = e^{\gamma t/2}q$, and rewrite the dynamics in terms of s. (c) How would you describe the system?

Solution

(a) We have

$$\frac{\partial L}{\partial \dot{q}} = e^{\gamma t}m\dot{q} ,$$

and

$$\frac{\partial L}{\partial q} = -e^{\gamma t}kq .$$

We then have

$$\ddot{q} = -\frac{k}{m}q - \gamma\dot{q} .$$

We recognize this as the *damped harmonic oscillator*. The damping term γ is a frictional effect that we successfully incorporated in the Lagrangian through explicit time dependence. This dependence on time of the Lagrangian breaks time translational invariance. Hence, energy is not conserved, which is a signature of frictional effects. In this case, energy would fall off exponentially in time.

(b) Applying the change of variable prescribed, we get

$$\ddot{s} = -\left(\frac{k}{m} - \frac{\gamma^2}{4}\right)s .$$

(c) The change of variable reduced the system to the undamped harmonic oscillator. Notice that the key idea was to absorb the explicit time dependence of the Lagrangian into the new variable. ■

★★ **Problem 6.5** A massive particle moves under the acceleration of gravity and without friction on the surface of an inverted cone of revolution with half angle α. (a) Find the

Lagrangian in polar coordinates. (b) Provide a complete analysis of the trajectory problem. Do not integrate the final orbit equation, but explore circular orbits in detail.

Solution

(a) Put the cone upside down and the mass inside. The Lagrangian without implementing the constraint is

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) - mgz\ .$$

The constraint is the shape of the cone

$$r = \pm z\tan\alpha \to +z\tan\alpha > 0$$

for $z > 0$. Eliminate the z coordinate using this constraint. We have

$$L = \frac{1}{2}m\dot{r}^2\left(1 + \cot^2\alpha\right) + \frac{1}{2}mr^2\dot{\theta}^2 - mgr\cot\alpha\ .$$

(b) Energy is conserved since there is no explicit time dependence in the Lagrangian. From Noether's theorem, we have

$$E = \frac{1}{2}m\dot{r}^2\left(1 + \cot^2\alpha\right) + \frac{1}{2}mr^2\dot{\theta}^2 + mgr\cot\alpha\ .$$

This is a central force problem. Angular momentum is conserved:

$$\left(mr^2\dot{\theta}\right)^{\cdot} = 0$$

due to rotational symmetry. Defining $mr^2\dot{\theta} = l =$ constant, we have

$$E = \frac{1}{2}m\dot{r}^2\left(1 + \cot^2\alpha\right) + \frac{1}{2}\frac{l^2}{mr^2} + mgr\cot\alpha = \frac{1}{2}\mu\dot{r}^2 + U_{eff}(r)$$

where $\mu \equiv m(1 + \cot^2\alpha)$. For circular motion, $\dot{r} = 0$. The initial speed is

$$v_0^2 = r^2\dot{\theta}^2 = \frac{l^2}{m^2r^2}$$

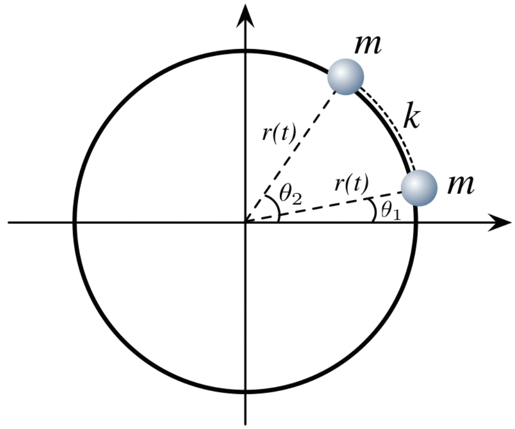
with $r = z_0\tan\alpha \equiv r_0$. For circular motion, we are the the minimum of the potential

$$\frac{\partial U_{eff}}{\partial r} = 0 \Rightarrow r_0^3 = \frac{l^2}{m^2 g\cot\alpha}\ .$$

Eliminating l using $l = mv_0r_0$ from above, we get

$$v_0^2 = gz_0\ .$$

■



⋆⋆⋆

Problem 6.6 A toy model for our expanding universe during the inflationary epoch consists of a circle of radius $r(t) = r_0 e^{\omega t}$ where we are confined on the one-dimensional world that is the circle. To probe the physics, imagine two point masses of identical mass m free to move on this circle without friction, connected by a spring of force-constant k and relaxed length zero, as depicted in the figure.

(a) Write the Lagrangian for the two-particle system in terms of the common radial coordinate r, and the two polar coordinates θ_1 and θ_2. Do *not* implement the radial constraint $r(t) = r_0 e^{\omega t}$ yet.

(b) Using a Lagrange multiplier for the radial constraint, write *four* equations describing the dynamics. In this process, show that

$$a_{1r} = 1 \quad , \quad a_{1t} = -\omega r \, .$$

(c) Consider the coordinate relabeling

$$\alpha \equiv \theta_1 + \theta_2 \quad , \quad \beta \equiv \theta_1 - \theta_2 \, .$$

Show that the equations of motion of part (b) for the two angle variables θ_1 and θ_2 can be rewritten in a decoupled form as

$$\ddot{\alpha} = C_1 \dot{\alpha} \, ;$$

$$\ddot{\beta} = C_2 \dot{\beta} + C_3 \beta \, ;$$

where C_1, C_2 and C_3 are constants that you will need to find.

(d) identify a symmetry transformation $\{\delta t, \delta\alpha, \delta\beta\}$ for this system. Find the associated conserved quantity. What would you call this conserved quantity?

(e) Then find $\alpha(t)$ and $\beta(t)$ using equations 6.186 and 6.187. Use the boundary conditions

$$\alpha(0) = \alpha_0 \quad , \quad \dot{\alpha}(0) = 0 \quad , \quad \beta(0) = 0 \quad , \quad \dot{\beta}(0) = C \quad .$$

What is the effect of the expansion on the dynamics? Note: This conclusion is the same as in the more realistic three-dimensional cosmological scenario.

(f) Find the force on the particles exerted by the expansion of the universe. Write this as a function of $\alpha(t)$, $\beta(t)$, and $r(t)$; and then show that its limiting form for later times in the evolution is given by

$$2\,m\,\omega^2\,r(t)\;.$$

Solution

(a) The Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}_1^2 + \dot{r}^2 + r^2\dot{\theta}_2^2) - \frac{1}{2}kr^2(\theta_1 - \theta_2)^2$$

(b)

$$\dot{r} = \omega r \Rightarrow dr - \omega r dt = 0 \Rightarrow a_{1r} = 1 \qquad a_{1t} = -\omega r$$

$$2m\ddot{r} = mr(\dot{\theta}_1^2 + \dot{\theta}_2^2) - kr(\theta_1 - \theta_2)^2 + \lambda_1 \qquad (\dagger\dagger)$$

$$(mr^2\dot{\theta}_1)^{\cdot} = -kr^2(\theta_1 - \theta_2) \qquad (*)$$

$$(mr^2\dot{\theta}_2)^{\cdot} = +kr^2(\theta_1 - \theta_2) \qquad (**)$$

$$\dot{r} = \omega r$$

(c) Subtract (*) and (**)

$$\Rightarrow (mr^2(\dot{\theta}_1 - \dot{\theta}_2))^{\cdot} = -2kr^2(\theta_1 - \theta_2)$$

Add (*) and (**)

$$(mr^2(\dot{\theta}_1 + \dot{\theta}_2))^{\cdot} = 0 \Rightarrow (r^2\dot{\alpha})^{\cdot} = 0 \qquad (r^2\dot{\beta})^{\cdot} = -2kr^2\beta$$

Expand

$$\ddot{\alpha} + 2m\dot{\alpha} = 0 \Rightarrow C_1 = -2\omega$$

$$\ddot{\beta} + 2k\beta + 2\omega\dot{\beta} = 0 \Rightarrow C_2 = -2\omega \quad C_3 = -2k$$

(d)

$$\{\delta t, \delta\alpha, \delta\beta\} = \{0, C, 0\}$$

$$\Rightarrow r^2\dot{\alpha} = \text{constant} \equiv k \rightarrow \text{angular momentum of center of mass}$$

(e)

$$\dot{\alpha} = \frac{k}{r^2} \qquad r = r_0 e^{\omega t} \qquad r(0) \equiv r_0$$

$$\dot{\alpha}(0) = \frac{k}{r_0^2} = 0 \Rightarrow k = 0 \Rightarrow \alpha(t) = \alpha_0 = \text{constant}$$

$$\ddot{\beta} + 2\omega\dot{\beta} + 2k\beta = 0 \qquad \text{expansion damps oscillations}$$

$$\text{Substitute} \quad \beta = e^{\gamma t}\sin(\omega' t) \quad \text{with choice} \quad \beta(0) = 0$$

$$\Rightarrow \gamma = -\omega \quad \text{and} \quad \omega' = \sqrt{2k - \omega^2}$$

$$\Rightarrow \beta(t) = \beta_0 e^{-\omega t} \sin \sqrt{2k - \omega^2}\, t$$

$$\dot{\beta}(0) = \beta_0 \sqrt{2k - \omega^2} \equiv C \Rightarrow \beta_0 = \frac{C}{\sqrt{2k - \omega^2}}$$

$$\Rightarrow \beta(t) = \frac{C}{\sqrt{2k - \omega^2}} e^{-\omega t} \sin \sqrt{2k - \omega^2}\, t$$

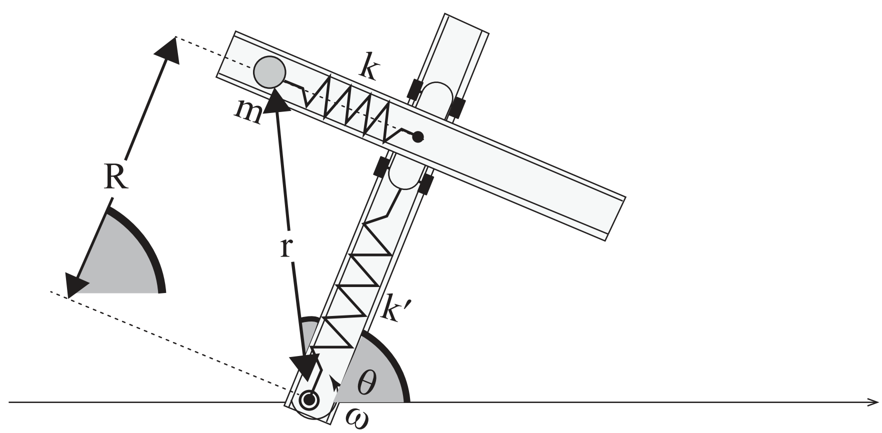
This assumes $2k > \omega^2$. Otherwise, we can have critically damped of overdamped solutions.

(f) From (††) above.

$$\lambda = 2m\omega^2 r - mr\frac{\dot{\beta}^2}{2} + kr\beta^2 \Rightarrow \lambda \to 2m\omega^2 r \quad \text{as} \quad t \to \infty$$

■

⋆⋆⋆ **Problem 6.7**



The figure above shows a mass m connected to a spring of force-constant k along a wooden track. The mass is restricted to move along this track without friction. The entire system is mounted on a toy wagon of zero mass resting on a track along a second frictionless beam. The wagon is connected by a spring of force-constant k' to an axle about which the whole apparatus is spinning with constant angular speed ω. The figure is a top down view, with gravity pointing into the page, and the rest length of each spring is zero. (a) First, write the Lagrangian of the system in terms of the four variables r, θ, R, and Θ shown on the Figure, without implementing any constraints. (b) Identify two constraint equations. Implement the one keeping the two tracks perpendicular to one another into the result of part (a) by eliminating R. Do *not* implement the constaint causing everything to spin at constant angular speed ω. (c) Introducing a Lagrange multiplier for the constraint having to do with the spin, write four differential equations describing the system. (d) Identify the force on the mass m due to the spin of the system, and find all conditions for which this force vanishes.

Solution

(a) Recalling that the mass of the wagon is zero, the kinetic energy is

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

and the potential energy is

$$U = \frac{1}{2}k' R^2 + \frac{1}{2}k(R^2 + r^2 - 2rR\cos(\theta - \Theta))$$

where we used the law of cosines in the second term.

(b) We have

$$r\cos(\theta - \Theta) = R \qquad \text{(track, perpend.)} \qquad \text{and} \quad \Theta = \omega t \qquad \text{(spin)}$$

$$\Rightarrow L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2}(k' - k)r^2\cos^2(\theta - \Theta) - \frac{1}{2}kr^2$$

(c)

$$\dot{\Theta} = \omega \Rightarrow a_{1\Theta} = 1 \qquad \text{(all other zero)}$$

$$\Rightarrow m\ddot{r} = mr\dot{\theta}^2 - kr - (k' - k)r\cos^2(\theta - \Theta)$$

$$(mr^2\dot{\theta})^{\cdot} = (k' - k)r\cos(\theta - \Theta)\sin(\theta - \Theta)$$

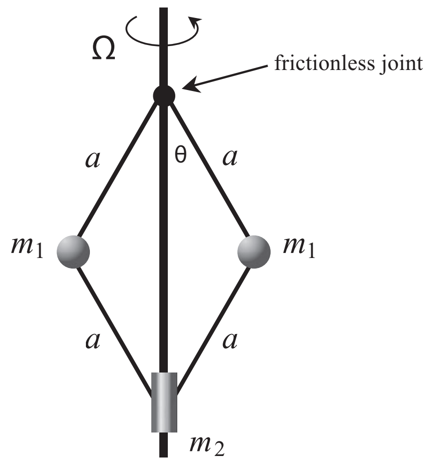
$$\lambda = (k' - k)r\cos(\theta - \Theta)\sin(\theta - \Theta)$$

(d)

$$\lambda = rF_\theta \qquad \text{vanishes when } k = k' \quad \text{or} \quad \theta - \Theta = \frac{n\pi}{2} \quad n \in \mathrm{z}$$

■

⋆⋆⋆ **Problem 6.8** Consider the system shown in the figure below. The particle of mass m_2 moves on a vertical axis without friction and the entire system rotates about this axis with a constant angular speed Ω. The frictionless joint near the top assures that the three masses always lie in the same plane; and the rods of length a are all rigid. (HINT: Use the origin of your coordinate system at the upper frictionless joint.)



(a) Find the equation of motion in terms of the single degree of freedom θ. (b) Using the method of Lagrange multipliers, find the torque on the masses m_1 due to the rotational motion. (c) Find a static solution in θ and identify the corresponding angle in terms of m_1, m_2, g, a, and Ω. Consider some limits/inequalities in your result and comment on whether they make sense. (d) Is the solution in part (c) stable? if so, what is the frequency of small oscillations about the configuration? (*Hint:* Use $\xi = \cos\theta$ and work on the Lagrangian instead of the equation of motion.)

Solution

(a)

$$L = \frac{1}{2}m_1(2)(\dot{r}^2 + \dot{r}^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2) + \frac{1}{2}m_2\dot{z}^2 + 2m_1 gr\cos\theta + m_2 gz$$

Now, identify constraint

$$r = a \qquad z = 2a\cos\theta$$

$$\Rightarrow L = a^2\dot{\theta}^2(m_1 + 2m_2\sin^2\theta) + m_1 a^2\sin^2\theta\dot{\varphi}^2 + 2(m_1 + m_2)ga\cos\theta$$

We still have one more constraint $\dot{\varphi} = \Omega$ which we implement in this part by substituting in L. We have then 1 degree of freedom, θ.

The equation of motion is

$$\ddot{\theta}(m_1 + 2m_2\sin^2\theta) = \sin\theta\cos\theta(m_1\Omega^2 - 2m_2\dot{\theta}^2) - (m_1 + m_2)\frac{g}{a}\sin\theta$$

(b) Now, delay the constraint

$$\dot{\varphi} = \Omega \Rightarrow d\varphi - \Omega dt = 0 \qquad a_{1\varphi} = 1 \quad a_1t = -\Omega \quad a_{1\theta} = 0$$

One Lagrange multiplier needed.

θ eq is unchanged, same as in (a). New φ eq.

$$(2m_1 a^2\sin^2\theta\dot{\varphi})^{\cdot} = \lambda = \tau$$

$$(= \mathbf{F}\frac{\partial \mathbf{r}}{\partial \varphi} = \mathbf{F}r\frac{\partial \hat{\mathbf{r}}}{\partial \varphi} = \mathbf{F}r\sin\theta\hat{\varphi} = F_\varphi a\sin\theta = \text{torque}) = (\mathbf{r}\times\mathbf{F})_z$$

$$\dot{\varphi} = \Omega \qquad \text{(constraint)}$$

(c)

$$\dot{\theta} = \ddot{\theta} = 0$$

From the equation of motion in (a)

$$0 = \sin\theta\cos\theta m_1\Omega^2 - (m_1+m_2)\frac{g}{a}\sin\theta$$

$$\Rightarrow \cos\theta_c = \frac{(m_1+m_2)}{m_1\Omega^2}\frac{g}{a}$$

$$\Omega \to \infty \Rightarrow \theta \to \pi/2$$

$$\text{need } \frac{(m_1+m_2)g}{m_1 a\Omega^2} < 1$$

(d)

$$\text{Write } \xi = \cos\theta \qquad (\xi_c = \cos\theta_c)$$

$$\dot{\xi} = -\sin\theta\dot{\theta} = -\sqrt{1-\xi^2}\dot{\theta} \Rightarrow \dot{\theta} = \frac{-\dot{\xi}}{\sqrt{1-\xi^2}}$$

$$\Rightarrow L = \dot{\xi}^2\frac{(m_1+2m_2(1-\xi^2))a^2}{1-\xi^2} + 2(m_1+m_2)ga\xi + m_1a^2\Omega^2(1-\xi^2)$$

Expand near $\xi = \xi_c$ at quadratic order.

$$L \simeq \frac{M}{2}\dot{\xi}^2 - \frac{k}{2}(\xi-\xi_c)^2 + m_1a^2\Omega^2(\xi_c^2+1)$$

$$M \equiv \frac{2(m_1+2m_2(1-\xi_c^2))a^2}{1-\xi_c^2}$$

$$k \equiv 2m_1a^2\Omega^2$$

Angular frequency:

$$\omega^2 = \frac{k}{M} = \frac{m_1\Omega^2(1-\xi_c^2)}{m_1+2m_2(1-\xi_c^2)}$$

Since $\xi_c < 1, \omega^2 > 0 \rightarrow$stable. ■

⋆⋆ **Problem 6.9** Find the equations of motion for the example in the text of a wheel chasing a moving target using the non-holonomic constraint.

Solution

$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{1}{2}\frac{I}{R^2}(\dot{x}^2+\dot{y}^2) + \frac{1}{2}I''\dot{\theta}^2 = \frac{1}{2}I'(\dot{x}^2+\dot{y}^2) + \frac{1}{2}I''\dot{\theta}^2\,.$$

Consider the last term if θ is dynamical.

$$\text{Constraint} \qquad \sin\theta dx - \cos\theta dy = 0 \qquad a_{1x} = \sin\theta \quad a_{1y} = -\cos\theta$$

$$\Rightarrow I'\ddot{x} = \lambda_1 \sin\theta \qquad (*)$$

$$I'\ddot{y} = -\lambda_1 \cos\theta \qquad (**)$$

$$\sin\theta\dot{x} = \cos\theta\dot{y} \Rightarrow \cos\theta\dot{\theta}\dot{x} + \sin\theta\ddot{x} = -\sin\theta\dot{\theta}\dot{y} + \cos\theta\ddot{y}$$

Substituting from (*) and (**)

$$\Rightarrow \lambda_1 = -I'(\sin\theta\dot{\theta}\dot{y} + \cos\theta\dot{\theta}\dot{x})$$

$$I'\ddot{x} = -I'\sin\theta(\sin\theta\dot{\theta}\dot{y} + \cos\theta\dot{\theta}\dot{x})$$

$$I'\ddot{y} = +I'\cos\theta(\sin\theta\dot{\theta}\dot{y} + \cos\theta\dot{\theta}\dot{x})$$

■

⋆⋆⋆ **Problem 6.10** Consider the example of the wheel from the example in the text, except that now we have no control over the wheel's steering except of course at time zero. We start the wheel at some position on the plane, give it an initial roll ω_0 and an initial spin $\dot{\theta}_0$. Describe the trajectory, assuming that the wheel does not tip away from the vertical.

Solution

Without loss of generality, assume $\theta_0 = 0$. We have $\dot{x}_0 \neq 0$ and $\dot{y}_0 = 0$.

$$R^2\omega_0^2 = \dot{x}_0^2 + \dot{y}_0^2$$

$$\lambda_1 = -I'(\sin\theta\dot{\theta}\dot{y} + \cos\theta\dot{\theta}\dot{x}) = -I'\dot{\theta}_0\dot{x}_0 = \text{constant.}$$

$$\Rightarrow I'\ddot{x} = -I'\dot{\theta}_0\dot{x}_0 \sin\theta = -\alpha\sin\theta \qquad (*)$$

$$I'\ddot{y} = +I'\dot{\theta}_0\dot{x}_0\cos\theta = +\alpha\cos\theta \qquad (**)$$

$$(\alpha \equiv I'\dot{\theta}_0\dot{x}_0)$$

Since θ is not dynamical, we consider the extra $\frac{1}{2}I''\dot{\theta}^2$ term in L

$$I''\ddot{\theta} = 0 \Rightarrow \dot{\theta} = \dot{\theta}_0 \Rightarrow \theta = \dot{\theta}_0 t$$

$$\Rightarrow \dot{y}(t) = \frac{\alpha}{\dot{\theta}_0}\sin(\dot{\theta}_0 t) + C_1' \qquad \text{integrating (**)}$$

$$\Rightarrow y(t) = -\frac{\alpha}{\dot{\theta}_0^2}\cos(\dot{\theta}_0 t) + C_1' t + C_2'$$

$$y(0) = 0 \quad \text{and} \quad \dot{y}(0) = 0 \Rightarrow C_1' = 0 \quad C_2' = \frac{\dot{x}_0}{\dot{\theta}_0}$$

$$\Rightarrow y(t) = \frac{\dot{x}_0}{\dot{\theta}_0}(1 - \cos(\dot{\theta}_0 t))$$

Similarly, we find

$$x(t) = \frac{\dot{x}_0}{\dot{\theta}_0}\sin(\dot{\theta}_0 t) \Rightarrow x^2 + \left(y - \frac{\dot{x}_0}{\dot{\theta}_0}\right)^2 = \frac{\dot{x}_0^2}{\dot{\theta}_0^2}$$

■

⋆⋆ **Problem 6.11** Consider a particle of mass m moving in two dimensions in the x-y plane, constrained to a rail-track whose shape is described by an arbitrary function $y = f(x)$. There is *no gravity* acting on the particle.

(a) Write the Lagrangian in terms of the x degree of freedom only.

(b) Consider some general transformation of the form

$$\delta x = g(x) \quad , \quad \delta t = 0 \; ;$$

where $g(x)$ is an arbitrary function of x. Assuming that this transformation is a symmetry of the system such that $\delta S = 0$, show that it implies the following differential equation relating $f(x)$ and $g(x)$

$$\frac{g'}{g} = -\frac{1}{2(1+f'^2)}\frac{d}{dx}\left(f'^2\right) \; ;$$

where prime stands for derivative with respect to x (*not* t).

(c) Write a general expression for the associated conserved charge in terms of $f(x)$, $g(x)$, and $\dot{x}$.

(d) We will now specify a certain $g(x)$, and try to find the laws of physics obeying the prescribed symmetry; *i.e.*, for given $g(x)$, we want to find the shape of the rail-track $f(x)$. Let

$$g(x) = \frac{g_0}{\sqrt{x}} \; ;$$

where g_0 is a constant. Find the corresponding $f(x)$ such that this $g(x)$ yields a symmetry. Sketch the shape of the rail-track. (*Hint*: $h(x) = f'^2$.)

Solution

(a)

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\dot{x}^2(1 + f'^2)$$

(b)

$$\delta x = g = \Delta x \quad \delta t = 0$$

$$\delta S = \frac{\partial L}{\partial \dot{x}}\Delta\dot{x} + \frac{\partial L}{\partial x}\Delta x = m\dot{x}(1+f'^2)\dot{g} + m\dot{x}^2 f' f'' g = 0$$

$$\Rightarrow (1+f'^2)g' + gf'f'' = 0$$

$$\frac{g'}{g} = -\frac{f'f''}{1+f'^2}$$

(c)

$$Q = \frac{\partial L}{\partial \dot{x}} \Delta x = m\dot{x}(1 + f'^2) g$$

(d)

$$g' = -\frac{1}{2} g_0 x^{-3/2} \qquad \frac{g'}{g} = -\frac{1}{2} \frac{x^{-3/2}}{x^{-1/2}} = -\frac{1}{2x}$$

$$+\frac{1}{x}(1 + f'^2) = \frac{d}{dx}(f'^2)$$

$$\frac{1+h}{x} = h' \Rightarrow \int \frac{dx}{x} = \int \frac{dh}{1+h}$$

$$\ln x = \ln(1+h) \qquad x = x_0(1+h)$$

$$h = \frac{x}{x_0} - 1 = f'^2 \Rightarrow f' = \sqrt{\frac{x}{x_0} - 1}$$

$$\int df = \int dx \sqrt{\frac{x}{x_0} - 1} = \int x_0 du\, u^{1/2} \qquad du = \frac{dx}{x_0}$$

$$f = x_0 \frac{2}{3} u^{3/2} = \frac{2x_0}{3} \left(\frac{x}{x_0} - 1 \right)^{3/2} + f_0$$

■

★★ **Problem 6.12** One of the most important symmetries in Nature is that of *scale invariance*. This symmetry is very common (e.g. arises whenever a substance undergoes a phase transition), fundamental (e.g. it is at the foundation of the concept of *renormalization group* for which a physics Nobel Prize was awarded in 1982), and entertaining (as you will now see in this problem).

Consider the action

$$S = \int dt \sqrt{h} \dot{q}^2$$

of two degrees of freedom $h(t)$ and $q(t)$.

(a) Show that the following transformation (known as a scale transformation or dilatation)

$$\delta q = \alpha q \ , \ \ \delta h = -2\alpha h \ , \ \ \delta t = \alpha t$$

is a symmetry of this system.

(b) Find the resulting constant of motion.

Solution

(a) The transformation is

$$\delta q = \alpha q = \Delta q + \alpha t \dot{q} \, ,$$

$$\delta h = -2\alpha h = \Delta h + \alpha t \dot{h} \, ,$$

and

$$\delta t = \alpha t \, .$$

These allow us to find Δq and Δh. We then have

$$\delta S = \int dt \left[\frac{1}{2\sqrt{h}} \dot{q}^2 \left(-2\alpha h - \alpha t \dot{h} \right) + 2\sqrt{h}\dot{q} \left(\alpha \dot{q} - \alpha \dot{q} - \alpha t \ddot{q} \right) + \frac{d}{dt} \left(\sqrt{h}\dot{q}^2 \alpha t \right) \right] = 0 \, .$$

Hence, we have a symmetry.

(b) We use Noether's theorem to get

$$Q = 2\sqrt{h}\dot{q} \left(\alpha q - \alpha t \dot{q} \right) + \sqrt{h}\dot{q}^2 \alpha t = \sqrt{h} \left(\left(q^2 \right)^{\cdot} - t\dot{q}^2 \right) .$$

■

⋆⋆⋆ **Problem 6.13** A massive particle moves under the acceleration of gravity and without friction on the surface of an inverted cone of revolution with half angle α.

(a) Find the Lagrangian in polar coordinates.

(b) Provide a complete analysis of the trajectory problem. Use Noether charge when useful.

Solution

(a)

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\varphi}^2 + \dot{z}^2) - mgz$$

$$\text{constraint} \qquad \frac{r}{z} = \tan\alpha \Rightarrow \dot{z} = \frac{\dot{r}}{\tan\alpha}$$

$$L = \frac{1}{2} m \left(\dot{r}^2 \left(\frac{1}{\sin^2\alpha} \right) + r^2 \dot{\varphi}^2 \right) - \frac{mgr}{\tan\alpha}$$

(b) Angular momentum constraint

$$(mr^2 \dot{\varphi})^{\cdot} = 0 \Rightarrow \dot{\varphi} = \frac{\ell}{mr^2}$$

Energy conserved

$$E = \frac{1}{2} m \frac{\dot{r}^2}{\sin^2\alpha} + \frac{1}{2} m r^2 \dot{\varphi}^2 + \frac{mgn}{\tan\alpha}$$

$$= \frac{1}{2} m' \dot{r}^2 + U_{\text{eff}}(r)$$

where

$$m' \equiv \frac{m}{\sin^2\alpha}$$

$$U_{\text{eff}}(r) = \frac{\ell^2}{2m' \sin^2\alpha r^2} + m' \cos^2\alpha \, g \, r$$

$$U'_{\text{eff}}(r_0) = 0 \Rightarrow r_0 = \left(\frac{\ell^2}{m'^2 \sin^2\alpha \cos^2\alpha} \right)^{1/3} \qquad \text{circular orbit is stable}$$

We can also integrate for $\dot{r}$

$$\pm\sqrt{\frac{2E}{m'} - U_{\text{eff}}(r)\frac{2}{m'}} = \dot{r}$$

$$\int dt = \int \frac{dr}{\sqrt{\frac{2E}{m'} - U_{\text{eff}}(r)\frac{2}{m'}}}$$

which then gives $r(t)$. ■

★★ **Problem 6.14** For the two body central-force problem with a Newtonian potential, the effective two-dimensional orbit dynamics can be described by the Lagrangian

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + \frac{k}{r} = \frac{1}{2}\mu\left(\dot{x}^2 + \dot{y}^2\right) + \frac{k}{\sqrt{x^2 + y^2}}$$

where $k > 0$ and we have chosen to use Cartesian coordinates.

(a) Show that the equations of motion become

$$\mu\ddot{x} = -k\frac{x}{(x^2 + y^2)^{3/2}} \quad , \quad \mu\ddot{y} = -k\frac{y}{(x^2 + y^2)^{3/2}} \; .$$

(b) Consider the rotation

$$\delta x = \alpha\, y \quad , \quad \delta y = -\alpha\, x \quad , \quad \delta t = 0$$

for small α. Show that this is a symmetry of the action.

Solution

(a) We have

$$\left(\frac{\partial L}{\partial \dot{x}}\right)^{\cdot} = \frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x} \Rightarrow \mu\ddot{x} = -k\frac{x}{r^3}$$

$$\left(\frac{\partial L}{\partial \dot{y}}\right)^{\cdot} = \frac{\partial L}{\partial y} = -\frac{\partial V}{\partial y} \Rightarrow \mu\ddot{y} = -k\frac{y}{r^3}$$

(b)

$$\Delta x = \delta x = \alpha y \qquad \Delta y = \delta y = -\alpha x \qquad \text{since } \delta t = 0$$

$$\delta S = \int dt\left(\frac{\partial L}{\partial x}\Delta x + \frac{\partial L}{\partial y}\Delta y + \frac{\partial L}{\partial \dot{x}}\Delta\dot{x} + \frac{\partial L}{\partial \dot{y}}\Delta\dot{y}\right)$$

$$= \int dt\left(\frac{-kx}{r^3}\alpha y + \frac{ky}{r^3}\alpha x + \mu\dot{x}\alpha\dot{y} - \mu\dot{y}\alpha\dot{x}\right) = 0$$

For small α, $\cos\alpha \sim 1$, $\sin\alpha \sim \alpha$

$$\Rightarrow x' \simeq x + y\alpha \Rightarrow \delta x = x' - x = \alpha y$$

$$y' = -x\alpha + y \Rightarrow \delta y = y' - y = -\alpha x$$

■

★★ **Problem 6.15** In the previous problem show that the conserved Noether charge associated with the symmetry 6.197 is indeed the angular momentum $|\mathbf{r} \times \mu\mathbf{v}|$, which is naturally entirely in the z direction.

Solution

$$Q = \frac{\partial L}{\partial \dot{x}}\Delta x + \frac{\partial L}{\partial \dot{y}}\Delta y = \mu\dot{x}\alpha y + \mu\dot{y}(-\alpha x) = \alpha\mu(\dot{x}y - \dot{y}x)$$

Comparing this to

$$|\mathbf{r} \times \mu\mathbf{v}| = \mu|(x\hat{x} + y\hat{y}) \times (\dot{x}\hat{x} + \dot{y}\hat{y})| = \mu|x\dot{y}\hat{z} - \dot{x}y\hat{z}| = \mu(x\dot{y} - \dot{x}y) \propto Q$$

As expected. ■

★★ **Problem 6.16** The two body central-force problem we have been dealing with in the previous two problems also has another unexpected and amazing symmetry. Consider the transformation

$$\delta x = -\frac{\beta}{2}\mu\, y\, \dot{y} \quad , \quad \delta y = \frac{\beta}{2}\mu\, (2\, x\, \dot{y} - y\, \dot{x}) \quad , \delta t = 0$$

for some constant β. This horrific velocity-dependent transformation is a symmetry if and only if the equations of motion 6.196 are satisfied – unlike other symmetries we've seen where the equations of motion need not be satisfied. It is said that it is an *on-shell symmetry* Show that the change in the Lagrangian resulting from this transformation is given by.

$$\delta L = \beta\, \mu\, k\, \frac{d}{dt}\left(\frac{x}{\sqrt{x^2 + y^2}}\right) .$$

Therefore, it's a total derivative and generates a symmetry under our generalized definition of a symmetry. (*Hint*: You will need to use Eqs. 6.196 to get this result.)

Solution

$$\delta L = -\frac{kx}{r^3}\Delta x - \frac{ky}{r^3}\Delta y + \mu\dot{x}\Delta x + \mu\dot{y}\Delta\dot{y} = \beta\mu\left[-\frac{k}{2}\frac{yx\dot{y}}{r^3} + \frac{k}{2}\frac{y^2\dot{x}}{r^3} - \frac{\dot{x}y}{2}\mu\ddot{y} + \dot{y}x\mu\ddot{y} - \frac{\dot{y}y}{2}\mu\ddot{x}\right]$$

$$= \frac{1}{2}\left(\frac{ky}{r^3} - \mu\ddot{y}\right)(\dot{x}y - \dot{y}x)\,\beta + \frac{\mu\dot{y}}{2}(x\ddot{y} - y\ddot{x})\beta \qquad \text{Use e.o.m to eliminate } \ddot{x}, \ddot{y}$$

$$= \mu\beta\frac{ky}{r^3}(\dot{x}y - \dot{y}x) = \mu\beta\frac{d}{dt}\left(\frac{kx}{r}\right)$$

As needed. ■

★ **Problem 6.17** In the previous problem show that the conserved charge associated with the symmetry is

$$Q \propto \mu^2 x\dot{y}^2 - \mu^2 y\dot{x}\dot{y} - \mu k\frac{x}{\sqrt{x^2 + y^2}} .$$

Solution

$$Q = \frac{\partial L}{\partial \dot{x}} \Delta x + \frac{\partial L}{\partial \dot{y}} \Delta x = \mu^2 \beta (\dot{y}^2 x - \dot{x}\dot{y}y)$$

$$\Rightarrow A_x = Q - K \quad \text{where} \quad K = \mu\beta \frac{kx}{r}$$

Noether charge is then

$$\propto \mu^2(\dot{y}^2 x - \dot{x}\dot{y}y) - \mu k \frac{x}{r}$$

■

★★ **Problem 6.18** The hidden symmetry of the previous few problems is part of a two-fold transformation – one of which is given by (6.198) and another similar one that we have not shown; together, they result in the conservation of a vector known as the Laplace-Runge-Lenz vector

$$\mathbf{A} = \mu\mathbf{v} \times (\mathbf{r} \times \mu\mathbf{v}) - \mu k \frac{\mathbf{r}}{r}$$

Show that Eq. 6.200 is the x-component of this more general vector quantity. (HINT: You may find it useful to use the identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}$.)

Solution

$$\mathbf{A} = \mu\mathbf{v} \times (\mathbf{r} \times \mu\mathbf{v}) - \mu k \frac{\mathbf{r}}{r}$$

$$A_x = \hat{x} \cdot (\mu\mathbf{v} \times (\mathbf{r} \times \mu\mathbf{v}) - \mu k \frac{x}{r} \qquad \hat{x}(\mu\mathbf{v} \times (\mathbf{r} \times \mu\mathbf{v})) = \hat{x}\mathbf{r}(\mu^2 v^2) - \mu\mathbf{v}\hat{x}(\mu\mathbf{v}\mathbf{r})$$

$$A_x = x\mu^2(\dot{x}^2 + \dot{y}^2) - \mu^2\dot{x}(\dot{x}x + \dot{y}y) - \mu\frac{kx}{r} = \mu^2\dot{x}^2 x + \mu^2\dot{y}^2 x - \mu^2\dot{x}^2 x - \mu^2\dot{x}\dot{y}y - \mu\frac{kx}{r}$$

$$= \mu^2(\dot{y}^2 x - \dot{x}\dot{y}y) - \mu\frac{kx}{r}$$

as in (e). For the record, the other component is conserved because of the symmetry

$$\delta x = m\frac{\beta}{2}(2\dot{x}y - x\dot{y})$$

$$\delta y = -m\frac{\beta}{2}x\dot{x}$$

■

★★ **Problem 6.19** Show using (6.201) that $d\mathbf{A}/dt = 0$. Draw an elliptical orbit in the x-y plane and show on it the Laplace-Runge-Lenz vector $\mathbf{A}$. The existence of this conserved vector quantity is the reason why one can smoothly deform ellipses into a circle without changing the energy of the system. Mathematically, this additional hidden symmetry implies that the Newtonian problem is equivalent to a free particle on a three-dimensional sphere embedded in an abstract four-dimensional world. It is believed that this is a mathematical accident; no physical significance of this fourth dimension has yet been identified...

Solution

$$\mathbf{A} = \mu\mathbf{v} \times (\mathbf{r} \times \mu\mathbf{v}) - \mu k \frac{\mathbf{r}}{r}$$

$$\frac{d\mathbf{A}}{dt} = \mu\dot{\mathbf{v}} \times (\mathbf{r} \times \mu\mathbf{v}) + \mu\mathbf{v} \times (\mathbf{v} \times \mu\mathbf{v}) + \mu\mathbf{v} \times (\mathbf{r} \times \mu\dot{\mathbf{v}}) - \mu k \frac{\mathbf{v}}{r} + \frac{\mu k}{r^2}\mathbf{r}\dot{r}$$

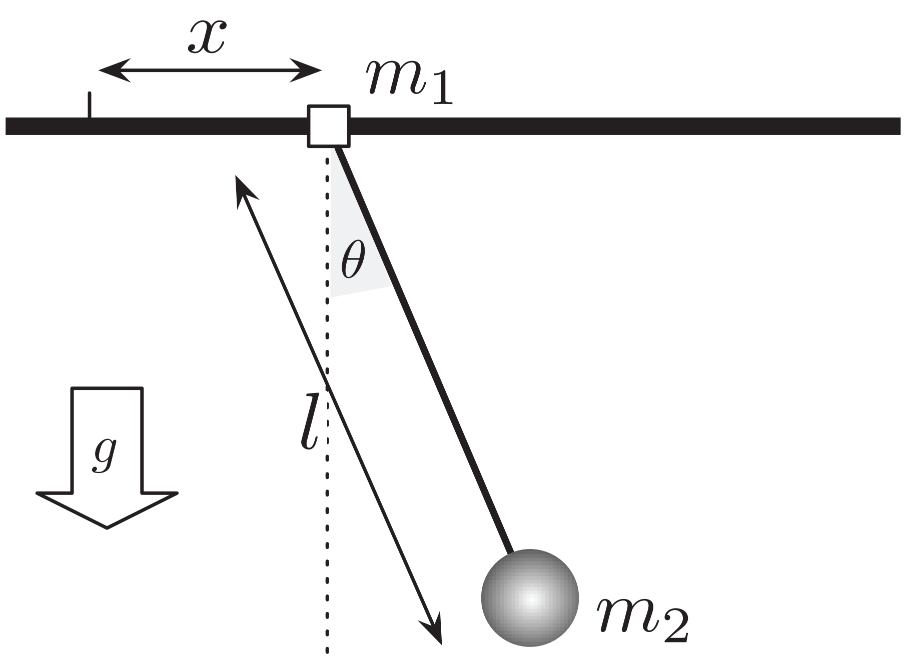
Using

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

$$\frac{d\mathbf{A}}{dt} = \mu^2\mathbf{r}(\dot{\mathbf{v}}.\mathbf{v}) - \mu^2\mathbf{v}(\dot{\mathbf{v}}.\mathbf{r}) - \mu\frac{k\mathbf{v}}{r} + \frac{\mu k}{r^2}\dot{r}\mathbf{r} - \frac{\mu k}{r^3}\mathbf{r}(\mathbf{r} \cdot \mathbf{v}) + \frac{kr^2}{r^3}\mu\mathbf{v} = 0$$

using $\mu\dot{\mathbf{v}} = -\frac{k}{r^3}\mathbf{r}$, since $\mathbf{v} \cdot \mathbf{r} = \dot{r}r$. ■

★★ **Problem 6.20** Consider a simple pendulum of mass m_2 and arm length l having its pivot on a point of support of mass m_1 that is free to move horizontally on a frictionless rail.



Sliding pendulum.

(a) Find the Lagrangian of the system in terms of the two degrees of freedom x and θ shown on the figure. Do *NOT* assume small displacements. (b) Identify two symmetries and the two corresponding conservation laws. Write two first-order differential equations that describe the dynamics of the two degrees of freedom x and θ. Correspondingly, write a single nasty integral for $\theta(t)$.

Solution

(a) The Lagrangian is

$$L = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(\dot{X}^2 + \dot{Y}^2) - m_2 gY$$

where X and Y are the coordinates of m_2 with respect to the origin shown in the figure. From geometry, we have

$$X = x + \ell\sin\theta \qquad Y = -\ell\cos\theta$$

$$\Rightarrow \dot{X} = \dot{x} + \ell\cos\theta\dot{\theta} \qquad \dot{Y} = +\ell\sin\theta\dot{\theta}$$

$$L = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(\dot{x}^2 + \ell^2\dot{\theta}^2 + 2\ell\cos\theta\dot{\theta}\dot{x}) + m_2 g\ell\cos\theta$$

(b) Translation in x and in time

$$\Rightarrow p_x = \frac{\partial L}{\partial\dot{x}} = \text{constant} = (m_1 + m_2)\dot{x} + m_2\ell\cos\theta\dot{\theta}$$

$$H = \frac{\partial L}{\partial\dot{x}}\dot{x} + \frac{\partial L}{\partial\dot{\theta}}\dot{\theta} - L = (m_1 + m_2)\dot{x}^2 + m_2\ell\cos\theta\dot{\theta}\dot{x} + m_2\ell^2\dot{\theta}^2 + m_2\ell\cos\theta\dot{x}\dot{\theta} - L$$

$$= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2\ell^2\dot{\theta}^2 + m_2\ell\cos\theta\dot{x}\dot{\theta} - m_2 g\ell\cos\theta = \text{constant}$$

$$\Rightarrow p_x = (m_1 + m_2)\dot{x} + m_2\ell\cos\theta\dot{\theta}$$

$$H = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2\ell^2\dot{\theta}^2 + m_2\ell\cos\theta\dot{x}\dot{\theta} - m_2 g\ell\cos\theta$$

$$\Rightarrow \frac{p_x - (m_1 + m_2)\dot{x}}{m_2\ell\cos\theta} = \dot{\theta} \rightarrow \text{Substitute in } H \text{ and solve for } \dot{x} \rightarrow \text{integrate.}$$

■

7 Gravitation

7.1 Problems and Solutions

⋆ **Problem 7.1** Two satellites of equal mass are each in a circular orbit around the earth. The orbit of satellite A has radius r_A, and the orbit of satellite B has radius $r_B = 2r_A$. Find the ratio of their (a) speeds (b) periods (c) kinetic energies (d) potential energies (e) total energies.

Solution

Newton's second law for a mass m in a circular orbit is $-GMm/r^2 = -mv^2/r$, where M is the mass of the earth. Therefore

$$\frac{GM}{r^2} = \frac{v^2}{r} = r\left(\frac{2\pi}{T}\right)^2.$$

where T is the orbital period, using $v = 2\pi r/T$. Therefore

$$\frac{GM/r_B^2}{GM/r_A^2} = \frac{v_B^2/r_B}{v_A^2/r_A} = \frac{r_B/T_B^2}{r_A/T_A^2}.$$

That is,

$$\left(\frac{r_A}{r_B}\right)^2 = \frac{r_A}{r_B}\left(\frac{v_B}{v_A}\right)^2 = \frac{r_B}{r_A}\left(\frac{T_A}{T_B}\right)^2.$$

Therefore with $r_B = 2r_A$ we find that (a) $v_B/v_A = 1/\sqrt{2}$ (b) $T_B/T_A = 2\sqrt{2}$. Then also (c)

$$\frac{K.E._B}{K.E._A} = \frac{v_B^2}{v_A^2} = \frac{1}{2}.$$

and (d)

$$\frac{U_B}{U_A} = \frac{-GMm/r_A}{-GMm/r_B} = \frac{r_B}{r_A} = 2.$$

and finally (e)

$$\frac{GM/r_B - 2GM/r_B}{GM/r_A - 2GM/r_a} = \frac{-GM/r_B}{-GM/r_A} = \frac{r_A}{r_B} = \frac{1}{2}.$$

■

⋆ **Problem 7.2** Halley's comet passes through earth's orbit every 76 years. Make a close estimate of the maximum distance Halley's comet gets from the sun.

Solution

From Kepler's third law,

$$\left(\frac{T_{\text{Halley}}}{T_{\text{earth}}}\right)^2 = \left(\frac{a_{\text{Halley}}}{a_{\text{earth}}}\right)^3.$$

where T is the period and a is the semimajor axis of the orbit. Therefore

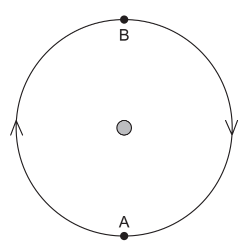
$$a_{\text{Halley}} = a_{\text{earth}}\left(\frac{T_{\text{Halley}}}{T_{\text{earth}}}\right)^{2/3} = a_{\text{earth}}(76)^{2/3} \cong 18\, a_{\text{earth}}.$$

so Halley's orbit is quite eccentric. We know that at perihelion its orbit gets within the orbit of the earth, so the maximum distance the orbit gets from the sun is nearly $2a_{\text{Halley}} \cong (35 \text{ or } 36)a_{\text{earth}}$. The semimajor axis of earth's orbit is $a_{\text{earth}} \cong 150 \times 10^6$ km, so the maximum distance Halley's comet gets from the sun is $\cong 5.4 \times 10^9$ km. This is larger than the orbital radius of Neptune. (The actual maximum distance is 5.2×10^9 km.) ■

⋆⋆ **Problem 7.3** Two astronauts are in the same circular orbit of radius R around the earth, 180° apart. Astronaut A has two cheese sandwiches, while Astronaut B has none. How can A throw a cheese sandwich to B? In terms of the astronaut's period of rotation about the earth, how long does it take the sandwich to arrive at B? What is the semi-major axis of the sandwich's orbit? (There are many solutions to this problem, assuming that A can throw the sandwich with arbitrary speeds.)

Solution

The initial common circular orbit is shown below.



If A throws a sandwich directly forward (i.e., toward the left in the figure) the sandwich will enter an elliptical orbit with perigee at A and apogee beyond that radius. This orbit will have a larger semimajor axis and longer period. We want B to be at A's initial position when the sandwich arrives, so

$$T_{sandwich} = T_{astronauts} \times (3/2,\ 5/2,\ 7/2, ...) = (n + 1/2)T_{astronauts} \quad n = 1, 2, 3, ...$$

Now from Kepler's third law,

$$\left(\frac{a_{sandwich}}{a_{astronauts}}\right)^{3/2} = \frac{T_{sandwich}}{T_{astronaut}} = n + 1/2$$

so

$$a_{sandwich} = (n + 1/2)^{2/3} R_{astronaut} \quad n = 1, 2, 3, \ldots$$

In addition to these solutions, A could for example throw directly *backward* to her own motion. With the right velocity, this could put the sandwich into an elliptical orbit with period $T_{astronaut}/2$, with $a_{sandwich} = R_{astronaut}/2^{2/3}$. Or if she throws backward harder, she could place the sandwich into a reverse circular orbit, so B would catch the sandwich in time $T_{astronaut}/4$, where the catch would be one quarter of the way around the circle. ■

⋆ **Problem 7.4** Suppose that the gravitational force exerted by the sun on the planets were inverse r-squared, but not proportional to the planet masses. Would Kepler's third law still be valid in this case?

Solution

No, not in general. For circular orbits $F = -k/r^2 = ma = -mr\omega^2$, so the orbital period T would obey $T^2 = (4\pi^2 m/k)r^3$. Therefore Kepler's third law would be valid in general only if k were proportional to m. ■

⋆ **Problem 7.5** Planets in a hypothetical solar system all move in circular orbits, and the ratio of the periods of any two orbits is equal to the ratio of their orbital radii *squared*. How does the central force depend on the distance from this sun?

Solution

Suppose that the central force has the form of an attractive power law: That is, $F = -kr^n$ for some values of the constants k and n. Then $-kr^n = ma = -mr\omega^2 = -mr(2\pi/T)^2$. That gives $T^2 = (4\pi^2 m/k)r^{1-n}$. That is, for any two orbits

$$\frac{T_1}{T_2} = \left(\frac{r_1}{r_2}\right)^{1-n} = \left(\frac{r_1}{r_2}\right)^2$$

if we assume that the constant k is proportional to the planet's mass m. Otherwise it won't work in general. From the equation it is clear that $n = -1$. So the force must have the form $F \propto -m/r$, an inverse first-power radial force. ■

⋆ **Problem 7.6** An astronaut is marooned in a powerless spaceship in circular orbit around the asteroid *Vesta*. The astronaut reasons that puncturing a small hole through the spaceship's outer surface into an internal water tank will lead to a jet action of escaping water vapor expanding into space. Which way should the jet be aimed so the spacecraft will most likely reach the surface of Vesta? Assume that the initial orbital radius is much larger than the radius of Vesta. (In Isaac Asimov's first published story *Marooned off Vesta*, the jet was oriented differently, but the ship reached the surface anyway.)

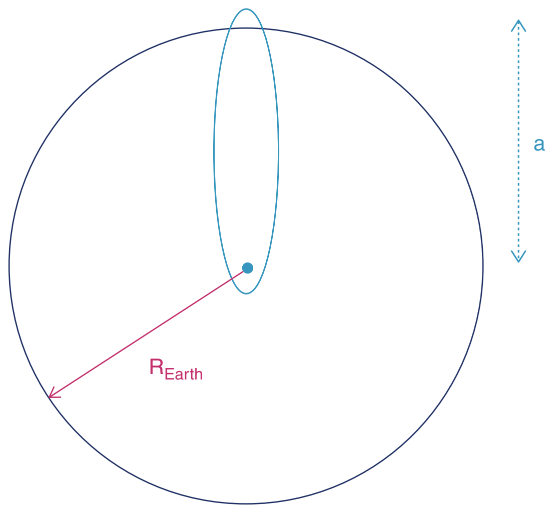
Solution

Under the circumstances, the spaceship cannot reach Vesta's surface unless the angular momentum of the ship is greatly reduced. So the best way to aim the jet is *forward*, in the direction of the ship's motion. This removes the maximum amount of angular momentum, allowing the ship to fall to Vesta. ■

★ **Problem 7.7** A thrown baseball travels in a small piece of an elliptical orbit before it strikes the ground. What is the semi-major axis of the ellipse? (Neglect air resistance.)

Solution

The figure below shows that the elliptical orbit of the ball is extremely eccentric. One of its foci is at the center of the earth, and this focus is very close to the orbit's perigee. That is, the major axis of the ellipse is only slightly greater than the radius of the earth. Therefore the semi-major axis is essentially $a = R_{Earth}/2$.



■

★ **Problem 7.8** Assume that the period of elliptical orbits around the sun depends only upon G, M (the sun's mass), and a, the semi-major axis of the orbit. Prove Kepler's third law using dimensional arguments alone.

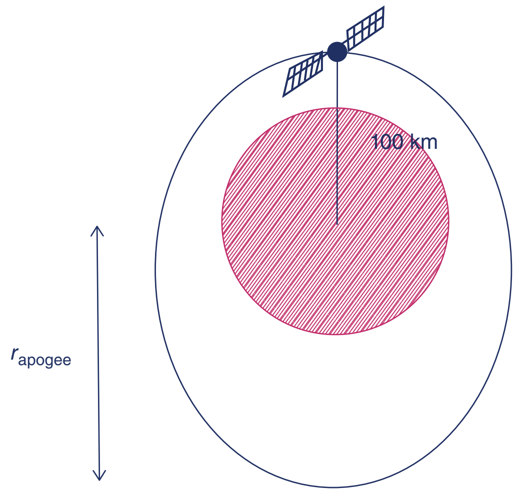
Solution

We assume that the period $T = T(G, M, a)$ where these quantities have the folllowing dimensions in terms of time τ, mass M, and length L: $[T] = \tau$, $[M] = M$, $[a] = L$, and $[G] = [Fr^2/M^2] = L^3/MT^2$. Therefore to find a formula for a quantity whose only dimension is time, to eliminate M we need the product GM, with $[GM] = L^3/T^2$, and then to eliminate L we need GM/a^3, with $[GM/a^3] = 1/\tau^2$. Then to get dimensions of time, $T \propto \sqrt{a^3/GM} \propto a^{3/2}$, so $T^2 \propto a^3$. That is, we have recovered Kepler's third law using dimensional analysis alone. ■

★ **Problem 7.9** A spy satellite designed to peer closely at a particular house every day at noon has a 24-hour period, and a perigee of 100 km directly above the house. What is the altitude of the satellite at apogee? (Earth's radius is 6400 km.)

Solution

A sketch of the orbit is shown.



The semimajor axis of the elliptical orbit is

$$a = \frac{1}{2}(r_{\text{perigee}} + r_{\text{apogee}}) = \left(\frac{\sqrt{GM}T}{2\pi}\right)^{2/3}$$

so

$$r_{\text{apogee}} = 2\left[\frac{(GM)^{1/3}T^{2/3}}{(2\pi)^{2/3}}\right] - 6500 \text{ km}$$

(Note that $r_{\text{earth}} \cong 6400$ km.). Substituting in the numbers, $G = 6.67 \times 10^{-11}$ N m^2/kg^2, $M = 5.97 \times 10^{24}$ kg, we find that

$$r_{\text{apogee}} \cong 78,000 \text{ km}.$$

Subtracting the radius of the earth, the maximum altitude of the satellite is $\cong$ 71,600 km. ■

★★ **Problem 7.10** Show that the kinetic energy

$$K.E = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2$$

of a system of two particles can be written in terms of their center-of-mass velocity $\dot{\mathbf{R}}_{\text{cm}}$ and relative velocity $\dot{\mathbf{r}}$ as

$$K.E. = \frac{1}{2}M\dot{\mathbf{R}}_{\text{cm}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2$$

where $M = m_1 + m_2$ is the total mass and $\mu = m_1m_2/M$ is the reduced mass of the system.

Solution

Here

$$\mathbf{R}_{CM} = \frac{m_1\mathbf{r_1} + m_2\mathbf{r_2}}{M} \quad \text{and} \quad \mathbf{r} = \mathbf{r_2} - \mathbf{r_1}$$

$$\text{so} \quad \frac{1}{2}M\dot{\mathbf{R}}^2_{\mathbf{CM}} + \frac{1}{2}\mu\dot{\mathbf{r}}^2 = \frac{1}{2}M(\frac{m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2}{M})^2 + \frac{1}{2}\mu(\dot{\mathbf{r}}_2 - \dot{\mathbf{r}}_1)^2$$

$$= \frac{1}{2M}(m_1^2\dot{r_1}^2 + 2m_1m_2\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_2 + m_2^2\dot{r}_2^2) + \frac{\mu}{2}(\dot{r}_2^2 - 2\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_2 + \dot{r}_1^2)$$

$$= (\frac{m_1^2}{2M} + \frac{\mu}{2})\dot{r}_1^2 + (\frac{2m_1m_2}{2M} - \frac{2\mu}{2})\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_2 + (\frac{m_2^2}{2M} + \frac{\mu}{2})\dot{r}_2^2$$

$$= (\frac{m_1^2 + m_1m_2}{2M})\dot{r}_1^2 + (\frac{m_1m_2 - m_1m_2}{M})\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_2 + (\frac{m_2^2 + m_1m_2}{2M})\dot{r}_2^2$$

$$= \frac{1}{2}m_1\dot{r_1}^2 + 0 + \frac{1}{2}m_2\dot{r}_2^2 \qquad \text{Using } M = m_1 + m_2,\ \mu = \frac{m_1m_2}{M}$$

■

★★ **Problem 7.11** Show that the shape $r(\varphi)$ for a central spring force ellipse takes the standard form $r^2 = a^2b^2/(b^2\cos^2\varphi + a^2\sin^2\varphi)$ if (in equation 7.37) we use the plus sign in the denominator and choose $\varphi_0 = \pi/4$.

Solution

The equation referred to, with the + sign and with $\varphi_0 = \pi/4$, is

$$r^2(\varphi) = \frac{\ell^2/m}{E + (\sqrt{E^2 - k\ell^2/m})\sin(2\varphi - \pi/2)} = \frac{\ell^2/m}{E - (\sqrt{E^2 - k\ell^2/m})\cos 2\varphi}.$$

Now use the identities $\sin^2(\varphi) + \cos^2(\varphi) = 1$ and $\cos 2\varphi = \cos^2\varphi - \sin^2\varphi$. Then we can write

$$\begin{aligned} r^2(\varphi) &= \frac{\ell^2/m}{E(\sin^2(\varphi) + \cos^2(\varphi)) - (\sqrt{E^2 - k\ell^2/m})(\cos^2\varphi - \sin^2\varphi)} \\ &= \frac{\ell^2/m}{(E - \sqrt{E^2 - k\ell^2/m})\cos^2\varphi + (E + \sqrt{E^2 - k\ell^2/m})\sin^2\varphi} \\ &= \frac{a^2b^2}{b^2\cos^2\varphi + a^2\sin^2\varphi} \equiv \frac{1}{(1/a^2)\cos^2\varphi + (1/b^2)\sin^2\varphi} \end{aligned}$$

where

$$a^2 = \frac{\ell^2/m}{E - \sqrt{E^2 - k\ell^2/m}} \quad \text{and} \quad b^2 = \frac{\ell^2/m}{E + \sqrt{E^2 - k\ell^2/m}}.$$

Note that a and b are the semimajor and semiminor axes of the ellipse, given here in terms of the physical parameters E, ℓ, k, and m. ■

★ **Problem 7.12** Show that the period of a particle that moves in a circular orbit close to the surface of a sphere depends only upon G and the average density ρ of the sphere. Find what this period would be for *any* sphere having an average density equal to that of water. (The sphere consisting of the planet Jupiter nearly qualifies!)

Solution

From $F = -GMm/r^2 = ma = -mr\omega^2 = -mr(2\pi/T)^2$, we find that the period is

$$T = 2\pi\sqrt{\frac{r^3}{GM}} = 2\pi\sqrt{\frac{r^3}{G((4/3)\pi r^3)}} = 2\pi\sqrt{\frac{3}{4\pi G\rho}},$$

depending only upon G and ρ, the density. In particular, if $\rho = 1\ \text{g/cm}^3 = 1000\ \text{kg/m}^3$,

$$T = 2\pi\sqrt{\frac{3}{4\pi(6.67\times10^{-11}\times1000)}}\ \text{s} = 11.9\times10^3\ \text{s} = 3.3\ \text{hrs}.$$

■

★ **Problem 7.13** (a) Communication satellites are placed into geosynchronous orbits; that is, they typically orbit in earth's equatorial plane, with a period of 24 hours. What is the radius of this orbit, and what is the altitude of the satellite above Earth's surface? (b) A satellite is to be placed in a synchronous orbit around the planet Jupiter to study the famous "red spot". What is the altitude of this orbit above the "surface" of Jupiter? (The rotation period of Jupiter is 9.9 hours, its mass is about 320 earth masses, and its radius is about 11 times that of earth.)

Solution

Using $F = ma$, we have $-GMm/r^2 = -mr\omega^2 = -mr(2\pi/T)^2$ where M is the mass of the earth and T is the period of the orbit. Therefore the radius of the orbit around the earth is

$$\begin{aligned} r_e &= \left(\frac{GMT^2}{4\pi^2}\right)^{1/3} = \left(\frac{(6.67\times10^{-11})(6\times10^{24})(24)^2(3600)^2}{4\pi^2}\right)^{1/3}\ \text{meters} \\ &= 4.23\times10^4\ \text{km} \end{aligned}$$

so the altitude of the satellite above earth's surface is $h = r_e - r_E = 4.23\times10^4$ km – 6.4×10^3 km = 3.6×10^4 km. Now to find the orbital radius of such a satellite around Jupiter, we can calculate the ratio

$$\frac{r_j}{r_e} = \left(\frac{M_J}{M_E}\right)^{1/3}\left(\frac{T_J}{T_E}\right)^{2/3} = (320)^{1/3}(9.9/24)^{2/3} = 3.8.$$

That is, the radius of the orbit around Jupiter is about 3.8 $r_e = 16.1\times10^4$ km. Therefore the altitude about Jupiter's "surface" is 16.1×10^4 km – $11(6.4\times10^3)$ km $\cong 9.0\times10^4$ km.

■

★ **Problem 7.14** The perihelion and aphelion of the asteroid *Apollo* are 0.964×10^8 km and 3.473×10^8 km from the sun, respectively. Apollo therefore swings in and out through Earth's orbit. Find (a) the semi-major axis (b) the period of Apollo's orbit in years, given earth's semimajor axis $a_E = 149.6\times10^6$ km. (Apollo is only one of many "Apollo asteroids" that cross earth's orbit. Some have struck the earth in the past, and others will strike in the future unless we find a way to prevent it.)

Solution

(a) The major axis of Apollo's orbit is $2a = (0.964 + 3.473) \times 10^8$ km, so its semimajor axis is half this much, $a = 2.219 \times 10^8$ km. (b) From Kepler's third law,

$$\left(\frac{T_{\text{Apollo}}}{T_{\text{earth}}}\right)^2 = \left(\frac{a_{\text{Apollo}}}{a_{\text{earth}}}\right)^3 = \left(\frac{2.219}{1.496}\right)^3 = 3.262.$$

Therefore $T_{\text{Apollo}} = 1.8$ yrs. ■

★★ **Problem 7.15** The time it takes for a probe of mass μ to move from one radius to another under the influence of a central spring force was shown in the chapter to be

$$t(r) = \pm\sqrt{\frac{\mu}{2}}\int_{r_0}^{r}\frac{r\,dr}{\sqrt{Er^2 - kr^4/2 - \ell^2/2\mu}},$$

where E is the energy, k is the spring constant, and ℓ is the angular momentum. Evaluate the integral in general, and find (in terms of given parameters) how long it takes the probe to go from the maximum to the minimum value of r.

Solution

Let $u = r^2$, so that $du = 2rdr$. The integral then becomes

$$\begin{aligned} t(u) &= \pm\frac{1}{2}\sqrt{\frac{\mu}{2}}\int_{u_0}^{u}\frac{du}{\sqrt{-ku^2/2 + Eu - \ell^2/2\mu}} = \mp\frac{1}{2}\sqrt{\frac{\mu}{k}}\sin^{-1}\left(\frac{-ku + E}{\sqrt{E^2 - k\ell^2/\mu}}\right)\Bigg|_{u_0}^{u}. \\ &= \mp\frac{1}{2}\sqrt{\frac{\mu}{k}}\sin^{-1}\left(\frac{E - kr^2}{\sqrt{E^2 - k\ell^2/\mu}}\right)\Bigg|_{r_0}^{r} \end{aligned}$$

The minimum and maximum values of r occur when the quantity $-kr^4/2 + Er^2 - \ell^2/2\mu = 0$, that is, where (by the quadratic formula)

$$r^2 = \frac{E}{k} \mp \frac{\sqrt{E^2 - k\ell^2/\mu}}{k}.$$

so if $r_0 = r_{min}$ and $r = r_{max}$, the time to go from one to another is

$$\begin{aligned} &t(r_{min} \to r_{max}) \\ &= \mp\frac{1}{2}\sqrt{\frac{\mu}{k}}\left[\sin^{-1}\frac{E - (E + \sqrt{E^2 - k\ell^2/\mu}}{\sqrt{E^2 - l\ell^2/\mu}} - \sin^{-1}\frac{E - (E - \sqrt{E^2 - k\ell^2/\mu}}{\sqrt{E^2 - l\ell^2/\mu}}\right] \\ &= \mp\frac{1}{2}\sqrt{\frac{\mu}{k}}\left[\sin^{-1}(-1) - \sin^{-1}(+1)\right] = \pm\frac{\pi}{2}\sqrt{\frac{\mu}{k}}. \end{aligned}$$

That is, for the SHO the time it takes to go from a maximum to a minimum value of r is

$$t = \frac{\pi}{2}\sqrt{\frac{\mu}{k}}$$

which doesn't depend upon the energy! Note that if there are two minima and two maxima in every orbit, the period of oscillation is four times as large, i.e., $2\pi\sqrt{\mu/k}$. ■

★★ **Problem 7.16** (a) Evaluate the integral in equation 7.26 to find $t(r)$ for a particle moving in a central gravitational field. (b) From the results, derive the equation for the period $T = (2\pi/\sqrt{GM})a^{3/2}$ in terms of the semi-major axis a for particles moving in elliptical orbits around a central mass.

Solution

(a) We begin with

$$t(r) = \pm\sqrt{\frac{\mu}{2}}\int_{r_0}^{r}\frac{r\,dr}{\sqrt{Er^2 + GM\mu r - \ell^2/2\mu}}.$$

Define $X = Er^2 + GM\mu r - \ell^2/2\mu$. From integral tables, we learn that

$$t(r) = \pm\sqrt{\frac{\mu}{2}}\frac{1}{E}\left[\sqrt{Er^2 + GM\mu r - \ell^2/2\mu} + \frac{GM\mu}{2\sqrt{-E}}\sin^{-1}\frac{2Er + GM\mu}{\sqrt{(GM\mu)^2 + 2E\ell^2/\mu}}\right]_{r_0}^{r}$$

noting that $E < 0$ for bound orbits.

(b) To find the orbital period, we will find the time to move from periapse (r_p) to apoapse (r_a), and double the result. Note that $X = 0$ at both r_p and r_a, so

$$t(p \to a) = \pm\sqrt{\frac{\mu}{2}}\frac{GM\mu}{2E\sqrt{-E}}\left(\sin^{-1}\frac{2Er_a + GM\mu}{\sqrt{(GM\mu)^2 + 2E\ell^2/\mu}} - \sin^{-1}\frac{2Er_p + GM\mu}{\sqrt{(GM\mu)^2 + 2E\ell^2/\mu}}\right).$$

But note that $X = 0$ at both r_p and r_a: That is, by the quadratic equation,

$$r_{a,p} = \frac{-GM\mu \pm \sqrt{(GM\mu)^2 + 2E\ell^2/\mu}}{2E},$$

so

$$2Er_{a,p} + GM\mu = \pm\sqrt{(GM\mu)^2 + 2E\ell^2/\mu}.$$

That is,

$$t(p \to a) = \frac{GM\mu^{3/2}}{2^{3/2}(-E)^{3/2}}[\sin^{-1}(1) - \sin^{-1}(-1)] = GM\left(\frac{\mu}{2(-E)}\right)^{3/2}$$
$$\pi = (\pi GM)(\mu/(-2E))^{3/2}.$$

But $E = -GM\mu/2a$, as shown in the text, so altogether the period of the orbit is

$$T = 2t(p \to a) = (2\pi/\sqrt{GM})a^{3/2}$$

■

★★ **Problem 7.17** The sun moves at a speed $v_S = 220$ km/s in a circular orbit of radius $r_S = 30,000$ light years around the center of the Milky Way galaxy. The earth requires $T_E = 1$ year to orbit the sun, at a radius of 1.50×10^{11} m. (a) Using this information, find a formula for the total mass responsible for keeping the sun in its orbit, as a multiple of the sun's mass M_0, in terms also of the parameters v_S, r_S, T_E, and r_E. Note that G is not needed here! (b) Find this mass numerically.

Solution

(a) For circular orbits, $F = -GMm/r^2 = ma = -mr\omega^2 = -mr(2\pi/T)^2$, so $2\pi/T)^2 = GM/r^3$. For the earth around the sun, $(2\pi/T_e)^2 = GM_e/r_e^3$, and for the sun around the galactic nucleus, $(2\pi/T_s)^2 = GM_{Gal}/r_s^3$. Divide these two equations to find that

$$\frac{M_{Gal}}{M_s} = \left(\frac{T_e}{T_s}\right)^2 \left(\frac{r_s}{r_e}\right)^3.$$

Now $T_s = 2\pi r_s/v_s$, so

$$\frac{M_{Gal}}{M_s} = \left(\frac{r_s}{r_e}\right)^3 T_e^2 \left(\frac{v_s}{2\pi r_s}\right)^2 = \frac{T_e^2 r_s v_s^2}{4\pi^2 r_E^3}.$$

Given $T_1 = 1$ yr, $r_e = 1.50 \times 10^{11}$ m, $r_s = 3 \times 10^4 c\cdot$ yrs, and $v_s = 2.2 \times 10^5$ m/s, we find $M_{gal} \cong 10^{11} M_s$. ∎

★★ **Problem 7.18** The two stars in a double-star system circle one another gravitationally, with period T. If they are suddenly stopped in their orbits and allowed to fall together, show that they will collide after a time $T/4\sqrt{2}$.

Solution

The equivalent one-body problem has a single star of reduced mass $\mu = m_1 m_2/(m_1 + m_2)$ circling a stationary central mass $M = m_1 + m_2$. In circular orbit, Newton's second law gives

$$\frac{GM\mu}{r_0^2} = \mu r_0 \omega^2 = \mu r_0 (2\pi/T)^2 \quad \text{so} \quad T = \frac{2\pi}{\sqrt{GM}} r_0^{3/2}$$

where r_0 is the orbital radius of μ. Now when μ is stopped, it falls straight towards M. Its path is then essentially half of an *extremely* elliptical orbit, with foci at the endpoints (periastron and apastron) of the orbit. That is, the path has length $2a$, where a is the semimajor axis of the ellipse. Therefore we can use Kepler's third law to find

$$T_{ellipse} = T_{circle}\left(\frac{a}{r_0}\right)^{3/2} = T_{circle}\left(\frac{r_0/2}{r_0}\right)^{3/2} = \frac{T_{circle}}{2\sqrt{2}}.$$

The time to fall, which corresponds to half the full-orbit period, is therefore $t = T_{circle}/4\sqrt{2}$, as claimed. ∎

★★ **Problem 7.19** A particle is subjected to an attractive central spring force $F = -kr$. Show, *using Cartesian coordinates*, that the particle moves in an elliptical orbit, with the force center at the *center* of the eilipse, rather than at one focus of the ellipse.

Solution

In Cartesian coordinates, with unit vectors $\hat{x}$ and $\hat{y}$, the force is

$$\mathbf{F} = -k(x\hat{x} + y\hat{y}) = m\mathbf{a} = m(\ddot{x}\hat{x} + \ddot{y}\hat{y}).$$

Therefore the equations of motion are

$$\ddot{x} + \frac{k}{m}x = 0 \quad \text{and} \quad \ddot{y} + \frac{k}{m}y = 0,$$

both simple harmonic oscillator equations, with solutions

$$x = A_1 \sin(\omega t + \varphi_1) \quad \text{and} \quad y = A_2 \sin(\omega t + \varphi_2)$$

where $\omega = \sqrt{k/m}$ and the other constants are arbitrary. At some time on the orbit there will be a maximum distance from the origin, since the orbit is bounded. We orient the x axis in this direction, so that $x = x_{max} = A_1$ and $\varphi_1 = \pi/2$ so that $x(t) = A_1 \cos \omega t$. At this point $y = 0$, so in general $y = A_2 \sin \omega t$. Thus

$$\left(\frac{x}{A_1}\right)^2 + \left(\frac{y}{A_2}\right)^2 = \cos^2 \omega t + \sin^2 \omega t = 1,$$

the equation of an ellipse in Cartesian coordinates. Obviously the center of the ellipse as located at $x = y = 0$. ∎

★★ **Problem 7.20** Use equation 7.32 to show that if the central force on a particle is $F = 0$, the particle moves in a straight line.

Solution

The equation is

$$\theta = \pm \frac{\ell}{\sqrt{2m}} \int^r \frac{dr/r^2}{\sqrt{E - \ell^2/2mr^2 - U(r)}}.$$

If $F = 0$ we can take $U = 0$ without loss of generality. So

$$\theta = \pm \frac{\ell}{\sqrt{2m}} \int^r \frac{dr/r}{\sqrt{Er^2 - \ell^2/2m}}.$$

Now substitute $r = A \sec \varphi$, where A is an arbitrary constant. Then $dr = A \sec \varphi \tan \varphi d\varphi$:

$$\begin{aligned}\theta &= \pm \frac{\ell}{\sqrt{2m}} \int^\varphi \frac{\sec \varphi \tan \varphi d\varphi / \sec \varphi}{\sqrt{EA^2 \sec^2 \varphi - \ell^2/2m}} \\ &= \pm \frac{\ell}{\sqrt{2m}} \int^\varphi \frac{\tan \varphi d\varphi}{\sqrt{EA^2}\sqrt{\sec^2 \varphi - 1}}.\end{aligned}$$

Now choose A by letting $A^2 = \ell^2/2mE$. This is to simplify the equation. Thus

$$\theta = \pm \frac{\ell}{\sqrt{2m}} \int \frac{\tan \varphi d\varphi}{\sqrt{\frac{\ell^2}{2m}} \tan \varphi} = \pm(\varphi - \varphi_0).$$

where φ_0 is the constant of integration. That is,

$$\theta = \pm(\sec^{-1}\left(\frac{r}{\ell/\sqrt{2mE}} + \text{constant}\right)).$$

so

$$\frac{\sqrt{2mE}r}{\ell} = \pm \sec(\theta - \theta_0) = \pm \frac{1}{\cos(\theta - \theta_0)}.$$

Inverting,

$$r \cos(\theta - \theta_0) = \pm \frac{\ell}{\sqrt{2mE}} = \text{constant}.$$

or

$$r[\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0] \equiv x\cos\theta_0 + y\sin\theta_0 = \text{constant}.$$

which is the equation of an arbitrary straight line in Cartesian coordinates. ■

★★ **Problem 7.21** Find the central force law $F(r)$ for which a particle can move in a spiral orbit $r = k\theta^2$, where k is a constant.

Solution

For central forces both energy and angular momentum are conserved, so $\ell = mr^2\dot{\theta}$ and $E = (1/2)m(\dot{r}^2 + r^2\dot{\theta}^2) + U(r)$ are both constants of the motion. Now $r = k\theta^2$, so $\theta = \pm\sqrt{r/k}$, and so $\dot{\theta} = \pm(1/2\sqrt{k}r^{-1/2}\dot{r}$. Therefore $\ell = \pm(mr^2/2\sqrt{k}r^{-1/2}\dot{r}$. Inverting, we find that $\dot{r} = \pm(2\sqrt{k}\ell/m)r^{-3/2}$. Eliminating $\dot{r}$ in the energy equation,

$$E = \frac{2k\ell^2}{mr^3}. + \frac{\ell^2}{2mr^2}. + U(r).$$

so

$$U(r) = E - \frac{\ell^2}{2m}\left(\frac{1}{r^2} + \frac{4k}{r^3}\right).$$

Now we can calculate the force,

$$F \equiv -\frac{dU(r)}{dr} = -\alpha\left(\frac{1}{r^3} - \frac{6k}{r^4}\right)$$

where $\alpha = \ell^2/m$ is an arbitrary constant. ■

★★ **Problem 7.22** Find two second integrals of motion for a particle of mass m in the case $F(r) = -k/r^3$, where k is a constant. Describe the shape of the trajectories, assuming that the angular momentum $\ell > \sqrt{km}$.

Solution

The two second integrals of motion correspond to the conservation of energy E and angular momentum ℓ,

$$E = T + U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - k/2r^2 \quad \text{and} \quad \ell = mr^2\dot{\theta}$$

Eliminating $\dot{\theta}$, we have $E + (1/2)m\dot{r}^2 + \ell^2/(2mr^2) - k/2r^2$. Then from Eq. 7.32 we have

$$\theta(r) = \pm\frac{\ell}{\sqrt{2m}}\int\frac{dr/r^2}{\sqrt{E - (1/2)(\ell^2/mr^2 - k/r^2)}} = \pm\frac{\ell}{2mE}\int\frac{dr/r}{\sqrt{r^2 - a^2}}$$

where $a = \sqrt{(\ell^2/m - k)/2E}$. This is a standard integral, giving

$$\theta(r) = \pm\frac{\ell}{a\sqrt{2mE}}\cos^{-1}|a/r| + \theta_0$$

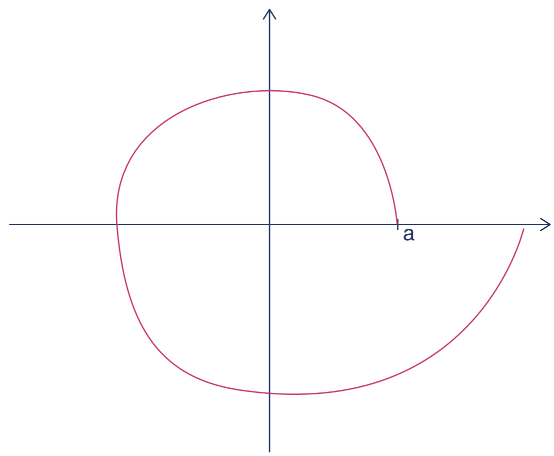
where θ_0 is a constant of integration. Inverting this equation,

$$\frac{a}{r} = \cos\left(\frac{\sqrt{2mE}}{\ell}a(\theta - \theta_0)\right) = \cos\left(\frac{\sqrt{2mE}}{\ell}\sqrt{\frac{\ell^2/m - k}{2E}}(\theta - \theta_0)\right)$$

so

$$r\cos\left(\sqrt{1-km/\ell^2}(\theta-\theta_0)\right) = a \equiv \sqrt{\frac{\ell^2/m-k}{2E}}.$$

If $k=0$ this is the equation of a straight line, as we would expect. For $\ell > \sqrt{km}$ the orbit is unbound as shown below.



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★★★ **Problem 7.23** A particle of mass m is subject to a central force $F(r) = -GMm/r^2 - k/r^3$, where k is a positive constant. That is, the particle experiences an inverse-cubed attractive force as well as a gravitational force. Show that if k is less than some limiting value, the motion is that of a precessing ellipse. What is this limiting value, in terms of m and the particle's angular momentum?

Solution

The potential energy of the particle is $U(r) = -\int^r F(r)\,dr = -GMm/r - k/2r^2$. There are two first integrals of motion, $E = (1/2)m(\dot{r}^2 + r^2\dot{\theta}^2) - GMm/r - l/2r^2 = \text{constant}$, and $\ell = mr^2\dot{\theta} = \text{constant}$. Eliminate $\dot{\theta}$ between these two equations:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{(\ell^2-km)}{2mr^2} - \frac{GMm}{r}.$$

Substitute $u = 1/r$, so that by the chain rule

$$\dot{r} \equiv \frac{dr}{dt} = \frac{dr}{du}\frac{du}{d\theta}\frac{d\theta}{dt} = -\frac{1}{u^2}u'\frac{\ell}{mr^2} = -\frac{\ell}{m}u'$$

where $u' \equiv du/d\theta$. Therefore

$$E = \frac{m}{2}\left(\frac{\ell^2}{m^2}\right)u'^2 + \frac{(\ell^2-km)}{2m}u^2 - GMmu.$$

Now take the derivative $d/d\theta$ of the entire equation: this gives, after some rearranging, the equation

$$u'' + (1-km/\ell^2)u = GMm^2/\ell^2,$$

which is the simple harmonic oscillator equation with an inhomogeneous term on the right. The general solution of the homogeneous equation $u'' + (1 - km/\ell^2)u = 0$ is

$$u_C = A\cos(\sqrt{1 - mk/\ell^2}\theta - \delta)$$

for arbitrary A, δ. A particular solution of the full equation is $u_P = GMm^2/(\ell^2 - mk)$. The most general solution of the full equation is the sum of these,

$$u = u_C + u_P = A\cos(\sqrt{1 - mk/\ell^2}\theta - \delta) + \frac{GMm^2}{\ell^2 - mk}.$$

if $mk/\ell^2 < 1$, that is, if $k < \ell^2/m$. Solving for r,

$$r = \frac{1}{u} = \frac{1}{A\cos(\sqrt{1 - mk/\ell^2}\theta - \delta) + B} = \frac{\text{constant}}{1 + e\cos[\sqrt{1 - mk/\ell^2}\theta]}$$

where we have chosen $\delta = 0$ by orienting the coordinates appropriately. Here "e" is the eccentricity of the ellipse. (Note that the equation of an ellipse in polar coordinates is $r = \text{con}/(1 + e\cos\theta)$, valid in this problem as $k \to 0$. Perihelion for the ellipse occurs at minimum r, i.e., at $\theta = 0, 2\pi, 4\pi, ...$, always at the same location if $k = 0$. The new solution has a perihelion at $\theta = 0$, followed by the next one where $\sqrt{1 - mk/\ell^2}\theta = 2\pi$, etc. That is, at $\theta = 2\pi/\sqrt{1 - mk/\ell^2} > 2\pi$. The orbit therefore precesses by

$$\delta\theta = 2\pi\left[\frac{1}{\sqrt{1 - mk/\ell^2}} - 1\right].$$

in each revolution. Note that *if* k is very small, with $mk/\ell^2 << 1$, then $(1 - mk/\ell^2)^{-1/2} \cong 1 + mk/2\ell^2$ by the binomial approximation. The precession in this case is $\pi mk/\ell^2$ per revolution. ■

★★ **Problem 7.24** Find the allowed orbital shapes for a particle moving in a *repulsive* inverse-square central force. These shapes would apply to α-particles scattered by gold nuclei, for example, due to the repulsive Coulomb force between them.

Solution

For

$$U(r) = \frac{k}{r} \quad \text{with } k > 0$$

most of discussion in text goes through with $-GMm \to +K$

$$U_{\text{eff}}(r) \to \frac{\ell^2}{2mr^2} + \frac{k}{r} \Rightarrow r(\varphi) = -\frac{-\ell^2}{(1 + \epsilon\cos\varphi)km}$$

$\Rightarrow$ *no* circular or bound orbits $\to$ only hyperbolas

$$\epsilon = \sqrt{1 + \frac{2E\ell^2}{k^2 m}} \quad \text{and} \quad r_p = \frac{+\ell^2}{(-1 + \epsilon)km}$$

E is always $> 0 \Rightarrow \epsilon > 1$. (Parabolas, with $\epsilon = 1$, would need $E = 0 \Rightarrow \ell = 0, v = 0$, $r \to \infty$) Note $\cos\varphi < 0$

$$\Rightarrow r(\varphi) = \frac{\ell^2}{(-1 - \epsilon\cos\varphi)km} > 0$$

■

★★ **Problem 7.25** A particle moves in the field of a central force for which the potential energy is $U(r) = kr^n$, where both k and n are constants, positive, negative, or zero. For what range of k and n can the particle move in a stable, circular orbit at some radius?

Solution

Both the energy and angular momentum are constants of the motion, where

$$E = T + U = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + kr^n \quad \text{and} \quad \ell = mr^2\dot{\theta}.$$

Eliminate $\dot{\theta}$: This gives

$$E = \frac{1}{2}m\dot{r}^2 + U_{\text{eff}} \quad \text{where} \quad U_{\text{eff}} = kr^n + \frac{\ell^2}{2mr^2}.$$

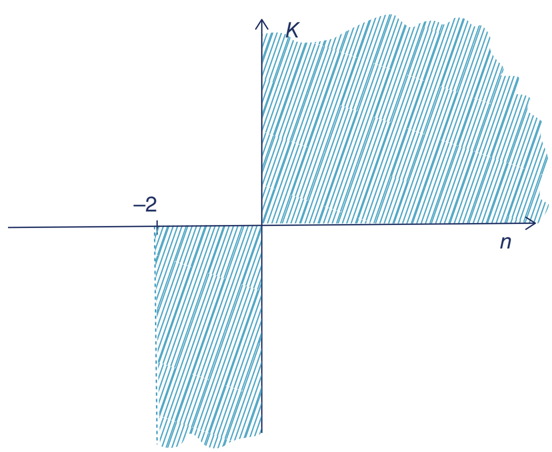
For a circular orbit we require that

$$U'_{\text{eff}} = knr^{n-1} - \frac{\ell^2}{mr^3} = 0, \quad \text{that is,} \quad r^{n+2} = \frac{\ell^2}{knm}.$$

and if the orbit is *stable* we require that there be a local *minimum* of the effective potential at the circular orbit radius. That is, $U''_{\text{eff}} > 0$ at the circular orbit. This means that

$$\begin{aligned} U''_{\text{eff}} &= kn(n-1)r^{n-2} + \frac{3\ell^2}{mr^4} = kn(n-1)\left(\frac{\ell^2}{knm}\right)^{(n-2)/(n+2)} + \frac{3\ell^2/m}{(\ell^2/knm)^{4/(n+2)}} \\ &= (kn)^{4/(n+2)}\left(\frac{\ell^2}{m}\right)^{(n-2)/(n+2)}(n+2) > 0 \ \text{ for stability.} \end{aligned}$$

Now we can require that the product $kn > 0$, to make the radius of the circle both real and positive. Therefore stability requires that $n + 2 > 0$. That is, we require that both $kn > 0$ and $n > -2$. A plot of the allowed values of k and n is shown below.



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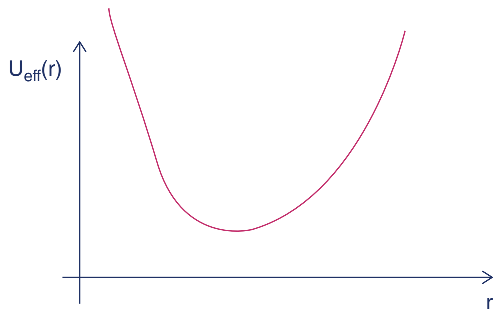
★★ **Problem 7.26** A particle of mass m and angular momentum ℓ moves in a central spring-like force field $F = -kr$. (a) Sketch the effective potential energy $U_{\text{eff}}(r)$. (b) Find the radius r_0 of circular orbits. (c) Find the period of small oscillations about this orbit, if the particle is perturbed slightly from it. (d) Compare with the period of rotation of the particle about the center of force. Is the orbit closed or open for such small oscillations?

Solution

(a) The Lagrangian is $L = (1/2)m(\dot{r}^2 + r^2\dot{\theta}^2) - (1/2)kr^2$. There are two first integrals, $E = (1/2)m(\dot{r}^2 + r^2\dot{\theta}^2) + (1/2)kr^2$ and $\ell = mr^2\dot{\theta}$. Eliminating $\dot{\theta}$,

$$E = \frac{1}{2}m\dot{r}^2 + U_{\text{eff}} \quad \text{where} \quad U_{\text{eff}} = \frac{\ell^2}{2mr^2} + \frac{1}{2}kr^2.$$

The effective potential is shown below.



(b) Circular orbits are at the minimum of U_{eff}: $dU_{\text{eff}}/dr = -\ell^2/mr^3 + kr = 0$ gives $r_{\text{circle}} = (\ell^2/mk)^{1/4}$.

(c) Small oscillations about the minimum have circular frequency

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{k_{\text{eff}}}{m}} = \sqrt{\frac{U''_{\text{eff}}(r_{\text{circle}})}{m}}.$$

where $U_{\text{eff}} = (3\ell^2/mr^4) + k = 4k$ at the circular orbit. Thus $\omega = 2\pi/T = \sqrt{4k/m}$. Therefore the period of small oscillations is

$$T = \frac{2\pi}{\omega} = \pi\sqrt{\frac{m}{k}}.$$

Using $F = ma$ for a circular orbit, we can find that the orbital period is $T_{\text{orbit}} = 2\pi\sqrt{m/k}$, so the period of small oscillations is *half* the orbital period. Thus the orbit is closed for small oscillations. ■

★★ **Problem 7.27** Find the period of small oscillations about a circular orbit for a planet of mass m and angular momentum ℓ around the sun. Compare with the period of the circular orbit itself. Is the orbit open or closed for such small oscillations?

Solution

(a) Energy and angular momentum are both conserved,

$$E = (1/2)m(\dot{r}^2 + r^2\dot{\theta}^2) - GMm/r \quad \text{and} \quad \ell = mr^2\dot{\theta}.$$

Eliminating $\dot{\theta}$,

$$E = \frac{1}{2}m\dot{r}^2 + U_{\text{eff}} \quad \text{where} \quad U_{\text{eff}} = \frac{\ell^2}{2mr^2} - \frac{GMm}{r}.$$

Circular orbits correspond to $U'_{\text{eff}} = 0$: This gives $r_0 = (\ell/m)^2/GM$. The period of the orbit follows from

$$F = -\frac{GMm}{r_0^2} = ma = -mr_0\omega^2 = -mr_0(2\pi/T_0)^2.$$

The result is the orbital period

$$T_0 = 2\pi\sqrt{\frac{r_0^3}{GM}} = \frac{2\pi(\ell/m)^3}{G^2M^2}.$$

The period T of small oscillations about the circular orbit can be found from

$$\omega = \sqrt{\frac{k_{\text{eff}}}{m}} = \sqrt{\frac{U''_{\text{eff}}}{m}} = \frac{2\pi}{T} \quad \text{where} \quad U''_{\text{eff}} = \frac{3\ell^2}{mr_0^4} - \frac{2GMm}{r^3}.$$

Evaluating this at the circular orbit radius, we find finally

$$T_{\text{osc}} = \frac{2\pi(\ell/m)^3}{G^2M^2}$$

exactly the same as the orbital period. Therefore the perturbed orbit is closed.

■

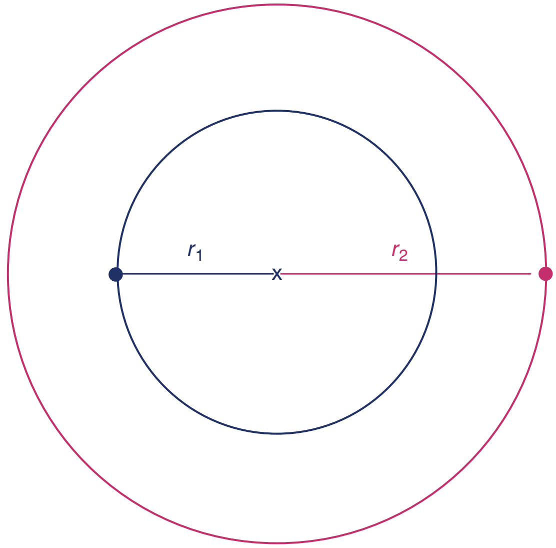
★★ **Problem 7.28** (a) A binary star system consists of two stars of masses m_1 and m_2 orbiting about one another. Suppose that the orbits of the two stars are circles of radii r_1 and r_2, centered on their center of mass. Show that the period of the orbital motion is given by

$$T^2 = \frac{4\pi^2}{G(m_1 + m_2)}(r_1 + r_2)^3.$$

(b) The binary system Cygnus X-1 consists of two stars orbiting about their common center of mass with orbital period 5.6 days. One of the stars is a supergiant with a mass 25 times that of the sun. The other star is believed to be a black hole with a mass of about 10 times the mass of the sun. From the information given, determine the distance between these stars, assuming that the orbits are circular.

Solution

(a) A diagram of the orbits is shown below. We know that $m_1r_1 = m_2r_2$, the condition that the center of mass (CM) remains at rest.



Also

$$F = -\frac{Gm_1m_2}{(r_1+r_2)^2} = ma = -m_1r_1\omega^2 = -m_1r_1\left(\frac{2\pi}{T}\right)^2$$

so

$$\begin{aligned} T^2 &= \frac{4\pi^2(r_1+r_2)^2}{G(m_1m_2)}m_1r_1 = \frac{4\pi^2(r_1+r_2)^2}{Gm_2}r_1 = \frac{4\pi^2(r_1+r_2)^2}{G(m_1+m_2)}\left(\frac{m_1+m_2}{m_2}r_1\right) \\ &= \frac{4\pi^2(r_1+r_2)^2}{G(m_1+m_2)}\left(\frac{m_1+m_2}{m_2}r_1\right) = \frac{4\pi^2}{G(m_1+m_2)}(r_1+r_2)^3 \end{aligned}$$

as we wanted to show. (b) Substituting in the numbers given, we find that the distance between the two stars is $r_1 + r_2 \cong 3 \times 10^7$ km. ■

★★ **Problem 7.29** A spacecraft is in a circular orbit of radius r about the earth. What is the minimum Δv the rocket engines must provide to allow the craft to escape from the earth, in terms of G, M_E, and r?

Solution

Initially the spacecraft's energy is $E_0 = \frac{1}{2}mv_0^2 - GM_Em/r < 0$. We need a Δv sufficient to achieve $E = 0$, in which case the craft will be in a parabolic orbit about the sun, coasting to infinity. That is, with Δv in the same direction as v_0, we have

$$\frac{1}{2}m(v_0+\Delta v)^2 - \frac{GM_Em}{r} = 0.$$

In the original circular orbit we have (from $F = ma$)

$$\frac{GM_Em}{r^2} = \frac{mv_0^2}{r}, \quad \text{so} \quad v_0 = \sqrt{\frac{GM_E}{r}}$$

Substituting this into the equation above, we have

$$\frac{1}{2}\left(\sqrt{\frac{GM_E}{r}}+\Delta v\right)^2 = \frac{GM_E}{r}.$$

Solving this for Δv, we find

$$\Delta v = (\sqrt{2}-1)\sqrt{\frac{GM_E}{r}}$$

which is the minimum required Δv. ■

⋆⋆ **Problem 7.30** A spacecraft departs from the earth. Which takes less rocket fuel: to escape from the solar system or to fall into the sun? (Assume the spacecraft has already escaped from the earth, and do not include possible gravitational assists from other planets.)

Solution

To drop into the sun, we must negate the earth's orbital velocity $v_e = 29.7$ km/s, neglecting the Δv needed to escape from the earth. Then the spacecraft will fall directly into the sun, because it has no tangential velocity. To barely escape from the entire Solar System, the craft must instead have net zero *energy* in the frame in the sun, which means it will be in a parabolic orbit in the sun's frame. To achieve zero energy, we must have the escape velocity in the sun's frame, where $(1/2)mv_{esc}^2 - GM_S m/r_e = 0$, that is, $v_{esc} = \sqrt{2GM_S/r_e}$. In the circular orbit of the sun, $v = \sqrt{GM_S/r_e}$, found from Newton's second law for circular orbits, $mv^2/r_e = GM_S m/r_e^2$. So the boost Δv needed by the rocket is $\Delta v = v_{esc} - v = (\sqrt{2}-1)\sqrt{GM_S/r_e} = 0.414$ (29.7 km/s), where we assume that the boost is in the same direction as the earth is moving in its orbit. That is, the boost needed to escape from the Solar System, is only 41.4% of the Δv needed to drop the spacecraft into the sun. Escape requires less fuel to accomplish. ■

⋆⋆ **Problem 7.31** After the engines of a 100 kg spacecraft have been shut down, the spacecraft is found to be a distance 10^7 m from the center of the earth, moving with a speed of 7000 m/s at an angle of 45° relative to a straight line from the earth to the spacecraft. (a) Calculate the total energy and angular momentum of the spacecraft. (b) Determine the semi-major axis and the eccentricity of the spacecraft's geocentric trajectory.

