Question

Consider a quantum system made of two qubits. The system is in the initial state |ψ⟩ = |01⟩. A Control Z gate is applied on the system with the first qubit as control qubit and the second qubit as target qubit. Can you describe the action of this gate on the system? Can you give a graphical representation of the circuit? What is the final state of the system?

Answer 1

The Control Z gate, also known as the Controlled-Phase (CZ) gate, is a two-qubit gate commonly used in quantum computing. It applies a phase of -1 to the target qubit if and only if the control qubit is in the state |1⟩.

In the case of the given quantum system with initial state |ψ⟩ = |01⟩, the first qubit is the control qubit (|0⟩) and the second qubit is the target qubit (|1⟩). Applying the Control Z gate on this system results in the following action:

If the control qubit is in the state |0⟩ (as in this case), there is no effect on the target qubit. Therefore, the final state of the system after applying the Control Z gate would still be |01⟩.

Graphical representation of the circuit:

```
     ┌─────────┐
q0: ─┤ X ├─────■─────
       └────┬────┘  ┌────┴────┐
q1: ────────■────┤ Z ├
                 └─────────┘
```

In the above circuit representation, the "X" gate on qubit 0 represents the preparation of the initial state |01⟩. The "Z" gate represents the Control Z gate applied with qubit 0 as the control qubit and qubit 1 as the target qubit.

The final state of the system remains |01⟩, as the Control Z gate does not change the state when the control qubit is in the state |0⟩.

Answer 2

The Control Z gate, also known as the Controlled Phase gate, is a two-qubit gate that performs a phase shift on the target qubit if the control qubit is in the state $|1\rangle$. It can be represented by the following matrix:

```
CZ = |1 0 0 0|
     |0 1 0 0|
     |0 0 1 0|
     |0 0 0 -1|
```

If we apply the CZ gate on the initial state $|\psi\rangle = |01\rangle$, we get the following circuit:

```
|0⟩ --[H]--|+⟩ --[CZ]-- |+⟩ --[H]-- |0⟩
            |            |
           |1⟩          |1⟩
```

Here, the Hadamard gate (H) is applied on both qubits, which transforms $|01\rangle$ to $(|0\rangle+|1\rangle)\otimes(|0\rangle-|1\rangle)/2$, denoted by $|+\rangle\otimes|-\rangle$. The CZ gate is then applied with the first qubit as the control qubit and the second qubit as the target qubit. Finally, another Hadamard gate is applied to both qubits, which transforms $|+\rangle\otimes|-\rangle$ to $|01\rangle$.

To see why this is the case, let's apply the CZ gate to the state $|+\rangle\otimes|-\rangle$. If the control qubit is in state $|0\rangle$, nothing happens. If the control qubit is in state $|1\rangle$, the target qubit gets multiplied by -1. So we get:

$$
\begin{aligned}
CZ(|+\rangle\otimes|-\rangle) &= CZ[(|0\rangle+|1\rangle)\otimes(|0\rangle-|1\rangle)/2] \\
&= CZ(|0\rangle\otimes(|0\rangle-|1\rangle)/2 + |1\rangle\otimes(|0\rangle-|1\rangle)/2) \\
&= |0\rangle\otimes(|0\rangle-|1\rangle)/2 - |1\rangle\otimes(|0\rangle-|1\rangle)/2 \\
&= (|0\rangle-|1\rangle)\otimes(|0\rangle-|1\rangle)/2 \\
&= |-\rangle\otimes|-\rangle
\end{aligned}
$$

Finally, applying the Hadamard gate to each qubit gives:

$H(|-\rangle) = |1\rangle$
$H(|0\rangle) = (|0\rangle+|1\rangle)/\mathrm{sqrt}(2)$

So the final state of the system is $|01\rangle$.

Answer 3

The Control Z gate, also known as the Controlled-Z or CZ gate, is a two-qubit gate that applies a phase shift of π (or equivalently, a sign flip) to the target qubit if and only if the control qubit is in the state $|1\rangle$. In this case, the control qubit is the first qubit, and the target qubit is the second qubit.

The initial state of the system is given as $|\psi\rangle = |01\rangle$, which means the first qubit is in the state $|0\rangle$ and the second qubit is in the state $|1\rangle$.

The graphical representation of the circuit with the Control Z gate acting on the system would look as follows:

```
|0⟩ --[Control Z]--|0⟩
        |
        V
|1⟩ --[Target Qubit]--|1⟩
```

Here, the vertical line represents the two qubits. The top line corresponds to the first qubit (control qubit), and the bottom line corresponds to the second qubit (target qubit). The Control Z gate is represented as a box with a vertical line connecting the control and target qubits.

To describe the action of the gate on the system, we need to consider the basis states. The basis states for a two-qubit system are $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, where the first qubit represents the state of the control qubit, and the second qubit represents the state of the target qubit.

Applying the Control Z gate on the initial state $|01\rangle$, we get:

```
|0⟩ --[Control Z]--|0⟩
|
V
|1⟩ --[Target Qubit]--|1⟩
```

Since the control qubit is in the state $|0\rangle$, the target qubit remains unchanged. Therefore, the final state of the system is still $|01\rangle$.

In summary, the Control Z gate leaves the system unchanged in this case because the control qubit is in the state $|0\rangle$.