Question

Consider a quantum system made of one qubit. The system is in the initial state $|\psi\rangle = 1/\sqrt{N}\ (a|0\rangle + b|1\rangle)$, where a and b are complex numbers and N is a normalization factor. A quantum gate Phase is applied on $|\psi\rangle$. Can you describe the action of this gate on the system? Can you give a graphical representation of the circuit? What is the final state of the system?

Answer 1

The Phase gate, often denoted as "S" or "P", is a basic single-qubit gate in quantum computing. It introduces a phase shift of $\pi/2$ (or 90 degrees) to the state $|1\rangle$ while leaving the state $|0\rangle$ unchanged. In matrix form, the Phase gate is represented as:

Phase = [[1, 0],
[0, i]],

where i is the imaginary unit.

To describe the action of the Phase gate on the given initial state $|\psi\rangle = 1/\sqrt{N}\ (a|0\rangle + b|1\rangle)$, we apply the gate to each component of the superposition:

Phase $(1/\sqrt{N}\ (a|0\rangle + b|1\rangle)) = 1/\sqrt{N}\ (a|0\rangle + ib|1\rangle)$,

where we have introduced the phase factor i to the $|1\rangle$ component.

For a graphical representation of the circuit, we can use the following quantum circuit diagram:

```
     ┌───┐
q_0: ┤ S ├
     └───┘
```

This circuit diagram shows a single qubit, labeled q_0, with the Phase gate, denoted as "S," applied to it.

Finally, to determine the final state of the system, we substitute the transformed components back into the initial state expression:

$|\psi\rangle = 1/\sqrt{N}\ (a|0\rangle + ib|1\rangle)$.

So, after the Phase gate is applied, the final state of the system is $|\psi\rangle = 1/\sqrt{N}\ (a|0\rangle + ib|1\rangle)$.

Answer 2
The Phase gate, denoted as P(θ), is a single-qubit gate that applies a phase shift of θ to the |1⟩ state. The matrix representation of this gate in the computational basis is:

```
P(θ) = [1   0  ]
       [0  e^(iθ)]
```

Thus, if we apply the Phase gate to the initial state |ψ⟩, we get:

$P(\theta)\ |\psi\rangle = 1/\sqrt{N}\ (\ a|0\rangle + e^{(i\theta)}\ b|1\rangle\ )$

This means that the gate applies a phase shift of θ to the |1⟩ state, while leaving the |0⟩ state unchanged.

A graphical representation of the circuit for this gate on a single qubit is:

|ψ⟩ ---[P(θ)]--- P(θ)|ψ⟩

where the gate is represented by the symbol [P(θ)].

The final state of the system after applying the Phase gate is:

$P(\theta)\ |\psi\rangle = 1/\sqrt{N}\ (\ a|0\rangle + e^{(i\theta)}\ b|1\rangle\ )$

which is a superposition of the |0⟩ and |1⟩ states with a phase shift of θ on the |1⟩ state.

Answer 3
The quantum gate you mentioned, the Phase gate (also known as the P gate or RZ gate), is a single-qubit gate commonly represented by the following matrix:

[1 0]
[0 e^(iθ)]

where θ is the phase angle. The gate acts on a qubit and applies a phase shift of θ to the state |1⟩ while leaving the state |0⟩ unchanged.

To describe the action of the Phase gate on the system in the initial state |ψ⟩ = 1/√N (a|0⟩ + b|1⟩), let's apply the gate to each basis state:

Phase|0⟩ = |0⟩ (since there is no phase shift on |0⟩)
Phase|1⟩ = e^(iθ)|1⟩

So, applying the Phase gate to the initial state |ψ⟩, we have:

Phase(|ψ⟩) = 1/√N (a|0⟩ + b(e^(iθ)|1⟩))

Now, to represent this circuit graphically, we can use the standard quantum circuit notation. The circuit for this scenario would look like this:

```
         ┌───────┐
q_0: ────────┤ P ├──
     ┌───────┴───────┴────┐
q_1: ┤ Initial State ├
     └─────────────────────────┘
```

In the circuit diagram above, the box labeled "P" represents the Phase gate applied to the qubit q_0. The qubit q_1 is prepared in the initial state |ψ⟩ = 1/√N (a|0⟩ + b|1⟩), which is represented by the "Initial State" box.

Finally, to determine the final state of the system after applying the Phase gate, we can rewrite the state as:

Phase(|ψ⟩) = 1/√N (a|0⟩ + b(e^(iθ)|1⟩))

The final state can be obtained by normalizing the coefficients:

Phase(|ψ⟩) = 1/√N' (a|0⟩ + b(e^(iθ)|1⟩))

where N' is the new normalization factor.