Solution

(a) The energy of the spacecraft is

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = m(\frac{v^2}{2} - \frac{GM}{r})$$

$$= 100\text{kg}\left[\frac{(7\times 10^3\text{m/s})^2}{2} - \frac{(2/3)\times 10^{-10}6\times 10^{24}\text{ m}^2}{10^7\text{ s}^2}\right] = -1.5\times 10^9\text{ J}$$

and the angular momentum is

$$|L| = |r\times p| = rp\sin 45^\circ = 10^7\text{ m}(100\text{ kg})7000\text{ m/s}(1/\sqrt{2}) = 5\times 10^{12}\text{ kg m}^2\text{/s}$$

(b) From Eq 7.56 the semimajor axis is

$$a = -\frac{GMm}{2E} = \frac{-(2/3) \times 10^{-10} \times 6 \times 10^{24} \times 100}{2(-1.5 \times 10^9)} = 1.5 \times 10^7 \text{ m}.$$

Now

$$\epsilon = \sqrt{1 + \frac{2EL^2}{G^2M^2m^3}} \quad \text{where} \quad \frac{2EL^2}{(GM)^2m^3} = \frac{2(-1.5 \times 10^9)(5 \times 10^{12})^2}{(\frac{2}{3} \times 10^{-10} 6 \times 10^{24})^2 10^6} = -0.469$$

$$\Rightarrow \epsilon = \sqrt{1 - 0.469} = 0.73 \quad \text{to two significant figures}$$

■

★★ **Problem 7.32** A 100 kg spacecraft is in circular orbit around the earth, with orbital radius 10^4 km and with speed 6.32 km/s. It is desired to turn on the rocket engines to accelerate the spacecraft up to a speed so that it will escape the earth and coast out to Jupiter. Use a value of 1.5×10^8 km for the radius of earth's orbit, 7.8×10^8 km for Jupiter's orbital radius, and a value of 30 km/s for the velocity of the earth. Determine (a) the semi-major axis of the Hohmann transfer orbit to Jupiter; (b) the travel time to Jupiter; (c) the heliocentric velocity of the spacecraft as it leaves the earth; (d) the minimum Δv required from the engines to inject the spacecraft into the transfer orbit.

Solution

(a) The major axis of the transfer orbit is $2a = r_E + r_J$, so $a = (r_E + r_J)/2 = 4.65 \times 10^8$ km. (b) Kepler's third law $T^2 \propto a^3$ gives $T/(1 \text{ yr}) = (4.65/1.5)^{3/2} = 5.46$ for a full orbit, so the travel time to Jupiter is $t = T/2 = 2.7$ years. (c) For the initial heliocentric velocity, the vis-viva equation gives

$$v = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)} = 38 \text{ km/s}.$$

(d) Given $v_0 = 6.32$ km/s in the parking orbit around the earth (in earth's rest frame) and with $v_\infty = v - v_E$ = 38 km/s – 29.7 km/s = 8.3 km/s, we have

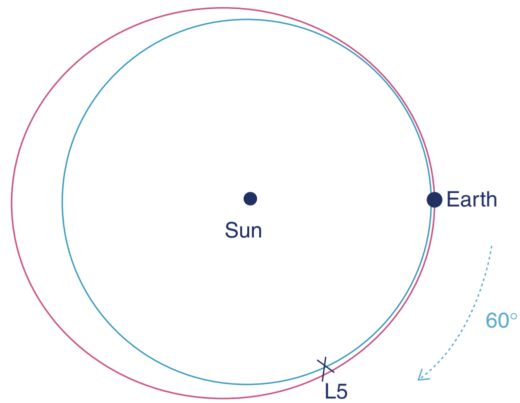
$$\Delta v = \sqrt{v_\infty^2 + 2v_0^2} - v_0 = \sqrt{(8.3)^2 + 2(6.32)^2} - 6.32 = 5.9 \text{ km/s}.$$

This is the required boost out of its initial orbit around the earth to insert the spacecraft into a Hohmann transfer orbit to Jupiter. ■

★★ **Problem 7.33** The earth-sun L5 Lagrange point is a point of stable equilibrium that trails the earth in its heliocentric orbit by 60° as the earth (and spacecraft) orbit the sun. Some gravity wave experimenters want to set up a gravity wave experiment at this point. The simplest trajectory from earth puts the spacecraft on an elliptical orbit with a period slightly longer than one year, so that, when the spacecraft returns to perihelion, the L5 point will be there. (a) Show that the period of this orbit is 14 months. (b) What is the semi-major axis of this elliptical orbit? (c) What is the perihelion speed of the spacecraft in this orbit? (d) When the spacecraft finally reaches the L5 point, how much velocity will it have to lose (using its engines) to settle into a circular heliocentric orbit at the L5 point?

Solution

Diagrams of the spacecraft's orbit are shown below.



(a) Sixty degrees is 1/6 of a complete orbit, and 1/6 of an earth orbit around the sun requires 2 months. So L5 will be at the present location of earth in 1 year + 2 months = 14 months later. (b) By Kepler's third law, $(T/T_E)^2 = (a/r_E)^3$, so

$$a_{\text{ellipse}} = r_{\text{earth}}(T/T_E)^{2/3} = r_{\text{earth}}(14/12)^{2/3} = 1.108 r_e \cong 165.8 \times 10^6 \text{ km}$$

(c) The vis-viva equation will answer this question. It is (using Standard International Units)

$$v^2 = GM_S\left(\frac{2}{r} - \frac{1}{a}\right) = (6.67 \times 10^{-11})(1.99 \times 10^{30})\left(\frac{2}{r_E} - \frac{1}{1.108\, r_E}\right)$$

which gives $v = 31$ km/s. (d) The speed of an object in a circular orbit around the sun at earth's orbital radius is 29.7 km/s. So the spacecraft must lose about 1 km/s. ■

⋆⋆⋆ **Problem 7.34** In *Stranger in a Strange Land*, Robert Heinlein claims that travelers to Mars spent 258 days on the journey out, the same for return, "plus 455 days waiting at Mars while the planets crawled back into positions for the return orbit." Show that travelers *would* have to wait about 455 days, if both earth-Mars journeys were by Hohmann transfer orbits.

Solution

The spacecraft transfer orbit has its perihelion at the earth and its aphelion at Mars. The orbit is half of a full elliptical orbit of the sun, and has a semimajor axis $a = (r_e + r_M)/2 = (1.000 + 1.524)$ AU = 1.262 AU in terms of the (essentially) circular orbit radii of the earth and Mars. Here 1 AU = 1 astronomical unit = average distance of the earth from the sun. The travel time is that of a half orbit, $T = T_S/2$ where T_S is the period of a total spacecraft orbit. Now by Kepler's third law,

$$\frac{T_S}{T_E} = \left(\frac{a_S}{a_e}\right)^{3/2} = (1.262)^{3/2}$$

so the travel time to Mars is

$$t \cong \frac{1}{2}(365 \text{ days})(1.262)^{3/2} \cong 258 \text{ days}$$

as Heinlein claims. To reach Mars, the spacecraft must be sent off at a point in earth's orbit such that Mars will be at the correct location in its orbit 258 days later. Similarly, when returning there are only certain times when a launch can be made so that earth will be at the right position in its orbit. Let is specify the angle θ_e of the earth in its orbit around the Sun by $\theta_e = (2\pi/T_e)t$, where $T_e = 365$ days and $t = 0$ when the initial launch from the earth takes place. Similarly, the angle of Mars in its orbit is

$$\theta_M = \theta_{0M} + \left(\frac{2\pi}{T_M}\right) t$$

where θ_{0M} is the initial angle of Mars in its orbit, i.e., the angle of Mars in its orbit when the spacecraft first leaves earth, and $T_M = 687$ days is Mars's orbital period. Now we need to find θ_{0M}. At time $t = 258$ days, Mars has to be at $\theta_M = \pi$, since that is where the spacecraft will be upon arrival at Mars's orbit. That is,

$$\pi = \theta_{0M} + \frac{2\pi}{T_M}(258\text{d}), \text{ so } \theta_{0M} = \pi - 2\pi\frac{258}{687} = 0.249\pi \text{ radians.}$$

Therefore in general, the angles of the earth and Mars are

$$\theta_e = \left(\frac{2\pi}{365}\right) t \text{ and } \theta_M = 0.249\pi + \left(\frac{2\pi}{287}\right) t$$

where t is in days. In particular, when the spacecraft arrives at Mars,

$$\theta_e = 2\pi\left(\frac{258}{365}\right) = 1.41\pi \text{ radians and } \theta_M = \pi \text{ radians.}$$

Now we can figure out the trip home. Let T_W be the wait time on Mars before the trip home can start. At this time,

$$\theta_e = 2\pi\left(\frac{258 + T_W}{365}\right) \text{ and } \theta_M = .249\pi + 2\pi\left(\frac{258 + T_W}{687}.\right)$$

Then when the spacecraft arrives home,

$$\theta_e = 2\pi\left(\frac{258 + T_W + 258}{365}\right) = \theta_M \text{ (at blast off) } + n\pi \text{ where } n = 1, 3, 5, \ldots$$

That is,

$$\theta_e = 2\pi\left(\frac{516 + T_W}{365}\right) = 0.249\pi + 2\pi\left(\frac{258 + T_W}{687}\right) + n\pi$$

Solving for T_W, we find

$$T_W = -713 + 391n.$$

Now obviously T_W must be positive, so $n = 1$ is not possible. But $n = 3$ *is* possible. Therefore the smallest possible time to wait on Mars for the return trip is

$$T_W = -713 + 391 \times 3 = 460 \text{ days}$$

which is close to Heinlein's 455 days. That is, one needs to wait on Mars for about a year and a quarter. ■

★★ **Problem 7.35** A spacecraft approaches Mars at the end of its Hohmann transfer orbit. (a) What is its velocity in the sun's frame, before Mars's gravity has had an appreciable influence on it? (b) What Δv must be given to the spacecraft to insert it directly from the transfer orbit into a circular orbit of radius 6000 km around Mars?

Solution

(a) The spacecraft's speed when in elliptical transfer orbit near the earth is 32.7 km/s. We can find its speed when near Mars by conserving angular momentum. So

$$r_M v_M = r_e v_e \Rightarrow v_M = v_e\left(\frac{r_e}{r_M}\right) = 32.7\frac{\text{km}}{s}\left(\frac{1\text{ au}}{1.52\text{ au}}\right)$$

$v_M = 21.5$ km/s (less than Mars's orbital velocity)

(b) The orbital velocity of Mars is 24 km/s; picture it rotating counterclockwise around the sun, at the right-most point in its orbit as drawn, with the spacecraft approaching from below (i.e., behind) in the sun's frame. However, since Mercury is traveling faster than the spacecraft, it will approach Mars in Mars' frame from above. As it approaches Mars we want to capture the spacecraft into a counter-clock wise orbit about Mars.

$$\text{Here} \quad v_\infty = 24\text{ km/s} - 21.5\text{ km/s} = 2.5\text{ km/s}$$

And by energy conservation Mars's frame,

$$\frac{1}{2}mv_\infty^2 = \frac{1}{2}m(v_0 + \Delta v)^2 - \frac{GM_m m}{v_0}$$

where r_0 and v_0 are the radius and speed in Mass orbit, so

$$(v_0 + \Delta v)^2 = v_\infty^2 + \frac{2GM_m}{r_0} = (2.5\text{ km/s})^2 + \frac{2((2/3)\times 10^{-10})(6.4\times 10^{23})}{6\times 10^6}$$

$$= (6.25\times 10^6\frac{\text{m}^2}{\text{s}^2}) + 1.42\times 10^7\text{ m}^2/\text{s}^2 = 20.5\times 10^6\text{ m}^2/\text{s}^2$$

$$(v_0 + \Delta v) = 4.5\times 10^3\text{ m/s} = 2.8\times 10^3\text{ m/s} + \Delta v$$

$$\text{so} \quad \Delta v = (4.5 - 2.7)\text{km/s} = 1.8\text{ km/s}$$

in a direction to *slow down* the spacecraft in Mars' frame to insert itself into an orbit around Mars. ■

★★ **Problem 7.36** A spacecraft parked in circular low-earth orbit 200 km above the ground is to travel out to a circular geosynchronous orbit, of period 24 hours, where it will remain. (a) What initial Δv is required to insert the spacecraft into the transfer orbit? (b) What final Δv is required to enter the synchronous orbit from the transfer orbit?

Solution

For circular orbits $v = \sqrt{GM_e/r}$, where, for low-earth orbits

$$r_{LEO} = (6370 + 200)\text{ km} = 6570\text{ km}$$

and for a geosynchronous orbit

$$r_{GEO} = 42,200\text{ km}.$$

So $\quad v_{LEO} = \sqrt{\dfrac{GM_R}{r_{LEO}}} = 7.8\text{ km/s} \qquad v_{GEO} = \dfrac{2\pi r_{GEO}}{p} = \dfrac{2\pi(4.22\times 10^7\text{ m})}{24(3600)s} = 3.07\text{ km/s}$

The transfer orbit has semimajor axis

$$a = \frac{r_{LEO} + r_{GEO}}{2} \qquad a = \frac{(6570 + 42,200)\text{ km}}{2} = 24,400\text{ km}$$

$$E = -\frac{GMm}{2a} = \frac{1}{2}mv^2 - \frac{GMm}{r} \Rightarrow v_0 \text{ for transfer orbit obeys } v_0^2 = \frac{2GM}{r_0} - \frac{GM}{a}$$

$$v_0^2 = GM(\frac{2}{r_0} - \frac{1}{a}) = GM(\frac{2}{6570\text{ km}} - \frac{1}{24,400\text{ km}})$$

$$v_0 = 1.03\times 10^4\text{ m/s} = 10.3\text{ km/s}$$

(a) So the initial

$$\Delta v = 10.3\text{ km/s} - 7.8\text{ km/s} = 2.5\text{ km/s}$$

to insert the spacecraft into the transfer orbit. When the craft arrives at r_{GEO}, its velocity is

$$v_{GEO} = v_{LEO}(\frac{r_{LEO}}{r_{GEO}})$$

by conservation of angular momentum.

$$= 10.3\text{ km/s}(\frac{6570\text{ km}}{42,200\text{ km}}) = 1.60\frac{\text{km}}{s}.$$

(b) To insert it into the geosynchronous orbit at speed 3.07 km/s requires a

$$\Delta v = (3.07 - 1.60)\text{ km/s} \qquad \Delta v = 1.47\text{ km/s} \cong 1.5\text{ km/s}$$

■

★★ **Problem 7.37** A spacecraft is in a circular parking orbit 300 km above earth's surface. What is the transfer-orbit travel time out to the moon's orbit, and what are the two $\Delta v's$ needed? Neglect the moon's gravity.

Solution

The semimajor axis of the transfer orbit is $a = \frac{r_{LEO}+r_m}{2}$

$$r_{LEO} = (6370 + 300)\text{ km} = 6670\text{ km} \qquad \text{(LEO = low earth orbit)}$$

$$r_m = 384,000\text{ km} \quad \text{so} \quad a = \frac{(384,000 + 6670)\text{ km}}{2} \cong \frac{390,700}{2}\text{ km} = 195,000\text{ km}$$

So by Kepler's third law, the time to reach the moon is about

$$T = T_{\text{craft}}/2 = \frac{1}{2}(a/r_m)^{3/2}T_m = \frac{1}{2}(\frac{195}{384})^{3/2}27.5 \text{ days} \cong 4.9 \text{ days}$$

Orbital velocities

$$mv^2/r = GMm/r^2 \Rightarrow v = \sqrt{\frac{GM}{r}}$$

$$\text{Low-earth orbit} \quad v_{LEO} = \sqrt{\frac{2/3 \times 10^{-10}\ 6 \times 10^{24}}{6.67 \times 10^6}} = 0.77 \times 10^4 \text{ m/s} \cong 7.7 \text{ km/s}$$

$$\text{Moon's orbit} \quad v_{\text{moon}} = \sqrt{\frac{2/3 \times 10^{-10}\ 6 \times 10^{24}}{3.84 \times 10^8}} = 1.02 \times 10^3 \text{ m/s} \cong 1.0 \text{ km/s}$$

Transfer orbit

$$E = -\frac{GMm}{2a} = \frac{1}{2}mv^2 - \frac{GMm}{r} \Rightarrow r^2 = GM(\frac{2}{r} - \frac{1}{a})$$

At perigee $r = 6670$ km

$$\text{so} \quad v = \sqrt{\frac{2}{3} \times 10^{-10} \times 6 \times 10^{24}\left(\frac{2}{6.67 \times 10^6} - \frac{1}{195 \times 100}\right)}$$

$$= \sqrt{4 \times 10^{14-6}(\frac{2}{6067} - \frac{1}{195})} \simeq 1.1 \times 10^4 \text{ m/s}$$

$$v_p = 11 \text{ km/s}$$

At apogee $r = 384,000$ km. Using conservation of angular momentum,

$$v_a = \left(\frac{r_p}{r_a}\right) \quad , \quad v_p = \left(\frac{6670}{384,000}\right) \qquad v_a = 0.19 \text{ km/s}$$

Therefore

$$\Delta v_{\text{initial}} = 11 \text{ km/s} - 7.7 \text{ km/s} = 3.3 \text{ km/s}$$

to go from low-earth orbit to the transfer orbit

$$\Delta v_{\text{final}} = 1.0 \text{ km/s} - 0.19 \text{ km/s} = 0.8 \text{ km/s}$$

to insert into moon's orbit from the transfer orbit. This neglects the moon's gravity. Both of these Δv's accelerate the spacecraft. ■

⋆⋆ **Problem 7.38** A spacecraft is sent from the earth to Jupiter by a Hohmann transfer orbit. (a) What is the semi-major axis of the transfer ellipse? (b) How long does it take the spacecraft to reach Jupiter? (c) If the spacecraft actually leaves from a circular parking orbit around the earth of radius 7000 km, find the rocket Δv required to insert the spacecraft into the transfer orbit.

Solution

(a) The semimajor axis of the transfer orbit is

$$a = \frac{r_e + r_J}{2} = \frac{1.50 + 7.78}{2} \times 10^8 \text{ km} = 4.64 \times 10^8 \text{ km}$$

(b) The transfer time (using Kepler's third law) is therefore

$$T = \frac{T_c}{2} = \frac{1}{2}(\frac{a}{r_e})^{3/2} T_e = 2.72 \text{ years}$$

(c) The spacecraft speed in the sun's frame, just as it enters the transfer orbit, follows from the vis-viva equation

$$v = \sqrt{GM(\frac{2}{r} - \frac{1}{a})} \text{ where } r = 1.50 \times 10^{11} \text{ m} \quad a = 4.64 \times 10^{11} \text{ m}$$

$$\Rightarrow v = \sqrt{\frac{2}{3} \times 10^{-10} \left(\frac{2}{1.5 \times 10^{11}} - \frac{1}{4.64 \times 10^{11}} \right) \times 2 \times 10^{30}} = 38.5 \text{ km/s}$$

The earth has velocity 29.7 km/s in its orbit. Add to that the ship speed in the parking orbit at radius 7000 km, which is

$$v_0 = \sqrt{GM_e/r_0} = 7.5 \text{ km/s}.$$

So now combine earth's orbital velocity with that of the parking orbit velocity, we have

$$v_\infty = v - v_e = 38.5 \text{ km/s} - 29.7 \text{ km/s} = 8.8 \text{ km/s}$$

and then

$$\Delta v = \sqrt{v_\infty^2 + 2v_0^2} - v_0 = \sqrt{(8.8)^2 + 2(7.5)^2} - 7.5 = 13.8 - 7.5 = 6.3 \text{ km/s}$$

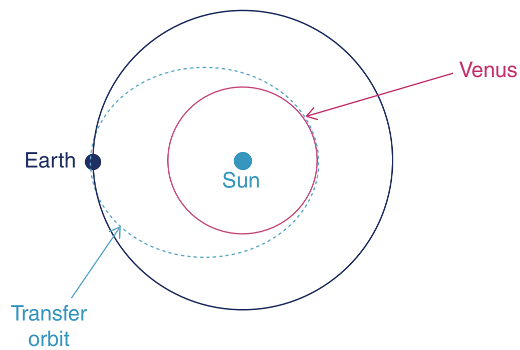
■

★★★ **Problem 7.39** Find the Hohmann transfer-orbit time to Venus, and the $\Delta v's$ needed to leave an earth parking orbit of radius 7000 km and later to enter a parking orbit around Venus, also of $r = 7000$ km. Sketch the journey, showing the orbit directions and the directions in which the rocket engine must be fired.

Solution

Assume that the orbit of the earth around the sun is clockwise. Then a sketch of the transfer orbit C is shown below. Note that its semimajor axis is

$$a_C = \frac{r_E + r_V}{2} = \frac{1.50 + 1.08}{2} \times 10^8 \text{ km} = 1.29 \times 10^8 \text{ km}.$$



From Kepler's third law, the time it takes to travel by the transfer orbit is therefore

$$t_C = \frac{T_C}{2} = \frac{1}{2}\left(\frac{a_C}{r_e}\right)^{3/2} T_E = \frac{1}{2}\left(\frac{1.29}{1.50}\right)^{3/2} T_e = 0.399 \text{ yrs} = 146 \text{ days}.$$

Then to drop toward Venus, we need to *slow down* the spacecraft in the sun's frame, so its velocity is *less* than earth's in the CW direction. So there are two strategies: We can place the craft in a CCW orbit around the earth, and blast off when the craft is moving opposite to earth's velocity around the sun, or place the craft in a CW orbit around the earth, and again blast off when the craft is moving opposite to earth's velocity around the sun. We can find v, the initial velocity of the craft in the sun's frame, by using the vis-viva equation

$$v = \sqrt{GM_S}\left(\frac{2}{r} - \frac{1}{a}\right)^{\frac{1}{2}} = \sqrt{6.67 \times 10^{-11}(1.99 \times 10^{30})}\left(\frac{2}{1.50 \times 10^{11}} - \frac{1}{1.29 \times 10^{11}}\right)^{\frac{1}{2}}$$
$$= 27.2 \text{ km/s}.$$

The velocity of the earth around the sun is $v_e = 29.7$ km/s, so $v_\infty = (29.7 - 27.2)$ km/s = 2.5 km/s, which is the velocity the craft must achieve relative to the earth, after it has escaped from the earth. This velocity is backwards to the earth's motion. The speed of the craft in earth orbit at radius 7000 km from the center of the Earth is $v_0 = 7.5$ km/s, which we can easily find using a side calculation. So then

$$\Delta v = \sqrt{v_\infty^2 + 2v_0^2} - v_0 = \left(\sqrt{(2.5)^2 + 2(7.5)^2} - 7.5\right) \text{ km/s} = 3.4 \text{ km/s}$$

which is the rocket boost needed from earth orbit to inject into the Hohmann transfer orbit.

Now when the spacecraft reaches the orbit of Venus, the speed of the spacecraft in the sun's frame is

$$v = \sqrt{GM_S}\left(\frac{2}{r_V} - \frac{1}{a}\right)^{1/2}.$$

so

$$\frac{v_{at\ Venus}}{v_{at\ earth}} = \left(\frac{\frac{2}{r_V} - \frac{1}{a}}{\frac{2}{r_E} - \frac{1}{a}}\right)^{1/2} = 1.39.$$

Therefore $v_{at\ Venus} = 1.39\ v_{at\ earth} = (1.39)(17.2\ \mathrm{km/s}) = 37.8\ \mathrm{km/s}$. This is faster than Venus's orbit around the sun, so it is necessary to slow down the spacecraft by firing its retrorocket. We need to slow it down so it can enter a parking orbit around Venus.

Picture Venus moving to the right in the sun's frame, with velocity v_V. Therefore the spacecraft's velocity is $v_{at\ Venus}$(in the sun's frame) = $v_V + v_\infty$, where v_∞ is the velocity of the craft in Venus's frame, when it is still far from Venus. From conservation of energy in Venus's frame,

$$\frac{1}{2}mv_\infty^2 = \frac{1}{2}m(v_0 + \Delta v)^2 - \frac{GM_V m}{r_0},$$

where v_0, r_0 are the velocity and radius of a circular parking orbit around Venus and Δv is the rocket burn needed to slow the craft to enter the parking orbit. So solving for Δv, $\Delta v = \sqrt{v_\infty^2 = 2v_0^2} - v_0$, where $v_\infty = v_{at\ Venus}$ (in the sun's frame) – v_V = 37.8 km/s – 35.0 km/s = 2.8 km/s. Here we have found v_V from $v_v = 2\pi r_V/T_V$, so

$$\frac{v_V}{v_e} = \frac{r_V}{r_e}\frac{T_e}{T_V} = \frac{r_V}{r_e}\left(\frac{r_e}{r_V}\right)^{3/2} = \left(\frac{r_e}{r_V}\right)^{1/2} = \left(\frac{1.50}{1.08}\right)^{1/2} = 1.179$$

We can find the speed v_0 in a 7000 km radius orbit around Venus using $F = ma$; the result is $v_0 = 6.82$ m/s. So finally

$$\Delta v = \sqrt{v_\infty^2 + 2v_0^2} - v_0 = 3.2\ \mathrm{km/s}.$$

Therefore the total boosts are (Δv at earth) + (Δv at Venus) = 3.4 km/s + 3.2 km/s = 6.6 km/s. Enough fuel has to be carried on board to effect these two $\Delta v's$. ■

⋆ **Problem 7.40** Consider an astronaut standing on a weighing scale within a spacecraft. The scale by definition reads the normal force exerted by the scale on the astronaut (or, by Newton's third law, the force exerted on the scale by the astronaut.) By the principle of equivalence, the astronaut can't tell whether the spacecraft is (a) sitting at rest on the ground in uniform gravity g, or (b) is in gravity-free space, with uniform acceleration a numerically equal to the gravity g in case (a). Show that in one case the measured weight will be proportional to the inertial mass of the astronaut, and in the other case proportional to the astronaut's gravitational mass. So if the principle of equivalence is valid, these two types of mass must have equal magnitudes.

Solution

There are two forces on the astronaut when the astronaut and scale are at rest on the earth: A normal force N upward due to the scale, and a gravitational force $m_G g$ downward due to gravity, where m_{G} is the gravitational mass. The astronaut is not accelerating, so these forces must balance: i.e., $m_{\mathrm{G}}g = N \equiv$ measured weight, proportional to the gravitational mass. But in an accelerating spacecraft far from real gravity, there is only a single force acting on the astronaut, the normal force due to the scale. So the total force N is upward, which must equal $m_I a$ by Newton's second law. Here m_{I} is the inertial mass of the astronaut. So in this case $N = m_{\mathrm{I}}a \equiv$ weight in the accelerating spaceship. So if $a = g$ numerically, and if by the equivalence principle one cannot tell whether one is standing at rest on the ground or accelerating uniformly upward, clearly $m_{\mathrm{I}} = m_{\mathrm{G}}$. ■

⋆⋆⋆ **Problem 7.41 Bertrand's theorem:** In Section 7.5 we stated a powerful theorem that asserts that the only potentials for which all bounded orbits are closed are: $U_{\text{eff}} \propto r^2$ and $U_{\text{eff}} \propto r^{-1}$. To prove this theorem, let us proceed in steps. If a potential is to have bound orbits, the effective potential must have a minimum since a bound orbit is a dip in the effective potential. The minimum is at $r = R$ given by

$$U'(R) = \frac{\ell^2}{\mu R^3}$$

as shown in equation 7.22. This corresponds to a circular orbit which is stable if

$$U''(R) + \frac{3}{R} U'(R) > 0$$

as shown in equation 7.23.

Consider perturbing this circular orbit so that we now have an r_{min} and an r_{max} about $r = R$. Define the apsidal angle $\Delta\varphi$ as the angle between the point on the perturbed orbit at r_{min} and the point at r_{max}. Assume $(R - r_{\text{min}}/R \ll 1$ and $(r_{\text{max}} - R)/R \ll 1$. Note that closed orbits require

$$\Delta\varphi = 2\pi \frac{m}{n}$$

for integer m and n and for all R.

- Show that

$$\Delta\varphi = \pi \sqrt{\frac{U'(R)}{3\,U'(R) + R\,U''(R)}} .$$

  Notice that the argument under the square root is always positive by virtue of the stability of the original circular orbit.
- In general, any potential $U(r)$ can be expanded in terms of positive and negative powers of r, with the possibility of a logarithmic term

$$U(r) = \sum_{\text{n}=-\infty}^{\infty} \frac{a_n}{r^n} + a \ln r .$$

  Show that, to have the apsidal angle independent of r, we must have: $U(r) \propto r^{-\alpha}$ for $\alpha < 2$ and $\alpha \neq 0$, or $U(r) \propto \ln r$. Show that the value of $\Delta\varphi$ is then

$$\Delta\varphi = \frac{\pi}{\sqrt{2-\alpha}} ,$$

  where the logarithmic case corresponds to $\alpha = 0$ in this equation.
- Show that if $\lim_{\text{r}\to\infty} U(r) = \infty$, we must have $\lim_{\text{E}\to\infty} \Delta\varphi = \pi/2$. This corresponds to the case $\alpha < 0$. We then must have

$$\Delta\varphi = \frac{\pi}{\sqrt{2-\alpha}} = \frac{\pi}{2} ,$$

  or $\alpha = -2$, thus proving one of the two cases of the theorem.

- Show that for the case $0 \le \alpha$, we can consider $\lim_{\mathrm{E}\to 0} \Delta\varphi = \pi/(2-\alpha)$. This then implies

$$\Delta\varphi = \frac{\pi}{\sqrt{2-\alpha}} = \frac{\pi}{2-\alpha}$$

which leaves only the possibility $\alpha = 1$, completing the proof of the theorem.

Solution

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r) \qquad \dot{r} = \frac{dr}{d\varphi}\dot{\varphi} = r'\frac{\ell}{\mu r^2}$$

$$E = \frac{1}{2}\mu(\frac{\ell}{\mu r^2})^2 r'^2 + \frac{\ell^2}{2\mu r^2} + U(r)$$

Change variable

$$u \equiv \frac{1}{r} \Rightarrow r' = -\frac{1}{u^2}u'$$

$$\Rightarrow E = \frac{\ell^2}{2\mu}u'^2 + \frac{\ell^2}{2\mu}u^2 + U(\frac{1}{u})$$

Expand $U_{\text{eff}}(\frac{1}{u})$ around circular orbit of radius $u = 1/R$

$$U_{\text{eff}}(\frac{1}{u}) \simeq U_{\text{eff}}(R) + \frac{dU_{\text{eff}}(R)}{du}(u - \frac{1}{R}) + \frac{1}{2}\frac{d^2U_{\text{eff}}(R)}{du^2}(u - \frac{1}{R})^2$$

$$\frac{dU_{\text{eff}}}{du} = \frac{dr}{du}\frac{dU_{\text{eff}}}{dr} = -\frac{1}{u^2}(U'(r) - \frac{\ell^2}{\mu r^2})$$

$$\frac{d^2U_{\text{eff}}}{du^2} = \frac{dr}{du}\frac{d}{dr}\left(\frac{dU_{\text{eff}}}{du}\right) = -\frac{1}{u^2}\frac{d}{dr}\left(-r^2(U'(r) - \frac{\ell^2}{\mu r^3})\right)$$

$$= -r^2\left[-2r(U'(r) - \frac{\ell^2}{\mu r^3}) - r^2(U''(r) + \frac{3\ell^2}{\mu r^4})\right]$$

$$\frac{d^2U_{\text{eff}}(R)}{du^2} = +R^3\left[2U'(R) - \frac{2\ell^2}{\mu R^3} + RU''(R) + \frac{3\ell^2}{\mu R^3}\right] = 2U'(R)R^3 + \frac{\ell^2}{\mu} + R^4U''(R)$$

$$\text{From} \quad \left.\frac{dU_{\text{eff}}}{du}\right|_R = 0 \Rightarrow U'(R) = \frac{\ell^2}{\mu R^3}$$

$$U''_{\text{eff}}(R) = (\frac{3}{R}U'(R) + U''(R))R^4 \Rightarrow U_{\text{eff}}(\frac{1}{u}) \simeq U_{\text{eff}}(R) + \frac{1}{2}R^4(\frac{3}{R}U' + U'')(u - \frac{1}{R})^2$$

$$E = \frac{\ell^2}{2\mu}u'^2 + U_{\text{eff}}(\frac{1}{u}) \cong \frac{1}{2}R^3U'(R)u'^2 + \frac{1}{2}R^4(\frac{3}{R}U'(R) + U''(R))(u - \frac{1}{R})^2$$

$$\Rightarrow \Omega^2 = \frac{R^4(\frac{3}{R}U' + U'')}{R^3U'} = \frac{3U' + RU''}{U'} \Rightarrow \Omega = \sqrt{\frac{3U' + RU''}{U'}}$$

Between r_{min} and r_{max}, we have

$$\Omega\Delta\varphi = \pi \Rightarrow \Delta\varphi = \frac{\pi}{\Omega} \quad \text{as needed.}$$

Write $w \equiv U'$ (Note: the prime on U means the derivative with respect to r, not u). We need

$$\Omega = \sqrt{3 + R\frac{w'}{w}} = \text{constant} \equiv \sqrt{\beta} \quad \beta > 0$$

$$\text{or} \quad \frac{Rw'}{w} = \beta - 3 \Rightarrow w(r) = kr^{\beta-3} \Rightarrow U(r) = -\frac{k}{\alpha}r^{-\alpha} \Rightarrow \Omega = \sqrt{\beta} = \sqrt{2-\alpha}$$

$$\Rightarrow \Delta\varphi = \frac{\pi}{\sqrt{2-\alpha}}$$

Now, let's take limiting cases. For $\alpha < 0$, we need the case $E \to +\infty \Rightarrow U_{max} \to \infty$.

$$E = \frac{\ell^2}{2\mu}U_{max}^2 - \frac{k}{\alpha}U_{max}^{\alpha} \to \frac{\ell^2}{2\mu}U_{max}^2$$

$$E = \frac{\ell^2}{2\mu}U'^2 + \frac{U^2}{U_m^2}E - \frac{k}{a}(\frac{U}{U_m})^{\alpha}(\frac{2\mu E}{\ell^2})^{\alpha/2} \quad \text{with } E \to +\infty \text{ unless } \frac{U}{U_m} \to 0 \Rightarrow U_{min} = 0$$

$$E \to \frac{\ell^2}{2\mu}U'^2 + \frac{\ell^2}{2\mu}U^2$$

with bound orbit between $U_{min} = 0$ and

$$U_{max} \Rightarrow \Omega\Delta\varphi = \frac{\pi}{2} \quad \text{(quarter cycle in harmonic oscillation)}$$

$$\text{with} \quad \Omega = \frac{\frac{\ell^2}{2\mu}}{\frac{\ell^2}{2\mu}} = 1 \Rightarrow \frac{\pi}{2} = \frac{\pi}{\sqrt{2-\alpha}} \Rightarrow \alpha = -2 \Rightarrow U(r) \sim r^2$$

Next, consider case $2 > \alpha > 0$. Consier limiting case $E = 0$ (barely bounded)

$$E = 0 = \frac{\ell^2}{2\mu}U'^2 + \frac{\ell^2}{2\mu}U^2 - \frac{k}{a}U^{\alpha} \Rightarrow \frac{k}{\alpha} = \frac{\ell^2}{2\mu}U^{-\alpha}U'^2 + \frac{\ell^2}{2\mu}U^{2-\alpha}$$

Change variable to $x^2 = U^{2-\alpha}$

$$\Rightarrow (2-\alpha)U^{1-\alpha}U' = 2xx' \Rightarrow \frac{k}{\alpha} = \frac{\ell^2}{2\mu}\frac{4}{(2-\alpha)^2}x'^2 + \frac{\ell^2}{2\mu}x^2$$

$$r_{max} = \infty \Rightarrow U_{min} = 0 \Rightarrow x_{min} = 0$$

$$\Rightarrow \text{Oscillation cycle } \Omega\Delta\varphi = \frac{\pi}{2} \quad (\text{quarter with } \Omega^2 = \frac{(2-\alpha)^2}{4})$$

$$\Rightarrow \frac{2-\alpha}{2}\Delta\varphi = \frac{\pi}{2} \Rightarrow \Delta\varphi = \frac{\pi}{2-\alpha}$$

$$\text{But} \quad \Delta\varphi = \frac{\pi}{\sqrt{2-\alpha}} = \frac{\pi}{2-\alpha} \Rightarrow \alpha = 1 \Rightarrow U(r) \sim \frac{1}{r}$$

Hence, the only potentials for which all orbits are closed are

$$U(r) \sim r^{\alpha} \quad \text{and} \quad U(r) \sim \frac{1}{r}$$

■

★★ **Problem 7.42** The luminous matter we observe in our Milky Way galaxy is only about 5% of the galaxy's total mass: The rest is called "dark matter," which seems to act upon all matter gravitationally but in no other way. As a rough approximation, we can therefore neglect luminous matter entirely as a source of gravity in understanding the dynamics of the galaxy. Along with many other stars, both much closer and much farther from the galactic center, we all circle about the center of the galaxy with about the same velocity 220 km/s. Our solar system in particular is 8.5 kiloparsecs from the galactic center (1 psc = 3.26 light years.) (a) From this information, how must the dark-matter density ρ for this range of orbital radii depend upon r, the distance of an orbiting star from the galactic center? (b) The dark-matter density in the vicinity of the sun is thought to be $\rho_0 \simeq 0.3$ GeV/c^2 per cm^3. Assuming now that the radial dependence of density $\rho(r)$ found in part (a) is valid all the way to the center of the galaxy, what is the total mass of dark matter within the orbit of our sun as a multiple of one solar mass, where $M_{\text{sun}} = 2 \times 10^{30}$ kg? (c) Suppose several rogue stars are in highly non-circular orbits around the galactic center, perhaps as a result of collisions with one another. Which (if any) of Kepler's laws would then still be correct for these stars? Explain. (d) Consider a proposal that the radial dependence of dark-matter density as found in part (a) might still be valid for arbitrarily large distances from the center. Show that in fact this is *not* possible, and explain why.

Solution

(a) Neglecting luminous matter, and assuming the dark matter's roughly spherical, we have

$$\frac{GM_{\text{init}}\, m}{r^2} = mr\omega^2 = mr(2\pi/T)^2 = \frac{mv^2}{r}$$

We take v ~constant, so $M_{\text{initial}} \propto r^2/r \sim r$ where $M_{\text{initial}} = \int_0^r \rho(r)dV$ is the galactic mass within the orbit of a star circling at radius in the galaxy. Here $dV = 4\pi r^2 dr$, so

$$M_{\text{initial}} \propto r \propto \int_0^r \rho(r) r^2 dr$$

Let $\rho(r) \propto r^n$, so

$$r \propto \int_0^r r^{n+2} dr \propto \frac{r^{n+3}}{n+3} \Rightarrow n = -2.$$

That is, $\rho \propto 1/r^2$

(b)

$$M(r) = \int_0^r \rho(r)dV = \int_0^r \frac{\alpha}{r^2} 4\pi r^2 dr = 4\pi\alpha r$$

$$\rho_0 = \frac{\alpha}{r_0^2} = \frac{\alpha}{(8.5 \text{ kpc})^2} = 0.3\ \frac{\text{GeV/c}^2}{\text{cm}^3} \Rightarrow \alpha = (0.3\ \frac{\text{GeV/c}^2}{\text{cm}^3})(8.3 \text{ kpc})^2$$

$$\text{so} \quad M(r) = 4\pi(\frac{0.3 \text{ GeV}}{\text{cm}^3\text{c}^2})(8.5 \text{ kpc})^3 = \text{mass inside orbit of solar system}$$

$$1kpc = 10^3 \text{ pc} = 3.26 \times 10^3 \text{ light-years} = 3.09 \times 10^{19} \text{ m} \quad 1 \text{ GeV/c}^2 = 1.60 \times 10^{-10} \text{ J/c}^2$$

$$\text{so} \quad M(r) = 1.21 \times 10^{41} \text{ kg} = 6 \times 10^{10} \text{ solar masses}$$

(c) Assuming no more collisions, only Kepler's second law would still be obeyed, as it is for any central force, due to conservation of angular momentum.

(d) If $\rho \propto 1/r^2$ out to arbitrary radii, then

$$M(r) = \int_0^r \rho(r) 4\pi r^2 dr \propto \int_0^r dr \ \to \infty \quad \text{as} \quad r \to \infty$$

Our galaxy does not have such unbounded mass, because that would show up as attracting neighboring galaxies much more than is possible from observations. ■

⋆ **Problem 7.43** Within the solar system itself it is often thought that the density of unseen dark matter is quite uniform, with mass density $\rho_0 \simeq 0.3$ GeV/c^2 per cm^3 (the mass equivalent of about 1 proton per three cubic centimeters.) The sun itself has mass $M_0 = 2 \times 10^{30}$ kg. (a) What fraction of a solar mass within the radius of earth's orbit might one expect in the form of dark matter? (The average radius of earth's orbit is 150×10^6 km.) (b) Would Kepler's second law still be valid for orbits of comets within the solar system? Explain. (c) Would Kepler's third law still be valid for the planets?

Solution

The supposed density of dark matter is (a)

$$\rho_0 = \text{constant} \simeq 0.3 \frac{\text{GeV/c}^2}{\text{cm}^3} \sim 5.4 \times 10^{-28} \text{kg/cm}^3$$

$$\text{fraction} = \frac{\rho_0 \frac{4}{3}\pi r_E^3}{M_0} \quad \text{where } r_E \text{ is the radius of Earth's orbit, } (1.4 \times 10^{13} \text{cm})$$

So the fraction is

$$\frac{7.6 \times 10^{12} \text{kg}}{2.0 \times 10^{30} \text{kg}} \simeq 3.8 \times 10^{-18} \Rightarrow \text{which is tiny and negligible.}$$

(b) Kepler's 2nd law follows from angular momentum conservation, which holds for any radial force. A uniform distribution of dark matter would not change that.

(c) The third law depends on a $1/r^2$ force law. A uniform density of mass leads to a linear force law $\propto r$ by Gauss's law. To see if this effect is significant, we need to replace r_E in (a) with the radius of the solar system,

$$s_E \to 6 \times 10^{14} \text{cm} \Rightarrow \text{fraction} \simeq 10^{-15} - 10^{-14}$$

which is still a negligible effect. ■

⋆ **Problem 7.44** Communications satellites are typically placed in orbits of radius r_{CS} circling the earth once per day. The 24 or so GPS (Global Positioning System) satellites are placed in one of six orbital planes, with each satellite circling the earth *twice* per day. (a) Find the radius of their orbits as a fraction of r_{CS}. (b) Low-earth orbit satellites typically have orbital periods of about 90 minutes. Find their radii as a fraction of r_{CS}.

Solution

We can use Kepler's third law $T^2 \propto a^3$ where for circular orbits the semi major axis $a = r$, the orbital radius. Thus

$$\left(\frac{T}{T_{cs}}\right)^2 = \left(\frac{r}{r_{cs}}\right)^3 \Rightarrow \left(\frac{r}{r_{cs}}\right) = \left(\frac{T}{T_{cs}}\right)^{2/3}$$

(a)

$$\text{If} \quad T = \frac{1}{2}T_{cs}, \text{ then} \quad r/r_{cs} = (\frac{1}{2})^{2/3} = \frac{1}{1.59} = 0.63 = \frac{r}{r_{cs}}$$

(b)

$$\frac{T}{T_{cs}} = \frac{115}{24} = \frac{3}{48} = \frac{1}{16}$$

$$\frac{r}{r_{cs}} = (T/T_{cs})^{2/3} = \frac{1}{6.35} = 0.16$$

to two significant figures. ■

⋆⋆⋆ **Problem 7.45** Trajectory specialists plan to send a spacecraft to Saturn requiring a gravitational assist by Jupiter. In Jupiter's rest frame the spacecraft's velocity will be turned 90° as it flies by, as illustrated in Figure 7.15(a). (a) If the nearest point on the spacecraft's path is a distance of $2R_J = 140,000$ km from the center of Jupiter, how fast (in km/s) is the spacecraft's speed in Jupiter's frame when it is at this nearest point? (b) In Jupiter's frame what is v_0 (as shown in the figure), the spacecraft's speed (in km/s) both long before and long after its encounter with Jupiter (but not so long before or after that its distance from the sun has changed appreciably)? (c) Note that long after the encounter, in the sun's frame the velocity of the spacecraft is $v_0 + v_J$ along the direction of Jupiter's motion around the sun, as illustrated in Fig. 7.15(b). Is this velocity sufficient to take the spacecraft out to the orbit of Saturn? Explain. (Useful data: Jupiter has mass $M_J = 1.9 \times 10^{27}$ kg and radius $R_J = 70,000$ km. Its average orbital radius and velocity around the sun are about 780 $\times 10^6$ km and 13 km/s, respectively. The average orbital radius of Saturn is 1.4 $\times 10^9$ km. Newton's gravitational constant is $G = 6.67 \times 10^{-11}\text{m}^3/(\text{ kg s}^2.))$

Solution

The radius of a hyperbolic orbit is $r = r_p(1+\epsilon)/(1+\epsilon\cos\varphi)$, where the eccentricity $\epsilon > 1$. Note that $r = r_p$, the radius at periapse, occurs here at $\varphi = 0$, and also that $r \to \infty$ at $\varphi = \pm(3\pi/4)$. At these angles $\cos\varphi = -1/\sqrt{2}$, so we find $\epsilon = \sqrt{(2)}$. From Eq. 7.60, the energy is

$$E = \frac{GMm(\epsilon - 1)}{2r_p} = T + U = \frac{1}{2}mv^2 - \frac{GM_{\mathrm{J}}m}{r}.$$

Solving for v at periapse, we find $v_{\mathrm{p}} = 47$ km/s. (b) As $r \to \infty$, $E \to \frac{1}{2}mv_\infty^2$. Then solving for v_∞, v_∞ = 19.5 km/s. (c) In the sun's frame the spacecraft's velocity would be $v_0 + v_J$ = 19.5 km/s + 13 km/s = 32.5 km/s. A Hohmann transfer orbit would have semimajor axis $a = (r_J + r_s)/2 = (7.8 + 14)/2 \times 10^8$ km = 1.09×10^9 km. In the sun's frame the energy per unit mass of this orbit is $(E/m) = -GM_{\mathrm{sun}}/2a = -6.1 \times 10^7$ J/kg.

At its beginning at Jupiter, the spacecraft has energy per unit mass

$$(E/m)_{\text{at Jupiter}} = \frac{1}{2}v^2 - \frac{GM_{\mathrm{sun}}}{r_J} = (5.28 - 1.71) \times 10^8 \text{ J/kg} = +3.6 \times 10^8 \text{ J/kg}.$$

So the initial energy of the spacecraft at Jupiter is more than enough to inject it into a Hohmann transfer orbit to Saturn. ■

⋆ **Problem 7.46** Show that the Virial Theorem is correct for a planet in circular orbit around the sun.

Solution

For potentials $U = -k/r$, the virial theorem predicts

$$\langle T \rangle = -\frac{1}{2}\langle U \rangle.$$

For a circular orbit

$$GMm/r^2 = mv^2/r \Rightarrow v^2 = GM/r$$

so the kinetic energy T is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}\frac{GMm}{r} = -\frac{1}{2}(-\frac{GMm}{r}) = -\frac{1}{2}U$$

Therefore for circular orbits $T = -\frac{1}{2}U$ at all times, so

$$\langle T \rangle = -\frac{1}{2}\langle U \rangle$$

as the virial theorem predicts. ■

⋆⋆ **Problem 7.47** Show that the Virial Theorem is correct for a particle of mass m free to move in a plane, and attached to one end of a Hooke's-law spring exerting the force $F = -kr$, if the particle is in (a) a circular orbit (b) an elliptical orbit.

Solution

The parameter $k = +1$ in the Virial Theorem for spring forces, so $\langle T \rangle = \langle U \rangle$.

(a) In circular orbit

$$\frac{mv^2}{r} = kr \quad \text{i.e. } F = ma$$

$$\text{so} \quad \langle T \rangle = \left\langle \frac{1}{2}mv^2 \right\rangle = \frac{1}{2}\left\langle kr^2 \right\rangle = \langle U \rangle$$

so the virial theorem is correct for a circular orbit due to a spring force.

(b) For an elliptical orbit we can arrange to make

$$x = A\sin\omega t \quad \text{and} \quad y = B\cos\omega t$$

$$\text{so} \quad \dot{x} = \omega A\cos\omega t \qquad \dot{y} = -\omega B\sin\omega t.$$

The average potential energy is

$$\langle U\rangle = \left\langle \frac{1}{2}k(x^2+y^2)\right\rangle = \frac{k}{2}\left[A^2\left\langle \sin^2\omega t\right\rangle + B^2\left\langle \cos^2\omega t\right\rangle\right]$$

Over a complete cycle

$$\left\langle \sin^2\omega t\right\rangle = \left\langle \cos^2\omega t\right\rangle$$

$$\text{so} \quad \langle U\rangle = \frac{k}{2}(A^2+B^2)\left\langle \sin^2\omega t\right\rangle$$

The average K.E. is

$$\langle T\rangle = \frac{1}{2}m\left\langle \hat{x}^2+\hat{y}^2\right\rangle = \frac{1}{2}m\left[A^2\omega^2\left\langle \cos^2\omega t\right\rangle + B^2\omega^2\left\langle \sin^2\omega t\right\rangle\right]$$

$$= \frac{1}{2}m(A^2+B^2)\omega^2\left\langle \sin^2\omega t\right\rangle$$

But $m\omega^2 = k$, so $\langle T\rangle = \langle U\rangle$ for an elliptical orbit of a spring mass system. ∎

★★ **Problem 7.48** Suppose that in studying a particular globular cluster containing 10^5 stars, whose average mass is that of our sun, astronomers find that the total kinetic energy of the stars is 10 times that of the magnitude of their total potential energy. (a) Estimate the amount of dark matter in the cluster, expressed in solar masses. (b) Then assume there is no such thing as dark matter, but that the potential energy between two stars has the form $U(r) = -\alpha r^n$ where $n \neq -1$. Is there a value of n such that the virial theorem would be satisfied without dark matter?

Solution

(a) We are given that

$$\langle T\rangle_{\text{stars}} = 10|\langle U\rangle_{\text{stars}}| \quad \text{where} \quad T_{\text{stars}} = \sum \frac{1}{2}mv^2,\ U_{\text{stars}} = -Gm^2\sum\frac{1}{r},$$

which disagrees with the virial theorem. But suppose the total mass is larger than we observe by a factor α. Then $\langle T\rangle_{total} = \alpha\langle T\rangle_{\text{stars}}$ and $\langle U\rangle_{total} = \alpha^2\langle U\rangle_{\text{stars}}$ so if the total kinetic and potential energies obey the virial theorem, with gravity the binding force, we have

$$\langle T\rangle_{\text{total}} = \frac{1}{2}|\langle U\rangle_{\text{total}}|$$

or

$$\alpha\langle T\rangle_{\text{stars}} = \frac{1}{2}\alpha^2|\langle U\rangle_{\text{stars}}| = \alpha\times 10|\langle U\rangle_{\text{stars}}\cdot$$

where the final equality implies that $\alpha^2/2 = 10\alpha$, or $\alpha = 20$. That is, the total dark matter mass must be 19 times as large as the mass of all the stars in the cluster, which means a total dark matter mass of 19×10^5 solar masses.

(b) The virial theorem states that, if the potential energy has the form $U = \alpha r^{n'+1}$, then

$$\langle T \rangle_{total} = \frac{n'+1}{2} \langle U \rangle$$

In this problem we are given that $U(r) = -\alpha r^n$, so we must choose $n = n' + 1$ and reverse the sign of any α. Therefore

$$\langle T \rangle = \frac{n+2}{n} \langle U \rangle \quad \text{observe} \quad \frac{n+2}{n} = 10 \text{ or } -10$$

$$\Rightarrow n = 18 \quad \text{or} \quad n + 2 = -20 \quad n = -22$$

Two choices, $n = 18$ or $n = -22$, both work with the problem statement as given. Either could give a bound state with the proper choice of the sign of α, chosen to give a potential minimum at $r = 0$. ■

⋆ **Problem 7.49** The cover of this book shows the paths of a number of stars orbiting a massive object named Sagittarius A-Star (Sgr A* for short) at the center of our Milky Way galaxy. One of these stars, called "S2," has an orbit whose period is 16.05 years, a semimajor axis of 970 au, and a periastron distance of 120 au. Assuming that S2 follows a Keplerian elliptical orbit, what is its orbital (a) eccentricity and (b) semiminor axis? (c) Most importantly, according to these observations what is the mass of Sgr A*, expressed as a multiple of the mass of the sun? (Note that 1 au is the average distance of the earth from the sun.)

Solution

First, note that star "S2" is called "SO2" in a different catalog, which should have been pointed out in the problem statement. Two different teams of astrophysicists, working for many years, have studied this star and many others that orbit the central massive object. The leaders of the two teams, Andrea Ghez and Reinhard Genzel, each received a 2020 Nobel Prize in Physics for accumulating strong evidence that Sag A* is a black hole.

We are given the period

$$T = 16.05 \text{ years} \quad a = 970 \text{ au} \quad r_p = 120 \text{ au}$$

Therefore (a)

$$r_p = a(1-\epsilon) \Rightarrow 1 - \frac{r_p}{a} = \epsilon \Rightarrow \epsilon \simeq 0.88.$$

(b) Also

$$b = a\sqrt{1-\epsilon^2} \simeq 461 \text{ au}$$

(c)

$$T = \frac{2\pi}{\sqrt{GM}} a^{3/2} \Rightarrow GM = (\frac{2\pi}{T})^2 a^3 \quad GM_\odot = (\frac{2\pi}{T_\oplus})^2 (1)^3$$

Taking the ratio

$$\Rightarrow \frac{M}{M_\odot} = \frac{T_\oplus^2}{T^2} a^3 \quad \text{where} \quad T_\oplus = 1 \text{ year}$$

$$T = 16.05 \text{ years and } a = 970 \text{ au} \Rightarrow \frac{M}{M_\odot} \simeq 3.5 \times 10^6.$$

More accurate calculations show that Sag A* has a mass somewhat in excess of 4×10^6 solar masses. ■

★★ **Problem 7.50** (a) Using the observed characteristics of Star S2's orbit as given in the preceding problem, and assuming it moves in a Keplerian elliptical orbit, find the speed of the star at periastron as a percentage of the speed of light. (b) If its orbit happened to be oriented so that at periastron S2 were moving directly towards the earth, by what factor would its light be blue-shifted due to the Doppler effect?

Solution

(a) We can use the vis-viva equation $v_p^2 = GM\left(\frac{2}{r_p} - \frac{1}{a}\right)$ together with the equation $T = 2\pi a^{3/2}/\sqrt{GM}$ for the peri to eliminate GM and derive the equation

$$v_p = \frac{2\pi a}{T}\sqrt{\frac{2a}{r_p} - 1}.$$

The values of the quantities on the right are given in the previous problem statement. Using them, we find $v_p/c = 0.006$.

(b) The relativistic Doppler shift factor for an approaching source is then

$$\sqrt{\frac{1 + v/c}{1 - v/c}} \simeq 1 + v/c = 1.006$$

where we have used the binomial approximation. [Note that it is tempting to use

$$\frac{mv_p^2}{r_p} = \frac{GMm}{r_p^2}$$

here to find v_p, but that equation applies only to circular orbits, which ours is not. Note that the vis-viva equation reduces to the simpler form $r_p = a$ in the case of circular orbits. ■

8 Electromagnetism

8.1 Problems and Solutions

★★ **Problem 8.1** Consider an infinite wire carrying a constant linear charge density λ_0. Write the Lagrangian of a probe charge Q in the vicinity, and find its trajectory.

Solution

Gauss's law

$$\oint \mathbf{E} \cdot d\mathbf{A} = 4\pi Q_{\text{enc}} \Rightarrow E(2\pi h)r = 4\pi\lambda h$$

for a cylindrical Gaussian surface with height h and radius r. Therefore

$$\Rightarrow \mathbf{E} = \frac{2\lambda}{r}\hat{r} \Rightarrow V = -\int \mathbf{E}\, d\ell = -2\lambda \ln r + \text{constant} \Rightarrow L = \frac{1}{2}m(\dot{r}^2 + m\dot{\varphi}^2 + \dot{z}^2) + 2Q\lambda \ln r$$

Note there are three conserved quantities for three symmetries,

$$(mr^2\dot{\varphi})^{\cdot} = 0 \Rightarrow \dot{\varphi} = \frac{\ell}{mr^2}$$

$$(m\dot{z})^{\cdot} = 0 \Rightarrow \dot{z} = \frac{p_z}{m}$$

$$E = \frac{1}{2}m(\dot{r}^2 + m\dot{\varphi}^2 + \dot{z}^2) - 2Q\lambda \ln r$$

$$\frac{dr}{d\varphi} = \frac{\dot{r}}{\dot{\varphi}} = \pm\frac{mr^2}{\ell}\sqrt{\frac{2E}{m} - \frac{\ell^2}{m^2r^2} - \frac{(p_z)^2}{m^2} + \frac{4Q\lambda}{m}\ln r}$$

■

★★ **Problem 8.2** Consider the oscillating Paul trap potential

$$U(z, \rho) = \frac{U_0 + U_1 \cos \Omega t}{\rho_0^2 + 2z_0^2}\left(2z^2 + \left(\rho_0^2 - \rho^2\right)\right)$$

written in cylindrical coordinates. (a) Show that this potential satisfies Laplace's equation. (b) Consider a point particle of charge Q in this potential. Analyze the dynamics using a Lagrangian and show that the particle is trapped.

Solution
(a) In cylindrical coordinates, the Laplacian is

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}. \textit{ Therefore for the potential given,}$$

$$\nabla^2 U = \frac{1}{\rho}\frac{\partial}{\partial rho}(-2\rho^2) + 4 = 0$$

(b) The Lagrangian is then

$$L = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2) - Qf(t)(2z^2 + \rho_0^2 - \rho^2) \quad \text{where} \quad f(t) \equiv \frac{U_0 + U_1\cos\Omega t}{\rho_0^2 + 2z_0^2}$$

Equation of motion

$$m\ddot{\rho} = m\rho\dot{\varphi}^2 + Qf(t)2\rho$$

$$(mr^2\dot{\varphi})^{\cdot} = 0 \Rightarrow \dot{\varphi} = \frac{\ell}{m\rho^2}$$

$$m\ddot{z} = -Qf(t)4z$$

Energy not conserved since there is t-dependence $\frac{\partial L}{\partial t} \neq 0$. ■

⋆ **Problem 8.3** Show that the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ is a consistent gauge condition.

Solution

$$\text{Say} \quad \nabla \cdot \mathbf{A}' \neq 0$$

$$\text{Write} \quad \mathbf{A} = \mathbf{A}' + \nabla\Lambda \Rightarrow \nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A}' + \nabla^2\Lambda$$

Then solve the Poisson equation

$$\nabla^2\Lambda = -\nabla \cdot \mathbf{A}' \neq 0$$

solution guaranteed given boundary conditions $\Rightarrow \nabla \cdot \mathbf{A} = 0$. ■

⋆ **Problem 8.4** Find the residual gauge freedom in the Coulomb gauge.

Solution
If

$$\nabla \cdot \mathbf{A} = 0$$

$$\text{Consider} \quad \mathbf{A}' = \mathbf{A} + \nabla\Lambda \Rightarrow \nabla \cdot \mathbf{A}' = \nabla \cdot \mathbf{A} + \nabla^2\Lambda = \nabla^2\Lambda \equiv 0$$

Any Λ satisfying $\nabla^2\Lambda = 0$ preserves gauge condition. ■

⋆ **Problem 8.5** Show that the Lorentz gauge $\partial_\mu A_\nu \eta_{\mu\nu} = 0$ is a consistent gauge condition.

Solution

$$\text{Say} \quad \partial_\mu A^{\mu'} = 0$$

$$\text{write} \quad A^\mu = A^{\mu'} + \partial^\mu \Lambda \Rightarrow \partial_\mu A^\mu = \partial_\mu A^{\mu'} + \partial_\mu \partial^\mu \Lambda \equiv 0$$

Choose Λ such that

$$\partial_\mu \partial^\mu \Lambda = -\partial_\mu A^{\mu'} \neq 0$$

Solution exists given boundary conditions. ■

⋆ **Problem 8.6** Find the residual gauge freedom in the Lorentz gauge.

Solution

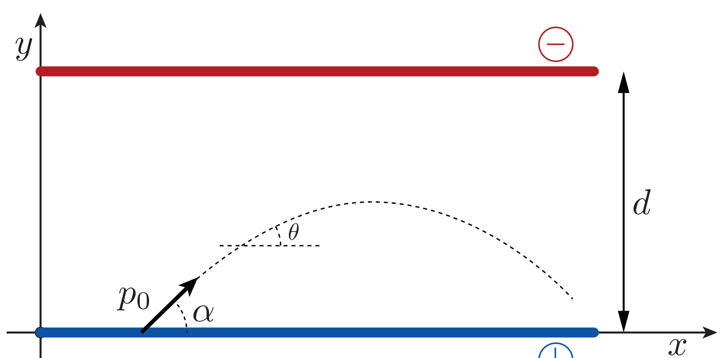
$$\text{Say} \quad \partial_\mu A^\mu = 0$$

$$A^{\mu'} = A^\mu + \partial^\mu \Lambda$$

$$\partial_\mu A^{\mu'} = \partial_\mu A^\mu + \partial_\mu \partial^\mu \Lambda \equiv 0$$

Any Λ satisfying $\partial_\mu \partial^\mu \Lambda = 0$ preserves gauge condition. ■

⋆⋆⋆ **Problem 8.7** An ultrarelativistic electron with $v \sim c$ and momentum p_0 enters a region between the two plates of a capacitor as shown in the figure. The plate separation is d and a voltage V is applied to the plates.



(a) Show that

$$\frac{d}{dt}(p_0 \cos\alpha \, \tan\theta) = -\frac{e\,V}{d}$$

where θ is the time dependent angle the electron makes with the horizontal axis during its trek. (b) Write a differential equation for $y(x)$ assuming that

$$\frac{1}{c}\frac{d}{dt} \sim \frac{d}{dl}$$

where $dl = \sqrt{dx^2 + dy^2}$. (c) Find the trajectory $y(x)$ by solving the differential equation from part (b).

Solution

(a) The equations of motion are

$$(*)\frac{d}{dt}(\mathrm{p}^x) = \frac{d}{dt}(p\cos\theta) = 0$$

and

$$(**)\frac{d}{dt}(\mathrm{p}^y) = \frac{d}{dt}(p\sin\theta) = -\frac{eV}{d} \qquad \text{since} \quad \mathbf{E} = \frac{V}{d}\hat{\mathbf{y}}$$

From (*), we have

$$p\cos\theta = \text{constant} = p_0\cos\alpha \Rightarrow p = \frac{p_0\cos\alpha}{\cos\theta}$$

Substituting in (**), we find

$$\frac{d}{dt}(p_0\cos\alpha\tan\theta) = -\frac{eV}{d}$$

(b) We have

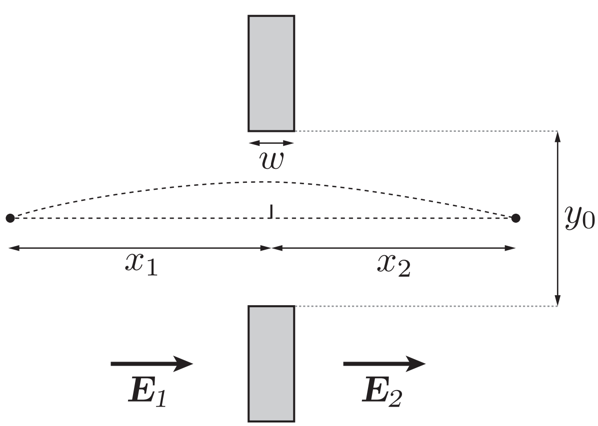
$$\tan\theta = \frac{dy}{dx} \quad \text{and} \quad \frac{d}{dt} \cong c\frac{d}{dx\sqrt{1+(\frac{dy}{dx})^2}} \Rightarrow \frac{d^2y}{dx^2} = \frac{-eV}{cp_0 d\cos\alpha}\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

(c) Then because $d^2y/dx^2 \propto \sqrt{1+(dy/dx)^2}$, we note that $dy/dx = \sinh k(x-\alpha)$ solves the equation, since

$$1+\sinh^2 = \cosh^2 \Rightarrow y(x) = y_0 + \frac{1}{k}\cosh(k(x-\alpha)) \quad \text{where} \quad k \equiv \frac{eV}{cp_0 d\cos\alpha}.$$

■

★★ **Problem 8.8** Charged particles are accelerated through a potential difference V_0 before falling onto a lens consisting of an aperture of height y_0 and thickness w, as shown in the figure.



There is a uniform electric field $\mathbf{E}_1$ and $\mathbf{E}_2$ on the left and right of the aperture respectively, as depicted. The figure also shows the trajectory of a charged particle of charge q emerging a distance x_1 from the aperture on the left and focusing a distance x_2 on the right. Assume $V_0 \gg E_1 x_1$ and $E_2 x_2$, and x_1 and $x_2 \gg y_0$. (a) Using $\nabla \cdot \mathbf{E} = 0$, show that inside the aperture, we have

$$E^x \simeq (E_2 - E_1)w \quad , \quad E^y \simeq -\frac{E_2 - E_1}{w} y \, .$$

(b) Show that

$$\frac{1}{x_1} + \frac{1}{x_2} \simeq \frac{E_2 - E_1}{2\, V_0}$$

so that the aperture functions as a lens for charged particles.

Solution

(a) From

$$\nabla \cdot \mathbf{E} = 0 = \frac{\partial \mathrm{E}^x}{\partial x} + \frac{\partial \mathrm{E}^y}{\partial y}$$

and expecting

$$\frac{\partial \mathrm{E}^x}{\partial x} \sim \frac{E_2 - E_1}{w}, \quad \text{we get } \frac{\partial \mathrm{E}^y}{\partial y} = -\frac{E_2 - E_1}{w} \Rightarrow \mathrm{E}^y \sim -\frac{E_2 - E_1}{w} y$$

(b) A charged particle of charge q will experience a force in the y-direction.

$$\mathrm{F}^y \sim \frac{-q(E_2 - E_1)y}{w} \sim \frac{\Delta \mathrm{p}^y}{\Delta t} = \frac{\Delta \mathrm{p}^y}{w/v} \Rightarrow \Delta \mathrm{p}^y \sim -\frac{q(E_2 - E_1)y}{v}$$

Then, beam is then deflected by

$$-\frac{\Delta \mathrm{p}^y}{p} \sim \Delta \phi$$

We know

$$\theta_1 + \theta_2 = \Delta \phi \quad \text{and} \quad \frac{y}{x_1} \cong \theta_1 \quad \frac{y}{x_2} \cong \theta_2$$

$$\Rightarrow \frac{1}{x_1} + \frac{1}{x_2} = -\frac{\Delta \mathrm{p}^y}{\mathrm{p}^y} = +\frac{q(E_2 - E_1)}{p\, v} = \frac{E_2 - E_1}{2 v_0}$$

where $q v_0 = p^2/2m$ by energy conservation. ∎

★★ **Problem 8.9** A charged particle is circling a magnetic field that gradually increases in magnitude from B_1 to B_2 as the particle advances along the field lines. Show that the particle will be reflected if

$$v_{0\|} \leq v_{0\perp} \sqrt{\frac{B_2}{B_1} - 1}$$

where $v_{0\|}$ and $v_{0\perp}$ are the components of the particle's velocity parallel and perpendicular to the magnetic field.

Solution

Using cylindrical coordinates, the Lagrangian is

$$L = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2) + \frac{q}{c}\frac{1}{2}B_{1,2}\rho(\rho\dot{\varphi})$$

where

$$\mathbf{B}_{1,2} = B_{1,2}\hat{z} \Rightarrow \mathbf{A}_{1,2} = \frac{1}{2}\mathbf{B}_{1,2} \times \mathbf{r} = \frac{1}{2}B_{1,2}\rho\hat{\varphi}$$

We then have

$$\frac{\partial L}{\partial \dot{\varphi}} = \text{constant} \equiv \ell = m\rho^2\dot{\varphi} + \frac{q}{2c}\rho^2 B_{1,2}$$

The circular motion is such that

$$\dot{\varphi} \cong -\frac{qB_{1,2}}{mc}$$

We also have the Hamilton constant

$$\Rightarrow H = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2) = \text{constant} \Rightarrow \frac{1}{2}m(v_{\parallel}^2 + \rho^2\dot{\varphi}^2) = \frac{1}{2}m(v_{0\parallel}^2 + v_{0\perp}^2) \quad (*)$$

But we also have

$$\ell = m\rho^2\dot{\varphi} + \frac{q}{2c}\rho^2 B_{1,2} = m\rho^2\left(-\frac{qB_{1,2}}{mc}\right) + \frac{q}{2c}\rho^2 B_{1,2} = -\frac{q\rho^2 B_{1,2}}{2c} = m\rho^2\dot{\varphi}$$

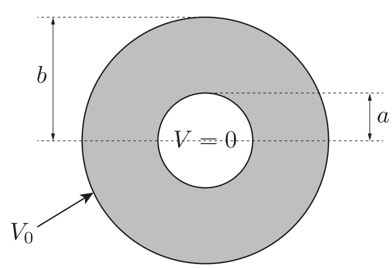
$$\Rightarrow m\rho^2\dot{\varphi} = m\rho_0 v_{0\perp}$$

For reflection, we must have $\nu_{11} = 0$.

$$(*) \Rightarrow \frac{v_{0\parallel}^2}{v_{0\perp}^2} = \frac{\rho_0^2}{\rho^2} - 1 = \frac{B}{B_1} - 1 \Rightarrow \frac{v_{0\parallel}}{v_{0\perp}} \leqslant \sqrt{\frac{B_2}{B_1} - 1}$$

■

★★ **Problem 8.10** A coaxial cable has a grounded center and a voltage V_0 on the rim, as shown in the figure.



A uniform magnetic field B_0 lies along the cylindrical axis of symmetry. Electrons propagate from the center to the rim. Find the mininum V_0 so that current can flow from the center to the rim.

Solution

We have in cylindrical coordinates

$$L = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2) - \frac{e}{2c}B_0\rho^2\dot{\varphi} + eV(\rho)$$

where

$$\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r} = \frac{1}{2}B\rho\varphi$$

We then have

$$\frac{\partial L}{\partial \dot{\varphi}} = \ell = \text{constant} = m\rho^2\dot{\varphi} + \frac{q}{2c}\rho^2 B \quad (*)$$

$$H = \text{constant} = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2) - eV(\rho)$$

Solve for $\dot{\varphi}$ in (*) and substitute in H

$$\Rightarrow m\rho^2\dot{\varphi} = \frac{eB}{2c}(\rho^2 - a^2)$$

where we used the boundary cond. at $\dot{\varphi} = 0$ at $\rho = a$

$$H = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\frac{e^2B^2}{4\,rc^2}\frac{(\rho^2 - a^2)^2}{m^2\rho^4} - eV(\rho)$$

We need $H = 0$ if $\dot{\rho} = 0$ at $\rho = a$ and $V(a) = 0$. And we wnat $\dot{\rho} = 0$ at $\rho = b$ for the critical point.

$$\Rightarrow V(b) = \frac{1}{8mc^2b^2}eB^2(b^2 - a^2)^2 = V_0 \quad \text{minimum}$$

■

★★★ **Problem 8.11** Neutrons have zero charge but carry a magnetic dipole moment μ. As a result, they are subject to a magnetic force given by $\mathbf{F} = (\mu \cdot \nabla)\,\mathbf{B}$. A beam of neutrons with $\mu = \mu\hat{\mathbf{x}}$ is moving along the z direction into a region of magnetic field $\mathbf{B}$. Find a simple form for $\mathbf{B}$ capable of *focusing* the beam and prevent it from dispersing. This means that small disturbances in the beam profile would not grow. HINT: For any vector field $\mathbf{V}$ satisfying $\nabla \cdot \mathbf{V} = 0$, we can write $\mathbf{V} = \nabla f$ for a function f satisfying $\nabla^2 f = 0$.

Solution

Following the hint, we want $\mathbf{B} = \nabla f$ with $\nabla^2 f = 0$ in the x-y plane. That is

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

We also want the force on the neutron to be the kind that restores deflections; i.e., small displacements from $x = y = 0$ should give harmonic stable oscillations.

$$\mathrm{F}^x = \mu\frac{\partial}{\partial x}\mathrm{B}^x \sim -x \Rightarrow \mu\frac{\partial^2}{\partial x^2}f \sim -x \Rightarrow f \sim -x^3 + g(y)x + h(y)$$

$$\mathrm{F}^y = \mu\frac{\partial}{\partial x}\mathrm{B}^y \sim -y \Rightarrow \mu\frac{\partial^2}{\partial x \partial y}f \sim -y$$

$$g' \sim -y \qquad g \sim -y^2 \Rightarrow f \sim C_1 x^3 + C_2 y^2 x + h(y)$$

Now we need

$$\nabla^2 f = 0 = 6C_1 x + 2C_2 x + h'' = 0 \Rightarrow 3C_1 = C_2 \quad \text{and} \quad h'' = 0 \Rightarrow f = C_1 x^3 + 3C_1 y^2 x + C_3 y + C_4$$

$$\Rightarrow \mathrm{B}^x = 3C_1 x^2 + 3C_1 y^2 \quad \text{and} \quad \mathrm{B}^y = 6C_1 yx + C_3$$

■

★★ **Problem 8.12** A magnetic monopole is a particle that casts out a radial magnetic field satisfying $\nabla \cdot \mathbf{B} = 4\pi q_m \delta(\mathbf{r})$ where q_m is the *magnetic charge* of the monopole. A non-relativistic *electrically charged* particle of charge q is moving near a magnetic monopole of magnetic charge q_m. (a) Show that the magnetic field from the monopole takes the form

$$\mathbf{B} = \frac{q_m}{r^2}\hat{\mathbf{r}}\,.$$

(b) Write the equations of motion of the electrically charged particle assuming that the magnetic monopole remains stationary. (c) Find the constants of motion; in particular show that the so-called *Fierz vector*

$$\mathbf{Z} = \mathbf{r} \times \mathbf{p} - q\,q_m\,\frac{\mathbf{r}}{r}$$

remains constant.

Solution

(a) Using a spherical gaussian surface centered on the monopole, we get

$$\nabla \cdot \mathbf{B} = 4\pi q_m \delta(\mathbf{r}) \Rightarrow \oint \mathbf{B} \cdot d\mathbf{A} = 4\pi q_m \int \delta(\mathbf{r})\mathrm{dVol} = q_m \Rightarrow \mathbf{B} = \frac{4\pi q_m}{4\pi r^2}\hat{r}$$

in spherical coordinates

(b) Note that, since $\nabla \cdot \mathbf{B} \neq 0$, the existence of a vector potential $\mathbf{B} \neq \nabla \times \mathbf{A}$ is non-trivial (it's possible to find an $\mathbf{A}$ that exists locally but not globally). We then used $\mathbf{F} = m\mathbf{a}$ directly

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v} \times \mathbf{B} = \frac{qq_m}{r^2}\mathbf{v} \times \hat{\mathbf{r}}$$

(c) The force law $q\mathbf{v} \times \mathbf{B}$ still does not work $\Rightarrow$ Kinetic energy is conserved

$$E = \frac{1}{2}mv^2 = \text{constant}$$

For the other conserved quantity, we might guess that a modified angular momentum is conserved based on our experience with the $q\mathbf{v} \times \mathbf{B}$ force law within Lagrangian. We check

$$\frac{d\mathbf{z}}{dt} = \mathbf{v} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}} - \frac{qq_m\mathbf{v}}{r} + \frac{qq_m}{r^2}\dot{r}r\hat{\mathbf{r}} = 0$$

$$\dot{\mathbf{p}} \propto \mathbf{v} \times \mathbf{r} \to (r\mathbf{v} - v_r\mathbf{r})\frac{qq_m}{r^2}$$

■

⋆⋆⋆ **Problem 8.13** Consider a charged *relativistic* particle of charge q and mass m moving in a cylindrically symmetric magnetic field with $\mathrm{B}^{\varphi} = 0$. (a) Show that this general setup can be described with a vector potential that has one non-zero component $\mathrm{A}^{\varphi}(\rho, z)$. (b) Write the equations of motion in cylindrical coordinates. (c) Consider circular orbits only and show that this implies that we need $\mathrm{B}^{\rho} = 0$. Then find the form of B^{z} needed to achieve circular orbits.

Solution

(a) Using cylindrical coordinates we have

$$B^{\rho} = \frac{1}{\rho}\frac{\partial A^2}{\partial \varphi} - \frac{\partial A^{\varphi}}{\partial z} \qquad B^{\varphi} = \frac{\partial A\rho}{\partial z} - \frac{\partial A^z}{\partial \rho} = 0$$

$$B^z = \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}(\rho A^{\varphi}) - \frac{\partial A^{\rho}}{\partial \varphi}\right)$$

We can choose a gauge where

$$A^{\rho} = A^z = 0 \Rightarrow B^{\rho} = -\frac{\partial A^{\varphi}}{\partial z} \qquad B^z = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A^{\varphi})$$

where $A^{\varphi}(\rho, z)$ by symmetry.

(b) We have

$$L = -mc^2(1 - \frac{1}{c^2}(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2))^{1/2} - \frac{q}{c}\rho\dot{\varphi}A^{\varphi}(\rho, z)$$

Equation of ρ:

$$\frac{\partial L}{\partial \dot{\rho}} = +\frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}\frac{\dot{\rho}}{c^2}$$

$$(*) \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\rho}\right) = \frac{d}{dt}\left(\frac{m\dot\rho}{\sqrt{1-v^2/c^2}}\right) = \frac{\partial L}{\partial \rho} = \frac{m\rho\dot\varphi^2}{\sqrt{1-v^2/c^2}} - \frac{q}{c}\dot\varphi A^\varphi - \frac{q}{c}\rho\dot\varphi\frac{\partial A^\varphi}{\partial \rho}$$

Equation of φ:

$$(**) \quad \frac{d}{dt}\left(\frac{m\rho^2\dot\varphi}{\sqrt{1-v^2/c^2}} - \frac{q}{c}\rho A^\varphi\right) = 0$$

Equation of z:

$$\frac{m\ddot z}{\sqrt{1-v^2/c^2}} = -\frac{q}{c}\rho\dot\varphi\frac{\partial A^\varphi}{\partial z}$$

(c) For circular orbit, we need

$$\dot\rho = 0 \quad \text{and} \quad \frac{\partial A^\varphi}{\partial z} = 0 \Rightarrow B^\rho = 0 \quad \text{and} \quad A^\varphi(\rho)$$

From (*), we have:

$$\frac{m\rho\dot\varphi^2}{\sqrt{1-v^2/c^2}} = +\frac{q}{c}\dot\varphi A^\varphi + \frac{q}{c}\rho\dot\varphi\frac{\partial A^\varphi}{\partial \rho} \qquad (***)$$

From (**), we have:

$$\frac{m\rho^2\dot\varphi^2}{\sqrt{1-v^2/c^2}} - \frac{q}{c}\rho A^\varphi = \text{constant} = \ell \Rightarrow \frac{m\rho\dot\varphi^2}{\sqrt{1-v^2/c^2}} - \frac{q}{c}A^\varphi = \frac{\ell}{\rho}$$

$$(***) \Rightarrow \frac{m\rho\dot\varphi^2}{\sqrt{1-v^2/c^2}} = +\frac{q}{c}A^\varphi + \frac{q}{c}\rho\frac{\partial A^\varphi}{\partial \rho} = +\frac{q}{c}\frac{\partial}{\partial\rho}(\rho A^\varphi)$$

Taking the ratio, we get

$$1 = \frac{\frac{q}{c}A^\varphi + \frac{\ell}{\rho}}{+\frac{q}{c}\frac{\partial}{\partial\rho}(\rho A^\varphi)}$$

$$\text{or} \quad \frac{\partial}{\partial\rho}(\rho A^\varphi) = +A^\varphi + \frac{c\ell}{q\rho} \Rightarrow \rho\frac{\partial A^\varphi}{\partial\rho} = \frac{c\ell}{q\rho}$$

$$\Rightarrow A^\varphi = -\frac{c\ell}{q\rho} + \text{constant} \Rightarrow B^z = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\left(-\frac{c\ell}{q\rho} + k\right)\right) = \frac{k}{\rho}$$

$$\mathbf{B} = \frac{k}{\rho}\hat z$$

ℓ independent. ∎

★★ **Problem 8.14** For the previous problem, find the angular speed by which the particle spins about the magnetic field in terms of the radius of the circular orbit ρ and other constants in the problem.

Solution

$$\frac{m\rho\dot{\varphi}}{\sqrt{1-v^2/c^2}} = \frac{q}{c}\frac{\partial}{\partial\rho}(\rho A^{\varphi}) = \frac{q}{c}\frac{\partial}{\partial\rho}\left(\rho\left(1-\frac{c\ell}{q\rho}+k\right)\right) = \frac{q}{c}k$$

$$\Rightarrow \frac{(m\rho\dot{\varphi})^2}{1-\frac{\rho^2\dot{\varphi}^2}{c^2}} = \frac{q^2}{c^2}k^2$$

$$\Rightarrow m^2\rho^2\dot{\varphi}^2 = \frac{q^2k^2}{c^2} - \frac{\rho^2}{c^2}\dot{\varphi}^2\frac{q^2k^2}{c^2}$$

$$\Rightarrow \dot{\varphi}^2\left(m^2\rho^2 + \frac{q^2k^2}{c^4}\rho^2\right) = \frac{q^2k^2}{c^2}$$

$$\Rightarrow \dot{\varphi} = \frac{qk/c}{\rho\sqrt{m^2+\frac{q^2k^2}{c^2}}} = \frac{1}{\rho}\frac{1}{\sqrt{1+\frac{m^2c^2}{q^2k^2}}}$$

■

⋆ **Problem 8.15** A cyclotron is made of sheet metal in the form of an empty tuna-fish can, set on a table with a flat-side down and then sliced from above through its center into two D-shaped pieces. The two "Dees" are then separated slightly so there is a small gap between them. A high-frequency alternating voltage is applied to the Dees, so they are always oppositely charged. At peak voltage there is therefore an electric field in the gap from the positive to the negative Dee that can accelerate charges across the gap. There is also a constant and uniform magnetic field applied vertically, *i.e.*, perpendicular to the Dees, supplied by a large external electromagnet. Therefore after a charged particle has been accelerated across the gap it enters a Dee, where it follows a semicircular path due to the magnetic field and maintains constant speed because the electric field inside a Dee is negligible. By the time the charged particle has completed a semicircle it arrives back at the gap, but by now the charges on the two Dees have been reversed, so the particle is again accelerated in the gap, entering the previous Dee and then executing a larger semicircular path this time because it is moving faster. As the particle moves faster and faster the semicircular paths increase in radius, so in effect the particle moves in a spiraling path until it reaches the outer edge of the machine, where by then it has achieved a very large kinetic energy due to the multiple accelerations it has received by repeatedly passing through the gap. It is then deflected out of the cyclotron where it causes a high-energy collision with other particles at rest in the lab. (a) Assuming that the charged particle is nonrelativistic, show that its kinetic energy by the time it reaches the outer radius R of the cyclotron is $T = q^2B^2R^2/2mc^2$, where q and m are the particle's charge and mass, B is the magnetic field, and R is the outer radius of the cyclotron. (b) If we want to accelerate protons to a kinetic energy of 16 MeV, what must be the applied magnetic field B (in Gauss) if the diameter of the cyclotron is 152 cm? Protons have mass energy mc^2 = 938 MeV and charge $q = 4.8 \times 10^{-10}$ esu. Note that 1 eV = 1.602 $\times 10^{-12}$ ergs = 1.602 $\times 10^{-19}$ Joules. In Gaussian units B is measured in "Gauss," and in Standard International (SI) units B is measured in "Teslas," where 1 Tesla = 10^4 Gauss.

Solution

(a) Slowly spirally outward, the protons move in a nearly circular orbit as they reach the outer radius R, with force

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}/c$$

and acceleration v^2/R. So $F = ma$ gives

$$qvB/c = mv^2/R, \quad \text{so} \quad v = qBR/mc.$$

Therefore the kinetic energy is

$$T = \frac{1}{2}mv^2 = \frac{q^2B^2R^2}{2mc^2}$$

(b) Therefore

$$B = \left(\frac{2mc^2}{q^2R^2}\right)^{1/2} T^{1/2} = \frac{(2 \times 938\,\text{MeV})^{1/2}(16\,\text{MeV})^{1/2}}{4.8 \times 10^{-10}(152\,\text{cm}/2)}$$

$$= \frac{173\,\text{MeV}}{365} \times 10^{10} \times \left(\frac{10^6\,\text{eV}}{\text{MeV}}\right) \frac{1.602 \times 10^{-12}\,\text{ergs}}{eV}$$

$$= 0.76 \times 10^4\,\text{Gauss} = 7,600\,\text{Gauss} = 0.76\,\text{Teslas}$$

■

⋆⋆ **Problem 8.16** A nice feature of the cyclotron described in the preceding problem is that the alternating current frequency applied to the "Dees" is a constant $\omega = qB/mc$ for nonrelativistic particles, regardless of their energy, so the circulating particles will arrive at the gaps at just the right time. No matter the radius at which a particle orbits, the time it takes to travel between two gap encounters is exactly the same. (a) Show that this is no longer true for relativistic particles. Find a new expression for ω in terms of q, B, m, c, and $\gamma \equiv (1 - \beta^2)^{-1/2} \equiv (1 - v^2/c^2)^{-1/2}$. (b) How might one design an "isochronous cyclotron," in which relativistic protons will still reach the gaps at the correct time, with the same constant-frequency alternating current applied to the Dees? (c) The TRIUMF isochronous cyclotron has a proton outer orbital radius of 7.9 m, where the protons have a kinetic energy of 510 MeV. What is the magnetic field strength at the outer orbit? (d) How fast are these protons moving, expressed as a fraction of the speed of light?

Solution

(a) For relativistic particles it is still true that

$$\mathbf{F} = d\mathbf{p}/dt$$

but now $\mathbf{p} = \gamma m\mathbf{v}$. Therefore

$$\frac{d\mathbf{p}}{dt} = \gamma m\frac{d\mathbf{v}}{dt} + \frac{d\gamma}{dt}m\mathbf{v},$$

while the magnetic force still has the form

$$\mathbf{F}_B = (q/c)\mathbf{v} \times \mathbf{B}$$

But $\mathbf{v} \perp \mathbf{B}$, so

$$F_B = (q/c)vB = \gamma m \left| \frac{d\mathbf{v}}{dt} \right|$$

so

$$\frac{d\gamma}{dt} m\mathbf{v} \text{ is } \perp \text{ to } q\frac{\mathbf{v} \times \mathbf{B}}{c}$$

As the particle moves in a nearly circular orbit, the change in its velocity is $\Delta v = v\Delta\theta$, in a direction perpendicular to $\mathbf{v}$, so

$$\Delta v = v\Delta\theta = v\frac{\Delta s}{r} = v\frac{v\Delta t}{r} = \frac{v^2}{r}\Delta t$$

just as for nonrelativistic motion, so $a = v^2/r$. Therefore

$$\frac{q}{c}vB = \gamma m\frac{v^2}{R} \Rightarrow v = R\omega = \frac{qB/c}{\gamma m/R} = \frac{qBR}{\gamma mc}$$

so $\omega = qB/\gamma mc$ less than for non-relativistic motion by the factor $1/\gamma$.

(b) By increasing the magnetic field with time, so B/γ remains constant as γ increases.

(c) The magnetic field is:

$$B = \frac{\gamma mc\omega}{q} = \frac{\gamma mc}{q}\frac{v}{R}.$$

Also

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \Rightarrow \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}}, \quad \text{so} \quad B = \frac{mc^2}{qR}\sqrt{\gamma^2 - 1}.$$

But

$$T = (\gamma - 1)mc^2, \quad \text{so} \quad \gamma = 1 + T/mc^2 \Rightarrow B = \frac{mc^2}{qR}\sqrt{\left(\frac{T}{mc^2}\right)^2 + \frac{2T}{mc^2}}$$

Note that

$$T/mc^2 = 510\,\text{MeV}/938\,\text{MeV} = 0.54$$

$$\text{so} \quad B = \frac{(938\,\text{MeV}) \times 10^6\,\text{eV/MeV} \times 1.6 \times 10^{-12}\,\text{ergs/eV}}{4.8 \times 10^{-10}\,\text{eso} \times 790\,\text{cm}}\sqrt{(.54)^2 + 2(-54)}$$

$$= 0.396 \times 10^4\sqrt{0.292 + 1.08} = 4,600\,\text{Gauss} = 4.6 \times 10^3\,\text{Gauss} = 0.46\,\text{Tesla}$$

(d)

$$\frac{T}{mc^2} = \frac{510\,\text{MeV}}{928\,\text{MeV}} = 0.54 = (\gamma - 1) \Rightarrow \gamma = 1.54$$

$$\frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} = \sqrt{1 - 0.42} = 0.76$$

■

⋆ **Problem 8.17** Several problems are encountered in trying to scale up cyclotrons to produce increasingly energetic protons. One of them is that the external magnets have to be made larger and larger, which is prohibitively expensive and ultimately becomes completely unfeasible. A newer generation of machines called synchrotrons were therefore invented in which protons can circulate at constant radius, so the magnets only need to cover a much smaller area. (a) In that case, how can relativistic protons be accelerated to higher and higher speeds if their orbital radius remains constant? (b) The Large Hadron Collider (LHC) of CERN (Organisation Européenne pour la Recherche Nucléair) accelerates protons up to total energies as large as 7.0 TeV (1 TeV = 10^3 GeV = 10^6 MeV) or perhaps even larger. The circumference of the proton path is 27 km, lying in an underground tunnel near Geneva, partly in Switzerland and partly in France. What magnetic field B is required in this case?

Solution

The protons obey $\mathbf{F} = d\mathbf{p}/dt$ with an inward force $F = \frac{q}{c}vB$ and where $\mathbf{p} = \gamma m\mathbf{v}$ is the momentum. The time derivative of $\mathbf{p}$ is

$$d\mathbf{p}/dt = \frac{d\gamma}{dt}m\mathbf{v} + \gamma m d\mathbf{v}/dt$$

where the inward component is

$$\gamma m\, d\mathbf{v}/dt = \gamma m\, v^2/R.$$

Note that

$$\Delta v = v\Delta\theta = v\frac{\Delta s}{R} = v\left(\frac{v\Delta t}{R}\right) = \frac{v^2}{R}\Delta t$$

as the proton moves in a circular arc of radius R. Therefore

$$(q/c)vB = \gamma mv^2/R \Rightarrow v = \frac{qBR}{\gamma mc}$$

Therefore

$$B = \frac{\gamma mcv}{qR}$$

At 7.0 TeV, a proton of min-energy 938 MeV is moving at nearly the speed of light. Also,

$$\gamma = \frac{E}{mc^2} = \frac{7.0 \times 10^6\,\text{MeV}}{938\,\text{MeV}V} \cong 7.5 \times 10^3$$

Therefore

$$B = \frac{\gamma mc^2(v/c)}{qR} = \frac{E \times 1 \times 2\pi}{4.8 \times 10^{-10}\,\text{esu C}} = \frac{7.0 \times 10^{12}\,\text{eV} \times 1.60 \times 10^{-12}\,\text{ergs/ eV} \times 2\pi}{4.8 \times 10^{-10}\,\text{esu} \times 27\,\text{km} \times 10^5\,\text{cm/km}}$$

$$= .54 \times 10^5\,\text{Gauss} = 54,000\,\text{Gauss} = 5.4\,\text{Teslas}$$

■

⋆ **Problem 8.18** Consider two inertial frames $\mathcal{O}$ and $\mathcal{O}'$ where $\mathcal{O}'$ is moving with velocity $\mathbf{v}$ relative to $\mathcal{O}$. We split all three-vectors in components parallel and perpendicular to the direction of the Lorentz boost, $\mathbf{v}$: for example, we have $\mathbf{E} = \mathbf{E}_{\parallel} + \mathbf{E}_{\perp}$. Show that the Lorentz transformations of the electric and magnetic fields given in the text, equations 8.35 and 8.36 can be written instead as

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} \quad , \quad \mathbf{E}'_{\perp} = \gamma\left(\mathbf{E}_{\perp} + ((\mathbf{v}/c) \times \mathbf{B})_{\perp}\right)$$
$$\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel} \quad , \quad \mathbf{B}'_{\perp} = \gamma\left(\mathbf{B}_{\perp} - (\mathbf{v}/c \times \mathbf{E})_{\perp}\right)$$

Solution

$$\mathbf{E}' = \gamma\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) + (1-\gamma)\frac{\mathbf{E}\cdot\mathbf{v}}{v^2}\mathbf{v}$$

Therefore

$$\mathbf{E}'_{\parallel} = \gamma\left(\mathbf{E}_{\parallel} + \left(\frac{\mathbf{v}}{c} \times \mathbf{B}\right)_{\parallel}\right) + (1-\gamma)\frac{\mathbf{E}_{\parallel} v}{v^2} v$$

$$\mathbf{E}'_{\parallel} = \gamma\mathbf{E}_{\parallel} + (1-\gamma)\mathbf{E}_{\parallel} = \mathbf{E}_{\parallel} \quad \text{since} \quad \frac{\mathbf{v}}{c} \times B \quad \text{is} \perp \text{to } \mathbf{v}$$

Also

$$\mathbf{E}'_{\perp} = \gamma\left(\mathbf{E}_{\perp} + \left(\frac{\mathbf{v}}{c} \times \mathbf{B}\right)_{\perp}\right) \quad \text{since} \quad \mathbf{E}_{\perp}\cdot\mathbf{v} = 0 \quad \text{by definition}$$

Similarly, Eq 8.36 is

$$\mathbf{B}' = \gamma(\mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E}) + (1-\gamma)\frac{\mathbf{B}\cdot\mathbf{v}}{v^2}\mathbf{v}$$

so

$$\mathbf{B}'_{\parallel} = \gamma\mathbf{B}_{\parallel} + (1-\gamma)\frac{Bv}{v^2}\mathbf{v} = \mathbf{B}_{\parallel} \qquad \text{since} \quad \frac{\mathbf{v}}{c} \times \mathbf{E} \text{ is} \perp \text{to } \mathbf{v}$$

and

$$\mathbf{B}'_{\perp} = \gamma\left(\mathbf{B}_{\perp} - \left(\frac{\mathbf{v}}{c} \times \mathbf{E}\right)_{\perp}\right)$$

since $\mathbf{B}_{\perp}\cdot\mathbf{v} = 0$ by definition. ■

⋆⋆⋆ **Problem 8.19** We discovered in the text that the scalar and vector potentials are components of a four vector $A^{\mu} = (\phi, \mathbf{A})$. In this problem, we will take as given the existence of this four-vector potential A^{μ} and, using the known Lorentz transformation of a four-vector and the relations of A^{μ} to $\mathbf{E}$ and $\mathbf{B}$, we want to *derive* the Lorentz transformations of $\mathbf{E}$ and $\mathbf{B}$. Consider two inertial frames $\mathcal{O}$ and $\mathcal{O}'$ where $\mathcal{O}'$ is moving with velocity $\mathbf{v}$ relative to $\mathcal{O}$. We split all three-vectors in components parallel and perpendicular to the direction of the Lorentz boost, $\mathbf{v}$: for example, we have $\mathbf{E} = \mathbf{E}_{\parallel} + \mathbf{E}_{\perp}$. Note that the gradient vector can also be decomposed as $\boldsymbol{\nabla} = \boldsymbol{\nabla}_{\parallel} + \boldsymbol{\nabla}_{\perp}$. (a) First show that $\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}$. (b) Show next that $\boldsymbol{\nabla}'_{\parallel} = \gamma\left(\boldsymbol{\nabla}_{\parallel} + (\mathbf{v}/c^2)(\partial/\partial t)\right)$. (c) Finally, show that $\mathbf{B}'_{\perp} = \gamma\left(\mathbf{B}_{\perp} - ((\mathbf{v}/c) \times \mathbf{E})_{\perp}\right)$ as in the previous problem.

Solution

The Lorentz transformation for the four-vector $(\phi, \mathbf{A})$ is

$$\phi = \gamma\left(\phi' + \frac{v}{c}A^{x'}\right), \quad A^x = \gamma\left(A^{x'} + \frac{v}{c}\phi'\right), \quad A^y = A^{y'}, \quad A^z = A^{z'}$$

Now take the curl

$$\begin{aligned}\mathbf{B} = \nabla \times \mathbf{A} &= (\hat{\mathbf{x}}\partial_x + \hat{\mathbf{y}}\partial_y + \hat{\mathbf{z}}\partial_z) \times (A^x\hat{\mathbf{x}} + A^y\hat{\mathbf{y}} + A^z\hat{\mathbf{z}}) \\ &= (\partial_y A^z - \partial_z A^y)\hat{\mathbf{x}} + (\partial_z A^x - \partial_x A^z)\hat{\mathbf{y}} + (\partial_x A^y - \partial_y A^x)\hat{\mathbf{z}} \\ &\equiv B^x\hat{\mathbf{x}} + B^y\hat{\mathbf{y}} + B^z\hat{\mathbf{z}}\ .\end{aligned}$$

So

$$B^x = \partial_z A^z - \partial_z A^y = \partial_y A^{z'} - \partial_z A^{y'} = B'^x.$$

(a) That is $\mathbf{B}'_{\|} = \mathbf{B}_{\|}$, since the x direction is parallel to $\mathbf{v}$.
(b) From multivariable calculus

$$\frac{\partial}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial x'}\frac{\partial}{\partial t}$$

or in other nutation,

$$\nabla'_{\|} = \frac{\partial x}{\partial x'}\nabla_{\|} + \frac{\partial t}{\partial x'}\frac{\partial}{\partial t}.$$

The Lorentz transformation equation

$$x = \gamma(x' + vt') \quad \text{and} \quad t = \gamma(t' + \frac{v}{c^2}x')$$

show that

$$\partial x/\partial x' = \gamma \quad \text{and} \quad \partial t/\partial x' = \gamma v/c^2.$$

Therefore

$$\nabla'_{\|} = \gamma(\nabla_{\|} + (\mathbf{v}/c^2)\partial/\partial t) \quad \text{as claimed.}$$

(c)

$$\begin{aligned}B'^y = (\Delta' \times \mathbf{A}')^y = \partial A^{x'}/partialz' - \partial A^{z'}/\partial x' &= \partial_z\gamma\left(A^x - \frac{v}{c}\phi\right) - \gamma\left(\partial_x + \frac{v}{c^2}\partial t\right)A^z \\ &= \gamma\left[\left(\frac{\partial A^x}{\partial z} - \frac{\partial A^z}{\partial x}\right) - \frac{v}{c}\left(\frac{\partial\phi}{\partial z} + \frac{1}{c}\frac{\partial A^z}{\partial t}\right)\right] \\ &= \gamma\left[B^y + \frac{v^x}{c}E^z\right] = \gamma\left[B^y - (\mathbf{v}/c \times \mathbf{E})^y\right]\end{aligned}$$

Similarly,

$$B^{z'} = \gamma\left[B^z - (v/c \times \mathbf{E})^z\right].$$

Altogether, since the y and z directions are $\perp$ to $\mathbf{v}$, we have

$$\mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - (\mathbf{v}/c \times \mathbf{E})_{\perp})$$

as claimed. ■

★★★ **Problem 8.20** In the previous problem, you derived the Lorentz transformations of the $\mathbf{B}$ field starting with the assumption that the scalar and vector potentials are components of a four vector $A^\mu = (\phi, \mathbf{A})$. Using a similar approach, derive the Lorentz transformation of the electric field $\mathbf{E}$; show that you get $\mathbf{E}'_{\|} = \mathbf{E}_{\|}$ and $\mathbf{E}'_{\perp} = \gamma\,(\mathbf{E}_{\perp} + ((\mathbf{v}/c) \times \mathbf{B})_{\perp})$. Note that this is a more involved computation than in the previous problem.

Solution

The electric field in the primed frame is the x direction (which is the direction parallel to $\mathbf{v}$) is

$$E^{x'} = -\frac{\partial \phi'}{\partial x'} - \frac{1}{c}\frac{\partial A^{x'}}{\partial t'}$$

in terms of ϕ' and $\mathbf{A}'$. Here $(\phi', \mathbf{A}')$ is a four-vector, so

$$\phi' = \gamma(\phi - v/c\, A_x)$$

and

$$A^{x'} = \gamma(A^x - (v/c)\phi)$$

Therefore,

$$E^{x'} = -\frac{\partial}{\partial x'}\left[\gamma(\phi - (v/c)A^{x'})\right] - \frac{1}{c}\frac{\partial}{\partial t'}(\gamma(A^x - (v/c)\phi)).$$

Also

$$\frac{\partial}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial x'}\frac{\partial}{\partial t}$$

$$= \gamma(\partial/\partial x) + \gamma(v/c^2)\frac{\partial}{\partial t},$$

since

$$x = \gamma(x' + vt') \quad \text{and} \quad t = \gamma(t' + v/c^2\, x')$$

Similarly

$$\frac{\partial}{\partial t'} = \frac{\partial x}{\partial t'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} = \gamma v\frac{\partial}{\partial x} + \gamma\frac{\partial}{\partial t}$$

Therefore

$$E^{x'} = -\left(\gamma\frac{\partial}{\partial x} + \gamma v/c^2\frac{\partial}{\partial t}\right)(\gamma(\phi - v/cA^x)) - \frac{1}{c}\left[\gamma v\frac{\partial}{\partial x} + \gamma\frac{\partial}{\partial t}\right][\gamma(A^x - v/c\,\phi)]$$

$$-\gamma^2(\frac{\partial}{\partial x} + v/c^2\frac{\partial}{\partial t})(\phi - (v/c)A^x) - \gamma^2(\frac{v}{c}\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t})(A^x - \frac{v}{c}\phi)$$

$$= -\gamma^2\left[\frac{\partial\phi}{\partial x} - \frac{v}{c}\frac{\partial A^x}{\partial x} + \frac{v}{c^2}\left(\frac{\partial\phi}{\partial t} - \frac{v}{c}\frac{\partial A^x}{\partial t}\right) + \frac{v}{c}\left(\frac{\partial A^x}{\partial x} - \frac{v}{c}\frac{\partial\phi}{\partial x}\right) + \frac{1}{c}\left(\frac{\partial A^x}{\partial t} - \frac{v}{c}\frac{\partial\phi}{\partial t}\right)\right]$$

$$= -\gamma^2(1 - v^2/c^2)\frac{\partial\phi}{\partial x} - \gamma^2\left(\frac{\partial A^x}{\partial t}\right)\left(\frac{1}{c} - \frac{v^2}{c^3}\right)$$

$$= +\gamma^2(1 - v^2/c^2)\left[-\frac{\partial \phi}{\partial x} - \frac{1}{c}\frac{\partial A^x}{\partial t} = E^x\right]$$

since

$$\gamma^2 = (1 - v^2/c^2)^{-1}. \quad \text{Therefore} \quad E'_{\|} = E_{\|}$$

Similarly,

$$E^{y'} = -\frac{\partial \phi'}{\partial y'} - \frac{1}{c}\frac{\partial A^{y'}}{\partial t'}$$

where

$$\phi' = \gamma(\phi - (v/c)A^x) \quad \text{and} \quad A^{y'} = A^y$$

Also

$$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y} \quad \text{and} \quad \frac{\partial}{\partial t'} = \frac{\partial x}{\partial t'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} = \gamma v\frac{\partial}{\partial x} + \gamma\frac{\partial}{\partial t}.$$

Therefore

$$E^{y'} = -\frac{\partial}{\partial y}\gamma(\phi - (v/c)A^x) - \frac{1}{c}(\gamma v\frac{\partial}{\partial x} + \gamma\frac{\partial}{\partial t})A^y$$

$$= -\gamma\left[\frac{\partial \phi}{\partial y} - \frac{v}{c}\frac{\partial A^x}{\partial y} + \frac{v}{c}\frac{\partial A^y}{\partial x} + \frac{\partial A_y}{\partial t}\right]$$

$$= -\gamma\frac{\mathbf{v}}{c}\left(\frac{\partial A^y}{\partial x} - \frac{\partial A^x}{\partial y}\right) + \gamma\left(-\frac{\partial \phi}{\partial y} - \frac{\partial A^y}{\partial t}\right)$$

$$E'_y = \gamma\left[E^y + \left(\frac{\mathbf{v}}{c} \times \mathbf{B}\right)^y\right]$$

since

$$\left(\frac{\mathbf{v}}{c} \times \mathbf{B}\right)^y = \frac{v^x}{c} \cdot B^z = \frac{v}{c}\left[\frac{\partial A^x}{\partial y} - \frac{\partial A^y}{\partial x}\right]$$

In a simmilar way one can show that

$$E^{z'} = \gamma\left[E^z + \left(\frac{\mathbf{v}}{c} \times \mathbf{B}\right)^z\right]$$

so altogether

$$\mathbf{E}'_{\perp} = \gamma\left[\mathbf{E}_{\perp} + (\frac{\mathbf{v}}{c} \times \mathbf{B})_{\perp}\right]$$

■

★★ **Problem 8.21** Using the Lorentz transformations of the $\mathbf{E}$ and $\mathbf{B}$ fields, show that $E^2 - B^2$ is a Lorentz invariant; that is, show that $E'^2 - B'^2 = E^2 - B^2$.

Solution

Given

$$\mathbf{E}' = \gamma\left[\mathbf{E} + \mathbf{v}/c \times \mathbf{B}\right] + (1 - \gamma)\left(\frac{\mathbf{E} \cdot \mathbf{v}}{v^2}\right)\mathbf{v}$$

and

$$\mathbf{B}' = \gamma\left[\mathbf{B} + \mathbf{v}/c \times \mathbf{E}\right] + (1-\gamma)\left(\frac{\mathbf{B}\cdot\mathbf{v}}{v^2}\right)\mathbf{v}$$

Then

$$\begin{aligned}\mathbf{E}'^2 = \mathbf{E}'\cdot\mathbf{E} &= \gamma^2\left[\mathbf{E} + \mathbf{v}/c \times \mathbf{B}\right]^2 + (1-\gamma)^2\left(\frac{\mathbf{E}\cdot\mathbf{v}}{v^2}\right)^2 v^2 \\ &\quad + 2\gamma(1-\gamma)\left[\mathbf{E} + \mathbf{v}/c \times \mathbf{B}\right]\cdot\left(\frac{\mathbf{E}\cdot\mathbf{v}}{v^2}\right)\mathbf{v} \\ &= \gamma^2\left[\mathbf{E}^2 + (\mathbf{v}/c\times\mathbf{B})^2 + 2\mathbf{E}\cdot(\mathbf{v}/c\times\mathbf{B})\right] + (1-\gamma)^2\left(\frac{\mathbf{E}\cdot\mathbf{v}}{v^2}\right)^2 v^2 \\ &\quad + 2\gamma(1-\gamma)\mathbf{E}\cdot\mathbf{v}\left(\frac{\mathbf{E}\cdot\mathbf{v}}{v^2}\right)\end{aligned}$$

Note

$$\left(\frac{\mathbf{v}}{c}\times\mathbf{B}\right)\cdot\mathbf{v} = 0$$

since

$$\left(\frac{\mathbf{v}}{c}\times\mathbf{B}\right) \text{ is } \perp \text{ to } \mathbf{v}.$$

Also note that

$$\mathbf{E}\cdot(\mathbf{v}/c\times\mathbf{B}) = \mathbf{v}/c\cdot(\mathbf{B}\times\mathbf{E})$$

Vector identity

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})$$

Also

$$(1-\gamma)^2 + 2\gamma(1-\gamma) = (1-\gamma)\left[1-\gamma+2\gamma\right] = 1-\gamma^2$$

Therefore

$$\mathbf{E}'^2 = \gamma^2\left[\mathbf{E}^2 + \left(\frac{\mathbf{v}}{c}\times\mathbf{B}\right)^2 + \frac{2\mathbf{v}}{c}\cdot(\mathbf{B}\times\mathbf{E})\right] + (1-\gamma^2)\frac{(\mathbf{E}\cdot\mathbf{v})^2}{v^2}$$

Similarly,

$$\mathbf{B}'^2 = \gamma^2\left[\mathbf{B}^2 + \left(\frac{\mathbf{v}}{c}\times\mathbf{E}\right)^2 + \frac{2\mathbf{v}}{c}\cdot(\mathbf{E}\times\mathbf{B})\right] + (1-\gamma^2)\frac{(\mathbf{B}\cdot\mathbf{v})^2}{v^2}$$

so

$$\begin{aligned}\mathbf{E}'^2 - \mathbf{B}'^2 &= \gamma^2((\mathbf{E}^2 - \mathbf{B}^2) + \left(\frac{\mathbf{v}}{c}\times\mathbf{B}\right)^2 - \left(\frac{\mathbf{v}}{c}\times\mathbf{E}\right)^2 \\ &\quad + \frac{2\mathbf{v}}{c}\left[(\mathbf{B}\times\mathbf{E}) + (\mathbf{E}\times\mathbf{B})\right] + (1-\gamma^2)\left[\frac{(\mathbf{E}\cdot\mathbf{v})^2 - (\mathbf{B}\cdot\mathbf{v})^2}{v^2}\right])\end{aligned}$$

Note that

$$\mathbf{B}\times\mathbf{E} = -\mathbf{E}\times\mathbf{B}$$

so two terms cancel. Also without loss of generality assume $\mathbf{v} = v\hat{x}$. Then

$$\left(\frac{\mathbf{v}}{c} \times \mathbf{B}\right) = \frac{v}{c}(B^y\hat{\mathbf{z}} - B^z\hat{\mathbf{y}})$$

so

$$\left(\frac{\mathbf{v}}{c} \times \mathbf{B}\right)^2 = \frac{v^2}{c^2}((B^y)^2 + (B^z)^2)$$

and similarly

$$\left(\frac{\mathbf{v}}{c} \times \mathbf{E}\right)^2 = \frac{v^2}{c^2}((E^y)^2 + (E^z)^2)$$

So

$$\mathbf{E}'^2 - \mathbf{B}'^2 = \gamma^2 \left[\mathbf{E}^2 - \mathbf{B}^2 - \frac{v^2}{c^2}((E^y)^2 + (E^z)^2 - (B^y)^2 - (B^z)^2)\right] + (1-\gamma^2)((E^x)^2 - (B^x)^2)$$

$$= \gamma^2 \left[\mathbf{E}^2 - \mathbf{B}^2 - \frac{v^2}{c^2}(\mathbf{E}^2 - \mathbf{B}^2) + \frac{v^2}{c^2}((E^y)^2 - (E^x)^2) - ((E^y)^2 - (B^x)^2)\right] + ((E^x)^2 - (B^x)^2)$$

$$= \mathbf{E}^2 - \mathbf{B}^2 + ((E^x)^2 - (B^x)^2)\left[1 - \gamma^2(1 - v^2/c^2)\right] = \mathbf{E}^2 - \mathbf{B}^2$$

as claimed. ■

★★ **Problem 8.22** Using the Lorentz transformations of the $\mathbf{E}$ and $\mathbf{B}$ fields, show that $\mathbf{E} \cdot \mathbf{B}$ is a Lorentz invariant; that is, show that $\mathbf{E}' \cdot \mathbf{B}' = \mathbf{E} \cdot \mathbf{B}$.

Solution

Give

$$\mathbf{E}' = \gamma(\mathbf{E} + \mathbf{v}/c \times \mathbf{B}) + (1-\gamma)\left(\frac{\mathbf{E} \cdot \mathbf{v}}{v^2}\right)\mathbf{v}$$

$$\mathbf{B}' = \gamma(\mathbf{B} - \mathbf{v}/c \times \mathbf{E}) + (1-\gamma)\left(\frac{\mathbf{B} \cdot \mathbf{v}}{v^2}\right)\mathbf{v}$$

Then

$$\mathbf{E}' - \mathbf{B}' = \gamma^2\left[\mathbf{E} + \mathbf{v}/c \times \mathbf{B}\right] \cdot \left[\mathbf{B} - \mathbf{v}/c \times \mathbf{E}\right] + (1-\gamma)^2\left(\frac{\mathbf{E} \cdot \mathbf{v}}{v^2}\right)\left(\frac{\mathbf{B} \cdot \mathbf{v}}{v^2}\right)v^2$$

$$+\gamma\left[\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right] \cdot (1-\gamma)\mathbf{v}\left(\frac{\mathbf{B} \cdot \mathbf{v}}{v^2}\right) + \gamma(1-\gamma)\left(\mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E}\right) \cdot \mathbf{v}\left(\frac{\mathbf{E} \cdot \mathbf{v}}{v^2}\right).$$

Now

$$\mathbf{v} \cdot \left(\frac{\mathbf{v}}{c} \times \mathbf{B}\right) = \mathbf{v} \cdot \left(\frac{\mathbf{v}}{c} \times \mathbf{E}\right) = 0$$

($\mathbf{v}$ is $\perp$ to $(\frac{\mathbf{v}}{c} \times \mathbf{B})$) And without loss of generality we can let $\mathbf{v} = v\hat{x}$. Therefore

$$\mathbf{E}' \cdot \mathbf{B}' = \gamma^2\left[\mathbf{E} \cdot \mathbf{B} - \left(\frac{\mathbf{v}}{c} \times \mathbf{B}\right) \cdot \left(\frac{\mathbf{v}}{c} \times \mathbf{E}\right)\right]$$

$$+(1-\gamma)^2 E^x B^x + \gamma(1-\gamma)\left[E^x B^y + B^y E^x\right]$$

Now

$$\left(\frac{v^y}{c}\times\mathbf{B}\right)\cdot\left(\frac{v^x}{c}\times\mathbf{E}\right)=\frac{v^x\hat{\mathbf{x}}}{c}\times(B^y\hat{\mathbf{y}}+B^z\hat{\mathbf{z}})\cdot\frac{v^x}{c}\hat{\mathbf{x}}\times(E^y\hat{\mathbf{y}}+E^z\hat{\mathbf{z}})$$

$$=\left(\frac{v}{c}\right)^2[B^z\hat{\mathbf{z}}-B^z\hat{\mathbf{y}}]\cdot[E^y\hat{\mathbf{z}}-E^z\hat{\mathbf{y}}]=\frac{v^2}{c^2}(E^yB^y+E^zB^z)$$

so

$$\mathbf{E}'\cdot\mathbf{B}'=\gamma^2\left[\mathbf{E}\cdot\mathbf{B}-\frac{v^2}{c^2}(E^yB^y+E^zB^z)\right]+\left[(1-\gamma)^2+2\gamma(1-\gamma)\right]E^xB^z$$

$$=\gamma\left[E^xB^x+E^yB^y+E^zB^z-v^2/c^2(E^yB^y+E^zB^z)+(1-\gamma^2)E^xB^x\right]$$

$$=E^xB^x+\gamma^2(1-v^2/c^2)(E^yB^y+E^zB^z)$$

$$=E^xB^x+E^yB^y+E^zB^z=\mathbf{E}\cdot\mathbf{B}$$

■

★★ **Problem 8.23** Show that the action of a relativistic charged particle (8.43) is invariant under a gauge transformation

Solution

$$S=-mc^2\int dt\sqrt{1-\frac{v^2}{c^2}}+q\int dt(-\phi+\mathbf{A}\cdot\frac{\mathbf{v}}{c})$$

The gauge transformation is

$$\mathbf{A}\to\mathbf{A}+\nabla f$$

and

$$\phi\to\phi-\frac{1}{c}\frac{\partial f}{\partial z}$$

The second term in the action then transforms as

$$q\int dt\left(-\phi+\mathbf{A}\cdot\frac{\mathbf{v}}{c}\right)\to q\int dt\left(-\phi+\frac{1}{c}\frac{\partial f}{\partial t}+\mathbf{A}\cdot\frac{\mathbf{v}}{c}+\nabla f\cdot\frac{\mathbf{v}}{c}\right)$$

But

$$\frac{1}{c}\frac{\partial f}{\partial t}+\nabla f\cdot\frac{\mathbf{v}}{c}=\frac{1}{c}\frac{\partial f}{\partial t}+\frac{\partial f}{\partial x^i}\frac{\dot{x}^i}{c}=\frac{1}{c}\frac{df}{dt}$$

by the chain rule on $f(t,x,y,z)$. Hence, the change in the action is

$$\Delta S=q\int dt\frac{df}{dt}$$

which can be integrated to give f evaluated at the boundaries of time. This cannot change the equations of motion $\to\Delta S=0$. ■

★★★ **Problem 8.24** Using Noether's theorem , find the conserved quantity that results from the invariance of the action (8.43) under a gauge transformation of the four-vector potential. For this, consider an infinitesimal but arbitrary gauge transformation.

Solution

The gauge transformation is

$$\phi \to \phi - \frac{1}{c}\frac{\partial f}{\partial t} = \phi', \qquad \mathbf{A} \to \mathbf{A} + \nabla f = \mathbf{A}'$$

so that

$$\Delta\phi = \phi' - \phi = -\frac{1}{c}\frac{\partial f}{\partial t} \qquad \Delta\mathbf{A} = \mathbf{A}' - \mathbf{A} = \nabla f$$

and

$$\Delta\mathbf{r} = 0 \qquad \delta t = 0$$

$$\Delta\left(q\int dt\,\left(-\phi + \mathbf{A}\cdot\frac{\mathbf{v}}{c}\right)\right) = q\int dt\,\left(-\Delta\phi + \Delta\mathbf{A}\cdot\frac{\mathbf{v}}{c}\right) = q\int dt\,\left(\frac{1}{c}\frac{\partial f}{\partial t} + \nabla f\cdot\frac{\mathbf{v}}{c}\right) = 0$$

by symmetry.

Now, we want to extract a differential equation for the charged particle that implies conservation. Write

$$q = \int d^3x\, q\delta^3(\mathbf{r})$$

where $\mathbf{r}(t)$ is the location of the particle.

$$\Rightarrow \int d^4x\,\left(q\delta^3(\mathbf{r})\frac{1}{c}\frac{\partial f}{\partial t} + q\delta^3(\mathbf{r})\nabla f\cdot\frac{\mathbf{v}}{c}\right) = 0$$

Define the charge density

$$q\delta^3(\mathbf{r}) \equiv \rho \Rightarrow \int d^4x\,\left(\rho\frac{1}{c}\frac{\partial f}{\partial t} + \rho\nabla f\cdot\frac{\mathbf{v}}{c}\right) = -\int d^4x\,\left(\frac{1}{c}\frac{\partial}{\partial t}\rho + \frac{1}{c}\nabla\cdot(\rho\mathbf{v})\right)f$$

where we used integration by parts to move the derivatives on ρ and $\rho\mathbf{v}$ (dropping a boundary term which evaluates ρ at ∞ where charges vanish.)

$$\Rightarrow \int d^4x f\left(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right) = 0$$

for ant f.

$$\Rightarrow \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

which identifies ρ as charge density and $\mathbf{J} = \rho\mathbf{v}$ as current density. ■

★★ **Problem 8.25** Derive the equations of motion resulting from the action of a relativistic charged particle (8.43) and verify that you get the Lorentz force law.

Solution

$$S = -mc^2 \int dt \sqrt{1 - \frac{v^2}{c^2}} + q \int dt \left(-\phi + \mathbf{A} \cdot \frac{\mathbf{v}}{c} \right)$$

$$\frac{\partial L}{\partial \dot{x}^2} = m\gamma_v \dot{x}^i + \frac{q}{c} A^i = p^i + \frac{q}{c} A^i$$

$$\frac{\partial L}{\partial x^i} = -q \frac{\partial \phi}{\partial x^i} + q \frac{\partial \mathbf{A}}{\partial x^i} \cdot \frac{\mathbf{v}}{c}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{d}{dt} p^i + \frac{q}{c} \frac{dA^i}{dt} = \frac{\partial L}{\partial x^i} = -q \frac{\partial \phi}{\partial x^i} + q \frac{\partial \mathbf{A}}{\partial x^i} \cdot \frac{\mathbf{v}}{c}$$

But

$$\frac{dA^i}{dt} = \frac{\partial A^i}{\partial t} + \frac{\partial A^i}{\partial x^j} \frac{dx^j}{dt}$$

by the chain rule.

$$\Rightarrow \frac{dp^i}{dt} = -\frac{q}{c} \frac{\partial A^i}{\partial t} - q \frac{\partial \phi}{\partial x^i} - \frac{q}{c} \frac{\partial A^i}{\partial x^j} \dot{x}^j + \frac{q}{c} \frac{\partial \mathbf{A}}{\partial x^i} \cdot \mathbf{v}$$

We know

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$

and

$$\mathbf{E} = -\boldsymbol{\nabla} \phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$$

Furthermore, consider the term

$$-\frac{q}{c} \frac{\partial A^i}{\partial x^j} \dot{x}^j + \frac{q}{c} \frac{\partial \mathbf{A}}{\partial x^2} \cdot \mathbf{v}$$

for say $i = x$

$$\Rightarrow -\frac{q}{c} \frac{\partial A^x}{\partial x^j} \dot{x}^j + \frac{q}{c} \frac{\partial \mathbf{A}}{\partial x} \cdot \mathbf{v}$$

$$= -\frac{q}{c} \frac{\partial A^x}{\partial x} \dot{x} - \frac{q}{c} \frac{\partial A^x}{\partial y} \dot{y} - \frac{q}{c} \frac{\partial A^x}{\partial z} \dot{z} + \frac{q}{c} \frac{\partial A^x}{\partial x} \dot{x} + \frac{q}{c} \frac{\partial A^y}{\partial x} \dot{y} + \frac{q}{c} \frac{\partial A^z}{\partial x} \dot{z}$$

$$= -\frac{q}{c} \dot{y} \left(\frac{\partial A^x}{\partial y} - \frac{\partial A^y}{\partial x} \right) + \frac{q}{c} \dot{z} \left(\frac{\partial A^z}{\partial x} - \frac{\partial A^x}{\partial z} \right)$$

$$= -\frac{q}{c} \dot{y} (\boldsymbol{\nabla} \times \mathbf{A})^z + \frac{q}{c} \dot{z} (\boldsymbol{\nabla} \times \mathbf{A})^y$$

$$= -\frac{q}{c} \dot{y} B^z + \frac{q}{c} \dot{z} B^y = -\frac{q}{c} \left(\mathbf{v} \times \mathbf{B} \right)^x$$

so that

$$\frac{dp^x}{dt} = qE^x - q \left(\frac{\mathbf{v}}{c} \times \mathbf{B} \right)^x$$

Similarly, we get

$$\frac{dp^y}{dt} = qE^y + q\left(\frac{\mathbf{v}}{c} \times \mathbf{B}\right)^y$$

$$\frac{dp^z}{dt} = qE^z + q\left(\frac{\mathbf{v}}{c} \times \mathbf{B}\right)^z$$

■

★★ **Problem 8.26** Show that Maxwell's equations given by 8.16 imply the *wave equations*

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho_Q$$

and

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J}\,.$$

To do this, you will need the vector identity $\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{A} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2\mathbf{A}$. You will also need to *fix* the gauge freedom using $\boldsymbol{\nabla} \cdot \mathbf{A} + (1/c)\partial\phi/\partial t = 0$. The latter is allowed due to the gauge symmetry discussed in the text.

Solution

We start from

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 0 \qquad \boldsymbol{\nabla} \cdot \mathbf{B} = 0$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{1}{2}\frac{\partial\mathbf{B}}{\partial t} \qquad \boldsymbol{\nabla} \times \mathbf{B} = \frac{1}{2}\frac{\partial\mathbf{E}}{\partial t}$$

with

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \qquad \mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{1}{2}\frac{\partial\mathbf{A}}{\partial t}\,.$$

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{A} = \frac{1}{c}\frac{\partial}{\partial t}\left(-\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right)$$

$$\Rightarrow \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \boldsymbol{\nabla}^2\mathbf{A} = -\boldsymbol{\nabla}\left(\frac{1}{c}\frac{\partial\phi}{\partial t}\right) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A}$$

$$\Rightarrow \boldsymbol{\nabla}^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} = \boldsymbol{\nabla}\left(\frac{1}{c}\frac{\partial\phi}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{A}\right)$$

Using the gauge symmetry, fix the gauge

$$\frac{1}{c}\frac{\partial\phi}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{A} = 0 \Rightarrow \boldsymbol{\nabla}^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = 0$$

as needed. Now substitute ϕ and $\mathbf{A}$ in (1):

$$\boldsymbol{\nabla} \cdot \left(-\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right) = 0 \Rightarrow -\boldsymbol{\nabla}^2\phi - \frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \mathbf{A}) = 0$$

Using the gauge fixing condition

$$\nabla \cdot \mathbf{A} = -\frac{1}{2}\frac{\partial \phi}{\partial t} \Rightarrow -\nabla^2 \phi + \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0$$

as needed. ■

⋆ **Problem 8.27** Using the Lorentz transformation of the four-vector potential A^μ and the wave equations from the previous problem, deduce the Lorentz transformations of charge density ρ_Q and current density $\mathbf{J}$.

Solution

The 4-vector potential is

$$A^\mu = (\phi, \mathbf{A})$$

We can then write the wave equation as

$$\eta^{\alpha\beta}\partial_\alpha \partial_\beta A^\mu = -\frac{4\pi}{c} j^\mu$$

where j^μ is a 4-vector current. (The $-\frac{4\pi}{c}$ is conventional so that current density is mapped onto j^i below)

Comparing with the wave equations, we identify

$$j^0 = \rho_Q c$$

and

$$\underset{i=1,2,3}{j^i} = J^i.$$

■

⋆⋆ **Problem 8.28** A relativistic particle with charge Q and mass M moves in the presence of a uniform electric field $\mathbf{E} = E_0\hat{\boldsymbol{z}}$. The initial energy is K_0 and the momentum is p_0 in the $\hat{\boldsymbol{y}}$ direction. Show that the trajectory in the y-z plane is described by

$$z = \frac{K_0 + Mc^2}{QE_0}\cosh\left(\frac{QE_0}{p_0 c}y\right) .$$

Solution

The Lagrangian is

$$L = -Mc^2\sqrt{1 - \frac{v^2}{c^2}} + QE_0 z$$

where we set

$$\mathbf{E} = E_0\hat{z} = -\nabla\phi$$

to find

$$\phi = -E_0 z + \phi_0$$

We have three conservation laws:

$$\mathrm{p}^x = \gamma_v M\dot{x} = \text{constant}$$

$$\mathrm{p}^y = \gamma_v M\dot{y} = \text{constant}$$

$$H = \dot{x}^i \frac{\partial L}{\partial \dot{x}^i} - L = M\gamma_v v^2 + \frac{Mc^2}{\gamma_v} - QE_0 z + \phi_0 = \text{constant}$$

The motion is in the $y - z$ plane given the initial conditions $\Rightarrow \mathrm{p}^x = 0 = \dot{x}$.

We then have

$$\gamma_v M\dot{y} = \mathrm{p}^y = p_0$$

$$\gamma_v Mc^2 (\frac{v^2}{c^2} + \frac{1}{\gamma_v^2}) - QE_0 z + \phi_0 = H$$

$$\Rightarrow \gamma_v Mc^2 - QE_0 z + \phi_0 = H \equiv Mc^2 + K_0 + \phi_0$$

we chose the initial condition at $z = 0$. We have a set of coupled differential equations:

$$\gamma_v M\dot{y} = p_0$$

$$\gamma_v Mc^2 - QE_0 z = Mc^2 + K_0$$

with

$$\gamma_0 = \frac{1}{\sqrt{1 - (\dot{y}^2 + \dot{z}^2)/c^2}}$$

We want dz/dy, so we look for the ratio $\dot{z}/\dot{y} = dz/dy$. Dividing out γ_v from the two equations

$$\Rightarrow \frac{\dot{y}}{c} = \frac{p_0 c}{QE_0 z + Mc^2 + K_0}$$

We can now find $\dot{z}$:

$$\gamma_v M\dot{y} = p_0 \Rightarrow M^2\dot{y}^2 = p_0^2 \left(1 - \frac{\dot{y}^2}{c^2} - \frac{\dot{z}^2}{c^2}\right)$$

$$\frac{\dot{z}^2}{c^2} = -\frac{\dot{y}^2}{c^2}\left(\frac{M^2c^2}{p_0^2} + 1\right) + 1$$

Substituting for $\dot{y}$, we get

$$\frac{\dot{z}}{c} = \pm\sqrt{1 - \frac{p_0^2c^2 + M^2c^4}{(QE_0 z + Mc^2 + K_0)^2}}$$

We now write

$$\frac{dz}{dy} = \frac{\dot{z}}{\dot{y}} = \sqrt{\frac{(QE_0 z + Mc^2 + K_0)^2}{p_0^2c^2} - \frac{p_0^2c^2 + M^2c^4}{p_0^2c^2}}$$

which we can integrate. Note that

$$(Mc^2 + K_0)^2 = M^2c^4 + p_0^2c^2$$

from the relativistic dispersion relation

$$\Rightarrow \frac{dz}{dy} = \sqrt{\left(\frac{QE_0}{p_0c}z + \frac{Mc^2 + K_0}{p_0c}\right)^2 - \left(\frac{Mc^2 + K_0}{p_0c}\right)^2}$$

Shift the z coordinate for convenience

$$\Rightarrow \frac{dz}{dy} = \sqrt{\left(\frac{qE_0}{p_0c}z\right)^2 - \left(\frac{Mc^2 + K_0}{p_0c}\right)^2} = \frac{Mc^2 + K_0}{p_0c}\sqrt{\left(\frac{qE_0}{Mc^2 + K_0}z\right)^2 - 1}$$

$$\Rightarrow z(y) = \frac{Mc^2 + K_0}{qE_0}\cosh\left(\frac{qE_0}{p_0c}y\right)$$

■

★★ **Problem 8.29** A relativistic particle of charge Q and mass M is moving in uniform circular motion bound by a radial potential. We learned from equation (8.167) that the charge will lose energy to electromagnetic radiation.

Assuming that this loss of energy is slow, we can describe the particle as gradually spiraling towards $r = 0$ while maintaining constant angular momentum. Apply this treatment to the ground state of the Hydrogen atom, where the atomic radius is about one Angstrom (10^{-10} m); and estimate the time it takes for the electron to crash into the nucleus. Are you surprised? Why does this not happen?

Solution

We have

$$\frac{dE}{dt} = \frac{2}{3}\frac{q^2}{m^2c^3}\gamma^2\left(\frac{d\mathbf{p}}{dt}\right)^2$$

Relativistic angular momentum follow from the Lagrangian

$$L = -mc^2\sqrt{1 - \frac{1}{c^2}(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2)} + q\phi \Rightarrow \ell = \frac{\partial L}{\partial\dot{\varphi}} = \gamma m\rho^2\dot{\varphi} \Rightarrow \dot{\varphi} = \frac{\ell}{\gamma m\rho^2}$$

While the momentum is

$$\mathrm{p}^{\rho} = \mathrm{p}^{z} = 0$$

and

$$\mathrm{p}^{\varphi} = \gamma m\rho\dot{\varphi}$$

For circular motion, we have

$$\frac{d\mathbf{p}}{dt} = -p\dot{\varphi}\hat{\rho}$$

We then have

$$\frac{dE}{dt} = \frac{2}{3}\frac{q^2}{m^2c^3}\gamma^2p^2\dot{\varphi}^2$$

$$\frac{dE}{dt} = \frac{2}{3}\frac{q^2}{m^4c^3}\frac{\ell^4}{\rho^6}$$

assuming $\ell \sim$ constant. The energy is adiabotically of order

$$E \sim \frac{q^2}{\rho} \Rightarrow \frac{q^2}{\rho^2}\frac{d\rho}{dt} \sim \frac{q^2}{m^4c^3}\frac{\ell^4}{\rho^6} \Rightarrow \frac{d\rho}{dt} \sim \frac{\ell^4}{m^4c^3}\frac{1}{\rho^4}$$

We can then estimate the time T for collapse as

$$T \sim \rho_0^5 \frac{m^4c^3}{\ell^4}$$

when ρ_0 is the initial radius of the atom. For hydrogen atom,

$$m \sim 9 \times 10^{-31}\text{kg}$$

$$\rho_0 \sim 10^{-10}\text{m}$$

$$\rho \sim h \sim 1 \times 10^{-34} m^2\text{kg/s}$$

$$\Rightarrow T \sim 10^{-9} s \sim 1 \text{ ns}$$

The atom would collapse in a nanosecond. In reality, the situation is very quantum mechanical and the ground state is stabilized so that the Heisenberg uncertainty principle is saturated. ■

★★ **Problem 8.30** A particle of charge Q and mass M moves through a region of uniform magnetic and gravitational fields described by constant field vectors $\mathbf{B}$ and $\mathbf{g}$ respectively. Show that the particle will have a drift velocity given by $M c\,(\mathbf{g} \times \mathbf{B})/Q B^2$.

Solution

The Lagrangian is

$$L = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{Q}{2c}B_0 x\dot{y} - \frac{Q}{2c}B_0 y\dot{x} - Mg\hat{\mathbf{g}} \cdot \mathbf{r}$$

where $\mathbf{B} == B_0\hat{z}$ without loss of generality and $\hat{g}$ is the unit vector in the direction of $\mathbf{g}$. We can also orient the coordinates about the z-axis so that $\mathbf{g}$ lies in say the x-z plane. We denote the angle between $\mathbf{g}$ and $\mathbf{B}$ as θ.

$$\Rightarrow \hat{\mathbf{g}} \cdot \mathbf{r} = (\cos\theta\hat{z} + \sin\theta\hat{x}) \cdot \mathbf{r} = z\cos\theta + x\sin\theta$$

$$\Rightarrow L = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{Q}{2c}B_0 x\dot{y} - \frac{Q}{2c}B_0 y\dot{x} - Mg(z\cos\theta + x\sin\theta)$$

There are no cyclic coordinates, but H is conserved

$$\frac{\partial L}{\partial \dot{x}} = M\dot{x} - \frac{Q}{2c}B_0 y \qquad \frac{\partial L}{\partial \dot{y}} = M\dot{y} + \frac{QB_0}{2c}x \qquad \frac{\partial L}{\partial \dot{z}} = M\dot{z}$$

$$\begin{aligned} H = {} & M\dot{x}^2 - \frac{QB_0}{2c}y\dot{x} + M\dot{y}^2 + \frac{QB_0}{2c}x\dot{y} + M\dot{z}^2 - \frac{1}{2}M(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\ & - \frac{QB_0}{2c}x\dot{y} + \frac{QB_0}{2c}y\dot{x} + Mg(z\cos\theta + x\sin\theta) \\ = {} & \frac{1}{2}M(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + Mg(z\cos\theta + x\sin\theta) = \text{constant} \end{aligned}$$

The equations of motion become:

$$M\dot{v}^x - \frac{QB_0}{2c}v^y - Mg\sin\theta$$

$$M\dot{v}^y = -\frac{QB_0}{c}v^x$$

$$M\dot{v}^z = -Mg\cos\theta$$

We can decouple the first two equations by differentiating

$$M\ddot{v}^y = -\frac{QB_0}{c}\dot{v}^x = -\frac{QB_0}{c}\left(\frac{QB_0}{Mc}v^y - g\sin\theta\right) \Rightarrow \ddot{v}^y = -\frac{Q^2B_0^2}{M^2c^2}\left(v^y - \frac{Mgc}{QB_0}\sin\theta\right)$$

$$\Rightarrow v^y(t) = A_1\cos(\omega t + A_2) + C$$

$$\dot{v}^y(t) = -A_1\omega\sin(\omega t + A_2)$$

$$\ddot{v}^y(t) = -A_1\omega^2\cos(\omega t + A_2)$$

$$v^y(t) = A_1\cos(\omega t + A_2) + \frac{g\sin\theta}{\omega}$$

where $\omega = \frac{QB_0}{Mc}$. We then get $v^x(t)$

$$v^x = -\frac{Mc}{QB_0}\dot{v}^y = +\frac{1}{\omega}A_1\omega\sin(\omega t + A_2)$$

We then see that the drift is in the y direction, i.e. in the direction of $\mathbf{g}\times\mathbf{B_0}$ with magnitude

$$\frac{g\sin\theta}{\omega}$$

The $\sin\theta$ corresponds to the cross product of $\mathbf{y}$ and $\mathbf{B_0}$:

$$\mathbf{v}_{\text{drift}} = \frac{Mc\mathbf{g}\times\mathbf{B_0}}{QB_0^2} \Rightarrow v_{\text{drift}} = \frac{cMgB_0\sin\theta}{QB_0^2} = \frac{Mgc\sin\theta}{QB_0} = \frac{g\sin\theta}{\omega}$$

■

⋆⋆ **Problem 8.31** A particle of charge Q and mass M starts at the origin of the coordinate system with initial speed v_0 in the $\hat{z}$ direction. There are uniform electric and magnetic fields E and B in the $\hat{x}$ direction. Find the location of the particle when it has reached one half of its maximum value in z for the first time.

Solution

$$\mathbf{v} = v_0\hat{z}$$

with Lagrangian

$$L = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + qE_0x + \frac{q}{2c}B_0y\dot{z} - \frac{q}{2c}B_0z\dot{y}$$

There are no cyclic coordinates, but H is conserved

$$\frac{\partial L}{\partial \dot{x}} = M\dot{x} \qquad \frac{\partial L}{\partial \dot{y}} = M\dot{y} - \frac{q}{2c}B_0 z \qquad \frac{\partial L}{\partial \dot{z}} = M\dot{z} + \frac{q}{2c}B_0 y$$

$$\Rightarrow H = M\dot{x}^2 + M\dot{y}^2 - \frac{q}{2c}B_0 z\dot{y} + M\dot{z}^2 + \frac{q}{2c}B_0 y\dot{z} - \frac{1}{2}M(\dot{x}^2+\dot{y}^2+\dot{z}^2) - qE_0 x - \frac{q}{2c}B_0 y\dot{z} + \frac{q}{2c}B_0 z\dot{y}$$

$$= \frac{1}{2}M(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - qE_0 x = \text{constant}$$

The equations of motion become:

$$M\dot{v}^x = qE_0$$

$$M\dot{v}^y - \frac{q}{2c}B_0 v^z = +\frac{q}{2c}B_0 v^z \Rightarrow M\dot{v}^y = \frac{qB_0 v^z}{c}$$

$$M\dot{v}^z - \frac{q}{2c}B_0 v^y = -\frac{q}{2c}B_0 v^y \Rightarrow M\dot{v}^z = -\frac{qB_0 v^y}{c}$$

We can decouple the last 2 equations:

$$M\ddot{v}^y = \frac{qB_0}{c}\dot{v}^z = -\frac{qB_0}{c}\frac{qB_0}{Mc}v^y \Rightarrow \ddot{v}^y = -\left(\frac{qB_0}{Mc}\right)^2 v^y$$

Similarly

$$\ddot{v}^z = -\left(\frac{qB_0}{Mc}\right)^2 v^z \Rightarrow v^z(t) = A\cos(\omega t) + B\sin(\omega t)$$

where $\omega = qB_0/Mc$ and

$$v^z(0) = v_0 \Rightarrow A = v_0$$

And

$$v^y = A'\cos(\omega t) + B'\sin(\omega t)$$

with

$$v^y(0) = 0 \Rightarrow A' = 0 \Rightarrow v^z(t) = v_0\cos(\omega t) + B\sin(\omega t) \qquad v^y(t) = B'\sin(\omega t)$$

But we must have

$$M\dot{v}^y = \frac{qB_0}{c}v^z \Rightarrow M\omega B'\cos(\omega t) = \frac{qB_0}{c}(v_0\cos(\omega t) + B\sin(\omega t))$$

$$\Rightarrow B = 0 \quad \text{and} \quad M\omega B' = qB_0 v_0/c$$

$$\Rightarrow v^z(t) = v_0\cos(\omega t) \qquad v^y(t) = v_0\sin(\omega t)$$

Integrating once more, we get

$$\dot{z} = v_0\cos(\omega t) \Rightarrow z(t) = +\frac{v_0}{\omega}\sin(\omega t) + z_0$$

with $z = 0$ since $z(0) = 0$. Hence, the half distance along z is $\frac{v_0}{2\omega}$ ∎

9 Accelerating Frames

9.1 Problems and Solutions

⋆⋆ **Problem 9.1** A satellite is in a polar orbit around the earth, passing successively over the north and south poles (see Figure 9.21). As we stand on the ground, what is the motion of the satellite as we see it from our rotating frame?

Solution

We are given that the satellite moves in a circular orbit in an inertial frame in which earth rotates toward the east. In fact, earth turns through angle $\psi = \Omega t$ in time t, where Ω is its angular velocity. So if we erect a spherical coordinate system (r, θ, φ) in the inertial frame with origin at earth's center and z axis pointing out of the north pole, and a second system (r', θ', φ') fixed to the rotating earth with the same origin and z axis, then the radius $r = r'$ and polar angle $\theta = \theta'$ of the satellite are the same in both systems, but the azimuthal angle $\varphi' = \varphi - \Omega t$ due to earth's rotation, as illustrated in Figure 9.21.

If the satellite orbits with radius r_0 in the (x, z) plane of the inertial frame, its inertial coordinates are $(r, \theta, \varphi) = (r_0, \omega t, 0)$ as a function of time, starting from the north pole moving south, and $(r_0, \pi - \omega t, \pi)$ from the south pole back to the north, where ω is the angular velocity of the satellite's motion. The corresponding coordinates in the spherical system fixed to the rotating earth are $(r', \theta', \varphi') = (r_0, \omega t, -\Omega t)$ from north to south and $(r_0, \pi - \omega t, \pi - \Omega t)$ back to the north. The path of the satellite over earth's surface for a particular choice of ω therefore looks as shown in Figure 9.21. ■

⋆ **Problem 9.2** The string on a helium balloon is attached inside a car at rest, as shown in Figure 9.22(a). If the car accelerates forward, does the balloon tilt forward or backward? If a is the car's acceleration and g is the gravitational field, what is the balloon's tilt angle from the vertical when it comes to equilibrium?

Solution

By the principle of equivalence, a frame that accelerates uniformly *forward* is equivalent to a frame at rest in which a uniform gravitational field is added *backward*. Therefore we can pretend that the car is at rest, and in addition to the usual gravity downward we add an extra gravity to the rear, equal in magnitude to the actual acceleration of the car. The vector sum of the real gravity and this pseudogravity is an effective gravity pointed downward and to the rear. The balloon floats "upward," opposite to the direction of the effective gravity.

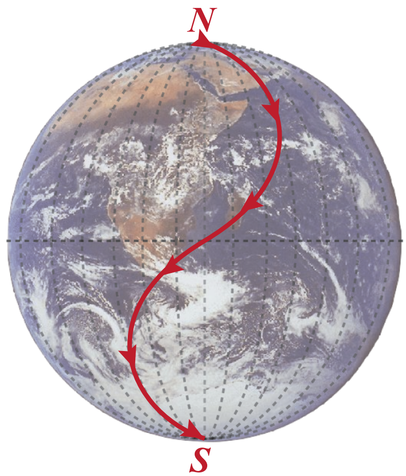


Fig. 9.21 Path of a satellite in a polar orbit, shown in the rotating frame in which earth is at rest. Dashed lines represent the equator and longitude lines.

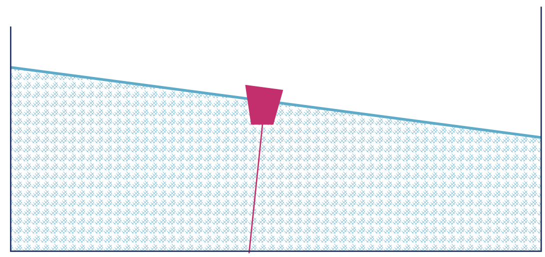
Fig. 9.22 (a) a balloon in a car (b) a cork in a fishtank

Therefore inside the car the balloon tilts forward, toward the windshield. The tilt angle relative to the vertical will be $\theta = \tan^{-1}(a/g)$. ■

⋆ **Problem 9.3** A cork floats in a fish tank half full of water; it is attached to the bottom of the tank by a stretched rubber band, as shown in Figure 9.22(b). If the tank and contents are uniformly accelerated to the right, sketch the water surface, cork, and rubber band after the water has stopped sloshing back and forth and the system has come to equilibrium.

Solution

From the Principle of Equivalence, a uniform acceleration to the right is equivalent to a uniform gravitation to the left. That is, the total effective gravity on the system is the vector sum of "real" gravity *downward* and an artificial gravity to the *left*. The surface of the water will be normal to this total effective gravity, and the cork will float to the new "upward," as illustrated in the figure below



■

⋆ **Problem 9.4** In the rotating frame of the earth, stars appear to orbit in circles, with a period of 24 hours. Show that the centrifugal and Coriolis pseudoforces acting together provide the net force needed in the rotating frame to cause a star to orbit as described.

Solution

The sum of the two pseudoforces is

$$\mathbf{F} = -m\,\boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r} - 2m\,\boldsymbol{\omega} \times \mathbf{v}.$$

In this situation $\boldsymbol{\omega}$ is perpendicular to both $\mathbf{r}$ and $\mathbf{v}$, so from calculating the cross products the first (centrifugal) term is directed outward, while the second (Coriolis) term is directed inward. Furthermore, for circular motion $v = r\omega$, so the net force outward is

$$F = m\omega^2 r - 2m\omega^2 r = -m\omega^2 r.$$

The acceleration of a particle in circular motion is $-v^2/r = -r\omega^2$, so $F = ma$ gives $F = -m\omega^2 r = ma = m(-\omega^2 r)$, which is obviously correct. ■

⋆ **Problem 9.5** A cylindrical space colony rotates about its symmetry axis with period 62.8 seconds. If the effective gravity felt by colonists standing on the inner rim is one earth "gee", what is the radius of the colony? What then is the percentage difference in the effective gravity acting upon the head and on the feet of a two-meter tall colonist?

Solution

Using cylindrical coordinates, the effective gravity $g_{\text{eff}} = v^2/\rho = \rho\omega^2 = 4\pi^2\rho/T^2 = g_{\text{earth}}$, where T is the period. Therefore the radius of the colony is

$$\rho = \frac{T^2}{4\pi^2}\,g_{\text{earth}} = \frac{(6.28)^2 \times 100}{4(3.14)^2}\,9.8\ \text{m} = 980\ \text{m}.$$

Note that T is the same throughout, so

$$\frac{\Delta g_{\text{eff}}}{g_{\text{eff}}} = \frac{4\pi^2\Delta\rho/T^2}{4\pi^2\rho/T^2} = \frac{\Delta\rho}{\rho} = \frac{2}{980} \cong 0.2\%$$

■

⋆ **Problem 9.6** (a) A uniformly rotating merry-go-round spins clockwise as seen from above. A rider stands at the rotation axis, and then slowly walks radially outward toward the rim. How will the centrifugal and Coriolis pseudoforces affect her? How will she have to lean to keep from falling over? (b) Now suppose the merry-go-round is spinning up, starting from rest and spinning faster and faster in the clockwise sense. The rider again starts at the center and walks outward. What new effect will the rider notice, and how will that affect how she has to lean to avoid falling?

Solution

(a) As the rider walks outward, the centrifugal pseudoforce will become increasingly important, so she will tend to be "thrown outward," toward the rim. That means she will have to lean backward slightly as she walks, to avoid falling over. The Coriolis pseudoforce will also act upon her, due to her velocity. As she walks outward this pseudoforce will push on her toward her left, so to keep walking straight outwards, she will have to lean slightly to her right. Altogether, she will have to lean a bit backward and to her right to avoid falling over as she walks. ■

⋆ **Problem 9.7** In *Rendezvous with Rama* by Arthur C. Clarke, observers inside a cylindrical spaceship view a waterfall, which originates at one of the endcaps at a point halfway between the rotation axis and rim, and then "falls" to the rim. The spaceship is rotating clockwise about its symmetry axis as seen in an external inertial frame, looking along the entire spaceship axis at the waterfall in the distance. Explain why the water does not fall straight as seen by people within the ship. Which way does it bend?

Solution

It "falls" to the rim due to the centrifugal pseudoforce, and bends due to the Coriolis pseudoforce $-2m\boldsymbol{\omega} \times \mathbf{v}$. Looking along the axis of symmetry, the $\boldsymbol{\omega}$ vector is pointed away from the viewer, so if $\mathbf{v}$ is "downward," $\boldsymbol{\omega} \times \mathbf{v}$ is to the left as seen by the viewer. Therefore $-2m\boldsymbol{\omega} \times \mathbf{v}$ is to the right, which is the direction of the bend. In other words, its shape is similar to that of a comma reflected left to right. ■

⋆⋆ **Problem 9.8** A train runs around the inside of the outer rim of a cylindrical space colony of radius R and angular velocity ω along its symmetry axis. How would the effective gravity on the passengers depend upon the train's speed v relative to the rim (a) if the train travels *in* the rotation direction of the rim? (b) if it travels in the *opposite* direction? (c) If you were designing a train system, which way would you make the train run?

Solution

The effective gravity on a passenger is g_{eff}, whose magnitude is defined by the normal force $N = mg_{\text{eff}}$ of the chair seat on the passenger when the passenger is sitting on the train. Then we can choose to find N either in the inertial frame, in which the space colony is seen to rotate, or in the rotating frame in which the colony is at rest. The answer must be the same in each case, since N could be found by standing on a scale, which would read the same number no matter what frame of reference is used to view it.

First let us find N as calculated in the inertial frame. In that case there are no pseudoforces, so N is the *only* force acting on the passenger. As seen in the inertial frame the passenger is moving at speed $v_{in} = v \pm R\omega$, depending upon whether the train is traveling in the rotation direction (plus sign) or opposite to the rotation direction (minus sign.) The passenger is generally traveling in a circle as seen in the inertial frame, so has an acceleration v_{in}^2/R towards the axis of rotation. Thus, since N is the only force, we have $N = mg_{\text{eff}} = ma = mv_{in}^2/R = m[v \pm R\omega]^2/R$, and so the effective gravity experienced by the passenger is

$$g_{\text{eff}} = \frac{[v \pm R\omega]^2}{R} = \frac{v^2}{R} + \omega^2 R \pm 2\omega v$$

for travel in or opposite to the rotation direction of the space colony, respectively.

Alternatively, we can find N by calculating in the rotating frame of the colony. We can then compare it with the result just obtained. In addition to the normal force N, two pseudoforces now act upon the passenger. The centrifugal pseudoforce is outward, away from the symmetry axis, equal to $m\omega^2 R$. The Coriolis pseudoforce is $-2m\omega \times \mathbf{v}$, with magnitude $2m\omega v$ and direction *outward* if the train is moving *in* the rotation direction of the colony, and *inward* if the train is moving in the opposite direction. Therefore the net inward force (including pseudoforces) is $N - m\omega^2 R \mp 2m\omega v = ma = mv^2/R$, since in the rotating frame the train is moving in a circle with speed v, so has inward acceleration v^2/R. The minus (plus) sign obtains if the train is moving in (opposite to) the rotation direction. Therefore

$$g_{\text{eff}} \equiv \frac{N}{m} = \frac{v^2}{R} + \omega^2 R \pm 2\omega v.$$

This is the same result found by calculating in the inertial frame, as expected. ■

⋆ **Problem 9.9** Why don't we notice Coriolis effects when we walk, drive cars, or throw baseballs? In contrast, why may Coriolis effects be significant for long-range artillery or moving air masses?

Solution

The Coriolis force on earth is normally very small over short times or distances, but its cumulative effects can be observed over longer times or distances, such as in weather patterns. ■

⋆⋆ **Problem 9.10** In 1914 there was a World War I naval battle between British and German battle cruisers near the Falkland Islands, at 52° south latitude (i. e., $\lambda = -52°$.) Guns on the British ships fired 12-inch shells at German ships up to about 15 km distant. The great majority of the shells missed their targets, due to the constant rolling of the ships, defensive maneuvers by the Germans, and perhaps other factors. After the battle (which the British won) another possible reason was offered: Coriolis deflections. The story goes that the British were used to battles in the northern hemisphere, where projectiles deflect toward the right, and aimed their guns incorrectly for the Falklands battle, where projectiles deflect toward the left. There seems to be some controversy over whether or not Coriolis effects were important in the battle. The purpose of this problem is to estimate their magnitude.

The British guns reportedly had a muzzle velocity of 823 m/s. (a) For a target 15 km away, and pretending there was no air resistance, what must have been the elevation angle (the angle up from the horizontal) of the guns? Note that the guns were apparently limited to elevation angles of 15° or less.) (b) By about how much would the shells have missed their target due to the sideways Coriolis effect? (Note that *if* the British used gun-aiming tables appropriate for 52° *north* latitude, which is appropriate for the north Atlantic, then the miss distance of the shells would have been about twice as much as the southern hemisphere deflection alone.)

Solution

(a) Pretending there was no air resistance, an easily-derived projectile motion formula for the range is $R = (v_0^2/g)\sin 2\alpha$, in terms of the muzzle velocity v_0 and elevation angle α. For $R = 15$ km and the given velocity, we can solve for α: This gives two solutions: $\alpha = 6.26$ degrees and 83.7 degrees. The second solution is not possible because the guns could not be raised that high. (b) The magnitude of the *deflection* is the same in whatever direction the shells are projected. If they are shot toward the east, for example, then with a Cartesian coordinate system x, y, z in the directions of east, north, and "up", respectively, the zeroth-order shell velocity is $v_0 \cos\alpha\, \hat{\mathbf{x}}$, and the Coriolis acceleration is

$$\mathbf{a}_{\text{Cor}} = -2\omega\times\mathbf{v} = -2\omega v_0\cos\alpha[\cos\lambda\,\hat{\mathbf{y}}+\sin\lambda\,\hat{\mathbf{z}}]\times\hat{\mathbf{x}} = -2\omega v_0\cos\alpha[\sin\lambda(\hat{\mathbf{y}})-\cos\lambda(\hat{\mathbf{z}})].$$

We are looking here for the sideways deflection, which is the component in the y direction. The Coriolis acceleration in this direction is

$$\ddot{y} = -2\omega v_0\cos\alpha\sin\lambda \quad \text{so} \quad y = -\omega v_0\cos\alpha\sin\lambda t^2.$$

where we have integrated twice with respect to time and set $y = 0$ and $\dot{y} = 0$ initially. The shell arrives at its target at time $t = R/(v_0)\cos\alpha)$. Substituting in numbers, including $\lambda = -52°$, we find a northward deflection of $d \cong 16$ m. This does not seem like a serious deflection, when compared with errors caused by the rolling, pitching, and yawing of the ships on both sides. Note that air resistance is actually quite important in long-range trajectories, so all numbers calculated while ignoring it must be crude estimates at best. ■

⋆⋆⋆ **Problem 9.11** In World War I the German army set up an enormous cannon (which they called the "Paris gun") to fire shells at Paris, 120 km away from the cannon situated at a point NNE of Paris. The muzzle velocity was 1640 m/s. (a) Neglecting both air resistance and the Coriolis effect, find two solutions for the elevation angle of the gun, assuming the altitudes of the launching and target points were the same. (b) With these same assumptions, for each of the two possible elevation angles, how long would the shell have taken to reach its target, and what maximum altitude would it have achieved? Compare your results with the actual flight time 182 s and maximum altitude 42 km. What do you think is the primary reason for the large discrepancies? (c) With the same assumptions as in part (a), calculate the Coriolis deflection of the shell aimed at Paris for the larger elevation-angle solution. Be sure to include the Coriolis deflections due to both the horizontal and vertical components of the shell's velocity. This results might give at least a very rough estimate for the actual Coriolis deflection. For simplicity, assume the shell began traveling due south. Is its deflection toward the east or toward the west?

Solution

(a) Pretending that there was no air resistance, the range, in terms of the initial velocity and elevation angle α of the gun, is $R = (v_0^2/g)\sin 2\alpha$ from elementary kinematics. Solving for α, using information given in the problem, we find two solutions, $\alpha \cong 13°$ and $\alpha \cong 77°$.

(b) If $\alpha = 13°$ the flight time of the shell would have been $t_f = R/(v_0 \cos\alpha) \cong 75$ s, and its maximum altitude would have been $h = (t_f/2)(v_0 \sin\alpha - gt_f/2) \cong 7.5$ km. If $\alpha = 77°$ the flight time of the shell would have been about 163 s , and its maximum altitude would have been about 149 km. Neither of these results is anywhere close to the actual data for the trajectory. The primary reason for the discrepancy is that air resistance was actually very important in the trajectory of the shell. The actual elevation angle used would have been somewhere between 13° and 77°, and the trajectory would no longer be parabolic, symmetric about the halfway point; its highest point would be closer to the target than to the cannon, and (with the shell losing speed due to air drag) it would fall more steeply downward on the target.

For a shell fired southward at 77° above the horizontal, the Coriolis acceleration is

$$\begin{aligned}\mathbf{a}_{\rm Cor} &= -2\boldsymbol{\omega}\times\mathbf{v} = -2\omega[(\cos\lambda\hat{\mathbf{y}} + \sin\lambda\hat{\mathbf{z}}\times[-v_0\cos\alpha\hat{\mathbf{y}} + (v_0\sin\alpha - gt)\hat{\mathbf{z}}]\\ &= -2\omega[\cos\lambda(v_0\sin\alpha - gt) + \sin\lambda v_0\cos\alpha]\hat{\mathbf{x}}.\end{aligned}$$

Integrating twice over time, and using initial conditions $x(0) = 0, \dot{x}(0) = 0$, at the time t_f when the shell reaches the target we have a deflection

$$d = -2\omega[\cos\lambda(v_0\sin\alpha t_f^2/2 - gt_f^3/6) + \sin\lambda v_0\cos\alpha t_f^2/2.$$

We can calculate this deflection numerically, using $v_0 = 1640$ m/s, $t_f = 325$ s, $\alpha = 77°$, and $\lambda = 48.9°$. The result is $d \cong -4.9$ km, indicating a deflection westward of about 5 km. This may be a somewhat exaggerated estimate of the deflection for the actual trajectory, but a deflection of a kilometer or so would not be unreasonable. The Coriolis effect was taken into account by the German army. ■

★★ **Problem 9.12** A satellite in low-earth orbit with a 90-minute period passes over the north pole, headed south along the 0° line of longitude passing through Greenwich, England. (a) What is its longitude when it reaches the latitude of Greenwich ($\lambda = 50°$)? (b) When it reaches the equator, what angle does its trajectory make with the equator as seen by an observer on earth? (c) How close does it come to passing over the south pole?

Solution

(a) This problem is easiest to solve in an inertial frame in which the earth is seen to rotate toward the east. The azimuthal angle on the earth is $\varphi' = \varphi - \Omega t = -\Omega t$ since the azimuthal angle φ in the inertial frame stays at zero. Here $\Omega = 2\pi/T = 2\pi/24$ hours, and $t = (\theta/2\pi)\cdot$ 90 minutes, where θ is the polar angle of the satellite down from the north pole. If the latitude is 50°, the angle $\theta = 40°$ from the north pole. Therefore $t = (40/360) \times 1.5$ hrs, and so

$$\varphi = -\frac{2\pi}{24\text{ hrs}}\frac{1.5\text{ hrs}}{9} = -0.0436\text{ radians} = -2.5\text{ degrees W longitude}.$$

(b) The equator moves at speed $v_{\rm eq} 2\pi R/T = 1675$ km/hr toward the east, as seen in an inertial frame at rest relative to the center of the earth. Here $T = 24$ hrs is earth's rotational period. Meanwhile, the satellite moves at speed v_s, where (by Newton's second law) $mv_s^2/r = GMm/r^2 \Rightarrow v_s = \sqrt{GM/r}$. In terms of $T_s = 90$ minutes, it is also true that $v_s = 2\pi r/T_s$. Eliminating the satellite's orbital radius r between these two equations for v_s, we find that

$$v_s = \left(\frac{2\pi GM}{T_s}\right)^{1/3} = 7.8\ \mathrm{km/s} = 28,100\ \mathrm{kkm/hr}.$$

We therefore know both the velocity (eastward) of earth's equator and the velocity (southward) of the satellite, both measured in an inertial frame. The angle of the satellite relative to the equator, as seen on earth, is therefore $\psi = \tan^{-1}\frac{v_s}{v_{\rm eq}} = \tan^{-1}\frac{28,100}{1675} = 86.5$ degrees. (c) It passes exactly over the south pole. (Well, that is for an exactly spherical earth, which of course is incorrect.) ■

⋆ **Problem 9.13** (a) Find the centrifugal acceleration of a particle on the earth's surface at the equator, due to earth's rotation, as a fraction of the gravitational field g at that point. (b) Do the same for the centrifugal acceleration due to the motion of earth around the sun. Note that this acceleration is small compared with that due to the axial rotation.

Solution

(a) The centrifugal acceleration at the equator has magnitude $\omega^2 R$, where the angular velocity of the earth's rotation is $\omega = 2\pi/24\mathrm{hrs} = 7.3 \times 10^{-5}\ \mathrm{s}^{-1}$ and the radius of the earth is $R = 6.4 \times 10^6$ m. Therefore the centrifugal acceleration at the equator is

$$a_c = (7.3 \times 10^{-5})^2 (6.4 \times 10^6)\mathrm{m/s}^2 = 0.034\ \mathrm{m/s}^2.$$

Therefore $a_c/g_E = 0.034/9.8 \cong 0.0035$.

(b) The centrifugal acceleration related to earth's rotation about the sun is $a_c = \omega^2 R$, where $\omega = 2\pi/1\ \mathrm{yr} = 2.0 \times 10^{-7}\mathrm{s}^{-1}$ and $R = 150 \times 10^9$ m. Therefore

$$a_c = 6.0 \times 10^{-3}\mathrm{m/s}^2 \ \text{ and so } \ a_c/g_E \cong 0.00061$$

■

⋆⋆ **Problem 9.14** Suppose we flatten and smooth out the ice at the south pole, and place a hockey puck at rest on the ice exactly at the pole. We then give it a small velocity, initially along longitude 0°. Pretend that there is no friction between the puck and the ice, and that there is no air resistance either. (a) If it reaches a final point 90° longitude west when it is 100.0 m from the pole, what was its initial speed? (b) At this final point, what is its speed relative to the ice? (c) What force or pseudoforce is responsible for the increased speed?

Solution

(a) Let v_0 be the puck's initial speed. Then the time it takes the puck to reach 100 m from the pole is $t = 100\ \mathrm{m}/v_0$, since the puck moves in a straight line in the inertial frame in which the earth is seen to turn. A 90° rotation of the earth, which is 1/4 of a complete rotation, requires 24/4 hours = 6 hrs = 21,600 seconds, so this time must be $t = 21,600s = 100m/v_0$.

Therefore $v_0 = 0.00463$ m/s = 0.463 cm/s. This is both its initial speed and final speed in the inertial frame. (In the inertial frame neither gravity nor the normal force do work on the puck.) (b) Relative to the ice, however, the puck speeds up because it now has both a northward velocity of 0.463 cm/s and a westward velocity of $r\omega$, where $r = 100$ m and $\omega = 2\pi/(24 \text{ hrs}) = 7.26 \times 10^{-5}\ \text{s}^{-1}$. Thus the western velocity is 0.0726 cm/s. The puck's total speed relative to the ice is therefore $\sqrt{(.463)^2 + (0.073)^2} = 0.469$ cm/s, only a slight increase. (c) In the rotating frame, which is the rest frame of the ice, the only forces with horizontal components are the centrifugal and Coriolis pseudoforces. Coriolis pseudoforces do no work, however, so cannot be responsible for increasing the kinetic energy of the puck. It is the centrifugal pseudoforce $m\omega^2 r$ that does work in this frame. As the distance r of the puck from the pole increases, $m\omega^2 r$ slowly increases, and does positive work on the puck, causing an increase in its kinetic energy. ■

⋆⋆ **Problem 9.15** A merry-go-round has a 5 m radius and rotates with a 10 s period. If one "gee" is the gravitational force/mass experienced by a person standing still on the earth, how many gees are felt by a person walking from the center toward the rim of the merry-go-round at velocity 1 m/s in a straight line, due to the Coriolis pseudoforce? At what sideways angle is the person likely to lean while walking? How many gees are felt when the person is standing 3 m from the center, due to the centrifugal pseudoforce? At what angle is the person likely to lean backwards at this point?

Solution

The Coriolis pseudoforce has magnitude $2m\omega v = 2m(2\pi/T)v$ in this case, since ω is perpendicular to the velocity. Here T is the rotational period of the merry-go-round. Therefore the Coriolis acceleration relative to g_E is

$$\frac{a_{cor}}{g_e} = \frac{4\pi}{(10\text{s})}\frac{1\ \text{m/s}}{(9.8\ \text{m/s}^2}) = 0.13.$$

The Coriolis deflection is to the right or left, depending upon the spin direction of the merry-go-round. The tangent of the lean angle relative to the vertical is $\tan\theta = 0.13$, so the lean angle is $\theta = \tan^{-1}(0.13) = 7.4°$ from the vertical, as caused by the Coriolis effect.

The centrifugal pseudoforce in this case is $m\omega^2 r = m(2\pi/T)^2 r$, so

$$\frac{a_{cen}}{g_e} = \frac{4\pi^2}{(10\text{s})^2}\frac{3\text{m}}{9.8\ \text{m/s}^2} = 0.12.$$

The lean angle relative to the vertical in this case is $\theta = \tan^{-1} 0.12 = 6.8°$ from the vertical.

Altogether, the person walking as described on the merry-go-round would be likely to lean (relative to the vertical) 7.4° left or right (depending upon the rotation direction) and 6.8° backward, to keep from falling. When standing still at 3 m from the center, there would be only the 6.8° backward lean. ■

⋆⋆ **Problem 9.16** A ball is dropped from height h by someone standing still on the earth's equator. (a) Does it fall to the east or west of a point just beneath the position from which it was dropped? (b) When it strikes the ground, how far is the ball from the point originally directly beneath it, in terms of $g, h,$ and ω, earth's angular velocity? (Pretend there is no air resistance.) (c) Explain the direction found in (a) using conservation of the ball's angular momentum in the inertial frame in which earth rotates toward the east.

Solution

(a) Place Cartesian coordinates with origin at the ground, just beneath the ball. Let the x direction be eastward, the y direction be northward, and the z direction be "upward." Then the earth's angular velocity vector is $\boldsymbol{\omega} = \omega(\cos\lambda\hat{\mathbf{y}} + \sin\lambda\hat{\mathbf{z}})$ where λ is the latitude. At the equator $\lambda = 0$, so $\boldsymbol{\omega} = \omega\hat{\mathbf{y}}$. The ball's velocity vector is $\mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}$. Therefore the Coriolis pseudoforce in the rest-frame of the earth is

$$\mathbf{F}_{\text{Cor}} = -2m\boldsymbol{\omega} \times \mathbf{v} = -2m\omega\hat{\mathbf{y}} \times (v_x\hat{\mathbf{x}} + v_z\hat{\mathbf{z}}) \cong -2m\omega v_z(\hat{\mathbf{y}} \times \hat{\mathbf{z}}) = -2m\omega v_z\hat{\mathbf{x}}$$

since we expect $|v_x| << |v_z|$ as the ball falls from rest. Now v_z is negative, so the force is in the positive x direction, i.e., toward the east. Therefore the ball falls slightly toward the east from where it was dropped.

(b) Using $F = ma$, we have $2m\omega(gt) = m dv_x/dt$, so the velocity toward the east as a function of time is $v_x = \omega g t^2$. Therefore the eastward displacement is $x = \omega g t^3/3$. The time it takes to reach the ground is the solution to $h = (1/2)gt^2$, i. e. $t = \sqrt{2h/g}$. Therefore the eastward displacement is

$$d = \frac{2\sqrt{2}}{3}\omega\left(\frac{h^3}{g}\right)^{1/2}$$

(c) In the inertial frame in which the earth is seen to rotate, the ball is moving eastward initially, due to the earth's rotation. This gives it an angular momentum toward the north. As it falls it conserves this angular momentum, so as its radius from the center of the earth decreases, its eastward velocity must increase. Therefore we expect it to fall to the east of its starting point. ■

★★ **Problem 9.17** A ball at a point on the earth with latitude λ is thrown vertically upward to a small altitude h. (a) Does the ball fall to the east or west of its starting point? (b) Show that the ball strikes the ground a distance $(4/3)\Omega\cos\lambda(2\,h/g)^{3/2}$ from its starting point. (c) Explain qualiltatively the ball's path using conservation of angular momentum in the inertial frame in which the earth rotates eastward.

Solution

(a) We have

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \text{ and } \boldsymbol{\Omega} = \Omega\sin\lambda\hat{\mathbf{z}} - \Omega\cos\lambda\hat{\mathbf{x}},$$

so the Lagrangian is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz + \frac{1}{2}m(\boldsymbol{\Omega} \times \mathbf{r})^2 + m\mathbf{v} \cdot (\boldsymbol{\Omega} \times \mathbf{r}) \text{ dropping the small Euler term.}$$

Also, because Ω is very small, we drop terms with Ω^2.

$$\Rightarrow L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - m\Omega\dot{x}y\sin\lambda - m\Omega\dot{z}y\cos\lambda + m\Omega\dot{y}x\sin\lambda + m\Omega\dot{y}z\cos\lambda - mgz$$

Since motion is largest in the z-direction, the terms $m\Omega\dot{x}y\sin\lambda$ and $m\Omega\dot{y}x\sin\lambda$ have been struck. The result is

$$\Rightarrow m\ddot{x} = 0$$

$$m\ddot{y} + m\Omega\cos\lambda\dot{z} = -m\Omega\cos\lambda\dot{z}$$

$$m\ddot{z} - m\Omega\cos\lambda\dot{y} = -mg + m\Omega\dot{y}\cos\lambda$$

We then have, to an excellent approximation,

$$\ddot{x} = 0 \qquad \ddot{z} = -g \qquad \ddot{y} = -2\Omega\cos\lambda\dot{z}$$

Now the x-dynamics $\Rightarrow \quad x = 0$

$$z = v_0 t - \frac{1}{2}gt^2$$

$$\Rightarrow \ddot{y} = -2\Omega\cos\lambda(v_0 - gt)$$

$$\Rightarrow y(t) = -2\Omega\cos\lambda(\frac{v_0 t^2}{2} - \frac{gt^3}{6})$$

If T is half of the time for the round trip,

$$\Rightarrow v_0 = gT \qquad \text{full time is 2T}$$

On the way up

$$\Delta y = y(2T) = -\frac{4}{3}\Omega g\cos\lambda T^3$$

To write it in terms of height h,

$$v_0^2 = 2gh = g^2T^2 \Rightarrow T \simeq \sqrt{\frac{2h}{g}}$$

$$\Rightarrow \Delta y = -\frac{4}{3}\Omega g\cos\lambda(\frac{2h}{g})^{3/2} < 0 \Rightarrow \text{Westward}$$

(b) As the ball is thrown, it naturally has an eastward velocity component as seen in an inertial frame, due to the earth's rotation. As it rises up, this v_E decreases, since $v_E r \sim$ constant by angular momentum conservation. This means that the earth rotates faster eastbound under the ball and when the ball hits the ground, it is on the western side of its starting point. ■

⋆ **Problem 9.18** Show that the usual formula $P = 2\pi\sqrt{R/g}$ for the period of small-amplitude oscillations of a pendulum of length R becomes instead $P = 2\pi\sqrt{R/g}(\sqrt{m_\mathrm{I}/m_\mathrm{G}})$ if the inertial and gravitational masses of the pendulum bob differ. (Newton himself built pendulums with plumb bobs made of different materials. He would swing two of them side by side, both with the same length R, to see if he could detect a difference in period apart from experimental errors. He could not. Nevertheless, it is interesting that he conceived of the possibility they might be different.)

Solution

The Lagrangian of the pendulum is $L = T - U = (1/2)m_I R^2\dot{\theta}^2 + m_G gR\cos\theta$, where θ is the angle of the pendulum relative to the vertical. Note that the kinetic energy is related to the inertial mass, while the gravitational force, and therefore the potential, is related to the gravitational mass. Substituting this into the Lagrange equations and using $\sin\theta \cong \theta$ for small angles, gives

$$\ddot{\theta} + (g/R)(m_G/m_I)\theta = 0.$$

The result follows. ■

★★ **Problem 9.19** If an artillery shell is fired a short distance from a point on earth's surface at latitude λ, with speed v_0 and an angle of inclination α to the horizontal, show that (pretending there is no air resistance) its lateral deflection when it strikes the ground is

$$d = (4v_0^3/g^2)\omega\sin\lambda(\sin^2\alpha\cos\alpha),$$

where ω is the earth's angular velocity.

Solution

Let us define a set of Cartesian coordinates with x toward the east, y toward the north, and z upwards, away from the surface. Then

$$\mathbf{F}_{\text{Cor}} = -2m\boldsymbol{\omega}\times\mathbf{v} = 2m\omega[\mathbf{i}(\sin\lambda\dot{y} - \cos\lambda\dot{z}) - \mathbf{j}\sin\lambda\dot{x} + \mathbf{k}\cos\lambda\dot{x}],$$

which is the force causing the deflection. We are not interested in any vertical deflection, so we can neglect the last term, which is in the z direction. For simplicity, suppose the cannonball is shot eastward (the deflection is independent of direction.) The important velocity is therefore $\dot{x}$, so we take

$$\mathbf{F}_{\text{Cor}} = 2m\omega(-\mathbf{j}\sin\lambda\dot{x}) = -2m\omega(\sin\lambda)v_o\cos\alpha\,\mathbf{j},$$

a force toward the south. The shell has a horizontal (x) component of velocity $v_0\cos\alpha$. That is, the Coriolis acceleration of the shell is

$$\mathbf{a}_{\text{Cor}} = -2\omega(\sin\lambda)v_o\cos\alpha\mathbf{j}, \quad \text{so} \quad \ddot{y} = 2\omega(\sin\lambda)v_o\cos\alpha,$$

a constant for short distances. Therefore we can use the constant-acceleration formula

$$y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2 = 0 + 0 + \frac{1}{2}a_y t^2$$

to get a deflection toward the south of

$$d = \frac{1}{2}|a_y|t^2 = \omega(\sin\lambda)v_0\cos\alpha t^2.$$

The initial vertical velocity component is $v_0\sin\alpha$, so $z = v_0\sin\alpha t - (1/2)gt^2$, neglecting the very small vertical component of the Coriolis force. Now $z = 0$ at $t = 2v_0\sin\alpha/g$, so

$$d = (4v_0^3/g^2)\omega\sin\lambda(\sin^2\alpha\cos\alpha)$$

as claimed. ■

★★ **Problem 9.20** An artillery shell is projected due north from a point at latitude λ at an angle of 45° to the horizontal, and aimed at a target whose distance is D, where D is small compared with earth's radius. (a) Show that due to the Coriolis pseudoforce, and neglecting air resistance, the shell will miss its target by a distance

$$d = \left(\frac{2D^3}{g}\right)^{1/2} \omega(\sin\lambda - \frac{1}{3}\cos\lambda)$$

where ω is earth's angular velocity. (b) Evaluate this distance for $\lambda = 30°$ and $D = 50$ km. (c) What is the physical reason for the deviation to the east near the north pole, but to the west both on the equator and near the south pole?

Solution

We will use Cartesian coordinates where the origin is at the point where the shell is launched, and where the x, y, and z directions correspond to east, north, and "up," respectively. Then the sum of the gravitational force and Coriolis pseudoforce on the shell is

$$\mathbf{F} = -mg\hat{\mathbf{z}} - 2m\boldsymbol{\omega} \times \mathbf{v} = -mg\hat{\mathbf{z}} - 2m\omega(\cos\lambda\hat{\mathbf{y}} + \sin\lambda\hat{\mathbf{z}}) \times (v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\mathbf{z}).$$

The primary motions of the shell are northward and up and down, so that to a good approximation we can neglect v_x. Therefore taking the cross products, we find

$$\mathbf{F} = -mg\hat{\mathbf{z}} - 2m\omega(\cos\lambda v_z - \sin\lambda v_y)\hat{\mathbf{x}} = m(\ddot{x}\hat{\mathbf{x}} + \ddot{z}\hat{\mathbf{z}})$$

The equation in the z direction is $\ddot{z} = -g$, with solution $z = v_{0z}t - (1/2)gt^2$ where $v_{0z} = v_0 \sin 45° = v_0/\sqrt{2}$. The shell starts and ends at $z = 0$, so ends at $t_f = \sqrt{2}v_0/g$. The range of the shell is therefore $D = v_{0x}t_f = v_0^2/g$.

The equation in the x direction is

$$\ddot{x} = -2\omega[\cos\lambda(v_0/\sqrt{2} - gt) - \sin\lambda v_0/\sqrt{2}]$$

Integrating twice over time,

$$\begin{aligned} \dot{x} &= -2\omega[\cos\lambda(v_0 t/\sqrt{2} - (1/2)gt^2) - \sin\lambda v_0 t/\sqrt{2} \\ x &= -2\omega[\cos\lambda(v_0 t^2/2\sqrt{2} - (1/6)gt^3) - \sin\lambda v_0 t^2/2\sqrt{2} \end{aligned} \tag{9.1}$$

Substituting in our previous results $t_f = \sqrt{2}v_0/g$ and $v_0 = \sqrt{Dg}$, we find the total deflection in the x direction is

$$d = \left(\frac{2D^3}{g}\right)^{1/2} \omega(\sin\lambda - \frac{1}{3}\cos\lambda)$$

as claimed in the problem statement. (b) For $\lambda = 30°$ and $D = 50$ km, we have

$$d = \left(\frac{2(50,000\text{ m})^3}{9.8\text{ m/s}^2}\right)^{1/2} (7.3 \times 10^{-5}\text{s}^{-1}(1/2 - \frac{1}{3}\sqrt{3}/2) \cong 0.16\text{ km}.$$

That is, for a range of 50 km the displacement in this case is about 160 m eastward, eastward because d is positive.

(c) First, note that in the southern hemisphere, where $\lambda < 0$, both terms in the expression for d are negative, so the deflection is always westward. This is true also at the equator and in the northern hemisphere for latitudes less than $\tan^{-1}(1/3)$. For higher latitudes, including the north pole, the expression for d is positive, so the deflection is eastward. What physics is responsible for this odd behavior?

The key is angular momentum conservation in the inertial frame in which the earth is seen to rotate toward the east, since the projectile experiences no torque about its rotation axis. The projectile also does two things during its path: It changes latitude and it changes altitude. The latitude effect is that as a projectile heads north while it is in the southern hemisphere, it increases its radius from the rotation axis, so to conserve angular momentum it must decrease its angular velocity about that axis, which means that it drifts westward. The opposite is true in the northern hemisphere: as the projectile heads north there, it decreases its distance from the rotation axis, so its angular velocity must increase to make up for it, meaning that it drifts eastward. But in addition to the latitude effect there is also an altitude effect. While the projectile is above ground level its distance from the rotation axis is greater than when at the ground, so it must decrease its angular velocity, which means that it drifts westward. So altogether, if the projectile is in the southern hemisphere, both latitude and altitude effects make it drift westward; at the equator there is no latitude effect but there is still an altitude effect making it drift westward; and for low latitudes in the northern hemisphere the latitude effect, which now makes the projectile tend to drift eastward, is small enough that the westward drift caused by the altitude effect carries the day. For higher latitudes in the northern hemisphere the latitude effect finally dominates, and the projectile drifts eastward. ■

⋆ **Problem 9.21** In a rotating cylindrical space colony of radius R and angular velocity ω about its axis of rotation, a ball of mass m is thrown with speed $v = \omega R/2)$ from a point halfway between the rotation axis and rim, in a direction exactly opposite to the rotation direction, as seen by colonists. (a) State the nature of its subsequent path in the inertial frame in which the cylinder is seen to rotate. (b) Sketch the ball's path in the rotating frame of the colony, and show that this path is predicted by the pseudoforces acting upon it.

Solution

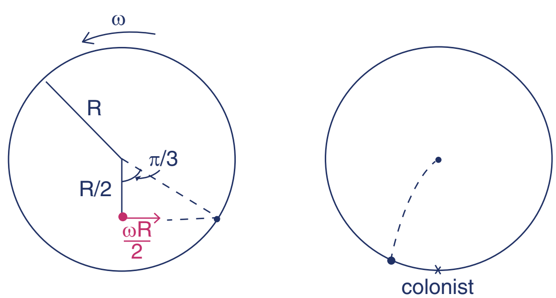
(a) The ball starts at rest and remains at rest in the inertial frame. (b) It travels in a circle of radius $R/2$ in the rotating frame, centered on the rotation axis, with period $T = 2\pi/\omega$. Therefore it is accelerating toward the center with $a = v^2/r = (R/2)\omega^2$. This acceleration is "caused" by an outward centrifugal pseudoforce $m\omega^2(R/2)$ and an inward Coriolis pseudoforce $2m\omega v = 2m\omega(R/2)\omega = mR\omega^2$. The sum of these two pseudoforces is $F_{\text{net}} = m(R/2)\omega^2$ inward, which is consistent with the required inward acceleration. ■

⋆ **Problem 9.22** A ball is released from rest in the frame of a rotating cylindrical space colony, at a point halfway between the rotation axis and rim. (a) Sketch the subsequent path of the ball as seen by an inertial observer who sees the colony rotating counterclockwise with angular velocity ω. (b) If the colonist is directly "beneath" the ball when it is released (*i.e.*, at the rim at a point along a line connecting the rotation axis and release point), how far must the colonist run to catch the ball, in terms of the colony radius R? Sketch the path of the ball as seen by the colonist.

Solution

(a) Note that the ball moves to the right with speed $R\omega/2$ in the inertial frame, as shown in the figure below. When released, its distance from the rotation axis is $R/2$, while when it strikes the rim its distance from the rotation axis is R. The ball moves in a straight line in the inertial frame, because there are no forces on it. The angle between the starting and ending points, with vertex at the axis, is $\pi/3$, so it strikes the rim a distance $\pi R/3$ around the circumference. The flight time of the ball is $t = \text{distance/speed} = R\sin 60^\circ/(R\omega/2) = (\sqrt{3}/2)R/(R\omega/2) = \sqrt{3}/\omega$.

(b) While the ball is in transit the colony rim turns through a distance $R\omega t = R\omega\sqrt{3}/\omega = \sqrt{3}R$. Therefore as the ball is seen to "fall" in the rotating frame, it does not fall vertically, but strikes the rim a distance $\pi R/3 - \sqrt{3}R = (\pi/3 - \sqrt{3})R \cong -0.68R$ away from a point directly beneath its starting point. The minus sign means that it falls to the *left* as seen by colonists. This is the distance directly beneath the starting point the colonist must run to catch the ball. A sketch of the ball's path is shown below.



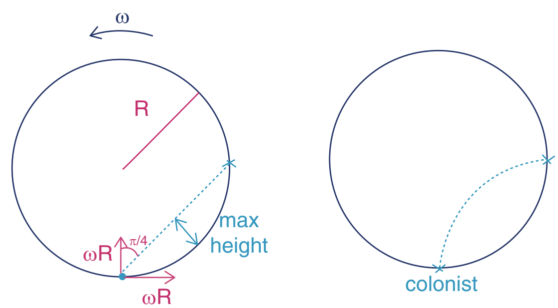
(a) (b)

■

★ **Problem 9.23** A cylindrical space colony of radius R rotates with angular velocity ω about its symmetry axis. A colonist standing on the rim throws a ball straight "up" (*i.e.*, aimed at the rotation axis) with speed $v = R\omega$ from the colonist's point of view. (a) Sketch the subsequent path of the ball as seen by an inertial observer to whom the colony is rotating counterclockwise. *Hint*: First find the initial velocity of the ball in the inertial frame. (b) Sketch the ball's path as seen in the colony frame. (c) How far around the rim must the colonist run to catch the ball?

Solution

(a) In the inertial frame, the ball has (in the figure below) a vertical velocity $R\omega$ due to being thrown, and a horizontal velocity $R\omega$ to the right due to the rim rotation. Therefore it moves at a 45° angle in the inertial frame, as shown at the left below.



(b) We can map this result into the rotating frame, showing that the ball's trajectory is as shown at the right above. Note that the "height" it achieves (the distance away from the rim) is the same in both cases, and that it "falls" back to the rim before the colony has turned 1/4 of the way around.

(c) From the inertial-frame point of view, the ball strikes the rim a quarter of the way around the circle, which is a distance $2\pi R/4 = \pi R/2$ from its starting point, which requires a time $R/v_{\text{vertical}} = R/(R\omega) = 1/\omega$. During this time the rim turns by the distance $v_{\text{tangential}} t = (R\omega)/\omega = R$. Therefore the colonist must run a distance

$$d = \frac{\pi R}{2} - R = \left(\frac{\pi}{2} - 1\right) R \cong 0.57R$$

to catch the ball. Note that the ball's velocity in the rotating frame is perpendicular to the rim both when the ball is thrown and when it is caught. ■

⋆⋆ **Problem 9.24** For the film "2001 Space Odyssey," director Stanley Kubrick had a giant centrifuge constructed, of diameter 11.6 m. On the movie set, motors rotated the centrifuge about a horizontal axis, like a Ferris wheel. This was the home for fictional astronauts on their long journey to the planet Jupiter, providing artificial gravity throughout the trip.

(a) In one scene, astronaut Dr. Frank Poole is seen jogging all the way around the circumference of the centrifuge, requiring about 25 s to do so. What was the rotational period of the centrifuge on the movie set, and how fast was he jogging?

(b) In the movie, it appears that Poole is jogging in a gravity approximately the same as on earth. (No surprise!) Assuming that $g_{\text{effective}} = g_{\text{earth}}$ while standing at rest on the centrifuge rim, and that the centrifuge was actually rotating on the spaceship en route to Jupiter, what would the rotational period of the centrifuge have to be?

(c) Suppose the movie astronaut was 1.9 m tall. By what percentage less would the artificial gravity be on his head than on his feet, just standing on the centrifuge rim?

(d) Would it make any difference if the fictional astronaut were jogging *in* the direction of rotation or *opposite* to it? If so, what would be the effect of jogging in the two directions?

Solution

(a) The period T was approximately 25 s, because on the set he would always be at the bottom of the centrifuge while jogging. His speed was $v \cong 2\pi R/T \cong 6.28(5.8)\text{m}/(25\text{ s}) \cong 1.46\text{ m/s} \cong 3.3\text{ mi/hr}$.

(b) In the rotating frame of the centrifuge, if he is standing at rest there is a normal force $N \equiv mg_{\text{eff}}$, positive inward, as well as a centrifugal pseudoforce $m\omega^2 R$ (positive outward). Therefore when at rest, he is not accelerating relative to the centrifuge, and so we have $g_{\text{eff}} = \omega^2 R$. Therefore $\omega = \sqrt{g_{\text{eff}}/R} = \sqrt{9.8/5.8}$ rad/s $\cong$ 1.3 rad/s. The period is $T = 2\pi/\omega \cong 4.8$ s. That is, to produce an effective gravity equal to that on earth, this centrifuge would have to spin quite fast. If one would be satisfied by achieving only 1/6 earth g's, which is the gravity on the moon, then the period of rotation could be six times larger, or about 29 seconds.

(c) The artificial gravity is linearly proportional to the radius r from the axis of rotation. Therefore the ratio of the artificial gravity at his head to that at his feet is (5.8 – 1.9)/5.8 = 0.67. This is a 33% reduction at his head relative to his feet.

(d) It would make a difference. While jogging at speed v, two additional effects come into play. First, he is now jogging in a circle in the frame of the centrifuge, so he has a centripetal acceleration v^2/R. Second, there is now a Coriolis pseudoforce in addition to the centrifugal pseudoforce. When jogging *in* the centrifuge's rotation direction, the magnitude of the Coriolis pseudoforce is $2m\omega v$, and its direction is outward, away from the central axis. Therefore the total outward pseudoforce is $m(\omega^2 R + 2\omega v)$. When jogging *opposite to* the centrifuge's rotation direction, the Coriolis pseudoforce switches signs, so the total outward pseudoforce is less, $m(\omega^2 R - 2\omega v)$. There is also a force inward, the normal force N of the centrifuge rim on his feet, directed inward. If we average over the constantly changing normal force as he jogs, we can take N to be constant. Setting $F = ma$, with positive direction inward, we have

$$N - m(\omega^2 R \pm 2\omega v) = mv^2/R$$

so the average normal force on his feet is

$$N = m(\omega^2 R \pm 2\omega v) + mv^2/R$$

where the upper (plus) sign holds for jogging in the direction of rotation, while the lower (minus) sign holds for jogging opposite to the direction of motion. Obviously the normal force is greater in the former case, so generally speaking it would be harder to run in the rotation direction than against it. Suppose for example we take the system already described, with $R = 5.8$ m, $v = 1.46$ m/s, $\omega = 1.3$ rad/s. Then the average normal force is $N \cong (10.2 \pm 3.8\text{m/s}^2)m$ = (14.0 m/s^2) m while jogging in the direction of rotation, but only (6.4 m/s^2) m while jogging opposite to the direction of rotation. This would obviously be noticeable; it would be much easier in this case to jog opposite to rather than in the direction of rotation. If we define an effective gravity while jogging by $N = mg_{\text{eff}}$, it would change from (14.0 m/s^2) to (6.4 m/s^2). ■

★ **Problem 9.25** (a) Prove that there is no work done by the Coriolis pseudoforce acting on a particle moving in a rotating frame. (b) If the Coriolis pseudoforce were the *only* force acting on a particle, what could you conclude about the particle's speed in the rotating frame?

Solution

The Coriolis pseudoforce on a particle of mass m is $\mathbf{F}_{\text{Cor}} = -2m\boldsymbol{\omega} \times \mathbf{v}$ where $\mathbf{v}$ is the particle's velocity in the rotating frame. The rate at which work is done by a force is $dW/dt = \mathbf{F} \cdot \mathbf{v}$, so the work done by the Coriolis pseudoforce in particular is $-2m(\boldsymbol{\omega} \times \mathbf{v}) \cdot \mathbf{v} = 0$, since the cross product $\boldsymbol{\omega} \times \mathbf{v}$ is perpendicular to $\mathbf{v}$, and so the dot product of this quantity with $\mathbf{v}$ is zero. No work is being done on the particle by the Coriolis pseudoforce, so by the work-energy theorem the kinetic energy is unchanged, and so the speed cannot change either. The direction of motion will change in general, but not the speed. ■

★★ **Problem 9.26** Show that the formula $d\mathbf{A} = d\boldsymbol{\theta} \times \mathbf{A}$ for the change in an inertial frame of a vector $\mathbf{A}$ that is stationary in a rotating frame is still valid when $\mathbf{A}$ is not perpendicular to $\boldsymbol{\Omega}$ and when the tail of $\mathbf{A}$ is not situated at the rotation axis.

Solution

Consider projections of $\mathbf{A}$ and $\mathbf{A} + d\mathbf{A}$ onto a plane that is perpendicular to $\boldsymbol{\Omega} \rightarrow$ call these

$$\mathbf{A}' \quad \text{and} \quad \mathbf{A}' + d\mathbf{A}' = \mathbf{A}' + d\mathbf{A}.$$

Note that

$$|\mathbf{A}| = |\mathbf{A} + d\mathbf{A}|, \quad |\mathbf{A}' + d\mathbf{A}'| = |\mathbf{A} + d\mathbf{A}| \sin\alpha = A \sin\alpha$$

$$|\mathbf{A}'| = A\sin\alpha = |\mathbf{A}' + d\mathbf{A}'| \Rightarrow dA = A' d\theta = A \sin\alpha d\theta$$

Taking $d\theta$ to be parallel to $\boldsymbol{\Omega}$, by the right hand rule we see that

$$d\mathbf{A} // d\boldsymbol{\theta} \times \mathbf{A} \quad \Rightarrow \quad d\mathbf{A} = d\boldsymbol{\theta} \times \mathbf{A}$$

in general. ■

★ **Problem 9.27** A well-known actor and television crew, filming a travel documentary, were driving south in Africa when they were approached by a local citizen as they neared the equator. He offered (for a fee) to demonstrate the change in swirl direction of water in a hand basin. They all walked a few minutes north of the equator, he filled the basin with water, removed a plug in a hole in the middle of the basin, and sure enough, the water swirled counterclockwise as the water drained out. They then walked a few minutes south of the equator, repeated the experiment, and the water swirled clockwise this time as the water rushed out. It would have been interesting to learn how much the local man made demonstrating his skill over and over for equator-crossing tourists! How might the local man have produced the result?

Solution

The Coriolis pseudoforce is zero at the equator, and negligible over the entire range of the demonstrations described. It would be easy to pour water into the basin so it has some angular momentum in whichever sense you would like it to swirl as it drains through the central hole. The swirl might be unnoticeable when the water fills the basin, but as the water drains out and flows toward the center, the angular velocity of the swirl will increase greatly. Coriolis effects have nothing whatsoever to do with what was observed. ■

★★ **Problem 9.28** Prevailing winds in middle latitudes of the northern hemisphere are westerly, blowing from west to east at typical speed v. The tendency of the Coriolis pseudoforce to deflect the path southward is typically balanced by a horizontal pressure gradient that keeps the air flowing eastward. The pressure gradient is related through the ideal gas law to the north-south temperature gradient, with warmer air in the south. Find an expression for the wind speed in terms of the temperature gradient $\Delta T/\Delta x$, the latitude λ, the earth's angular velocity Ω, and any necessary gas constants. Estimate the typical temperature gradient and find the typical flow velocity. Are the results reasonable? Would you expect prevailing winds in comparable southern latitudes to be westerly or easterly?

Solution

Consider a thin slice of air flowing eastward. The mass of air in the sample is $\Delta m = \rho\Delta(\text{Vol}) = \rho A\Delta x$, where ρ is the density of air, A is the area of the sample on the north or south face of the sample, and Δx is the (very small) distance between the north and south faces. In the northern hemisphere, a Coriolis force

$$\Delta F_{\text{Cor}} = 2(\Delta m)\omega v \sin\lambda = 2\rho A\Delta x\omega v \sin\lambda$$

acts upon it, pushing on it toward the south. This southward force can be balanced by an air pressure differential, with higher pressure in the warmer south. The northward force due to the pressure differential Δp is ΔpA. Balancing the Coriolis force with this air-pressure force, we have

$$2\rho A\Delta x\omega v \sin\lambda = \Delta pA,$$

so that

$$\frac{\Delta p}{\Delta x} = 2\rho\omega v \sin\lambda.$$

The pressure differential is, in turn, caused by a temperature differential between the south and north faces. According to the ideal gas law, $pV = NkT$, where V is the volume of the sample, N is the number of molecules in that volume, k is Boltzmann's constant, and T is the temperature. Therefore $\Delta p = (N/V)k\Delta T = (\rho/m_0)k\Delta T$, where m_0 is the average mass of an air molecule. Solving for the flow velocity v, we have

$$v = \frac{k}{2m_0\omega\sin\lambda}\frac{\Delta T}{\Delta x}.$$

We can make a rough calculation of a typical flow velocity using (say) latitude $\lambda = 45^\circ$, $\Delta T = 7^\circ$ C over a north-south distance $\Delta x = 1000$ km, and $m_0 \cong 30 \times (1.67 \times 10^{-27}$ kg), larger than that of nitrogen, and smaller than that of oxygen. These particular numbers give $v \cong 19$ m/s ~ 42 miles/hour, a reasonable result. Actual jet stream velocities and directions vary, especially over land, because of temperature differences above deserts and mountain ranges for example. They do however typically travel from west to east, which is responsible for smaller typical airline travel times from the west coast to the east coast of North America, and larger travel times from east to west.

Midlatitude winds in the southern hemisphere should also be westerlies. In this case the higher temperatures are to the north, so a Coriolis force toward the left (north) could be compensated by higher atmospheric pressure at the north. ■

★★ **Problem 9.29** The Gulf Stream flows northward off the Florida coast, so tends to be deflected eastward. This causes the water level to rise on the eastern side, since the more stationary Atlantic waters cannot easily be moved aside. The higher waters on the eastern side provide the higher pressures needed to counteract the Coriolis force, so the stream is relatively undeflected. Looking northward, the stream looks as shown in Figure 9.23, with a greatly exaggerated eastern rise. Using a thin vertical slice of water and balancing the pressure and Coriolis forces upon it on the left and right, find an expression for the slope dy/dx of the surface in terms of the earth's angular velocity ω, the latitude λ, the acceleration of gravity g, and the stream velocity v. The westernmost islands of the Bahamas are only about 80 km from the east coast of Florida. Between them the Gulf Stream flows somewhat in excess of 1 m/s, and the sea level is about 0.5 m higher at the Bahamas. Are these measurements consistent with your results?

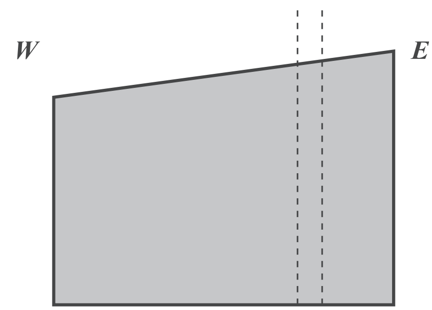


Fig. 9.23 Tilt of the northward-flowing gulf stream surface, looking north

Solution

A thin, vertical slab of water of height y, east-west thickness Δx, and north-south length ℓ has mass $\Delta m = \rho\Delta V = \rho(y\ell\Delta x)$ where ρ is the density. As it flows northward the Coriolis pseudoforce on it is $2(\Delta m)\omega v\sin\lambda = 2\rho(y\ell\Delta x)\omega v\sin\lambda$, pushing eastward. Counteracting this force is a greater pressure on the right, caused by an increased height of the water on the east side of the slab. That is, if the height of the water on the west face of the slab is y, the height on the east face is $y + \Delta y$, which means a pressure increase of $\Delta p = \rho g\Delta y$ everywhere on the east face of the slab. That is, the net pressure force counteracting the Coriolis effect is $\Delta p y\ell = \rho g\Delta y y\ell$, the pressure difference multiplied by the area of the slab.

Balancing the two forces, we have

$$2\rho(y\ell\Delta x)\omega v\sin\lambda = \rho g\Delta y y\ell, \quad \text{so} \quad \frac{\Delta y}{\Delta x} = \frac{2\omega v}{g}\sin\lambda$$

providing the slope upward from west to east across the slab.

The Bahamas have a latitude of $\lambda \cong 25°$ and the westernmost islands are about 80 km from the east coast of Florida. Therefore taking Δx = 80 km, we predict an ocean height rise toward the east of

$$\Delta y \cong \frac{2\omega v}{g}\sin\lambda\Delta x \cong \frac{2\times 7.3\times 10^{-5}\times 1}{9.8}(0.42)(80,000) \cong 0.50 \text{ m}.$$

which is indeed consistent. ■

★★★ **Problem 9.30** A spherical asteroid of radius R and uniform density ρ rotates with angular velocity Ω about an axis through its center. Visiting astronauts drill a smooth hole from one point on the equator clear through the asteroid's center to a point on the opposite side. (a) If an astronaut falls into one end of the hole, how long does it take her to reach the opposite side? (b) During the trip she is pressed against the side of the hole with a force N. Find N in terms of W, her weight on the asteroid's surface at one of the poles, as a function of her distance r from the center of the asteroid. Can N exceed W?

Solution

(a) In the rotating (non-inertial) frame of the asteroid, the net force on the falling astronaut is

$$\mathbf{F} = -\frac{GM(r)m}{r^2}\hat{\mathbf{r}} + m\Omega^2\mathbf{r} - 2m\boldsymbol{\Omega}\times\mathbf{v}$$

plus a normal force $\mathbf{N}$ if the astronaut touches the side of the hole. Here $M(r) = M(r/R)^3$ is the asteroid mass within radius r, and M and R are the asteroid's total mass and radius. Therefore we can write

$$\mathbf{F} = -\frac{GMm}{R^3}\mathbf{r} + m\Omega^2\mathbf{r} - 2m\boldsymbol{\Omega}\times\mathbf{v} + \mathbf{N}.$$

The Coriolis and normal forces are perpendicular to the direction of motion, so in the radial direction

$$\left(-\frac{GMm}{R^3} + m\Omega^2\right) r = m\ddot{r} \quad\Rightarrow\quad \ddot{r} + \left(\frac{GM}{R^3} - \Omega^2\right) r = 0$$

which is the simple harmonic oscillator equation, with solution $r = R\cos\omega t$ where $\omega = \sqrt{\frac{GM}{R^3} - \Omega^2}$, and the fact that $r = R$ at $t = 0$. We will rename the dependent variable "x", since x can be negative as the astronaut falls through the center of the asteroid to the other side. That is, we take $x = R\cos\omega t$. In terms of the period of oscillation T, the time it takes to reach the other end of the hole is

$$t = \frac{T}{2} = \frac{1}{2}\frac{2\pi}{\omega} = \frac{\pi}{\sqrt{\frac{GM}{R^3} - \Omega^2}}.$$

Here $M = (4/3)\pi R^3\rho$, so

$$t = \frac{\pi}{\sqrt{\frac{GM}{R^3} - \Omega^2}} = \frac{\pi}{\sqrt{\frac{(4/3)\pi G\rho}{R^3} - \Omega^2}}.$$

(b) The Coriolis force pushes the astronaut against the side of the hole, and the hole pushes back. That is, $N = 2m\Omega v = 2m\Omega\dot{x}$. The magnitude of the normal force is

$$|N| = 2m\Omega\omega R\sin\omega t = 2m\Omega\omega R\sqrt{1-\cos^2\omega t} = 2m\Omega\omega R\sqrt{1 - r^2/R^2}.$$

The astronaut's weight is $W = mg = m(GM/R^2)$, at either pole. Therefore as a function of W and r,

$$N = 2W\frac{R^3}{GM}\Omega\sqrt{\frac{GM}{R^3} - \Omega^2}\,\sqrt{1 - r^2/R^2}$$

which is obviously a maximum if $r = 0$, at the center of the hole.

Now can N exceed W? Note that at $r = 0$,

$$\frac{N}{W} = 2\frac{R^3\Omega}{GM}\sqrt{\frac{GM}{R^3} - \Omega^2} = 2Z^{1/2}\sqrt{1 - Z}$$

where $Z \equiv R^3\Omega^2/GM$. So N/W is maximized when $Z^{1/2}\sqrt{1 - Z}$ is maximized, which is at $Z = 1/2$. In that case $N/W = 1$, so the answer is "**no**", N cannot exceed W. ∎

⋆ **Problem 9.31** (a) Find the radius of a toroidal space colony spinning once every two minutes, if the effective gravity for colonists living within the torus is 10 m/s^2. (b) Arriving tourists dock at the central hub, and are then transported to the torus by an elevator running through a "spoke" of the wheel. The elevator runs at constant speed, except for brief periods of acceleration at the beginning and end, and requires one minute for the journey. Plot quantitatively the centrifugal and Coriolis accelerations of a rider as a function of time while the elevator is running at constant speed, as a fraction of 10 m/s^2. Would the Coriolis acceleration be noticeable?

Solution

The effective gravity for the colonists is $\omega^2 R$, so if T is the period,

$$R = g_{\rm eff}/\omega^2 = g_{\rm eff}(T/2\pi)^2 = 10\ \mathrm{m/s^2}(120\mathrm{s})/6.28)^2 = 3.65\ \mathrm{km}.$$

The two accelerations are $\omega^2 r$ and $2m\omega v$, where $v = 3.65$ km/(60 s) and $r = vt$. ∎

⋆⋆ **Problem 9.32** A cylindrical space station rotating with angular velocity Ω contains an atmosphere with molecular weight M and temperature T. Show that if P_0 is the atmospheric pressure at the rotation axis, the pressure at radius ρ is $P = P_0 \exp(M\Omega^2\rho^2/2RT)$, where R is the ideal gas constant. If the station has a radius of 100 m, an effective rim gravity 10 m/s^2, and an oxygen atmosphere at temperature 300 K, what is the ratio of the rim pressure to that at the rotation axis? Would this difference be important to inhabitants who travel from the rim to the axis?

Solution

We have three fictitious forces acting on an element of the gas, which are

$$-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m(\boldsymbol{\omega} \times \mathbf{v}_{rest}) - m(\dot{\boldsymbol{\omega}} \times \mathbf{r})$$

$$-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \to g = \Omega^2\rho \quad \text{outward}$$

$$-2m(\boldsymbol{\omega} \times \mathbf{v}_{\rm rest}) \to 0 \text{ since } \mathbf{v}_{\rm rest} = 0$$

For a static gas at equilibrium,

$$-m(\dot{\boldsymbol{\omega}} \times \mathbf{r}) \to \text{small since } \dot{\omega} \simeq 0$$

Balance forces in $\hat{\rho}$ direction

$$-P(\rho+\Delta\rho)\rho\Delta\varphi\Delta z + P(\rho)\rho\Delta\varphi\Delta z + \Omega^2\rho n\rho\Delta\varphi\Delta\rho\Delta z$$

Write

$$P(\rho+\Delta\rho) \simeq P(\rho) + P'\Delta\rho$$

$$\Rightarrow P' = \Omega^2\rho n = \Omega^2\rho\frac{pM}{RT}$$

Ideal gas law

$$P = (\frac{M}{n})^{-1}RT$$

$$\int\frac{dP}{:P} = \frac{\Omega^2 M}{RT}\int\rho d\rho$$

$$\Rightarrow P(\rho) = P_0 e^{\frac{M\Omega^2\rho^2}{2RT}}$$

where

$$P_0 = P(\rho=0) \quad R = 8.314 \text{ J/mol K} \quad T = 300 \text{ K}$$

$$A = 100\text{m} \quad g = 10 \text{ m/s}^2 \quad \Omega^2 A = 10 \text{ m/s}^2 \Rightarrow \Omega = 0.1 \text{ s}^{-1}$$

$$\Rightarrow P(A)/P_0 = e^{\frac{M\Omega^2 A^2}{2RT}} \simeq 1$$

$$M_{O_2} \simeq 2.7\times 10^{-26} \text{ kg}$$

■

★★ **Problem 9.33** An astronaut is stranded in space above the earth, in the same orbit as a Space Station, but 200 m behind it. Both are circling 280 km above earth's surface in a 90 minute orbit. The astronaut and spacesuit together have a mass of 100 kg. (a) In what direction, and with what speed, can the astronaut throw a one-kilogram wrench so that the recoil will allow the astronaut to reach the vehicle and safety? (b) How long will it take the astronaut to arrive? (c) Sketch the trajectory of the astronaut and of the wrench after the throw, in the rest frame of the Station, and then sketch the trajectory of each in the non-rotating, inertial frame in which the Station orbits the earth.

Solution

If the astronaut throws the wrench *toward the Station*, she will recoil, and then follow the trajectory described by $x = (4v_0/\omega)\sin\omega t - 3v_0 t$, $y = -(2v_0/\omega)(1-\cos\omega t)$. She leaves at $t = 0$, and arrives at $\omega t = 2\pi$, which is the time when y is again zero. Here $\omega = 2\pi/T$, where T is the Station's period in earth's orbit. Therefore $t = T \Rightarrow x = -3v_0 T$ upon arrival, equal to - 200 m. For the Station, $Mr\omega^2 = Mr(2\pi/T)^2 = GmM_e/r^2$, where M_e is the mass of the earth. Therefore $T = 2\pi r^{3/2}/\sqrt{GM_e}$ where r = (6370 + 280) km = 6650 km. One can then verify that $T \cong 90$ minutes. Therefore $3v_0 T = 200$ m implies that $v_0 = 200$ m/ $(3\times 90$ m) = 0.012 m/s. That is her speed, a very slow drift. She must throw the wrench, which is only 1% of her mass, with 100 times her final speed, namely 1.2 m/s, which is about 2.7 mi/hr. She can certainly throw the wrench that fast. (b) She will require 90 minutes to arrive back at the Station. ■

⋆⋆⋆ **Problem 9.34** Only the centrifugal and Coriolis pseudoforces act upon a particular projectile moving within a rotating cylindrical space colony of radius R. (a) Find the differential equations of motion of the projectile in the rotating frame, using Cartesian coordinates centered on the rotation axis. (b) Decouple and solve the differential equations, to find expressions for $x(t)$ and $y(t)$ in terms of four arbitrary constants of integration. (c) Evaluate the constants in terms of the initial conditions $x_0, y_0, v_{x_0}, v_{y_0}$. (d) A colonist on the rim at $(x_0, y_0) = (R, 0)$ throws a ball toward the rotation axis with velocity $(v_{x_0}, v_{y_0}) = (-v_0, 0)$. Find a general expression for the time at which the ball returns to the rim, and show that in the limit as v_0 becomes small, the time agrees with what you would expect in a uniform gravitational field $g = R\Omega^2$.

Solution

(a) The Lagrangian is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}m(\mathbf{\Omega} \times \mathbf{r})^2 + m\mathbf{v} \cdot (\mathbf{\Omega} \times \mathbf{r})$$

where

$$\mathbf{r} = x\hat{x} + y\hat{y} \qquad \mathbf{\Omega} = \Omega\hat{z}$$

$$\Rightarrow L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}m\Omega^2(x^2 + y^2) - m\Omega x\dot{y} - m\Omega y\dot{x}$$

$$\Rightarrow \ddot{x} = \Omega^2 x + 2\Omega\dot{y} \qquad (*)$$

$$\ddot{y} = \Omega^2 y - 2\Omega\dot{x} \qquad (**)$$

(b) Write

$$A = x + iy \qquad A^* = x - iy$$

Multiply (**) by i and add to $(*)$

$$\Rightarrow \ddot{A} = \Omega^2 A - 2i\Omega\dot{A}$$

Write

$$A(t) = F(t)e^{-i\Omega t}$$

$$\Rightarrow \ddot{F} = 0 \Rightarrow F(t) = C_1 t + C_2$$

where C_1 and C_2 are complex constants $\Rightarrow$ 4 real constants of integration as expected.

$$\Rightarrow A(t) = (C_1 t + C_2)e^{-i\Omega t}$$

$$x(t) = \frac{A + A^*}{2} \qquad y(t) = \frac{A - A^*}{2i}$$

(c)

$$A(0) = x_0 + iy_0 = C_2 \qquad \dot{A}(0) = C_1 - i\Omega C_2 = v_{x_0} + iv_{y_0}$$

$$\Rightarrow C_1 = i\Omega(x_0 + iy_0) + v_{x_0} + iv_{y_0} = (v_{x_0} - \Omega y_0) + i(v_{y_0} + \Omega x_0)$$

$$\Rightarrow A(t) = [(v_{x_0} - \Omega y_0)t + i(v_{y_0} + \Omega x_0)t + x_0 + iy_0] \times e^{-i\Omega t}$$

Writing

$$e^{-i\Omega t} = \cos(\Omega t) - i\sin(\Omega t)$$

$$\Rightarrow x(t) = ((v_{x_0} - \Omega y_0)t + x_0)\cos(\Omega t) + ((v_{y_0} + \Omega x_0)t + y_0)\sin(\Omega t)$$

$$y(t) = ((v_{y_0} - \Omega x_0)t + y_0)\cos(\Omega t) - ((v_{x_0} - \Omega y_0)t + x_0)\sin(\Omega t)$$

(b)

$$(x_0, y_0) = (R, 0) \qquad (v_{x_0}, v_{y_0}) = (-v_0, 0)$$

$$x(t) = (-v_0 t + R)\cos(\Omega t) + \Omega Rt\sin(\Omega t)$$

$$y(t) = \Omega Rt\cos(\Omega t) - (-v_0 t + R)\sin(\Omega t)$$

Consider

$$x^2 + y^2 = (R - tv_0)^2 + R^2\Omega^2 t^2 = R^2 \Rightarrow t = \frac{2Rv_0}{v_0^2 + R^2\Omega^2}$$

If

$$v_0 \ll R\Omega \Rightarrow t = \frac{2Rv_0}{R^2\Omega^2} = \frac{2v_0}{R\Omega^2} = \frac{2v_0}{g}$$

In normal gravity, we have

$$\Delta z = 0 = v_0 t - \frac{1}{2}gt^2 \Rightarrow t = \frac{2v_0}{g}$$

■

⋆⋆ **Problem 9.35** A uniform electric charge density ρ fills a very long stationary cylinder. (a) Show from Gauss's law $\oint \mathbf{E} \cdot d\mathbf{S} = 4\pi q_{\text{in}}$ that the electric field within the cylinder is $\mathbf{E} = 2\pi\rho\boldsymbol{\rho}$, where $\boldsymbol{\rho}$ is the radius vector out from the symmetry axis. Here q_{in} is the charge within an appropriate Gaussian surface . (b) A uniform magnetic field $\mathbf{B} = B_0\hat{\hat{z}}$ is created in the same region of space. Including the effects of both $\mathbf{E}$ and $\mathbf{B}$, find an expression for the force exerted on a test charge q placed within the cylinder. (c) Show that if the test charge has the proper charge/mass ratio q/m, there exists a *rotating* frame in which the charge is bound to move just as in part (b) with *no* electromagnetic fields at all. Find this ratio q/m, and the angular velocity ω of the rotating frame.

Solution

(a) It is convenient to use cylindrical coordinates ρ, θ, z here, so to avoid confusion with two different uses of the symbol ρ, we will denote the charge density by ρ_e and the radial coordinate by simply ρ. Gauss's law is

$$\oint \mathbf{E} \cdot d\mathbf{A} = 4\pi q_{in}$$

so at the surface of a coaxial cylindrical Gaussian surface within the stationary cylinder, ρ = constant, so we have

$$E 2\pi \rho h = 4\pi \rho_e \pi \rho^2 h \Rightarrow E = 2\pi \rho_e \rho \Rightarrow \mathbf{E} = 2\pi \rho_e \boldsymbol{\rho}$$

where h is the (arbitrary) length of the Gaussian surface. (b) Now taking into account both the electric and magnetic forces, the total force on a test charge q is

$$\mathbf{F}_{\mathbf{E\&M}} = q\mathbf{E} + q\frac{\mathbf{v}}{c} \times \mathbf{B} = 2\pi q \rho_e \rho + q\frac{\mathbf{v}}{c} \times \mathbf{B} \quad \text{with } \mathbf{B} = B_0 \hat{\mathbf{z}}.$$

(c) Now consider instead a rotating frame about the central axis in which the only "forces" on a particle with charge q and mass m are the centrifugal and Coriolis pseudoforces

$$\mathbf{F}_{Rot} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \rho) - 2m\boldsymbol{\omega} \times \mathbf{v}$$

We will suppose that the angular velocity of rotation is $\boldsymbol{\omega} = \omega_0 \hat{\mathbf{z}}$ so that after taking all the cross-products, $\mathbf{F}_{Rot}$ reduces to

$$\mathbf{F}_{Rot} = m\omega_0^2 \boldsymbol{\rho} + 2m\omega_0 (\mathbf{v} \times \hat{\mathbf{z}}).$$

Comparing this force with the electromagnetic force calculated before, we see that they have similar forms, and so if constants are chosen appropriately the motion of a charge in a rotating frame can mimic motion in the nonrotating electromagnetic case described before. In fact, the two forces match if $2\pi q \rho_e = m\omega_0^2$ and $qB_0/c = 2m\omega_0$. Stated another way, they match if we choose $\omega_0 = 4\pi c \rho_e / B_0$ and the ratio $q/m = 8\pi c^2 \rho_e / B_0^2$. ■

⋆⋆ **Problem 9.36** Show that for the problem of a spacecraft rendezvous and docking discussed in the text, the dynamics in the z direction decouples from that in the x-y plane. What is the equation of motion of the z coordinate?

Solution

Adding in the z-dynamics means that

$$\Delta \mathbf{r} \to x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

$$\Rightarrow \mathbf{r} \to x\hat{\mathbf{x}} + (r_0 + y)\hat{\mathbf{y}} + z\hat{\mathbf{z}} \Rightarrow r^2 \to r_0^2 + 2r_0 y + x^2 + y^2 + z^2$$

$$(*) \qquad \simeq r_0^2 + 2r_0 y + x^2 + y^2 \qquad \text{for small z}$$

There is only a simple change In the Lagrangian,

$$L \to L_{\text{text}} + \frac{1}{2} m\dot{z}^2$$

And there is no contribution from the 2nd line of equation (9.60), because

$$\omega\hat{\mathbf{z}} \times (z\hat{\mathbf{z}}) = 0.$$

There is also no contribution from 3rd line of (9.60), because

$$\boldsymbol{\Omega} \times \mathbf{r} \to \omega\hat{\mathbf{z}} \times (z\hat{\mathbf{z}}) \to 0$$

And the contribution from 4th line negligible, because of (*)

Therefore we have free dynamics in the z direction, decoupled from the x and y dynamics,

$$\ddot{z} = 0$$

■

★★★ **Problem 9.37** *Space visits for everyone?* An alternative way to visit space has been proposed: A space station of mass m is tethered to one end of a long cable and the other end of the cable is attached to a point on the earth's equator, a distance R from the center of the earth. In the rotating frame of the earth, three forces act on the station: the centrifugal pseudoforce $mr\omega^2$ outward, and earth's gravity GMm/r^2 and the cable tension T inward, where r is the distance of the station from the center of the earth. As one goes to larger radii the centrifugal pseudoforce grows while gravity decreases, so there must be a radius r_0 where these forces balance, so that the station will remain in place. Then one could ride an elevator up the cable and get some spectacular views and experience zero $g's$ without using any rocket fuel. (a) Assuming $T = 0$, find the distance r_0 from the earth's center to the station where the forces balance, in terms of G, M, and ω, the angular velocity of the earth's rotation. (b) Of course the cable will require some tension $T(r)$ if an elevator is to travel up and down along it. This might be achieved by placing the space station at a somewhat greater distance $r_0 + \Delta r_0$ from the earth's center, requiring a positive downward tension force for it to stay in place. Let the cable have uniform mass per unit length λ. Then show that the tension $T(r)$ obeys the equation

$$\frac{dT}{dr} = \lambda\left(\frac{GM}{r^2} - r\omega^2\right).$$

(c) At what radius is the tension in the cable a maximum? (d) Find the tension T_s in the cable just where it is attached to the space station. Assume here that $\Delta r_0 << r_0$, and so keep only first-order terms in Δr_0. Express your answer in terms of m, ω, and Δr_0. (e) Find a general expression for the tension $T(r)$ anywhere along the cable, in terms of $T_s, \lambda, \omega, r_0, G, M$, and r. (f) In particular, what is the cable tension at $r = r_0$? (g) At what radius is the cable tension a *minimum*? (h) Find the minimum value of Δr_0 required, in terms of other given parameters, so that the cable will never have a negative tension anywhere along its length, because that would cause it to buckle.

Solution

(a) Balancing gravity and the centrifugal pseudoforce for a particle a distance r from the center of the earth,

$$\frac{GMm}{r_0^2} = mr_0\omega^2 \Rightarrow r_0^3 = \frac{GM}{\omega^2} \Rightarrow r_0 = (\frac{GM}{\omega^2})^{1/3},$$

which is the same as the radius of geosynchronous orbits. (b) Looking at a small slice Δr of the cable, the upward forces on the slice are $T + \Delta T$ and $\Delta mr\omega^2$, and the downward forces are T and $GM\Delta m/r^2$ where $\Delta m = \lambda\Delta r$. Balancing these in equilibrium,

$$T(r+\Delta r) - T(r) + \lambda\Delta r r\omega^2 - \lambda\Delta r\frac{GM}{r^2} = 0$$

$$\Rightarrow \frac{T(r+\Delta r) - T(r)}{\Delta r} \xrightarrow{\Delta r\to 0} \frac{dT}{dr} = \lambda(\frac{GM}{r^2} - r\omega^2)$$

as claimed.

(c) Note that $dT/dr > 0$ for $r < r_0$ and $\frac{dT}{dr} < 0$ for $r > r_0$, and $\frac{dT}{dr} = 0$ at $r_0(GM/\omega^2)^{1/3}$. So the maximum tension is at $r = r_0$, the radius of geosynchronous orbits.

(d) At the space station of mass m, at location $r = r_0+\Delta r$ where $\Delta r \ll r_0$, in equilibrium we have the tension at the station

$$T_s = m(r_0 + \Delta r_0)\omega^2 - GMm/(r_0 + \Delta r_0)^2 = m\omega^2\Delta r_0 + 2GMm/\Delta r_0/r_0^3$$

where in the last step we used the binomial approximation. So finally, using the result of part (a), we have

$$T_s = 3m\omega^2\Delta r_0$$

where m is the mass of the station, a distance Δr_0 from geosynchronous radius r_0.

(e) Now to find the tension T everywhere along the cable, integrate the differential equation found in part (b). That is,

$$T(r) = \lambda\int dr\left(\frac{GM}{r^2} - r\omega^2\right) + C$$

where C is a constant of integration. Performing the integration,

$$T(r) = \lambda\left[-\frac{GM}{r} - \frac{1}{2}\omega^2 r^2\right] + C.$$

which must be equal to T_s at $r = r_0+\Delta r_0$. Solving for C, using the binomial approximation to keep no more than first-order terms in Δr_0, we find that

$$C = [3m\Delta r_0 + (3/2)\lambda r + 0^2]\omega^2.$$

Therefore in general the tension along the cable, as a function of position, is

$$T(r) = \left(3m\Delta r_0 + \frac{3}{2}\lambda r_0^2\right)\omega^2 - \lambda\left(\frac{GM}{r} + \frac{1}{2}\omega^2 r^2\right).$$

(f) At $r = r_0$, we find that $T(r_0) = 3m\omega^2\Delta r_0$, the same tension as at the space station to first order in Δr_0.

(g) The minimum tension occurs where

$$\frac{GM}{r} + \frac{1}{2}\omega^2 r^2$$

is at a maximum, which is at ground level $r = R$. At that point

$$T(R) = (3m\omega^2 \Delta r_0 + (3/2)\lambda r_0^2)\omega^2 - \lambda\left(\frac{GM}{R} + \frac{1}{2}\omega^2 R^2\right).$$

(h) To make $T \geqslant 0$ everywhere, it is sufficient to make $T(R) \geqslant 0$. This is then a condition on Δr_0, which reduces to

$$\Delta r_0 \geqslant \frac{\lambda}{3m\omega^2}\left[\frac{GM}{R} + \frac{1}{2}\omega^2 R^2 - \frac{3}{2}r_0^2\omega^2\right].$$

where again Δr_0 is the distance of the space station outside the geosynchronous orbit location, m is the mass of the station, λ = mass/length of the cable, R is the radius of the earth, and ω is its angular velocity of rotation. ■

10 From Black Holes to Random Forces

10.1 Problems and Solutions

⋆⋆ **Problem 10.1** Verify the particular solutions given of the inhomogeneous first-order equations for the perihelion precession, as given in equation 10.34.

Solution

The first equation is

$$u_1'' + u_1 = 3A(1+\epsilon^2/2),$$

with solution

$$u_1^{(1)} = 3A(1+\epsilon^2/2).$$

Given $u_1^{(1)}$ we have $u_1^{(1)''} = 0$, so the solution is obviously correct. The next equation is

$$u_1'' + u_1 = 3A(\epsilon^2/2)\cos 2\varphi$$

with a claimed solution

$$u_1^{(3)} = -(\epsilon^2 A/2)\cos 2\varphi.$$

For the claimed solution,

$$u_1^{(3)'} = \epsilon^2 A \sin 2\varphi$$

and

$$u_1^{(3)''} = 2\epsilon^2 A \cos 2\varphi.$$

Then

$$u_1'' + u_1 = 2\epsilon^2 A\cos 2\varphi - (\epsilon^2 A/2)\cos 2\varphi = \frac{3}{2}\epsilon^2 A\cos 2\varphi$$

which also satisfies the equation. Finally, we have

$$u_1'' + u_1 = 6A\epsilon\cos\varphi$$

with claimed solution

$$u_1^{(2)} = 3\epsilon A\varphi\sin\varphi.$$

If so,

$$u_1'^{(2)} = 3\epsilon A(\sin\varphi + \varphi\cos\varphi)$$

and

$$u_1''^{(2)} = 3\epsilon A(\cos\varphi + \cos\varphi - \varphi\sin\varphi) = 3\epsilon A(2\cos\varphi - \varphi\sin\varphi).$$

Then

$$u_1''^{(2)} + u_1^{(2)} = 3\epsilon A(2\cos\varphi - \varphi\sin\varphi) + 3\epsilon A\varphi\sin\varphi = 6A\epsilon\cos\varphi$$

which also satisfies the equation. Therefore we have verified that the three particular solutions are correct. ■

⋆⋆ **Problem 10.2** The metric of flat, Minkowski spacetime in Cartesian coordinates is $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$. Show that the geodesics of particles in this spacetime correspond to motion in straight lines at constant speed.

Solution

Geodesics obey $\delta S = 0$, where

$$S = \int \left[-c^2\dot{t}^2 + \dot{x}^2 + \dot{y}^2 + \dot{z}^2\right] d\tau \quad \text{with} \quad \dot{t} \equiv \frac{dt}{d\tau}, \quad \dot{x} = \frac{dx}{d\tau}, \text{etc.}$$

The Euler-Lagrange equation for t is

$$\frac{d}{d\tau}\frac{\partial I}{\partial \dot{t}} - \frac{\partial I}{\partial t} = 0 \Rightarrow \frac{\partial I}{\partial \dot{t}} = -2c^2\dot{t} = \text{con} \equiv -2c^2\alpha$$

and for x is

$$\frac{d}{d\tau}\frac{\partial I}{\partial \dot{x}} - \frac{\partial I}{\partial x} = 0 \Rightarrow \frac{\partial I}{\partial \dot{x}} = 2\dot{x} = \text{con} \equiv 2\dot{x}_0$$

with similar equations for y and z. Therefore,

$$\frac{dx}{d\tau} = \dot{x}_0 \quad \frac{dy}{d\tau} = \dot{y}_0 \quad \frac{dz}{d\tau} = \dot{z}_0$$

or

$$\frac{dx}{dt} = \frac{dx}{d\tau}\Big/\frac{dt}{d\tau} = \dot{x}_0/\alpha = \text{constant, etc.}$$

All velocities are constant, which corresponds to motion in a straight line at constant speed. ■

⋆⋆ **Problem 10.3** The geodesic problem in the Schwarzschild geometry is to make stationary the functional $S = \int I\, d\tau$, where

$$I = \sqrt{(1 - 2\mathcal{M}/r)c^2\dot{t}^2 - (1 - 2\mathcal{M}/r)^{-1}\dot{r}^2 - r^2\dot{\varphi}^2}$$

where $\dot{t} = dt/d\tau$, etc. Use this integrand in the Euler-Lagrange equations to show that one obtains exactly the same differential equations of motion in the end if the square root is removed; *i.e.*, if we instead make stationary the functional $S' = \int I^2\, d\tau$. You may use the fact that $(1 - 2\mathcal{M}/r)c^2\dot{t}^2 - (1 - 2\mathcal{M}/r)^{-1}\dot{r}^2 - r^2d\dot{\varphi}^2 = c^2$, a constant along the particle path. The result is important because it is often much easier to use I^2 in the integrand rather than I.

Solution

Using I as integrand, we know that $\delta S = 0$ gives the equation

$$\frac{d}{d\tau}\frac{\partial I}{\partial \dot{q}_i} - \frac{\partial I}{\partial q_i} = 0,$$

where

$$q_i = (t, r, \varphi) \quad \text{and} \quad \dot{q}_i = (dt/d\tau, dr/d\tau, d\varphi/d\tau).$$

Using I^2 as integrand, we know that

$$\delta S' = 0 \quad \text{and} \quad \frac{d}{d\tau}\frac{\partial I^2}{\partial \dot{q}_i} - \frac{\partial I^2}{\partial q_i} = 0.$$

That is,

$$\frac{d}{d\tau}2I\frac{\partial I}{\partial \dot{q}_i} - 2I\frac{\partial I}{\partial q_i} = 0.$$

However, I is a constant along the particle path, so $\frac{dI}{d\tau} = 0$. Note that this does not mean that $\frac{\partial I}{\partial q_i}$ or $\frac{\partial I}{\partial \dot{q}_i}$ are zero. Therefore

$$\frac{d}{d\tau}I\frac{\partial I}{\partial \dot{q}_i} = I\frac{d}{d\tau}\frac{\partial I}{\partial \dot{q}_i}.$$

Also $I \neq 0$, so it follows that

$$2I(\frac{d}{d\tau}\frac{\partial I}{\partial \dot{q}_i} - \frac{\partial I}{\partial q_i}) = 0$$

implies that

$$\frac{d}{d\tau}\frac{\partial I}{\partial \dot{q}_i} - \frac{\partial I}{\partial q_i} = 0.$$

That is, if we begin with I^2 as the integrand, we end up with the same Euler-Lagrangian equation as if we had used I as the integrand. There is much less algebra if we use I^2. There is nothing special about the Schwarzschild metric here; we can do the same for any metric.

■

★★ **Problem 10.4** Show that there are no stable circular orbits of a particle in the Schwarzschild geometry with a radius less than $6GM/c^2$.

Solution

The effective potential energy (Eq.10.30) is

$$U_{\text{eff}} = -\frac{GMm}{r} + \frac{\ell^2}{2mr^2}(1 - \frac{2GM}{rc^2}).$$

Circular orbits are possible where

$$U'_{\text{eff}} = \frac{GMm}{r^2} - \frac{\ell^2}{2mr^3} + \frac{3\ell^2 GM}{mc^2 r^4} = 0.$$

Multiplying by r^4 gives a quadratic equation, with solutions

$$r_\pm = \frac{\ell^2}{2GMm^2}\left[1 \pm \sqrt{1 - 12(\frac{GMm}{\ell c})^2}\right]$$

as long as $12(\frac{GMm}{\ell c})^2 \leqslant 1$.

U_{eff} has a shape, if $12(\frac{GMm}{\ell c})^2 < 1$, featuring a maximum at some radius r_- and a minimum at some larger radius r_+, so there is a stable orbit at r_+.

If $12(\frac{GMm}{\ell c})^2 = 1$, there is no longer a local potential minimum, so there is no stable orbit in this case. Therefore r must be larger than

$$r = \frac{\ell^2}{2GMm^2} = \frac{12(\frac{GMm}{c})^2}{2GMm^2} = \frac{6GM}{c^2}$$

to have a stable circular orbit. ■

★★ **Problem 10.5** Show from the effective potential corresponding to the Schwarzschild metric that if U_{eff} can be used for arbitrarily small radii, there are actually *two* radii at which a particle can be in a circular orbit. The outer radius corresponds to the usual stable, circular orbit such as a planet would have around the Sun. Find the radius of the inner circular orbit, and show that it is unstable, so that if the orbiting particle deviates slightly outward from this radius it will keep moving outward, and if it deviates slightly inward it will keep moving inward.

Solution

The effective potential energy (Eq. 10.30) is

$$U_{\text{eff}} = -\frac{GMm}{r} + \frac{\ell^2}{2mr^2}(1 - \frac{2GM}{rc^2}).$$

Circular orbits are possible where

$$U'_{\text{eff}} = \frac{GMm}{r^2} - \frac{\ell^2}{2mr^3} + \frac{3\ell^2 GM}{mc^2 r^4} = 0.$$

Multiplying by r^4 gives a quadratic equation, with solutions

$$r_\pm = \frac{\ell^2}{2GMm^2}\left[1 \pm \sqrt{1 - 12(\frac{GMm}{\ell c})^2}\right]$$

as long as $12(\frac{GMm}{\ell c})^2 \leqslant 1$.

The effective potential U_{eff} has a shape, if $12(\frac{GMm}{\ell c})^2 < 1$, which has a maximum at some radius r_- and a minimum at some larger radius r_+. Thus there is a stable orbit at r_+, located at this minimum in U_{eff}. The effective potential is a maximum at r_-, so if a particle in circular orbit at r_- were to deviate slightly in radius it will spiral outward if $r > r_+$, and spiral inward if $r < r_-$. This behavior is the same as in classical mechanics when a particle moves near a maximum in its potential energy. This can be shown analytically by letting $r = r_0(1 + \epsilon)$, where ϵ is small. Then the differential equations of motion that are linear in ϵ show that ϵ oscillates around $\epsilon = 0$ near a potential minimum, and grows exponentially near a potential maximum. ■

⋆⋆ **Problem 10.6** Kepler's second law for classical orbits states that planets sweep out equal areas in equal times. Is that still true in Schwarzschild spacetime, assuming orbital radii $r > 2GM/c^2$? (a) First suppose that "time" here means the coordinate time t in Schwarzschild coordinates. (b) Then suppose instead that "time" means the proper time τ of the planets themselves.

Solution

For the *classical* motion of a planet orbiting in a central force, the area dA of a narrow (triangular) slice of the orbit is $dA = (1/2)(\text{base} \times \text{height}) = (1/2)r \cdot rd\theta$, so that the rate at which area is swept out is $dA/dt = (1/2)r^2\dot{\theta}$, constant for any central force because angular momentum per unit mass ℓ of the planet is conserved in that case. This constancy of the "areal velocity" is equivalent to Kepler's statement that planets sweep out equal areas in equal times. For the general-relativistic motion of a planet in the Schwarzschild metric we still have angular momentum conservation, expressed as $\ell = r^2 d\theta/d\tau$ = constant, where now τ must be read on the planet's own clock, that is, the proper time. Therefore it is tempting to say that Kepler's equal area in equal time law is still valid in general relativity as long as the clocks measuring time are those carried on the planet, and not (say) those measuring coordinate time in the Schwarzschild metric. However, even that would not be true, because although the area of a thin slice of pie in the Euclidean space of classical physics, (extending from $r = 0$ out to the orbit) is $(1/2)r \cdot rd\theta$, that is not so in the curved spacetime of Schwarzschild. For example, the "height" of a thin slice of pie classical orbit is r, but in Schwarzschild spacetime a radial coordinate distance interval Δr is not the radial *physical* distance, which would be $\int_r^{r+\Delta r} dr(1 - 2\mathcal{M}/r)^{-1/2}$. If the central object is a sun of finite radius, the metric inside the sun would not be that of Schwarzschild, and if the central object is a black hole we could still use the Schwarzschild metric, but the coordinate r would no longer even be spacelike, so our picture of a thin slice of pie extending from $r = 0$ out to the orbit would make no sense. So in summary, in classical mechanics conservation of angular momentum is equivalent to "equal areas in equal time," but in the Schwarzschild metric angular momentum is still conserved, but the area swept out by an orbiting planet is quite different, and is not even defined if the Schwarzschild metric is valid all the way into $r = 0$. So the answers to parts (a) and (b) are both "*no*." ■

⋆⋆ **Problem 10.7** Earth's orbit has a semimajor axis $a = 1.496 \times 10^8$ km and eccentricity $\epsilon = 0.017$. Find the general relativistic precession of the earth's perihelion in seconds of arc per century.

Solution

The precession per revolution is

$$\delta = 6\pi GM/a(1 - \epsilon^2)c^2,$$

where M is the sun's mass, a is the semimajor axis of the planet's orbit, and ϵ is the eccentricity. For Mercury,

$$a_M = 5.8 \times 10^{10} \text{ m} \quad \text{and} \quad \epsilon = 0.2056$$

and for the earth

$$a_E = 14.96 \times 10^{11} \text{ m} \quad \text{and} \quad \epsilon = 0.017.$$

so

$$\delta E/\delta M = \frac{a_M(1-\epsilon_M^2)}{a_E(1-\epsilon_E^2)} = \frac{5.8}{14.96}\frac{(1-.2056^2)}{(1-.017^2)} = 0.37$$

The precession per century ratios

$$\frac{\text{precession}}{\text{century}} = \frac{\text{precession}}{\text{orbit}}(\text{ratio of orbits/century}) = 0.37(\frac{88 \text{ days}}{365 \text{ days}}) = 0.089$$

So earth's precession/century should be

$$0.089 \times 43'' \text{ arc/century} = 3.8''/\text{century}$$

due to general relativity, which agrees well with observation. ■

★★ **Problem 10.8** Sometimes more than one coordinate system can usefully describe the same spacetime geometry. This is true in particular for the Schwarzschild geometry surrounding a spherically symmetric mass M. The usual Schwarzschild metric is

$$ds^2 = -(1-2\mathcal{M}/r)c^2dt^2 + (1-2\mathcal{M}/r)^{-1}dr^2 + r^2 d\Omega^2$$

where $\mathcal{M} \equiv GM/c^2$ and $d\Omega^2 \equiv (d\theta^2 + \sin^2\theta d\varphi^2)$. The so-called "isotropic" metric, describing exactly the same spacetime, has the form

$$ds^2 = -(1-2\mathcal{M}/r)c^2dt^2 + e^{2u}(d\bar{r}^2 + \bar{r}^2 d\Omega^2),$$

with the same dt^2 term, while the other terms contain a new radial coordinate $\bar{r}$ instead of r, and where $u = u(\bar{r})$. (a) Find $\bar{r}$ in terms of r and $\mathcal{M}$, choosing a constant of integration so that $\bar{r} \to r$ as $r \to \infty$. (b) What is an advantage of using the isotropic metric?

Solution

(a) There are two requirements:

$$(1) \quad (1-\frac{2\mathcal{M}}{r})^{-1}dr^2 = e^{2u}d\bar{r}^2 \Rightarrow \frac{dr}{\sqrt{1-2\mathcal{M}/r}} = e^u d\bar{r}$$

$$(2) \quad r^2 = e^{2u}\bar{r}^2 \Rightarrow r = e^u\bar{r}.$$

Eliminate e^u between (1) and (2):

$$\int \frac{dr/r}{\sqrt{1-2\mathcal{M}/r}} = \frac{d\bar{r}}{\bar{r}}.$$

Integrate:

$$\int^{\bar{r}} \frac{d\bar{r}}{\bar{r}} = \int^r \frac{dr}{\sqrt{r^2-2\mathcal{M}r}}$$

so

$$\ln\bar{r} = \ln\left[\sqrt{r^2-2\mathcal{M}r} + (r-\mathcal{M})\right] + \ln C \equiv \ln C\left[\sqrt{r^2-2\mathcal{M}r} + r - \mathcal{M}\right]$$

where C is a constant of integration. So

$$\bar{r} = C\left[\sqrt{r^2 - 2\mathcal{M}r} + r - \mathcal{M}\right].$$

We want $\bar{r} \to r$ as $r \to \infty$, so $C = 1/2$. So finally

$$\bar{r} = \frac{1}{2}\left[\sqrt{r^2 - 2\mathcal{M}r} + (r - \mathcal{M})\right]$$

(b) Isotropic coordinates here are as close as one can get to local Euclidean coordinates

$$dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

An advantage is that in isotropic coordinates the ratio of physical distances in tangential and radial directions is

$$\ell_\theta/\ell_{\bar{r}} = e^u\bar{r}\Delta\theta/e^u\Delta\bar{r} = \bar{r}\Delta\theta/\Delta\bar{r},$$

just as in Euclidean geometry. In regular Schwarzschild coordinates, however,

$$\ell_\theta/\ell_r = r\sqrt{1 - \frac{2\mathcal{M}}{r}}\Delta\theta/\Delta r.$$

■

⋆ **Problem 10.9** The geometry on the surface of a sphere is noneuclidean, so the circumference C and radius R of a circle drawn on the sphere do not obey $C = 2\pi R$, where for example the circumference is a constant-latitude path and the radius is drawn on the sphere down from the north pole along a constant-longitude path. Suppose we measure latitude by the angle, measured from the center of the sphere, between the north pole and constant-latitude path. (a) If the angle is 90°, what is the coefficient α in $C = \alpha R$? (b) If in effect *pi* were 3.00000 instead of 3.14156. What would be the angle in that case? (c) The feature that $C < 2\pi R$ is a property of a positively curved surface. In Euclidean geometry, given a line and a point exterior to the line, there is one and only one line through the given point that is parallel to the given line, parallel meaning that the two lines never meet. What is the analog statement for a positively curved surface?

Solution

(a) If $\theta = 90°$, the "radius" of the circle is one-quarter of a circle of constant longitude, i.e. $\frac{2\pi R}{4} = \pi R/2$. The circumference is the distance around the equator $2\pi R$, so

$$2\pi R = \alpha(\pi R/2) \Rightarrow \alpha = 4$$

(instead of 2π, as in a flat circle).

(b) π has cancelled out here, so $\alpha = 4$ still.

(c) The analog is that there are no lines parallel to the given line. Great circles will cross in two places. That is, if a point is identified outside a great circle, a geodesic through that point will intersect the initial great circle twice. ■

⋆⋆ **Problem 10.10** Before the age of relativity, some people calculated that light would be deflected by the sun in a classical model in which light consists of particles of tiny mass m moving at speed c, pulled by the sun's Newtonian gravity. Find in that case the approximate deflection of a light beam in terms of any or all of m, c, G, M, the sun's mass, and R, the distance of closest approach of the light beam from the sun's center. Compare your result with the actual deflection of light in the Schwarzschild spacetime as derived in the chapter.

Solution

We suppose that a small mass m can travel at speed c toward the right, and that a gravitational force GMm/r^2 acts upon it due to the sun. The component of the sun's force perpendicular to the direction of motion of m is

$$\frac{GMm}{r^2}\cos\theta = \frac{GMm}{R^2}\cos^3\theta$$

since $\cos\theta = R/r$, where R is the distance of closest approach. The resulting acceleration of m in the perpendicular direction is $a_\perp = \frac{dv_t}{dt}$. But the infinitesimal deflection of m is

$$d\delta = dv_\perp, \text{ so } a_\perp = \frac{dv_\perp}{dt} = c\frac{d\delta}{dt}.$$

Also

$$\tan\theta = x/R, \text{ so } \sec^2\theta d\theta/dt = \frac{dx/dt}{R} = \frac{c}{R},$$

so $F = ma$ gives

$$\frac{GMm}{R^2}\cos^3\theta = mc\frac{d\delta}{dt} = mc(\frac{d\theta}{dt})\frac{d\delta}{d\theta} = \frac{mc^2}{R}\cos^2\theta\frac{d\delta}{d\theta}.$$

Therefore

$$d\delta/d\theta = \frac{GM}{Rc^2}\cos\theta.$$

The net deflection will be

$$\delta = \int d\delta = \frac{GM}{Rc^2}\int_{-\pi/2}^{\pi/2} d\theta\cos\theta = \frac{GM}{Rc^2}\sin\theta\Big|_{\pi/2}^{\pi/2} = \frac{2GM}{Rc^2}$$

which is half the general relativistic result. ■

★★ **Problem 10.11** (a) Find the escape velocity $dr/d\tau$ of a particle of mass m starting from rest at radius $r_0 = 4GM/c^2$ in a Schwarzschild spacetime of mass M, where τ is read on the particle's own clock. (b) Then find the escape velocity $dr/d\tau$ of the particle, starting at rest from the same point, where now τ is read on a clock that remains at rest at r_0.

Solution

(a) From Eq. 12.28, if a particle moves radially only, so the angular momentum $\ell = 0$, then

$$\frac{1}{2}m(\frac{dr}{d\tau_p})^2 - \frac{GMm}{r} = E,$$

where $d\tau_p$ is the infinitesimal time on the particle's clock. The escape energy is $E = 0$, so the corresponding escape velocity is

$$dr/d\tau_p = \sqrt{\frac{2GM}{r_0}} = \sqrt{\frac{2GM}{4GM/c^2}} = c/\sqrt{2},$$

where τ_p is the particle's proper time.

(b) From Eq. 10.25 we have

$$\dot{t} = dt/d\tau_p = \epsilon(1 - \frac{2M}{r})^{-1}$$

where

$$\frac{Mc^2}{2}(\epsilon^2 - 1) = E = 0 \Rightarrow \epsilon = 1$$

and so

$$dt/d\tau_p = (1 - \frac{2M}{r_0})^{-1} = (1 - \frac{2GM}{c^2(4GM/c^2))})^{-1} = 2$$

where t is the coordinate time and τ_p is the proper time of the particle which has the escape velocity. From Eq. 10.9, clocks that remain at rest at r_0 read

$$\Delta\tau_r = \sqrt{1 - \frac{2M}{r_0}}\Delta t = \sqrt{1 - \frac{2GM/c^2}{4GM/c^2}}\Delta t = \Delta t/\sqrt{2}$$

so

$$\frac{d\tau_{rest}}{dt} = \frac{1}{\sqrt{2}}.$$

Therefore

$$dr/d\tau_{rest} = (\frac{dr}{dt})\frac{dt}{d\tau_{rest}}$$

$$\frac{dr}{d\tau_{rest}} = \frac{dr}{dt}\sqrt{2} = \frac{dr}{d\tau_p}\frac{d\tau_p}{dt}\sqrt{2} = \frac{c}{\sqrt{2}}\frac{1}{2}\sqrt{2} = c/2$$

■

★★ **Problem 10.12** Tachyons are hypothetical particles (never observed, at least so far) that always travel faster than light. Therefore in general relativistic spacetimes they would follow spacelike (rather than timelike or null) geodesics. Prove that the deflection of such a particle in passing by the sun would be less than that for light.

Solution

The Lagrangian for the motion of a particle in the equatorial plane can be taken as

$$L = (1 - \frac{2M}{r})c^2\dot{t}^2 - (1 - \frac{2M}{r})^{-1}\dot{r}^2 - r^2\dot{\varphi}^2,$$

where

$$\dot{t} = dt/d\lambda, \quad \dot{r} = dr/d\lambda, \quad \dot{\varphi} = d\varphi/d\lambda,$$

where λ is a parameter along the path (called an "affine parameter"). The resulting Lagrange equations for t and φ are

$$\dot{t} = (1 - \frac{2M}{r})^{-1}\epsilon, \quad \dot{\varphi} = \ell/r^2$$

where ϵ, ℓ are constants. Therefore

$$L = (1 - \frac{2M}{r})^{-1} c^2 \epsilon^2 - (1 - \frac{2M}{r})^{-1} \dot{r}^2 - \ell^2/r^2 = \begin{cases} c^2 & \text{timelike paths} \\ 0 & \text{null paths} \\ -c^2 & \text{spacelike paths} \end{cases}$$

Rewriting,

$$\dot{r}^2 - c^2\epsilon^2 + \frac{\ell^2}{r^2}(1 - \frac{2M}{r}) + (1 - \frac{2M}{r}) \begin{cases} c^2 \\ 0 \\ -c^2 \end{cases} = 0$$

Now let $u = 1/r$ (as in Example 10.6), so

$$\dot{r} \equiv dr/d\lambda = -\frac{1}{u^2}\frac{du}{d\lambda} = -\frac{1}{u^2}\frac{du}{d\varphi}\frac{d\varphi}{d\lambda} = -\frac{1}{u^2}u'\frac{\ell}{r^2} = -\ell u',$$

where $u' = du/d\varphi$. Therefore

$$\ell^2 u'^2 - c^2\epsilon^2 + \ell^2 u^2(1 - 2Mu) + (1 - 2Mu) \begin{cases} c^2 \\ 0 \\ -c^2 \end{cases} = 0$$

Differentiate this last equation with respect to φ:

$$2\ell^2 u' u'' + 2\ell^2 u u' - 2M\ell^2(3u^2 u') - 2Mu' \begin{cases} c^2 \\ 0 \\ -c^2 \end{cases} = 0$$

Cancel out $2\ell^2 u'$:

$$u'' + u - 3Mu^2 - \frac{M}{\ell^2} \begin{cases} c^2 \\ 0 \\ -c^2 \end{cases} = 0$$

$$u'' + u = 3Mu^2 + \frac{Mc^2}{\ell^2} \begin{cases} 1 & \text{timelike} \\ 0 & \text{null} \\ -1 & \text{spacelike} \end{cases}$$

Now using a perturbation expansion (as in Example 10.6) let $u = u_0 + \epsilon u_1$ where u_0 is the straight-line solution valid for $M = 0$. ($u_0'' + u_0 = 0$). So

$$u_0'' + u_0 + \epsilon(u_1'' + u_1) = 3M(u_0 + \epsilon u_1)^2 + \frac{Mc^2}{\ell^2} \begin{cases} 1 \\ 0 \\ -1 \end{cases} \cong 3Mu_0^2 + \frac{Mc^2}{\ell^2} \begin{cases} 1 \\ 0 \\ -1 \end{cases}$$

dropping the ϵu_1 term, which is extremely small. Here $u_0 = A\cos\varphi$, a straight line with closest approach at $\varphi = 0$. So we have

$$\epsilon(u_1'' + u_1) \cong 3MA^2\cos^2\varphi + \frac{Mc^2}{\ell^2}\begin{cases}1\\0\\-1\end{cases}.$$

As in Example 10.6, let $\epsilon = \frac{3M}{r_0}$ so

$$\epsilon(u_1'' + u_1) \simeq \epsilon r_0 A^2\cos^2\varphi + \frac{\epsilon r_0 c^2}{3\ell^2}\begin{cases}1\\0\\-1\end{cases}$$

so

$$u_1'' + u_1 = r_0 A^2\cos^2\varphi + \frac{r_0 c^2}{3\ell^2}\begin{cases}1 & \text{timelike}\\0 & \text{null}\\-1 & \text{spacelike}\end{cases}$$

Now without solving this equation (although that is easy) we can see what happens. For the null case we know the path bends around the central mass. For timelike path, like those of massive particles, the right-hand side is increased, so the bending (i.e. the departure from a straight-line path) must be larger, as we know it is. But for tachyons, which follow spacelike paths, the right-hand side is smaller, indicating that the bending will be smaller. ■

⋆ **Problem 10.13** Consider two concentric coplanar circles in the Schwarzschild metric surrounding the sun, with measured circumferences C_1 and C_2. In terms of C_1 and C_2, find an expression for (a) the radial coordinate distance Δr between them. (b) the radial measured distance between them.

Solution

The spatial (r, ϕ) two-surface around the equator of the Schwarzschild geometry is

$$ds^2 = (1 - \frac{2M}{r})^{-1}dr^2 + r^2 d\phi^2 \quad (\theta = \pi/2)$$

A measured distance around the circumference is

$$\int ds_\phi = r\int d\phi = 2\pi r$$

so $s_\phi = 2\pi r$, just as in a Euclidean space. So $C_1 = 2\pi r_1$ and $C_2 = 2\pi r_2$.

(a) The radial coordinate distance between them is $\Delta r = r_2 - r_1$.

(b) The radial measured distance is

$$\int_{r_1}^{r_2} ds_r = \int_{r_1}^{r_2}(1 - \frac{2M}{r})^{-1/2}dr$$

Fig. 10.1 This picture was taken by the Hubble telescope over the course of ten consecutive days. The image contains about 10,000 galaxies located at a distance of about 13 billion light-years. All the galaxies in the picture are moving away from us because of the expansion of the universe. Credits: R. Williams (STScI), the Hubble Deep Field Team and NASA Horizon Telescope Collaboration.

so

$$\Delta s_r = \int_{r_1}^{r_2} \frac{dr}{\sqrt{1-\frac{2M}{r}}} = \int_{r_1}^{r_2} \frac{dr\, r}{\sqrt{r^2-2Mr}} = \left[\sqrt{r^2-2Mr} + M\int \frac{dr}{\sqrt{r^2-2Mr}}\right]_{r_1}^{r_2}$$

$$= \left[\sqrt{r^2-2Mr} + M\ln|2\sqrt{r^2-2Mr}+2r-2M|\right]_{r_1}^{r_2}$$

$$\Delta s_r = \sqrt{r^2-2Mr}\Big|_{r_1}^{r_2} + M\ln\left[2\sqrt{r_2^2-2Mr_2}+2(r_2-M)\right] - M\ln\left[2\sqrt{r_1^2-2Mr_1}+2(r_1-M)\right]$$

$$= \sqrt{r_2^2-2Mr_2} - \sqrt{r_1^2-2Mr_1} + M\ln\left|\frac{\sqrt{r_2^2-2Mr_2}+(r_2-M)}{\sqrt{r_1^2-2Mr_1}+(r_1-M)}\right|$$

Valid for $r_2 > r_1 > 2M$. Note that $\Delta s_r \to r_2 - r_1$ as $M \to 0$. ■

⋆⋆⋆ **Problem 10.14** The Robertson-Walker metrics

$$ds^2 = -c^2dt^2 + a(t)^2\left[\frac{dr^2}{1-k(r/R)^2} + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2\right]$$

are applicable to universes that are both spatially homogenous and isotropic: That is, they have no preferred positions or directions. The spacetimes also feature a universal time t and a constant R with dimensions of length. There are three possible choices for the constant k: $k = 1, 0$, or -1, which correspond to three-dimensional spatial geometries that have constant positive curvature ($k = +1$), constant negative curvature ($k = -1$), or that are flat ($k = 0$.) Here $a(t)$ is called the "scale factor" of the universe; If $a(t)$ grows with time distant galaxies become farther apart, or if $a(t)$ shrinks distant galaxies come closer together. The function $a(t)$ can be found using Einstein's field equations of general relativity, given the kind of matter, radiation, or other quantities that live in the universe. The result is the "Friedmann" equations, in which the universe is filled with a material having uniform mass density ρ and uniform pressure p; we can also add "dark energy" as represented by the constant Λ. There are then two independent equations for $a(t)$:

$$(1)\quad \frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2} = \frac{8}{3}\pi G\rho + \frac{\Lambda}{3}c^2$$

and

$$(2)\quad \frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2} = \frac{8\pi Gp}{c^2} + \Lambda c^2$$

where overdots represent time derivatives and G is Newton's gravitational constant. (a) First suppose the universe is spatially flat, with $k = 0$, and that the cosmological constant Λ is also zero. Also suppose the energy density consists entirely of mass density ρ, which decreases as the universe expands so that $\rho a(t)^3 = \rho_0 a_0^3$ where ρ_0 and a_0 are the current mass density and scale factor. In that case solve the Friedmann equations to find $a(t)$ in terms of t, G, ρ_0, and a_0. It is thought that this is a good approximation to the situation for our universe in most of its history so far. It is called the "matter dominated" period. (b) Repeat part (a) except suppose the energy density consists entirely of photons in thermal equilibrium, in which case the energy density obeys $\rho a(t)^4 = \rho_0 a_0^4$. This situation is thought to be a good approximation for our universe for a hundred thousand years or so early on, and is called the "radiation dominated" period. (c) Finally, repeat part (a) for the case $k = 0, \rho = 0, p = 0$, but $\Lambda =$ constant > 0. This may be a good approximation to our universe for a brief time after the big bang began; it is called the "inflationary" period for reasons that will be apparent from the solution. It may also be a good approximation for our universe in the distant future. (d) At the current time the universe seems to be behaving as though it were driven by both dust-like matter and the cosmological constant Λ. Sketch a graph of $a(t)$ vs t extending from times long ago to times in the distant future, showing what happens as the universe gradually transitions from one form of dominance to the other.

Solution

(a) With $k = 0, \Lambda = 0$, Eq. (1) becomes

$$\dot{a}^2/a^2 = \frac{8\pi G\rho}{3} = \frac{8}{3}\pi G\rho_0 a_0^3/a^3$$

so

$$\dot{a} \equiv \frac{da}{dt} = \sqrt{\frac{8}{3}\pi G\rho_0 a_0^3}\ /\ \sqrt{a}.$$

Rewriting,

$$\sqrt{a}da = \sqrt{\frac{8}{3}\pi G\rho_0 a_0^3}dt \Rightarrow \frac{2}{3}a^{3/2} = (\frac{8}{3}\pi G\rho_0 a_0^3)^{1/2}\ t$$

$$a(t) = (3/2)^{2/3}(\frac{8}{3}\pi G\rho_0 a_0^3)^{1/3} t^{2/3} = a_0(6\pi G\rho_0)^{1/3}\, t^{2/3}$$

(b) With

$$\rho a^4 = \rho_0 a_0^4,$$

Equation (1) becomes

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho_0 a_0^4}{3a^4} \Rightarrow \dot{a} = \sqrt{\frac{8}{3}\pi G\rho_0 a_0^4}/a$$

so

$$\int ada = \frac{a^2}{2} = \sqrt{\frac{8}{3}\pi G\rho_0 a_0^4}\, t$$

$$a(t) = (\frac{32}{3}\pi G\rho_0 a_0^4)^{1/4} t^{1/2} = a_0(\frac{32}{3}\pi G\rho_0)^{1/4} t^{1/2}$$

(c) With $k = 0, \rho = 0, p = 0, \Lambda > 0$, Eq. (1) becomes

$$\dot{a}^2/a^2 = \Lambda c^2/3 \Rightarrow \dot{a} = \sqrt{\frac{\Lambda}{3}} ca \Rightarrow a = a_0 e^{\sqrt{\Lambda/3}ct}.$$

This satisfies Eq. (2) as well. ■

★★ **Problem 10.15** Inspired from equations 10.77,write gravitational field vectors describing a gravitational wave of angular frequency ω propagating in vacuum in the positive z direction, specifying both the 'electric' and 'magnetic' field vectors. Assume the 'electric' gravitational field amplitude is given by g_0.

Solution

The structure of the equations are Maxwell-like, so we expect wave solutions.

$$\begin{cases} \nabla \cdot \mathbf{g} = 0 \\ \nabla \cdot \mathbf{b} = 0 \end{cases} \Rightarrow \text{Transverse waves}$$

$$\begin{cases} \nabla \times \mathbf{g} = -\frac{1}{c}\frac{\partial \mathbf{b}}{\partial t} \\ \nabla \times \mathbf{b} = \frac{1}{c}\frac{\partial \mathbf{g}}{\partial t} \end{cases}$$

$\Rightarrow$ the wave speed is c. Here $\mathbf{g}$ and $\mathbf{b}$ are orthogonal as given by the right hand rule. Also

$$k|\mathbf{g}| = \frac{1}{c}\omega|\mathbf{b}| \Rightarrow |\mathbf{g}| = |\mathbf{b}|$$

Therefore we have

$$\mathbf{g} = \mathbf{g}_0 \cos(kz - \omega t) \text{ with } \frac{\omega}{k} = c$$

$$\mathbf{b} = \hat{\mathbf{z}} \times \mathbf{g}_0 \cos(kz - \omega t)$$

■

Fig. 10.2 The supermassive black hole at the center of the M87 galaxy. This is the first image of a black hole every captured. The black hole is 6.5 million times more massive than our sun and is 53 million light-years away. Image credit: Event Horizon Telescope Collaboration.

⋆⋆ **Problem 10.16** Show that there is exactly one radius at which a light beam can move in a circular orbit around a spherical black hole, and find this radius. Then show that the orbit is unstable, by showing that a tangential beam beginning at a slightly larger radius will spiral outward and never return, and a tangential beam beginning at a slightly smaller radius will spiral into the black hole.

Solution

In Example 10 − 6 we found that a light ray obeys $u'' + u = 3\mathcal{M}u^2$ around a black hole, where

$$\mathcal{M} = \frac{GM}{c^2}, \quad u = 1/r, \text{ and } u'' = d^2u/d\varphi^2.$$

For a circular orbit u =constant, so $u = 1/3\mathcal{M}$ or $r = 3\mathcal{M} = 3GM/c^2$, the only radius at which light can travel in a circle. Now let $a = \frac{1}{3\mathcal{M}}(1+\delta)$ where $|\delta| << 1$. Substituting into the differential equation,

$$\frac{1}{3\mathcal{M}}\delta'' + \frac{1}{3\mathcal{M}}(1+\delta) = \frac{3\mathcal{M}}{(3\mathcal{M})^2}(1+2\delta)$$

for terms up to first order in δ. This reduces to $\delta'' - \delta = 0$, with solutions

$$\delta = ae^{\varphi} + be^{-\varphi}.$$

If the ray starts off tangentially, we have

$$\delta' = ae^{\varphi} - be^{-\varphi} = 0$$

at $\varphi = 0$, so $a = b$, and

$$\delta = a(e^{\varphi} + e^{-\varphi}) = 2a\cosh\varphi.$$

Now if initially $\delta > 0$ (so $r_0 < 3\mathcal{M}$) then δ becomes larger (r becomes smaller) as φ increases or decreases. The ray spirals inward. If initially $\delta < 0$ (so $r_0 > 3M$) a is negative and δ becomes more positive, and r increases, meaning the ray spirals outward. The orbit is unstable. ■

★★ **Problem 10.17** Write the Lagrangian of a charged particle in terms of potentials in the case where we use the static gauge condition, and show that it appears to be different than the Lagrangian in the absence of any gauge fixing. Then show that even though the Lagrangian is different, the Lagrange equations of motion are the same.

Solution

We have

$$L = \frac{1}{2}mv^2 - q\phi + \frac{q}{c}\mathbf{A}\cdot\mathbf{v}$$

$$\Rightarrow \frac{\partial L}{\partial \dot{x}} = m\dot{x} + q\frac{A_x}{c} \quad \frac{\partial L}{\partial x} = \frac{q}{c}\frac{\partial \mathbf{A}}{\partial x}\cdot\mathbf{v}$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) = m\ddot{x} + \frac{q}{c}(\partial_x A_x\dot{x} + \partial_y A_x\dot{y} + \partial_z A_z\dot{z} + \frac{\partial A_x}{\partial t})$$

$$= \frac{\partial L}{\partial x} = \frac{q}{c}(\partial_x A_x\dot{x} + \partial_x A_y\dot{y} + \partial_x A_z\dot{z})$$

where we used the chain rule and we write $\partial_x = \frac{\partial}{\partial x}$, etc.

$$\Rightarrow m\ddot{x} = \frac{q}{c}(\partial_x A_y\dot{y} - \partial_y A_x\dot{y} + \partial_x A_z\dot{z} - \partial_z A_x\dot{z} - \frac{\partial A_x}{\partial t})$$

We have

$$E_x = -\frac{1}{c}\frac{\partial A_x}{\partial t} \text{ and } \mathbf{B} = \nabla\times\mathbf{A}$$

$$\Rightarrow m\ddot{x} = -\frac{q}{c}\frac{\partial A_x}{\partial t} + \frac{q}{c}\left[\dot{y}(\partial_x A_y - \partial_y A_x) + \dot{z}(\partial_z A_z - \partial_z A_x)\right]$$

$$= qE^x + \frac{q}{c}(\dot{y}\mathrm{B}^z - \dot{z}\mathrm{B}^y) = qE^x + \frac{q}{c}(\mathbf{v}\times\mathbf{B})^x$$

Similarly for $\ddot{y}$ and $\ddot{z}$. ■

★★ **Problem 10.18** Show that for a Gaussian probability distribution

$$p(x) = \frac{e^{-\frac{(x-x_0)^2}{2a^2}}}{\sqrt{2\pi a^2}},$$

all the moments are given by

$$\langle (x - x_0)^n \rangle = 1 \times 3 \times 5 \times (n-1) \times a^n$$

for even n, and are zero otherwise. Hence the Gaussian distribution is entirely characterized by its mean x_0 and deviation a.

Solution

Given that

$$p(x) = \frac{e^{-\frac{(x-x_0)^2}{2\,a^2}}}{\sqrt{2\,\pi\,a^2}},$$

the moments are

$$\langle (x - x_0)^n \rangle = \int_{-\infty}^{\infty} p(x)(x - x_0)^n dx$$

For n odd, the integral is zero by symmetry. For n even

$$\langle (x - x_0)^n \rangle = \lim_{t \to 1} (-1a^2 \frac{\partial}{\partial t})^{n/2} \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-x_0)^2}{2\,a^2}}}{\sqrt{2\,\pi\,a^2}}$$

$$\begin{aligned}
&= 2a^2(\frac{1}{2}) = a^2 && n = 2\\
&= 2a^2(\frac{3}{2}) = 3a^2 && n = 4\\
&= 2a^2(\frac{5}{2}) = 5a^2 && n = 6\\
&\vdots\\
&= (n-1)a^2 && n
\end{aligned}$$

■

★★ **Problem 10.19** (a) Show that for any probability distribution, if we compute the generating function $Z(\beta) \equiv \langle e^{\beta X} \rangle$ for arbitrary β and X being the stochastic variable, we can compute all moments using

$$\langle X^n \rangle = \lim_{\beta \to 0} \left(\frac{d}{d\beta} \right)^n Z(\beta) \,.$$

(b) Use the generating function to compute the moments of a Gaussian stochastic variable
(c) Use the generating function to compute the moments of a stochastic variable with uniform probability distribution: $p(x) = 1/(2a)$ for $(x_0 - a) \leq x \leq (x_0 + a)$, and $p(x) = 0$ otherwise.

Solution

We have

$$z(\beta) = \langle e^{\beta x} \rangle$$

$$\langle X^n \rangle = \int x^n p(x) dx$$

$$Z(\beta) = \int e^{\beta x} p(x) dx$$

$$(\frac{\partial}{\partial \beta})^n Z(\beta) = \int x^n e^{\beta x} p(x) dx$$

$$\lim_{\beta \to 0} \quad \to \int x^n p(x) dx = \langle X^n \rangle$$

■

★★★ **Problem 10.20** For the stochastic equation studied in the text, show that

$$\overline{X(t)^2} = \frac{\sigma^2}{\alpha^2} \left(t + \frac{1}{2\,\alpha} e^{-2\alpha t} \right) \to \frac{\sigma^2}{\alpha^2}\, t$$

Solution

$$\frac{d\overline{X}}{dt} = \overline{V} = 0 \text{ since } \overline{C} = 0$$

$$\frac{d}{dt}\overline{X^2} = 2\overline{XV}$$

To find $\overline{X(t)V(t)}$ we write

$$\overline{X(t)V(t)} = \overline{X(0)V(t)} + \int_0^t \overline{V(s)V(t)} ds$$

$$= \overline{X(0)C}e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)}\overline{X(0)f(s)} ds + \frac{\sigma^2}{2\alpha} \int_0^t e^{-\alpha(t+s)}(e^{2\alpha s} - 1) ds$$

$$= \frac{\sigma^2}{2\alpha} \int_0^t \left[e^{-\alpha(t-s)} - e^{-\alpha(t+s)} \right] ds$$

$$= \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})^2$$

$$\Rightarrow \frac{d}{dt}\overline{X^2} = \frac{\sigma^2}{\alpha}(1 - e^{-2\alpha t})^2$$

$$\Rightarrow \overline{X^2} = \frac{\sigma^2}{\alpha^2}(t + \frac{2}{\alpha}e^{-\alpha t} - \frac{1}{2\alpha}e^{-2\alpha t}) \xrightarrow{t \to \infty} \frac{\sigma^2}{\alpha}t$$

■

★★ **Problem 10.21** Using the generating function Z introduced in an earlier problem, show that: (a) If X is a stochastic variable with a gaussian distribution with mean x_0 and variance σ^2, then $a + bX$ is a stochastic variable with a Gaussian distribution with mean $a + b x_0$ and variance $b^2\sigma^2$. (b) If X and Y are Gaussian stochastic variables with means x_0 and y_0

respectively, and variances σ_X^2 and σ_Y^2, then $X+Y$ is a Gaussian stochastic variable with mean x_0+y_0 and variance $\sigma_X^2+\sigma_Y^2$.

Solution

(a) Start with

$$Z(\beta)=\left\langle e^{\beta X}\right\rangle=\int_{-\infty}^{\infty} e^{\beta x}\frac{e^{-\frac{(x-x_0)^3}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dx=e^{\beta x_0+\beta^2\frac{\sigma^2}{2}}$$

Now consider the generating function for $a+bX$

$$Z_{a+bX}(\beta)=\left\langle e^{\beta(a+bX)}\right\rangle=e^{\beta a}\left\langle e^{\beta bX}\right\rangle=e^{\beta a}Z(\beta b)$$

$$=e^{\beta a}e^{\beta bx_0+\frac{\sigma^2}{2}\beta^2b^2}=e^{\beta(a+bx_0)+\beta^2\frac{\sigma^2b^2}{2}}$$

This is the generating function for a variable with gaussian profile with mean $a+bx_0$ and variance σ^2b^2.

(b)

$$Z_{X+Y}(\beta)=\left\langle e^{\beta(X+Y)}\right\rangle=\left\langle e^{\beta X}e^{\beta Y}\right\rangle=\left\langle e^{\beta X}\right\rangle\left\langle e^{\beta Y}\right\rangle$$

Since X and Y are not correlated

$$=Z_X(\beta)Z_Y(\beta)=e^{\beta x_0+\beta^2\frac{\sigma_X{}^2}{2}}e^{\beta y_0+\beta^2\frac{\sigma_Y{}^2}{2}}=e^{\beta(x_0+y_0)}e^{\beta^2\frac{\sigma X^2+\sigma Y^2}{2}}$$

This is a gaussian generating function with mean x_0+y_0 and variance $\sigma_X^2+\sigma_Y^2$. ■

★★★ **Problem 10.22** Show that if the initial condition C of the linear stochastic differential equation introduced in the text has a Gaussian distribution, so does the solution of the stochastic differential equation. HINT: You might want to compute generating functions as in previous problems.

Solution

The expression implies that the variable $X(t)$ at t is a sum of

$$\alpha X,\quad X+\alpha,\quad X+f$$

where α is nonrandom, and f is random white noise. We need to show that the generating functions for each of these are gaussian generating functions.

$$Z_{\alpha X}(\beta)=\left\langle e^{\beta\alpha X}\right\rangle=Z(\beta\alpha),$$

which is gaussian with mean βx_0 and variance $\beta^2\sigma^2$.

$$Z_{X+\alpha}(\beta)=\left\langle e^{\beta(X+\alpha)}\right\rangle=e^{\beta\alpha}\left\langle e^{\beta X}\right\rangle$$

$$Z_{X+f}=\left\langle e^{\beta(X+f)}\right\rangle=\left\langle e^{\beta X}e^{\beta f}\right\rangle=\left\langle e^{\beta X}\right\rangle\left\langle e^{\beta f}\right\rangle=\left\langle e^{\beta X}\right\rangle$$

Since $\bar{f}=0$.

gaussian with mean shifted by α. Since

$$Z(\beta) = e^{\beta x_0 + \beta^2 \frac{\sigma^2}{2}}$$

for a gaussian with mean x_0 and variance σ^2 (see previous problem).

This implies that $X(t)$ is a gaussian when we note also that the sum of gaussians is a gaussian (see problem 10.14(b)). ■

⋆⋆ **Problem 10.23** Show that the case of a particle executing a random walk as described by the statistical moments of its position computed in the text, the probability function $p(x,t)$ satisfies the so-called **diffusion equation**

$$\frac{\partial p}{\partial t} = D\,\frac{\partial^2 p}{\partial x^2}$$

where the constant D is called the diffusion coefficient.

Solution

The probability function is

$$p(x,t) = \frac{e^{-\frac{(x-x_0)^2}{2\,\sigma^2 t}\,\alpha^2}}{\sqrt{2\,\pi\,\frac{\sigma^2 t}{\alpha^2}}} \quad \text{since} \quad \bar{X}^2 = \frac{\sigma^2}{\alpha^2}t$$

$$\Rightarrow \frac{\partial p}{\partial t} = \frac{\alpha}{2\sqrt{2\pi}\sigma^3}\,\frac{(x-x_0)^2\alpha^2 - t\sigma^2}{t^{5/2}} e^{-\frac{(x-x_0)^2\alpha^2}{2\sigma^2 t}}$$

$$\frac{\partial p}{\partial x} = -(\frac{\alpha^2}{t\sigma^2})^{3/2}\left(\frac{x-x_0}{\sqrt{2\pi}}\right)e^{-\frac{(x-x_0)^2\alpha^2}{2\sigma^2 t}}$$

$$\frac{\partial^2 p}{\partial x^2} = \frac{\alpha^3}{\sqrt{2\pi}\sigma^5}\,\frac{(x-x_0)^2\alpha^2 - t\sigma^2}{t^{5/2}} e^{-\frac{(x-x_0)^2\alpha^2}{2\sigma^2 t}}$$

Therefore finally

$$\frac{\frac{\partial p}{\partial t}}{\frac{\partial^2 p}{\partial x^2}} \equiv D = \frac{\sigma^2}{2\alpha^2}$$

■

⋆ **Problem 10.24** From statistical mechanics, for each degree of freedom q of a free system in thermal equilibrium at temperature T, the corresponding thermal fluctuations of $\dot{q}$ is given by

$$\left\langle \frac{1}{2} m\,\dot{q}^2 \right\rangle = \frac{k\,T}{2}\,.$$

Here m is the mass associated in the kinetic energy expression written in terms of q. Mapping this setup on the random brownian motion dynamics elaborated in the text, find a relation between T, σ, and α; that is, a relation between temperature, random force strength, and friction. This is a form of the **fluctuation-dissipation theorem**.

Solution

$$\frac{1}{2}m\bar{V}^2 = \frac{1}{2}m\frac{\sigma^2}{2\alpha} = \frac{1}{2}kT \Rightarrow \frac{\sigma^2}{2\alpha} = \frac{kT}{m}$$

■

⋆ **Problem 10.25** A team of researchers has long tracked the path of a star named S2 that orbits the supermassive black hole Sagittarius A* at the center of our Milky Way galaxy. (The orbit is one of those shown on the cover of this book.) The orbital period of S2 is 16.05 years, its semimajor axis is 970 au, where 1 au is the average distance of the earth from the sun, and its orbital eccentricity is 0.88. The mass of the central black hole is approximately 4 million solar masses, where one solar mass is 2×10^{30} kg. The researchers also found that the highly elliptical orbit of S2 precesses by approximately 12 minutes of arc per revolution. Is this value consistent with the predictions of general relativity for a Schwarzschild spacetime?

Solution

We are given

$$a = 970 \text{ au}, \quad \epsilon = 0.88 \text{ and } M = 4 \times 10^6 M_\odot$$

General relativity then predicts a precession in units of radians/revolution

$$\Delta = \frac{6\pi GM}{c^2 a(1-\epsilon^2)}.$$

For the planet Mercury, the precession is 5.04×10^{-7} radians/revolution, as given in the chapter, where $a = 5.8 \times 10^{10}$ m $= 0.387$ au, since the mean radius of the earth's orbit is 1 au = 150×10^6 km. So now we can take the ratio

$$\frac{\Delta_{S2}}{\Delta_{Merc}} = \frac{M_{bh}}{M_\odot}\frac{a_{Merc}}{a_{S2}}\frac{1-\epsilon^2_{Merc}}{1-\epsilon^2_{S2}}$$

which turns out to equal 6.77×10^3. Therefore general relativity predicts

$$\Delta_{S2} = 6,770\Delta_{Merc} = 11.7 \text{ minutes of arc/revolution}$$

which is consistent with observations. ■

⋆⋆ **Problem 10.26** Suppose that the orbit of Star S2, as described in the preceding problem, lies in a plane that is perpendicular to our line of sight. Then at periastron, when S2 is a distance 120 au from the central black hole, there will be both a transverse Doppler effect and a gravitational redshift for light from S2 to earth. (a) Find the sum of these frequency shifts. (The result has been confirmed by observations. The transverse Doppler effect is described in Problem 2.51.) (b) Calculate the Schwarzschild radius (the Event Horizon radius) of the central Sgr A* black hole, and compare it with the periastron distance of S2. (The actual Event Horizon radius differs somewhat from the Schwarzschild radius because Sgr A* is undoubtedly rotating.)

Solution

(a) Gravitational redshift would be

$$\frac{\nu_1}{\nu_2} = \sqrt{\frac{1 - 2GM/r_2c^2}{1 - 2GM/r_1c^2}} \Rightarrow \frac{\nu_\infty}{\nu_p} = \sqrt{1 - \frac{2GM}{r_pc^2}} \simeq 0.9997$$

where $M \simeq 8 \times 10^{36}$kg and $r_p = 1.8 \times 10^{13}$m. The transverse Doppler effect gives

$$\frac{\nu_\infty}{\nu_p} = \sqrt{1 - \frac{V^2}{c^2}} \simeq 0.9998$$

where

$$\frac{V}{c} = \frac{GM}{cr_p} \simeq 0.02$$

The total effect is

$$\frac{\nu_\infty}{\nu_p} \simeq 0.9997 \times 0.9998 \simeq 0.9995$$

(b)

$$R_0 = \frac{2GM}{c^2} \simeq 0.08 \text{ au.}$$

The periastron of the orbit is much larger than this. ■

Part III

11 Hamiltonian Formulation

11.1 Problems and Solutions

⋆ **Problem 11.1** Find the Legendre transform $B(x,z)$ of the function $A(x,y) = x^4 - (y+a)^4$, and verify that $-\partial A/\partial x = \partial B/\partial x$.

Solution
Given

$$A(x,y) = x^4 - (y+a)^4,$$

then

$$z = \frac{\partial A(x,y)}{\partial y} = -4(y+a)^3,$$

so

$$(y+a) = -(z/4)^{1/3} \text{ check: } (y+a)^3 = -z/4$$

$$y = -a - (z/4)^{1/3}$$

Now

$$B(x,z) = zy(x,z) - A(x,y(x,z)) = -az - z(z/4)^{1/3} - x^4 + (y+a)^4$$

$$= -az - z^{4/3}/4^{1/3} - x^4 + (z/4)^{4/3} = -az - x^4 + (z/4)^{4/3} - (z/4)^{4/3} \cdot 4$$

$$B(x,z) = -x^4 - az - 3(z/4)^{4/3} = -x^4 - az - 3/4\, z^{4/3}/4^{1/3}$$

Note that

$$\partial A/\partial x = 4x^3, \text{ and } \frac{\partial B}{\partial x} = -4x^3$$

so

$$-\frac{\partial A}{\partial x} = \frac{\partial B}{\partial x}.$$

■

⋆ **Problem 11.2** In thermodynamics the enthalpy H (no relation to the Hamiltonian H) is a function of the entropy S and pressure P such that $\partial H/\partial S = T$ and $\partial H/\partial P = V$, so that

$$dH = TdS + VdP$$

where T is the temperature and V the volume. The enthalpy is particularly useful in isentropic and isobaric processes, because if the process is isentropic or isobaric, one of the two terms on the right vanishes. But suppose we wanted to deal with *isothermal* and isobaric processes, by constructing a function of T and P alone. Define such a function, in terms of H, T, and S, using a Legendre transformation. (The defined function G is called the Gibbs free energy.)

Solution

Start with enthalpy $H = H(S, P)$, so then

$$dH = \frac{\partial H}{\partial S} dS + \frac{\partial H}{\partial P} dP,$$

where $\frac{\partial H}{\partial S} = T$ and $\frac{\partial H}{\partial P} = -V$.

That is,

$$dH = TdS + VdP.$$

Now we transform to the Gibbs free energy $G = G(P, T)$ by a Legendre transformation, subtracting the product of the new variable (T) and the one we are getting rid of, (S). That is,

$$G(P, T) = H(S, P) - TS.$$

Then

$$dG(P, T) = \frac{\partial G}{\partial P} dP + \frac{\partial G}{\partial T} dT$$

$$= dH(S, P) - TdS - SdT = TdS + VdP - TdS - SdT = VdP - SdT$$

$$\Rightarrow \frac{\partial G}{\partial P} = V \text{ and } \frac{\partial G}{\partial T} = -S.$$

■

⋆ **Problem 11.3** In thermodynamics, for a system such as an enclosed gas, the internal energy $U(S, V)$ can be expressed in terms of the independent variables of entropy S and volume V, such that $dU = TdS - PdV$, where T is the temperature and P the pressure. Suppose we want to find a related function in which the volume is to be eliminated in favor of the pressure, using a Legendre transformation. (a) Which is the passive variable, and which are the active variables? (b) Find an expression for the new function in terms of U, P, and V. (The result is the enthalpy H or its negative, where the enthalpy H is unrelated to the Hamiltonian H.)

Solution

(a) We begin with $U(S, V)$, with

$$dU = \frac{\partial U}{\partial S} dS + \frac{\partial U}{\partial V} dV = TdS - PdV$$

where $T = \frac{\partial U}{dS}$ and $P = -\frac{\partial U}{\partial V}$. We then seek a Legendre transform to obtain the enthalpy $H(S, P)$, with the volume eliminated in favor of the pressure. So S is the passive variable, while V and P are the active variables.

(b) The Legendre transformation is

$$H(S, P) = U(S, V) + PV,$$

so then

$$\begin{aligned} dH = \frac{\partial H}{\partial S} dS + \frac{\partial H}{\partial P} dP &= dU(S, V) + PdV + VdP \\ &= TdS - PdV + PdV + VdP = TdS + VdP \end{aligned}$$

In this case we had to add the product of the active variables. ■

★★ **Problem 11.4** The energy of a relativistic free particle is the Hamiltonian $H = \sqrt{p^2c^2 + m^2c^4}$ in terms of the particle's momentum and mass. (a) Using one of Hamilton's equations in one dimension, find the particle's velocity v in terms of its momentum and mass. (b) Invert the result to find the momentum p in terms of the velocity and the mass. (c) Then find the free-particle Lagrangian for a relativistic particle using the Legendre transform

$$L(v) = pv - H.$$

(d) Beginning with the same Hamiltonian, generalize parts (a), (b) and (c) to a relativistic particle free to move in three dimensions.

Solution

(a) One of Hamilton's equations is

$$\frac{\partial H}{\partial p} = \dot{x} = v,$$

so

$$v = \frac{1}{2}(p^2c^2 + m^2c^4)^{-1/2} 2pc^2.$$

Therefore

$$v = \frac{pc^2}{\sqrt{p^2c^2 + m^2c^4}}$$

(b) Inverting to find an expression for the momentum,

$$v^2(p^2c^2 + m^2c^4) = p^2c^4$$

so

$$p^2c^4\left[c^2 - v^2\right] = m^2v^2c^4.$$

Then finally

$$p^2 = \frac{m^2v^2c^2}{c^2(1 - v^2/c^2)} \qquad p = \frac{mv}{\sqrt{1 - v^2/c^2}}$$

(c) Now to find the Lagrangian,

$$L(v) = pv - H = \frac{mv^2}{\sqrt{1 - v^2/c^2}} - \sqrt{p^2c^2 + m^2c^4}$$

$$= \frac{mv^2}{\sqrt{1 - v^2/c^2}} - \sqrt{\frac{m^2v^2c^2 + m^2c^4(1 - v^2/c^2)}{1 - v^2/c^2}}$$

$$= \frac{mv^2 - mc^2}{\sqrt{1 - v^2/c^2}} = \frac{-mc^2(1 - v^2/c^2)}{\sqrt{1 - v^2/c^2}} = -mc^2\sqrt{1 - v^2/c^2}$$

(d) The Hamilton equations are

$$\frac{\partial H}{\partial p_x} = \dot{x}, \quad \frac{\partial H}{\partial p_y} = \dot{y}, \quad \frac{\partial H}{\partial p_z} = \dot{z}$$

$$\Rightarrow v_x = \frac{p_x\, c^2}{\sqrt{p^2c^2 + m^2c^4}}, \quad v_y = \frac{p_y\, c^2}{\sqrt{p^2c^2 + m^2c^4}}, \quad v_z = \frac{p_z\, c^2}{\sqrt{p^2c^2 + m^2c^4}}$$

Inverting,

$$p_x = \frac{mv_x}{\sqrt{1 - v^2/c^2}}, \quad p_y = \frac{mv_y}{\sqrt{1 - v^2/c^2}}, \quad p_z = \frac{mv_z}{\sqrt{1 - v^2/c^2}}$$

Now

$$L = \sum p_i v_i - H = \frac{m(v_x^2 + v_y^2 + v_z^2)}{\sqrt{1 - v^2/c^2}} - \sqrt{p^2c^2 + m^2c^4}$$

$$= \frac{mv^2}{\sqrt{1 - v^2/c^2}} - \sqrt{\frac{m^2v^2c^2 + m^2c^4(1 - v^2/c^2)}{1 - v^2/c^2}} = \frac{mv^2 - mc^2}{\sqrt{1 - v^2/c^2}} = -mc^2\sqrt{1 - v^2/c^2}$$

where in three dimensions

$$v^2 = v_x^2 + v_y^2 + v_z^2$$

■

⋆ **Problem 11.5** The Lagrangian for a particular system is

$$L = \dot{x}^2 + a\dot{y} + b\dot{x}\dot{z},$$

where a and b are constants. Find the Hamiltonian, identify any conserved quantities, and write out Hamilton's equations of motion for the system.

Solution

The canonical momenta are

$$p_x = \frac{\partial L}{\partial \dot{x}} 2\dot{x} + b\dot{z}, \quad p_y = \frac{\partial L}{\partial \dot{y}} = a, \quad p_z = \frac{\partial L}{\partial \dot{z}} = b\dot{x}.$$

Therefore the Hamiltonian is

$$H = \sum \dot{x}_i p_i - L = (2\dot{x}^2 + b\dot{x}\hat{z}) + a\dot{y} + b\dot{x}\dot{z} - (\dot{x}^2 + a\dot{y} + b\dot{x}\dot{z}) = \dot{x}^2 + b\dot{x}\dot{z}$$

$$= (\frac{p_z}{b})^2 + p_z(\frac{p_x - 2\dot{x}}{b}) = \frac{p_z^2}{b^2} + \frac{p_z}{b}(p_x - 2\frac{p_z}{b})$$

$$H = -\frac{p_z^2}{b^2} + \frac{p_x p_z}{b}$$

$$\frac{\partial H}{\partial x} = 0 = -\dot{p}_x \quad \frac{\partial H}{\partial y} = 0 = -\dot{p}_y \quad \frac{\partial H}{\partial z} = 0 = -\dot{p}_z$$

so p_x, p_y, and p_z are all conserved.

$$\frac{\partial H}{\partial p_x} = \frac{p_z}{b} = \dot{x}$$

($p_z = b\dot{x}$ as we already know)

$$\frac{\partial H}{\partial p_y} = 0 = \dot{y} \Rightarrow y = \text{constant}$$

$$\frac{\partial H}{\partial p_z} = -\frac{2p_z}{b^2} + \frac{p_x}{b} = \dot{z} = \text{constant.}$$

■

★ **Problem 11.6** A system with two degrees of freedom has the Lagrangian

$$L = \dot{q}_1^2 + \alpha\dot{q}_1\dot{q}_2 + \beta q_2^2/2,$$

where α, and β are constants. Find the Hamiltonian, identify any conserved quantities, and write out Hamilton's equations of motion.

Solution

The canonical momenta are

$$p_1 = \partial L/\partial\dot{q}_i = 2\dot{q}_1 + \alpha\dot{q}_2$$

and

$$p_2 = \partial L/\partial\dot{q}_2 = \alpha\dot{q}_1.$$

Therefore the Hamiltonian is

$$H = \sum \dot{q}_i p_i - L = \dot{q}_1 p_1 + \dot{q}_2 p_2 - L = 2\dot{q}_1^2 + \alpha\dot{q}_1\dot{q}_2 + \alpha\dot{q}_1\dot{q}_2 - \dot{q}_1^2 - \alpha\dot{q}_1\dot{q}_2 - \beta q_2^2/2$$

Now

$$\dot{q}_1 = p_2/\alpha \text{ and } \dot{q}_2 = \frac{1}{\alpha}(p_1 - 2\dot{q}_1) = p_1 - \frac{2}{\alpha}(\frac{p_z}{\alpha}) = \frac{1}{\alpha}(p_1 - 2p_2/\alpha),$$

so eliminate $\dot{q}_1$ and $\dot{q}_2$ in H. That is,

$$H = \dot{q}_1^2 + \alpha\dot{q}_1\dot{q}_2 - \frac{\beta}{2}q_2^2 = p_2^2/\alpha^2 + \alpha\frac{p}{\alpha^2}(p_1 - 2p_2/\alpha) - \frac{\beta}{2}q_2^2$$

$$H = \frac{1}{\alpha}p_1 p_2 - \frac{p_2^2}{\alpha^2} - \frac{\beta}{2}q_2^2$$

$$\frac{\partial H}{\partial p_1} = \frac{p_2}{\alpha} = \dot{q}_1 \quad \frac{\partial H}{\partial p_2} = \frac{p_1}{\alpha} - \frac{2p_2}{\alpha^2} = \dot{q}_2$$

$$\frac{\partial H}{\partial q_1} = 0 = -\dot{p}_1$$

so p_1 is conserved.

$$\frac{\partial H}{\partial q_2} = -\beta q_2 = -\dot{p}_2$$

so p_2 is not conserved.

$$p_1 = \text{constant} \quad \frac{dp_2}{dt} = \beta q_2$$

■

⋆ **Problem 11.7** Write the Hamiltonian and find Hamilton's equations of motion for a simple pendulum of length ℓ and mass m. Sketch the constant-H contours in the θ, p_θ phase plane.

Solution

The kinetic and potential energies are

$$T = \frac{1}{2}m(\ell\dot{\theta})^2 \quad U = -mg\ell\cos\theta$$

so

$$L = T - U = \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell\cos\theta$$

The canonical momentum is

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = m\ell^2\dot{\theta}$$

So the Hamiltonian is

$$H = \dot{\theta}p_\theta - L = \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\cos\theta \text{ or } H = \frac{p_\theta^2}{2m\ell^2} - mgl\cos\theta$$

in terms of $p_\theta,\ \theta$.

Hamilton's equations are

$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} \Rightarrow p_\theta = m\ell^2\dot{\theta}$$

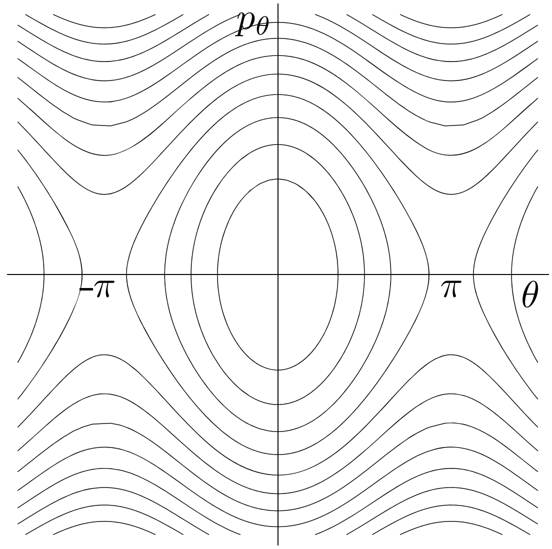
and

$$\frac{\partial H}{\partial\theta} = -\dot{p}_\theta \Rightarrow mg\ell\sin\theta = -\dot{p}_\theta = -m\ell^2\ddot{\theta},$$

which is the pendulum equation

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0.$$

These paths are all possible only for a massless rod (rather than a string) as support for the bob. Above a certain angle a string would collapse.



■

★★ **Problem 11.8** (a) Write the Hamiltonian for a spherical pendulum of length ℓ and mass m, using the polar angle θ and azimuthal angle φ as generalized coordinates. (b) Then write out Hamilton's equations of motion, and identify two first-integrals of motion. (c) Find a first-order differential equation of motion involving θ alone and its first time derivative. (d) Sketch contours of constant H in the θ, p_θ phase plane, and use it to identify the types of motion one expects.

Solution

(a)

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\ell^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) \quad U = -mg\ell\cos\theta$$

so

$$L = \frac{1}{2}m\ell^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) + mg\ell\cos\theta$$

The momenta are

$$p_\theta = m\ell^2\dot{\theta} \text{ and } p_\varphi = m\ell^2\sin^2\theta\dot{\varphi}.$$

So

$$H = \sum p_i\dot{q}_i - L = m\ell^2\dot{\theta}^2 + m\ell^2\sin^2\theta\dot{p}^2 - L = \frac{1}{2}m\ell^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) - mg\ell\cos\theta$$

$$H = \frac{p_\theta^2}{2m\ell^2} + \frac{p_\varphi^2}{2m\ell^2\sin^2\theta} - mg\ell\cos\theta$$

(b)

$$\frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m\ell^2 r} = \dot{\theta} \text{ and } \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{m\ell^2\sin^2\theta} = \dot{\varphi},$$

confirming what we already know. Also

$$\frac{\partial H}{\partial \theta} = -\frac{p_\varphi^2 \cos\theta}{m\ell^2 \sin^2\theta} + mg\ell \sin\theta = -\dot{p}_\theta$$

$$\frac{\partial H}{\partial \varphi} = -\dot{p}_\varphi = 0 \Rightarrow p_\varphi \text{ is conserved.}$$

This is conservation of angular momentum in the ϕ direction.

(c) H is also conserved, because L is not an explicit function of time. Note that $H = E$ as well. So, with $p_\theta = m\ell^2\dot{\theta}$, we can write

$$E = H = \frac{m^2\ell^4\dot{\theta}^2}{2m\ell^2} + \frac{p_\varphi^2}{2m\ell^2\sin^2\theta} - mg\ell\cos\theta$$

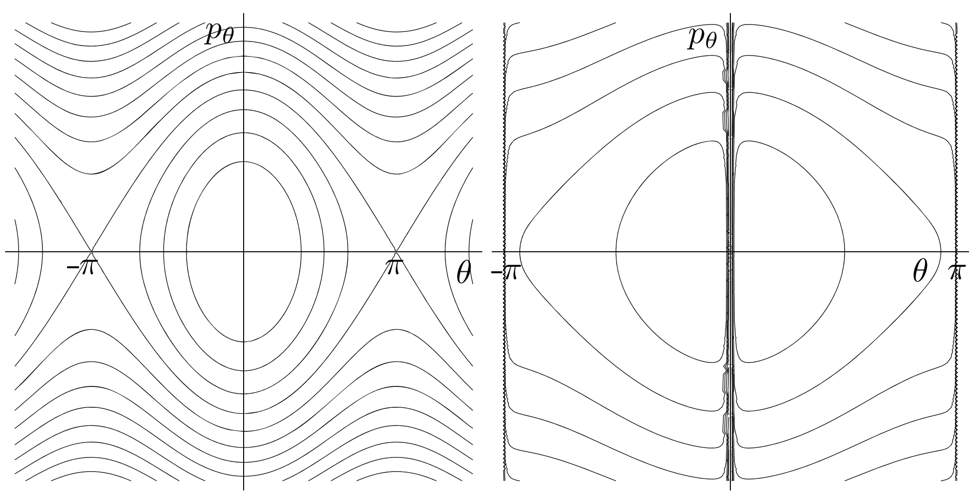
which (with E =constant, p_φ =constant) is a first-order d.e. involving θ and $\dot{\theta}$ alone among variables.

(d)

$$p_\theta = \pm\sqrt{2m\ell^2}\left[H + mg\ell\cos\theta - \frac{p_\varphi^2}{2m\ell^2\sin^2\theta}\right]^{1/2}$$

If $p_\phi = 0$, we have an ordinary plane pendulum. If $p_\phi \neq 0$, pendulum never reaches $(\theta = 0)$ if $p_\theta \neq 0$. Note the exact shape follows the equation

$$\frac{p_\theta^2}{2m\ell^2} = H + mg\ell\cos\theta - \frac{p_\phi^2}{2m\ell^2\sin^2\theta}$$



■

⋆ **Problem 11.9** A Hamiltonian with one degree of freedom has the form

$$H = \frac{p^2}{2m} + \frac{kq^2}{2} - 2aq^3\sin\alpha t$$

where m, k, a, and α are constants. Find the Lagrangian corresponding to this Hamiltonian. Write out both Hamilton's equations and Lagrange's equations, and show directly that they are equivalent.

Solution

$$L = L(q, \dot{q}, t) \quad p = \frac{\partial L}{\partial \dot{q}} \quad H = p\dot{q} - L.$$

So

$$L = -H(q, p, t) + p\dot{q}$$

Suppose $p = m\dot{q}$. Then

$$L = m\dot{q}^2 - \frac{p^2}{2m} - \frac{kq^2}{2} + 2aq^3 \sin \alpha t = m\dot{q}^2 - \frac{1}{2}m\dot{q}^2 - \frac{kq^2}{2} + 2aq^3 \sin \alpha t$$

$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2 + 2aq^3 \sin \alpha t$$

If so,

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}$$

$$H = p\dot{q} - L = m\dot{q}^2 - \frac{1}{2}m\dot{q}^2 + \frac{1}{2}kq^2 - 2aq^3 \sin \alpha t = \frac{p^2}{2m} + \frac{1}{2}kq^2 - 2aq^3 \sin \alpha t$$

and $\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$.

Lagrange equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \Rightarrow m\ddot{q} - \left[-kq + 6aq^2 \sin \alpha t\right] = 0 \Rightarrow m\ddot{q} + kq - 6aq^2 \sin \alpha t = 0$$

Hamilton equations:

$$\frac{\partial H}{\partial p} = \frac{p}{m} = \dot{q} \quad \frac{\partial H}{\partial q} = kq - 6aq^2 \sin \alpha t = -\dot{p}$$

So H's equations give

$$\dot{q} = m\ddot{q} = -kq + 6aq^2 \sin \alpha t \text{ or } m\ddot{q} + kq - 6aq^2 \sin \alpha t = 0,$$

the same as the Lagrange equation. ∎

★★ **Problem 11.10** A particle of mass m slides on the inside of a frictionless vertically-oriented cone of semi-vertical angle α. (a) Find the Hamiltonian H of the particle, using generalized coordinates r, the distance of the particle from the vertex of the cone, and φ, the azimuthal angle. (b) Write down two first-integrals of motion, and identify their physical meaning. (c) Show that a stable circular (constant - r) orbit is possible, and find its value of r for given angular momentum p_φ. (d) Find the frequency of small oscillations ω_{osc} about this circular motion, and compare it with the frequency of rotation ω_{circle}. (e) Is there a value of the tilt-angle α for which the two frequencies are equal? What is the physical significance of the equality?

Solution

(a) The Lagrangian is

$$L = T - U = \frac{1}{2}m(\dot{r}^2 + r^2 \sin^2 \alpha \dot{\phi}^2) - mgr\cos\alpha$$

so

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad \text{and} \quad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2 \sin^2 \alpha \dot{\phi}$$

so

$$H = \sum \dot{q}_i p_i - L = m\dot{r}^2 + mr^2 \sin^2 \alpha \dot{\phi}^2 - \frac{1}{2}m(\dot{r}^2 + r^2 \sin^2 \alpha \dot{\phi}^2) + mgr\cos\alpha$$

$$= \frac{1}{2}m(\dot{r}^2 + r^2 \sin^2 \alpha \dot{\phi}^2) + mgr\cos\alpha$$

$$H = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2 \sin^2 \alpha} + mgr\cos\alpha$$

(b) Note that ϕ is a cyclic coordinate, so $p_\phi = mr^2 \sin^2 \alpha \dot{\phi}$ is conserved, which is the angular momentum about the vertical axis. Also the time t is absent from L, so $H = E$ is conserved as well.

(c) Write

$$H = p_r^2/2m + U_{\text{eff}}(r), \ \text{where } U_{\text{eff}}(r) = \frac{p_\phi^2}{2mr^2 \sin^2 \alpha} + mgr\cos\alpha$$

U_{eff} has a minimum where

$$dU_{\text{eff}}/dr = 0.$$

So there is a stable circular orbit at

$$r_0 = \left(\frac{p_\phi^2}{m^2 g \sin^2 \alpha \cos\alpha}\right)^{1/3}.$$

(d) The rotation frequency of this orbit is

$$\omega_{rot} \equiv \dot{\phi} = \frac{p_\phi}{mr_0^2 \sin^2 \alpha} = \frac{p_\phi}{m \sin^2 \alpha}\left(\frac{m^2 g \sin^2 \alpha \cos\alpha}{p_\phi^2}\right)^{2/3} = \left(\frac{mg^2 \cos^2 \alpha}{\sin^2 \alpha p_\phi}\right)^{1/3}.$$

Now to find the frequency of small oscillations about the circular orbit, we calculate $\omega_{osc} = \sqrt{\frac{k_{\text{eff}}}{m}}$, where

$$k_{\text{eff}} \equiv \left.\frac{d^2 U_{\text{eff}}}{dr^2}\right|_{r=r_0}.$$

Here

$$U'_{\text{eff}} = -\frac{p_\phi^2}{mr^3 \sin^2 \alpha} + mg\cos\alpha \ \text{and} \ U''_{\text{eff}} = -\frac{3p_\phi^2}{mr^4 \sin^2 \alpha}.$$

So

$$U''_{\text{eff}}|_{r=r_0} = \frac{3p_\phi^2}{m \sin^2 \alpha}\left(\frac{m^2 g \sin^2 \alpha \cos\alpha}{p_\phi^2}\right)^{4/3} = 3\left(\frac{m^5 g^4 \sin^2 \alpha \cos^4 \alpha}{p_\phi^2}\right)^{1/3}$$

Thus

$$\omega_{osc} = \sqrt{\frac{U''_{\text{eff}}\big|_{r=r_0}}{m}} = \sqrt{3}\left(\frac{mg^2 \sin\alpha \cos^2\alpha}{p_\phi}\right)^{1/3}.$$

Therefore

$$\frac{\omega_{osc}}{\omega_{circle}} = \sqrt{3}\sin\alpha$$

(e) They are equal if $\sin\alpha = 1/\sqrt{3}$, in which case the orbit is closed. ■

⋆ **Problem 11.11** A particle of mass m is attracted to the origin by a force of magnitude k/r^2. Using plane polar coordinates, find the Hamiltonian and Hamilton's equations of motion. Sketch constant-H contours in the (r, p_r) phase plane.

Solution

The motion will be in a plane, so we use r and θ as coordinates. The kinetic and potential energies are

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \quad U = -k/r \quad (k > 0)$$

so

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + k/r$$

The canonical momenta are

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \quad p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

The Hamiltonian is

$$H = \sum \dot{q}_i p_i - L$$

$$H = m\dot{r}^2 + mr^2\dot{\theta}^2 - \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - k/r$$

$$= \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - k/r = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - k/r$$

Hamilton's equations are

$$\frac{\partial H}{\partial p_r} = \dot{r} \quad \left(\frac{p_r}{m} = \dot{r}\right)$$

$$\frac{\partial H}{\partial r} = -\dot{p}_r \quad \left(-\frac{p_\theta^2}{mr^3} + \frac{k}{r^2} = -\dot{p}_r\right)$$

$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} \quad \left(\frac{p_\theta}{mr^2} = \dot{\theta}\right)$$

$$\frac{\partial H}{\partial \theta} = -\dot{p}_\theta \quad (\dot{p}_\theta = 0 \quad p_\theta = \text{constant})$$

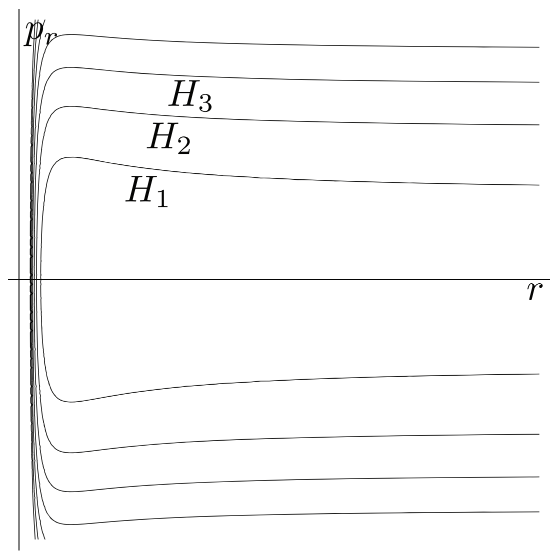
Combining, we have

$$\dot{p}_r = \frac{p_\theta^2}{mr^3} - \frac{k}{r^2} = m\ddot{r}$$

For fixed $H > 0$,

$$p_r^2/2m = H + k/r - \frac{p_\theta^2}{2mr^2}$$

$$p_r = \pm\sqrt{2m(H + k/r - \frac{p_\theta^2}{2mr^2})}$$



■

★★ **Problem 11.12** A double pendulum consists of two strings of equal length ℓ and two bobs of equal mass m. The upper string is attached to the ceiling, while the lower end is attached to the first bob. One end of the lower string is attached to the first bob, while the other end is attached to the second bob. Using generalized coordinates θ_1 (the angle of the upper string relative to the vertical) and θ_2 (the angle of the lower string relative to the vertical), find (a) the Lagrangian of the system (Hint: It can be tricky to find the kinetic energy of the lower bob in terms of the angles and their time derivatives. Use Cartesian coordinates initially; then convert these to generalized coordinates) (b) the canonical momenta (c) the Hamiltonian in terms of the angles and their first derivatives. Are there any constants of the motion? If so, what are they, and why are they constants? (Note that to go on and find the motion of the system using Hamilton's equations, one must first write $H(\theta_1, \theta_2, p_{\theta_1}, p_{\theta_1})$, without $\dot{\theta}_1$ and $\dot{\theta}_2$. This step, and the next step of solving the equations, involves a lot of algebra. This illustrates the fact that in somewhat complicated problems one could long since have written out Lagrange's equations and solved them, by the time one has even written out the Hamiltonian in canonical form.)

Solution

(a) Upper mass:

$$T_U = \frac{1}{2}m\ell^2\dot{\theta}_1^2, \quad U_U = -mg\ell\cos\theta_1$$

Lower mass:

$$T_L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

where

$$x = \ell(\sin\theta_1 + \sin\theta_2) \quad y = -\ell(\cos\theta_1 + \cos\theta_2) \quad z = 0$$

So

$$\begin{aligned}
T_L &= \frac{1}{2}m\ell^2\left[(\cos\theta_1\dot{\theta}_1 + \cos\theta_2\dot{\theta}_2)^2 + (\sin\theta_1\dot{\theta}_1 + \sin\theta_2\dot{\theta}_2)^2\right] \\
&= \frac{1}{2}m\ell^2\left[\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)\dot{\theta}_1\dot{\theta}_2\right] \\
&= \frac{1}{2}m\ell^2\left[\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2\right]
\end{aligned}$$

$$U_L = -mg\ell(\cos\theta_1 + \cos\theta_2)$$

So altogether, the Lagrangian of the system is

$$\begin{aligned}
L = T - U &= \frac{1}{2}m\ell^2\dot{\theta}_1^2 + mg\ell\cos\theta_1 + \frac{1}{2}m\ell^2\left[\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2\right] \\
&\quad + mg\ell(\cos\theta_1 + \cos\theta_2) \\
&= m\ell^2\dot{\theta}_1^2 + \frac{1}{2}m\ell^2\dot{\theta}_2^2 + m\ell^2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 + mg\ell(2\cos\theta_1 + \cos\theta_2) \\
&= m\ell^2\left[\dot{\theta}_1^2 + \frac{1}{2}\dot{\theta}_2^2 + \cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 + g/\ell(2\cos\theta_1 + \cos\theta_2)\right]
\end{aligned}$$

(b)

$$p_{\theta_1} = \frac{\partial L}{\partial\dot{\theta}_1} = 2m\ell^2\dot{\theta}_1 + m\ell^2\cos(\theta_1 - \theta_2)\dot{\theta}_2$$

$$p_{\theta_2} = \frac{\partial L}{\partial\dot{\theta}_2} = m\ell^2\dot{\theta}_2 + m\ell^2\cos(\theta_1 - \theta_2)\dot{\theta}_1$$

(c)

$$H = \sum p_i\dot{q}_i - L$$

$$\begin{aligned}
&= 2m\ell^2\dot{\theta}_1^2 + m\ell^2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 + m\ell\dot{\theta}_2^2 + m\ell^2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 \\
&\quad - m\ell^2\left[\dot{\theta}_1^2 + \frac{1}{2}\dot{\theta}_2^2 + \cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 + g/\ell(2\cos\theta_1 + \cos\theta_2)\right]
\end{aligned}$$

$$H = m\ell^2\dot{\theta}_1^2 + \frac{1}{2}m\ell\dot{\theta}_2^2 + m\ell^2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 - mg\ell(2\cos\theta_1 + \cos\theta_2)$$

in terms of $\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2$. ■

⋆ **Problem 11.13** A double Atwood's machine consists of two massless pulleys, each of radius R, some massless string, and three weights, with masses m_1, m_2, and m_3. The axis of pulley 1 is supported by a strut from the ceiling. A piece is string of length ℓ_1 is slung over the pulley, and one end of the string is attached to weight m_1 while the other end is attached to the axis of pulley 2. A second string of length ℓ_2 is slung over pulley 2; one end is attached to m_2 and the other to m_3. The strings are inextendible, but otherwise the weights and pulley 2 are free to move vertically. Let x be the distance of m_1 below the axis of pulley 1, and y be the distance of m_2 below the axis of pulley 2. (a) Find the Lagrangian $L(x, y, \dot{x}, \dot{y})$. (b) Find the canonical momenta p_x and p_y, in terms of $\dot{x}$ and $\dot{y}$. (c) Find the Hamiltonian of the system in terms of $x, y, \dot{x}$, and $\dot{y}$. (Note that to go on and find the motion of the system using Hamilton's equations, one must first write $H(x, y, p_x, p_y)$, without $\dot{x}$ and $\dot{y}$. This step, and the next step of solving the equations, involves a lot of algebra. This illustrates the fact that in somewhat complicated problems one could long since have written out Lagrange's equations and solved them, by the time one has even written out the Hamiltonian in canonical form.)

Solution

(a) Measure gravitational potential energy from the axis of pulley 1. Then the kinetic energy of m_1 is

$$T_1 = \frac{1}{2} m_1 \dot{x}^2$$

and its potential energy is

$$U_1 = -m_1 g x.$$

Similarly,

$$T_2 = \frac{1}{2} m_2 \left[\frac{d}{dt} (\ell_1 - x - \pi R_1 + y) \right]^2$$

and

$$U_2 = -m_2 g \left[\ell_1 - x - \pi R_1 + y \right].$$

Also

$$T_3 = \frac{1}{2} m_3 \left[\frac{d}{dt} (\ell_1 - x - \pi R_1 + \ell_2 - y - \pi R_2) \right]^2$$

and

$$U_3 = -m_3 g \left[\ell_1 - x - \pi R_1 + \ell_2 - y - \pi R_2 \right].$$

Altogether,

$$T_1 + T_2 + T_3 = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \left[-\dot{x} + \dot{y} \right]^2 + \frac{1}{2} m_3 \left[-\dot{x} - \dot{y} \right]^2$$

and

$$U_1 + U_2 + U_3 = -m_1 g x - m_2 g \left[\ell_1 - \pi R_1 - x + y \right] - m_3 g \left[\ell_1 + \ell_2 - \pi (R_1 + R_2) - x - y \right]$$

Dropping constants in U, which make no difference in the end,

$$L = T - U = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left[-\dot{x}+\dot{y}\right]^2 + \frac{1}{2}m_3\left[-\dot{x}-\dot{y}\right]^2$$

$$+ m_1 gx + m_2 g(-x+y) + m_3 g(-x-y)$$

Rewrite L:

$$L = (\frac{m_1}{2}+\frac{m_2}{2}+\frac{m_3}{2})\dot{x}^2+(\frac{m_2}{2}+\frac{m_3}{2})\dot{y}^2-m_2(\dot{x}\dot{y})+m_3\dot{x}\dot{y}+(m_1-m_2-m_3)gx+(m_2-m_3)gy$$

$$= \frac{1}{2}(m_1+m_2+m_3)\dot{x}^2+\frac{1}{2}(m_2+m_3)\dot{y}^2+(m_3-m_2)\dot{x}\dot{y}+(m_1-m_2-m_3)gx+(m_2-m_3)gy.$$

(b) So the canonical momenta are

$$p_x = \frac{\partial L}{\partial \dot{x}} = (m_1+m_2+m_3)\dot{x} + (m_3-m_2)\dot{y}$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = (m_2+m_3)\dot{y} + (m_3-m_2)\dot{x}$$

(c)The Hamiltonian is therefore

$$H = \sum \dot{q}_i p_i - L = (m_1+m_2+m_3)\dot{x}^2 + (m_3-m_2)\dot{x}\dot{y} + (m_2+m_3)\dot{y}^2 + (m_3-m_2)\dot{x}\dot{y}$$

$$-\frac{1}{2}(m_1+m_2+m_3)\dot{x}^2-\frac{1}{2}(m_2+m_3)\dot{y}^2-(m_3-m_2)\dot{x}\dot{y}-(m_1-m_2-m_3)gx-(m_2-m_3)gy.$$

$$H = \frac{1}{2}(m_1+m_2+m_3)\dot{x}^2+(m_3-m_2)\dot{x}\dot{y}+\frac{1}{2}(m_2+m_3)\dot{y}^2-(m_1-m_2-m_3)gx-(m_2-m_3)gy.$$

in terms of $x, y, \dot{x}, \dot{y}$. This expression for H is not in canonical form, so one could not use Hamilton's equations with it as it stands. ■

⋆ **Problem 11.14** A massless unstretchable string is slung over a massless pulley. A weight of mass $2m$ is attached to one end of the string and a weight of mass m is attached to the other end. One end of a spring of force constant k is attached beneath m, and a second weight of mass m is hung on the spring. Using the distance x of the weight $2m$ beneath the pulley and the stretch y of the spring as generalized coordinates, find the Hamiltonian of the system. (a) Show that one of the two coordinates is ignorable (*i.e.*, cyclic.) To what symmetry does this correspond? (b) If the system is released from rest with $y(0) = 0$, find $x(t)$ and $y(t)$.

Solution

The total kinetic energy is

$$T = \frac{1}{2}(2m)\dot{x}^2 + \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m(\dot{y}-\dot{x})^2$$

and the total potential energy, measured from the upper pulley, is

$$U = -(2m)gx - mg(\ell - x) + -mg(\ell - x + y) + \frac{1}{2}ky^2 = -mgy + \frac{1}{2}ky^2 + \text{constants}$$

so

$$L = T - U = \frac{3}{2}m\dot{x}^2 + \frac{1}{2}m(\dot{y}-\dot{x})^2 + mgy - \frac{1}{2}ky^2$$

dropping constants, which would make no difference. The canonical momenta are

$$p_x = \partial L/\partial \dot{x} = 3m\dot{x} - m(\dot{y}-\dot{x})^2 = 4m\dot{x} - m\dot{y}$$

and

$$p_y = \frac{\partial L}{\partial \dot{y}} = m(\dot{y}-\dot{x}), \text{ so } H = \sum \dot{q}_i p_i - L,$$

giving

$$H = 4m\dot{x}^2 - m\dot{x}\dot{y} + m\dot{y}^2 - m\dot{x}\dot{y} - \left[\frac{3}{2}m\dot{x}^2 + \frac{1}{2}m(\dot{y}-\dot{x})^2 + mgy - \frac{1}{2}ky^2\right]$$

$$= (4m - \frac{3}{2}m - \frac{1}{2}m)\dot{x}^2 - (2m-m)\dot{x}\dot{y} + (m - \frac{1}{2}m)\dot{y}^2 - mgy + \frac{1}{2}k\dot{y}^2$$

$$H = 2m\dot{x}^2 - m\dot{x}\dot{y} + \frac{1}{2}m\dot{y}^2 - mgy + \frac{1}{2}ky^2 = E.$$

Here $H = E$ is conserved because L is not an explicit function of time.

(a) The coordinate x is cyclic, so $p_x = 4m\dot{x} - m\dot{y}$ is conserved. The coordinate x is cyclic, because U is independent of x, due to the fact that the total mass is the same on each side of the pulley.

(b) At time $t = 0$, $y(0) = 0$ and $\dot{x}(0)$ and $\dot{y}(0) = 0$. Therefore $p_x = 0$ nad $H = 0$. So at all times, $\dot{x} = \dot{y}/4$, and so

$$2m\frac{\dot{y}^2}{16} - \frac{m\dot{y}^2}{4} + \frac{1}{2}m\dot{y}^2 - mgy + \frac{1}{2}ky^2 = E = 0.$$

Then

$$\frac{3m}{8}\dot{y}^2 - mgy + \frac{1}{2}ky^2 = 0$$

$$\dot{y}^2 = \frac{8}{3}(gy - \frac{1}{2}\frac{k}{m}y^2) \quad \dot{y} = \pm\sqrt{\frac{8}{3}(gy - \frac{1}{2}(\frac{k}{m}))y^2}.$$

$$\Rightarrow \int \frac{dy}{\sqrt{gy - \frac{1}{2}(\frac{k}{m}))y^2}} = \pm\sqrt{\frac{8}{3}}\, t + \text{constant.} = -\sqrt{\frac{2m}{k}}\sin^{-1}(1 - \frac{ky}{mg})$$

so

$$y = \frac{mg}{k}\left[1 \mp \sin(\omega t + C)\right] \text{ where } \omega = \sqrt{\frac{4k}{3m}}$$

$$\dot{y} = \mp\frac{mg}{k}\omega\cos(\omega t + C)$$

$$\dot{y} = - \text{ at } t = 0 \Rightarrow C = \pi/2$$

$$y = 0 \text{ at } t = 0 \Rightarrow \text{minus sign in y expression.}$$

Thus

$$y(t) = \frac{mg}{k}\left[1 - \cos\omega t\right]$$

$$x(t) = \frac{mg}{4k}\left[1 - \cos\omega t\right] + x_0 \quad \omega = \sqrt{\frac{4k}{3m}}$$

which satisfies the initial conditions with $y(0) = \dot{y}(0) = 0, x(0) = x_0, \dot{x}(0) = 0$. ■

★ **Problem 11.15** (a) A particle is free to move only in the x direction, subject to the potential energy $U = U_0 e^{-\alpha x^2}$, where α and U_0 are positive constants. Sketch constant-Hamiltonian curves in a phase diagram, including values of H with $H < U_0, H = U_0$, and $H > U_0$. (b) Repeat part (a) if $U_0 < 0$ and $\alpha > 0$, for values of H including those with $0 > H > U_0, H = 0$, and $H > 0$.

Solution

We have here $H = p^2/2m + U(x)$ where $U = U_0 e^{-\alpha x^2}$, from which it is straightforward to sketch constant-H curves on phase diagrams with axes x and p. In the interests of saving a lot of space we leave this to the reader. ■

★★ **Problem 11.16** A cyclic coordinate q_k is a coordinate absent from the Lagrangian (even though $\dot{q}_k$ is present in L.) (a) Show that a cyclic coordinate is likewise absent from the Hamiltonian. (b) Show from the Hamiltonian formalism that the momentum p_k canonical to a cyclic coordinate q_k is conserved, so $p_k = \alpha = \text{constant}$. Therefore one can ignore both q_k and p_k in the Hamiltonian. This led E. J. Routh to suggest a procedure for dealing with problems having cyclic coordinates. He carries out a transformation from the $q, \dot{q}$ basis to the q, p basis only for the cyclic coordinates, finding their equations of motion in the Hamiltonian form, and then uses Lagrange's equations for the noncyclic coordinates. Denote the cyclic coordinates by $q_{\mathrm{s}+1} ... q_n$; then define the *Routhian* as

$$R(q_1, ...q_n \ ; \ \dot{q}_1....\dot{q}_s \ ; \ p_{\mathrm{s}+1}....p_n \ ; \ t) = \sum_{\mathrm{i=s+1}}^{n} p_i \dot{q}_1 - L$$

Show then (using R rather than H) that one obtains Hamilton-type equations for the $n - s$ cyclic coordinates, while (using R rather than L) one obtains Lagrange-type equations for the non-cyclic coordinates. The Hamilton-type equations are trivial, showing that the momenta canonical to the cyclic coordinates are constants of the motion. In this procedure one can in effect "ignore" the cyclic coordinates, so "cyclic" coordinates are also "ignorable" coordinates.

Solution

(a) The Hamiltonian is

$$H = \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L$$

and we know

$$\frac{\partial L}{\partial q_\ell} = 0 \Rightarrow \frac{\partial H}{\partial q_\ell} = \dot{q}_k \frac{\partial^2 L}{\partial \dot{q}_k \partial q_\ell} - \frac{\partial L}{\partial q_\ell} = 0 - 0 = 0$$

(b) Hamiltonian equations imply

$$\dot{p}_k = -\frac{\partial H}{\partial q_k} = 0 \Rightarrow p_k = \text{constant}$$

(c) We start with L and the equations of motion

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_k}) = \frac{\partial L}{\partial q_k} \quad k = 1\dots n$$

Let $k = 1\dots s$ be non-cyclic coordinates. Let $\ell = s+1\dots n$ be cyclic coordinates. We write

$$dL = \frac{\partial L}{\partial q_k}dq_k + \frac{\partial L}{\partial q_\ell}dq_\ell + \frac{\partial L}{\partial \dot{q}_k}d\dot{q}_k + \frac{\partial L}{\partial \dot{q}_\ell}dq_\ell + \frac{\partial L}{\partial t}$$

$$= \frac{\partial L}{\partial q_k}dq_k + \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_\ell})dq_\ell + \frac{\partial L}{\partial \dot{q}_k}d\dot{q}_k + p\ell dq_\ell + \frac{\partial L}{\partial t}$$

where we use the equations of motion for q_ℓ and $p_\ell = \frac{\partial L}{\partial \dot{q}_\ell}$.

We then have

$$dL = \frac{\partial L}{\partial q_k}dq_k + \frac{\partial L}{\partial \dot{q}_k}d\dot{q}_k + p_\ell d\dot{q}_\ell + \dot{p}_\ell dq_\ell + \frac{\partial L}{\partial t}$$

$$= \frac{\partial L}{\partial q_k}dq_k + \frac{\partial L}{\partial \dot{q}_k}d\dot{q}_k + d(p_\ell \dot{q}_\ell) - \dot{q}_\ell dp_\ell + \dot{p}_\ell dq_\ell + \frac{\partial L}{\partial t}$$

$$\Rightarrow d(L - p_\ell \dot{q}_\ell) = \frac{\partial L}{\partial q_k}dq_k + \frac{\partial L}{\partial \dot{q}_k}d\dot{q}_k - \dot{q}_\ell dp_\ell + \dot{p}_\ell dq_\ell + \frac{\partial L}{\partial t}$$

where we used

$$d(p_\ell \dot{q}_\ell) = dp_\ell \dot{q}_\ell + p_\ell d\dot{q}_\ell$$

Defining the Routhian

$$R = -L + p_\ell \dot{q}_\ell$$

$$\Rightarrow dR = -\frac{\partial L}{\partial q_k}dq_k - \frac{\partial L}{\partial \dot{q}_k}d\dot{q}_k + \dot{q}_\ell dp_\ell - \dot{p}_\ell dq_\ell - \frac{\partial L}{\partial t}$$

$$= \frac{\partial R}{\partial q_\ell}dq_\ell + \frac{\partial R}{\partial p_\ell}dp_\ell + \frac{\partial R}{\partial q_k}dq_k + \frac{\partial R}{\partial \dot{q}_k}d\dot{q}_k + \frac{\partial R}{\partial t}$$

Equating coefficients of differentials, we get

$$\dot{p}_\ell = -\frac{\partial R}{\partial q_\ell}, \quad \dot{q}_\ell = \frac{\partial R}{\partial p_\ell}, \quad \frac{\partial R}{\partial t} = -\frac{\partial L}{\partial t}$$

and

$$-\frac{\partial L}{\partial q_k} = \frac{\partial R}{\partial q_k}, \quad -\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial R}{\partial \dot{q}_k}$$

which imply

$$\frac{d}{dt}(\frac{\partial R}{\partial \dot{q}_k}) = \frac{\partial R}{\partial q_k}$$

So, R acts as L for non-cyclic q_k but it acts as H for cyclic q_ℓ

$$\dot{p}_\ell = -\frac{\partial R}{\partial q_\ell} = 0 \Rightarrow p_\ell = \text{constant}$$

■

★★ **Problem 11.17** Show that the Poisson bracket of two constants of the motion is itself a constant of the motion, even when the constants depend explicitly on time.

Solution

Say $F_1(q,p,t)$ and $F_2(q,p,t)$ are two constants of motion

$$\Rightarrow \dot{F}_1 = \{F_1, H\} + \frac{\partial F_1}{\partial t} = 0$$

$$\dot{F}_2 = \{F_2, H\} + \frac{\partial F_2}{\partial t} = 0$$

Construct $F_3 = \{F_1, F_2\}$

$$\Rightarrow \dot{F}_3 = \{\{F_1, F_2\}, H\} + \frac{\partial}{\partial t}\{F_1, F_2\}$$

Using the identity

$$\{A, \{BC\}\} = \{A, B\}C + B\{A, C\} \text{ and } \{A, B\} = -\{B, A\}$$

we get

$$\dot{F}_3 = -\{H, \{F_1, F_2\}\} + \{\frac{\partial F_1}{\partial t}, F_2\} + \{F_1, \frac{\partial F_2}{\partial t}\}$$

$$= -\{H, \{F_1, F_2\}\} - \{\{F, H\}, F_2\} - \{F_1, \{F_2, H\}\}$$

$$= -\{H, \{F_1, F_2\}\} - \{F_2, \{H, F_1\}\} - \{F_1, \{F_2, H\}\} = 0$$

by the Jacobi identity. ■

★ **Problem 11.18** Prove the anticommutativity and distributivity of Poisson brackets by showing that (a) $\{A,B\}_{\text{q,p}} = -\{B,A\}_{\text{q,p}}$ (b) $\{A,B+C\}_{\text{q,p}} = \{A,B\}_{\text{q,p}} + \{A,C\}_{\text{q,p}}$.

Solution

(a)

$$\{A, B\} = \frac{\partial A}{\partial q_k}\frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_k}\frac{\partial B}{\partial q_k} = -\frac{\partial B}{\partial q_k}\frac{\partial A}{\partial p_k} + \frac{\partial B}{\partial p_k}\frac{\partial A}{\partial q_k} = -\{A, B\}$$

(b)

$$\{A, B+C\} = \frac{\partial A}{\partial q_k}\frac{\partial (B+C)}{\partial p_k} - \frac{\partial A}{\partial p_k}\frac{\partial (B+C)}{\partial q_k} = \frac{\partial A}{\partial q_k}\frac{\partial B}{\partial p_k} + \frac{\partial A}{\partial q_k}\frac{\partial C}{\partial p_k} - \frac{\partial A}{\partial p_k}\frac{\partial B}{\partial q_k} - \frac{\partial A}{\partial p_k}\frac{\partial C}{\partial q_k}$$

$$= \{A, B\} + \{A, C\}$$

■

⋆ **Problem 11.19** Show that Hamilton's equations of motion can be written in terms of Poisson brackets as

$$\dot{q} = \{q, H\}_{\mathrm{q,p}} \ , \quad \dot{p} = \{p, H\}_{\mathrm{q,p}} \, .$$

Solution

$$\dot{q} = \left\{q, H\right\} = \frac{\partial q}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial q}{\partial p}\frac{\partial H}{\partial q} = \frac{\partial H}{\partial p}$$

$$\dot{p} = \left\{p, H\right\} = \frac{\partial p}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial p}{\partial p}\frac{\partial H}{\partial q} = \frac{\partial H}{\partial q}$$

■

⋆⋆ **Problem 11.20** A Hamiltonian has the form

$$H = q_1 p_1 - q_2 p_2 + a q_1^2 - b q_2^2,$$

where a and b are constants. (a) Using the method of Poisson brackets, show that

$$f_1 \equiv q_1 q_2 \quad \text{and} \quad f_2 \equiv \frac{1}{q_1}(p_2 + b q_2)$$

are constants of the motion. (b) Then show that $\{f_1, f_2\}$ is also a constant of the motion. (c) Is H itself constant? Check by finding q_1, q_2, p_1, and p_2 as explicit functions of time.

Solution

$$H = q_1 p_1 - q_2 p_2 + a q_1^2 - b q_2^2$$

(a)

$$\dot{f}_1 = \frac{\partial f_1}{\partial t} + \{f_1, H\} = \{f, H\} = \{q_1, q_2, H\}$$

$$= \frac{\partial(q_1, q_2)}{\partial q_1}\frac{\partial H}{\partial p_1} + \frac{\partial(q_1, q_2)}{\partial q_2}\frac{\partial H}{\partial p_2} = q_2(q_1) + q_1(-q_2) = 0$$

$$\dot{f}_2 = \frac{\partial f_2}{\partial t} + \{f_2, H\} = \left\{\frac{1}{q_1}(p_2 + b q_2), H\right\}$$

$$= \frac{\partial}{\partial q_1}\left(\frac{1}{q_1}(p_2 + b q_2)\right)\frac{\partial H}{\partial p_1} + \frac{\partial}{\partial q_2}\left(\frac{1}{q_1}(p_2 + b q_2)\right)\frac{\partial H}{\partial p_2} - \frac{\partial}{\partial p_2}\left(\frac{1}{q_1}(p_2 + b q_2)\right)\frac{\partial H}{\partial q_2}$$

$$= -\frac{1}{q_1}(p_2 + b q_2)(q_1) + \left(\frac{b}{q_1}\right)(-q_2) - \left(\frac{1}{q_1}\right)(-p_2 - 2 b q_2) = 0$$

(b)

$$\{f_1, f_2\} = \left\{q_1, q_2, \frac{1}{q_1}(p_2 + b q_2)\right\}$$

$$= \frac{\partial(q_1, q_2)}{\partial q_1}\frac{\partial}{\partial p_1}\left(\frac{1}{q_1}(p_2 + b q_2)\right) + \frac{\partial}{\partial q_2}(q_1 q_2)\frac{\partial}{\partial p_2}\frac{1}{q_1}(p_2 + b q_2)$$

$$= q_2(0) + q_1 \frac{1}{q_1} = 1$$

which is a constant of motion, obviously.

(c)

$$\dot{H} = \frac{\partial H}{\partial t} + \{H, H\} = 0 + 0 = 0$$

Yes, H is constant.

$$\dot{q}_1 = \{q_1, H\} = \frac{\partial H}{\partial p_1} = q_1$$

$$\dot{q}_2 = \{q_2, H\} = \frac{\partial H}{\partial p_2} = -q_2$$

$$\dot{p}_1 = \{p_1, H\} = -\frac{\partial H}{\partial q_1} = -p_1 - 2aq_1$$

$$\dot{p}_2 = \{p_2, H\} = -\frac{\partial H}{\partial q_2} = +p_2 + 2bq_2$$

$$\Rightarrow q_1(t) = C_1 e^t \quad q_2(t) = C_2 e^{-t}$$

$$p_1(t) = -aC_1 e^t + C_3 e^{-t} \text{ where } C_1, C_2, C_3, C_4 \text{ are constants}$$

$$p_2 = -bC_2 e^{-t} + C_4 e^t$$

We then substitute these in H.

$$H = C_1 e^t(-aC_1 e^t + C_3 e^{-t}) - C_2 e^{-t}(-bC_2 e^{-t} + C_5 e^t) + aC_1^2 e^{2t} - bC_2^2 e^{-2t}$$
$$= -aC_1^2 e^{2t} + C_1 C_3 + bC_2^2 e^{-2t} - C_2 C_5 + aC_1^2 e^{2t} - bC_2^2 e^{-2t} = C_1 C_3 - C_2 C_5$$

■

★★★ **Problem 11.21** Show, using the Poisson bracket formalism, that the *Laplace-Runge-Lenz vector*

$$\mathbf{A} \equiv \mathbf{p} \times \mathbf{L} - \frac{mk\mathbf{r}}{r}$$

is a constant of the motion for the Kepler problem of a particle moving in the central inverse-square force field $F = -k/r^2$. Here $\mathbf{p}$ is the particle's momentum, and $\mathbf{L}$ is its angular momentum. HINT: Write $L^k = \varepsilon^{kln} x^l p^n$ and you might need to use the identity $\varepsilon^{ijk} \varepsilon^{ilm} = \delta^{jl}\delta^{km} - \delta^{jm}\delta^{kl}$.

Solution

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - \frac{mk\mathbf{r}}{r}$$

And the Hamiltonian is

$$H = \frac{\mathbf{p}^2}{2m} - \frac{k}{r} \text{ with } \mathbf{L} = \mathbf{r} \times \mathbf{p}$$

We have

$$\dot{\mathbf{A}} = \frac{\partial \mathbf{A}}{\partial t} + \{\mathbf{A}, H\} = \{\mathbf{A}, H\}$$

We then need to compute the Poisson bracket of any component of $\mathbf{A}$ and H. We write

$$A^2 = \epsilon^{ijk} p^j L^k - \frac{mkx^i}{r}$$

where ϵ^{ijk} is the 3 dimensional totally antisymmetric tensor. And

$$L^k = \epsilon^{k\ell n} x^\ell p^n$$

We then need to compute

$$\left\{\epsilon^{ijk} p^j \epsilon^{k\ell n} x^\ell p^n - \frac{mk}{r} x^i, \frac{p^{a^2}}{2m} - \frac{k}{r}\right\} = \epsilon^{ijk}\epsilon^{k\ell n}\left\{p^j x^\ell p^n, \frac{p^{a^2}}{2m} - \frac{k}{r}\right\} - mk\left\{\frac{x^i}{r}, \frac{p^{a^2}}{2m}\right\}$$

where we used the fact that

$$\{x^i, x^j\} = \{p^i, p^j\} = 0$$

$$= \epsilon^{ijk}\epsilon^{k\ell n}\left(\frac{\partial(p^j x^\ell p^n)}{\partial x^b}\frac{\partial(\frac{p^{a^2}}{2m} - \frac{k}{r})}{\partial p^b} - \frac{\partial(p^j x^\ell p^n)}{\partial p^b}\frac{\partial(\frac{p^{a^2}}{2m} - \frac{k}{r})}{\partial x^b}\right)$$

$$-mk\left(\frac{\partial(x^2/r)}{\partial x^b}\frac{\partial(p^{a^2}/2m)}{\partial p^b}\right)$$

$$= \epsilon^{ijk}\epsilon^{k\ell n}\left(\delta^{\ell b} p^j p^n \frac{p^b}{m} - (\delta^{jb} x^\ell p^n + \delta^{nb} p^j x^p)\left(\frac{kx^b}{r^3}\right)\right) - mk\left(\frac{\delta^{ib}}{r} - \frac{x^i x^b}{r^3}\right)\frac{p^b}{m}$$

where

$$\frac{\partial}{\partial x^b}\left(\frac{1}{r}\right) = -\frac{x^b}{r^3}$$

We then have

$$= \epsilon^{ijk}\epsilon^{kbn} p^j p^n p^b - \frac{k\epsilon^{ijk}\epsilon^{k\ell n} x^\ell x^b p^n}{r^3} - \frac{k\epsilon^{ijk}\epsilon^{k\ell b} p^j x^\ell x^b}{r^3} - \frac{kp^i}{r} + \frac{kx^i x^b p^b}{r^3}$$

The first term vanishes because $\epsilon^{kbn} p^n p^b = 0$ by antisymmetry of ϵ^{kbn} in band n, i.e.

$$\epsilon^{kbn} p^n p^b = \epsilon^{kbn} p^b p^n = -\epsilon^{kbn} p^n p^b \to 0$$

n and b relabeled by antisymmetry of ϵ^{kbn}. Since it equlas negative of itself. Similarly for the third term since $\epsilon^{k\ell b} x^\ell x^b = 0$.

$$\Rightarrow \dot{\mathbf{A}} = -\frac{k\epsilon^{ibk}\epsilon^{k\ell n} x^\ell x^b p^n}{r^3} - -\frac{kp^i}{r} + \frac{kx^i x^b p^b}{r^3} = -\frac{k}{r^3}(\delta^{i\ell}\delta^{bn} - \delta^{in}\delta^{b\ell}) x^\ell x^b p^n - \frac{kp^i}{r} + \frac{kx^i x^b p^b}{r^3}$$

$$= -\frac{k}{r^3} x^i x^n p^n + \frac{k}{r^3} x^b x^b p^i - \frac{kp^i}{r} + \frac{kx^i x^b p^b}{r^3} = 0 \text{ as needed.}$$

■

★ **Problem 11.22** A beam of protons with a circular cross-section of radius r_0 moves within a linear accelerator oriented in the x direction. Suppose that the transverse momentum components (p_y, p_z) of the beam are distributed uniformly in momentum space, in a circle of radius p_0. If a magnetic lens system at the end of the accelerator focusses the beam into a small circular spot of radius r_1, find, using Liouville's theorem, the corresponding distribution of the beam in momentum space. Here what may be a desirable focussing of the beam in position-space has the often unfortunate consequence of broadening the momentum distribution.

Solution

Phase space volume is preserved according to Liouville's theorem:

$$\text{Initial volume } = \frac{4}{3}\pi r_0^3 \pi p_0^2 \Delta p_x$$

$$\text{Final volume } = \frac{4}{3}\pi r_1^3 \pi p_1^2 \Delta p_x$$

where Δp_x is the speed of momenta in the x-direction, and we assume that the acceleration is momentum independent so the Δp_x does not change.

$$\Rightarrow \frac{4}{3}\pi r_0^3 \pi p_0^2 \Delta p_x = \frac{4}{3}\pi r_1^3 \pi p_1^2 \Delta p_x$$

$$\Rightarrow \frac{r_0^3 p_0^3}{r_1^3} = p_1^2$$

$$\Rightarrow p_1 = p_0 (\frac{r_0}{r_1})^{3/2} > p_0$$

so that the momenta in the $y - z$ directions spread out. ■

★★ **Problem 11.23** A large number of particles, each of mass m, move in response to a uniform gravitational field g in the negative z direction. At time $t = 0$, they are all located within the corners of a rectangle in (z, p_z) phase space, whose positions are: (1) $z = z_0, p_z = p_0$, (2) $z = z_0 + \Delta z, p_z = p_0$, (3) $z = z_0, p_z = p_0 + \Delta p$, and (4) $z = z_0 + \Delta z, p_z = p_0 + \Delta p$. By direct computation, find the area in phase space enclosed by these particles at times (a) $t = 0, (b) t = m\Delta z/p_0$, and (c) $t = 2m\Delta z/p_0$. Also show the shape of the region in phase space for cases (b) and (c).

Solution

(a) The x and y locations remain unchanged, but we have

$$z = z_0' + \frac{p_0}{m}t - \frac{1}{2}gt^2 \text{ and } p = p_0' - mgt$$

where $z_0 < z_0' < z_0 + \Delta z$ and $p_0 < p_0' < p_0 + \Delta p$. The phase space area that can change lies in the $z - p_z$ plane and is constant $\Delta z \Delta p$ =constant for all times. But the shape changes in time: Take the 4 corners of the region of interest in phase space;

$$(z_0, p_0), \quad (z_0 + \Delta z, p_0), \quad (z_0, p_0 + \Delta p), \quad \text{and} \quad (z_0 + \Delta z, p_0 + \Delta p).$$

Set z_0', p_0' equal to these 4 points.

We then find under time evolution:

$$(z_0, p_0) \rightarrow (z(t), p(t))$$

$$(z_0 + \Delta z, p_0) \rightarrow (z(t) + \Delta z, p(t))$$

$$(z_0, p_0 + \Delta p) \rightarrow (z(t) + \frac{\Delta p}{m} t, p(t) + \Delta p)$$

$$(z_0 + \Delta z, p_0 + \Delta p) \rightarrow (z(t) + \Delta z + \frac{\Delta p}{m} t, p(t) + \Delta p).$$

So, the rectangle evolves into a parallelogram with the same area, as required. ■

⋆⋆ **Problem 11.24** In an electron microscope, electrons scattered from an object of height z_0 are focused by a lens at distance D_0 from the object and form an image of height z_1 at a distance D_1 behind the lens. The aperture of the lens is A. Show by direct calculation that the area in the (z, p_z) phase plane occupied by electrons leaving the object (and destined to pass through the lens) is the same as the phase area occupied by electrons arriving at the image. Assume that $z_0 \ll D_0$ and $z_1 \ll D_1$. (from *Mechanics*, 3rd edition, by Keith R. Symon.)

Solution

Note that if $A \ll D_0$ and $A \ll D_1$, then $\theta_0 \simeq \frac{A/2}{D_0}$ and $\theta_1 \simeq \frac{A/2}{D_1}$. We also assume that $z_0 \ll D_0$ and $z_1 \ll D_1$. The initial spread in z_0 is $\Delta z_0 = z_0$ (between the head and tail of the object) and the final spread is $\Delta z_1 = z_1$. If the magnitude of the momentum of an electron is p, then

$$p_z = p \sin \theta_0 \simeq p\theta_0$$

(since $\theta_0 \ll 1$) so the spread in p_z is

$$(\Delta p_z)_0 = 2p\theta_0 = pA/D_0$$

and similarly

$$(\Delta p_z)_z \simeq pA/D_1.$$

So the initial phase plane area is $z_0 \frac{pA}{D_0}$ and the final phase plane area is $z_1 \frac{pA}{D_1}$. However, from the geometry (or lens formulas), $z_0/D_0 \simeq z/D$ from similar triangle. So the phase plane areas are

$$A_0 = pA \frac{z_0}{D_0} = p^A \frac{z_1}{D_1} = A_1,$$

and so the phase plane area remains the same. ■

⋆ **Problem 11.25** Show directly that the transformation

$$Q = \ln\left(\frac{1}{q} \sin p\right) \quad P = q \cot p$$

is canonical.

Solution

$$\{Q,P\} = \left\{\ln\frac{\sin p}{q}, q\cot p\right\}$$

$$= \frac{\partial}{\partial q}(\ln\frac{\sin p}{q})\frac{\partial}{\partial p}(q\cot p) - \frac{\partial}{\partial p}(\ln\frac{\sin p}{q})\frac{\partial}{\partial q}(q\cot p)$$

$$= \frac{q}{\sin p}(-\frac{\sin p}{q^2})q(-\csc^2 p) - \frac{q}{\sin p}\frac{\cos p}{q}\cot p$$

$$= \frac{1}{\sin^2 p} - \frac{\cos^2 p}{\sin^2 p} = 1$$

as needed. ■

⋆⋆ **Problem 11.26** Show that if the Hamiltonian and some quantity Q are both constants of the motion, then the n^{th} partial derivative of Q with respect to time must also be a constant of the motion.

Solution

We know

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0$$

and

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + \{Q,H\} = 0$$

We want to compute

$$\frac{d}{dt}(\frac{\partial^n Q}{\partial t^n}) \Rightarrow \frac{d}{dt}(\frac{\partial^n Q}{\partial t^n}) = \frac{\partial^{n+1}Q}{\partial t^{n+1}} + \left\{\frac{\partial^n Q}{\partial t^n}, H\right\}$$

But from

$$\frac{\partial Q}{\partial t} + \{Q,H\} = 0$$

We can apply the operator $\frac{\partial^n}{\partial t^n}$

$$\Rightarrow \frac{d}{dt}(\frac{\partial^{n+1}Q}{\partial t^{n+1}}) + \left\{\frac{\partial^n Q}{\partial t^n}, H\right\} = 0$$

since

$$\frac{\partial H}{\partial t} = 0 \Rightarrow \frac{d}{dt}(\frac{\partial^n Q}{\partial t^n}) = 0$$

■

⋆ **Problem 11.27** Prove the Jacobi identity for Poisson brackets,

$$\left\{A, \{B,C\}_{\mathrm{q,p}}\right\}_{\mathrm{q,p}} + \left\{B, \{C,A\}_{\mathrm{q,p}}\right\}_{\mathrm{q,p}} + \left\{C, \{A,B\}_{\mathrm{q,p}}\right\}_{\mathrm{q,p}} = 0.$$

Solution

$$\{A,\{B,C\}\} = \frac{\partial A}{\partial q}\frac{\partial}{\partial q}\{B,C\} - \frac{\partial A}{\partial p}\frac{\partial\{B,C\}}{\partial q}$$

$$= \frac{\partial A}{\partial q}\frac{\partial}{\partial q}\left(\frac{\partial B}{\partial q}\frac{\partial C}{\partial p} - \frac{\partial B}{\partial p}\frac{\partial C}{\partial q}\right) - \frac{\partial A}{\partial p}\frac{\partial}{\partial q}\left(\frac{\partial B}{\partial q}\frac{\partial C}{\partial p} - \frac{\partial B}{\partial p}\frac{\partial C}{\partial q}\right)$$

$$= \frac{\partial A}{\partial q}\frac{\partial^2 B}{\partial p\partial q}\frac{\partial C}{\partial p} + \frac{\partial A}{\partial q}\frac{\partial B}{\partial q}\frac{\partial^2 C}{\partial p^2} - \frac{\partial A}{\partial q}\frac{\partial^2 B}{\partial p^2}\frac{\partial C}{\partial q} - \frac{\partial A}{\partial q}\frac{\partial B}{\partial p}\frac{\partial^2 C}{\partial q\partial p}$$

$$-\frac{\partial A}{\partial p}\frac{\partial^2 B}{\partial q^2}\frac{\partial C}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}\frac{\partial^2 C}{\partial q\partial p} + \frac{\partial A}{\partial p}\frac{\partial^2 B}{\partial q\partial p}\frac{\partial C}{\partial q} + \frac{\partial A}{\partial p}\frac{\partial B}{\partial p}\frac{\partial^2 C}{\partial q^2}$$

Now, notice that is we add to this the same expression with $A,B,C \to B,C,A$, then the first term will cancel with the sixth term, and similarly all terms cancel pairwise when we also add $A,B,C \to C,A,B$. ■

★ **Problem 11.28** (a) Find the Hamiltonian for a projectile of mass m moving in a uniform gravitational field g, using coordinates x,y. (b) Then find Hamilton's equations of motion and solve them.

Solution

(a) The Hamiltonian is

$$H = \frac{p^{x^2}}{2m} + \frac{p^{y^2}}{2m} + mgy$$

where we have used the Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mgy.$$

(b) We then have

$$\dot{x} = \frac{\partial H}{\partial p^x} = \frac{p^x}{m} \quad \dot{y} = \frac{\partial H}{\partial p^y} = \frac{p^y}{m}$$

$$\dot{p}^x = -\frac{\partial H}{\partial x} = 0 \quad \dot{p}^y = -\frac{\partial H}{\partial y} = -mg$$

Solving the second set of equations

$$p^x = \text{constant} = p_0^x \quad p^y = -mgt + p_0^y$$

$$\Rightarrow \dot{x} = \frac{p^x}{m} = \frac{p_0^x}{m} \Rightarrow x = \frac{p_0^x}{m} + x_0$$

and

$$\dot{y} = \frac{p^y}{m} = -gt + \frac{p_0^y}{m} \Rightarrow y = y_0 + \frac{p_0^y}{m}t - \frac{1}{2}gt^2$$

■

★★ **Problem 11.29** (a) Find the Hamiltonian for a projectile of mass m moving in a force field with potential energy $U(\rho, \varphi, z)$, where ρ, φ, z are cylindrical coordinates. (b) Find Hamilton's equations of motion. (c) Solve them as far as possible if $U = U(\rho)$ alone.

Solution

(a) The Lagrangian is

$$L = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2) - U(\rho, \varphi, z)$$

$$\Rightarrow p_\rho = \frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho} \quad p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m\rho^2\dot{\varphi} \quad p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

$$\Rightarrow H = p_\rho\dot{\rho} + p_\varphi\dot{\varphi} + p_z\dot{z} - L = \frac{p_\rho^2}{m} + \frac{p_\varphi^2}{m\rho^2} + \frac{p_z^2}{m} - L$$

$$= \frac{p_\rho^2}{2m} + \frac{p_\varphi^2}{2m\rho^2} + \frac{p_z^2}{2m} + U(\rho, \varphi, z)$$

(b) The equations of motion become:

$$\dot{\rho} = \frac{\partial H}{\partial p_\rho} = \frac{p_\rho}{m} \quad \dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{m\rho^2} \quad \dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}$$

and

$$\dot{p}_\rho = -\frac{\partial H}{\partial \rho} = +\frac{p_\varphi^2}{m\rho^3} - \frac{\partial U}{\partial \rho} \quad \dot{p}_\ell = -\frac{\partial H}{\partial \varphi} = -\frac{\partial U}{\partial \varphi} \quad \dot{p}_z = -\frac{\partial H}{\partial z} = -\frac{\partial U}{\partial z}$$

(c) We now have

$$U = U(\rho) \Rightarrow \dot{p}_\varphi = 0 \text{ and } \dot{p}_z = 0 \Rightarrow p_\varphi = \text{constant and } p_z = \text{constant}$$

Then we have to solve

$$\dot{p}_\rho = \frac{p_\varphi^2}{m\rho^3} - \frac{\partial U}{\partial \rho}$$

along with $\dot{\rho} = \frac{p_\rho}{m}$ A better strategy is to use another constant of motion:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0 \Rightarrow H = \frac{p_\rho^2}{2m} + \frac{p_\varphi^2}{2m\rho^2} + \frac{p_z^2}{2m} + U(\rho) = \text{constant}$$

$$\Rightarrow p_\rho = \pm\sqrt{2mH - \frac{p_\varphi^2}{\rho^2} - p_z^2 - 2mU}$$

where H, p_φ, p_z are constants.

To find $\rho(t)$, we use

$$\dot{\rho} = \frac{p_\rho}{m} = \pm\sqrt{2mH - \frac{p_\rho^2}{\rho^2} - p_z^2 - 2mU(\rho)}$$

and integrate for $\rho(t)$, which ten gives us $p_\rho(t)$ as well.

Finally, to find $\varphi(t)$, we integrate

$$\dot{\varphi} = \frac{p_\varphi}{m\rho(t)^2}$$

where $\rho(t)$ is obtained above and to find $z(t)$, we integrate $\dot{z} = \frac{p_z}{m}$. ∎

★ **Problem 11.30** Consider a particle of mass m with relativistic Hamiltonian $H = \sqrt{p^2c^2 + m^2c^4} + U(x, y, z)$ where U is its relativistic potential energy. Find the particle's equations of motion.

Solution

$$H = \sqrt{p^2c^2 + m^2c^4} + U$$

The equations of motion are:

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x c^2}{\sqrt{p^2c^2 + m^2c^4}}$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y c^2}{\sqrt{p^2c^2 + m^2c^4}}$$

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z c^2}{\sqrt{p^2c^2 + m^2c^4}}$$

and

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -\frac{\partial U}{\partial x} \quad \dot{p}_y = -\frac{\partial H}{\partial y} = -\frac{\partial U}{\partial y} \quad \dot{p}_z = -\frac{\partial H}{\partial z} = -\frac{\partial U}{\partial z}$$

∎

★ **Problem 11.31** We found Hamilton's equations by starting with the Lagrangian $L(q_i, \dot{q}_i, t)$ and using a Legendre transformation to define the Hamiltonian $H(q_i, p_i, t)$. Now starting with the Hamiltonian and Hamilton's equations, use a reverse Legendre transformation to define L, and show that one obtains the Lagrange equations.

Solution

We start with $H(q, p, t)$ and we want to Legendre transform to $q, \dot{q}$. So, going from $p \to \dot{p}$, we write $\dot{q}_a = \frac{\partial H}{\partial p_a}$ which we can use to solve for p_a in terms of q_a and $\dot{q}_a$. We then substitute $p_a(q, \dot{q}, t)$ into

$$L \equiv p_a\dot{q}_a - H$$

We then write

$$dL = \frac{\partial L}{\partial \dot{q}_a}d\dot{q}_a + \frac{\partial L}{\partial q_a}dq_a + \frac{\partial L}{\partial t}dt$$

$$= dpa\dot{q}_a + pad\dot{q}_a - \frac{\partial H}{\partial p_a}dp_a - \frac{\partial H}{\partial q_a}dq_a - \frac{\partial H}{\partial t}dt$$

$$= p_a d\dot{q}_a + \dot{p}_a dq_a - \frac{\partial H}{\partial t}dt$$

where we used $\frac{\partial H}{\partial p_a} = \dot{q}_a$ and $-\frac{\partial H}{\partial q_a} = \dot{p}_a$. We can express the left-hand side as

$$\Rightarrow \left(-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_a}\right) + \frac{\partial L}{\partial q_a}\right) dq_a + \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_a} dq_a\right) + \frac{\partial L}{\partial t} dt$$

$$= \frac{d}{dt}(p_a dq_a) - \frac{\partial H}{\partial t} dt$$

We then identify

$$p_a = \frac{\partial L}{\partial \dot{q}_a}, \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}, \text{ and } \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_a}\right) = \frac{\partial L}{\partial q_a}$$

as needed. ■

★ **Problem 11.32** Suppose that for some situations the coordinates p, q are canonical. Show that the transformed coordinates $P = \frac{1}{2}(p^2 + q^2), Q = \tan^{-1}(q/p)$ are also canonical.

Solution

We know $\{q, p\} = 1$. We now check

$$\{Q, P\}_{p,q} = \left\{\tan^{-1}\frac{p}{q}, \frac{1}{2}(p^2 + q^2)\right\}_{p,q}$$

$$= \frac{\partial}{\partial q}(\tan^{-1}\frac{p}{q})\frac{\partial}{\partial p}(\frac{1}{2}(p^2 + q^2)) - \frac{\partial}{\partial p}(\tan^{-1}\frac{q}{p})\frac{\partial}{\partial q}(\frac{1}{2}(p^2 + q^2))$$

$$= \frac{1}{1 + \frac{q^2}{p^2}}(\frac{1}{p})p - \frac{1}{1 + \frac{q^2}{p^2}}(-\frac{q}{p^2})q$$

$$= \frac{p^2}{q^2 + p^2} + \frac{q^2}{q^2 + p^2} = 1$$

as needed. ■

★★ **Problem 11.33** Prove that if one makes two successive canonical transformations, the result is also canonical.

Solution

Say $q, p \to Q, P$ is canonical

$$\Rightarrow \{Q, P\}_{q,p} = 1$$

with

$$\{q, p\} = 1,$$

then $Q, P \to Q', P'$ is canonical

$$\Rightarrow \{Q', P'\}_{Q,P} = 1$$

We want to show that $q, p \to Q', P'$ is canonical or

$$\{Q', P'\}_{q,p} = 1$$

$$\{Q', P'\}_{q,p} = \frac{\partial Q'}{\partial q}\frac{\partial P'}{\partial p} - \frac{\partial Q'}{\partial p}\frac{\partial P'}{\partial q}$$

By the chain rule

$$\frac{\partial Q'}{\partial q} = \frac{\partial Q'}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial Q'}{\partial p}\frac{\partial P}{\partial q}$$

Similarly

$$\frac{\partial}{\partial p} \to \frac{\partial Q}{\partial p}\frac{\partial}{\partial Q} + \frac{\partial P}{\partial p}\frac{\partial}{\partial p}$$

We then get

$$\begin{aligned}\{Q', P'\}_{q,p} = &\left(\frac{\partial Q'}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial Q'}{\partial p}\frac{\partial P}{\partial q}\right)\left(\frac{\partial P'}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial P'}{\partial p}\frac{\partial P}{\partial p}\right)\\ &- \left(\frac{\partial Q'}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial Q'}{\partial p}\frac{\partial P}{\partial p}\right)\left(\frac{\partial P'}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial P'}{\partial P}\frac{\partial P}{\partial q}\right)\end{aligned}$$

Only the cross-terms survive:

$$\begin{aligned}\{Q', P'\}_{q,p} &= \frac{\partial Q'}{\partial Q}\frac{\partial P'}{\partial p}\left(\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}\right) + \frac{\partial Q'}{\partial P}\frac{\partial P'}{\partial Q}\left(\frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} - \frac{\partial P}{\partial p}\frac{\partial Q}{\partial q}\right)\\ &= \frac{\partial Q'}{\partial Q}\frac{\partial P'}{\partial p}\{Q, P\}_{q,p} - \frac{\partial Q'}{\partial P}\frac{\partial P'}{\partial Q}\{Q, P\}_{q,p}\\ &= \frac{\partial Q'}{\partial Q}\frac{\partial P'}{\partial p} - \frac{\partial Q'}{\partial P}\frac{\partial P'}{\partial Q} = \{Q', P'\}_{Q,P} = 1\end{aligned}$$

as needed. ■

⋆ **Problem 11.34** Prove that the Poisson bracket is invariant under a canonical transformation.

Solution

We want to show

$$\{f(p,q), g(p,q)\}_{q,p} = \{f(P,Q), g(P,Q)\}_{Q,P}$$

We start with

$$\{f, g\}_{q,p} = Det\left[\frac{\partial(f,g)}{\partial(g,p)}\right]$$

We also know that

$$Det\left[\frac{\partial(g,p)}{\partial(Q,P)}\right] = 1$$

$$\{f, g\}_{q,p} = Det\left[\frac{\partial(f,g)}{\partial(g,p)}\right] Det\left[\frac{\partial(g,p)}{\partial(Q,P)}\right] = Det\left[\frac{\partial(f,g)}{\partial(g,p)}\frac{\partial(g,p)}{\partial(Q,P)}\right],$$

since the determinant of a product of matrices is the product of the determinants. But

$$\frac{\partial(f,g)}{\partial(g,p)}\frac{\partial(g,p)}{\partial(Q,P)} = \frac{\partial(f,g)}{\partial(Q,P)}$$

by the chain rule. For example,

$$\begin{pmatrix} \frac{\partial f}{\partial q} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial q} & \frac{\partial g}{\partial p} \end{pmatrix} \begin{pmatrix} \frac{\partial q}{\partial Q} & \frac{\partial q}{\partial P} \\ \frac{\partial p}{\partial Q} & \frac{\partial p}{\partial P} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial q}\frac{\partial q}{\partial Q} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial Q} & \frac{\partial f}{\partial q}\frac{\partial q}{\partial P} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial P} \\ \frac{\partial g}{\partial q}\frac{\partial q}{\partial Q} + \frac{\partial g}{\partial p}\frac{\partial p}{\partial Q} & \frac{\partial g}{\partial q}\frac{\partial q}{\partial P} + \frac{\partial g}{\partial p}\frac{\partial p}{\partial P} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial f}{\partial Q} & \frac{\partial f}{\partial P} \\ \frac{\partial g}{\partial Q} & \frac{\partial g}{\partial P} \end{pmatrix}$$

This generalizes to $2N$ phase space coordinates by block diagonal multiplication

$$\Rightarrow \{f,g\}_{q,p} = Det\left[\frac{\partial(f,g)}{\partial(Q,P)}\right] = \{f,g\}_{Q,P}\,.$$

$\Rightarrow$ The Poisson bracket is preserved by canonical transformations. ■

⋆⋆ **Problem 11.35** A plane pendulum consists of a rod of length R and negligible mass supporting a plumb bob of mass m that swings back and forth in a uniform gravitational field g. The point of support at the top end of the rod is forced to oscillate vertically up and down with $y = A\cos\omega t$. Using the angle θ of the rod from the vertical as the coordinate, (a) find the Lagrangian of the bob. (b) Find the Hamiltonian H. Is $H = E$, the energy? Is either one or both conserved? (c) Write out Hamilton's equations of motion.

Solution

(a) The Lagrangian is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

However, we have

$$x = R\sin\theta \text{ and } y = Y - R\cos\theta$$

where Y is the vertical location of the anchor.

$$Y = A\cos(\omega t) \text{ given } \Rightarrow \dot{x} = R\cos\theta\,\dot{\theta}, \quad \dot{y} = \dot{Y} + R\sin\theta\,\dot{\theta} = -A\omega\sin(\omega t) + R\sin\theta\,\dot{\theta}$$

We then have

$$L = \frac{1}{2}m(R^2\dot{\theta}^2 + A^2\omega^2\sin^2(\omega t) - 2A\omega R\sin(\omega t)\sin\theta\,\dot{\theta}) - mg(A\cos(\omega t) - R\cos\theta)$$

(b) To find the Hamiltonian, we Legendre transform such that $\dot{\theta} \to p_\theta$

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = mR^2\dot{\theta} - mA\omega R\sin(\omega t)\sin\theta$$

$$\Rightarrow \dot{\theta} = \frac{p_\theta}{mR^2} + \frac{A\omega}{R}\sin(\omega t)\sin\theta$$

$$H = \dot{\theta}p_\theta - L = \frac{mR^2}{2}\left(\frac{p_\theta}{mR^2} + \frac{A\omega}{R}\sin(\omega t)\sin\theta\right)^2 - \frac{1}{2}mA^2\omega^2\sin^2(\omega t) + mg(A\cos(\omega t) - R\cos\theta)$$

$$= \frac{p_\theta^2}{2mR^2} - \frac{mA^2\omega^2}{2}\sin^2(\omega t)\cos^2\theta + \frac{A\omega}{R}\sin(\omega t)\sin\theta p_\theta + mg(A\cos(\omega t) - R\cos\theta)$$

We have

$$\frac{\partial H}{\partial t} \neq 0 \Rightarrow \frac{\partial H}{\partial t} \neq 0$$

So H is not conserved. Also, $H \neq E$ as an external agent inputs energy into system.

(c) The equations of motion for θ, p_θ are:

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mR^2} + \frac{A\omega}{R}\sin(\omega t)\sin\theta$$

and

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = mA^2\omega^2\sin^2(\omega t)\cos\theta\sin\theta + \frac{A\omega}{R}\sin(\omega t)\cos\theta p_\theta + mgR\sin\theta$$

■

★★ **Problem 11.36** A plane pendulum consists of a string supporting a plumb bob of mass m free to swing in a vertical plane and free to swing subject to uniform gravity g. The upper end of the string is threaded through a hole in the ceiling and steadily pulled upward, so the length of the string beneath the point in the ceiling is $\ell(t) = \ell_0 - \alpha t$, where α is a positive constant. (a) Find the Lagrangian of the plumb bob. (b) Find its Hamiltonian H. Is $H = E$, the energy of the bob? (c) Write out Hamilton's equations of motion. (d) Solve them assuming $l(t)$ is changing slowly and the angle of the pendulum remains small.

Solution

(a) The Lagrangian is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

with

$$x = \ell(t)\sin\theta \quad y = -\ell(t)\cos\theta$$

with

$$\ell(t) = \ell_0 - \alpha t$$

$$\Rightarrow \dot{x} = \ell\sin\theta + \ell\cos\theta\,\dot{\theta} = -\alpha\sin\theta + \ell\cos\theta\,\dot{\theta}$$

$$\dot{y} = \alpha\cos\theta + \ell\sin\theta\,\dot{\theta}$$

$$\Rightarrow L = \frac{1}{2}m(\ell^2\cos^2\theta\,\dot{\theta}^2 + \alpha^2\sin\theta - 2\alpha\ell\sin\theta\cos\theta\,\dot{\theta} + \ell^2\sin^2\theta\,\dot{\theta}^2$$
$$+ \alpha^2\cos^2\theta + 2\alpha\ell\sin\theta\cos\theta\,\dot{\theta}) + mg\ell\cos\theta$$

$$= \frac{1}{2}m(\ell^2\dot{\theta}^2 + \alpha^2) + mg\ell\cos\theta$$

(b) The Hamiltonian is given in terms of θ and p_θ

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta} \Rightarrow \dot{\theta} = p_\theta/m\ell^2$$

$$\Rightarrow H = \dot{\theta} p_\theta - L = \frac{p_\theta^2}{m\ell^2} - \frac{1}{2} m\left(\frac{p_\theta^2}{m^2\ell^2} + \alpha^2\right) - mg\ell\cos\theta$$

$$= \frac{p_\theta^2}{2m\ell^2} - \frac{1}{2} m\alpha^2 - mg\ell\cos\theta$$

Note that $\frac{\partial H}{\partial t} \neq 0$ since $\ell = \ell_0 - \alpha t \Rightarrow \frac{\partial H}{\partial t} \neq 0 \Rightarrow$ H is not conserved.

$$E = \frac{p_\theta^2}{2m\ell^2} + \frac{1}{2} m\alpha^2 - mg\ell\cos\theta \neq H$$

(c) The equations of motion for θ and p_θ are

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m\ell^2}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mg\ell\sin\theta = -mg(\ell_0 - \alpha t)\sin\theta$$

To decouple the equations, look at

$$\ddot{\theta} = \frac{\dot{p}_\theta}{m\ell^2} - \frac{2p_\theta}{m\ell^3}\dot{\ell} = \frac{1}{m\ell^2}(-mg(\ell_0 - \alpha t)\sin\theta) + \frac{2\alpha}{m\ell^3} m\ell^2\dot{\theta} = -\frac{g}{\ell(t)}\sin\theta + \frac{2\alpha}{\ell(t)}\dot{\theta}$$

with

$$\ell(t) = \ell_0 - \alpha t$$

$$\Rightarrow \ddot{\theta} - \frac{2\alpha}{\ell(t)}\dot{\theta} + \frac{g}{\ell(t)}\sin\theta = 0$$

Notice that the changing length of the pendulum has a damping effect on the motion.

(d) If we assume $\ell(t)$ changes little over a period of small oscillations where $\sin\theta \sim \theta$, we have

$$\ddot{\theta} - \frac{2\alpha}{\ell}\dot{\theta} + \frac{g}{\ell}\theta = 0$$

which implies oscillator motion with angular frequency $\omega \cong \sqrt{g/\rho}$ and damping parameter $2\alpha/\rho$. ■

★★ **Problem 11.37** At time $t = 0$ a large number of particles, each of mass m, is strung out along the x axis from $x = 0$ to $x = \Delta x$, with momenta p_x varying from $p = p_0$ to $p = p_0 + \Delta p$. No forces act on the particles and they do not collide. (a) Show that the points representing these particles fill a rectangle in the x, p_x phase plane, and sketch it identifying the four points at the corners of the rectangle with their positions and momenta. (b) Sketch the locations of the same particles in the phase plane some time t_1 later, where $t_1 > mx_0/p_0$. (c) What then is the shape of the area on the phase plane occupied by all of these particles? (d) Prove that the area of the occupied region at t_1 is the same as it was at $t = 0$. (Note that if the number of points and the area are both unchanged, then the average density of points is also unchanged, in accord with the Liouville theorem.)

Solution

(a) The particles evolve along straight lines

$$x = x_0' + \frac{p_0'}{m}t \text{ where } x_0' \in (0, \Delta x) \text{ and } p_0' \in (p_0, p_0 + \Delta p)$$

while

$$p_x = p_0' = \text{constant}$$

The corners of the region in the $x - p^x$ plane is:

$$\left(\frac{p_0 t}{m}, p_0\right)$$

$$\left(\Delta x + \frac{p_0 t}{m}, p_0\right)$$

$$\left(\frac{(p_0 + \Delta p)t}{m}, p_0 + \Delta p\right)$$

$$\left(\Delta x + \frac{(p_0 + \Delta p)t}{m}, p_0 + \Delta p\right)$$

At $t = 0$, this is a rectangle.

(b) At $t = t_1 > mx_0/p_0$, we have the corners:

$$\left(\frac{p_0 t}{m}, p_0\right)$$

$$\left(\Delta x + \frac{p_0 t}{m}, p_0\right)$$

$$\left(\frac{(p_0 + \Delta p)t}{m}, p_0 + \Delta p\right)$$

$$\left(\Delta x + \frac{(p_0 + \Delta p)t}{m}, p_0 + \Delta p\right)$$

(c) The shape at t_1 is a parallelogram.

(d) The area of the parallelogram at $t = t_1$ is $\Delta p \Delta x$ as expected. ■

★★★ **Problem 11.38** Any spherically symmetric function of the canonical coordinate and momentum of a particle can depend only on r^2, p^2, and $\mathbf{r} \cdot \mathbf{p}$. Show that the Poisson bracket of any such function f with a component of the particle's angular momentum is zero. In particular, show that $\{L_z, f\} = 0$, where $L_z = (\mathbf{r} \times \mathbf{p})_z$.

Solution

$$f(n^2, p^2, \mathbf{r} \cdot \mathbf{p}) = f(a, b, c)$$

$$L_z = xp^y - yp^x$$

$$\{L_z, f\} = \{xp^y - yp^x, f\}$$

$$= \{xp^y, f\} - \{yp^x, f\} = -\{f, xp^y\} + \{f, yp^x\}$$

$$= -\{f, x\} p^y - x\{f, p^y\} + \{f, y\} p^x + y\{f, p^x\}$$

$$= +\frac{\partial f}{\partial p^x} p^y - x\frac{\partial f}{\partial y} - \frac{\partial f}{\partial p^y} p^x + y\frac{\partial f}{\partial x}$$

But we have

$$\frac{\partial f}{\partial p^x} = \frac{\partial f}{\partial b}\frac{\partial b}{\partial p^x} + \frac{\partial f}{\partial c}\frac{\partial c}{\partial p^x} = \frac{\partial f}{\partial b} 2p^x + \frac{\partial f}{\partial c} x$$

$$\frac{\partial f}{\partial p^y} = \frac{\partial f}{\partial b} 2p^y + \frac{\partial f}{\partial c} y$$

and

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial a}\frac{\partial a}{\partial x} + \frac{\partial f}{\partial c}\frac{\partial c}{\partial x} = \frac{\partial f}{\partial a} 2x + \frac{\partial f}{\partial c} p^x$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial a} 2y + \frac{\partial f}{\partial c} p^y$$

We then have

$$\{L_z, f\} = p^y\left(\frac{\partial f}{\partial b} 2p^x + \frac{\partial f}{\partial c} x\right) - x\left(\frac{\partial f}{\partial a} 2y + \frac{\partial f}{\partial c} p^y\right) - p^x\left(\frac{\partial f}{\partial b} 2p^y + \frac{\partial f}{\partial c} y\right) + y\left(\frac{\partial f}{\partial a} 2x + \frac{\partial f}{\partial c} p^x\right)$$

$$= \frac{\partial f}{\partial b}(2p^x p^y - 2p^x p^y) + \frac{\partial f}{\partial a}(-2xy + 2xy) + \frac{\partial f}{\partial c}(p^y x - xp^y - p^x y + yp^x) = 0$$

as needed. ■

★★ **Problem 11.39** Write the Hamiltonian of a free particle of mass m in a reference frame that is rotating uniformly with angular velocity $\boldsymbol{\omega}$ with respect to an inertial frame.

Solution

$$L = \frac{1}{2} m\dot{\mathbf{r}}^2 = \frac{1}{2} m\mathbf{v}_{in}^2$$

But from Chapter 9, we have

$$\mathbf{v}_{in} = \mathbf{v}_{rot} + \boldsymbol{\omega} \times \mathbf{r}$$

$$\Rightarrow L = \frac{1}{2} m(\mathbf{v}_{rot} + \boldsymbol{\omega} \times \mathbf{r})^2 = \frac{1}{2} m(\mathbf{v}_{rot}^2 + 2\mathbf{v}_{rot} \cdot (\boldsymbol{\omega} \times \mathbf{r}) + (\boldsymbol{\omega} \times \mathbf{r}) \cdot (\boldsymbol{\omega} \times \mathbf{r}))$$

$$= \frac{1}{2} m\mathbf{v}_{rot}^2 + m\mathbf{v}_{rot} \cdot (\boldsymbol{\omega} \times \mathbf{r}) + \frac{1}{2} m\omega^2 r^2 - \frac{1}{2} m(\boldsymbol{\omega} \cdot \mathbf{r})^2$$

where we used

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d})$$

We write $\mathbf{v}_{rot} = (\dot{x}, \dot{y}, \dot{z})$. We then have

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + m(\boldsymbol{\omega} \times \mathbf{r})^x$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} + m(\boldsymbol{\omega} \times \mathbf{r})^y$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} + m(\boldsymbol{\omega} \times \mathbf{r})^z$$

$$\Rightarrow \mathbf{p}_{rot} = m\mathbf{v}_{rot} + m(\boldsymbol{\omega} \times \mathbf{r})$$

$$\Rightarrow H = \dot{x}p_x + \dot{y}p_y + \dot{z}p_z - L = m\mathbf{v}_{rot}^2 + m\mathbf{v}_{rot} \cdot (\boldsymbol{\omega} \times \mathbf{r}) - \frac{1}{2}m\mathbf{v}_{rot}^2 - m\mathbf{v}_{rot} \cdot (\boldsymbol{\omega} \times \mathbf{r})$$
$$- \frac{1}{2}m\omega^2 r^2 + \frac{1}{2}m(\boldsymbol{\omega} \cdot \mathbf{r})^2$$

$$= \frac{1}{2}m\mathbf{v}_{rot}^2 - \frac{1}{2}m\omega^2 r^2 + \frac{1}{2}m(\boldsymbol{\omega} \cdot \mathbf{r})^2$$

with

$$\mathbf{v}_{rot} = \frac{\mathbf{p}_{rot}}{m} - (\boldsymbol{\omega} \times \mathbf{r})$$

$$\Rightarrow H = \frac{1}{2m}(\mathbf{p}_{rot} - m(\boldsymbol{\omega} \times \mathbf{r}))^2 - \frac{1}{2}m\omega^2 r^2 + \frac{1}{2}m(\omega \cdot \mathbf{r})^2$$

■

12 Rigid-Body Dynamics

12.1 Problems and Solutions

⋆ **Problem 12.1** Three objects, starting from rest at the same altitude, roll without slipping down an inclined plane. One is a ring of mass M and radius R; another is a uniform-density disk of mass $2M$ and radius R, and the last is a uniform-density sphere of mass M and radius $2R$. In what order do they reach the bottom of the inclined plane?

Solution

All three objects have moments of inertia of the form $I = kmr^2$ where $k = 1$ (ring) $k = \frac{1}{2}$ (disk) $k = 2/5$ (sphere). In rolling to the bottom, all lose the same gravitational potential energy per unit mass, so each gains the same kinetic energy/mass, shared between translational and rotational. So

$$\frac{mgh}{m} = \frac{\frac{1}{2}mv^2 + \frac{1}{2}(kmr^2\omega^2)}{m} \Rightarrow gh = \frac{1}{2}(v^2 + kv^2)$$

since $v = r\omega$ for rolling without slipping. So $v = \sqrt{\frac{2gh}{1+k}}$ for each. The larger k is, the smaller v is, so the longer that object takes to roll to the bottom. So any sphere beats any disk, and any disk beats any ring. The sphere is first, the disk is second, the ring is last. ■

⋆⋆ **Problem 12.2** A cylindrical pole is inserted into a frozen lake so the pole stands vertically. One end of a rope is attached to a point on the surface of the pole near where it enters the ice, and the rope is then laid out in a straight line on the surface. An ice skater with initial velocity $\mathbf{v_0}$ approaches the opposite end of the rope, moving perpendicular to the rope. As she reaches the rope she grabs it and holds on, so the rope winds up around the pole. (a) When the rope is half wound up, what is her speed, assuming there is no friction between the rope or herself and the ice? (b) Has there been any change in her kinetic energy ? If so, identify what positive or negative work has been done on her, and by what source. (c) Has there been any change in her angular momentum? If so, identify the source of the torque done upon her.

Solution

If angular momentum ℓ is conserved, $\ell = mvr$ means that if r is cut in half, v must double to $2v_0$. If energy is conserved, $E = \frac{1}{2}mv^2$ (there is no potential energy here) so then $v = v_0$.

Both cannot be true. The answer is $v = v_0$ by energy conservation, since there is no friction and nothing is doing work on the skater. (The skater, just holding on, does no work).

Angular momentum is *not* conserved, because as the rope winds up, the pole exerts a torque tending to reduce the angular momentum. That is, the torque $\mathbf{N} = \mathbf{r}_{pole} \times (m_{\text{skate}} \mathbf{v}_{\text{skate}}) = d\ell/dt$. (Note that as the rope winds up, it pulls on the skater in a direction that is not quite towards the center of the pole, but always toward a surface of the pole. So the force on the skater is not quite central.) ■

⋆ **Problem 12.3** Humanity collectively uses energy at the average rate of about 18 Terawatts. (a) At that rate, after one year how long would the length of the day have increased if during that year we were able to power our activities purely by harnessing the rotational kinetic energy of the earth? (The earth has moment of inertia $I = 0.33MR^2$ about the north-south polar axis.) (b) The moment of inertia of a sphere is often given as $I = (2/5)MR^2$. What is the primary reason why this is incorrect for the earth?

Solution

(a) The energy humans use in one year is

$$18 \times 10^{12} \frac{\text{J}}{\text{s}} \cdot 1 \text{ year} = 18 \times 10^{12} \frac{\text{J}}{\text{s}} \cdot 3.16 \times 10^7 \text{ s} = 5.7 \times 10^{20} \text{J}.$$

The energy in earth's rotation is

$$E = \frac{1}{2} I_{earth} \omega^2 = \frac{1}{2}(0.33MR^2)(2\pi/T)^2 = 2\pi^2(0.33)MR^2/T^2.$$

A slight change in period dT changes E by

$$\Delta E = -2\pi^2(0.33)MR^2(2\Delta T/T^3) = 5.7 \times 10^{20} \text{ J}.$$

$$\Delta T = -11.5 \times 10^{-5} \text{ s} \approx 10^{-4} \text{ s}$$

using

$$M = 6.0 \times 10^{24} \text{ kg} \qquad R = 6,400 \text{ km} \qquad T = 24 \text{ hrs}$$

(b) The density of the earth is larger in the core than near the surface. ■

⋆ **Problem 12.4** In a supernova explosion, the core of a heavy star collapses and the outer layers are blown away. Before collapse, suppose the core of a given star has twice the mass of the sun and the same radius as the sun, and rotates with period 20 days. The core collapses in a few seconds to become a neutron star of radius 20 km. (a) Estimate its new period of rotation. (b) Estimate the ratio of the final rotational kinetic energy to the initial rotational kinetic energy of the core. What could account for the change?

Solution

(a) The angular momentum of the core remains constant if we assume it does not interact appreciably with the outer layers. So

$$\ell = I_0 \omega_0 = I_f \omega_f$$

or

$$I_0/T_0 = I_f/T_f$$

since $\omega = 2\pi/T$, where T =period. So

$$T_f = T_0(I_f/I_0) = T_0(\frac{R_f}{R_0})^2 = 20 \text{ days}(\frac{20 \text{ km}}{7\times10^5 \text{ km}})^2 = 1.4\times10^{-3} \text{ s} \sim 1 \text{ ms} \text{ to one sig fig.}$$

(b)

$$\frac{K_f}{K_i} - \frac{\frac{1}{2}I_f\omega_f^2}{\frac{1}{2}I_0\omega_0^2} = (\frac{R_f}{R_0})^2(\frac{T_0}{T_f})^2 \sim 2.4\times10^9.$$

The core collapses, so loses gravitational potential energy. ■

★ **Problem 12.5** In some theoretical models of pulsars, which are rotating neutron stars, the braking torque slowing the pulsar's spin rate is proportional to the n^{th} power of the pulsar's angular velocity Ω; that is, $\dot{\Omega} = -K\Omega^n$, where K is a constant. (a) Find a formula for the time rate of change of the pulsar period $\dot{P}$ in terms of P itself and the constants n and K. (b) For the Vela and Crab pulsars, at least, the product $P\dot{P}$ = constant. What is their braking index n?

Solution

(a)

$$P = \frac{2\pi}{\Omega},$$

so

$$\dot{P} = -\frac{2\pi}{\Omega^2}\dot{\Omega} = \frac{2\pi}{\Omega^2}k\Omega^n = 2\pi k\Omega^{n-2}$$

$$\dot{P} = 2\pi k\left(\frac{2\pi}{P}\right)^{n-2} = (2\pi)^{n-1}kP^{2-n}$$

(b)

$$P\dot{P} = (2\pi)^{n-1}kp^{3-n} = \text{constant if } n = 3$$

■

★★ **Problem 12.6** Tidal effects of the moon on the earth have caused the earth's rotation rate to slow, thus reducing the spin angular momentum of the earth leading to an increase in earth's day by 0.1 s in the past 3800 years. This reduction has been made up for by an increase in the orbital angular momentum of the moon around the earth. Therefore how much farther is the moon now from the earth than it was 3800 years ago? (Note that the moment of inertia of the earth about its axis is $I = 0.33MR^2$ and that the mean earth-moon distance is 3.8×10^5 km.)

Solution

The change in earth's spin angular momentum is

$$\Delta S = \Delta(I\omega) = I\Delta\omega.$$

Here

$$\Delta\omega = \Delta\left(\frac{2\pi}{T}\right) = -\frac{2\pi}{T^2}\Delta T$$

for small changes in period T. So

$$\Delta S = -\frac{2\pi I}{T^2}\Delta T.$$

The orbital angular momentum of the moon about the earth is

$$\ell = rM_{\text{moon}}v_{\text{moon}} \Rightarrow \Delta\ell = \Delta(rMv)_{\text{moon}},$$

assuming a circular orbit. Here v and r are related by

$$\frac{GM_{\text{earth}}m_{\text{moon}}}{r^2} = \frac{m_{\text{moon}}v^2}{r} \Rightarrow v = \sqrt{GM_{\text{earth}}/r}, so$$

$$\ell = M_{\text{moon}}\sqrt{GM_{\text{earth}}r} \Rightarrow \Delta\ell = M_{\text{moon}}\sqrt{GM_{\text{earth}}}\frac{1}{2}r^{-1/2}\Delta r = \frac{1}{2}M_{\text{moon}}\sqrt{\frac{GM_{\text{earth}}}{r}}\Delta r.$$

Conserving overall angular momentum,

$$\Delta S + \Delta\ell = 0 \Rightarrow \Delta r = \frac{4\pi}{T^2}\frac{(0.33)M_{\text{earth}}R_e^2\Delta T}{M_{\text{moon}}\sqrt{GM_{\text{earth}}/r}} \approx 180 \text{ m}$$

Using

$$R_e = 6370 \text{ km} \quad M_{\text{earth}} = 6.0\times10^{24} \text{ kg} \quad M_{\text{moon}} = 7.35\times10^{22} \text{ kg} \quad r = 3.8\times10^8 \text{ m}$$

■

⋆ **Problem 12.7** Compute the moment of inertia matrix of a solid circular cylinder of height H and base radius R, and of uniform mass density $\rho = \rho_0$. In this expression, the cylinder is arranged so that its symmetry axis is along the z axis and its top cap sits on the x, y plane; *i.e.*, the cylinder extends from $z = -H$ to $z = 0$. Compute all entries of the moment of inertia matrix with respect to the origin in this configuration.

Solution

Slice the cylinder into horizontal disks. Each disk has radius R and thickness Δz_1, and has $\Delta I_{zz} = \frac{1}{2}\Delta mR^2$ where $\Delta m = \rho_0\Delta z\pi R^2$. The contributions of each disk to the total

$$I_{zz} = \sum\Delta I_{zz} = \frac{1}{2}\int_{z=-H}^{0} dmR^2 = \rho_0\pi R^2\int_{z=-H}^{0} dz \quad I_{zz} = \frac{\rho_0}{2}\pi R^4 H.$$

The total mass of the cylinder is

$$M = \rho_0 V \quad n = \rho_0\cdot H\cdot\pi R^2.$$

So

$$I_{zz} = \frac{1}{2}\frac{M}{\pi R^2 H}\times\pi R^4 H = \frac{1}{2}MR^2.$$

Now to find I_{xx} and I_{yy}, again consider thin disks. For each thin disk,

$$\Delta I_{xx} = \frac{1}{4}\Delta m R^2$$

for an x axis in the plane of the disk itself. For the axis through the top face of the cylinder,

$$\Delta I_{xx} = \frac{1}{2}\Delta m R^2 + \Delta m|z|^2$$

by the perpendicular axis theorem for thin lamina. So

$$I_{xx}{}^{tot} = \sum \Delta m\left(R^2/4 + |z|^2\right) = \sum \rho_0 \Delta z \pi R^2 \left(\frac{R^2}{4} + z^2\right)$$

$$I_{xx} = \int_{-H}^{0} \rho_0 \pi R^2 dz \left(\frac{R^2}{4} + z^2\right) = \rho_0 \pi R^2 \left[\frac{R^2}{4} z\Big|_{-H}^{0} + \frac{z^3}{3}\Big|_{-H}^{0}\right]$$

$$= \rho_0 \pi R^2 \left[\frac{R^2}{4} H + \frac{H^3}{3}\right] = \left(\frac{M}{\pi R^2 H}\right) \pi R^2 \left(\frac{R^2}{4} H + \frac{H^3}{3}\right)$$

$$= M\left(\frac{R^2}{4} + \frac{H^2}{3}\right) = \frac{1}{4} M R^2 + \frac{1}{3} M H^2$$

By symmetry

$$I_{yy} = I_{xx}$$

so

$$I_{xx} = I_{yy} = \frac{1}{4} M R^2 + \frac{1}{3} M H^2 \qquad I_{zz} = \frac{1}{2} M R^2.$$

The off-diagonal elements are zero, since (say)

$$I_{xy} = \int dm(-xy) = 0,$$

because for every Δm with positive xy, there is another Δm with negative xy. The same goes for I_{xz} and I_{yz}. So altogether,

$$\begin{pmatrix} \frac{1}{2} M R^2 + \frac{1}{3} M H^2 & 0 & 0 \\ 0 & \frac{1}{2} M R^2 + \frac{1}{3} M H^2 & 0 \\ 0 & 0 & \frac{1}{2} M R^2 \end{pmatrix}$$

■

★★ **Problem 12.8** A rod of length ℓ and mass m is attached to a pivot on one end. The rim of a disc of radius R and mass M is attached to its other end in such a way that the disc can pivot in the same plane in which the rod is restricted to swing. Find the Lagrangian and equations of motion.

Solution

The moment of inertia about the pivot is

$$I_{\text{rod}} + I_{\text{disk}} = \frac{1}{3}m\ell^2 + \left(\frac{1}{2}mR^2 + M(\ell + R)^2\right)$$

where

$$I_{\text{disk}} = \frac{1}{2}MR^2 + M(\ell + R)^2$$

by the parallel-axis theorem, since $\frac{1}{2}MR^2$ is the moment of inertia of the disk about its center, and $M(\ell + R)^2$ is what needs to be added to get I_{disk} about the pivot. Therefore the kinetic energy is

$$T = \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}\left(\frac{1}{3}m\ell^2 + \frac{1}{2}MR^2 + M(\ell + R)^2\right)\dot{\theta}^2$$

The potential energy measured from the altitude of the pivot is

$$U = (-mg\ell/2 - Mg(\ell + R))\cos\theta.$$

So the Lagrangian is

$$L = T - U = \frac{1}{2}\left(\frac{1}{3}m\ell^2 + \frac{1}{2}MR^2 + M(\ell + R)^2\right)\dot{\theta}^2 + \left(\frac{mg\ell}{2} + Mg(\ell + R)\right)\cos\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = \left(\frac{1}{3}m\ell^2 + \frac{1}{2}MR^2 + M(\ell + R)^2\right)\dot{\theta} \quad , \quad \frac{\partial L}{\partial \theta} = -\left(\frac{mg\ell}{2} + Mg(\ell + R)\right)\sin\theta$$

Lagrange's equation is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

So

$$\left(\frac{1}{3}m\ell^2 + \frac{1}{2}MR^2 + M(\ell + R)^2\right)\ddot{\theta} + \left(\frac{mg\ell}{2} + Mg(\ell + R)\right)\sin\theta = 0$$

Check in the case $R = 0$:

$$\left(\frac{1}{3}m + M\right)\ell^2\ddot{\theta} + (\frac{1}{2}m + M)g\ell\sin\theta = 0.$$

If also $m = 0$,

$$\ddot{\theta} + g/\ell\sin\theta = 0$$

■

★★ **Problem 12.9** (a) Using Euler angles, write the constraint of rolling without slipping for a sphere of radius R moving on a flat surface. (b) Write the Lagrangian and equations of motion using Lagrange multipliers. (c) Show that the rotational and translational kinetic energies are independently conserved.

Solution

(a) Rolling without slipping corresponds to the point of contact with the ground moving with the spin. In the body frame, let the position vector from the center of mass to the point of contact be $\mathbf{r} = -R\hat{\mathbf{z}}'$, where R is the radius of the wheel, and $\hat{\mathbf{z}}'$ is the vertical unit vector in the body frame. We want $\boldsymbol{\omega} \times \mathbf{r} = \mathbf{v}_p$, expressed in the body frame, where $\mathbf{v}_p$ is the velocity of the point of contact in the $x' - y'$ plane

$$\Rightarrow \mathbf{v}_p = \dot{x}'\hat{\mathbf{x}}' + \dot{y}'\hat{\mathbf{y}}'$$

We then have

$$(\dot{\varphi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi, \dot{\varphi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi, \dot{\varphi}\cos\theta + \dot{\varphi})$$

$$x(0,0,-R) = (\dot{x}'.\dot{y}',0) = (-\dot{x}, -\dot{y}, 0)$$

where we used the fact that a displacement of the point of contact with velocity $(\dot{x}', \dot{y}', 0)$ in the body frame corresponds to a velocity in the opposite direction in the lab frame

$$\Rightarrow R(\dot{\varphi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi) = -\dot{x} \qquad \text{(constraint 1)}$$

$$-R(\dot{\varphi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi) = -\dot{y} \qquad \text{(constraint 2)}$$

We also need

$$\dot{\varphi}\cos\theta + \dot{\varphi} = 0$$

so that the object does not spin about the point of contact. (constraint 3)

(b) We now write the Lagrangian

$$L = T_{\text{rot}} + T_{\text{trans}} = \frac{1}{2}I((\omega^x)^2 + (\omega^y)^2 + (\omega^z)^2 + \frac{1}{2}M(\dot{x}^2 + \dot{y}^2))$$

$$= \frac{1}{2}I(\dot{\theta}^2 + \dot{\varphi}^2 + \dot{\psi}^2 + 2\dot{\psi}\dot{\varphi}\cos\theta) + \frac{1}{2}M(\dot{x}^2 + \dot{y}^2)$$

for 5 variables $(\theta, \varphi, \psi, x, y)$ with 3 Lagrange multipliers $\lambda_1, \lambda_2, \lambda_3$ for the 3 constraints from (a). The constraint coefficients are read off from (a):

$$a_{1x} = 1 \quad a_{1\varphi} = R\sin\theta\sin\psi \quad a_{1\theta} = R\cos\psi \quad a_{2y} = 1$$

$$a_{2\varphi} = -2R\sin\theta\cos\psi \quad a_{1\theta} = R\sin\psi \quad a_{3\varphi} = \cos\theta \quad a_{3\psi} = +1$$

with all other $a's$ zero.

The equations of motion become:

$$x: \quad M\ddot{x} = \lambda_1$$

$$y: \quad M\ddot{y} = \lambda_2$$

$$\theta: \quad I\ddot{\theta} = -I\dot{\psi}\dot{\varphi}\sin\theta + \lambda_1 R\cos\psi + \lambda_2 R\sin\psi$$

$$\varphi: \quad \frac{d}{dt}(I\dot{\varphi} + I\dot{\psi}\cos\theta) = \lambda_1 R\sin\theta\sin\psi - \lambda_2 R\sin\theta\cos\psi$$

$$\psi: \quad \frac{d}{dt}(I\dot{\psi} + I\dot{\varphi}\cos\theta) = \lambda_3$$

(c) Let's implement the constraints directly to check energy conservation (without the baggage of multipliers): Notice that, from the constraints

$$\dot{x}^2 + \dot{y}^2 = R^2\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2R^2$$

and

$$\dot{\psi} = -\dot{\varphi}\cos\theta$$

So, we can eliminate x, y and ψ easily:

$$L=\frac{1}{2}I(\dot{\theta}^2+\dot{\varphi}^2+\dot{\varphi}^2\cos^2\theta-2\dot{\varphi}^2\cos^2\theta)+\frac{1}{2}MR^2(\dot{\varphi}^2\sin^2\theta+\dot{\theta}^2)=\frac{1}{2}(I+MR^2)(\dot{\theta}^2+\dot{\varphi}^2\sin^2\theta)$$

The Hamiltonian is then given by

$$p_\theta = (I+MR^2)\dot{\theta} \quad p_\varphi = (I+MR^2)\dot{\varphi}\sin^2\theta$$

$$H = \dot{\theta}p_\theta + \dot{\varphi}p_\varphi - L = \frac{1}{2}(I+MR^2)(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) = \text{constant} = L$$

But this implies

$$\dot{x}^2 + \dot{y}^2 = \text{constant} \Rightarrow T_{\text{trans}} = \text{constant} \Rightarrow T_{\text{rot}} = \text{constant. respectively.}$$

$$L = T_{\text{trans}} + T_{\text{rot}} = \text{constant.}$$

■

⋆ **Problem 12.10** (a) Find the moment of inertia I_{zz} of a thin disk of mass m and radius R about an axis through its center and perpendicular to the plane of the disk. (b) What are I_{xx} and I_{yy} in this case? (c) A solid cylinder of mass M, radius R, and length L can be considered to be a stack of disks. Use the parallel axis theorem to help find the principal moments of inertia of the cylinder whose origin is at the center of the cylinder.

Solution

(a) We have

$$I_{zz} = \int_{r=0}^{R} dmr^2 = \sigma \int_0^R dr(2\pi r)r^2$$

where

$$\sigma = \frac{\text{mass}}{\text{area}} = \frac{M}{\pi R^2}$$

so

$$I_{zz} = (\frac{M}{\pi R^2})2\pi \int_0^R drr^3 = \frac{2M}{R^2}\frac{R^4}{4} = \frac{1}{2}MR^2$$

(b) By the perpendicular axis theorem

$$I_{xx} + I_{zz} = I_{zz}$$

However, by symmetry $I_{xx} = I_{yy}$. So

$$I_{xx} = I_{yy} = \frac{I_{zz}}{2} = \frac{1}{4}MR^2$$

(c) For each disk

$$\Delta I_{zz} = \frac{1}{2}\Delta MR^2.$$

So for the entire cylinder

$$I_{zz} = \frac{1}{2}\sum \Delta mR^2 = \frac{1}{2}MR^2$$

For a disk at $z = 0$,

$$\Delta I_{xx} = \frac{1}{4}\Delta mR^2.$$

By the parallel-axis theorem, a disk a distance z from the origin has

$$\Delta I_{xx} = \frac{1}{4}\Delta mR^2 + \Delta mz^2 = \rho\left[\frac{1}{4}(\pi R^2)\Delta zR^2 + \pi R^2\Delta zz^2\right]$$

$$= \rho \cdot \pi R^2(\frac{1}{4}R^2\Delta z + z^2\Delta z)$$

$$I_{xx} = (\frac{M}{\pi R^2 L})\pi R^2 \int_{-L/2}^{L/2} (\frac{1}{4}R^2 + z^2)dz = \frac{M}{L}\left[\frac{1}{4}R^2 L + z^3/3\Big|_{-L/2}^{L/2}\right]$$

$$I_{xx} = I_{yy} = \frac{1}{4}MR^2 + \frac{M}{3L}((\frac{L}{2})^3 - (-\frac{L}{2})^3) = \frac{1}{4}MR^2 + \frac{ML^2}{12}$$

■

⋆ **Problem 12.11** A private plane has a single propeller in front, which rotates in the clockwise sense as seen by the pilot. Flying horizontally, the pilot causes the tail rudder to extend out to the left from the plane's flight direction. (a) If the plane is ultralight and the propeller is large, heavy, and rotates fast, what is the primary response of the plane? (b) If instead the plane is heavy and the propeller is small, light, and rotates slowly, what then is the plane's primary response?

Solution

(a) Look down upon the plane from above, with the plane moving towards the right. Then if the plane is ultralight we can ignore it and concentrate on the response of the propeller alone. The torque $\mathbf{N}$ on it (due to the force on the rudder) is directed out of the page: that is, (from $\mathbf{N} = d\boldsymbol{\ell}/dt$) the $\Delta\boldsymbol{\ell}$ is out of the page. The subsequent $\boldsymbol{\ell} = \boldsymbol{\ell}_0 + \Delta\boldsymbol{\ell}$ forces the nose of the airplane up, out of the page.

(b) If the plane is heavy and the propeller is light, we ignore the response of the propeller. The torque is still directed out of the page, forcing the plane to begin rotating CCW. That is, the plane turns leftward as seen by the pilot. ■

★★ **Problem 12.12** A uniform-density cone has mass M, base radius R, and height H. Find its inertia matrix if the origin is at the center of the circular base in the x, y plane, the axis of symmetry is along the z axis, and the apex of the cone is at positive z.

Solution

Slice the cone into horizontal disks of thickness Δz and radius r, where

$$r = \frac{R}{H}(H - z)$$

The disk has moment of inertia

$$\Delta I_{zz} = \frac{1}{2}\Delta m r^2 = \frac{1}{2}\rho\pi r^2 \Delta z \cdot r^2$$

where

$$\rho = \frac{M}{\text{volume}} = \frac{M}{\frac{1}{3}(\pi R^2)H}$$

$$\Delta I_{zz} = \frac{\pi}{2}\rho \Delta z r^4$$

so the total I_{zz} for the cone is

$$I_{zz} = \frac{\pi}{2}\rho \int dz r^4 = \frac{\pi}{2}\rho \int_0^H dz (\frac{R}{H})^4 (H - z)^4 = \frac{\pi}{2}\frac{M}{\frac{1}{3}(\pi R^2)H}(\frac{R}{H})^H \int_0^H dz (H^2 - 2Hz + z^2)^2$$

$$= \frac{3M}{2}\frac{R^2}{H^5}\left[\int_0^H dz (H^4 + 4H^2z^2 + z^4 - 4H^3 z + 2H^2z^2 - 4HZ^3)\right]$$

$$= \frac{3MR^2}{2H^5}\left[H^5 + \frac{4H^2H^3}{3} + \frac{H^5}{5} - \frac{4H^5}{2} + \frac{2H^5}{3} - \frac{4H^5}{4}\right]$$

$$= \frac{3}{2}MR^2\left[1 + 2 + \frac{1}{5} - 2 - 1\right] = \frac{3}{10}MR^2$$

Now to find I_{xx}, again slice the cone into horizontal disks of thickness Δz and radius

$$r = \frac{R}{H}(H - z).$$

For an axis passing through a disk,

$$I_{xx} = \frac{1}{4}\Delta m r^2,$$

so by the parallel-axis theorem

$$I_{xx} = \frac{1}{4}\Delta m r^2 + \Delta m z^2$$

about an axis through the base of the cone. So for the entire cone,

$$I_{xx} = \int dm\left(\frac{r^2}{4}+z^2\right) = \int_{z=0}^{H} \rho dz(\pi r^2)\left(\frac{r^2}{4}+z^2\right)$$
$$= \rho\pi(\frac{R}{H})^2\int_{z=0}^{H} dz(H-z)^2\left(\frac{1}{4}(\frac{R^2}{H^2})(H-z)^2+z^2\right))$$

$$= \rho\pi(\frac{R}{H})^2\int_{z=0}^{H} dz(H^2-2Hz+z^2)\left[\frac{R^2}{4H^2}(H^2-2Hz+z^2)+z^2\right]$$

$$= \rho\pi(\frac{R}{H})^2\int_{z=0}^{H} dz\left\{R^2(H^2-2Hz+z^2)(\frac{1}{4}-\frac{z}{2H}+\frac{z^2}{4H^2})+(H^2z^2-2Hz^3+z^4)\right\}$$

$$= \rho\pi(\frac{R}{H})^2\int_{z=0}^{H} dz\left\{R^2\left[\frac{H^2}{4}-\frac{Hz}{2}+\frac{z^2}{4}-\frac{Hz}{z}+z^2-\frac{z^3}{2H}+\frac{z^2}{4}-\frac{z^3}{2H}+\frac{z^4}{4H^2}\right]\right.$$
$$\left. + \ H^2z^2-2Hz^3+z^4\right\}$$

Finally, we have

$$I_{xx} = \frac{3}{20}MR^2+\frac{1}{10}MH^2 = I_{yy} \text{ by symmetry.}$$

The off-diagonal elements include $I_{xy} = \int(-xy) = 0$, by another symmetry argument. That is, for every Δm with $xy > 0$, there is a Δm with $xy < 0$. The same goes for other off-diagonal products. In summary, the moment of inertial matrix is diagonal, with $I_{xx} = I_{yy}$ as given above, and I_{zz} also as given above. ■

⋆ **Problem 12.13** The Crab Nebula is a bright, reddish nebula consisting of the debris from a supernova explosion observed on earth in 1054 AD. The estimated total power it emits, mostly in X-rays, UV, and visible light, is of order 10^{31} W. The nebula harbors a pulsar in its center, which emits a pulsed light signal. Pulsars are rotating neutron stars, having a mass comparable to that of the sun (2.0×10^{30} kg) but a radius of only about 10 km. The period between successive pulses (the rotational period of the star) is $P = 0.033091$ s, which slowly increases with time, $dP/dt = 4.42 \times 10^{-13}$ s/s. Is it possible that the decreasing rotational energy of the star is the ultimate source of the energy observed in radiation?

Solution

A simple neutron star model is a constant-density sphere, with

$$I = \frac{2}{5}MR^2.$$

The kinetic energy is then

$$T = \frac{1}{2}I\omega^2 = \frac{1}{5}MR^2\omega^2 = \frac{1}{5}MR^2(\frac{2\pi}{P})^2 = \frac{4\pi^2}{5}\frac{MR^2}{P^2}.$$

If P changes with time, so does the K.E., "T". In fact,

$$\frac{dT}{dt} = -\frac{8\pi^2}{5}\frac{MR^2}{P^3}\frac{dP}{dt} = -\frac{8\pi^2}{5}\frac{(2\cdot 10^{30}\text{kg})(10^4\text{m})^2}{(3.3\cdot 10^{-2}\text{s})^3}(4.4\cdot 10^{-13}\frac{\text{s}}{\text{s}})$$

$$= \frac{-8\pi^2(2)(4.4)}{5(3.3)^3}10^{30+8-13+6}\text{kg m}^2/\text{s}^2$$

$$\cong -4\cdot 10^{31}\text{ Joules/s} \equiv -4\cdot 10^{31}\text{ Watts}$$

So all the observed emission could easily have come from the pulsar's spin-down. Some other mechanism, such as the emission of high-speed particles, presumably removes most of the energy. Also the pulsar mass and radius are not well-known, so we would not expect close agreement. ■

★★ **Problem 12.14** A cylindrical space station is a hollow cylinder of mass M, radius R, and length D, and endcaps of negligible mass. It spins about its symmetry axis (z axis) with angular velocity ω_0. (a) Find its inertia matrix about its center. A meteor of mass m and velocity v_0, moving in the x direction, strikes the station very near one of the endcaps, and bounces directly back with velocity $-v_0/2$. After the collision, find the station's (b) CM velocity (c) angular momentum, both magnitude and direction. (d) Show that subsequently the symmetry axis of the station rotates about the angular momentum vector, so the station wobbles as seen by an outside inertial observer. Find the period of this rotation.

Solution

(a) Consider a thin slice of width Δz. If the slice is at $z = 0$, then

$$\Delta I_{zz} = \Delta m R^2 = \int dm(x^2+y^2) \quad \Delta I_{xx} = \int dm(y^2+z^2) \quad \Delta I_{yy} = \int dm(x^2+z^2)$$

So

$$\Delta I_{xx} + \Delta I_{yy} = \int dm(y^2+x^2) = \Delta I_{zz}.$$

By symmetry

$$\Delta I_{xx} = \Delta I_{yy} = \frac{1}{2}\Delta I_{zz} = \frac{1}{2}\Delta m R^2,$$

where Δm = slice mass. If the slice is at arbitrary z, by the parallel axis theorem

$$\Delta I_{xx} = \Delta I_{yy} = \frac{1}{2}\Delta m R^2 + \Delta m z^2.$$

So for the entire station,

$$I_{zz} = \int \Delta m R^2 = mR^2$$

$$I_{xx} = I_{yy} = \int \Delta m(R^2/2 + z^2) = \frac{MR^2}{2} + \int_{-D/2}^{D/2} \rho_A(z^2)(2\pi R dz)$$

where ρ_A is the area density,

$$M = 2\pi RL\rho_A.$$

Therefore

$$I_{xx} = I_{yy} = \frac{MR^2}{2} + \rho_A \frac{2\pi R}{3}\left[\frac{D^3}{8} + \frac{D^3}{8}\right] = \frac{MR^2}{2} + \rho_A \frac{2\pi RD^3}{3 \cdot 4} = \frac{MR^2}{2} + \frac{M}{12}L^2 \text{ so altogether}$$

$$I = \begin{pmatrix} \frac{M}{2}(R^2 + \frac{1}{6}D^2) & 0 & 0 \\ 0 & \frac{M}{2}(R^2 + \frac{1}{6}D^2) & 0 \\ 0 & 0 & MR^2 \end{pmatrix}$$

(b) Conserve momentum for the system (station + meteor). Before the collision:

$$\mathbf{p}_{\text{stat}} + \mathbf{p}_{\text{meteor}} = 0 + mv_0\hat{x}$$

After:

$$\mathbf{p}_{\text{station}} + \mathbf{p}_{\text{meteor}} = \mathbf{p}_{\text{station}} - mv_0/2\hat{x}.$$

Conserving momentum,

$$\mathbf{p}_{\text{station}} = \frac{3mv_0}{2}\hat{x} \text{ after}$$

(c) Take the center of the station as origin

$$\mathbf{L}_{\text{before}} = \mathbf{L}_{\text{station}} + \mathbf{L}_{\text{meteor}} = I_{zz}\boldsymbol{\omega} + (\mathbf{r} \times \mathbf{p})_{\text{meteor}} = MR^2\omega\hat{z} + \frac{D}{2}mv_0\hat{y}$$

$$\mathbf{L}_{\text{after}} = \mathbf{L}_{\text{station,after}} - \frac{D}{2}(\frac{mv_0}{2})\hat{y}$$

so, conserving $\mathbf{L}$,

$$\mathbf{L}_{f,\text{station}} = mR^2\omega\hat{z} + \frac{3}{4}mv_0D\hat{y} \text{ with magnitude}$$

$$|\mathbf{L}_{f,\text{station}}| = \sqrt{(MR^2\omega)^2 + (3/4mv_0D)^2}$$

Then according to Eq 12.176, the spin rate of tumble in an inertial frame is $\dot{\phi} = L/I_{xx}$ so the period of wobble is

$$p = \frac{2\pi}{\dot{\varphi}} = \frac{2\pi I_{xx}}{L} = \frac{\pi M(R^2 + 1/6D^2)}{\sqrt{(MR^2\omega)^2 + (3/4mv_0L)^2}}$$

■

★★ **Problem 12.15** (a) Find all elements of the principal moment of inertia matrix for a thin uniform rod of mass Δm and length D if the rod is oriented along the x axis and the origin of coordinates is at the center of the rod. (b) Use the parallel-axis theorem, the perpendicular axis theorem for thin lamina, and the result of part (a) to find the principal moment of inertia matrix for a thin square of side D and total mass ΔM that is perpendicular to the z axis, with the origin of coordinates at the center of the square. (c) Find the principal moment of inertia matrix for a cube of edge length D and mass M. (d) Find the moment of inertia for the cube about an axis parallel to one of the axes in part (c) and which is oriented along the middle of one face of the cube. (e) Find the moment of inertia for the cube about an axis parallel to one of the axes in part (c) and which is oriented along the length of one corner of the cube.

Solution

(a) Elements of the matrix include

$$I_{xx} = \int dm(y^2 + z^2) \simeq 0$$

$$I_{yy} = \int dm(x^2 + z^2) = \frac{\Delta M}{D} \int_{-D/2}^{D/2} dx x^2 = \frac{\Delta M}{D} \frac{x^3}{3}\Big|_{D/2}^{D/2} = \frac{1}{12} MD^2$$

$$I_{zz} = \int dm(x^2 + y^2) = \frac{1}{12} \Delta MD^2$$

$$I_{xy} = \int dm(-xy) = 0 \quad I_{xz} = \int dm(-xz) = 0 \text{ etc}$$

Altogether

$$I = \Delta MD^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/12 & 0 \\ 0 & 0 & 1/12 \end{pmatrix}$$

(b) Let the lamina lie in the x, y plane. It can be made from thin rods lying along the y direction of width Δx. If a given rod is a distance x from the origin, by the parallel-axis theorem its moment of inertia about the origin is

$$\Delta I_{zz} = \frac{1}{12} \Delta MD^2 + \Delta Mx^2 = \sigma D \Delta x \left(\frac{1}{12} D^2 + x^2\right)$$

where $\sigma = \frac{M}{D^2}$, the mass density (mass/area) of the square, so

$$I_{zz} = \sum \Delta I_{zz} = \sigma D \int_{-D/2}^{D/2} dx \left(\frac{1}{12} D^2 + x^2\right) = \sigma D \left[\frac{1}{12} D^2 \cdot D + \frac{x^3}{3}\Big|_{-D/2}^{D/2}\right]$$

$$= \frac{M}{D} \left[\frac{1}{12} D^3 + \frac{1}{12} D^3\right] = \frac{1}{6} MD^2$$

Now because the square is thin, we can use the perpendicular axis theorem, so

$$I_{xx} + I_{yy} = I_{zz} = \frac{1}{6} MD^2$$

By symmetry $I_{xx} = I_{yy}$, so

$$I_{xx} = I_{yy} = \frac{1}{12} MD^2.$$

The off-diagonal elements are

$$I_{xy} = \int dm(-xy)$$

But xy is as often negative as positive, so the integral $= 0$. Also $I_{xz} = 0$ since $z = 0$. So altogether the moment of inertia matrix is

$$I = \Delta M D^2 \begin{pmatrix} 1/12 & 0 & 0 \\ 0 & 1/12 & 0 \\ 0 & 0 & 1/6 \end{pmatrix}$$

if the square lies in the $z = 0$ plane.

(c) The cube can be made from a stack of square plates $\perp$ to z. So

$$I_{zz} = \sum (\Delta I_{zz})_{\text{plates}} = (\sum \Delta M) D^2 \cdot \frac{1}{6} = \frac{1}{6} M D^2$$

where now M is the mass of the cube. Now from symmetry

$$I_{xx} = I_{yy} = I_{zz} = \frac{1}{6} M D^2$$

The off-diagonal elements like

$$I_{xy} = \int dm(-xy) = 0$$

since the products xy, xz, yz are as often negative as positive. So the matrix is

$$I = MD^2 \begin{pmatrix} 1/6 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & 1/6 \end{pmatrix}$$

(d)

$$I = I_{zz} + M(D/2)^2 = \frac{1}{6} MD^2 + \frac{1}{4} MD^2 = \frac{5}{12} MD^2$$

by the parallel-axis theorem.

(e)

$$I = \frac{1}{6} MD^2 + M(\frac{D\sqrt{2}}{2})^2 = \frac{1}{6} MD^2 + \frac{1}{2} MD^2 = \frac{2}{3} MD^2$$

by the parallel-axis theorem. ■

⋆⋆ **Problem 12.16** (a) Find the principal moments of inertia for a thin disk of mass Δm and radius R, if its mass density is uniform, the origin of coordinates is at the center of the disk, the x and y axes are in the plane of the disk, and the z axis is perpendicular to the disk. (b) Use this result to help find the principal moments of inertia of a uniform-density sphere of mass M and radius R_0, with origin at the center of the sphere. (c) The moment of inertia for rotation about the symmetry axis of a ring of mass Δm and radius r is $I = \Delta m r^2$. Use this fact to help find the moment of inertia about a symmetry axis for a thin spherical shell of mass ΔM and radius R, with origin at the center of the shell. (d) Use the result of part (c) to find the principal moments of inertia of a solid, uniform-density sphere of mass M and radius R_0. Compare with the result of part (b).

Solution

(a)

$$\Delta I_{zz} = \int dmr^2 = \sigma \int_0^R dr 2\pi r \cdot r^2 \quad \Delta I_{zz} = (\frac{\Delta m}{\pi R^2})\frac{2\pi R^4}{4} = \frac{1}{2}\Delta m R^2.$$

By the perpendicular axis theorem

$$I_{xx} + I_{yy} = I_{zz},$$

for thin lamina, as this is. By symmetry $\Delta I_{xx} = \Delta I_{yy}$, so

$$\Delta I_{xx} = \Delta I_{yy} = \frac{1}{4}\Delta m R^2$$

(b) Slice the sphere into disks of thickness Δz and radius $r = \sqrt{R^2 - z^2}$. ΔI_{zz} for a disk at height $\pi/2$ is then

$$\Delta I_{zz} = \frac{1}{2}\Delta m r^2 = \frac{1}{2}\Delta m(R^2 - z^2)$$

where

$$\Delta m = \rho \Delta z \pi r^2 = \rho \Delta z \pi (R^2 - z^2).$$

Therefore

$$\Delta I_{zz} = \int_{-R}^{R} \frac{1}{2}\rho dz \pi (R^2 - z^2)^2 = \frac{1}{2}\rho\pi \int_{-R}^{R} dz(R^4 - 2R^2z^2 + z^4)$$

$$= \frac{\pi}{R}(\frac{M}{4/3\pi R^3})\left[R^4 z - 2R^2\frac{z^3}{3} + \frac{z^5}{5}\right]_{-R}^{R}$$

$$= \frac{3M}{8R^3}\left[2R^5 - \frac{4}{3}R^5 + \frac{2}{3}R^5\right] = \frac{3}{8}MR^2(\frac{30 - 20 + 6}{15}) = \frac{2}{5}MR^2$$

By symmetry,

$$I_{xx} = I_{yy} = I_{zz} = \frac{2}{5}MR^2$$

(c) A thin spherical shell has mass density

$$\rho = \frac{\Delta M}{4\pi R^2 \Delta R}$$

where ΔM is the shell's mass and ΔR is its thickness. Slice the shell horizontally, splitting the shell into rings of radius r, where $r = R\sin\theta$. The volume of the ring is

$$\Delta V = (2\pi R \sin\theta)\Delta R \cdot R\Delta\theta.$$

So the moment of inertia of the ring about the z axis is

$$\Delta I_{zz} = \Delta m r^2 = \rho \Delta V R^2 \sin^2\theta$$

Therefore the moment of inertia of the entire shell is

$$I_{zz} = \int dI_{zz} = \rho \int \Delta V R^2 \sin^2\theta = \rho \cdot 2\pi R^2 \Delta R \cdot R^2 \int_0^\pi \sin^3\theta d\theta$$

$$= (\frac{\Delta M}{4\pi R^2 \Delta R}) \cdot 2\pi R^4 \Delta R \int_0^\pi d\theta \sin\theta(1-\cos^2\theta)$$

$$= \frac{\Delta M R^2}{2}\left[-\cos\theta\Big|_0^\pi + \frac{\cos^3\theta}{3}\Big|_0^\pi\right]$$

$$= \frac{\Delta M}{2}\left[2 - \frac{2}{3}\right] R^2$$

$I_{zz} = \frac{2}{3}\Delta M R^2$ where ΔM is the mass of the shell. (by symmetry $I_{xx} = I_{yy} = \frac{2}{3}\Delta M R^2$ as well.)

(d) For a solid uniform-density sphere, the total

$$I_{zz} = \int_0^R \frac{2}{3} \cdot dmr^2$$

summing over all shells with $0 \leqslant r \leqslant R$.

$$= \frac{2}{3}\rho \int_0^R (dr 4\pi r^2) r^2 = \frac{2}{3}\left(\frac{M}{-1/3\pi R^3}\right)\frac{4\pi r^5}{5}\Big|_0^R$$

$$= \frac{1}{2} \cdot \frac{4}{5} M R^2 = \frac{2}{5} M R^2,$$

just as we found in part (b). Then by symmetry,

$$I_{xx} = I_{yy} = I_{zz}$$

by symmetry. ■

★ **Problem 12.17** Prove that none of the principal moments of inertia of a rigid body can be larger than the sum of the other two.

Solution

Let the principal axes be in the x, y, z directions. Then the moment of inertia matrix is

$$I = \begin{pmatrix} \sum_i m_i(y_i^2 + z_i^2) & 0 & 0 \\ 0 & \sum_i m_i(x_i^2 + \dot{z}_i^2) & 0 \\ 0 & 0 & \sum_i m_i(x_i^2 + y_i^2) \end{pmatrix}$$

so that

$$I_{yy} + I_{zz} = \sum_i m_i(y_i^2 + z_i^2 + 2x_i^2) = I_{xx} + 2\sum_\ell m_i x_i^2 \geqslant I_{xx}.$$

Therefore

$$I_{xx} \leqslant I_{yy} + I_{zz}.$$

Similarly, it is easy to show that

$$I_{yy} \leqslant I_{xx} + I_{zz}$$

and

$$I_{zz} \leqslant I_{xx} + I_{yy}$$

The reason is that $\sum m_i x_i^2 \geqslant 0$, $\sum m_i y_\ell^2 \geqslant 0$, $\sum m_i z_i^2 \geqslant 0$. ■

★ **Problem 12.18** (a) Find all elements of the moment of inertia matrix for a cube of mass M and edge length ℓ using its principal axes. (b) Then find all elements of the moment of inertia matrix for the cube if the axes have been turned by 45° about the original z axis.

Solution

(a) We have

$$I_{zz} = \sum m_i(x_i^2 + y_i^2) = \rho \int dv(x^2 + y^2) = \rho \int dxdydz(x^2 + y^2).$$

$$= \frac{M}{\ell^3} \int_{-\ell/2}^{\ell/2} dz \int \int dxdy(x^2 + y^2) = \frac{M}{\ell^2} \int dx \int dy(x^2 + y^2)$$

$$= \frac{M}{\ell^2} \left[\ell \cdot \int_{-\ell/2}^{\ell/2} dyy^2 + \ell \int_{-\ell/2}^{\ell/2} dxx^2 \right] = \frac{M}{\ell} 2 \left[\frac{x^3}{3} \Big|_{-\ell/2}^{\ell/2} \right]$$

$$= \frac{2M}{3\ell} \left[\frac{\ell^3}{8} - (-\frac{\ell^3}{8}) \right] = \frac{4M}{3\ell} \frac{\ell^3}{8} = \frac{1}{6} M\ell^2$$

By symmetry,

$$I_{xx} = I_{yy} = I_{zz},$$

so

$$I = \frac{1}{6} M\ell^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) The rotation matrix is

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{R}$$

So

$$I' = \mathbb{R} I \mathbb{R}^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ +\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{1}{6} M\ell^2$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ +\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{1}{6} M\ell^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{1}{6} M\ell^2$$

The same as the non-rotated moment of inertia. ■

⋆ **Problem 12.19** If the entire human race were to leave their current habitats, estimate how much the length of the day would be changed if (a) they gathered at the equator; (b) they gathered at the poles.

Solution

We will conserve angular momentum of the earth plus people. The earth has $I_e = 0.33MR^2$, a bit smaller than that for a uniform-density sphere, because the earth is most dense in the center. Here

$$M = 6 \times 10^{24} \text{ kg and } R = 6.4 \times 10^6 \text{ m},$$

so

$$I_e \simeq 0.33(6 \times 10^{24} \text{ kg})(6.4 \times 10^6)^2 \text{ kg m}^2 = 8.1 \times 10^{37} \text{ kg m}^2$$

The earth has $\sim 7 \times 10^9$ people, with average mass ~ 65 kg. We will suppose on average that they are at 30° latitude, north or south. A ring of people at that latitude would have

$$I_{\text{people}} = M(R\cos 30°)^2 = (7 \times 10^9)65 \text{ kg}\left[6.4 \times 10^6 \text{ m}\frac{\sqrt{3}}{2}\right]^2 \cong 1.4 \times 10^{25} \text{ kg m}^2$$

(a) If everyone is at the equator

$$I_{\text{people}} = 1.4 \times 10^{25} \text{ kg m}^2 \times (\frac{2}{\sqrt{3}})^2 = 2.61 \times 10^{35} \text{ kg m}^2$$

Conserving total angular momentum, because there is no external torque, we have

$$(I_e + I_{\text{people at }30°}\omega_0 = (I_e + I_{\text{people at }0°})\omega_f$$

$$\omega_f/\omega_0 = \frac{I_e + I_{30°}}{I_e + I_{0°}} \cong 1 + \frac{I_{30°}}{I_e} - \frac{I_{0°}}{I_e} = \frac{T_0}{T_f}$$

(ratio of periods) so

$$T_f \simeq T_0\left[1 + \frac{I_{0°} - I_{30°}}{I_e}\right] = 24 \text{ hrs}\left[1 + \frac{\frac{1}{3}(1.4 \times 10^{25})}{8.1 \times 10^{37}}\right]$$

$$= 24 \text{ hrs} + 24 \text{ hrs}(5.8 \times 10^{-14}) = 24 \text{ hrs} + 5.0 \times 10^{-9} s$$

If everyone moves to the equator, the length of the day would increase by ~ 5 nanoseconds.

(b) If everyone is at the poles, the people have no angular momentum, so

$$(I_e + I_{\text{people at }0°})\omega_0 = I_e\omega_f \Rightarrow \frac{\omega_0}{\omega_f} = \frac{I_e}{I_e + I_{30°}} = \frac{T_f}{T_0}$$

$$T_f \cong T_0(1 + \frac{I_{30°}}{I_e})^{-1} \simeq T_0(1 - \frac{I_{30°}}{I_0}) = 24 \text{ hrs} - 24 \text{ hrs}\frac{1.4 \times 10^{25}}{8.1 \times 10^{37}} = 24 - 15 \times 10^{-9}\ s$$

So if everyone moves to the poles, the length of the day would shorten by ~ 15 ns. ■

⋆ **Problem 12.20** Consider a square plane lamina with coordinate axes x, y in the plane with origin at the center of the square and which are perpendicular to edges of the square. If the moment of inertia about each of these two axes is I_0, what are the moments of inertia about axes x' and y' in the plane turned about the z axis by a 30° angle relative to the original two axes?

Solution

Given $I_{xx} = I_{yy} \equiv I_0$. By the perpendicular axis theorem for plane lamina, here

$$I_{zz} = I_{xx} + I_{yy} = 2I_0.$$

If we rotate about the z axis by 30°, we don't change I_{zz}, so

$$I_{zz} = 2I_0 = I'_{xx} + I'_{yy}$$

Now look at the new x' and y' axes: By symmetry, it is clearly still true that $I_{x'x'} = I_{y'y'}$, and by the perpendicular axis theorem for plane lamina

$$I_{x'x'} + I_{y'y'} = I_{zz} = 2I_0.$$

Here about the new axes

$$I_{x'x'} = I_{y'y'} = I_0$$

and this is not only true for a 30° rotation, but for any orthogonal rotation as well. ■

⋆⋆ **Problem 12.21** In the text we found the total angular velocity vector in the body frame of a rigid body in terms of the Euler angles and their time derivatives,

$$\begin{aligned}\boldsymbol{\omega} &= (\omega^{x'}, \omega^{y'}, \omega^{z'}) \\ &= \left(\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi, \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi, \dot{\varphi}\cos\theta + \dot{\psi}\right).\end{aligned}$$

Show then that in the laboratory frame

$$\begin{aligned}\boldsymbol{\omega} &= (\omega^{x}, \omega^{y}, \omega^{z}) \\ &= (\dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi, -\dot{\psi}\sin\theta\cos\varphi + \dot{\theta}\sin\varphi, \dot{\psi}\cos\theta + \dot{\varphi}).\end{aligned}$$

Solution

$$\begin{aligned}&(\omega^{x'}, \omega^{y'}, \omega^{z'}) \\ &= \left(\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi, \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi, \dot{\varphi}\cos\theta + \dot{\psi}\right)\end{aligned}$$

We know

$$\begin{pmatrix}\omega^x \\ \omega^y \\ \omega^z\end{pmatrix} = R_1(-\varphi)R_2(-\theta)R_3(-\psi) = \begin{pmatrix}\omega^{x'} \\ \omega^{y'} \\ \omega^{z'}\end{pmatrix}$$

$$= \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \omega^{x'} \\ \omega^{y'} \\ \omega^{z'} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi\omega^{x'} - \sin\psi\omega^{y'} \\ \sin\psi\omega^{x'} + \cos\psi\omega^{y'} \\ \omega^{z'} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\psi\omega^{x'} - \sin\psi\omega^{y'} \\ \sin\psi\cos\theta\omega^{x'} + \cos\psi\cos\theta\omega^{y'} - \sin\theta\omega^{z'} \\ \sin\psi\sin\theta\omega^{x'} + \cos\psi\sin\theta\omega^{y'} + \cos\theta\omega^{z'} \end{pmatrix}$$

$$=$$

$$\begin{pmatrix} \cos\varphi\cos\psi\omega^{x'} - \cos\varphi\sin\psi\omega^{y'} - \sin\varphi\sin\psi\cos\theta\omega^{x'} - \sin\varphi\cos\psi\cos\theta\omega^{y'} + \sin\varphi\sin\theta\omega^{z'} \\ \sin\varphi\cos\psi\omega^{x'} - \sin\varphi\sin\psi\omega^{y'} + \cos\varphi\sin\psi\cos\theta\omega^{x'} + \cos\varphi\cos\psi\cos\theta\omega^{y'} - \cos\varphi\sin\theta\omega^{z'} \\ \sin\psi\sin\theta\omega^{x'} + \cos\psi\sin\theta\omega^{y'} + \cos\theta\omega^{z'} \end{pmatrix}$$

$$\Rightarrow \omega^x = \cos\varphi\cos\psi\sin\theta\sin\psi\dot\varphi + \cos\varphi\cos^2\psi\dot\theta - \sin\varphi\sin^2\psi\cos\theta\sin\theta\dot\varphi$$

$$- \sin\varphi\sin\psi\cos\theta\cos\psi\dot\theta - \cos\varphi\sin\psi\sin\theta\cos\psi\dot\varphi + \cos\varphi\sin^2\psi\dot\theta$$

$$- \sin\varphi\cos^2\psi\cos\theta\sin\theta\dot\varphi + \sin\varphi\cos\psi\cos\theta\dot\theta\sin\psi + \sin\varphi\sin\theta\cos\theta\dot\varphi + \sin\varphi\sin\theta\dot\psi$$

$$= \cos\varphi\dot\theta + \sin\varphi\sin\theta\dot\psi$$

And similarly for

$$\omega^y = -\dot\psi\sin\theta\cos\varphi + \dot\theta\sin\varphi \text{ and } \omega^z = \dot\psi\cos\theta + \dot\varphi$$

■

⋆⋆⋆ **Problem 12.22** An equilateral triangle of mass M and side-length L is cut from uniform-density sheet metal. (a) Draw the triangle along with the three perpendicular bisectors, each of which extends from the middle of a side to the opposite vertex. Show that each bisector has length $\sqrt{3}L/2$. (b) Explain why the center of mass of the triangle must be located at the point where the three perpendicular bisectors intersect. Let this point be the origin. (c) Let the z axis be perpendicular to the triangle, the y axis be along one of the perpendicular bisectors, and the x axis be perpendicular to both. Find the moment of inertia matrix for the triangle in these coordinates.

Solution

(a) The length of a perpendicular bisector to any side is

$$L\sin 60^\circ = L\sqrt{3}/2.$$

(b) Any other point would be asymmetric between the three sides of the triangle. Apart from this symmetry argument, one can show that using this point at origin, the integral

$\int dmy = 0$, where y is a symmetry axis in the plane of the triangle, and similarly $\int dmx = 0$, where the x, y coordinates are both in the plane of the triangle.

(c) It is easy to show that the origin is a distance

$$2/3 \times \frac{L\sqrt{3}}{2} = L\sqrt{3}/3$$

from each vertex, and a distance

$$1/3 \times L\sqrt{3}/2 = L\sqrt{3}/6$$

from the center of each side. Furthermore, a thin strip parallel to the x-axis has width dy and length $2/3(L - \sqrt{3}y)$, so has area

$$dA = 2/3(L - \sqrt{3}y)dy.$$

Summing over these strips, the total area is

$$A = \int dA = \int_{-L\sqrt{3}/6}^{L\sqrt{3}/3} 2/3(L - \sqrt{3}y)dy = \frac{1}{2}L(L\sqrt{3}/2) = \frac{\sqrt{3}}{4}L^2 = \frac{1}{2}\text{Base} \cdot \text{height}$$

First, we calculate

$$I_{xx} = \int dmy^2 = \int \sigma dAy^2$$

where

$$\sigma = \text{mass/area} = M/(\frac{\sqrt{3}}{4}L^2) = 4M/\sqrt{3}L^2.$$

Also

$$dA = 2/3(L - \sqrt{3}y)dy$$

as given before. The limits on the integral are from $y = -L\sqrt{3}/6$ to $L\sqrt{3}/6$. The result is $I_{xx} = \frac{1}{24}ML^2$.

Then we can calculate $I_{yy} = \int dmx^2$. One way to do this is to note that in rotating about the y axis each of the strips shown before is like a rod rotating about its midpoint, which has moment of inertia $\frac{1}{12}\Delta m\ell^2$ where $\ell = 2/3(L - \sqrt{3}y)$. So

$$I_{yy} = \int \frac{1}{12}(\frac{M}{A}dy)\left[\frac{2}{3} \times (L - \sqrt{3}y)\right]^3$$

between limits $y = -L\sqrt{3}/6$ and $L\sqrt{3}/3$. The result is that

$$I_{yy} = \frac{1}{24}ML^2.$$

Finally, I_{zz} can be found from the perpendicular axis theorem for plane lamina,

$$I_{zz} = I_{xx} + I_{yy} = \frac{1}{12}ML^2.$$

It can also be calculated directly,

$$I_{zz} = \int dm(x^2 + y^2),$$

as a check. The off-diagonal elements are

$$I_{xy} = -\int dm(xy), \quad I_{xz} = -\int dm(xz), \quad I_{yz} = -\int dmyz.$$

The latter two integrals we take to be zero, since the triangle is "thin" at $z = 0$. The first integral is zero by a symmetry argument: For any given value of y, the contribution of mass with this value of $y = y_0$ are

$$-\int dm(xy_0) = -y_0 \int dmx = 0,$$

since for every mass element Δm at a positive value of x there is a mass element at a negative value of x. They cancel each other in the integral, so overall, the moment of inertia matrix is

$$\left(\frac{ML^2}{24}\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

■

★★ **Problem 12.23** Suppose for a given set of axes the moment of inertia matrix is

$$\left(\frac{m\ell^2}{24}\right)\begin{vmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{vmatrix}.$$

(a) Find the principal moments of inertia. (b) About what axis, and by what angle, should the original coordinate axes be turned to arrive at the principal axes?

Solution

(a) To diagonalize, we find the eigenvalues I in the secular equation

$$\frac{m\ell^2}{24}\begin{vmatrix} 1-I & -1 & 0 \\ -1 & 1-I & 0 \\ 0 & 0 & 2-I \end{vmatrix} = 0$$

That is,

$$\frac{m\ell^2}{24}(2-I) \times \begin{vmatrix} 1-I & -1 \\ -1 & 1-I \end{vmatrix} = \frac{m\ell^2}{24}(2-I)\left[(1-I)^2 - 1\right]$$

$$= \frac{m\ell^2}{24}(2-I)\left[I^2 - 2I\right] = -\frac{m\ell^2}{24}(2-I)^2 I.$$

So finally

$$I = \{\frac{m\ell^2}{12}, \frac{m\ell^2}{12}, 0\}$$

(b) From the symmetries in the matrix, a good guess could be to rotate by 45° about the z axis, since I_{zz} is already diagonal. So try

$$R = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\frac{m\ell^2}{24}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{m\ell^2}{24}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -\sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 2 \end{pmatrix} = \frac{m\ell^2}{24}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

It works, so the answer is to rotate by 45° about the z axis. ■

⋆⋆ **Problem 12.24** Using the rotation matrices appropriate for each of the three Euler angles, find the overall 3×3 rotation matrix for arbitrary rotations in terms of the angles φ, θ, and ψ, applied in the prescribed order.

Solution

The overall rotation matrix is

$$R = R_3(\psi)R_2(\theta)R_1(\phi)$$

$$= \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\cos\theta\sin\phi & \cos\theta\cos\phi & \sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\psi\cos\phi - \sin\psi\cos\theta\sin\phi & \cos\psi\sin\phi + \sin\psi\cos\theta\cos\phi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\psi\cos\theta\sin\phi & -\sin\psi\sin\phi + \cos\phi\cos\theta\cos\phi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

■

⋆⋆⋆ **Problem 12.25** A rigid body has principal moments of inertia $I_{xx} = I_0, I_{yy} = I_{zz} = 2I_0/3$. (a) Find all elements of the moment of inertia matrix in a reference frame that has been rotated by 30° about the z axis in the counterclockwise sense relative to the initial axes. (b) In this new (primed) frame the moment of inertia matrix has the form

$$\begin{pmatrix} I'_{xx} & I'_{xy} & I'_{xz} \\ I'_{yx} & I'_{yy} & I'_{yz} \\ I'_{zx} & I'_{zy} & I'_{zz} \end{pmatrix}$$

where the nine entries were found in part (a). Now pretending that you do not already know the answer, diagonalize this matrix to find the principal moments of inertia (That is, subtract I from each of the diagonal elements in the matrix, and then set the determinant of the resulting matrix equal to zero. This will give a cubic equation in I, which when solved will give the three principal moments of inertia.

Solution

(a) The rotation matrix is

$$\begin{pmatrix} \cos 30 & \sin 30 & 0 \\ -\sin 30 & \cos 30 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & 1/2 & 0 \\ -1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So $I' = RIR^{-1} =$

$$\begin{pmatrix} \sqrt{3}/2 & 1/2 & 0 \\ -1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I_0 & 0 & 0 \\ 0 & 2I_0/3 & 0 \\ 0 & 0 & 2I_0/3 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & 1/2 & 0 \\ -1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{3}/2 & 1/2 & 0 \\ -1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2\, I_0 & -1/2 I_0 & 0 \\ I_0/3 & I_0/\sqrt{3} & 0 \\ 0 & 0 & 2I_0/3 \end{pmatrix}$$

$$= \begin{pmatrix} 11/12\, I_0 & -\frac{1}{4\sqrt{3}} I_0 & 0 \\ -I_0/(4\sqrt{3}) & 3/4\, I_0 & 0 \\ 0 & 0 & 2I_0/3 \end{pmatrix}$$

(b) Now to find the principal moments of inertia from this matrix, set the determinant

$$\begin{vmatrix} 11/12\, I_0 - I & -\frac{1}{4\sqrt{3}} I_0 & 0 \\ -I_0/(4\sqrt{3}) & 3/4\, I_0 - I & 0 \\ 0 & 0 & 2I_0/3 - I \end{vmatrix} = 0$$

That is,

$$(\frac{2I_0}{3} - I) \left[(\frac{11}{12} I_0 - I)(\frac{3}{4} I_0 - I) - \frac{1}{16 \times 3} I_0^2 \right] = 0$$

with solutions $I = 2I_0/3$ and

$$(\frac{11}{12}I_0 - I)(\frac{3}{4}I_0 - I) - \frac{1}{48}I_0^2 = 0$$

$$\text{i.e. } \frac{2}{3}I_0^2 - \frac{5}{3}I_0 I + I^2 = 0 \Rightarrow (I - I_0)(I - \frac{2}{3}I_0) = 0$$

Altogether three solutions,

$$I = 2I_0/3, \quad I = 2I_0/3, \quad I = I_0$$

The same three we started with. ■

⋆ **Problem 12.26** Show that any antisymmetric part of the moment of inertia matrix of a rigid body does not contribute to the body's equations of motion. Therefore we may safely assume that the moment of inertia matrix is symmetric.

Solution

The relevant part of the Lagrangian is

$$\frac{1}{2}\omega^T I \omega = \frac{1}{2}\omega^i I_{ij}\omega^j$$

Write

$$I_{ij} = \frac{I_{ij}}{2} + \frac{I_{ij}}{2} = \frac{I_{ij}}{2} + \frac{I_{ji}}{2} - \frac{I_{ji}}{2} + \frac{I_{ij}}{2} = \frac{1}{2}(I_{ij} + I_{ji}) + \frac{1}{2}(I_{ij} - I_{ji}) = I^s_{ij} + I^A_{ij}$$

where

$$I^s_{ij} = I^s_{ji} \text{ and } I^A_{ij} = -I^A_{ji}$$

We then have

$$\frac{1}{2}\omega^i I_{ij}\omega^j = \frac{1}{2}\omega^i I^s_{ij}\omega^j + \frac{1}{2}\omega^i I^A_{ij}\omega^j$$

Focus on the antisymmetric part

$$\frac{1}{2}\omega^i I^A_{ij}\omega^j = \frac{1}{2}\omega^j I^A_{ji}\omega^i \text{ (relabeling i} \Leftrightarrow j)$$

$$= \frac{1}{2}\omega^i I^A_{ji}\omega^j \text{ (commutativity of multiplication)}$$

$$= -\frac{1}{2}\omega^i I^A_{ij}\omega^j \text{ (antisymmetry of } I^A_{ij})$$

$$\Rightarrow \frac{1}{2}\omega^i I^A_{ij}\omega^j = 0$$

as needed.

Hence, the antisymmetric part of I drops out of the Lagrangian (and equations of motion). ■

★ **Problem 12.27** (a) Write the Lagrangian for the Euler problem of a rigid body undergoing torque-free precession. (b) Write the equations of motion and show that they agree with those in the text.

Solution

(a) The Lagrangian is

$$L = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - MgZ + \frac{1}{2}I_1(\omega^{x'})^2 + \frac{1}{2}I_2(\omega^{y'})^2 + \frac{1}{2}I_3(\omega^{z'})^2$$

where

$$(\omega^{x'})^2 = \dot{\varphi}^2 \sin^2\theta \sin^2\psi + \dot{\theta}^2 \cos^2\psi + 2\dot{\varphi}\dot{\theta}\sin\theta\sin\psi\cos\psi$$

$$(\omega^{y'})^2 = \dot{\varphi}^2 \sin^2\theta \cos^2\psi + \dot{\theta}^2 \sin^2\psi - 2\dot{\varphi}\dot{\theta}\sin\theta\sin\psi\cos\psi$$

$$(\omega^{z'})^2 = \dot{\varphi}^2\cos^2\theta + \dot{\psi}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta$$

The rotational part becomes

$$\begin{aligned} L_{rot} &= \frac{1}{2}\dot{\varphi}^2\sin^2\theta(I_1\sin^2\psi + I_2\cos^2\psi) + \frac{1}{2}\dot{\theta}^2(I_1\cos^2\psi + I_2\sin^2\psi) \\ &+ \dot{\varphi}\dot{\theta}\sin\theta\sin\psi\cos\psi(I_1 - I_2) \\ &\quad + \frac{1}{2}I_3\dot{\varphi}^2\cos^2\theta + \frac{1}{2}I_3\dot{\psi}^2 + I_3\dot{\varphi}\dot{\psi}\cos\theta \end{aligned}$$

(b) For

$$I_1 = I_2 = I_3 \equiv I$$

$$L_{rot} = \frac{I}{2}\dot{\varphi}^2 + \frac{I}{2}\dot{\theta}^2 + \frac{1}{2}I\dot{\psi}^2 + I\dot{\varphi}\dot{\psi}\cos\theta$$

The equations of motion become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\varphi}}\right) = \frac{\partial L}{\partial\varphi} \Rightarrow \frac{d}{dt}(I\dot{\varphi} + I\dot{\psi}\cos\theta) = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{\partial L}{\partial\theta} \Rightarrow I\ddot{\theta} = -I\dot{\varphi}\dot{\psi}\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\psi}}\right) = \frac{\partial L}{\partial\psi} \Rightarrow \frac{d}{dt}(I\dot{\psi} + I\dot{\varphi}\cos\theta) = 0$$

$$\Rightarrow P_\varphi = I\dot{\varphi} + I\dot{\psi}\cos\theta = \text{constant}$$

$$P_\psi = I\dot{\psi} + I\dot{\varphi}\cos\theta = \text{constant}$$

$$\Rightarrow \dot{\varphi} = \frac{p_\varphi}{I\sin^2\theta} - \frac{\cos\theta}{\sin^2\theta}\frac{p_\psi}{I} \quad (*)$$

$$\dot{\psi} = -\frac{\cos\theta}{\sin^2\theta}\frac{p_\varphi}{I} + \frac{1}{\sin^2\theta}\frac{p_\psi}{I} \quad (**)$$

We also have the Hamiltonian H conserved

$$H = \dot{\varphi}p_\varphi + \dot{\psi}p_\psi + \dot{\theta}p_\theta - L = \frac{p_\varphi^2}{2I\sin^2\theta} + \frac{p_\psi^2}{2I\sin^2\theta} + \frac{1}{2}I\dot{\theta}^2 - \frac{p_\varphi p_\psi \cos\theta}{I\sin^2\theta}$$

$$\Rightarrow \dot{\theta} = \pm\sqrt{\frac{2H}{I} - \frac{1}{I^2\sin^2\theta}(p_\varphi^2 + p_\psi^2) + \frac{2p_\varphi p_\psi\cos\theta}{I^2\sin^2\theta}}$$

which can be integrated for $\theta(t)$. From this, we can integrate $(*)$ and $(**)$ for $\varphi(t)$ and $\psi(t)$ respectively. ■

⋆⋆ **Problem 12.28** Write the six equations of motion for the Lagrangian of a hoop attached to a spring from the example in the text.

Solution

A hoop attached to a spring has the Lagrangian

$$L = \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2\dot{Z}^2) + \frac{1}{4}MR^2(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) + \frac{1}{2}MR^2(\dot{\varphi} + \dot{\varphi}\cos\theta)^2 - MgZ$$

$$-\frac{1}{2}k(X^2 + Y^2 + Z^2) - \frac{1}{2}k(R^2 + 2Rx(\cos\theta\cos\varphi\sin\psi + \cos\psi\sin\varphi)$$

$$+2RY(\cos\theta\cos\varphi\cos\psi - \sin\varphi\sin\psi) - 2RZ\cos\varphi\sin\theta)$$

The equations of motion become:

$$M\ddot{x} = -kX - kR(\cos\theta\cos\varphi\sin\psi + \cos\psi\sin\varphi)$$

$$M\ddot{Y} = -kY - kR(\cos\theta\cos\varphi\cos\psi - \sin\varphi\sin\psi)$$

$$M\ddot{z} = -kZ + kR\cos\varphi\sin\theta - Mg$$

and for the orientation angles:

$$\theta : \frac{1}{2}MR^2\ddot{\theta} = \frac{1}{2}MR^2\dot{\varphi}^2\sin\theta\cos\theta - MR^2(\dot{\psi} + \dot{\varphi}\cos\theta)\sin\theta$$

$$+kRX\sin\theta\cos\varphi\sin\psi + kRY\sin\theta\cos\varphi\cos\psi + kRZ\cos\theta\cos\varphi$$

$$\varphi : \frac{d}{dt}\left(\frac{1}{2}MR^2\sin^2\theta\dot{\varphi} + MR^2(\dot{\psi} + \dot{\varphi}\cos\theta)\cos\theta\right)$$

$$= kRX(\cos\theta\sin\varphi\sin\psi - \cos\psi\cos\varphi) + kRY(\cos\theta\sin\varphi\cos\psi - \cos\psi\cos\varphi) - kRZ\sin\theta\sin\varphi$$

$$\psi : \frac{d}{dt}(MR^2(\dot{\psi} + \dot{\varphi}\cos\theta)) = -kRX(\cos\theta\cos\varphi\cos\psi - \sin\psi\sin\varphi)$$
$$+ kRY(\cos\theta\cos\varphi\sin\psi + \sin\varphi\cos\psi)$$

■

★★ **Problem 12.29** Show that the magnitude of the angular momentum vector for the torque-free rigid body dynamics case is given by $L = I_3 p_\psi / \cos\theta$.

Solution

From geometry, we have

$$L\cos\theta = L_3 = I_3 w_3 = I_3 p_\psi \Rightarrow L = \frac{I_3 p_\psi}{\cos\theta}$$

■

★ **Problem 12.30** Show that $\dot{u} = 0$ if $g = 0$ from equation 12.200.

Solution

If $g \to 0$, we have

$$\frac{\dot{u}^2}{2} + \frac{1}{2I^2}(p_\varphi - up_\psi)^2 + \frac{1}{2}\frac{2HI_3 - p_\psi^2}{II_3}(u^2 - 1) = 0$$

Taking $\theta = 0 \Rightarrow u = 1$, without loss of generality since there is no gravity. We find that

$$\frac{\dot{u}^2}{2} + \frac{1}{2I^2}(p_\varphi - p_\psi)^2 = 0$$

But

$$p_\varphi = I\dot{\varphi}\sin^2\theta + p_\psi\cos\theta = p_\psi \Rightarrow \dot{u} = 0$$

■

★ **Problem 12.31** Show that if

$$I_1 \geq I_2 \geq I_3$$

for a torque-free rigid body, we then have

$$\sqrt{2TI_3} \leq L \leq \sqrt{2TI_1}.$$

Solution

Note that

$$T = \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_2}$$

and

$$L^2 = L_1^2 + L_2^2 + L_3^2$$

If $I_1 \geqslant I_2 \geqslant I_3$, if we replace in T all I's by I_1, we get a smaller quantity.

$$T \geqslant \frac{L_1^2 + L_2^2 + L_3^2}{2I_1} = \frac{L^2}{2I_1}$$

Similarly, we replace all $I's$ in T with I_3, we get a bigger quantity.

$$\Rightarrow T \leqslant \frac{L_1^2 + L_2^2 + L_3^2}{2I_3} = \frac{L^2}{2I_3}$$

Hence

$$\sqrt{2I_3T} \leqslant L \text{ and } \sqrt{2I_1T} \geqslant L$$

or

$$\sqrt{2I_3T} \leqslant L \leqslant \sqrt{2I_1T}$$

as needed. ■

⋆ **Problem 12.32** Show that $u \rightarrow 1$ is a stable point for the gyroscope, and find the corresponding nutation. Show that there is a criticial angular momnetum $p_\psi = 2\sqrt{MglI}$.

Solution
The effective potential is

$$U_{\text{eff}}(u) = \frac{1}{2}\frac{(p_\varphi - p_\psi u)^2}{I^2} + \frac{1}{2}(\frac{2HI_3 - p_\psi^2}{II_3} = \frac{2Mgl_u}{I}) \times (u^2 - 1)$$

One of the roots would need to be at $u = 1$, so that $\dot{u} = 0$

$$\Rightarrow \frac{1}{2}\frac{(p_\varphi - p_\psi u)^2}{I} = 0 \Rightarrow p_\varphi = p_\psi$$

$$\Rightarrow U_{\text{eff}}(u) = \frac{1}{2I^2}p_\psi^2(u-1)^2 + \frac{1}{2}(\frac{2HI_3 - p_\psi^2}{II_3} - \frac{2Mgl_u}{I})(u^2 - 1)$$

$$= (u-1)\left[\frac{p_\psi^2}{2I^2}(u-1) + \frac{1}{2}(\frac{2HI_3 - p_\psi^2}{II_3} - \frac{2Mgl_u}{I})(u+1)\right]$$

If we want the gyroscope to stay at $v = 1$, we want $u = 1$ be a double root

$$\Rightarrow \frac{p_\psi^2}{2I^2}(u-1) + \frac{1}{2}(\frac{2HI_3 - p_\psi^2}{II_3} - \frac{2Mgl_u}{I})(u+1)\Big|_{u=1} = 0$$

$$\Rightarrow H = \frac{p_\psi^2}{2I_3} + Mgl$$

Then, with these initial conditions, $U_{\text{eff}}(u)$ the term

$$U_{\text{eff}}(u) = \frac{1}{2}(1-u)^2(\frac{p_\psi^2}{I^2} - \frac{2Mgl}{I}(1+u))$$

The third root u_1 is then

$$u_1 = \frac{p_\psi^2}{2MglI} - 1$$

If

$$\frac{p_\psi^2}{2MglI} - 1 > 1 \text{ or } \frac{p_\psi^2}{2MglI} > 2$$

then $u = 1$ is stable and the gyroscope stays at $u = 1$. However, if $\frac{p_\psi^2}{2MglI} < 2$, we have another root between $-1 < u_1 < 1$. $\Rightarrow$ we have nutation between $u - 1$ and u_1. ■

★★ **Problem 12.33** If we start a gyroscope at an angle $u(0) = u_0$ with $\dot{u}(0) = 0$ and non-zero $\dot{\psi}$ but zero $\dot{\varphi} = 0$, (a) show that the gyroscope nutates and find the maximum angle u_1 it reaches before bouncing back up. (b) Consider the case of a fast spin, where $p_\psi^2 \gg 2MglI$; find approximate forms for the two nutation angles and nutation frequency.

Solution

(a) From $u(0) = u_0$ and $\dot{u}(0) = 0$, we have

$$U_{\text{eff}}(u_0) = 0 \Rightarrow u_0$$

is one of the roots of U_{eff}. We also initially have

$$\dot{\varphi} = 0 \Rightarrow p_\varphi = p_\varphi u_0$$

The effective potential is:

$$U_{\text{eff}}(u) = \frac{1}{2}\frac{(p_\varphi - p_\varphi u)^2}{I^2} + \frac{1}{2}(\frac{2HI_3 - p_\psi^2}{I\,I_3} - \frac{2Mgl}{I}u)$$

$$\Rightarrow H - \frac{p_\psi^2}{2I_3} = Mglu_0$$

As $\dot{\theta}^2$ and $\dot{\varphi}^2$ increase, with H and p_ψ constant, we must have u decrease $\Rightarrow \theta$ increase.

Hence, the second root of $U_{\text{eff}}(u)$ between -1 and 1 is less than u_0. Let's call it u_1. Nutation will happen between u_0 and u_1. We can now factor $U_{\text{eff}}(u)$ as follows:

$$U_{\text{eff}}(0) = (u_0 - u)\left[\frac{2Mgl}{I}(1 - u^2) - \frac{p_\psi^2}{I^2}(u_0 - u)\right] = 0$$

$$\Rightarrow (1 - u_1)^2 - \frac{p_\psi^2}{2MglI}(u_0 - u_1) = 0$$

(b) We have

$$\frac{p_\psi^2}{2MglI} = \frac{I_3}{I}\frac{\frac{1}{2}I_3\omega_3^2}{Mgl} \gg 1$$

if I_3 is not much less than I and the gyroscope is spinning fast.

In this regime, we get

$$u_1 \simeq u_0 - \frac{2MglI}{p_\psi^2}(1 - u_0^2)$$

so that the nutation is small. To find the frequency of nutation in the regime when $u_1 \simeq u_0$ as above, we expand $U_{\text{eff}}(u)$ near u_0. We get

$$U_{\text{eff}}(u) \simeq \frac{1}{2}(\frac{p_\psi}{I})^2(u - u_0)^2$$

identifying

$$\Omega = \frac{p_\psi}{I} = \frac{I_3}{I}\omega_3.$$

■

★★ **Problem 12.34** A rigid body has an axis of symmetry, which we designate as axis 1. The principal moment of inertia about this axis is I_1, while the principal moments of inertia about the remaining two principal axes are $I_2 = I_3 \equiv I_0 \neq I_1$. (a) Write the Euler equations of rotational dynamics in terms of I_1, I_0, and the three angular velocities $\omega_1, \omega_2, \omega_3$. (b) Show that ω_1 is constant. (c) Find a second-order linear differential equation for ω_2 and another for ω_3. (d) Does either the magnitude or direction of this precession depend upon whether the rigid body is prolate (like an American football or rifle bullet) or oblate (like a saucer or frisbee)? (e) Prove that the symmetry axis of the rigid body is coplanar with the angular velocity vector $\boldsymbol{\omega}$ and with the angular momentum vector $\mathbf{L}$.

Solution

(a) The torque is assumed to be zero, so

We have

$$0 = (\frac{d\mathbf{L}}{dt})_{\text{body}} + \boldsymbol{\omega} \times \mathbf{L}$$

$$I\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3 = 0$$

$$I\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1 = (I_0 - I_1)\omega_3\omega_1$$

$$I\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2 = (I_1 - I_0)\omega_1\omega_2$$

(b) From (a), $\dot{\omega}_1 = 0 \rightarrow \omega_1 =$constant.

(c) We take derivatives of equations in (a)

$$I\ddot{\omega}_2 = (I_0 - I_1)\dot{\omega}_3\omega_1 = -\frac{(I_0 - I_1)^2}{I}\omega_1^2\omega_2$$

and

$$I\ddot{\omega}_3 = (I_1 - I_0)\omega_1\dot{\omega}_2 = -\frac{(I_1 - I_0)^2}{I}\omega_1^2\omega_3$$

We then have

$$\ddot{\omega}_2 = -\Omega^2\omega_2 \text{ and } \ddot{\omega}_3 = -\Omega^2\omega_3$$

where

$$\Omega^2 = \frac{(I_1 - I_0)^2}{I}\omega_1^2$$

(d) This affects whether $I_1 > I_0$ or $I_1 < I_0$. Ω is unaffected, but the sign of $\dot{\omega}_2$ and $\dot{\omega}_3$ would flip between these possibilities, $\Rightarrow$ spin direction flips.

(e) We want to show that $\mathbf{L}, \boldsymbol{\omega}$, and the body 1 axis lie in a plane.

$$L_1 = I_1\omega_1 \quad L_2 = I\omega_2 \quad L_3 = I\omega_3$$

$\Rightarrow$ In the $2 - 3$ plane, the projections of $\mathbf{L}$ and $\boldsymbol{\omega}$ are parallel.

$\Rightarrow$ $\mathbf{L}, \boldsymbol{\omega}$, and the 1 axis all lie in the same plane. ∎

⋆⋆ **Problem 12.35** Using Euler's equations, show that a rigid body rotating without applied torque has a total angular momentum whose magnitude is constant.

Solution

The Euler equations are

$$I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = 0$$

$$I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 = 0$$

$$I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = 0$$

Multiply the first equation by $I_1\omega_1$, the second by $I_2\omega_2$, and the third by $I_3\omega_3$. This gives

$$\frac{d}{dt}\frac{1}{2}I_1^2\omega_1^2 = I_1(I_2 - I_3)\omega_1\omega_2\omega_3$$

$$\frac{d}{dt}\frac{1}{2}I_2^2\omega_2^2 = I_2(I_3 - I_1)\omega_1\omega_2\omega_3$$

$$\frac{d}{dt}\frac{1}{2}I_3^2\omega_3^2 = I_3(I_1 - I_2)\omega_1\omega_2\omega_3$$

Add the three equations:

$$\frac{1}{2}\frac{d}{dt}(L_1^2 + L_2^2 + L_3^2) = 0,$$

where

$$L_1 = I_1\omega_1, \quad L_2 = I_2\omega_2, \quad L_3 = I_3\omega_3$$

Therefore, $L^2 = L_1^2 + L_2^2 + L_3^2$ is constant. ■

⋆⋆ **Problem 12.36** Find the product of two rotation matrices corresponding to successive rotations about (i) the x axis by angle α and (ii) the z axis by angle β, with (a) the x axis-rotation first, (b) the z axis-rotation first. (c) Then subtract the two results, to illustrate the fact that rotations do not generally commute. (d) By expanding sines and cosines for small angles up through terms of second order, illustrate the fact that infinitesimal rotations *do* commute if second-order effects are counted as negligible.

Solution

(a)

$$R_zR_x = \begin{pmatrix} \cos\beta & +\sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & +\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \cos\beta & +\sin\beta\cos\alpha & \sin\beta\sin\alpha \\ -\sin\beta & \cos\beta\cos\alpha & +\cos\beta\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}$$

(b)

$$R_xR_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & +\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & +\sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\beta & +\sin\beta & 0 \\ -\cos\alpha\sin\beta & \cos\alpha\cos\beta & +\sin\alpha \\ \sin\alpha\sin\beta & -\sin\alpha\cos\beta & \cos\alpha \end{pmatrix}$$

(c)

$$R_zR_x - R_xR_z = \begin{pmatrix} 0 & -\sin\beta(1-\cos\alpha) & \sin\beta\sin\alpha \\ -\sin\beta(1-\cos\alpha) & 0 & -\sin\alpha(1-\cos\beta) \\ -\sin\alpha\sin\beta & -\sin\alpha(1-\cos\beta) & 0 \end{pmatrix}$$

(d)

$$\sin\alpha \simeq \alpha \quad \cos\alpha \sim 1 - \alpha^2/2 \text{ etc, so}$$

$$R_zR_x - R_xR_z = \begin{pmatrix} 0 & -\beta\alpha^2/2 & \alpha\beta \\ -\beta\alpha^2/2 & 0 & -\alpha\beta^2/2 \\ -\alpha\beta & -\alpha\beta^2/2 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

if we drop second-order terms, etc. ■

⋆⋆ **Problem 12.37** In 2004 a satellite was launched into a circular polar orbit, 642 km above the earth's surface, containing an experiment called Gravity Probe B. The satellite remained in orbit for 16 months, flying successively over the north pole, the south pole, back over the north pole, etc. Four gyroscopes were on board, consisting of nearly perfectly-spherical, uniform-density, fused-quartz balls 1.9 cm in radius. All were made to spin at about 4200 rpm. Strenuous efforts were made to reduce any torques on the gyroscopes to near zero, so any precession or other drifts caused by them could be minimized. Then if Newton's theory of gravity were correct, the spin direction of the gyros would always point toward the same place in the sky, some particular distant star, for example. According to general relativity, however, there should be two very small drifts in the gyro spin directions: First, there is the "geodetic effect" in which the spin direction should drift slightly *forward* in the orbit (*i.e.*, in the north-south direction), still in the plane of the orbit. Second, there is the "frame dragging" effect, in which the spin direction of the gyro should slowly drift in a direction *perpendicular* to the plane of the orbit (*i.e.*, in the east-west direction.) It is only in polar orbit where the predicted geodetic and frame dragging effects are perpendicular to one another, allowing both to be measured. The predictions from general relativity were that the geodetic effect should lead to a drift of 6.6061 arcseconds/year, while the frame-dragging effect should be 0.0392 arcseconds/year. The data showed a drift in the plane of the orbit of 6.60 ± 0.0183 arcseconds/year and a perpendicular drift of 0.0372 ± 0.0072 arcseconds/year, in good agreement with the predictions. (a) The density of fused quartz is 2.2 g/cm^3. What were the principal moments of inertia of each gyro? (b) When spun at 4200 rpm, what was the angular momentum of each gyro? (c) What was its kinetic energy?

(This was sufficient to destroy the entire experiment if the gyro had touched its housing.)
(d) Suppose a tiny torque acted on one of the gyros, causing it to precess. Estimate the maximum torque allowable to keep the precession within the quoted errors in drift-rates given above.

Solution
(a) The moment of inertia of a uniform-density sphere is

$$I = 2/5MR^2 = 2/5(\rho \cdot \frac{4}{3}\pi R^3)R^2 = \frac{8}{15}\pi(2.2\frac{\text{g}}{\text{cm}^3})(1.9\text{ cm})^5$$

$$= 91.2\text{ g cm}^2 \times (\frac{1\text{ kg}}{1000\text{ g}})(\frac{1\text{ m}}{100\text{ cm}})^2 = 9.12 \times 10^{-6}\text{ kg m}^2,$$

the same about each principal axis.

(b) The angular momentum is

$$L = I\omega = 9.12 \times 10^{-6}\text{ kg m}^2 \cdot 4200\frac{\text{rev}}{\text{min}} \cdot \frac{2\pi\text{ rad}}{\text{rev}} \cdot \frac{1\text{ min}}{60\text{ seconds}}$$

$$= \frac{(9.12)(4.2)}{6.0} \times 10^3 \cdot 2\pi \cdot 10^{-6-1}\text{ kg m}^2\text{ s}^{-1} = 4.0 \times 10^{-3}\text{ kg m}^2/\text{s}$$

(c) The kinetic energy is

$$T = \frac{1}{2}I\omega^2 = L^2/2I = \frac{(4.0 \times 10^{-3}\text{ kg m}^2/s)^2}{2(9.12 \times 10^{-6}\text{ kg m}^2)} = 0.88 \times 10^0\text{ kg m}^2/\text{s}^2 = 0.88\text{ J}$$

(d) A torque could cause precession, so that the angular momentum L changes direction $\Delta\mathbf{L} = \mathbf{N}\Delta t$ where $\mathbf{N}$ is the torque. The rate of precession is

$$\frac{\Delta\theta}{\Delta t} = \Delta L/L\Delta t = N/L$$

so the torque is $N = L\frac{\Delta\theta}{\Delta t}$. The lower drift rate mentioned in the problem statement is 0.0372 arcseconds/year. So let us say that the maximum drift rate we could tolerate would be 10% of this amount, due to random torques. So the maximum torque we could tolerate would be

$$N = L(0.1(.0372)\text{arcsec/year})$$

$$= 4.0 \times 10^{-3}\text{ kg}\frac{\text{m}^2}{\text{s}}(0.0037)\frac{\text{arcsec}}{\text{year}}\left(\frac{\text{degree}}{3600\text{ arcsec}}\right)\left(\frac{2\pi\text{ radius}}{360\text{ degree}}\right)$$

$$= \frac{(4.0)(3.7)(2\pi)}{(3.16)(3.6)(3.6)}\frac{10^{-3-3-3-2}}{10^7}\text{kg}\frac{\text{m}^2}{\text{s}^2}$$

$$N \cong 2 \times 10^{-18}\text{ kg m}^2/\text{s}^2 = 2 \times 10^{-18}\text{ Newton-meters}$$

(we used the fact that there are about 3.16×10^7 sec/year.) ■

13 Coupled Oscillators

13.1 Problems and Solutions

★★ **Problem 13.1** Two blocks, of masses m and M, are connected by a single spring of force-constant k. The blocks are free to slide on a frictionless table. Beginning with the Lagrangian, find the oscillation frequency of the system in terms of k and the reduced mass $\mu \equiv mM/(m+M)$. Show that for the special case $M = m$, the frequency is what you would expect when the center of the spring remains at rest.

Solution

We will measure the coordinates x_1 and x_2 from the equilibrium points of m and M, respectively. Then the Lagrangian is

$$L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}M\dot{x}_2^2 - \frac{1}{2}k(x_2 - x_1)^2$$

Then using Lagrange's equations we can find the equations of motion of m and of M,

$$m\ddot{x}_1 - k(x_2 - x_1) = 0 \text{ and } M\ddot{x}_2 + k(x_2 - x_1) = 0$$

We can separate the variables by solving the first equation for x_2 and substituting the result into the second equation. That is, $x_2 = (m\ddot{x}_1 + kx_1)/k$, so after substitution and rearranging, we get

$$\ddddot{x}_1 + \frac{k}{\mu}\ddot{x}_1 = 0$$

where $\mu = mM/(m+M)$ is the reduced mass. This is a fourth-order differential equation, where the general solution would have four arbitrary constants. A solution is

$$x_1 = A + Bt + C\cos(\omega t + \varphi) \text{ where } \omega = \sqrt{k/\mu},$$

and A, B, C and φ are arbitrary. We can set $A = 0$ and $B = 0$ with an appropriate choice of coordinates and assuming that the CM is initially at rest. For the special case $m = M$, then $\mu = m/2$ and $\omega = \sqrt{2k/m}$. This is correct, since half a spring is twice as stiff as a full spring. (That is, each mass is then an equal distance from the CM of the system.) ■

★★ **Problem 13.2** Two blocks, of masses m and $2m$, are connected together linearly by three springs of equal force-constants k. The outer springs are also attached to stationary walls, while the middle spring connects the two masses. Find the normal mode frequencies.

Solution

The Lagrangian is

$$L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2 - \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2 - \frac{1}{2}k(x_2 - x_1)^2.$$

Then using the Lagrange equations for x_1 and x_2, we find the two equations of motion

$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0 \quad \text{and} \quad 2m\ddot{x}_2 + 2kx_2 - kx_1 = 0.$$

To find the normal modes, let $x_1 = Ae^{i\omega t}$ and $x_2 = Be^{i\omega t}$, or in matrix form

$$\begin{pmatrix} -m\omega^2 & -k \\ -k & -2m\omega^2 + 2k \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

with nontrivial solutions if the determinant

$$\begin{vmatrix} -m\omega^2 & -k \\ -k & -2m\omega^2 + 2k \end{vmatrix} = 0 \;\Rightarrow\; (m\omega^2 - 2k)(2m\omega^2 - 2k) - k^2 = 0$$

Solving this equation by the quadratic formula, we find that

$$\omega^2 = \frac{(3 \pm \sqrt{3})}{2}\frac{k}{m}.$$

Therefore the two normal mode frequencies are

$$\omega_+ = \sqrt{\frac{(3+\sqrt{3})k}{2m}} \quad \text{and} \quad \omega_- = \sqrt{\frac{(3-\sqrt{3})k}{2m}}.$$

■

⋆⋆ **Problem 13.3** Reconsider the problem of two equal-mass blocks and three springs, in a straight line with the outer springs attached to stationary walls. Now suppose the outer springs have the same force-constant k, while the central spring has force-constant $2k$. Find the eigenfrequencies and eigenvectors.

Solution

The Lagrangian in this case is

$$L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2 - \frac{1}{2}(2k)(x_2 - x_1)^2.$$

Then Lagrange's equations give us the equations of motion of the two masses,

$$m\ddot{x}_1 + kx_1 - 2k(x_2 - x_1) = 0 \quad \text{and} \quad m\ddot{x}_2 + kx_2 + 2k(x_2 - x_1) = 0.$$

or

$$m\ddot{x}_1 + 3kx_1 - 2kx_2 = 0 \quad \text{and} \quad m\ddot{x}_2 + 3kx_2 - 2kx_1 = 0.$$

Now to find the normal modes, we try $x_1 = Ae^{i\omega t}$ and $x_2 = Be^{i\omega t}$. This leads to the matrix equation

$$\begin{pmatrix} -m\omega^2 + 3k & -2k \\ -2k & -2m\omega^2 + 3k \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

There are nontrivial solutions if

$$\begin{vmatrix} -m\omega^2 + 3k & -2k \\ -2k & -m\omega^2 + 3k \end{vmatrix} = 0 \quad \Rightarrow \quad (-m\omega^2 + 3k)^2 - 4k^2 = 0$$

or $-m\omega^2 + 3k = \pm 2k$. The first solution is $-m\omega^2 = -k$, so $\omega_1 = \sqrt{k/m}$. The second solution gives $\omega_2 = \sqrt{5k/m}$. These are the eigenfrequencies.

Now suppose $\omega = \omega_1 = \sqrt{k/m}$: Then

$$\begin{pmatrix} -k + 3k & -2k \\ -2k & -k + 3k \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0 \quad \Rightarrow \quad 2kA - 2kB = 0 \quad \Rightarrow \quad A = B$$

That is, in this normal mode the two blocks slide back and forth together with equal amplitudes. The middle spring is never stretched or compressed, and so

$$e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Now instead suppose $\omega = \omega_2 = \sqrt{5k/m}$: Then

$$\begin{pmatrix} (-5+3)k & -2k \\ -2k & (-5+3)k \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0 \quad \Rightarrow \quad -2kA - 2kB = 0 \quad \Rightarrow \quad B = -A$$

In this normal mode the blocks oscillate *oppositely* with equal but opposite amplitudes, and so

$$e_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

■

⋆ **Problem 13.4** A hypothetical linear molecule of four atoms is free to move in three dimensions. How many degrees of freedom are there? How many translational modes? How many rotational modes? How many vibrational modes? Then suppose instead that the four atoms are all in the same plane but not lined up, still free to move in three dimensions. How many degrees of freedom are there in this case, and how many are there of each kind of translational, rotational, and vibrational modes?

Solution

In three dimensions there are $4 \times 3 = 12$ degrees of freedom. There are three translational modes, two rotational modes (note that rotation about the axis connecting all four does not count, because it is unobservable.). This leaves seven vibrational modes. If the four molecules are not lined up, there are still three translational modes, and now three rotational modes, leaving six vibrational modes. ■

⋆⋆ **Problem 13.5** Find the normal modes of oscillation for small-amplitude motions of a double pendulum (a lower mass m hanging from an upper mass M) where the pendulum lengths are equal. Find the normal mode frequencies and the amplitude ratios of M and m in each case. Let the generalized coordinates be θ_1, the angle of the upper mass relative to the vertical, and θ_2, the angle of the lower mass relative to the vertical.

Solution

The Cartesian components of the lower mass m are $x = \ell(\sin\theta_1 + \sin\theta_2)$, $y = -\ell(\cos\theta_1 + \cos\theta_2)$, so the kinetic energy of m alone is

$$\frac{1}{2}m(v_x^2 + v_y^2) = \frac{1}{2}m\ell^2[(\cos\theta_1\dot{\theta}_1 + \cos\theta_2\dot{\theta}_2)^2 + (\sin\theta_1\dot{\theta}_1 + \sin\theta_2\dot{\theta}_2)^2].$$

The total system Lagrangian is therefore

$$\begin{aligned} L &= \frac{1}{2}M\ell^2\dot{\theta}_1^2 + Mg\ell\cos\theta + \frac{1}{2}m\ell^2[\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2(\cos(\theta_1 - \theta_2)]\dot{\theta}_1\dot{\theta}_2 + mg\ell(\cos\theta_1 + \cos\theta_2) \\ &= \frac{1}{2}(M+m)\ell^2\dot{\theta}_1^2 + \frac{1}{2}m\ell^2\dot{\theta}_2^2 + m\ell^2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 + (M+m)g\ell\cos\theta_1 + mg\ell\cos\theta_2. \end{aligned}$$

where we have used the trig identity $\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 = \cos(\theta_1 - \theta_2)$.

Now we can use Lagrange's equations in θ_1 and θ_2 to find the equations of motion. The first of these is

$$\begin{aligned} (1)\quad (M+m)\ell^2\ddot{\theta}_1 + m\ell^2\cos(\theta_1 - \theta_2)\ddot{\theta}_2 - m\ell^2\sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)\dot{\theta}_2 \\ + m\ell^2\sin(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 + (M+m)g\ell\sin\theta_1 = 0 \end{aligned}$$

and the second is

$$\begin{aligned} (2)\quad m\ell^2\ddot{\theta}_2 + m\ell^2\cos(\theta_1 - \theta_2)\ddot{\theta}_1 - m\ell^2\sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)\dot{\theta}_1 \\ - m\ell^2\sin(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 + mg\ell\sin\theta_2 = 0 \end{aligned}$$

These equations are exact for any angles θ_1 and θ_2. But we can now specialize to small-amplitude motion, in which $\sin\theta \cong \theta$ and $\cos\theta \cong 1 - \theta^2/2$ through second-order terms in θ. That is, we can now drop all terms with more than two θs or $\dot{\theta}$s. This greatly simplifies the equations, to give the "*linearized*" equations

$$(1)\ (M+m)\ell^2\ddot{\theta}_1 + m\ell^2\ddot{\theta}_2 + (M+m)g\ell\theta_1 \cong 0$$

$$(2)\ m\ell^2\ddot{\theta}_2 + m\ell^2\ddot{\theta}_1 + mg\ell\theta_2 \cong 0.$$

For normal modes, let $\theta_1 = Ae^{i\omega t}$ and $\theta_2 = Be^{i\omega t}$. This gives the algebraic equations

$$(1)\ -(M+m)\ell^2\omega^2 A - m\ell^2\omega^2 B + (M+m)g\ell A = 0$$

$$(2)\ -m\ell^2\omega^2 B - m\ell^2\omega^2 A + mg\ell B = 0.$$

There are non-trivial solutions if the determinant of the coefficients vanishes:

$$\begin{vmatrix} (M+m)(-\ell^2\omega^2+g\ell) & -m\ell^2\omega^2 \\ -m\ell^2\omega^2 & m(-\ell^2\omega^2+g\ell) \end{vmatrix} = 0$$

which when evaluated gives a quadratic equation in ω^2. Using the quadratic formula, we find two solutions for ω^2,

$$\omega^2 = \frac{(M+m)g \pm \sqrt{(M+m)mg^2}}{M\ell}$$

So the two eigenfrequencies (i.e., normal mode frequencies) are

$$\omega_+ = \left[\left(\frac{M+m}{M}\right)\left(1+\sqrt{\frac{m}{M+m}}\right)\frac{g}{\ell}\right]^{1/2}$$

and

$$\omega_- = \left[\left(\frac{M+m}{M}\right)\left(1-\sqrt{\frac{m}{M+m}}\right)\frac{g}{\ell}\right]^{1/2}.$$

Now we want to find the relative amplitudes B/A for each of the two normal modes. First, we can find the quantity $g\ell - \ell^2\omega_\pm$ using the normal mode frequencies. By substitution and algebraic simplification we find that

$$g\ell - \ell^2\omega_\pm^2 = -\frac{g\ell m}{M}\left[1 \pm \sqrt{\frac{M+m}{m}}\right].$$

Substituting this quantity into the algebraic equations involving A and B, we find after rearranging and simplifying, that

$$\frac{B}{A} = \mp\sqrt{\frac{M+m}{m}}.$$

Summarizing, one mode has frequency and amplitude ratios

$$\omega_- = \left[\left(\frac{M+m}{M}\right)\left(1-\sqrt{\frac{m}{M+m}}\right)\frac{g}{\ell}\right]^{1/2} \qquad \frac{B}{A} = \sqrt{\frac{M+m}{m}}$$

so in this mode the two bobs move in the *same* direction with equal amplitudes. This is the lower-frequency mode.

The other mode has frequency and amplitude ratios

$$\omega_+ = \left[\left(\frac{M+m}{M}\right)\left(1+\sqrt{\frac{m}{M+m}}\right)\frac{g}{\ell}\right]^{1/2} \qquad \frac{B}{A} = -\sqrt{\frac{M+m}{m}}$$

so in this mode the two bobs move in *opposite* directions with equal amplitudes. This is the higher-frequency mode. ■

⋆⋆ **Problem 13.6** A uniform horizontal rod of mass m and length ℓ is supported against gravity by two identical springs, one at each end of the rod. Assuming the motion is confined to the vertical plane, find the normal modes and frequencies of the system. Then find the motion in case just one end of the rod is displaced from equilibrium and released from rest.

Solution

Neglecting gravity, which simply lowers the center of mass of the rod, and does not affect the dynamics, there are only two forces on the rod, $-ky_1$ and $-ky_2$, where y_1 is positive downward on the left end and y_2 is positive downward on the right end. Therefore Newton's second law gives $m\ddot{y}_{CM} = -k(y_1 + y_2)$ where $y_{CM} = (y_1 + y_2)/2$. It follows from $F = ma$ that

$$(\ddot{y}_1 + \ddot{y}_2) + \frac{2k}{m}(y_1 + y_2) = 0,$$

a simple harmonic oscillator equation in $(y_1 + y_2)$.

It is also true that the torque on the rod is the time rate of change of its angular momentum. Taking the origin to be the original rod midpoint, the net torque is $N = (k\ell/2)(y_2 - y_1)$, positive out of the page. The angular momentum of the rod is $L = I\dot{\theta} = \frac{1}{12}m\ell^2\dot{\theta}$, where $\dot{\theta} = (\dot{y}_1 - \dot{y}_2)/\ell$, also positive out of the page. So

$$k\frac{\ell}{2}(y_2 - y_1) = \frac{1}{12}m\ell^2\left(\frac{\ddot{y}_1 - \ddot{y}_2}{\ell}\right). \tag{13.1}$$

That is,

$$(\ddot{y}_1 - \ddot{y}_2) + \frac{6k}{m}(y_1 - y_2) = 0,$$

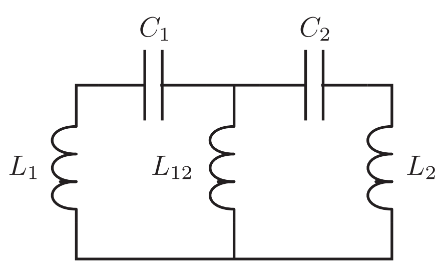
a simple harmonic oscillator equation in $(y_1 - y_2)$. Altogether, it is clear that the two normal modes are $Y_1 = y_1 + y_2$, with frequency $\omega_1 = \sqrt{2k/m}$, and $Y_2 = y_1 - y_2$, with frequency $\omega_2 = \sqrt{6k/m}$.

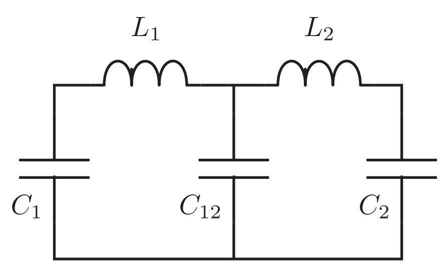
Choosing initial conditions $y_1(0) = 0, y_2(0) = a, \dot{y}_1(0) = \dot{y}_2(0) = 0$, the subsequent solutions are

$$y_1(t) = \frac{a}{2}(\cos\omega_1 t - \cos\omega_2 t), \quad y_2(t) = \frac{a}{2}(\cos\omega_1 t + \cos\omega_2 t).$$

■

★★ **Problem 13.7**





The voltage across a capacitor is $V_C = q/C$, where C is the capacitance and q is the charge on the capacitor. The voltage across an inductor is $V_L = LdI/dt$, where L is the inductance and I is the current through the inductor. A wire attached to a capacitor whose charge is changing carries a current $I = dq/dt$. The net voltage drop around any closed circuit is zero, so a simple electrical L, C circuit obeys $L\ddot{q} + q/C = 0$, and so oscillates with frequency $\omega = 1/\sqrt{LC}$. Find the normal mode oscillation frequencies and eigenvectors for each of the two-loop circuits shown, in the case $C_1 = C_2 \equiv C, L_1 = L_2 \equiv L, C_{12} = 2C$, and $L_{12} = 2L$.

Solution

For the circuit on the left in the diagram, let q_1 be the charge on C_1 and q_2 be the charge on C_2. Then moving clockwise around the loop at the left, we have

$$\frac{q_1}{C} + 2L(\ddot{q}_1 - \ddot{q}_2) + L\ddot{q}_1 = 0.$$

Also moving CW around the loop at the right,

$$\frac{q_2}{C} + 2L(\ddot{q}_2 - \ddot{q}_1) + L\ddot{q}_2 = 0.$$

That is,

$$3L\ddot{q}_1 - 2L\ddot{q}_2 + \frac{q_1}{C} = 0 \quad \text{and} \quad 3L\ddot{q}_2 - 2L\ddot{q}_1 + \frac{q_2}{C} = 0.$$

For normal modes, we set $q_1 = Ae^{i\omega t}$ and $q_2 = Be^{i\omega t}$. Then in matrix form,

$$\begin{pmatrix} -3\omega^2 L + 1/C & 2\omega^2 L \\ 2\omega^2 L & -3\omega^2 L + 1/C \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

There is a nontrivial solution if

$$\begin{vmatrix} -3\omega^2 L + 1/C & 2\omega^2 L \\ 2\omega^2 L & -3\omega^2 L + 1/C \end{vmatrix} = (-3\omega^2 L + 1/C)^2 - 4\omega^4 L^2 = 0$$

or

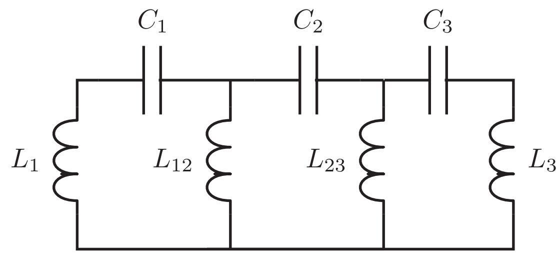
$$-3\omega^2 L + \frac{1}{C} = \pm 2\omega^2 L.$$

For each sign we get a normal mode frequency,

$$\omega_+ = \frac{1}{\sqrt{5LC}} \quad \text{and} \quad \omega_- = \frac{1}{\sqrt{LC}}$$

Now we can find the corresponding eigenvectors. Using $\omega_+ = 1/\sqrt{5LC}$, we can substitute into the algebraic equations to find that $B = -A$. In this normal mode the amplitudes are equal, and currents are in the opposite sense (that is, one current is moving clockwise while the other is counterclockwise, etc. There is therefore a large current through L_{12}. Inductors are analogous to mass, so the large engaged inductance is what makes the eigenfrequency low. Using $\omega_- = 1/\sqrt{LC}$, we find that $B = +A$, so in this case the amplitudes are equal and they have the same sense, with both currents moving clockwise, then both counterclockwise, etc. There is now no net current through L_{12}, so the frequency is higher than in the first normal mode. ■

★★ **Problem 13.8**



The voltage across a capacitor is $V_C = q/C$, where C is the capacitance and q is the charge on the capacitor. The voltage across an inductor is $V_L = LdI/dt$, where L is the inductance and I is the current through the inductor. A wire attached to a capacitor whose charge is changing carries a current $I = dq/dt$. The net voltage drop around any closed circuit is zero, so a simple electrical L, C circuit obeys $L\ddot{q} + q/C = 0$, and so oscillates with frequency $\omega = 1/\sqrt{LC}$. Find the normal mode oscillation frequencies of the three-loop circuit shown, for the case $C_1 = C_2 = C_3 = C$ and $L_1 = L_2 = L_{12} = L_{23} = L$.

Solution

If we move around each of the internal loops in the CW direction, we have

$$(1)\ \ \frac{q_1}{C} + L(\ddot{q}_1 - \ddot{q}_2) + L\ddot{q}_1 = 0$$

$$(2)\ \ \frac{q_2}{C} + L(\ddot{q}_2 - \ddot{q}_3) + L(\ddot{q}_2 - \ddot{q}_1) = 0$$

$$(3)\ \ \frac{q_3}{C} + L\ddot{q}_3 + L(\ddot{q}_3 - \ddot{q}_2) = 0$$

For normal modes we set $q_1 = Ae^{i\omega t}, q_2 = Be^{i\omega t}, q_3 = De^{i\omega t}$. Then we find

$$(1)\ \ \left(-2L\omega^2 + \frac{1}{C}\right) A + (L\omega^2)B = 0$$

$$(2)\ \ (L\omega^2)A + \left(-2L\omega^2 + \frac{1}{C}\right) B + (L\omega^2)D = 0$$

$$(3)\ \ (L\omega^2)B + \left(-2L\omega^2 + \frac{1}{C}\right) D = 0$$

There are nontrivial solutions if and only if the determinant of the coefficients vanishes: that is,

$$\begin{vmatrix} -2L\omega^2 + 1/C & L\omega^2 & 0 \\ L\omega^2 & -2L\omega^2 + 1/C & L\omega^2 \\ 0 & L\omega^2 & -2L\omega^2 + 1/C \end{vmatrix} = 0.$$

Expanding about the first row of the determinant, we find that

$$\left(-2L\omega^2 + \frac{1}{C}\right)\left[\left(-2L\omega^2 + \frac{1}{C}\right)^2 - 2(L\omega^2)^2\right] = 0,$$

which has three solutions for ω^2. The first of these, ω_1, comes from the first factor: This gives us the normal-mode frequency

$$\omega_1 = \frac{1}{\sqrt{2LC}}.$$

For the other two we have to expand the second factor and then use the quadratic formula for ω^2. This gives us the other two normal-mode frequencies

$$\omega_{2,3} = \sqrt{\frac{(1 \pm \sqrt{2}/2)}{LC}}.$$

These are the three normal-mode frequencies. ■

⋆⋆ **Problem 13.9** A block of mass M can move without friction on a horizontal rail. A simple pendulum of mass m and length ℓ hangs from the block. Find the normal mode frequencies for small-amplitude oscillations.

Solution

Let X be the distance of the center of the block from a fixed point at the same altitude, and hang the pendulum from the center of the block. Then the Cartesian coordinates of the plumb bob are $(x, y) = (X + \ell \sin\theta, -\ell\cos\theta)$, where θ is the angle of the pendulum relative to the vertical. Then the potential energy of the system is $U = mgy = -mg\ell\cos\theta$ and its kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}m\dot{X}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\dot{X}^2 + \frac{1}{2}m[(\dot{X} + \ell\cos\theta\dot{\theta})^2 + \ell^2\sin^2\theta\dot{\theta}^2] \\ &= \frac{1}{2}(M+m)\dot{X}^2 + m\ell\cos\theta\dot{X}\dot{\theta} + \frac{1}{2}m\ell^2\dot{\theta}^2. \end{aligned}$$

This expression for T would have been hard to guess directly using the generalized coordinates – it illustrates the importance of beginning with Cartesian coordinates in an inertial frame. Now we can assemble the Lagrangian for small oscillations, in which we use $\sin\theta \simeq \theta$ and $\cos\theta \simeq 1 - \theta^2/2$. The result is

$$L = \frac{1}{2}(M+m)\dot{X}^2 + m\ell\dot{X}\dot{\theta} + \frac{1}{2}m\ell^2\dot{\theta}^2 - \frac{1}{2}mg\ell\dot{\theta}^2$$

up to second order in small quantities and dropping unnecessary constants. Using Lagrange's equations for the two generalized coordinates X and θ, we find the two differential equations

$$(M+m)\ddot{X} + m\ell\ddot{\theta} = 0 \quad \text{and} \quad m\ell\ddot{X} + m\ell^2\ddot{\theta} + mg\ell\theta = 0$$

Eliminating $\ddot{X}$ between the two equations, we find that for small oscillations

$$\ddot{\theta} + \left(\frac{M+m}{M}\right)\frac{g}{\ell}\theta = 0,$$

which is the simple harmonic oscillator equation with frequency $\omega = \sqrt{\left(\frac{M+m}{M}\right)\frac{g}{\ell}}$. The other variable X oscillates with the same frequency. There could in addition be a uniform-velocity drift of the center of mass, with frequency $\omega = 0$. These then are the normal-mode frequencies, $\omega = 0$ and $\omega = \sqrt{\frac{M+m}{M}\frac{g}{\ell}}$. The first of these corresponds to a steady drift of the CM of the system, while in the second the CM stays at rest, the pendulum swings back and forth, while the block oscillates back and forth in the opposite direction, in such a way as to keep the overall CM at rest. ■

★★ **Problem 13.10** A block of mass M can move without friction on a horizontal rail. A horizontal spring of force-constant k connects one end of the block to a stationary wall. A simple pendulum of mass m and length ℓ hangs from the block. Find the normal mode frequencies for small-amplitude oscillations.

Solution

The Cartesian components of $M's$ position are $(x+\ell\sin\theta, -\ell\cos\theta)$, so its kinetic energy is

$$T = \frac{1}{2}(v_x^2 + v_y^2) = \frac{1}{2}m\left[(\dot{x} + \ell\cos\theta\dot{\theta})^2 + (\ell\sin\theta\,\dot{\theta})^2\right]$$

so the total Lagrangian of the system is

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left[\dot{x}^2 + 2\ell\cos\theta\dot{x}\dot{\theta} + \ell^2\dot{\theta}^2\right] - \frac{1}{2}kx^2 + mg\ell\cos\theta.$$

Then using the Lagrange equations to find the equations of motion for the generalized coordinates x and θ, we have

$$(M+m)\ddot{x} + m\ell(\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2) + kx = 0$$

and

$$m\ell\cos\theta\ddot{x} - m\ell\sin\theta\dot{\theta}\dot{x} + m\ell^2\ddot{\theta} + m\ell(\sin\theta\dot{x}\dot{\theta}) + mg\ell\sin\theta = 0.$$

These are nonlinear equations, so are difficult or impossible to solve except numerically. But for small amplitudes we can approximate $\sin\theta \cong \theta$ and $\cos\theta \cong 1$. Then the two equations become the linear equations

$$(M+m)\ddot{x} + m\ell\ddot{\theta} + kx \cong 0$$

and

$$m\ell\ddot{x} + m\ell^2\ddot{\theta} + mg\ell\theta \cong 0.$$

Now to find the normal modes, we let $x = Ae^{i\omega t}, \theta = Be^{i\omega t}$. Then we find, using matrix notation,

$$\begin{pmatrix} -(M+m)\omega^2 + k & -m\ell\omega^2 \\ -m\ell\omega^2 & -m\ell^2\omega^2 + mg\ell \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = 0$$

There is a nontrivial solution if the determinant of the coefficients of A and B is zero: That is, if

$$\left[-(M+m)\omega^2 + k\right]\left[-m\ell^2\omega^2 + mg\ell\right] - (m\ell\omega^2)^2 = 0.$$

This is a quadratic equation in ω^2. We can solve it using the quadratic formula, and then take the square root to get the two normal mode frequencies. The result is

$$\omega_{\pm} = \left\{ \frac{(M+m)g + k\ell}{2M\ell} \pm \frac{\sqrt{(M+m)^2 g^2 + k^2\ell^2 + 2(m-M)gk\ell}}{2M\ell} \right\}^{1/2} .$$

■

★★

Problem 13.11 The techniques used in this chapter can be extended to two- and three-dimensional systems. For example, we can find the normal-mode oscillations of a system of three equal masses *m* and three equal springs *k* in the configuration of an equilateral triangle, as shown in Figure 13.13(a).

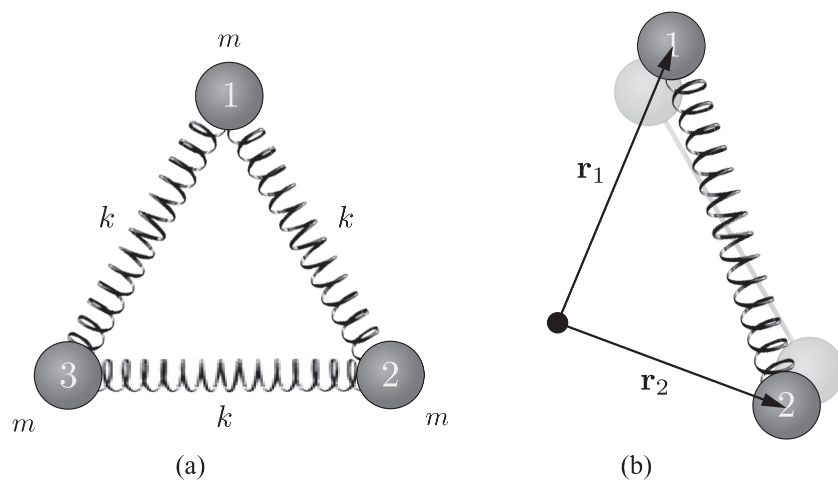


Fig. 13.13 (a) Three masses attached with identical springs and arranged in at the corners of an equilateral triangle. (b) Depiction of the stretching of one of the springs, along with two position vectors.

We will suppose the masses are free to move only in the plane of the triangle, so there are $3 \times 2 = 6$ degrees of freedom for this system. (a) How many of these modes are translational? rotational? vibrational? (b) Show that the mass matrix is given by

$$\hat{\mathbf{M}} = \begin{pmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & m & 0 & 0 \\ 0 & 0 & 0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 & 0 & m \end{pmatrix} ,$$

while the spring matrix takes the form

$$\hat{\mathbf{K}} = \begin{pmatrix} 2k & -k & -k & 0 & 0 & 0 \\ -k & 2k & -k & 0 & 0 & 0 \\ -k & -k & 2k & 0 & 0 & 0 \\ 0 & 0 & 0 & 2k & -k & -k \\ 0 & 0 & 0 & -k & 2k & -k \\ 0 & 0 & 0 & -k & -k & 2k \end{pmatrix} .$$

(c) Find the normal modes of vibrations; note that the matrices are block diagonal in that 3×3 sub-blocks do not mix. Note that, due to degeneracies, you will need to make choices for picking orthonormal eigenvectors. Show that, for one choice, the normal modes take the form shown in Figure 13.14.

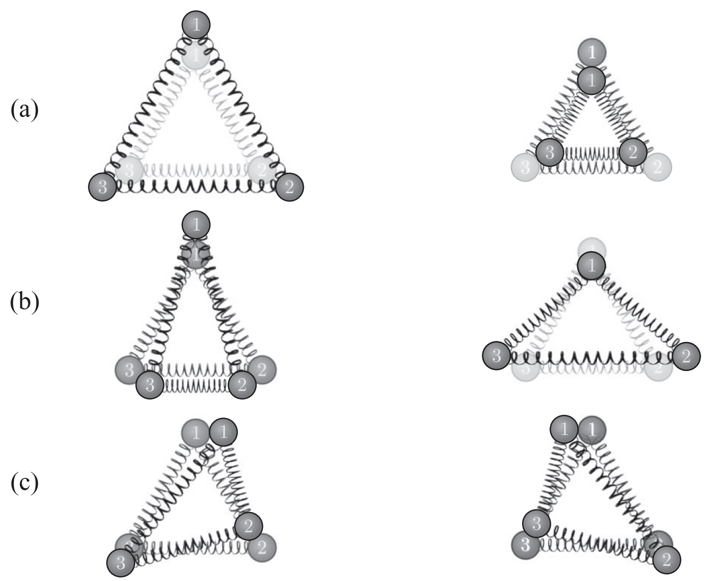


Fig. 13.14 The three vibrational modes of the triangular molecule.

Solution

Label the three masses 1, 2, and 3, beginning with No. 1 for the uppermost mass in the figure, then proceeding clockwise for Nos. 2 and 3. Describe displacements from the equilibrium positions by $x_1, y_1, x_2, y_2, x_3, y_3$. The kinetic energy of the system is then

$$T = \frac{1}{2}m[(\dot{x}_1^2 + \dot{y}_1^2) + (\dot{x}_2^2 + \dot{y}_2^2) + (\dot{x}_3^2 + \dot{y}_3^2)].$$

The potential energy stored in the spring attached between mass 1 and 2 is

$$U_{12} = \frac{1}{2}k[(x_1 - x_2)^2 + (y_1 - y_2)^2] .$$

To see this, note from Figure 13.13(b) that

$$\mathbf{r}_1 + \mathbf{R} - \mathbf{r}_2 = \mathbf{L} \Rightarrow |\mathbf{R} - \mathbf{L}|^2 = |\mathbf{r}_2 - \mathbf{r}_1|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 \; .$$

Similar expressions follow for the potential energies in the remaining two springs. The total Lagrangian of the system is therefore $L = T - U$, where

$$T = \frac{1}{2} m[(\dot{x}_1^2 + \dot{y}_1^2) + (\dot{x}_2^2 + \dot{y}_2^2) + (\dot{x}_3^2 + \dot{y}_3^2)]$$

and

$$\begin{aligned} U = \; & \frac{1}{2} k[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (x_2 - x_3)^2 + (y_2 - y_3)^2 \\ & + (x_3 - x_1)^2 + (y_3 - y_1)^2] \; . \end{aligned} \tag{13.2}$$

We can now identify the mass matrix $\hat{\mathbf{M}}$ and spring matrix $\hat{\mathbf{K}}$. Arranging the six dimensional vector space such that

$$q_k \to \begin{pmatrix} x_1 & x_2 & x_3 & y_1 & y_2 & y_3 \end{pmatrix} \; ,$$

we get first for the diagonal mass matrix

$$\hat{\mathbf{M}} = \begin{pmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & m & 0 & 0 \\ 0 & 0 & 0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 & 0 & m \end{pmatrix} \; .$$

And for the spring matrix, we end up with

$$\hat{\mathbf{K}} = \begin{pmatrix} 2k & -k & -k & 0 & 0 & 0 \\ -k & 2k & -k & 0 & 0 & 0 \\ -k & -k & 2k & 0 & 0 & 0 \\ 0 & 0 & 0 & 2k & -k & -k \\ 0 & 0 & 0 & -k & 2k & -k \\ 0 & 0 & 0 & -k & -k & 2k \end{pmatrix} \; .$$

We then have to solve the linear algebra problem

$$\left(\hat{\mathbf{K}} - \omega^2 \hat{\mathbf{M}}\right) \mathbf{b} = 0 \; ,$$

where we used

$$q_k(t) \to \left(b_{1x} e^{i\omega t} \;\; b_{2x} e^{i\omega t} \;\; b_{3x} e^{i\omega t} \;\; b_{1y} e^{i\omega t} \;\; b_{2y} e^{i\omega t} \;\; b_{3y} e^{i\omega t}\right) = \mathbf{b}\, e^{i\omega t}$$

Fortunately, our matrices have block-diagonal form: they consist of two 3 by 3 matrices that do not mix. Hence, we can deal with each of the two separate 3 by 3 blocks independently. This is why we arranged the vector space in such a way that the first three components correspond to the x coordinates while the last three to the y coordinates. The secular determinant for each block reads

$$\begin{vmatrix} -m\,\omega^2 + 2k & -k & -k \\ -k & -m\,\omega^2 + 2k & -k \\ -k & -k & -m\,\omega^2 + 2k \end{vmatrix} = 0.$$

Expanding the determinant, it is straightforward to show that there is a solution $\omega^2 = 0$ and a *double*-solution $\omega = \sqrt{3k/m}$, *i.e.*, we now have a scenario where the eigenvalues are degenerate. The first of these solutions, $\omega^2 = 0$, corresponds to the translational degrees of freedom, one in the x direction and one in the y direction. The second, when substituted into the algebraic equations, gives

$$-kb_{1x} - kb_{2x} - kb_{3x} = 0 \quad , \quad -kb_{1y} - kb_{2y} - kb_{3y} = 0$$

three times! Therefore the only guidance we have in finding normal-mode solutions are the equations

$$b_{1x} + b_{2x} + b_{3x} = 0 \quad \text{and} \quad b_{1y} + b_{2y} + b_{3y} = 0\,,$$

together with the knowledge that the oscillation frequency is $\omega = \sqrt{3k/m}$ for all modes. Remembering that the normalization condition gives us an additional relation we then have a total of three algebraic relations for six variables: b_{1x}, b_{2x}, b_{3x}, b_{1y}, b_{2y}, and b_{3y}. This means we have three independent modes. These modes are *degenerate*: all have the same eigenfrequency $\omega = \sqrt{3k/m}$. Note that for vibrational modes, the center of mass of the three masses remains at rest, both in the x and y directions. As mentioned earlier, when one has degeneracy, it is still possible to find an orthonormal set of normal modes that span the space of solutions. Let us see how this can be done.

When one has degeneracy, there are an infinite number of choices we might make for basis vectors that span the corresponding subspace. Having chosen a linearly independent set, any other mode with the same eigenvalue must be a linear combination of basis vectors from this set. Let us construct three degenerate normal modes for our system while driving our choices of modes by intuitive arguments. One type of oscillation that obviously repeats itself is one in which all three masses move in and out relative to the center of mass with equal amplitudes and in phase with one another, as shown in Figure 13.14(a). This is often called the "breathing" mode: It is as though the system was breathing in and out. The center of mass of the system, which is $2/3$ of the way from each vertex on a line from that vertex to the midpoint of the opposite side, can remain at rest, so there would not be a net linear momentum. There is also no *angular* momentum of the system about the center of mass; this is important, because the only degree of freedom allowed to have a net angular momentum is the rotational degree of freedom. Again, let the mass at the upper vertex in the diagram be designated No. 1, and the masses at the lower right and the lower left be No. 2 and No. 3, respectively, as shown in the Figure. Then as the system breathes in and out, b_{1x} is always zero, and also $b_{3x} = -b_{2x}$, and also $b_{2y} + b_{3y} = 2\,b_{2y} = -b_{1y}$. Using also the normalization condition, the first normal mode can be chosen as

$$\mathbf{e}^{(1)} = \frac{1}{\sqrt{m}} \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} \end{pmatrix} .$$

Now let us try to find a second normal mode in which mass No. 1 also moves purely in the y direction. If one exists, then it must still be true that $b_{3x} = -b_{2x}$ and $b_{2y} + b_{3y} = 2\,b_{2y} = -b_{1y}$. Therefore the second normal mode should take the form

$$\mathbf{e}^{(2)} = \begin{pmatrix} 0 & b_{2x} & -b_{2x} & b_{1y} & -\frac{b_{1y}}{2} & -\frac{b_{1y}}{2} \end{pmatrix} .$$

However, we also require that the second normal mode be *orthogonal* to the first normal mode, so the matrix product

$$\mathbf{e}^{(1)T} \cdot \mathbf{e}^{(2)} = 0 \Rightarrow b_{2x} + \frac{\sqrt{3}}{2} b_{1y} = 0 .$$

Along with the normalization conditon, this gives the second mode

$$\mathbf{e}^{(2)} = \frac{1}{\sqrt{m}} \begin{pmatrix} 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} \end{pmatrix} ,$$

which is sketched in Figure 13.14(b). It has the same frequency $\sqrt{3k/m}$, and it is normal to the breathing mode.

Note that in our second normal mode it is only mass No. 1 that moves away from and then toward the CM, while the other two masses do something different. Yet there is nothing unique about mass No. 1: We might equally well have chosen mass No. 2 as the one that moves away from and then toward the CM; the pattern of this mode is simply a 120° rotation of the pattern we chose initially. Calling this mode $\mathbf{e}^{(2b)}$, we have

$$\mathbf{e}^{(2b)} = \frac{1}{\sqrt{m}} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} .$$

Or we could have chosen mass No. 3 as the one that moves along a line intersecting the CM: the pattern of this mode amounts to a 240° rotation of our initial choice. Calling this mode $\mathbf{e}^{(2c)}$, we have

$$\mathbf{e}^{(2c)} = \frac{1}{\sqrt{m}} \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{2\sqrt{3}} \end{pmatrix} .$$

We might equally well have chosen either of these other modes, $\mathbf{e}^{(2b)}$ or $\mathbf{e}^{(2c)}$, as the second normal mode: Neither of them has a net linear momentum, for example, and neither has a net angular momentum. But perhaps one of these other modes could serve as our required *third* normal mode?

A clue that this cannot be true comes from a symmetry argument: If one of them can serve as the third normal mode, then why cannot the other be a normal mode as well? But that would give us a total of four vibrational normal modes, which is too many. In fact, it is easy to check that these alternative modes, although orthogonal to the breathing mode, are not orthogonal to our initial choice of the second normal mode, in which mass No. 1 moves along a line intersecting the CM. However, it could be that a *linear combination* of these two additional modes might serve as the third normal mode! By symmetry, the linear combination would have to weight these two additional modes equally, so that either their sum or their difference would be a candidate for the third normal mode. It is easy to show that the sum $\mathbf{e}^{(2c)} + \mathbf{e}^{(2b)}$ is *not* orthogonal to the second normal mode, but the difference between them, $\mathbf{e}^{(2c)} - \mathbf{e}^{(2b)}$, is indeed orthogonal to both the first and second normal modes, and so is our sought-for third normal mode. Properly normalized, this third normal mode is

$$\mathbf{e}^{(3)} = \frac{1}{\sqrt{m}}\begin{pmatrix}\frac{1}{\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & 0 & \frac{1}{2} & -\frac{1}{2}\end{pmatrix}.$$

and is depicted in Figure 13.14(c). Figure 13.14 shows all three normal modes we have chosen. They are not unique! Any oscillation of the system, which has no motion of the CM and no angular momentum about the CM, can be represented as a linear combination of these three. (b) Replace the third normal mode we have found here by an alternative. Show that your choice is orthogonal to the first two normal modes, that its CM does not move, and that it has no net angular momentum. ■

⋆ **Problem 13.12** In the previous problem, three degenerate normal modes were derived for the case of three equal masses at the vertices of an equilateral triangle, where the springs form the sides of the triangle. Any other oscillation in which the CM remains at rest and the system has no angular momentum must be a linear combination of these three modes. In particular, consider an oscillation identical to that of the second normal mode of the previous problem, except that it has been rotated to the right by 120°, so for example mass No. 2 at the lower right now oscillates directly toward and away from the CM, rather than mass No. 1. By symmetry with the second normal mode, this mode should be possible, and should have the same frequency. Find the linear combination of the normal modes of the previous problem which is equal to the oscillation described above.

Solution

We are looking at the mode

$$c = \frac{1}{\sqrt{m}}(-\frac{1}{2}, \frac{1}{2}, 0, -\frac{1}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}, \frac{1}{\sqrt{3}})$$

and we want to write it as a linear combination of

$$e^{(1)} = \frac{1}{\sqrt{m}}(0, \frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{3}}, -\frac{1}{2\sqrt{3}}. -\frac{1}{2\sqrt{3}})$$

$$e^{(2)} = \frac{1}{\sqrt{m}}(0, -\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{3}}, -\frac{1}{2\sqrt{3}}. -\frac{1}{2\sqrt{3}})$$

$$e^{(3)} = \frac{1}{\sqrt{m}}(\frac{1}{\sqrt{3}}, -\frac{1}{2\sqrt{3}}. -\frac{1}{2\sqrt{3}}, 0, \frac{1}{2}, -\frac{1}{2})$$

$$c = a_1 e^{(1)} + a_2 e^{(2)} + a_3 e^{(3)}$$

$$\Rightarrow a_1 = e^{(1)T} c = 0$$

$$a_2 = e^{(2)T} c = -\frac{1}{2}$$

$$a_3 = e^{(3)T} c = -\frac{\sqrt{3}}{2}$$

$$\Rightarrow c = -\frac{1}{2} e^{(2)} - \frac{\sqrt{3}}{2} e^{(3)}$$

■

★★★ **Problem 13.13** Starting from the matrix equation

$$(\hat{\mathbf{K}} - \omega_i^2 \hat{\mathbf{M}})\, \mathbf{b_i} = 0\,, \tag{13.3}$$

for the *i*th eigenvector, and knowing that $\hat{\mathbf{K}}$ and $\hat{\mathbf{M}}$ are symmetric and real, prove the following statements: (a) $\mathbf{b_j}^\dagger \hat{\mathbf{M}} \mathbf{b_j}$ is real and positive definite (no sum over j implied); (b) The eigenvalues ω_i^2 are real; (c) $\mathbf{b_j}^\dagger \hat{\mathbf{K}} \mathbf{b_j}$ is also real and positive definite if the system is stable (no sum over j implied); (d) $\mathbf{b_i}$ and $\mathbf{b_j}$ are orthogonal if $\omega_i^2 \neq \omega_j^2$. (e) The eigenvectors $\mathbf{b_i}$ are real up to a multiplicative constant.

Solution

(a) We start from

$$(\hat{\mathbf{K}} - \omega_i^2 \hat{\mathbf{M}})\, \mathbf{b}_i = 0 \text{ or } \hat{\mathbf{K}} \mathbf{b}_i = \omega_i^2 \hat{\mathbf{M}} \mathbf{b}_i \quad (*)$$

for the ith eigenvalue. Taking the complex conjugate, we have

$$\mathbf{b}_i^\dagger \hat{\mathbf{K}}^\dagger = (\omega_i^2)^* \mathbf{b}_i^\dagger \hat{\mathbf{M}}^\dagger \Rightarrow \mathbf{b}_i^\dagger \hat{\mathbf{K}} = (\omega_i^2)^* \mathbf{b}_i^\dagger \hat{\mathbf{M}} \quad (**)$$

where we know $\hat{\mathbf{K}}^\dagger = \hat{\mathbf{M}}$ and $\hat{\mathbf{M}}^\dagger = \hat{\mathbf{M}}$. Multiply (*) by $\mathbf{b}_j^\dagger$, the jth eigenvector from the left

$$\mathbf{b}_j^\dagger \hat{\mathbf{K}} \mathbf{b}_i = \omega_i^2 \mathbf{b}_j^\dagger \hat{\mathbf{M}} \mathbf{b}_i \quad (***)$$

Take (**) but for the jth eigenvalue

$$\mathbf{b}_j^\dagger \hat{\mathbf{K}} = (\omega_j^2)^* \mathbf{b}_j^\dagger \hat{\mathbf{M}}$$

and multiply by $\mathbf{b}_i$ from the right

$$\Rightarrow \mathbf{b}_j^\dagger \hat{\mathbf{K}} \mathbf{b}_i = (\omega_j^2)^* \mathbf{b}_j^\dagger \hat{\mathbf{M}} \mathbf{b}_i \quad (****)$$

Now subtract (***) from (****)

$$\Rightarrow 0 = (\omega_i^2 - (\omega_j^2)^* \mathbf{b}_j^\dagger \hat{\mathbf{M}} \mathbf{b}_i) \quad (*****)$$

Next, let's show that $\mathbf{b}_j^\dagger \hat{\mathbf{M}} \mathbf{b}_j$ is real and positive definite. Taking the complex conjugate

$$(\mathbf{b}_j^\dagger \hat{\mathbf{M}} \mathbf{b}_j)^* = \mathbf{b}_j^\dagger \hat{\mathbf{M}}^\dagger \mathbf{b}_j = \mathbf{b}_j^\dagger \hat{\mathbf{M}} \mathbf{b}_j \text{ no sum over } j$$

$$\Rightarrow \mathbf{b}_j^\dagger \hat{\mathbf{M}} \mathbf{b}_j \text{ is real.}$$

Now write $\mathbf{b}_j = \mathbf{b}_j^R + i\mathbf{b}_j^I$ (where $\mathbf{b}_j^R$ and $\mathbf{b}_j^I$ are real)

$$\Rightarrow \mathbf{b}_j^\dagger \hat{\mathbf{M}} \mathbf{b}_j = (\mathbf{b}_j^R)^T \hat{\mathbf{M}} \mathbf{b}_j^R + (\mathbf{b}_j^I)^T \hat{\mathbf{M}} \mathbf{b}_j^I + i(\mathbf{b}_j^R)^T \hat{\mathbf{M}} \mathbf{b}_j^I - i(\mathbf{b}_j^I)^T \hat{\mathbf{M}} \mathbf{b}_j^R$$

$$= (\mathbf{b}_j^R)^T \hat{\mathbf{M}} \mathbf{b}_j^R + (\mathbf{b}_j^I)^T \hat{\mathbf{M}} \mathbf{b}_j^I$$

where the imaginary part vanishes because $\hat{\mathbf{M}}^T = \hat{\mathbf{M}}$. But then $\mathbf{b}_j^\dagger \hat{\mathbf{M}} \mathbf{b}_j$ is proportioned to the kinetic energy which is always positive

$$\Rightarrow \mathbf{b}_j^\dagger \hat{\mathbf{M}} \mathbf{b}_j \text{ is real and positive definite}$$

(b) From (****), with $i = j$, we conclude $(\omega_2)^2$ are real.

(c) Going back to (***) with $i = j$,

$$\mathbf{b}_i^\dagger \hat{\mathbf{K}} \mathbf{b}_i = \omega_i^2 \mathbf{b}_i^\dagger \hat{\mathbf{M}} \mathbf{b}_i \text{ (no sum over } i)$$

$$\Rightarrow \omega_i^2 = \frac{\mathbf{b}_i^\dagger \hat{\mathbf{K}} \mathbf{b}_i}{\mathbf{b}_i^\dagger \hat{\mathbf{M}} \mathbf{b}_i}$$

with the denominator having been shown to be positive definite. The numerator $\mathbf{b}_i^\dagger \hat{\mathbf{K}} \mathbf{b}_i$ must be positive if we are near a minimum of the potential. Since ω_i^2 is real $\Rightarrow$ $\mathbf{b}_i^\dagger \hat{\mathbf{K}} \mathbf{b}_i$ must also be real.

(d) From (*****), for

$$\omega_i^2 \neq \omega_j^2 \Rightarrow \mathbf{b}_i^\dagger \hat{\mathbf{K}} \mathbf{b}_i = 0 \Rightarrow \text{ orthogonality follows}$$

(e) Now, we will show that the eigenvectors can be chosen to be real. Notice from (*) that the ratio of eigenvectors must be real. Since ω_i^2 is real, up to a common multiplicative factor, the eigenvectors must then be real. We typically fix the overall normalization so that $\mathbf{b}^T \hat{\mathbf{K}} \mathbf{b} = 1$. ■

★ **Problem 13.14** (a) The "6-12" potential energy $U(r) = -2a/r^6 + b/r^{12}$, where a and b are positive constants, is sometimes used to approximate the potential energy between two atoms in a diatomic molecule, where the atoms are separated by a distance r. Find the effective force constant and the angular frequency of small oscillations of a classical atom of mass m about the equilibrium point. (b) Repeat part (a) for the "Morse" potential energy $U(r) = D_e(1 - e^{a(r-r_0)})^2$, where D_e, a, and r_0 are constants.

Solution

(a) Given

$$U(r) = -2a/r^6 + b/r^{12},$$

its derivative is

$$U'(r) = 12a/r^7 - 12b/r^{13},$$

which is zero at $r_0 = (b/a)^{1/6}$. Also

$$U'' = -84a/r^8 + 156b/r^{14},$$

so

$$U''(r_0) = -84a(\frac{a}{b})^{8/6} + 156b(\frac{a}{b})^{14/6} = -84a^{7/3}b^{-4/3} + 156a^{7/3}b^{-4/3}$$
$$= 72a^{7/3}b^{-4/3} = k_{\text{eff}}.$$

So

$$\omega = \sqrt{k_{\text{eff}}/m} = 6\sqrt{2}a^{7/6}b^{-2/3}/\sqrt{m}$$

(b) For

$$U(r) = D_e(1 - e^{a(r-r_0)})^2,$$

$$U'(r) = -2D_e a e^{a(r-r_0)}(1 - e^{a(r-r_0)}) \quad (= 0 \text{ at } r = r_0)$$

Then

$$U'' = -2D_e a\left[e^{a(r-r_0)}(-ae^{a(r-r_0)}) + ae^{a(r-r_0)}(1 - e^{a(r-r_0)})\right]$$

$$U'' = 2D_e a^2 e^{a(r-r_0)}\left[e^{a(r-r_0)} - 1 + e^{a(r-r_0)}\right] = 2D_e a^2 e^{a(r-r_0)}\left[2e^{a(r-r_0)} - 1\right].$$

So $u''(r_0) = 2D_e a^2$. Thus

$$k_{\text{eff}} = U''(r_0) = 2D_e a^2$$

and

$$\omega = \sqrt{\frac{k_{\text{eff}}}{m}} = \sqrt{\frac{2D_e}{m}}a.$$

■

★★ **Problem 13.15** Consider an infinite number of masses m connected in a linear array to an infinite number of springs k. In equilibrium the masses are separated by distance a. Now allow small-amplitude transverse displacements of the masses, and take the limit as $a \to 0$, with an infinite number of infinitesimal masses and an infinite number of infinitesimal spring-constants, so that the shape of the array as a function of time and space is given by $\eta(t,x)$, where η is transverse to the direction of the array in equilibrium. Show that if the amplitude is very small, then $\eta(t,x)$ obeys a linear wave equation, whose solutions can be traveling or standing transverse waves.

Solution

Consider a large number of discrete masses m_ℓ a distance a apart, connected by springs of force-constant k. Pull them slightly in the transverse direction. The small angles θ are relative to the horizontal. The vertical component of force on m_i due to the spring connected to m_{i-1} is

$$F_y = -(ky_1 - ky_{i-1}) \cong -ka(\partial y/\partial x)_i$$

for small angles. The vertical force on m_i due to the spring connected to m_{i+1} is

$$ka\left[(\frac{\partial y}{\partial x})_{i+1} - (\frac{\partial y}{\partial x})_i\right] = ka^2\frac{\partial^2 y}{\partial x^2}$$

by Taylor expansion. So by Newton's second law

$$F_y = m\frac{\partial^2 y}{\partial t^2}, \quad \text{and} \quad ka^2\frac{\partial^2 y}{\partial x^2} = m\frac{\partial^2 y}{\partial t^2} \quad \text{or} \quad \frac{\partial^2 y}{\partial x^2} - (\frac{m}{ka^2})\frac{\partial^2 y}{\partial t^2} = 0.$$

To complete the transform to a continuum, let $m = \mu a$, where μ =mass/length, and let a "stiffness" symbol $\kappa = \lim_{a\to 0}(ka)$. (As $a \to 0, k \to \infty$ so $ka \to \kappa$.) So the wave equation for transverse waves is

$$\frac{\partial^2 y}{\partial x^2} - (\frac{\mu}{\kappa})\frac{\partial^2 y}{\partial t^2} = 0.$$

■

★★ **Problem 13.16** (a) With the help of the trig identities

$$\begin{aligned}\sin(A \pm B) &= \sin A\cos B \pm \cos A\sin B\\ \cos(A \pm B) &= \cos A\cos B \mp \sin A\sin B\end{aligned}$$

prove the following results, often useful in Fourier analysis, where m and n are positive integers with $m \neq n$:

$$\begin{aligned}(i) \int_{-\pi}^{\pi} d\theta \sin m\theta \sin n\theta &= 0\\ (ii) \int_{-\pi}^{\pi} d\theta \sin m\theta \cos n\theta &= 0\\ (iii) \int_{-\pi}^{\pi} d\theta \cos m\theta \cos n\theta &= 0\end{aligned}$$

(b) Evaluate the same three integrals for the case $m = n$. (c) Evaluate the same three integrals, for both $m \neq n$ and $m = n$, if the range of integration is $(0, \pi)$ instead of $(-\pi, \pi)$.

Solution

(a) From the identities we have

$$\sin A\sin B = \cos(A - B) - \cos A\cos B$$

$$\sin A\sin B = -\cos(A + B) + \cos A\cos B.$$

Add these equations:

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

So with $A = m\theta, B = n\theta$ and integrating,

$$\int_{-\pi}^{\pi} d\theta \sin m\theta \sin n\theta = \frac{1}{2}\int_{-\pi}^{\pi} \cos(n - m)\theta d\theta - \frac{1}{2}\int_{-\pi}^{\pi} \cos(n + m)\theta d\theta$$

$$= \frac{1}{2}\frac{\sin(n - m)\theta}{n - m}\Big|_{-\pi}^{\pi} - \frac{1}{2}\frac{\sin(n + m)\theta}{n + m}\Big|_{-\pi}^{\pi} = 0$$

if $m \neq n$. (Sine of an integer $\times\ \theta$ is zero at $\theta = \pi$ and at $\theta = -\pi$) Similarly,

$$\cos A\cos B = \cos(A + B) + \sin A\sin B = \cos(A - B) - \sin A\sin B$$

Add the equations:

$$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

So

$$\int_{-\pi}^{\pi} d\theta \cos m\theta \cos m\theta = \frac{1}{2}\int_{-\pi}^{\pi} \left[\cos(m+n)\theta + \cos(n-m)\theta\right] d\theta$$

$$= \frac{1}{2}\frac{\sin(n+m)\theta}{n+m}\bigg|_{-\pi}^{\pi} + \frac{1}{2}\frac{\sin(n-m)\theta}{n-m}\bigg|_{-\pi}^{\pi} = 0 \quad n \neq m$$

Next use

$$\sin A \cos B = \sin(A+B) - \cos A \sin B = \sin(A-B) + \cos A \sin B$$

Add

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

So

$$\sin n\theta \cos m\theta = \frac{1}{2}\left[\sin(n+m)\theta + \sin(n-m)\theta\right]$$

$$\int_{-\pi}^{\pi} d\theta \sin m\theta \cos m\theta = -\frac{1}{2}\left[\frac{\cos(n+m)\theta}{n+m} + \frac{\cos(n-m)\theta}{n-m}\right] d\theta$$

$$= -\frac{1}{2}\left[\frac{\cos(n+m)\pi}{n+m} - \frac{\cos(n+m)(-\pi)}{n+m} + \text{etc.}\right]$$

$$= -\frac{1}{2}\left[0+0\right] (\text{because } \frac{\cos(n+m)(-\pi)}{n+m} = \frac{\cos(n+m)\pi}{n+m})$$

so $\int_{-\pi}^{\pi} = 0, n \neq m$.

(b) If $n = m$, then

$$\int_{-\pi}^{\pi} d\theta \sin^2 n\theta = \int_{-\pi}^{\pi} d\theta \frac{1}{2}(1 - \cos 2n\theta) = \frac{1}{2}\int_{-\pi}^{\pi} d\theta - \frac{1}{2}\int_{-\pi}^{\pi} d\theta \cos 2n\theta = \frac{2\pi}{2} = \pi$$

$$\int_{-\pi}^{\pi} d\theta \cos^2 n\theta = \int_{-\pi}^{\pi} d\theta \frac{1}{2}(1 + \cos 2n\theta) = \frac{1}{2}\cdot 2\pi = \pi$$

$$\int_{-\pi}^{\pi} d\theta \sin n\theta \cos n\theta = \int_{\theta=-\pi}^{\pi} \frac{d(\sin n\theta)}{n} \sin n\theta = \frac{1}{n}\frac{\sin^2 n\theta}{2}\bigg|_{-\pi}^{\pi} = 0 \quad n \neq 0.$$

If $n = 0$, $\int = 0$ anyway, so

$$\int_{-\pi}^{\pi} d\theta \sin^2 n\theta = \int_{-\pi}^{\pi} d\theta \cos^2 n\theta = \pi$$

$$\int_{-\pi}^{\pi} d\theta \sin n\theta \cos n\theta = 0.$$

(c) Looking back at part (a), and changing the limits of integration to 0 and π, it is clear that both

$$\int_0^\pi d\theta \sin m\theta \sin n\theta = 0 \text{ and } \int_0^\pi d\theta \cos m\theta \cos n\theta = 0$$

for $m\&n$. From part (b) clearly (for $m = n$)

$$\int_0^\pi d\theta \sin^2 n\theta = \frac{\pi}{2} - \frac{1}{2}\int_0^\pi d\theta \cos 2n\theta = \pi/2 - \frac{1}{2}\frac{\sin 2n\theta}{2n}\bigg|_0^\pi = \pi/2.$$

Also

$$\int_0^\pi d\theta \cos^2 m\theta = \pi/2$$

The integral

$$\int_0^\pi d\theta \sin n\theta \cos m\theta \text{ (for } n \neq m)$$

$$= -\frac{1}{2}\left[\frac{\cos(n+m)\theta}{n+m}\bigg|_0^\pi + \frac{\cos(n-m)\theta}{n-m}\bigg|_0^\pi\right]$$

$$= -\frac{1}{2}\left[\frac{\cos(n+m)\pi}{n+m} - 1\right] - \frac{1}{2}\left[\frac{\cos(n-m)\theta}{n-m} - 1\right]$$

If $n + m$ is even, the first square bracket is zero. If $n + m$ is odd, the first square bracket term is $-\frac{1}{2}(-2) = 1$. If $n - m$ is even, the second square bracket term is zero. If $n - m$ is odd, the second square bracket term is 1.

So for $n \neq m$,

$$\int_0^\pi d\theta \sin m\theta \cos m\theta = \begin{cases} 0 & \text{if } n, m \text{ both even or both odd} \\ 2 & \text{if one is even and the other is odd, for } n \neq m \end{cases}$$

If $n = m$,

$$\int_0^\pi d\theta \sin n\theta \cos n\theta = \int_0^\pi \frac{d(\sin n\theta)}{n} \sin n\theta = \frac{1}{n}\frac{\sin^2 n\theta}{2} 0^\pi = 0$$

Summary:

For $m \neq n$,

$$\int_{-\pi}^\pi d\theta \sin m\theta \sin n\theta = \int_{-\pi}^\pi d\theta \cos m\theta \cos n\theta = 0.$$

$$\int_{-\pi}^\pi d\theta \sin m\theta \cos m\theta = 0.$$

$$\int_0^\pi d\theta \sin m\theta \sin n\theta = \int_0^\pi d\theta \cos m\theta \cos m\theta = 0$$

$$\int_0^\pi d\theta \sin m\theta \cos m\theta = \begin{cases} 0 & \text{if } n, m \text{ both even or both odd} \\ 2 & \text{if one is even and the other is odd, for } n \neq m \end{cases}$$

For $m = n$,

$$\int_{-\pi}^{\pi} d\theta \sin^2 n\theta = \int_{-\pi}^{\pi} d\theta \cos^2 n\theta = \pi$$

$$\int_{-\pi}^{\pi} = d\theta \sin n\theta \cos n\theta = 0$$

$$\int_{0}^{\pi} d\theta \sin^2 n\theta = \int_{0}^{\pi} d\theta \cos^2 n\theta = \pi/2$$

$$\int_{0}^{\pi} d\theta \sin n\theta \cos n\theta = 0$$

■

⋆ **Problem 13.17** A rod of length L is clamped at both ends $x = 0, L$ so that the displacement function obeys $\eta(t,0) = \eta(t,L) = 0$. Initially the displacement function is $\eta(0,x) = b\sin^2(\pi x/L)$ and $\partial\eta(t,x)/\partial t|_0 = 0$, where b is a positive constant. Find a Fourier-series representation of the solution of the wave equation at all future times.

Solution

As in Example 13.7, we can expand in terms of standing waves,

$$\eta(t,x) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} \cos\frac{n\pi vt}{L}$$

which satisfies the boundary conditions

$$\eta(t,0) = \eta(t,L) = 0$$

and the initial condition that $\eta(t,x)$ is a maximum at $t = 0$, we then have

$$\eta(0,x) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L},$$

a Fourier series. Multiply by $\sin\frac{m\pi x}{L}$ and integrate:

$$\sum_{n=1}^{\infty} C_n \int_0^L \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L} = C_m L/2$$

$$= \int_0^L dx \sin\frac{m\pi x}{L}(b\sin^2 \pi x/L).$$

Use the identity

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

to get

$$C_m = (\frac{b}{L}) \int_0^L dx \sin\frac{m\pi x}{L}(1-\cos\frac{2\pi x}{L}).$$

Now

$$\int_0^L dx \sin\frac{m\pi x}{L} = -\frac{\cos m\pi x/L}{m\pi/L}\Big|_0^L = \frac{L}{m\pi}(1-\cos m\pi)$$

$$\sin\frac{L}{m\pi} = \begin{cases} 0 & \text{if } m \text{ is even} \\ 2 & \text{if } m \text{ is odd} \end{cases}$$

and

$$\int_0^L dx \frac{m\pi x}{L}\cos\frac{2\pi x}{L} = -\left[\frac{\cos(\frac{m\pi}{L}-\frac{2\pi}{L})x}{2(\frac{m\pi}{L}-\frac{2\pi}{L})} + \frac{\cos(\frac{m\pi}{L}+\frac{2\pi}{L})x}{2(\frac{m\pi}{L}+\frac{2\pi}{L})}\right]_0^L$$

$$= -\left[\frac{(\cos(m-2)\pi-1)}{2(m-2)\pi/L} + \frac{(\cos(m+2)\pi-1)}{2(m+2)\pi/L}\right] \quad m \neq 2$$

$$= -\frac{L}{2\pi} = \begin{cases} 0 & m \text{ even} \\ \frac{-2}{m-2} - \frac{2}{m+2} & m \text{ odd} \end{cases} = \begin{cases} 0 & \text{if } m \text{ is even} \\ \frac{2Lm}{\pi(m^2-4)} & \text{if } m \text{ is odd} \end{cases}$$

(if $m = 2$,

$$\int_0^L dx \sin\frac{2\pi x}{L}\cos\frac{2\pi x}{L} = \frac{L}{2\pi}\int_0^{2\pi} dy \sin y \cos y = \frac{L}{2\pi}\frac{\sin^2 y}{2}\Big|_0^{2\pi} = 0)$$

So altogether,

$$C_m = \frac{b}{m\pi}\begin{cases} 0 & m \text{ even} \\ 2 & m \text{ odd} \end{cases} - \frac{2bm}{\pi(m^2-4)}\begin{cases} 0 & m \text{ even} \\ 1 & m \text{ odd} \end{cases} = \begin{cases} 0 & m \text{ even} \\ \frac{2b}{m\pi} - \frac{2bm}{\pi(m^2-4)} & m \text{ odd} \end{cases}$$

$$= \begin{cases} 0 & m \text{ even} \\ \frac{8b}{\pi m(4-m^2)} & m \text{ odd} \end{cases}$$

These are the Fourier coefficients in

$$\eta(0,x) = \sum_{n=1}^{\infty} C_n \sin\frac{m\pi x}{L}.$$

$$C_m = \frac{8b}{\pi}(\frac{1}{3}, -\frac{1}{15}, -\frac{1}{105}, \ldots)$$

■

⋆ **Problem 13.18** A rod of length L, with ends at $(x = 0, L)$, has an initial displacement function $\eta(0,x) = b$ for $0 \leq x \leq L/2$ and $\eta(0,x) = -b$ for $L/2 \leq x \leq L$, where b is a positive constant. At time $t = 0$ the derivative of $\eta(t,x)$ is $\partial\eta(t,x)\partial t|_0 = -v_0$ for $0 \leq x \leq L/2$ and equal to $+v_0$ for $L/2 \leq x \leq L$, where v_0 is a positive constant. Find a Fourier-series representation of the solution of the wave equation at all future times using the doubling trick.

Solution

We use the doubling trick

$$\eta(t, x+L) = \eta(tL - x)$$

which implies

$$\eta(t, x+2L) = \eta(t, x)$$

so that we can use Fourier series:

$$\eta(t,x) = \sum_n a_n e^{n2\pi ix/2L}$$

$$\sum_n a_n e^{\frac{\pi in}{L}(x+L)} = \sum_n a_n e^{\frac{\pi in}{L}(L-x)} \Rightarrow \sum_n a_n e^{i\pi nx/L} = \sum_n a_n e^{-i\pi nx/L}$$

$$\Rightarrow a_n = a_{-n}$$

$$\eta(t,x) = \sum_{n=-\infty}^{\infty} a_n e^{\pi inx/L}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n e^{\pi inx/L} + \sum_{n=-1}^{-\infty} a_n e^{\pi inx/L}$$

$$= a_0 + \sum_{n=1}^{\infty} (a_n e^{\pi inx/L} + a_{-n} e^{-\pi nx/L})$$

$$\boxed{\eta(t,x) = a_0(t) + 2\sum_{n=1}^{\infty} a_n(t)\cos(\pi nx/L)}$$

Now we substitute in the wave equation

$$-\frac{1}{v^2}\ddot{\eta} + \eta'' = 0$$

$$-\frac{1}{v^2}\ddot{a} - \frac{2}{v^2}\sum_{n=1}^{\infty}\ddot{a}_n\cos(\pi nx/L) - 2\sum_{n=1}^{\infty} a_n \cos(\pi nx/L)(\frac{\pi r}{L})^2 = 0$$

$$\Rightarrow \ddot{a}_0 = 0 \Rightarrow a_0 = A_0 t + B_0$$

$$n \geqslant 1 : \ddot{a}_n + v^2(\frac{\pi n}{L})^2 a_n = 0$$

$$\Rightarrow a_n(t) = A_n e^{\frac{i\pi nvt}{L}} + B_n e^{-\frac{i\pi nvt}{L}}$$

We first consider the general boundary conditions:

$$\eta(0,x) = f(x) \quad \dot{\eta}(0,x) = g(x)$$

$$\Rightarrow a_0(0) = 2\sum_{n=1}^{\infty} a_n(0)\cos(\pi xn/L) = f(x)$$

$$B_0 + 2\sum_{n=1}^{\infty}(A_n + B_n)\cos(\pi xn/L) = f(x)$$

$$A_0 + 2\sum_{n=1}^{\infty}\frac{i\pi nv}{L}(A_n - B_n)\cos(\pi xn/L) = g(x)$$

Using the orthonormality

$$\int_0^L \cos(\frac{\pi nx}{L})\cos(\frac{\pi mx}{L})dx = \delta_{mn}\frac{L}{2} \quad m,n \geqslant 1$$

$$\int_0^L \cos(\frac{\pi nx}{L}))dx = 0$$

$$\Rightarrow B_0 = \frac{1}{L}\int_0^L f(x)dx \quad A_0 = \frac{1}{L}\int_0^L g(x)dx$$

$$m \geqslant 1 : L(A_m + B_m) = \int_0^L f(x)\cos(\frac{\pi mx}{L})dx = \int_0^{L/2} b\cos(\frac{\pi mx}{L})dx - \int_{L/2}^L b\cos(\frac{\pi mx}{L})dx$$

$$i\pi mv(A_m - B_m) = \int_0^L g(x)\cos(\frac{\pi mx}{L})dx = -\int_0^{L/2} v_0\cos(\frac{\pi mx}{L})dx + \int_{L/2}^L v_0\cos(\frac{\pi mx}{L})dx$$

We evaluate the common integral

$$\int_0^{L/2}\cos(\frac{\pi mx}{L})dx - \int_{L/2}^L \cos(\frac{\pi mx}{L})dx = \frac{2L}{\pi m}\sin(\frac{\pi m}{2})$$

$$\Rightarrow A_m + B_m = \frac{2b}{\pi m}\sin(\frac{\pi m}{2})$$

$$A_m - B_m = \frac{iv_0}{\pi mv}\frac{2L}{\pi m}\sin(\frac{\pi m}{2}) = \frac{2iv_0L}{(\pi m)^2 v}\sin(\frac{\pi m}{2})$$

$$\Rightarrow A_m = \frac{1}{\pi m}\sin(\frac{\pi m}{2})\left[b + \frac{iv_0L}{\pi mv}\right]$$

$$B_m = \frac{\sin(\pi m/2)}{\pi m}\left[b - \frac{iv_0L}{\pi mv}\right]$$

and $A_0 = B_0 = 0$

We then substitute in $x(t,x)$ and simplify:

$$\Rightarrow \eta(t,x) = 2\sum_{n=1}^{\infty}\cos(\frac{\pi nx}{L})\left[\frac{\sin(\pi n/2)}{\pi n}(b + \frac{iv_0L}{\pi nc})e^{\frac{i\pi nvt}{L}} + \frac{\sin(\pi n/2)}{\pi n}(b - \frac{iv_0L}{\pi nc})e^{\frac{-i\pi nvt}{L}}\right]$$

$$= 4\sum_{n=1}^{\infty}\cos(\frac{\pi nx}{L})\frac{\sin(\pi n/2)}{(\pi n)}\left[b\cos(\frac{\pi nvt}{L}) - \frac{v_0L}{\pi nc}\sin(\frac{\pi nvt}{L})\right]$$

■

⋆ **Problem 13.19** A rod of length L, with ends at $(x = 0, L)$, has an initial displacement function $\eta(0,0) = \eta(0,L) = 0$ and $\eta(0,x) = b$ for $0 < x < L$, where b is a positive constant. (That is, η is discontinuous at the ends.) At time $t = 0$ the derivative of η with respect to time is $\partial\eta(t,x)/\partial t = 0$ for all x. Find a Fourier-series representation of the solution of the wave equation at all future times, for points $0 < x < L$.

Solution

Expand in terms of standing waves, as in Example 13.7.

$$\eta(t,x) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}\cos\frac{n\pi xt}{L}.$$

At $t = 0$,

$$\eta(0,x) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}$$

where the C_n are Fourier coefficients. Note $\eta = 0$ at $x = 0$ and $x = L$, as required. Also

$$\left.\frac{\partial\eta(t,x)}{\partial t}\right|_{t=0} = -\sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}\left(\frac{n\pi v}{L}\right)\sin\frac{n\pi vt}{L} = 0$$

at $t = 0$, as required. Now to find the C_n, multiply $\eta(0,x)$ by $\sin m\pi x/L$ and integrate.

$$\sum_{n=1}^{\infty} C_n \int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = C_m L/2$$

$$= b\int_0^L dx \sin\frac{m\pi x}{L} = -\frac{Lb}{m\pi}\cos\left.\frac{m\pi x}{L}\right|_0^L = \frac{-bL}{m\pi}[\cos m\pi - 1]$$

So

$$C_m = \frac{2b}{m\pi}(1 - \cos m\pi) = \frac{2b}{m\pi}\begin{pmatrix} 0\ m\ \text{even} \\ 2\ m\ \text{odd}\end{pmatrix}$$

$$C_m = \begin{matrix} 0\ m\ \text{even} \\ \frac{4b}{\pi m}\ m\ \text{odd}\ \left(\frac{4b}{\pi}, \frac{4b}{3\pi}, \cdots\right)\end{matrix}$$

$$\eta(0,x) = \sum_{n=1}^{\infty} \frac{4b}{\pi m}\sin\frac{n\pi x}{L} \quad \text{odd } n$$

■

⋆ **Problem 13.20** One end of a rod of length L is held at $x = 0$ while the other end is stretched from $x = L$ to $x = (1 + a)L$, where a is a constant. In this way an arbitrary point x in the rod is moved to $(1 + a)x$. Then at time $t = 0$ the rod is released. (a) What is the initial value of the displacement function $\eta(0,x)$? (b) Find $\eta(t,x)$. (c) Show that the velocity at the left end of the rod is either $2av$ or $-2av$, alternating between these vales with a time interval L/v, where v is the wave velocity in the rod. You might want to use the doubling trick from the text.

Solution

(a) The initial displacement is

$$\eta(0,x) = (1+a)x - x = ax \quad 0 \leqslant x \leqslant L$$

We also know $\dot{\eta}(0,x) = 0$

(b) We use the doubling trick

$$\eta(t, x+L) = \eta(t, L-x)$$

which implies

$$\eta(t, x+2L) = \eta(t,x)$$

so that we can use Fourier sereis:

$$\eta(t,x) = \sum_n a_n e^{n2\pi i x/2L}$$

$$\sum_n a_n e^{\frac{\pi i n(x+L)}{L}} = \sum_n a_n e^{\frac{n\pi i}{L}(L-x)} \Rightarrow \sum_n a_n e^{i\pi n x/L} = \sum_n a_n e^{-i\pi n x/L}$$

$$\Rightarrow a_n = a_{-n}$$

$$\eta(t,x) = \sum_{n=-\infty}^{\infty} a_n e^{\pi i n x/L}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n e^{i\pi n x/L} + \sum_{n=-1}^{-\infty} a_n e^{\frac{i\pi n x}{L}}$$

$$= a_0 + \sum_{n=1}^{\infty} (a_n e^{i\pi n x/L} + a_{-n} e^{-i\pi n x/L})$$

$$\boxed{\eta(t,x) = a_0(t) + 2\sum_{n=1}^{\infty} a_n(t)\cos(\pi n x/L)}$$

Now we substitute in the wave equation

$$-\frac{1}{v^2}\ddot{\eta} + \eta'' = 0$$

$$-\frac{1}{v^2}\ddot{a} - \frac{2}{v^2}\sum_{n=1}^{\infty} \ddot{a}_n \cos(\pi n x/L) - 2\sum_{n=1}^{\infty} a_n \cos(\pi n x/L)(\frac{\pi r}{L})^2 = 0$$

$$\Rightarrow \ddot{a}_0 = 0 \Rightarrow a_0 = A_0 t + B_0$$

$$n \geqslant 1 : \ddot{a}_n + v^2(\frac{\pi n}{L})^2 a_n = 0$$

$$\Rightarrow a_n(t) = A_n e^{\frac{i\pi n v t}{L}} + B_n e^{-\frac{i\pi n v t}{L}}$$

We first consider the general boundary conditions:

$$\eta(0,x) = f(x) \quad \dot{\eta}(0,x) = g(x)$$

$$\Rightarrow a_0(0) = 2\sum_{n=1}^{\infty} a_n(0)\cos(\pi xn/L) = f(x)$$

$$B_0 + 2\sum_{n=1}^{\infty}(A_n + B_n)\cos(\pi xn/L) = f(x)$$

$$A_0 + 2\sum_{n=1}^{\infty}\frac{i\pi nv}{L}(A_n - B_n)\cos(\pi xn/L) = g(x)$$

Using the orthonormality

$$\int_0^L \cos(\frac{\pi nx}{L})\cos(\frac{\pi mx}{L})dx = \delta_{mn}\frac{L}{2} \quad m,n \geqslant 1$$

$$\int_0^L \cos(\frac{\pi nx}{L}))dx = 0$$

$$\Rightarrow B_0 = \frac{1}{L}\int_0^L f(x)dx \quad A_0 = \frac{1}{L}\int_0^L g(x)dx$$

$$m \geqslant 1 : L(A_m + B_m) = \int_0^L f(x)\cos(\frac{\pi mx}{L})dx = \int_0^L ax\cos(\frac{\pi mx}{L})dx$$

$$i\pi mv(A_m - B_m) = \int_0^L g(x)\cos(\frac{\pi mx}{L})dx = 0 \Rightarrow A_m = B_m$$

$$A_m = \frac{a}{2L}\int_0^L x\cos(\frac{\pi mx}{L})dx = \frac{aL}{2\pi^2 m^2}((-1)^m - 1)$$

$$A_0 = 0 \quad B_0 = \frac{1}{L}\int_0^L axdx = \frac{aL}{2}$$

$$\Rightarrow \eta(t,x) = \frac{aL}{2} + 2\sum_{n=1}^{\infty}\cos(\frac{\pi nx}{L})\cos(\frac{\pi nvt}{L})\frac{aL}{\pi^2 n^2}((-1)^n - 1)$$

(c) We want $\dot{\eta}(t,0)$

$$\eta(t,0) = \frac{aL}{2} + 2\sum_{n=1}^{\infty}\cos(\frac{\pi nvt}{L})\frac{aL}{\pi^2 n^2}((-1)^n - 1)$$

$$\Rightarrow \dot{\eta}(t,0) = -2\sum_{n=1}^{\infty}\sin(\frac{\pi nvt}{L})\frac{av}{\pi n}((-1)^n - 1)$$

$$= -2av \sum_{n=1}^{\infty} \frac{\sin(\pi nvt/L)}{\pi n}((-1)^n - 1)$$

Periodic with $t \to t + \frac{2L}{v}$ oscillates between $\pm 2av$ with period $\frac{L}{v}$. ■

★ **Problem 13.21** An infinite rod has an initial square-pulse displacement function $\eta(0,x) = C$, a constant, for $|x| \le b$ and $\eta(0,x) = 0$ for $|x| > b$. (a) Find the displacement function $\eta(t,x)$ at later times, assuming all mass points in the rod are initially at rest. (b) Carry out a Fourier transform of $\eta(0,x)$ to determine $g(0,k)$, which shows the degree to which various wavelengths make up the square pulse.

Solution

(a) Given $\eta(0,x) = c$ for $|x| \leqslant b$ and $\eta(0,x) = 0$ for $|x| > b$.

$$\eta(t,x) = [f(x+vt) + f(x-vt)] \;\text{ so } \eta(0,x) = f(x).$$

Then

$$\eta(0,x) = \frac{1}{2}[C + C] = C$$

$$\eta(t,x) = \frac{1}{2}[(C+vt) + (C-vt)]$$

so an initial square wave becomes two square waves moving in opposite directions, each half the magnitude of the original.

(b)

$$g(0,k) = \frac{1}{2\pi}\int_{-b}^{b} dx\eta(0,x)e^{-ikx} = \frac{C}{2\pi}\int_{-b}^{b} dxe^{-ikx}$$

$$= \frac{C}{2\pi}\frac{e^{-ikx}}{-ik}\Big|_{-b}^{b} = -\frac{2}{2\pi ik}(e^{-ikb} - e^{ikb})$$

$$= \frac{C}{\pi k}(\frac{e^{-ikb} - e^{ikb}}{2i}) = \frac{C\sin(kb)}{\pi k}$$

■

★ **Problem 13.22** An infinite rod has an initial triangular-pulse displacement function $\eta(0,x) = C - |x|$ for $|x| < C$, where C is a constant, and zero otherwise. (a) Find the displacement function $\eta(t,x)$ at later times, assuming all mass points in the rod are initially at rest. (b) Carry out a Fourier transform of $\eta(0,x)$ to determine $g(0,k)$, which shows the degree to which various wavelengths make up the triangular pulse.

Solution

(a) Given $\eta(0,x) = C - |x|$ for $|x| < C$, $\eta(0,x) = 0$ otherwise. Then since all points in the rod are initially at rest, we have

$$\eta(t,x) = \frac{1}{2}[f(x+vt) + f(x-vt)]$$

so $\eta(0,x) = f(x)$. Then

$$\eta(t,x) = \frac{1}{2}\left[C - |x+vt| + C - |x-vt|\right]$$

which means that the original triangle splits into two triangles, each with half the total height, traveling in opposite directions at speed v.

(b)

$$g(0,k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\eta(0,x)e^{-ikx} = \frac{1}{2\pi}\left[\oint_{-C}^{0} dx(C+x)e^{-ikx} + \int_{0}^{C} dx(C-x)e^{-ikx}\right]$$

$$= \frac{1}{2\pi}\left[C\int_{-C}^{C} dxe^{-ikx} + \int_{-C}^{0} dxxe^{-ikx} - \int_{0}^{C} dxxe^{-ikx}\right]$$

$$= \frac{C}{\pi k}\left(\frac{e^{ikC} - e^{-ikC}}{2i}\right) - \frac{1}{\pi}\int_{0}^{C} dxx\left(\frac{e^{ikx} + e^{-ikx}}{2}\right)$$

$$= \frac{C}{\pi k}\sin kC - \frac{1}{\pi}\int_{0}^{C} dxx\cos kx = \frac{C}{\pi k}\sin kC - \frac{1}{\pi k^2}\int_{0}^{kC} dyy\cos y$$

$$g(0,k) = \frac{1-\cos(kC)}{\pi k^2}$$

■

★★ **Problem 13.23** An infinite rod has an initial Gaussian displacement function $\eta(0,x) = Ae^{-x^2/b^2}$, where A and b are constants. (a) Carry out a Fourier transform of $\eta(0,x)$, and show that the result is a Gaussian function in k space. (b) Then show that if the Gaussian in position space is narrow (with b small), then the Gaussian in k space is wide, and vice versa. (c) Define Δx as the distance between the two points on the position-space Gaussian for which $\eta(0,x)$ is half its maximum value. Similarly, define Δk as the distance in k space between the two points on $g(0,k)$ for which $g(0,k)$ is half its maximum value. Then find the product $\Delta x \cdot \Delta k$, and show that it is independent of b. *Hint*: The Fourier integrals can be evaluated by completing the square in the exponents.

Solution

(a) Given

$$\eta(0,x) = Ae^{-x^2/b^2},$$

its Fourier transform is

$$\omega(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\eta(0,x)e^{-ikx} = \frac{A}{2\pi}\int_{-\infty}^{\infty} dxe^{-x^2/b^2-ikx}.$$

We complete the square in the exponent, giving

$$\left(-\frac{x^2}{b^2} - ikx\right) = -\left(\frac{x}{b} + c\right)^2 + c^2$$

where we want to find the constant c. That is,

$$-(\frac{x^2}{b^2}+ikx) = -\frac{x^2}{b^2} - \frac{2cx}{b}$$

so

$$\frac{2c}{b} = ik \quad c = \frac{ikb}{2}.$$

Therefore

$$a(k) = \frac{A}{2\pi}\int_{-\infty}^{\infty} dx e^{-(\frac{x}{b}+\frac{ikb}{2})^2 - \frac{k^2b^2}{4}}.$$

Let

$$z = \frac{x}{b} + \frac{ikb}{2} \quad dz = dx/b$$

so then

$$a(k) = \frac{Ab}{2\pi}e^{-\frac{k^2b^2}{4}}\int_{-\infty}^{\infty} dz e^{-z^2} = \frac{Ab}{2\sqrt{\pi}}e^{-\frac{b^2k^2}{4}}$$

which is a Gaussian in k-space.

(b) If b is small, then

$$\eta(0,x) = Ae^{-x^2/b^2}$$

is narrow while its Fourier transform is

$$a(k) = \frac{Ab}{2\sqrt{\pi}}e^{-b^2k^2/4},$$

which is wide. Since in η the exponent is $(x/b)^2$, while in $a(k)$ the exponent is $(bk/2)^2$.

(c) η is half maximum when $\eta = A/2$, i.e. when

$$e^{-x^2/b^2} = 1/2 \quad 0 + x^2/b^2 = \ln 2 \quad x^2 = b^2 \ln 2$$

$$x = \pm b\sqrt{\ln 2}, \text{ so } \Delta x = 2b\sqrt{\ln 2}$$

In k-space the Gaussian is at half maximum when

$$e^{-b^2k^2/4} = \frac{1}{2} \text{ or } \frac{b^2k^2}{4} = \ln 2 \quad k^2 = \frac{4}{b^2}\ln 2$$

$$k = \pm\frac{2}{b}\sqrt{\ln 2} \Rightarrow \Delta k = \frac{4}{b}\sqrt{\ln 2}.$$

Thus

$$\Delta x \Delta k = 2b\sqrt{\ln 2}\frac{4}{b}\sqrt{\ln 2} = 8\ln 2$$

independent of b. Here $g(0,k) = a(k)$. ■

★ **Problem 13.24** At the end of the chapter we derived a general expression for waves $y(t,x)$ on a long string, in terms of the initial displacement $y(0,x) \equiv f(x)$ and velocity $\partial y(0,x)/\partial t \equiv g(x)$. Suppose that the initial displacement is $y(0,x) = f(x)$, where $f(x)$ is some given function. (a) What $g(x)$ would be required, in terms of $f(x)$, so that for any time $t > 0$, there is *only* a wave traveling to the right: $y(t,x) = f(x - vt)$? (b) Find this $g(x)$ in the special case that $f(x)$ is the Gaussian function $f(x) = Ae^{-x^2/b^2}$, where A and b are constants.

Solution

(a) We know that

$$\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = 0,$$

so if

$$y(t,x) = f(x - vt) \ \text{ then } \ y(0,x) = f(x)$$

as needed. Then

$$\frac{\partial y}{\partial t} = -vf'(x - vt)$$

$$\Rightarrow \frac{\partial y}{\partial t}(0,x) = -vf'(x) = g(x)$$

so $g(x) = -vf'(x)$.

(b) We need

$$g(x) = -vf'(x) = -v\frac{d}{dx}(Ae^{-x^2/b^2}) = \frac{2v}{b^2}Ae^{-x^2/b^2}x$$

■

★★★ **Problem 13.25** In the text we saw an example involving a non-diagonal mass matrix arising in the case of a single particle. In this problem, we will look at a similar scenario for two particles. Consider two interacting particles of mass m_1 and m_2 constrained to move in one dimension described by the Lagrangian

$$L = \frac{1}{2}m_1\,\dot{q}_1^2 + \frac{1}{2}m_2\,\dot{q}_2^2 - U(q_1 + q_2)\,.$$

The coordinates of the two particles are represented by q_1 and q_2 and the potential energy function is given by $U(Q) = \alpha\,Q^2/2$ for some constant α. The novelty here is that the potential between the two particles is not translationally invariant; it does not depend on the distance between the particles, $q \equiv q_1 - q_2$. Instead, the potential depends on the *sum* of the two coordinates $Q \equiv q_1 + q_2$. As a result, the usual coordinate transformation from q_1 and q_2 to the center of mass coordinate $Q_{cm} = (m_1 q_1 + m_2 q_2)/(m_1 + m_2)$ and the relative distance $q = q_1 - q_2$ is not very useful. Instead, we want to transform to $Q = q_1 + q_2$ and $q = q_1 - q_2$. (a) Show that the Lagrangian in the Q and q coordinates takes the form

$$L = \frac{1}{8}M\dot{Q}^2 + \frac{1}{8}M\dot{q}^2 + \frac{1}{4}m\,\dot{Q}\,\dot{q} - U(Q)\,,$$

where $M \equiv m_1 + m_2$ and $m \equiv m_1 - m_2$. (b) Find the eigenvalues and eigenvectors of small oscillations. Elaborate briefly on the meaning of each normal mode.

Solution

We have

$$\hat{\mathbf{M}} = \begin{pmatrix} \frac{M}{4} & \frac{m}{4} \\ \frac{m}{4} & \frac{M}{4} \end{pmatrix}$$

and

$$\hat{\mathbf{K}} = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}$$

from which it follows that the normal mode frequencies are

$$\omega_1^2 = 0$$

and

$$\omega_2^2 = \frac{4\,M\,\alpha}{M^2 - m^2},$$

while the corresponding eigenvectors are

$$\mathbf{e}_1 = \left(0, \frac{2}{\sqrt{M}}\right)$$

$$\mathbf{e}_2 = \frac{2}{\sqrt{M^2 - m^2}} \left(-\sqrt{M}, \frac{m}{\sqrt{M}}\right)$$

■

★★ **Problem 13.26** Consider a particle of mass m moving in three dimensions but constrained to the surface of the paraboloid $z = \alpha\left((x-1)^2 + (y-1)^2\right)$. The particle is also subject to the spring potential $U(x,y,z) = (1/2)\,k\,(x^2 + y^2)$. (a) Show that the Lagrangian of the system is given by

$$L = \frac{1}{2} m\,\dot{x}^2 \left(1 + 4\,\alpha^2\right) + \frac{1}{2} m\,\dot{y}^2 \left(1 + 4\,\alpha^2\right) + \frac{1}{2} 8\,m\,\alpha^2 \dot{x}\,\dot{y} - \frac{1}{2} k \left(x^2 + y^2\right)$$

to quadratic order in x, y, $\dot{x}$, and $\dot{y}$ – assuming the displacement from the origin is small. (b) Find the normal modes, eigenfrequencies and eigenvectors.

Solution

We have

$$\hat{\mathbf{M}} = m \begin{pmatrix} 1 + 4\,\alpha^2 & 4\,\alpha^2 \\ 4\,\alpha^2 & 1 + 4\,\alpha^2 \end{pmatrix}$$

and

$$\hat{\mathbf{K}} = k \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

from which we find

$$\omega_1^2 = \frac{k}{m}$$

$$\omega_2^2 = \frac{k}{m\,\left(1 + 8\,\alpha^2\right)}$$

$$\mathbf{e}_1 = (b, -b) = \frac{1}{\sqrt{2\,m}}\,(1, -1)$$

$$\mathbf{e}_2 = (b, b) = \frac{1}{\sqrt{2\,m\;(1 + 8\,\alpha^2)}}\,(1, 1)$$

■

14 Complex Systems

14.1 Problems and Solutions

⋆ **Problem 14.1** Consider the equation of motion

$$\frac{d^2u}{d\varphi^2} + u - \frac{1}{p} = 3\,\lambda\,u^2$$

where p and λ are constants. Find the solution using perturbation theory to first order in the small parameter λ. Assess whether your approximate solution is a good one by solving the problem using numerical techniques and comparing with your result from the first order perturbation method.

Solution

$$\frac{d^2u}{d\varphi^2} + u - \frac{1}{p} = 3\lambda u^2$$

We write

$$u = u_0 + \lambda u_1 + \ldots$$

$$u_0'' + \lambda u_1'' + u_0 + \lambda u_1 - \frac{1}{p} = 3\lambda(u_0 + \lambda u_1)^2$$

$$\lambda^0 : \quad u_0'' + u_0 - \frac{1}{p} = 0$$

$$\lambda^1 : \quad u_1'' + u_1 = 3u_0^2$$

$$\Rightarrow u_0 = \frac{1 - \cos(\varphi)}{p} + A\cos(\varphi) + B\sin(\varphi)$$

with

$$u_0(0) = A \quad u_0'(0) = B$$

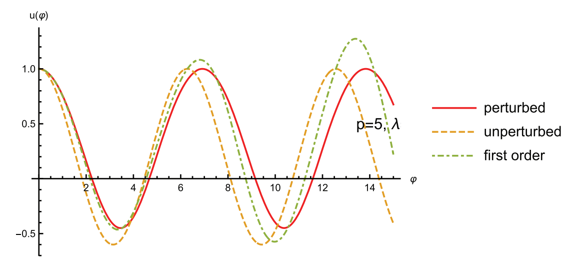
Then

$$u_1 = \int_0^{\varphi} 3v_0^2(\varphi')\sin(\varphi - \varphi')d\varphi'$$

$$= \frac{3}{2}\frac{(p(p-2)+3)}{p^2} - \frac{p(p-2)+4}{p^2}\cos\varphi - \frac{(p-1)^2}{2p^2}\cos 2\varphi$$
$$+ \frac{3(p-1)}{p^2}\varphi\sin\varphi$$

where we took $A = 1$ and $B = 0$ for concreteness and simplicity.

$$\Rightarrow u = \frac{1-\cos\varphi}{p} + \cos\varphi + \lambda u_1$$



■

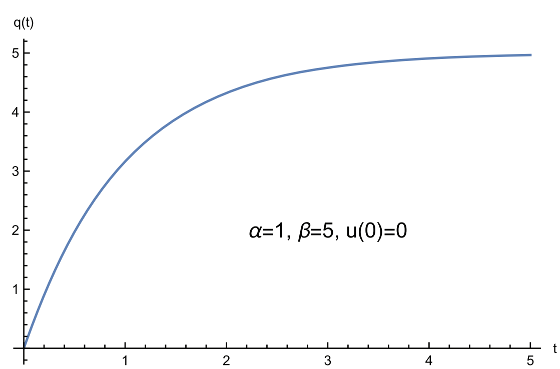
★ **Problem 14.2** Using numerical methods, solve the differential equation

$$\frac{dq}{dt} = -\alpha\, q + \beta.$$

Compare your results with the exact solution as a function of the discrete time-step you use, and the order/method of the algorithm you adopt. Try in particular contrasting 4th order Runge-Kutta with another algorithm of your choosing.

Solution

You get a curve that looks as follows:



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★ **Problem 14.3** Consider the logistic map discussed in the text. To gauge the density of bifurcations, one uses a measure of distance between fixed points as follows. Define $d = x^* - (1/2)$ as the distance between the fixed point $1/2$ and the nearest fixed point to it – labeled x^*. That is, you first find the r value corresponding to a convergence at value

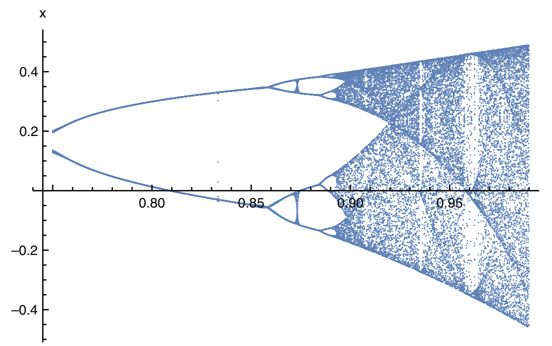
$x = 1/2$, then you identify the closest fixed point x^* to $1/2$ at this value of r, and compute the distance d. For example, at first period doubling, we have $d_1 = 0.3090\ldots$; then after the second, we have $d_2 = -0.1164\ldots$ We then define the parameter γ as

$$\gamma = \lim_{n\to\infty} -\frac{d_n}{d_{n+1}} .$$

Using numerical methods, compute γ and verify that it given by $\gamma = 2.502907\ldots$

Solution

We plot the map versus r and get



From this graph, we read off

$$d = [0.31, -0.12, 0.05, -0.02]$$

giving the sequence

$$\gamma = [2.58333, 2.4, 2.5]$$

■

★★ **Problem 14.4** Consider a particle in a Newtonian potential $V(r) = -k/r + \epsilon/r^n$ for some integer n. Using the alternate variable $u = 1/r$, (a) show that the radial equation of motion can be put into the form

$$\frac{d^2u}{d\varphi^2} + u = \frac{1}{p} + n\,\kappa\,u^{n+1}$$

where

$$p \equiv \frac{\ell^2}{\mu\,k} \quad , \quad \kappa \equiv \frac{\mu\,k}{\ell^2} .$$

μ is the reduced mass and ℓ is angular momentum. (b) We want to find $u(\varphi)$, the shape of the trajectory. Write numerical code that solves this equation, taking as input μ, k, ϵ, n, and ℓ. Study various scenarios, including (1) $n = 2$ and ϵ large, and (2) $n = 3$ with ϵ small.

Solution

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}^2) + \frac{k}{r} - \frac{\epsilon}{r^n}$$

$$\ell = \frac{\partial L}{\partial \dot{\varphi}} = \mu r^2 \dot{\varphi}$$

$$H = E = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} - \frac{k}{r} + \frac{\epsilon}{r^n}$$

$$\Rightarrow \dot{r} = \pm\sqrt{-\frac{\ell^2}{\mu^2 r^2} + \frac{2E}{\mu} + \frac{2k}{\mu r} - \frac{2E}{\mu r^n}}$$

$$\dot{\varphi} = \frac{\ell}{\mu r^2}$$

$$\frac{dr}{d\varphi} = \frac{\dot{r}}{\dot{\varphi}} = \sqrt{-r^2 + \frac{2k\mu}{\rho^2}r^3 + \frac{2E\mu}{\ell^2}r^4 - \frac{2\epsilon\mu}{\ell^2 r^{n-4}}}$$

Write $r = \frac{1}{u}$

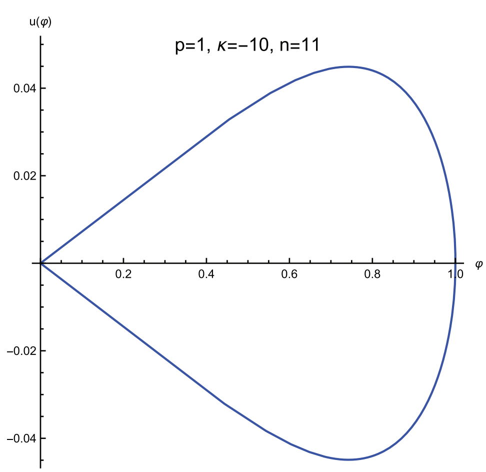
$$\Rightarrow \left(\frac{du}{d\varphi}\right)^2 = -u^2 + \frac{2k\mu}{\ell^2}u + \frac{2E\mu}{\ell^2} - \frac{2\epsilon\mu}{\ell^2}u^n$$

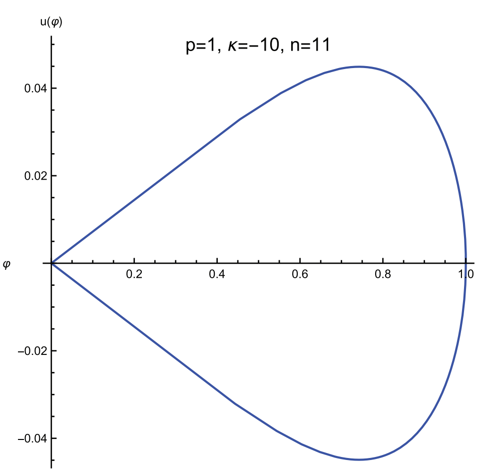
Take derivative with respect to φ

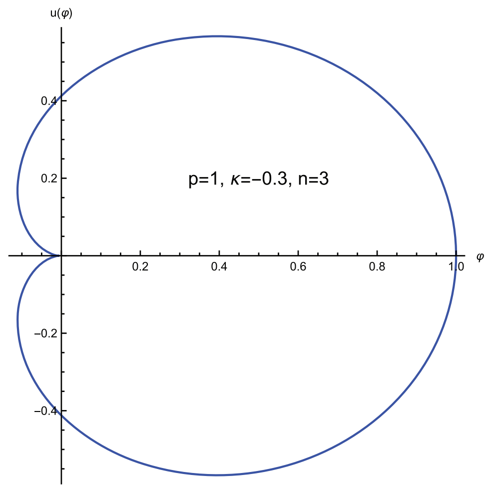
$$\Rightarrow \frac{d^2u}{d\varphi^2} + u = \frac{k\mu}{\ell^2} - \frac{n\varepsilon\mu}{\ell^2}u^{n-1}$$

Define:

$$p \equiv \frac{\ell^2}{k\mu} \qquad k \equiv \frac{\epsilon\mu}{\ell^2}$$







■

★★ **Problem 14.5** Consider the one dimensional harmonic oscillator with angular frequency ω perturbed by the small non-linear potential $\epsilon\, q^4$. (a) Find the solution using the perturbation technique introduced in the text to first order in the small perturbation; (b) Improve your solution from part (a) by implementing the technique outlined at the end of the perturbations section, writing a solution with angular frequency $\Omega = \omega + \epsilon\,\omega_1$. That is, your solution now depends on $s = \Omega t$ instead of $s = \omega t$. Fix ω_1 so that you cancel a term in the solution that is *not* periodic.

Solution

(a)

$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2 - \epsilon q^4$$

The equations of motion become:

$$\ddot{q} = -\omega^2 q - 4\frac{\epsilon}{m}q^3$$

We write the zeroth order solution with $q(0) = A$ and $\dot{q}(0) = 0$ as in the text

$$q_0(t) = A\cos(\omega t)$$

and introduce $Q = \frac{q}{A}$, $s \equiv \omega t$

$$\Rightarrow A\omega^2\ddot{Q} = -\omega^2 AQ - \frac{4\epsilon}{m}A^3Q^3$$

$$\Rightarrow \ddot{Q} = -Q - \xi Q^3$$

where $\xi = \frac{4\epsilon A^2}{m\omega^2}$ with $Q_0(s) = \cos(s)$ and $\dot{} \equiv \frac{d}{ds}$ henceforth

$$Q(s) = Q_0(s) + \xi Q_1(s) + \xi^2 Q_2(s) + \dots$$

$$\ddot{Q}_0 + \xi\ddot{Q}_1 + \xi^2\ddot{Q}_2 = -Q_0 - \xi Q_1 - \xi^2 Q_2 - \xi(Q_0 + \xi Q_1 + \xi^2 Q_2)^3 + \dots$$

$$\xi^0: \quad \ddot{Q}_0 = -Q_0 \text{ satisfied by construction}$$

$$\xi_1: \quad \ddot{Q}_1 = -Q_1 - Q_0^3 \Rightarrow Q_1(s) = \int_0^s -Q_0^3(s') \sin(s-s')ds'$$

$$\Rightarrow Q_1(s) = -\frac{1}{16}\sin(s)(6s + \sin(2s))$$

and

$$Q(s) \cong Q_0(s) + \xi Q_1(s)$$

(b)

$$\Omega = \omega + \omega_1 \xi \Rightarrow \frac{\Omega}{\omega} = 1 + \frac{\omega_1}{\omega}\xi \text{ with } s = \Omega t$$

$$\Rightarrow \Omega^2 \ddot{Q} = -\omega^2 Q - \frac{4\epsilon A^3}{m} Q^3 \Rightarrow \frac{\Omega^2}{\omega^2}\ddot{Q} = -Q - \xi Q^3$$

$$\left(1 + \frac{\omega_1}{\omega}\xi\right)^2 (\ddot{Q}_0 + \xi \ddot{Q}_1) = -Q_0 - \xi Q_1 - \xi Q_0^3 \text{ to linear order in } \xi$$

$$\Rightarrow \ddot{Q}_0 + 2\frac{\omega_1}{\omega}\xi \ddot{Q}_0 + \xi \ddot{Q}_1 = -Q_0 - \xi Q_1 - \xi Q_0^3$$

$$\Rightarrow \xi^1: \quad \ddot{Q}_1 + Q_1 + Q_0^3 - \frac{2\omega_1}{\omega_0} Q_0 = 0 \text{ where we used } \ddot{Q}_0 = -Q_0$$

$$\Rightarrow Q_1(s) = \int_0^s \left(-Q_0^3 + \frac{2\omega_1}{\omega_0} Q_0\right) \sin(s-s')ds'$$

$$= \frac{\omega_1}{\omega} s \sin(s) - \frac{3}{8} s \sin(s) - \frac{1}{16}\sin(s)\sin(2s)$$

We want to cancel the linear term

$$\Rightarrow \frac{\omega_1}{\omega} = \frac{3}{8}$$

$$\Rightarrow Q_1(s) = -\frac{1}{16}\sin(s)\sin(2s)$$

Putting things together, we get

$$q(t) = A\cos(\Omega t) - \frac{\epsilon A^3}{4m\omega^2}\sin(\Omega t)\sin(2\Omega t)$$

with

$$\Omega = \omega + \xi \frac{3}{8}\omega$$

■

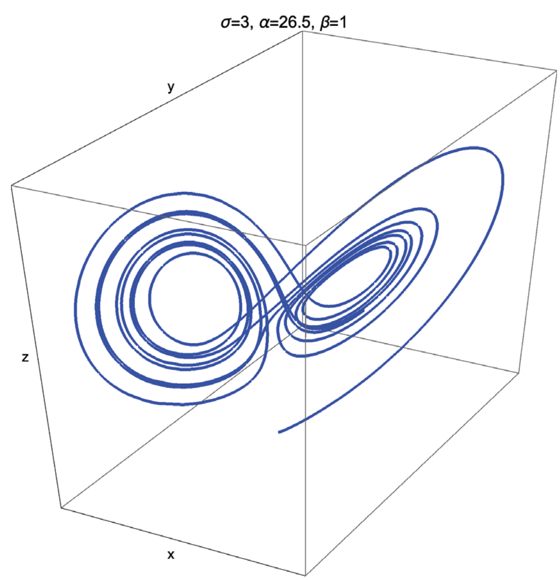
⋆ **Problem 14.6** The celebrated Lorentz attractor is described by the differential equations

$$\frac{dx}{dt} = -\sigma x + \sigma y \quad , \quad \frac{dy}{dt} = -xz + \alpha x - y \quad , \quad \frac{dz}{dt} = xy - \beta z \quad ,$$

and is used to described chaotic fluid dynamics involving heat flow. It is parameterized by α, β, and σ. (a) Solve this system of equations numerically and plot, for example, x vs y and z vs y. Determine the onset of chaos by testing super-sensitivity to initial conditions.

Solution

We get two basins of attractions as shown



You explore chaos by nudging the initial conditions by small amounts and looking for big changes in the trajectory in the future. ■

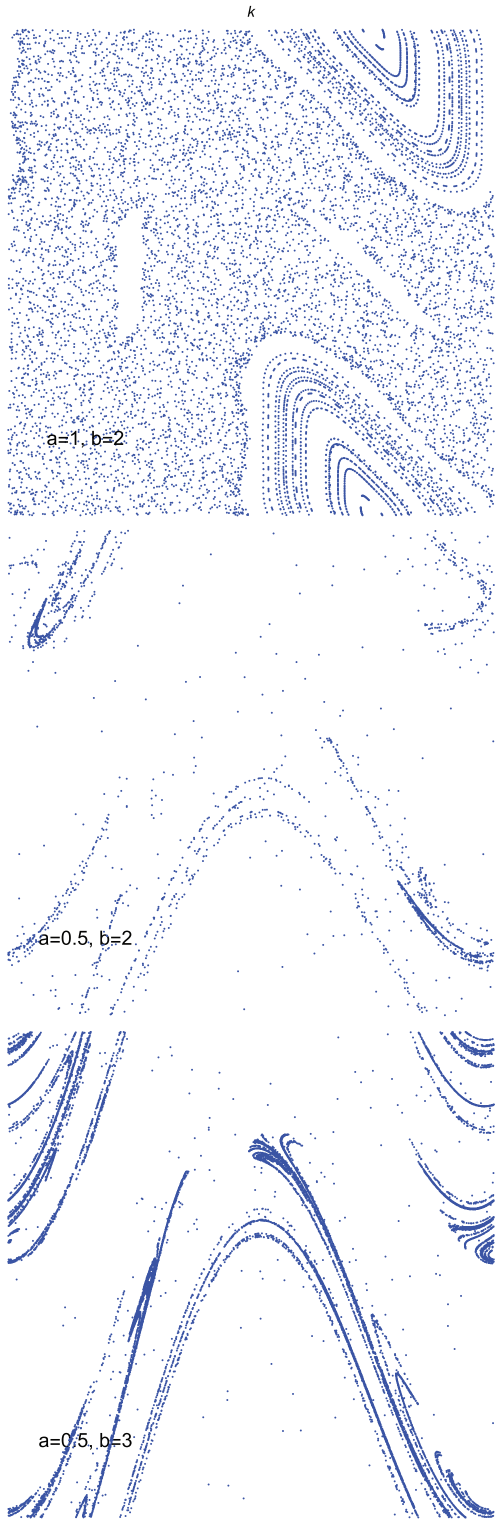
⋆⋆ **Problem 14.7** Consider the recursion relation

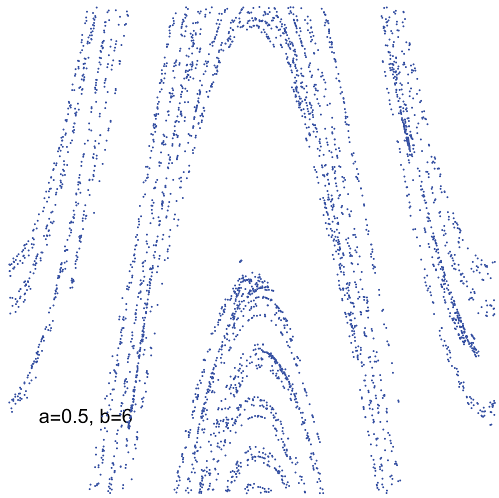
$$x_{n+1} = x_n + y_n \quad , \quad y_{n+1} = a\, y_n - b\, \cos(x_n + y_n) \, ,$$

where a and b are constants; this system is known as the *standard map*. Analyze the system as we did for the logistic map in the text. In particular, explore regions of the parameter space where (1) $a = 1$ and (2) $a = 1/2$ while varying b, (3) $b = 6$ near point $(3, 3)$.

Solution

We get the following sequence of sections:


k
a=1, b=2
a=0.5, b=2
a=0.5, b=3



■

★ **Problem 14.8** Show that the standard map of the previous problem described by

$$x_{n+1} = x_n + y_n \quad , \quad y_{n+1} = a\, y_n - b\, \cos(x_n + y_n)\ ,$$

can be obtained by discretizing time in the Hamiltonian equations of motion of the following physical system: a planar pendulum in the absence of gravity that is periodically kicked in a fixed direction with a fixed force. The phase space would be described by $\theta(t)$, the pendulum's angle from the vertical, and its canonical momentum $p_\theta(t)$ – which you will need to map to the discrete sequence given by x_n and y_n.

Solution

We have without the kicking

$$L = \frac{1}{2} m(\ell^2 \dot{\theta}^2) + F(t) \sin\theta$$

$$H = \frac{(p^\theta)^2}{2m\ell^2} - F(t) \sin\theta$$

where $p_\theta = m\ell^2\dot{\theta}$ and where $F(t)$ is α a fixed force to the right and θ is measured from the vertical) The relevant equations of motion are:

$$\dot{p}^\theta = -\frac{\partial H}{\partial \theta} = -F(t) \cos\theta$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p^\theta}{m\ell^2}$$

Notice that $F(t) \cos\theta$ would be a torque arising from a fixed force $F(t)$ acting to the right. We want

$$p^\theta(t) \to y_n \text{ with } \dot{p}^\theta = y_{n+1} - y_n$$

and

$$\theta(t) \to x_n \text{ with } \dot{x} = x_{n+1} - x_n$$

$$\Rightarrow x_{n+1} - x_n = \frac{y_n}{m\ell^2}$$

Rescale so that

$$\frac{y_n}{m\ell^2} \to y_n \Rightarrow x_{n+1} - x_n = y_n \text{ as needed.}$$

From the first equation (we can posit that F acts at each discrete time step)

$$y_{n+1} - y_n = -m\ell^2 F \cos(x_n)$$

if we replace

$$x_n \to x_{n+1} = x_n + y_n$$

set

$$b = m\ell^2 F \text{ and } a = 1$$

$\Rightarrow$ we get the Logistic map. ■

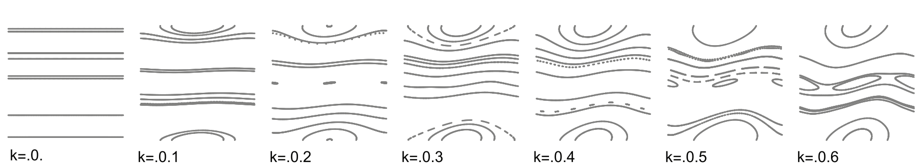
⋆⋆ **Problem 14.9** Consider the variant of the standard map described by the recursion relation

$$y_{n+1} = y_n + k \sin x_n \quad , \; x_{n+1} = x_n + y_{n+1}$$

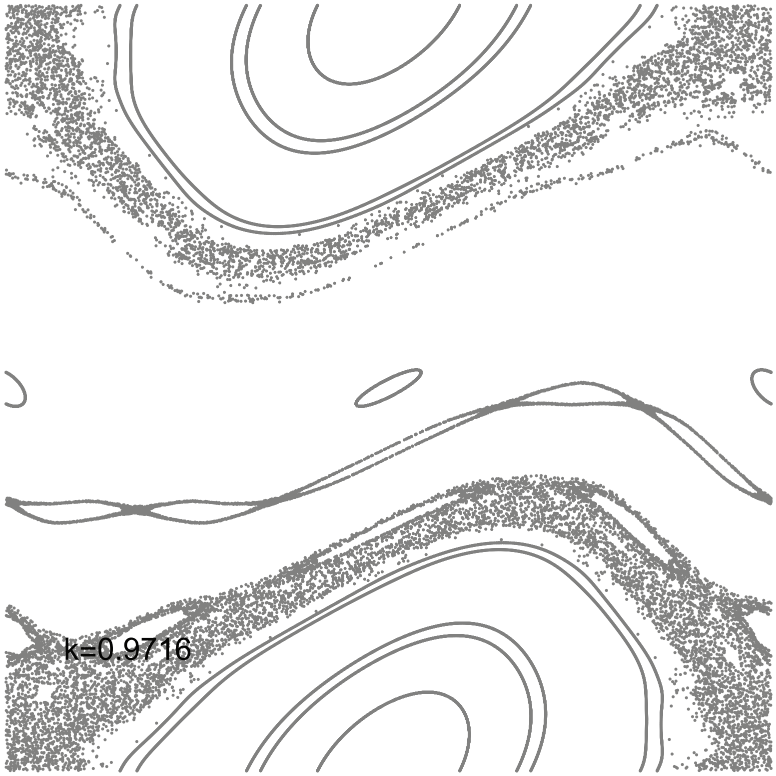
where k is a constant. (a) Study the distortion of the KAM tori as k is taken from $k = 0$ to $k = 0.6$. (b) Analyze the system when $k = 0.9716$. Compute the 'winding number' $\Omega \equiv \lim_{n\to\infty}(x_n - x_1)/n$ as a function of k.

Solution

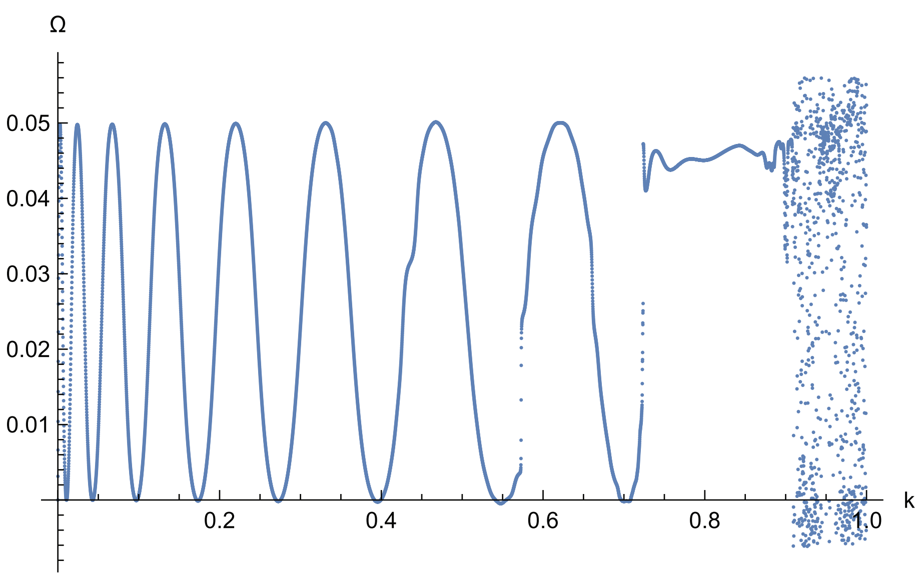
We get the following section maps for various values of k.



For $k = 0.9716$, we get



And the winding number takes the form



■

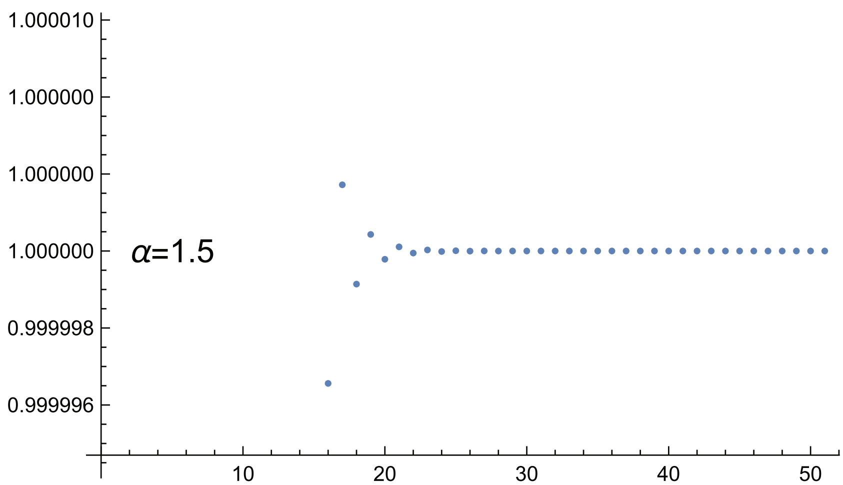
★★ **Problem 14.10** Consider the map given by

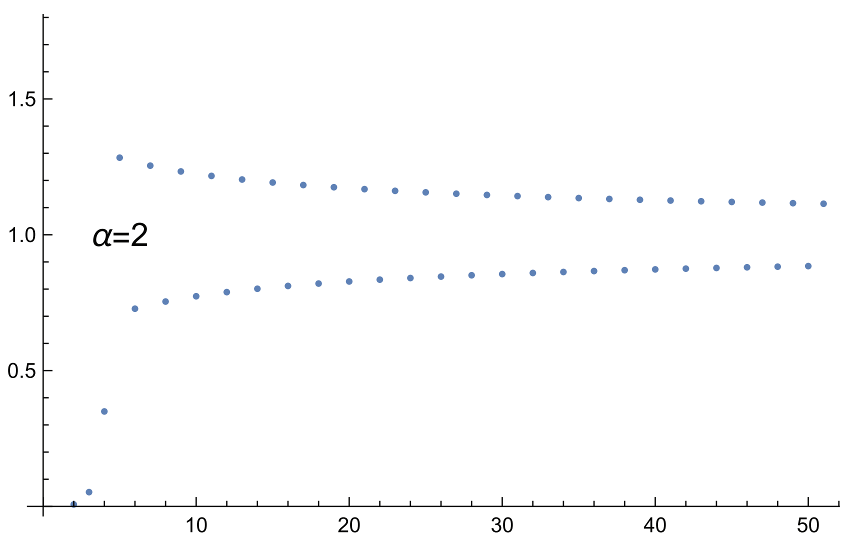
$$x_{n+1} = x_n e^{\alpha\,(1-x_n)}$$

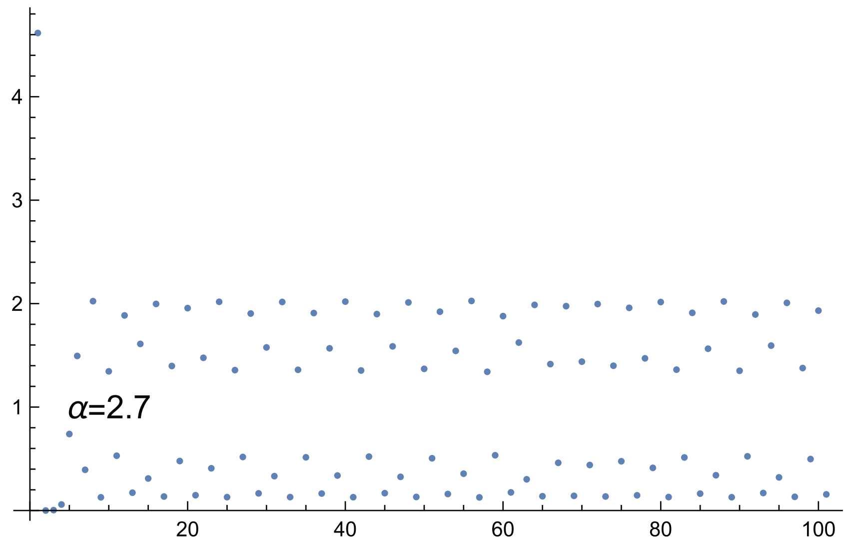
used to study population growth limited by disease. Analyze the system as done for the logistic map in the text, identifying the onset of chaos and bifurcations, if any. Consider in particular values of $\alpha = 1.5, 2, 2.7$.

Solution

Here are the plots one gets







■

★★ **Problem 14.11** Consider the map given by

$$x_{n+1} = \alpha \sin(\pi x_n)$$

where $0 < \alpha < 1$. Analyze the system as was done for the logistic map in the text, identifying the onset of chaos and bifurcations if any. Compute the parameter δ introduced in the text.

Solution

$$f(x_n) = \alpha \sin(\pi x_n)$$

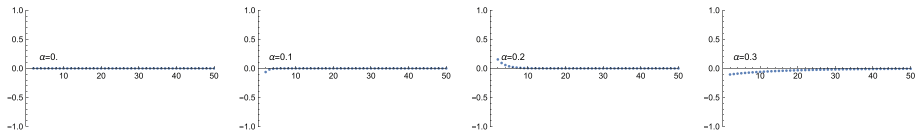
$$f'(x_n) = \pi\alpha \cos(\pi x_n)$$

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln|\pi\alpha \cos(\pi x_n)|$$

$$\delta = \lim_{k\to\infty} \frac{\alpha_k - \alpha_k - 1}{\alpha_{k+1} - \alpha_k}$$

k	α k
0	0.315
1	0.718
2	0.831
3	0.858
4	0.864

$\Rightarrow \delta$ sequence becomes 3.6, 4.2, 4.5 $\rightarrow$ approach 4.669 ... Onset of chaos where $\lambda > 0$, $\alpha \simeq 0.865\ldots$



■

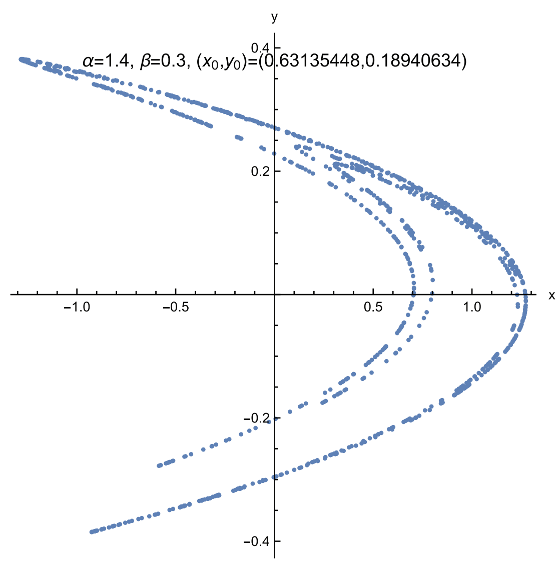
⋆⋆⋆ **Problem 14.12** Consider the two dimensional recursion

$$x_{n+1} = y_n + 1 - \alpha\, x_n^2 \quad , \quad y_{n+1} = \beta\, x_n \,,$$

introduced by Hénon to describe chaotic behavior in the trajectories of asteroids. Study the sequence for $\alpha = 1.4$ and $\beta = 0.3$ with the initial condition $x_0 = 0.63135448$ and $y_0 = 0.18940634$. Explore the parameter space for interesting features.

Solution

Here's a typical plot for this system:



■

⋆ **Problem 14.13** Consider the two dimensional map described by the recursions

$$\theta_{n+1} = \theta_n + 2\pi \frac{\beta^{3/2}}{r_n^{3/2}} \quad , \quad r_{n+1} = 2\, r_n - r_{n-1} - \alpha\, \frac{\cos\theta_n}{(r_n - \beta)^2}$$

parameterized by the constants α and β. This system arises in analyzing Saturn's rings produced by the influence of one its moons, Mimas. θ and r refer to the angular position and radius of a particle in the ring – averaged over a period of Mimas. (a) Verify numerically that a volume element in r-θ is preserved by the recursion. (b) Plot $(r_n \cos\theta_n, r_n \sin\theta_n)$ and identify bands of r_n where one has stable trajectories.

Solution

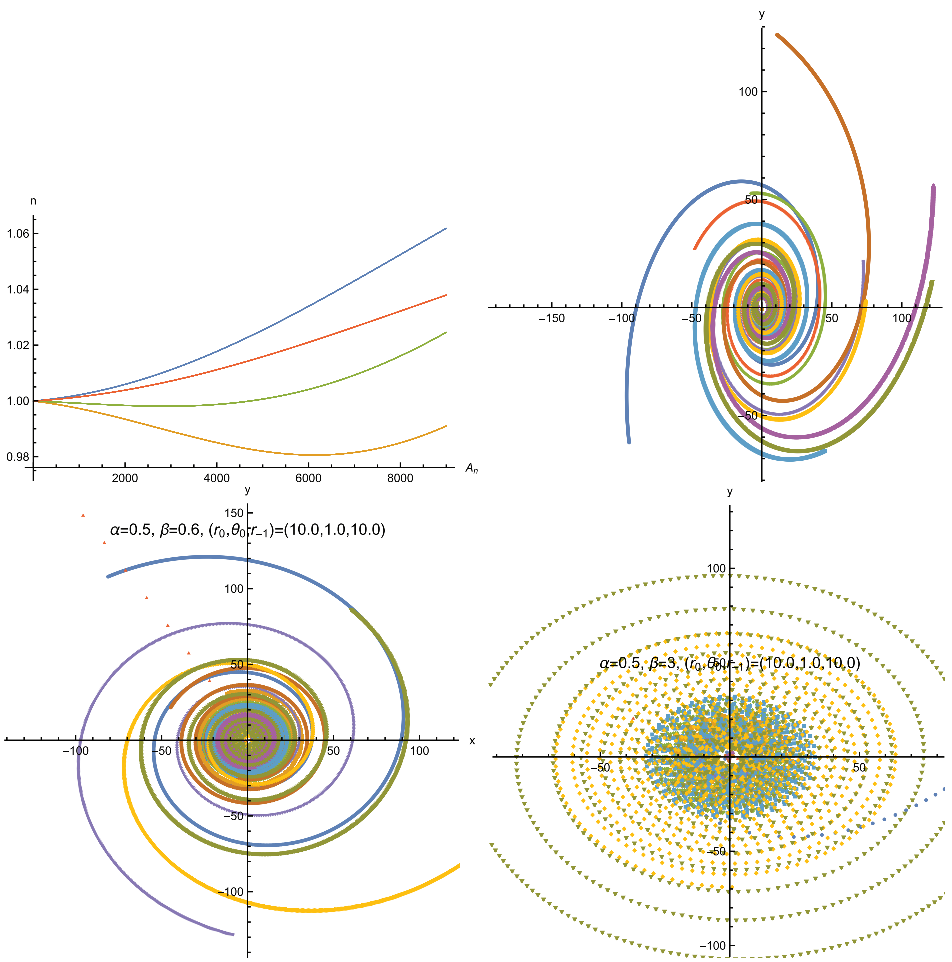
(a) To compute the area A_n, we use

$$A_n = \frac{1}{2}\left(\frac{r_n + r_{n-1}}{2}\right)^2 |\theta_n - \theta_{n-1}|$$

as a discrete form of

$$\Delta A = \frac{1}{2} r^2 \Delta\theta$$

Within numerical errors, we get stable A_n as shown for several initial conditions.



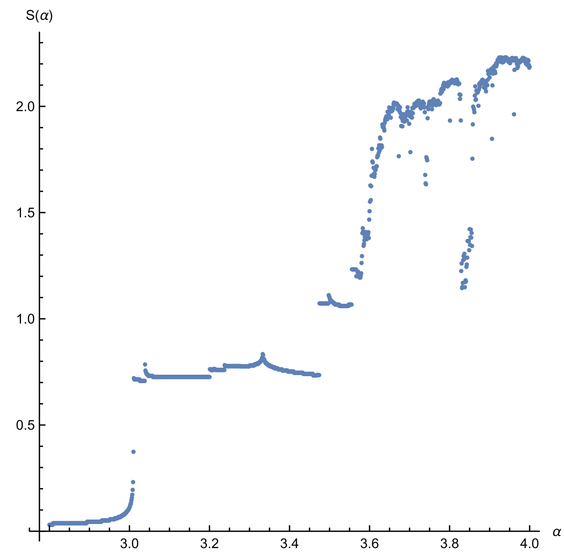
■

★★ **Problem 14.14** Consider the logistic map analyzed in the text. Divide the range $(0, 1)$ into N equal intervals and numerically compute the probabilities p_k that the recursion lands in the kth interval. Then compute the *entropy*

$$S \equiv -\sum_{k=1}^{N} p_k \ln p_k.$$

Do this so as to build up the function $S(\alpha)$ for the range of $2.8 < \alpha < 4$. Plot the function and correlate with the conclusions in the text.

Solution



■

★★ **Problem 14.15** Consider a magnetic compass needle with moment of inertia I and magnetic dipole moment $\boldsymbol{\mu}$, free to rotate in the x-y plane. Denote the polar angle by θ. A time dependent external magnetic field $\mathbf{B} = B_0 \cos \omega t \,\hat{\boldsymbol{x}}$ applies a torque given by $\boldsymbol{\mu} \times \mathbf{B}$ on the needle. (a) Write the equation of motion for θ. (b) Solve for $\theta(t)$ numerically and generate a Poincaré map by plotting discrete points $\theta(t = 2\pi n/\omega)$ for integer n. Verify the onset of chaos for $2\,B_0\mu/I > \omega^2$.

Solution

(a)

$$I\ddot{\theta} = \mu B_0 \cos(\omega t) \sin\theta$$

(b)

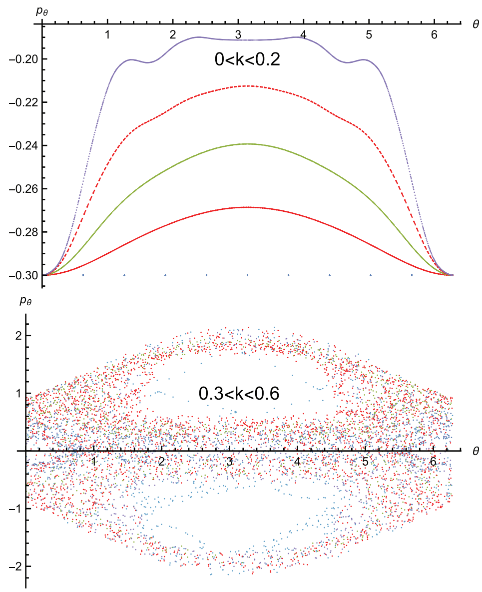
$$\ddot{\theta} = \frac{\mu B_0}{I} \sin\theta \cos(\omega t) = k \sin\theta \cos(\omega t) \text{ where } k \equiv \frac{\mu B_0}{I}$$

For chaos, we want $2k > \omega^2$. The Lagrangian for this system is then

$$L = \frac{1}{2} I\dot{\theta}^2 - k\cos\theta\cos(\omega t) \Rightarrow p^\theta = I\dot{\theta} \Rightarrow H = \frac{(p^\theta)^2}{2I} + k\cos\theta\cos(\omega t)$$

$$\dot{\theta} = \frac{\partial H}{\partial p^\theta} = \frac{p^\theta}{I}$$

$$\dot{p}^{\theta} = -\frac{\partial H}{\partial \theta} = k \sin\theta \cos(\omega t)$$



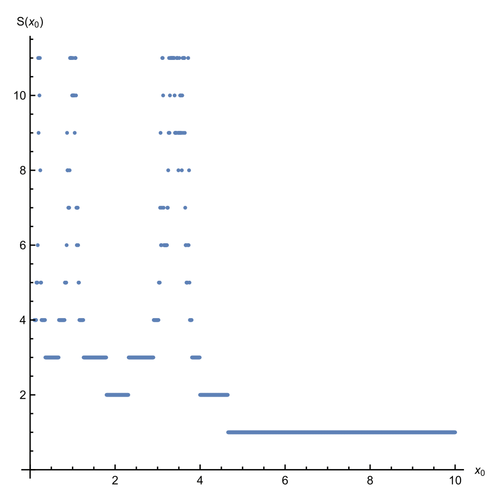
■

★★ **Problem 14.16** Consider the two dimensional map

$$x_{n+1} = \alpha \left(x_n - \frac{1}{4}(x_n + y_n)^2 \right) \quad , \quad y_{n+1} = \frac{1}{\alpha} \left(y_n + \frac{1}{4}(x_n + y_n)^2 \right)$$

that approximates the chaotic scattering behavior of a projectile off a region near the origin where it collides with a bunch of targets. Fix $\alpha = 5$ and y_0 to some small value near the origin. Then start with a bunch of values for x_0 near the origin but positive, and compute the number of steps $S(x_0)$ it takes for the projectile to leave the collision basin, say when $x_n < -5$. Plot $S(x_0)$.

Solution



■

★★★ **Problem 14.17** Consider the so-called *circle map* for the angular variable

$$\theta_{n+1} = \theta_n + r - \kappa \sin \theta_n.$$

Note that we have $\theta \sim \theta + 2\pi$. This system can approximately describe a damped driven pendulum with angle θ. (a) First consider the case where $\kappa = 0$. Using $\theta_0 = 2\pi \times 0.2$ and $r = 2\pi k$ where k is the ratio of two integers, check that the motion is periodic. Then try $r = 2\pi/\sqrt{2}$ and check periodicity. You can study periodicity by computing the 'winding number'

$$w = \lim_{k\to\infty} \frac{1}{2\pi k} \sum_{n=0}^{k-1} (\theta_{n+1} - \theta_n) .$$

For periodic or almost periodic motion, we would have $w \to r/(2\pi)$. (b) Now consider $\kappa = 1/2$ with $0 < r < 2\pi$, and explore what happens to the periodicity of the motion by computing w. (c) Study the case where $\kappa = 1$. Plot $w(r)$.

Solution

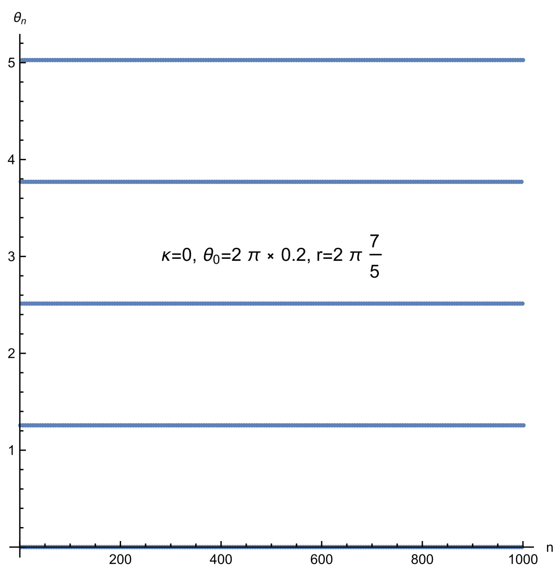
Computing ω, we get

$$\omega = \frac{r}{2\pi}$$

This is obvious for $k = 0$ from the recursion. For $k \neq 0$, we have

$$\omega \simeq \frac{r}{2\pi}$$

for finite but large k.



■

15 Seeds of Quantization

15.1 Problems and Solutions

⋆ **Problem 15.1** A ball of mass m is dropped from rest above the surface of an airless moon in essentially uniform gravity g. (a) If y is the vertical axis, show that the Hamilton-Jacobi equation for the ball is

$$\frac{1}{2m}\left(\frac{\partial S}{\partial y}\right)^2 + mgy + \frac{\partial S}{\partial t} = 0.$$

where S is Hamilton's principal function. (b) Then show that

$$S = \pm\frac{2\sqrt{2}}{3g\sqrt{m}}(C - mgy)^{3/2} - Ct$$

where C is a constant. (c) Then show, using a judicious choice of constants, that the equation of motion of the ball can be written

$$y = y_0 - \frac{1}{2}g(t - t_0)^2.$$

Solution

(a) The Hamiltonian is

$$H = \frac{p_y^2}{2m} + mgy,$$

so the Hamilton-Jacobi equation is

$$\frac{1}{2m}\left(\frac{\partial S}{\partial y}\right)^2 + mgy + \frac{\partial S}{\partial t} = 0$$

following Example 15.1.

(b) Separate variables, letting

$$S = f(y) + g(t),$$

so

$$\frac{1}{2m}\left(\frac{\partial f}{\partial y}\right)^2 + mgy = -\frac{dy}{dt} = C, \text{ a constant.}$$

Therefore

$$g(t) = -Ct \text{ and } \frac{df}{dy} = \pm\sqrt{2m(C - mgy)}.$$

So

$$f(y) = \pm\sqrt{2m}\int dy\sqrt{C - mgy} = \pm\sqrt{2m}(-\frac{1}{mg}\int du\sqrt{u})$$

where $u = C - mgy$. Integrating,

$$f(y) = \pm\frac{2\sqrt{2}}{3g\sqrt{m}}(C - mgy)^{3/2}$$

and so

$$S = f(y) + g(t) = \frac{\pm 2\sqrt{2}}{3g\sqrt{m}}(C - mgy)^{3/2} - Ct,$$

as claimed.

(c) Now

$$p_y = \frac{\partial S}{\partial y} = \mp\sqrt{2m}(C - mgy)^{1/2}$$

$$\Rightarrow \frac{p_y^2}{2m} + mgy = C \equiv E,$$

the energy. Now the new momentum is P, a constant, which we can identify with C. Then the new

$$Q = \frac{\partial S}{\partial C} = \pm\frac{1}{g}\sqrt{\frac{2}{m}}\sqrt{C - mgy} - t$$

Now we can also identify $Q \equiv -t_0$ since they are both constants.

$$\Rightarrow t_0 = \mp\frac{1}{g}\sqrt{\frac{2}{m}}(C - mgy)^{1/2} + t$$

and so

$$t - t_0 = \frac{\pm\sqrt{2/m}}{g}\sqrt{C - mgy}.$$

Therefore

$$E - mgy = \frac{1}{2}mg^2(t - t_0)^2.$$

It is natural to write

$$E = mgy_0 \Rightarrow y = y_0 - \frac{1}{2}g(t - t_0)^2$$

■

★★ **Problem 15.2** Starting from rest at time $t = 0$ and at altitude h_0, a block of mass M slides down a frictionless plane inclined at angle α to the horizontal. There is a uniform gravitational field g directed vertically downward. (a) Write the Hamilton-Jacobi equation for the block. (b) Solve the equation to find Hamilton's principal function S. (c) Find the equation of motion $s(t)$ for the block, where s is the distance along the incline, measured from the top of the incline.

Solution

(a) The kinetic and potential energies are

$$T = \frac{1}{2}M\dot{s}^2, \quad U = Mgh,$$

so

$$L = \frac{1}{2}M\dot{s}^2 - Mgh = \frac{1}{2}M\dot{s}^2 - Mg(h_0 - s\sin\alpha)$$

since $h_0 - h = s\sin\alpha$. The canonical momentum is

$$p = \frac{\partial L}{\partial \dot{s}} = M\dot{s},$$

so the Hamiltonian is

$$H = \frac{p^2}{2M} + Mg(h_0 - s\sin\alpha).$$

The Hamilton Jacobi equation is therefore (dropping the constant Mgh_0)

$$\frac{1}{2M}\left(\frac{\partial S}{\partial s}\right)^2 - Mgs\sin\alpha + \frac{\partial S}{\partial t} = 0.$$

(b) Separating variables,

$$S = f(s) + g(t) \Rightarrow \frac{1}{2M}\left(\frac{df}{ds}\right)^2 - Mgs\sin\alpha = -\frac{dg}{dt} = C, \text{ a constant.}$$

Thus

$$g(t) = -Ct \quad \& \quad \frac{df}{ds} = \pm\sqrt{2M}\sqrt{C + Mgs\sin\alpha},$$

so

$$f = \pm\sqrt{2M}\int ds\sqrt{C + Mgs\sin\alpha}.$$

Let

$$u = C + Mgs\sin\alpha \Rightarrow du = Mg\sin\alpha ds$$

so then

$$f = \frac{\pm\sqrt{2M}}{Mg\sin\alpha}\int du\sqrt{u} = \pm\sqrt{\frac{2}{M}}\frac{1}{g\sin\alpha}\frac{u^{3/2}}{3/2}$$

$$f = \pm\frac{2}{3}\sqrt{\frac{2}{M}}\frac{1}{g\sin\alpha}(C + Mgs\sin\alpha)^{3/2}.$$

Thus

$$S = f + g = \frac{2}{3}\sqrt{\frac{2}{M}}\frac{1}{g\sin\alpha}(C + Mgs\sin\alpha)^{3/2} - Ct.$$

Now

$$p = \frac{\partial S}{\partial s} = \sqrt{\frac{2}{M}}\frac{1}{g\sin\alpha}(C + Mgs\sin\alpha)^{1/2},$$

so

$$\frac{p^2}{2M} - Mgs\sin\alpha = \frac{2M^2g^2\sin^2\alpha}{Mg^2\sin^2\alpha}\left(\frac{C+Mgs\sin\alpha}{2M}\right) = Mgs\sin\alpha = C = E,$$

the energy, where now we measure potential energy from the starting point. We choose the new momentum to be $P = C$, so the new

$$Q = \frac{\partial S}{\partial C} = \sqrt{\frac{2}{M}}\frac{1}{g\sin\alpha}(C+Mgs\sin\alpha)^{1/2} - t$$

which is a constant also, which we call t_0. Thus

$$t - t_0 = \sqrt{\frac{2}{M}}\frac{1}{g\sin\alpha}(C+Mgs\sin\alpha)^{1/2}$$

so

$$\frac{Mg^2\sin^2\alpha}{2}(t-t_0)^2 = E + Mgs\sin\alpha.$$

Thus

$$S = \frac{1}{Mg\sin\alpha}\left[\frac{Mg^2\sin^2\alpha}{2}(t-t_0)^2 - \frac{E}{Mg\sin\alpha}\right]$$

$$S = \frac{g\sin\alpha}{2}(t-t_0)^2 - \frac{E}{(Mg\sin\alpha)^2}.$$

Now having chosen both T and U to be zero initially, $E = T + U = 0$ and therefore we have found that

$$S = \frac{1}{2}a(t-t_0)^2$$

where the acceleration is $a = g\sin\alpha$, which is correct. ■

⋆ **Problem 15.3** A spaceship drifts in gravity-free space. If its velocity is v_0 in the positive x direction at time $t = 0$, find its subsequent motion using the Hamilton-Jacobi method.

Solution

The Hamiltonian is $H = \frac{p_x^2}{2m}$, so the Hamilton-Jacobi equation is

$$\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{\partial S}{\partial t} = 0.$$

Let

$$S = f(x) + g(t),$$

so

$$\frac{1}{2m}\left(\frac{df}{dx}\right)^2 = -\frac{dg}{dt} = C, \text{ a constant.}$$

Therefore

$$g(t) = -Ct \text{ and } f(x) = \pm\sqrt{2mC}x,$$

where m is the ship's mass. Therefore

$$S = \pm\sqrt{2mC}x - Ct.$$

Now

$$p_x = \frac{\partial S}{\partial x} = \pm\sqrt{2mC}, \text{ so } \frac{p_x^2}{2m} = C = E,$$

the energy. The new

$$Q = \frac{\partial S}{\partial C} = \pm\sqrt{2m}\frac{1}{2}C^{-1/2}x - t$$

We identify $Q \equiv -t_0$, since they are both constants, so

$$t - t_0 = \pm\sqrt{\frac{m}{2C}}x \text{ or } x = \pm\sqrt{\frac{2C}{m}}(t - t_0).$$

We choose the plus sign since v_0 is positive, so

$$x = \sqrt{\frac{2E}{m}}(t - t_0).$$

Given

$$\frac{dx}{dt} = \sqrt{\frac{2E}{m}} = v_0,$$

if follows that

$$x = v_0(t - t_0).$$

■

★★ **Problem 15.4** A projectile is fired in a uniform gravitational field g with initial speed v_0 and angle θ_0 relative to the horizontal. Note that the Hamiltonian is

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + mgy.$$

Find the projectile's motion $x(t)$ and $y(t)$ using the Hamilton-Jacobi method.

Solution

The corresponding $H - J$ equation is

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2\right] + mgy + \frac{\partial S}{\partial t} = 0.$$

Let

$$S = S_x(x) + S_y(y) + S_t(t)$$

so

$$\frac{1}{2m}\left[\left(\frac{dS_x}{dx}\right)^2 + \left(\frac{dS_y}{dy}\right)^2\right] + mgy = -\frac{dS_t}{dt} = C_1,$$

a constant. Therefore $S_t(t) = -C_1 t$ and

$$\left(\frac{dS_x}{dx}\right)^2 = -\left(\frac{dS_y}{dy}\right)^2 - 2m^2 gy + 2mC_1 = C_2,$$

another constant. Therefore

$$\frac{dS_x}{dx} = \sqrt{C_2} \Rightarrow S_x = \sqrt{C_2}x$$

and also

$$\frac{dS_y}{dy} = \sqrt{2m^2 gy + 2mC_1 - C_2} \Rightarrow S_y = \int^y \sqrt{-2m^2 gy + 2mC_1 - C_2}dy$$

Therefore

$$S = \sqrt{C_2}x + \int^y \sqrt{-2m^2 gy + 2mC_1 - C_2}dy - C_1 t.$$

Now we can identify the new momenta as $P_1 = C_1$ and $P_2 = C_2$. The new coordinates are also constants.

$$Q_1 = \frac{\partial S}{\partial P_1} = \frac{\partial S}{\partial C_1} = \gamma_1$$

and

$$Q_2 = \frac{\partial S}{\partial P_2} = \frac{\partial S}{\partial C_2} = \gamma_2.$$

That is,

$$Q_1 = \frac{\partial S}{\partial C_1} = \frac{2m}{2}\int \frac{dy}{\sqrt{-2m^2 gy + (2mC_1 - C_2)}} - t = \gamma_1$$

Let

$$u = -2m^2 gy + (2mC_1 - C_2) \text{ so } du = -2m^2 gdy.$$

Therefore

$$\frac{m}{-2m^2 g}\int \frac{du}{\sqrt{u}} - t = -\frac{1}{2mg}\frac{u^{1/2}}{1/2} - t$$

$$= -\frac{1}{mg}\sqrt{-2m^2 gy + (2mC_1 - C_2)} - t = \gamma_1$$

So

$$-2m^2 gy + (2mC_1 - C_2) = m^2 g^2 (-t - \gamma_1)^2$$

$$y = \frac{(2mC_1 - C_2)}{2m^2 g} - \frac{m^2 g^2 (t + \gamma_1)^2}{2m^2 g} = \left(\frac{2mC_1 - C_2 - m^2 g^2 \gamma_1^2}{2m^2 g}\right) - g\gamma_1 t - \frac{1}{2}gt^2$$

$$y \equiv y_0 + v_{0\dot{y}} t - \frac{1}{2}gt^2$$

where

$$y_0 = \frac{(2mC_1 - C_2 - m^2 g^2 \gamma_1^2)}{2m^2 g} \text{ and } v_{0y} = -\gamma_1 g = v_0 \sin\theta.$$

Also

$$Q_2 = \frac{\partial S}{\partial C_2} = \frac{x}{2\sqrt{C_2}} - \frac{1}{2}\int^y \frac{dy}{\sqrt{-2m^2gy + (2mC_1 - C_2)}} = \gamma_2 = \frac{x}{2\sqrt{C_2}} - \frac{1}{2}(\gamma_1 + t)/m = \gamma_2$$

(from the expression for Q_1 above)

Solving for x,

$$x = 2\sqrt{C_2}\left[\gamma_2 + \frac{(\gamma_1 + t)}{m}\right] \equiv x_0 + v_0 \cos\theta t$$

where

$$\gamma_1 = -v_0 \sin\theta / g.$$

So we identify

$$x_0 = 2\sqrt{C_2}(\gamma_2 + \gamma_1/m) \quad v_0 \cos\theta = 2\sqrt{C_2}/m.$$

So in terms of initial conditions, $x_0, y_0, v_0 \cos\theta, v_0 \sin\theta$.

$$C_2 = \frac{m^2 v_0^2}{4}\cos^2\theta \quad \gamma_1 = -\frac{v_0 \sin\theta}{g}$$

$$\gamma_2 = \frac{x_0}{2\sqrt{C_2}} - \frac{\gamma_1}{m} = \frac{x_0}{m v_0 \cos\theta} + \frac{v_0 \sin\theta}{mg}$$

and finally

$$2mC_1 = 2m^2 g y_0 + C_2 + m^2 g^2 \gamma_1^2$$

$$C_1 = mgy_1 + \frac{mv_0^2}{2} = \text{ total energy } E\text{, a constant.}$$

In summary,

$$x = x_0 + v_0 \cos\theta t \quad y = y_0 + v_0 \sin\theta t - \frac{1}{2}gt^2$$

where x_0, v_0, y_0, and θ can be found in terms of C_1, C_2, γ_1, and γ_2. ■

★★ **Problem 15.5** A block m can slide along a frictionless table top in the x, y plane, subject to the forces exerted by one spring that lies along the x axis and has force-constant k_1, and another spring that lies along the y axis and has force-constant k_2. Assume the motions of m are so small that the springs remain essentially perpendicular to one another. Write the Hamiltonian, the Hamilton-Jacobi equation, and solve the equation to find the motions $x(t)$ and $y(t)$ of the block.

Solution

The Hamiltonian is

$$H = \frac{P_x^2}{2m} + \frac{P_y^2}{2m} + \frac{1}{2}k_1 x^2 + \frac{1}{2}k_2 y^2,$$

so the H-J equation is

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2\right]+\frac{1}{2}k_1x^2+\frac{1}{2}k_2y^2+\frac{\partial S}{\partial s}=0.$$

Separate variables by letting

$$S=S_x+S_y+S_t,$$

so

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2\right]+\frac{1}{2}k_1x^2+\frac{1}{2}k_2y^2=-\frac{\partial S_t}{\partial s}=C_1,$$

a constant. Thus $S_t=-C_1t$, and then

$$\frac{1}{2m}(\frac{dS_x}{dx})^2+\frac{1}{2}k_1x^2=-\frac{1}{2m}(\frac{dS_y}{dy})^2-\frac{1}{2}k_2y^2+C_1=C_2,$$

and the separation constant. Therefore

$$\frac{dS_x}{dx}=\pm\sqrt{2m(C_2-\frac{1}{2}k_1x^2)}$$

and

$$\frac{dS_y}{dy}=\pm\sqrt{2m\left[(C_1-C_2)-\frac{1}{2}k_2y^2\right]}.$$

Solving these,

$$S_x=\sqrt{2m}\int(C_2-\frac{1}{2}k_1x^2)^{1/2}dx$$

and

$$S_y=\sqrt{2m}\int\left[(C_1-C_2)-\frac{1}{2}k_2y^2\right]^{1/2}dy$$

where we have chosen the plus signs for simplicity.

So

$$S=S_x+S_y+S_z=\sqrt{2m}\int(C_2-\frac{1}{2}k_1x^2)^{1/2}dx+\sqrt{2m}\int\left[(C_1-C_2)-\frac{1}{2}k_2y^2\right]^{1/2}dy-C_1t.$$

We can identify the new momenta, P_1 and P_2, with the constants C_1 and C_2: That is $P_1=C_1$ and $P_2=C_2$. Then the new coordinates

$$Q_1=\frac{\partial S}{\partial P_1}=\frac{\partial S}{\partial C_1}\text{ and }Q_2=\frac{\partial S}{\partial P_2}=\frac{\partial S}{\partial C_2}$$

are also constants: $Q_1=\gamma_1$ and $Q_2=\gamma_2$. Now

$$Q_1=\frac{\partial S}{\partial C_1}=\sqrt{\frac{m}{2}}\int\frac{dy}{\left[(C_1-C_2)-\frac{1}{2}k_2y^2\right]^{1/2}}-t=\gamma_1.$$

Define

$$\sin\theta = \sqrt{\frac{k_2}{C_1 - C_2}}y, \text{ so } dy = \sqrt{\frac{C_1 - C_2}{k_2}}\cos\theta d\theta$$

Thus

$$Q_1 = \sqrt{\frac{m}{k_2}}\sin^{-1}\sqrt{\frac{k_2}{C_1 - C_2}}y - t = \gamma_1.$$

This gives

$$y(t) = \sqrt{\frac{C_1 - C_2}{k_2}}\sin(\omega_y t - \alpha) \equiv A_y \sin(\omega_y t - \alpha)$$

where

$$A_y \equiv \sqrt{\frac{C_1 - C_2}{k_2}}, \quad \omega_y = \sqrt{\frac{k_2}{m}}, \quad \alpha = \gamma_1\omega_y.$$

The other new coordinate is

$$Q_2 = \frac{\partial S}{\partial P_2} = \frac{\partial S}{\partial C_2} = \frac{\partial S_x}{\partial C_2} + \frac{\partial S_y}{\partial C_2} = \sqrt{\frac{m}{2}}\left[\int \frac{dx}{\sqrt{C_2 - \frac{1}{2}k_1x^2}} + \int \frac{dy}{\sqrt{(C_1 - C_2) - \frac{1}{2}k_2y^2}}\right]$$

$$= \sqrt{\frac{m}{k_1}}\sin^{-1}\sqrt{\frac{k_2}{C_2}}x + \sqrt{\frac{m}{k_2}}\sin^{-1}\sqrt{\frac{k_2}{C_1 - C_2}}y = \gamma_2,$$

a constant.

We have already found that

$$y = \sqrt{\frac{C_1 - C_2}{k_2}}\sin(\omega_y t - \alpha),$$

so

$$\sqrt{\frac{m}{k_1}}\sin^{-1}\sqrt{\frac{k_2}{C_2}}x = \gamma_2 - \sqrt{\frac{m}{k_2}}(\omega_y t - \alpha)$$

and so

$$x = \sqrt{\frac{C_2}{k_2}}\sin\left[\sqrt{\frac{k_1}{m}}\left(\gamma_2 - \sqrt{\frac{m}{k_2}}\alpha + \sqrt{\frac{m}{k_2}}\omega_y t\right)\right].$$

But

$$\omega_y = \sqrt{\frac{k_2}{m}} \text{ and } \alpha = \gamma_1\sqrt{\frac{k_2}{m}},$$

so

$$x(t) = \sqrt{\frac{C_2}{k_1}}, \quad \omega_x = \sqrt{\frac{k_1}{m}}, \quad \beta = \sqrt{\frac{k_1}{m}}(\gamma_2 - \gamma_1).$$

So altogether,

$$y(t) = A_y \sin(\omega_y t - \alpha) \text{ where } \omega_y = \sqrt{k_2/m}$$

$$x(t) = A_x \sin(\omega_x t - \beta) \text{ where } \omega_x = \sqrt{k_1/m}$$

The amplitudes A_x, A_y and the phase angles α, β are arbitrary, determined by initial conditions. ■

⋆ **Problem 15.6** A thin, stiff metal ring of radius R is placed in a vertical plane, and made to spin with constant angular velocity ω about a vertical axis that passes through the center of the ring. A bead of mass m is free to slide around the ring, with its position defined by its angle θ up from the bottom of the ring. (a) Find the Hamiltonian of the bead and write out the corresponding Hamilton-Jacobi equation. (b) Show that Hamilton's principal function can be separated into $S = S_\theta(\theta) + S_t(t)$, and find $S_t(t)$ explicitly and $S_\theta(\theta)$ as an integral over a function of θ.

Solution

The kinetic energy is

$$T = \frac{m}{2}(R^2 \sin^2\theta\omega^2 + R^2\dot{\theta}^2)$$

and the potential energy is

$$U = -mgR\cos\theta.$$

So the Lagrangian is

$$L = T - U = \frac{m}{2}(R^2 \sin^2\theta\omega^2 + R^2\dot{\theta}^2) + mgR\cos\theta$$

The canonical momentum is

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = mR^2\dot{\theta}.$$

So the Hamiltonian is

$$H = p_\theta\dot{\theta} - L = mR^2\dot{\theta}^2 - \frac{m}{2}(R^2\sin^2\theta\omega^2 + R^2\dot{\theta}^2) - mgR\cos\theta$$

$$= \frac{1}{2}mR^2\dot{\theta}^2 - \frac{1}{2}mR^2\omega^2\sin^2\theta - mgR\cos\theta$$

$$H = \frac{p_\theta^2}{2mR^2} - \frac{1}{2}mR^2\omega^2\sin^2\theta - mgR\cos\theta$$

Therefore the H-J equation is

$$\frac{1}{2mR^2}\left(\frac{\partial S}{\partial\theta}\right)^2 - \frac{1}{2}mR^2\omega^2\sin^2\theta - mgR\cos\theta + \frac{\partial S}{\partial t} = 0.$$

Let

$$S = S_\theta(\theta) + S_t(t):$$

$$\frac{1}{2mR^2}\left(\frac{dS_\theta}{d\theta}\right)^2 - \frac{1}{2}mR^2\omega^2\sin^2\theta - mgR\cos\theta = -\frac{dS_t}{dt} = C,$$

separating variables.

Therefore $S_t = -Ct$ and

$$\left(\frac{dS_\theta}{d\theta}\right)^2 = 2mR^2\left[\frac{1}{2}mR^2\omega^2\sin^2\theta + mgR\cos\theta + C\right]$$

So

$$S_\theta = \int d\theta\sqrt{2mR^2}\left[\frac{1}{2}mR^2\omega^2\sin^2\theta + mgR\cos\theta + C\right]^{1/2}$$

where C is a separation constant. ■

⋆⋆⋆ **Problem 15.7** The Hamiltonian for a particle of mass m, with arbitrary initial position and velocity, and subject to an inverse-square attractive force, can be written

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{k}{r}$$

where k is a positive constant. (a) Write the Hamilton-Jacobi equation for the particle. (b) By separating variables, show that Hamilton's principal function can be written in the form

$$S = S_r + S_\theta + S_t = S_r + C_1 t + C_2\theta$$

where C_1 and C_2 are constants, and S_r depends only upon r. (c) Write an expression for S_r in the form of an integral over r. (d) The new coordinate $Q_r = \partial S/\partial C_2 = \partial(S_r + S_\theta)/\partial C_2 = \alpha$, a constant, since the new coordinates in Hamilton-Jacobi theory are necessarily constants. Take this partial derivative (right through the integral sign!), to show that (with an appropriate choice of signs)

$$\theta - \alpha = \int \frac{C_2 dr}{r^2\sqrt{-2mC_1 + 2mk/r - C_2^2/r^2}}$$

(e) Evaluate the integral with the help of the substitution $u = 1/r$; then find an expression for $r(\theta)$. This gives the possible orbital shapes: circles, ellipses , parabolas, and hyperbolas.

Solution

(a) The Hamilton-Jacobi equation is

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial\theta}\right)^2 - \frac{k}{r} + \frac{\partial S}{\partial t}\right] = 0$$

(b) Separating variables,

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial\theta}\right)^2\right] - \frac{k}{r} = -\frac{\partial S}{\partial t} = -C_1,$$

a constant.

$$\Rightarrow S_t = +C_1 t \text{ and } \frac{r^2}{2m}\left(\frac{dS_r}{dr}\right)^2 + \frac{1}{2m}\left(\frac{dS_\theta}{d\theta}\right)^2 - kr + C_1 r^2 = 0$$

$$\text{or } \frac{r^2}{2m}\left(\frac{dS_r}{dr}\right)^2 - kr + C_1 r^2 = -\frac{1}{2m}\left(\frac{dS_\theta}{d\theta}\right)^2 = -\frac{C_2^2}{2m},$$

another separation constant. So $S_\theta = C_2\theta$. Therefore

$$S = S_r + S_\theta + S_t = S_r + C_1 t + C_2\theta.$$

(c) We have

$$\left(\frac{dS_r}{dr}\right)^2 = \frac{2m}{r^2}(kr - C_1 r^2 - C_2^2/2m)$$

so

$$S_r = \pm \int^r \sqrt{-2mC_1 + 2mk/r - C_2^2/r^2}\, dr$$

(d) The new coordinate

$$Q_r = \frac{\partial S}{\partial C_2} = \frac{\partial(S_r + S_\theta)}{\partial C_2} = \alpha,$$

which is a constant. So

$$\frac{\partial S}{\partial C_2} = \mp \int^r \frac{drC_2/r^2}{\sqrt{-2mC_1 + 2mk/r - C_2^2/r^2}} + \theta = \alpha.$$

Let

$$r = \frac{1}{u}, \quad dr = -\frac{1}{u^2}du.$$

Then

$$\pm \int \frac{C_2 du}{\sqrt{-2mC_1 + 2mku - C_2^2 u^2}} = \alpha - \theta$$

(e)

$$\mp \frac{C_2}{C_2} \sin^{-1}\left(\frac{-C_2^2 u + mk}{\sqrt{m^2k^2 - 2mC_1C_2^2}}\right) = \alpha - \theta$$

(Using integral tables). Therefore

$$C_2^2 u = mk \mp \sqrt{m^2k^2 - 2mC_1C_2^2}\sin(\alpha - \theta)$$

so

$$r = \frac{1}{u} = \frac{C_2^2}{mk \pm \sqrt{m^2k^2 - 2mC_1C_2^2}\sin(\alpha - \theta)}$$

$$r(\theta) = \frac{(C_2^2/mk)}{1 \pm \sqrt{m^2k^2 - 2mC_1C_2^2}\sin(\theta - \alpha)}$$

The paths under central gravity (from Ch. 7) are

$$r(\theta) = \frac{\ell^2/GMm^2}{1 + \varepsilon\cos\theta} = \frac{(C_2^2/mk)}{1 \pm \sqrt{m^2k^2 - 2mC_1C_2^2}\sin(\theta - \alpha)}$$

Choose $\alpha = -\pi/2$, $C_2 = \ell$, and the $+$ sign in the denominator. Also recall that $k = GMm$. Then the two expressions are the same, with

$$\varepsilon = \sqrt{1 + \frac{2E\ell^2}{G^2M^2m^2}}$$

if the constants $C_1 = -E_0$.

If $E < 0$ and $\varepsilon = 0$, circles. $0 < \varepsilon < 1$, ellipses.

If $E = 0$, $\varepsilon = 1$ parabolas.

If $E > 0$, $\varepsilon > 1$ hyperbolas ■

★ **Problem 15.8** Using the action-angle variables approach, find the oscillation frequency of a one-dimensional simple harmonic oscillator of mass m and force-constant k.

Solution

We have the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 = \text{constant} = E$$

We define

$$P \equiv J = \frac{1}{2\pi}\oint pdq \text{ where } p = \sqrt{2mE - m^2\omega^2 q^2}$$

The turning points are where $p = 0$

$$\Rightarrow \frac{1}{2}m\omega^2 q_0^2 = E \Rightarrow q_0 = \pm\sqrt{\frac{2E}{m\omega^2}}$$

We then have

$$J = 2 \times \frac{1}{2\pi}\int_{-q_0}^{+q_0} \sqrt{2mE - m^2\omega^2 q^2}dq$$

$$\Rightarrow J = \frac{E}{\omega} \Rightarrow \tilde{H} = \omega J$$

We then have the equation of motion

$$\dot{\Theta} = \frac{\partial \tilde{H}}{\partial J} = \omega$$

or

$$\Theta(t) = \omega t + \Theta$$

■

★★★ **Problem 15.9** Using the action-angle variables approach, find the frequencies of oscillation of a planet orbiting the sun, for both the radial and angular motions. What is the consequence of the fact that these frequencies turn out to be the same? You might need to use contour integration to evaluate an integral.

Solution

The Lagrangian is given by

$$L = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 - \frac{k}{r}$$

with

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2\dot{\theta} = \text{constant}$$

and

$$H = \frac{p_r^2}{2\mu} + \frac{p_\theta^2}{2\mu r^2} + \frac{k}{r} = \text{constant}$$

We define

$$J_\theta = \frac{1}{2\pi}\oint p_\theta d\theta = p_\theta$$

and

$$J_r = \frac{1}{2\pi}\oint p_r dr$$

Where we obtain p_r from energy conservation

$$\pm\sqrt{2\mu H - \frac{p_\theta^2}{r^2} - \frac{2\mu k}{r}} = p_r$$

$$\Rightarrow J_r = \frac{1}{2\pi}\oint\sqrt{2\mu H - \frac{J_\theta^2}{r^2} - \frac{2\mu k}{r}}dr$$

The integration can be done using contour integration

$$J_r = \frac{1}{2\pi}(\pi k\sqrt{\frac{2\mu}{-H}} - 2\pi J_\theta) = \frac{k}{2}\sqrt{\frac{2\mu}{-H}} - J_\theta$$

$$\Rightarrow \frac{2}{k}(J_r + J_\theta) = \sqrt{\frac{2\mu}{-H}}$$

$$\Rightarrow \tilde{H} = -\frac{\mu k^2}{2(J_r + J_\theta)^2}$$

$$\dot{\Theta} \equiv \frac{\partial \tilde{H}}{\partial J_\theta} = \omega_\Theta$$

$$\dot{R} \equiv \frac{\partial \tilde{H}}{\partial J_r} = \frac{\mu k^2}{(J_r + J_\theta)^3} = \omega_R = \omega_\Theta$$

The frequencies being the same ensures that orbits will close. ∎

★ **Problem 15.10** A particle of mass m moves in one dimension subject to a force F of constant magnitude, but directed toward the left for positive x and to the right for negative x. Thus the potential energy has the form $U = k|x|$ for some constant k. Using action-angle variables, find the frequency of oscillation as a function of the particle's energy.

Solution

We start with

$$H = \frac{p^3}{2m} + k|x| = -\text{constant}$$

Then we introduce

$$J = \frac{1}{2\pi}\oint pdx = \frac{1}{2\pi}\oint \sqrt{2mH - 2mk|x|}dx$$

where the turning points are at

$$p = 0 \Rightarrow x_{\pm} = \pm\frac{H}{k}$$

$$\Rightarrow J = 2 \times \frac{1}{2\pi}\int_{-\frac{H}{k}}^{+H/k} \sqrt{2m(H - k|x|)}dx$$

$$J = \frac{1}{\pi}\int_{-H/k}^{0} \sqrt{2m(H + kx)}dx + \frac{1}{\pi}\int_{0}^{+H/k} \sqrt{2m(H - kx)}dx = \frac{1}{\pi}\frac{4\sqrt{2m}}{3k}H^{3/2}$$

$$\Rightarrow \tilde{H} = (\frac{3\pi kJ}{2\sqrt{2m}})^{2/3}$$

$$\Rightarrow \dot{\Theta} = \frac{\partial\tilde{H}}{\partial J} = \frac{2}{3}\left(\frac{3\pi k}{4\sqrt{2\pi}}\right)^{2/3} J^{-1/3}$$

$$= \frac{2}{3}\left(\frac{3\pi k}{4\sqrt{2\pi}}\right)^{2/3}\left(\frac{3\pi k}{4\sqrt{2\pi}}\right)^{1/3} H^{-1/2}$$

$$= \frac{\pi k}{2\sqrt{2m}}H^{-1/2} = \omega$$

■

★★ **Problem 15.11** As an example of adiabatic invariance, consider the preceding problem in the case where the magnitude of the force F is slowly changed, i.e., slow relative to the oscillation period of the particle. Which (if any) of the following quantities remain constant in the adiabatic limit? (a) the oscillation amplitude? (b) the frequency of oscillation? (c) the energy?

Solution

We expect that J, the action angle variable, is constant in time when averaged over a period of oscillation. We have (from 15–10)

$$J = \frac{1}{\pi}\frac{4\sqrt{2m}}{3k(t)}H(t)^{3/2}$$

where H and k now depend on time.

$$\omega = \frac{\pi k(t)}{2\sqrt{2m}}H(t)^{-1/2} = \omega(t)$$

So, ω depends on time too.

Eliminating $k(t)$

$$\Rightarrow J = \frac{2}{3}\frac{H(t)}{\omega(t)}$$

$\Rightarrow$ $H(t)$ increases $\Rightarrow$ Amplitude increases. The frequency $\omega(t)$ also increases, so that the ratio H/ω is constant. ■

⋆ **Problem 15.12** A particle of mass m can move along the positive x axis only, subject to a constant force to the left. That is, there is an impenetrable wall at $x = 0$ preventing it from reaching negative x, and for positive x there is a constant force attracting the particle back towards the origin. Find the possible energy levels of the particle according to the "old quantum theory."

Solution

The potential energy is infinite for $x < 0$ and $U = kx$ for $x \geqslant 0$, where k is a positive constant (then $F = -\frac{dU}{dk} = -k$ as given) The old quantum mechanics states that $\oint pdx = nh$ where $E = \frac{p^2}{2m} + kx$ for $x \geqslant 0$. Therefore

$$p = \sqrt{2m(E - kx)}$$

So

$$\oint pdx = 2\int_0^{x_{\max}} dx\sqrt{2m(E-kx)} = 2\sqrt{2m}\int_0^{x_{\max}} \sqrt{E-kx}dx.$$

where $x_{\max} = E/k$. Let

$$u = E - kx \quad du = -kdx$$

so

$$nh = 2\sqrt{2m}\int_{u=E}^{0}\left(-\frac{du}{k}\right)\sqrt{u} = \frac{2\sqrt{2m}}{k}\int_0^E du\sqrt{u}$$

$$= \frac{2\sqrt{2m}}{k}\frac{u^{3/2}}{3/2}\Big|_0^E = \frac{4}{3}\frac{\sqrt{2m}}{k}E^{3/2},$$

$$E^{3/2} = nh\frac{3k}{4\sqrt{2m}} \quad E = n^{2/3}h^{2/3}\left(\frac{3}{4}\right)^{2/3}\frac{k^{2/3}}{(2m)^{1/3}}$$

$$E_n = n^{2/3}\left(\frac{9}{16}\frac{h^2k^2}{2m}\right)^{1/3} = n^{2/3}\left(\frac{9h^2k^2}{32m}\right)^{1/3} \qquad n = 0, 1, 2, \ldots$$

■

★ **Problem 15.13** A particle of mass m can move along the positive x axis only, subject to a Hooke's-law spring force $F = -kx$. That is, there is an impenetrable wall at $x = 0$ preventing it from reaching negative x, and for positive x there is a spring force attracting the particle back towards the origin. Find the allowed energies of the particle according to the "old quantum theory." Compare these energy levels with those for a particle moving *anywhere* on the x axis subject to a spring force $F = -kx$ attracting it to $x = 0$.

Solution

The potential energy is

$$U(x) = \infty \quad (x < 0)$$

$$U(x) = \frac{1}{2}kx^2 \quad (x \geqslant 0)$$

We have $\oint pdx = nh$ according to the old quantum theory. Now the classical energy is

$$E = p^2/2m + 1/2kx^2 \Rightarrow p = \sqrt{2m(E - \frac{1}{2}kx^2)}$$

So

$$\oint pdx = 2\int_0^{x_{\max}} \sqrt{2m(E - \frac{1}{2}kx^2)}dx = 2\sqrt{2mE}\int_0^{x_{\max}} dx\sqrt{1 - \frac{k}{2E}x^2}.$$

Let

$$u = \sqrt{\frac{k}{2E}}x \quad du = \sqrt{\frac{k}{2E}}dx,$$

so

$$nh = 2\sqrt{2mE}\sqrt{\frac{2E}{k}}\int_{u=0}^{1} du\sqrt{1-u^2}.$$

Let $u = \sin\theta$, $du = \cos\theta d\theta$.

$$nh = 4\sqrt{\frac{m}{k}} \equiv \int_0^{\pi/2} d\theta\cos^2\theta = 2\sqrt{\frac{m}{k}}E\int_0^{\pi/2} d\theta(1 + \cos 2\theta)$$

$$= 2\sqrt{\frac{m}{k}}E\left[\theta\Big|_0^{\pi/2} + \frac{\sin 2\theta}{2}\Big|_0^{\pi/2}\right] = \pi\sqrt{\frac{m}{k}}E.$$

So $E = \frac{nh}{\pi}\omega$, where $\omega = \sqrt{\frac{k}{m}}$ is the classical oscillation frequency. In terms of $h \equiv h/2\pi$, $E_n = 2nh\omega$ $n = 0, 1, 2, \ldots$. For a full SHO,

$$\oint pdx = 2\int_{-x_{\max}}^{x_{\max}}$$

instead of $2\int_0^{x_{\max}}$. So $\oint pdx =$ twice as large, hence $E_n = nh\omega$. There are two levels for every single level in the half SHO. ■

⋆ **Problem 15.14** A particle of mass m is confined to move inside a cubical box of side length L, with potential energy zero. Find the allowed energies of the particle according to the "old quantum theory."

Solution

$$E = p^2/2m \quad p^2 = p_x^2 + p_y^2 + p_z^2 = 2mE$$

$$\oint p_x dx = n_1 h \quad \oint p_y dy = n_2 h \quad \oint p_z dz = n_3 h$$

$$2\sqrt{2mE_x}L = n_1 h; \quad 2\sqrt{2mE_y}L = n_2 h; \quad 2\sqrt{2mE_z}L = n_3 h$$

$$E_x = \frac{n_1^2 h^2}{4 \cdot 2mL^2} \quad E_y = \frac{n_2^2 k^2}{4 \cdot 2mL^2} \quad E_z = \frac{n_3^2 h^2}{4 \cdot 2mL^2}$$

$$E = \frac{(n_1^2 + n_2 + n_3^2)h^2}{8mL^2}$$

where $n_1 = 0, 1, \cdots, n_2 = 0, 1, \cdots, n_3 = 0, 1, \cdots$ ∎

⋆⋆ **Problem 15.15** One end of a spring of rest-length zero and force-constant k is attached to a fixed point while the other end is attached to a ball of mass m, which is otherwise free to move as it likes in three-dimensional space. Find the allowed energies of the system according to the "old quantum theory."

Solution

$$E = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + \frac{1}{2}kr^2 = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + \frac{1}{2}k(x^2 + y^2 + z^2).$$

$$\Rightarrow p_x^2 = 2m\left(E_x - \frac{1}{2}kx^2\right) \quad p_y^2 = 2m\left(E_y - \frac{1}{2}ky^2\right) \quad p_z^2 = 2m\left(E_z - \frac{1}{2}kz^2\right)$$

where $E = E_x + E_y + E_z$. Now

$$\oint p_x dx = n_1 h \quad \oint p_y dy = n_2 h \quad \oint p_z dz = n_3 h.$$

where

$$p_x = \sqrt{2m(E_x - \frac{1}{2}kx^2)} \quad \text{etc.}$$

$$= \sqrt{2mE_x}\sqrt{1 - \frac{k}{2E_x}x^2} \text{ and}$$

$$\oint p_x dx = 2\int_{-x_{\max}}^{x_{\max}} dx\sqrt{2mEx}\sqrt{1 - \frac{kx^2}{2E_x}} \quad \text{etc.}$$

where $x_{\max} = \sqrt{\frac{2E_x}{k}}$. Thus

$$\oint p_x dx = 2\sqrt{2mE_x}\int_{-\sqrt{\frac{2E_x}{k}}}^{\sqrt{\frac{2E_x}{k}}} dx\sqrt{1-\frac{kx^2}{2E_y}} = n_1 h$$

Let $\sqrt{\frac{kx^2}{2E_x}} = \sin\theta$ so

$$\cos\theta d\theta = \sqrt{\frac{k}{2E_x}}dx$$

$$n_1 h = 2\sqrt{2mE_x}\int_{\theta=-\pi/2}^{\pi/2}\sqrt{\frac{2E_x}{k}}d\theta\cos^2\theta = 4\sqrt{\frac{m}{k}}E_x\frac{1}{2}\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta)d\theta$$

$$n_1 h = 2\sqrt{\frac{m}{k}}E_x\left[\pi + \frac{\sin 2\theta}{2}\Big|_{-\pi/2}^{\pi/2}\right] = 2\pi\sqrt{\frac{m}{k}}E_x.$$

That is,

$$E_x = \frac{n_1 h}{2\pi}\sqrt{\frac{k}{m}} = n_1\hbar\omega$$

where $\hbar \equiv k/2\pi$ and $\omega = \sqrt{\frac{k}{m}}$ is the classical angular frequency. Now altogether,

$$E = E_x + E_y + E_z = (n_1+n_2+n_3)\hbar\omega$$

$E = n\hbar\omega$ where $n = n_1+n_2+n_3$, with $n_1 = 0,1,2\ldots$, $n_2 = 0,1,2\ldots$, $n_3 = 0,1,2\ldots$ ■

★★ **Problem 15.16** From the classical point of view, the electron in a hydrogen atom moves under the influence of a central attractive force $F = -e^2/r^2$ caused by the proton nucleus. According to "old quantum theory" the phase integrals over r and θ are given by

$$\oint p_r\, dr = n_1 h \quad \text{and} \quad \oint p_\theta\, d\theta = n_2 h$$

where p_r and p_θ are the classical canonical momenta, h is Planck's constant and n_1 and n_2 are positive integers. Show that according to old quantum theory there are only a discrete set of possible energy levels, given by

$$E_n = -\frac{me^4}{2n^2\hbar^2}$$

where $\hbar \equiv h/2\pi$ and $n = n_1+n_2$.

Solution

The Lagrangian is

$$L = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - (-e^2/r) = \frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2) + e^2/r.$$

The canonical momenta are therefore

$$p_r \equiv \partial L/\partial\dot{r} = m\dot{r}, \quad p_\theta \equiv \partial L/\partial\dot{\theta} = mr^2\dot{\theta}.$$

Therefore

$$E = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{e^2}{r}.$$

Now p_θ is conserved, so

$$\oint p_\theta d\theta = 2\pi p_\theta = n_2 h.$$

Therefore

$$p_\theta = n_2 \frac{h}{2\pi} = n_2 \hbar$$

Then

$$p_r^2 = 2m \left[E + \frac{e^2}{r} - \frac{p_\theta^2}{2mr^2} \right]$$

and so

$$p_r = \frac{\sqrt{2m}}{r} \sqrt{Er^2 + e^2 r - p_\theta^2/2m}$$

and so

$$\oint p_r dr = \sqrt{2m} \times 2 \int_{r_{\text{min}}}^{r_{\text{max}}} \sqrt{Er^2 + e^2 r - p_\theta^2/2m}$$

First, we will find r_{max} and r_{min}, the radii for which

$$Er^2 + e^2 r - p_\theta^2/2m = 0.$$

By the quadratic equation,

$$r = \frac{-e^2 \pm \sqrt{e^4 + 2Ep_\theta^2/m}}{2E}.$$

However r must be > 0, so $E < 0$. Therefore

$$r_{\text{max min}} = \frac{e^2 \pm \sqrt{e^4 - 2|E|p_\theta^2/m}}{2|E|}.$$

Now the integral

$$\int \frac{dr}{r} \sqrt{X(r)}, \text{ where } X(r) \equiv a + br + cr^2$$

and

$$q \equiv 4ac - b^2, \text{ where } a = -\frac{p_\theta^2}{2m}, \quad b = e^2, \quad c = -|E|.$$

So

$$\int = \sqrt{X} + \frac{b}{2} \int \frac{dx}{\sqrt{X}} + a \int \frac{dx}{x\sqrt{X}}$$

(from integral tables)

$$\int = \sqrt{X} + \frac{b}{2} \left(-\frac{1}{\sqrt{-c}} \sin^{-1} \left(\frac{2cr + b}{\sqrt{-q}} \right) \right) + \frac{a}{\sqrt{-a}} \sin^{-1} \left(\frac{br + 2a}{r\sqrt{-q}} \right)$$

$$\int = \sqrt{-p_\theta^2/2m + e^2 r - |E|r^2} - \frac{e^2}{2\sqrt{|E|}} \sin^{-1}\left(\frac{-2|E|r + e^2}{\sqrt{e^4 - 2|E|p_\theta^2/m}}\right)$$

$$+ \frac{-(p_\theta^2/2m)}{\sqrt{p_\theta^2/2m}} \sin^{-1}\left(\frac{e^r - p_\theta^2/m}{r\sqrt{e^4 - 2|E|p_\theta^2/m}}\right).$$

Note that at $r = r_{\text{max}}$, the first term in the integral is zero. Also

$$-2|E|r + e^2 = -e^2 - \sqrt{e^4 - 2|E|p_\theta^2/m} + e^2 = -\sqrt{e^4 - 2|E|p_\theta^2/m}$$

and

$$e^2 r_{\text{max}} - p_\theta^2/m = \frac{e^4 + e^2\sqrt{e^4 - 2|E|p_\theta^2/m} - 2|E|p_\theta^2/m}{2|E|}$$

$$= \sqrt{e^4 - 2|E|p_\theta^2/m}\left[e^2 + \sqrt{e^4 - 2|E|p_\theta^2/m}\right]/2|E|$$

Therefore, evaluated at r_{max}, the integral is

$$-\frac{e^2}{2\sqrt{|E|}} \sin^{-1}(-1) - \sqrt{\frac{p_\theta^2}{2m}} \sin^{-1}(1) = \frac{\pi}{2}\left(\frac{e^2}{2\sqrt{|E|}} - \sqrt{\frac{p_\theta^2}{2m}}\right)$$

$$= \frac{\pi}{4\sqrt{|E|}}\left(e^2 - \sqrt{\frac{2|E|p_\theta^2}{m}}\right).$$

Similarly, evaluated at r_{min}, the integral is

$$-\frac{\pi}{4\sqrt{|E|}}\left(e^2 - \sqrt{\frac{2|E|p_\theta^2}{m}}\right)$$

So altogether

$$\oint p_r dr = 2\sqrt{2m}\left(\frac{\pi}{2\sqrt{|E|}}\right)\left(e^2 - \sqrt{\frac{2|E|p_\theta^2}{m}}\right) = n_1 h$$

Some algebra then given

$$E_n = -|E_n| = -\frac{me^4}{2n^2\hbar^2}$$

where $n = n_1 + n_2$. ■

★★ **Problem 15.17** If a quantum mechanical particle has definite energy E we can write its wave function in the form $\Psi(\mathbf{r}, t) = \psi(\mathbf{r})e^{-iEt/\hbar}$. (a) Substitute this into the full Schrödinger equation to show that the *time independent Schrödinger equation* for $\psi(\mathbf{r})$ may be written

$$\nabla^2\psi + \frac{2m}{\hbar^2}(E - U)\psi = 0$$

where both ψ and U are functions of position. (b) A particle of mass m is trapped inside a one-dimensional box of width L. The potential energy of the particle is zero for $0 < x < L$ and infinite otherwise. The wave function of the particle is zero outside the box and at $x = 0, L$. According to the one-dimensional Schrödinger equation, the lowest-energy "eigenfunction" ψ_1 for the particle is $\psi_1 = A \sin \pi x/L$ where A is a constant. Find the corresponding energy eigenvalue E_1. (c) Find all other energy eigenfunctions ψ_n of the particle, and the corresponding energy eigenvalues E_n.

Solution

(a) The full Schrödinger's equation is

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + U\Psi.$$

Substituting

$$\Psi = \psi(\mathbf{r}) e^{-iEt/\hbar}$$

gives

$$\nabla^2 \psi + \frac{2m}{\hbar^2}(E - U) = 0,$$

as claimed.

(b) Inside the box

$$\frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0$$

since $U = 0$. Substituting in

$$\psi_1 = A \sin \pi x/L$$

gives

$$-\frac{A\pi^2}{L^2} \sin \frac{\pi x}{L} + \frac{2mE_1}{\hbar^2} A \sin \frac{\pi x}{L} = 0 \Rightarrow E_1 = \frac{\pi^2 \hbar^2}{2mL^2}.$$

(c) Other solutions matching the boundary conditions $\psi = 0$ at $x = 0$, include

$$\psi = A_n \sin \frac{n\pi x}{L} \quad n = 1, 2, 3, \ldots$$

Substituting this into the time independent Schrödinger's equation gives

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2} \quad n = 1, 2, 3, \ldots$$

■

★★ **Problem 15.18** A particle of mass m is trapped inside a three-dimensional box with sides of length L. The potential energy of the particle is zero for $0 < x < L$, $0 < y < L$, $0 < z < L$ and infinite otherwise. Possible energy "eigenfunctions" $\psi_n(x, y, z)$ of the particle are zero outside the box and at every face. The lowest-energy eigenfunction ψ_1 for the particle is $\psi_1 = A \sin(\pi x/L) \sin(\pi y/L) \sin(\pi z/L)$ where A is a constant. (a) Find the corresponding energy eigenvalue E_1. (b) Find all other energy eigenfunctions ψ_n and corresponding energy eigenvalues E_n in terms of the "quantum number" n. (c) Compare these allowed energy values with those predicted by the "old quantum theory", showing that they agree in the limit of large n.

Solution

(a) The time-independent Schrödinger equation is

$$\nabla^2\psi + \frac{2m}{\hbar^2}(E-U)\psi = 0,$$

where here

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Substituting

$$\psi = A\sin\frac{\pi x}{L}\sin\frac{\pi y}{L}\sin\frac{\pi z}{L},$$

gives

$$-A\left[\left(\frac{\pi}{L}\right)^2 + \left(\frac{\pi}{L}\right)^2 + \left(\frac{\pi}{L}\right)^2\right] + \frac{2mE}{\hbar^2}A = 0,$$

so

$$E = \frac{3\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2,$$

which is the ground-state energy.

(b) Other solutions of the equation include

$$\psi_{n_1,n_2,n_3} = A_{n_1,n_2,n_3}\sin\frac{n_1\pi x}{L}\sin\frac{n_2\pi y}{L}\sin\frac{n_3\pi z}{L},$$

with corresponding energies

$$E_{n_1,n_2,n_3} = \frac{(n_1^2+n_2^2+n_3^3)}{2m}(\frac{\pi}{L})^2\hbar^2$$

where n_1, n_2, n_3 are integers $1, 2, 3, \ldots$

(c) The old QT predicts

$$\oint p_x dx = n_1 h, \quad \oint p_y dy = n_2 h, \quad \oint p_z dz = n_3 h$$

Here

$$E = \frac{p_x^2+p_y^2+p_z^2}{2m} \cong E_1 + E_2 + E_3.$$

where $E_1 = \frac{p_x^2}{2m}$ etc. So

$$\oint p_x dx = \sqrt{2mE_1}\oint dx = \sqrt{2mE_1} - 2L$$

so

$$2\sqrt{2mE_1}L = n_1 h \text{ or } 8mL^2E_1 = n_1^2 h_1^2$$

$$E_1 = \frac{n_1^2 h^2}{8mL^2}$$

$$E = E_1 + E_2 + E_3 = \frac{(n_1^2 + n_2^2 + n_3^3)h^2}{8mL^2}.$$

But $h \equiv 2\pi\hbar$, so

$$E = \frac{(n_1^2 + n_2^2 + n_3^3)\pi^2\hbar^2}{2mL^2}$$

same result by the old QT. Except then $n_1 = 0, 1, 2$ etc. ■

★★★ **Problem 15.19** The lowest-energy (*i.e.*, "ground state") eigenfunction of the electron in a hydrogen atom is spherically symmetric. (a) Using the time-independent Schrödinger equation, find this eigenfunction in terms of the radius r of the electron from the nucleus as origin. (b) Find also the electron's corresponding energy eigenvalue.

Solution

(a) The time-independent Schrödinger equation is

$$-(\frac{\hbar^2}{2m})\nabla^2\psi + U\psi = E\psi,$$

where $U = -e^2/r$ and in spherical coordinates the Laplacian is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\varphi}.\right)$$

We are searching for a spherically symmetric wave function, so the $\partial/\partial\theta$ and $\partial/\partial\varphi$ terms will vanish, leaving

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{d}{dr}(r^2\frac{d\psi}{dr})\right] - \frac{e^2}{r}\psi = E\psi.$$

As is often the case, we can solve this with trial error and educated guess work. We can try here $\psi = e^{-\alpha r}$, where α is a constant. Substituting in this trial solution,

$$+\frac{\hbar^2\alpha}{2mr^2}\frac{d}{dr}r^2e^{-\alpha r} - \frac{e^2}{r}e^{-\alpha r} \stackrel{?}{=} Ee^{-\alpha r}$$

$$\frac{\hbar^2\alpha}{2mr^2}\left[-r^2\alpha + 2r\right] - e^2/r \stackrel{?}{=} E \Rightarrow -\frac{\hbar^2\alpha^2}{2m} + \frac{\hbar^2\alpha}{mr} \stackrel{?}{=} \frac{e^2}{r} + E.$$

This is possible if

$$\frac{\hbar^2\alpha}{mr} = \frac{e^2}{r} \text{ and } E = -\frac{\hbar^2\alpha^2}{2m}.$$

That is, if

$$\alpha = \frac{me^2}{\hbar^2} \text{ and } E = -\frac{\hbar^2}{2m}\frac{m^2e^4}{\hbar^4} = -\frac{me^4}{2\hbar^2}$$

So (a) $\psi = Ae^{-\alpha r}$ where A is arbitrary (it can be found by normalization) and $\alpha = \frac{me^2}{\hbar^2}$.
(b) $E = -\frac{me^4}{2\hbar^2}$ ■

★★ **Problem 15.20** The ground state wave function of a one-dimensional simple harmonic oscillator of mass m and force-constant k is the Gaussian function $\psi(x) = Ae^{-\alpha x^2}$ where A is a normalization constant (adjusted to make $\int \psi^* \psi dx = 1$) and α is also a constant. (a) Using Schrödinger's equation, find α in terms of m, k, and $\hbar \equiv h/2\pi$. (b) Find the energy eigenvalue for this wave function, in terms of the same constants. (c) Compare with the energy predicted for this ground state by the "old quantum theory."

Solution

The time-independent Schrödinger equation is

$$\nabla^2\psi + \frac{2m}{\hbar^2}(E - U)\psi = 0 \text{ where } U = \frac{1}{2}kx^2.$$

and where we can take $\nabla^2 = \frac{\partial^2}{\partial x^2}$. Supposing

$$\psi(x) = Ae^{-\alpha x^2}, \quad \psi' = -2\alpha x A e^{-\alpha x^2}$$

and

$$\psi'' = -2\alpha A\left[e^{-\alpha x^2} + x(-2\alpha x)e^{-\alpha x^2}\right] = -2\alpha A\left[1 - 2\alpha x^2\right]e^{-\alpha x^2}$$

so, substituting into the Schrödinger equation,

$$-2\alpha A\left[1 - 2\alpha x^2\right]e^{-\alpha x^2} + \frac{2m}{\hbar^2}(E - \frac{1}{2}kx^2)Ae^{-\alpha x^2} = 0$$

It works if

$$-2\alpha A + \frac{2mE}{\hbar^2}A = 0 \quad (E = \frac{\alpha\hbar^2}{m})$$

(a) and

$$4\alpha^2 A x^2 = \frac{mkx^2}{\hbar^2}A \quad (\alpha^2 = \frac{mk}{4\hbar^2}) \Rightarrow \alpha = \frac{\sqrt{mk}}{2\hbar}$$

(b)

$$E = \frac{\alpha\hbar}{m} = \frac{\sqrt{mk}\hbar}{2m} = \frac{1}{2}\hbar\omega \text{ where } \omega = \sqrt{\frac{k}{m}}$$

is the classical frequency.

(c) In the "old quantum theory" the ground-state energy is $E = 0$. ■

★★ **Problem 15.21** The angular momentum vector of a particle of mass m is written as $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. Find the Poisson brackets of any two components of the angular momentum vector in Cartesian coordinates. Do this using the explicit representation of the Poisson bracket as derivatives with respect to canonical coordinates and momenta. Show that the result is given by $\{L^x, L^y\} = L^z$, $\{L^y, L^z\} = L^x$, and $\{L^z, L^x\} = L^y$ (*i.e.*, cyclic permutations of (xyz)). This is known as the angular momentum algebra. You might find it useful to write $L^i = \varepsilon^{ijk}x^j p^k$, and use identities involving the totally antisymmetric tensor ε^{ijk}.

Solution

We have

$$L^i = \varepsilon^{ijk} x^j p^k$$

$$\Rightarrow \{L^i, L^\ell\} = \{\varepsilon^{ijk} x^j p^k, \varepsilon^{\ell mn} x^m p^n\}$$

$$= \varepsilon^{ijk} x^j p^k \varepsilon^{\ell mn} x^m p^n \left(\frac{\partial(x^j p^k)}{\partial x^a} \frac{\partial(x^m p^n)}{\partial p^a} - \frac{\partial(x^j p^k)}{\partial p^a} \frac{\partial(x^m p^n)}{\partial x^a} \right)$$

$$= \varepsilon^{ijk} \varepsilon^{\ell mn} \left(\delta^{ja} \delta^{na} p^k x^m - \delta^{ka} \delta^{ma} x^j p^n \right)$$

$$= \varepsilon^{ijk} \varepsilon^{\ell mn} \left(\delta^{jn} p^k x^m - \delta^{km} x^j p^n \right)$$

$$= \varepsilon^{ijk} \varepsilon^{\ell mj} x^m p^k - \varepsilon^{ijk} \varepsilon^{\ell kn} x^j p^n$$

$$= -\varepsilon^{jik} \varepsilon^{j\ell m} x^m p^k + \varepsilon^{kij} \varepsilon^{k\ell n} x^j p^n$$

where we used the antisymmetry of ε^{ijk}. We now use the identity

$$\varepsilon^{ijk} \varepsilon^{imn} = \delta^{jm} \delta^{kn} - \delta^{jn} \delta^{kn}$$

$$\Rightarrow \{L^i, L^\ell\} = -(\delta^{i\ell} \delta^{km} - \delta^{im} \delta^{k\ell}) x^m p^k + (\delta^{i\ell} \delta^{jn} - \delta^{in} \delta^{j\ell}) x^j p^n$$

$$= -x^k p^k \delta^{i\ell} + x^i p^\ell + x^n p^n \delta^{i\ell} - x^\ell p^i$$

$$= x^i p^\ell - x^\ell p^i$$

But

$$\varepsilon^{i\ell k} L^k = \varepsilon^{i\ell k} \varepsilon^{kmn} x^m p^n$$

$$= (\delta^{im} \delta^{\ell n} - \delta^{in} \delta^{\ell m}) x^m p^n$$

$$= x^i p^\ell - x^\ell p^i$$

$$\Rightarrow \{L^i, L^\ell\} = \varepsilon^{i\ell k} L^k$$

as needed. ■

⋆ **Problem 15.22** (a) Repeat the previous problem but instead use only the four properties of the Poisson bracket and the facts that $\{x, p^x\} = \{y, p^y\} = \{z, p^z\} = 1$ while the other brackets of positions and momenta vanish. (b) Write the quantum version of this algebra using the quantization scheme described in the chapter.

Solution

(a) We have

$$L^i = \varepsilon^{ijk} x^j p^k$$

$$\Rightarrow \{L^i, L^\ell\} = \{\varepsilon^{ijk} x^j p^k, \varepsilon^{\ell mn} x^m p^n\} = \varepsilon^{ijk} \varepsilon^{\ell mn} \{x^j p^k, x^m p^n\}$$

But we know

$$\{A,BC\} = \{A,B\}C + B\{A,C\}$$

which can be applied repeatedly

$$\Rightarrow \{L^i, L^\ell\} = \varepsilon^{ijk}\varepsilon^{\ell mn}(\{x^j p^k, x^m\}p^n + x^m\{x^j p^k, p^n\})$$

$$= -\varepsilon^{ijk}\varepsilon^{\ell mn}(\{x^m, x^j p^k\}p^n + \{p^n, x^j p^k\}x^m)$$

$$= -\varepsilon^{ijk}\varepsilon^{\ell mn}((\{x^m, x^j\}p^k + x^j\{x^m, p^k\})p^n + (\{p^n, x^j\}p^k + \delta^j\{p^n, p^k\})x^m)$$

$$= -\varepsilon^{ijk}\varepsilon^{\ell mn}(\delta^{mk}x^j p^n - \delta^{jn}p^k x^m)$$

$$= -\varepsilon^{ijk}\varepsilon^{\ell kn}x^j p^n + \varepsilon^{ijk}\varepsilon^{\ell mj}p^k x^m$$

We now use the identity

$$\varepsilon^{ijk}\varepsilon^{imn} = \delta^{jm}\delta^{kn} - \delta^{jn}\delta^{kn}$$

and the antisymmetry of ε^{ijk}

$$\Rightarrow \{L^i, L^\ell\} = +\varepsilon^{kij}\varepsilon^{k\ell n}x^j p^n - \varepsilon^{jik}\varepsilon^{j\ell m}p^k x^m$$

$$= (\delta^{i\ell}\delta^{jn} - \delta^{in}\delta^{j\ell})x^j p^n - (\delta^{i\ell}\delta^{km} - \delta^{im}\delta^{k\ell})x^m p^k$$

$$= x^n p^n \delta^{i\ell} - x^\ell p^i - \delta^{i\ell}x^m p^m + x^i p^\ell = x^i p^\ell - x^\ell p^i$$

But

$$\varepsilon^{i\ell k}L^k = \varepsilon^{i\ell k}\varepsilon^{kmn}x^m p^n = (\delta^{im}\delta^{\ell n} - \delta^{in}\delta^{\ell m})x^m p^n = x^i p^\ell - x^\ell p^i$$

$$\Rightarrow \{L^i, L^\ell\} = \varepsilon^{i\ell k}L^k$$

as needed.

(b) We replace

$$\{\cdot,\cdot\} = \frac{1}{i\hbar}[\cdot,\cdot] \Rightarrow \left[\hat{L}^i, \hat{L}^\ell\right] = i\hbar\varepsilon^{i\ell k}L^k$$

■

⋆ **Problem 15.23** Using equation 11.140, find the generator of the transformation that can translate a function of the canonical momentum $f(p)$ by $p \to p + \epsilon$.

Solution

$$\delta A = \epsilon\{A, G\}$$

The transformation applied to $f(p)$ acts as

$$f(p) \to f(p+\epsilon) \Rightarrow \delta f = f(p+\epsilon) - f(p) = f(p) + \frac{\partial f}{\partial p}\epsilon - f(p) = \frac{\partial f}{\partial p}\epsilon$$

where we Taylor expanded $f(p+\epsilon)$ for small ϵ. We then need G such that

$$\delta f = \frac{\partial f}{\partial p}\epsilon = \epsilon\{f(p), G\}$$

$$\Rightarrow \frac{\partial f}{\partial p} = \frac{\partial f}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial G}{\partial q} = -\frac{\partial f}{\partial p}\frac{\partial G}{\partial q}$$

$$\Rightarrow \frac{\partial G}{\partial q} = -1$$

with

$$F_2(qP) = qP + \epsilon G(q, P, t)$$

The simplest solution is

$$G = -q$$

so that

$$F_2 = qP - \epsilon q \Rightarrow p = P - \epsilon \text{ and } Q = q \Rightarrow Q = q \text{ and } P = p + \epsilon \text{ as needed.}$$

■

★★ **Problem 15.24** Using equation 11.140, show that if we use $G = \epsilon L_x$ as a generator of a transformation (where L_x is the x-component of the angular momentum), we end up rotating the components of the position vector $\mathbf{r}$ by an infinitesimal angle ϵ about the x-axis. Show this by applying the generator onto an arbitrary function of position $A(\mathbf{r})$. Similarly, find the generators that rotate the position vector about the y and z-axes.

Solution

$$G = \varepsilon L^x$$

we have

$$A(\mathbf{r}) \to A(R\mathbf{r})$$

where

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\epsilon & \sin\epsilon \\ 0 & -\sin\epsilon & \cos\epsilon \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \epsilon \\ 0 & -\epsilon & 1 \end{pmatrix}$$

$$R\mathbf{r} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \epsilon \\ 0 & -\epsilon & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y + \epsilon z \\ -\epsilon y + z \end{pmatrix}$$

So, we have

$$A(\mathbf{r}) \to A(x, y + \epsilon z, z - \epsilon y)$$

$$\Rightarrow \delta A : A(R\mathbf{r}) - A(\mathbf{r}) = A(x, y + \epsilon z, z - \epsilon y) - A(x, y, z) \simeq \frac{\partial A}{\partial y}\epsilon z - \frac{\partial A}{\partial z}\epsilon y$$

where Taylor expanded $A(x, y+\epsilon z, z-\epsilon y)$ for small ε. We then need

$$\delta A = \epsilon \frac{\partial A}{\partial y} z - \epsilon \frac{\partial A}{\partial z} y \sim \{A, \varepsilon L^x\}$$

Let's check

$$\{A, \epsilon L^x\} = \{A, \varepsilon y p^z - \varepsilon z p^y\} = \varepsilon\{A, yp^z\} - \varepsilon\{A, zp^y\}$$

$$= \varepsilon \frac{\partial A}{\partial x^i}\frac{\partial (yp^z)}{\partial p^i} - \varepsilon \frac{\partial A}{\partial p^i}\frac{\partial (yp^z)}{\partial x^i} - \varepsilon \frac{\partial A}{\partial x^i}\frac{\partial (zp^y)}{\partial p^i} + \varepsilon \frac{\partial A}{\partial p^i}\frac{\partial (zp^y)}{\partial x^i}$$

$$= \varepsilon \frac{\partial A}{\partial z} y - \varepsilon \frac{\partial A}{\partial p^y} p^z - \varepsilon \frac{\partial A}{\partial y} z + \varepsilon \frac{\partial A}{\partial p^z} p^y$$

$$= \varepsilon\left(\frac{\partial A}{\partial z} y - \frac{\partial A}{\partial y} z\right)$$

We then have

$$\delta A = -\{A, \varepsilon L^x\} \text{ as needed.}$$

We can similarly show that

$$\delta A = -\{A, \varepsilon L^y\}$$

for rotation by ε about y-axis. and

$$\delta A = -\{A, \varepsilon L^z\}$$

for rotation by ε about z-axis. ■

★★ **Problem 15.25** Inspired by the previous problem, find the generators that rotate the momentum vector $\mathbf{p}$ about the x, y, and z axes by infinitesimal angles.

Solution

We have

$$A(\mathbf{p}) \to A(R\mathbf{p})$$

where

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\epsilon & \sin\epsilon \\ 0 & -\sin\epsilon & \cos\epsilon \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \epsilon \\ 0 & -\epsilon & 1 \end{pmatrix}$$

$$R\mathbf{r} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \epsilon \\ 0 & -\epsilon & 1 \end{pmatrix}\begin{pmatrix} p^x \\ p^y \\ p^z \end{pmatrix} = \begin{pmatrix} p^x \\ p^y + \epsilon p^z \\ -\epsilon p^y + p^z \end{pmatrix}$$

So, we have

$$A(\mathbf{p}) \to A(p^x, p^y + \epsilon p^z, p^z - \epsilon p^y)$$

$$\Rightarrow \delta A : A(R\mathbf{p}) - A(\mathbf{p}) = A(p^x, p^y + \epsilon p^z, p^z - \epsilon p^y) - A(p^x, p^y, p^z) \simeq \frac{\partial A}{\partial p^y}\epsilon p^z - \frac{\partial A}{\partial p^z}\epsilon p^y$$

where Taylor expanded

$$A(p^x, p^y + \epsilon p^z, p^z - \epsilon p^y)$$

for small ε.

Let's check

$$\{A, \varepsilon L^x\} = \{A, \varepsilon y p^z - \varepsilon z p^y\}$$

$$= \varepsilon\{A, yp^z\} - \varepsilon\{A, zp^y\}$$

$$= \varepsilon \frac{\partial A}{\partial x^i}\frac{\partial (yp^z)}{\partial p^i} - \varepsilon \frac{\partial A}{\partial p^i}\frac{\partial (yp^z)}{\partial x^i} - \varepsilon \frac{\partial A}{\partial x^i}\frac{\partial (zp^y)}{\partial p^i} + \varepsilon \frac{\partial A}{\partial p^i}\frac{\partial (zp^y)}{\partial x^i}$$

$$= \varepsilon \frac{\partial A}{\partial z} y - \varepsilon \frac{\partial A}{\partial p^y} p^z - \varepsilon \frac{\partial A}{\partial y} z + \varepsilon \frac{\partial A}{\partial p^z} p^y$$

$$= -\varepsilon \frac{\partial A}{\partial p^y} p^z + \varepsilon \frac{\partial A}{\partial p^z} p^y$$

$$\Rightarrow \delta A = -\{A, \varepsilon L^x\}$$

Similarly, for notation about y-axis, we get

$$\delta A = -\{A, \varepsilon L^y\}$$

and for rotation about z-axis, we get

$$\delta A = -\{A, \varepsilon L^z\}$$

■

⋆⋆ **Problem 15.26** Compute the Poisson bracket of any components of position or momentum with any component of angular momentum. Use the Poisson bracket representation as derivatives with respect to canonical coordinates and momenta. You might find it useful to write $L^i = \varepsilon^{ijk} x^j p^k$, and use identities involving the totally antisymmetric tensor ε^{ijk}.

Solution

We want $\{x^i, L^j\}$ and $\{p^i, L^j\}$ where

$$L^j = \varepsilon^{jk\ell} x^k p^\ell$$

we have

$$\{x^i, L^j\} = \varepsilon^{jk\ell}\{x^i, x^k p^\ell\}$$

$$= \varepsilon^{jk\ell}\left(\frac{\partial(x^i)}{\partial x^m}\frac{\partial(x^k p^\ell)}{\partial p^m} - \frac{\partial(x^i)}{\partial p^m}\frac{\partial(x^k p^\ell)}{\partial x^m}\right)$$

$$= \varepsilon^{jk\ell}(\delta^{im}\delta^{\ell m} x^k)$$

$$= \varepsilon^{jk\ell}\delta^{i\ell} x^k = \varepsilon^{jki} x^k = \varepsilon^{ijk} x^k$$

We also have

$$\{p^i, L^j\} = \varepsilon^{jk\ell}\{p^i, x^k p^\ell\}$$

$$= \varepsilon^{jk\ell}\left(\frac{\partial(p^i)}{\partial x^m}\frac{\partial(x^k p^\ell)}{\partial p^m} - \frac{\partial(p^i)}{\partial p^m}\frac{\partial(x^k p^\ell)}{\partial x^m}\right)$$

$$= \varepsilon^{jk\ell}(\delta^{im}\delta^{km}p^\ell)$$

$$= -\varepsilon^{jk\ell}\delta^{ik}p^\ell$$

$$= -\varepsilon^{jk\ell}p^\ell$$

$$= +\varepsilon^{jk\ell}p^\ell$$

■

★ **Problem 15.27** Repeat the previous problem but use only the four properties of the Poisson bracket and the particular Poisson brackets between the components of the position and momentum vectors. From this, deduce the corresponding commutation relations in quantum mechanics.

Solution

We want $\{x^i, L^j\}$ and $\{p^i, L^j\}$ where

$$L^j = \varepsilon^{jk\ell}x^k p^\ell$$

We have

$$\{x^i, L^j\} = \varepsilon^{jk\ell}\{x^i, x^k p^\ell\}$$

$$= \varepsilon^{jk\ell}(\{x^i, x^k\}p^\ell + x^k\{x^i, p^\ell\})$$

$$= \varepsilon^{jk\ell}x^k\delta^{i\ell} = \varepsilon^{jki}x^k = \varepsilon^{ijk}x^k$$

Similarly,

$$\{p^i, L^j\} = \varepsilon^{jk\ell}\{p^i, x^k p^\ell\}$$

$$= \varepsilon^{jk\ell}\left(\{p^i, x^k\}p^\ell + x^k\{p^i, p^\ell\}\right)$$

$$= -\varepsilon^{jk\ell}\{x^k, p^i\}p^\ell$$

$$= -\varepsilon^{jk\ell}\delta^{ki}p^\ell$$

$$= -\varepsilon^{ji\ell}p^\ell$$

$$= +\varepsilon^{ijk}p^\ell$$

To quantize, we write

$$\{\cdot,\cdot\} = \frac{[\cdot,\cdot]}{i\hbar} \Rightarrow \left[\hat{x}^i, \hat{L}^j\right] = i\hbar\varepsilon^{ijk}\hat{x}^k$$

and

$$\left[\hat{p}^i, \hat{L}^j\right] = i\hbar\varepsilon^{ijk}\hat{p}^k$$

■

★★ **Problem 15.28** Find the generator that performs a Galilean boost in the x-direction by an infinitesimal speed ϵ. Do this using equation 11.140, working backwards and considering expected effects of the transformation on position and momentum of a particle.

Solution

A Galilean boost corresponds to

$$x \to x + \epsilon t \qquad p^x \to p^x + m\epsilon$$

$$y \to y \quad \text{and} \quad p^y \to p^y$$

$$z \to z \qquad p^z \to p^z$$

For a function $A(\mathbf{r}, \mathbf{p})$, we get

$$\delta A = A(x + \epsilon t, y, z, p^x + m\epsilon, p^y, p^z) - A(x, y, z, p^x, p^y, p^z)$$

$$= \frac{\partial A}{\partial x}\epsilon t + \frac{\partial A}{\partial p^x} m\epsilon$$

where we Taylor expanded the first term for small ϵ.

We need

$$\delta A = \frac{\partial A}{\partial x}\epsilon t + \frac{\partial A}{\partial p^x} m\epsilon = \epsilon\{A, G\} = \epsilon\frac{\partial A}{\partial x}\frac{\partial G}{\partial p^x} - \frac{\partial A}{\partial p^x}\frac{\partial G}{\partial x}\epsilon$$

assuming $G(x, p^x)$ with no dependence on y, z, p^y, p^z as they are not needed for the matching. We then want $\frac{\partial G}{\partial p^x} = t$ and $\frac{\partial G}{\partial x} = -m$ with

$$F_2 = xP^x + \epsilon G(x, P^x)$$

Choose

$$G(x, P^x) = -mx + tP^x = -mx + tP^x$$

We then have

$$p^x = P^x + \epsilon\frac{\partial G}{\partial x} = P^x - m\epsilon$$

$$X = x + \epsilon\frac{\partial G}{\partial P^x} = x + \epsilon t$$

$$\Rightarrow P^x = p^x + mG \quad X = x + \epsilon t$$

as needed. ■

★★★ **Problem 15.29** (a) Find the generator that performs an infinitesimal scale transformation, where $\mathbf{r}' = (1+\epsilon)\mathbf{r}$, and similarly for momentum. (b) Find the brackets of this generator with the components of angular momentum of a particle.

Solution

(a) We have

$$\mathbf{r}' = (1+\epsilon)\mathbf{r} \Rightarrow \delta\mathbf{r} = \mathbf{r}' - \mathbf{r} = (1+\epsilon)\mathbf{r} - \mathbf{r} = \epsilon\mathbf{r}$$

For a function $A(\mathbf{r})$, we get

$$\delta A = A(\mathbf{r}') - A(\mathbf{r}) = A(\mathbf{r} + \epsilon\mathbf{r}) - A(\mathbf{r}) = \frac{\partial A}{\partial x^i}\epsilon x^i$$

where we Taylor expanded $A(\mathbf{r}')$ for small ϵ.

But we also have

$$\delta A = \epsilon\{A, G\} \Rightarrow \epsilon\frac{\partial A}{\partial x^i}x^i = \epsilon\left(\frac{\partial A}{\partial x^i}\frac{\partial G}{\partial p^i} - \frac{\partial A}{\partial p^i}\frac{\partial G}{\partial x^i}\right)$$

We can then take

$$G = p^i x^i \equiv D \Rightarrow F_2 = x^i P^i + \epsilon x^i P^i$$

$$\Rightarrow X^i = x^i + \epsilon\frac{\partial G}{\partial P^i} = (1+\epsilon)x^i$$

$$p^i = P^i + \epsilon\frac{\partial G}{\partial x^i} = P^i + \epsilon P^i$$

$$\Rightarrow X^i = (1+\epsilon)x^i \quad P^i = (1+\epsilon)^{-1}p^i = (1-\epsilon)p^i$$

Notice that P^i scales opposite to x^i. This is because $P^i = \frac{\partial L}{\partial \dot{x}^i}$ lives in a dual vector space of contravariant vectors (a topic beyond this book).

(b) We want

$$\{D, L^i\} = \{x^j p^j, \varepsilon^{i\ell k}x^\ell p^k\}$$

$$= \varepsilon^{i\ell k}\{x^j p^j, x^\ell, p^k\}$$

$$= \varepsilon^{i\ell k}(\{x^j p^j, x^\ell\}p^k + x^\ell\{x^j p^j, p^k\})$$

$$= \varepsilon^{i\ell k}(-\{x^\ell, x^j p^j\}p^k - x^\ell\{p^k, x^j p^j\})$$

$$= \varepsilon^{i\ell k}\left(-x^j\{x^\ell p^j\}p^k - x^\ell\{p^x, x^j\}p^j\right)$$

$$= \varepsilon^{i\ell k}(-x^j p^k \delta^{\ell j} + x^\ell p^j \delta^{kj})$$

$$= \varepsilon^{i\ell k}(-x^\ell p^k + x^\ell p^k)$$

$$= 0$$

■

★★★ **Problem 15.30** Show that

$$\hat{q}(t_0 + \Delta t) = e^{\frac{i}{\hbar}\hat{H}\Delta t}\hat{q}(t_0)e^{-\frac{i}{\hbar}\hat{H}\Delta t}$$

where we define

$$e^{-\frac{i}{\hbar}\hat{H}\Delta t} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-i\hat{H}}{\hbar}\right)^n$$

implies

$$\hat{q}(t_0 + \Delta t) = \hat{q}(t_0) + (-i)\frac{\Delta t}{\hbar}[\hat{q},\hat{H}] + (-i)^2\frac{1}{2!}\frac{\Delta t^2}{\hbar^2}[[\hat{q},\hat{H}],\hat{H}] + \cdots .$$

Do this by showing the pattern for the first few terms only.

Solution

We write

$$\hat{q}(t_0 + \Delta t) = e^{+\frac{i}{\hbar}\hat{H}\Delta t}\hat{q}(t_0)e^{-\frac{i}{\hbar}\hat{H}\Delta t}$$

$$\cong (1 + \frac{i}{\hbar}\hat{H}\Delta t - \frac{1}{2\hbar^2}\hat{H}^2\Delta t^2)\hat{q}(t_0)(1 - \frac{i}{\hbar}\hat{H}\Delta t - \frac{1}{2\hbar^2}\hat{H}^2\Delta t^2)$$

$$= \hat{q}(t_0) + \frac{i}{\hbar}\Delta t\hat{H}\hat{q}(t_0) - \frac{i}{\hbar}\Delta t\hat{q}(t_0)\hat{H} - \frac{1}{2\hbar^2}\Delta t^2\hat{H}^2\hat{q}(t_0)$$

$$-\frac{1}{2\hbar^2}\Delta t^2\hat{q}(t_0)\hat{H}^2 + \frac{1}{\hbar^2}\Delta t^2\hat{H}\hat{q}(t_0)\hat{H} + \mathcal{O}(\Delta t^3)$$

$$= \hat{q}(t_0) + \frac{(-i)}{\hbar}\Delta t\left[\hat{q}(t_0),\hat{H}\right] + \frac{(-i)}{\hbar}\Delta t\left[\left[\hat{q},\hat{H}\right],\hat{H}\right] + \mathcal{O}(\Delta t^3)$$

Where we verify

$$\left[\left[\hat{q},\hat{H}\right],\hat{H}\right] = \left[\hat{q}\hat{H}\right]\hat{H} - \hat{H}\left[\hat{q}\hat{H}\right]$$

$$= \hat{q}\hat{H}^2 - \hat{H}\hat{q}\hat{H} - \hat{H}\hat{q}\hat{H} + \hat{H}^2\hat{q}$$

$$= q\hat{H}^2 + \hat{H}^2\hat{q} - 2\hat{H}\hat{q}\hat{H}$$

as needed. ∎