VERIFIED ERROR BOUNDS FOR ALL EIGENVALUES AND EIGENVECTORS OF A MATRIX

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Abstract. A verification method is presented to compute error bounds for all eigenvectors and eigenvalues including clustered and/or multiple ones of a general, real or complex matrix. In case of a narrow cluster error bounds for an invariant subspace are computed because computation of a single eigenvector may be ill-posed.

Computer algebra and verification methods have in common that the computed results are correct with mathematical certainty. Unlike a computer algebra method, a verification method may fail in the sense that only partial or no inclusions at all are computed. That may happen for very ill-conditioned problems being too sensitive for the arithmetical precision in use. That cannot happen for computer algebra methods which are "never-failing" because potentially infinite precision is used. In turn, however, that may slow down computer algebra methods significantly and may impose limitations on the problem size. In contrast, verification methods solely use floating-point operations so that their computing time and treatable problem size is of the order of that of purely numerical algorithms.

For our problem it is proved that the union of the eigenvalue bounds contains the whole spectrum of the matrix, and bounds for corresponding invariant subspaces are computed. The computational complexity to compute inclusions of all eigenpairs of an $n \times n$-matrix is $\mathcal{O}(n^3)$.

Key words. Verification method, eigenvalue clusters, eigenvector, invariant subspace, all eigen-pairs, INTLAB

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1. Introduction. This note presents a verification method to compute error bounds for all eigenvectors and eigenvalues of a general, real or complex matrix. In case of clustered and/or multiple eigenvalues an inclusion of a basis of the corresponding invariant subspace is computed. This means that the computed bounds do contain the true result with mathematical certainty. In particular, it is proved that the computed bounds cover all eigenpairs of the matrix.

Verification methods [17, 22, 18] are completely rigorous, including all procedural and/or rounding errors, however, they are restricted to well-posed problems. For example, it is possible to prove that a small circle in the complex plane contains two eigenvalues of a matrix, but verification methods do not allow to prove that it contains a double eigenvalue because that problem is ill-posed: an arbitrarily small perturbation of the input matrix may have two simple eigenvalues. Similarly, it is not possible to prove reasonably sharp bounds for an eigenvector of an eigenvalue of algebraic multiplicity 2, regardless of the geometric multiplicity. That is even true for a symmetric input matrix where, in contrast, very sharp bounds of eigenvalues follow directly by perturbation bounds.

Verification methods should not be confused with computer algebra methods. The latter provide never-failing algorithms. This is true independent of the condition number because computer algebra methods potentially work in infinite precision. Both approaches have in common that the computed results are always mathematically correct. Verification methods intentionally use floating-point arithmetic for speed with the drawback that failures occur, meaning, that no inclusion can be computed. In a good verification method that happens only if the problem is too hard, i.e., ill-conditioned for the floating-point accuracy available.

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The first verification method for the algebraic eigenproblem is Krawczyk's paper [12] which applies his method for nonlinear problems [12] to $Ax - \lambda x = 0$ with some normalization of x. Krawczyk's method, however, is a refinement of given bounds. Moore proposed to use Brouwer's fixed point theorem [16] and proof of nonsingularity of some matrix to derive an existence test. Krawczyk's operator and Moore's ansatz are already contained in [10, p. 12f].

In [19] this method was improved in three ways: An interval iteration with so-called epsilon-inflation computes an inclusion if the problem is not too ill-conditioned, the proof of nonsingularity is omitted by requiring a self-mapping into the interior in Brouwer's fixed point theorem, and an inclusion for the error with respect to an approximation is computed rather than an inclusion of the solution itself. Those three techniques are today's standard for verification methods from the solution of linear systems to partial differential equations. For an overview see [22, 18].

Based on that, in [19] a verification method for one eigenvalue/eigenvector pair of a real or complex $n \times n$ matrix is introduced with complexity $\mathcal{O}(n^3)$. One might apply that method n times, but besides the complexity $\mathcal{O}(n^4)$ that fails for multiple eigenvalues and cannot guarantee that all eigenvalues are covered. Several publications concentrate on verified error bounds on one eigenpair, for example [28, 4, 5, 9, 26, 25]; in [2] a method is introduced for double eigenvalues.

Historically, the next step are verification methods for multiple eigenvalues and corresponding invariant subspaces introduced in [21]. Bounds are computed regardless of the Jordan structure, but for only one cluster. Verification methods for all eigenpairs of a symmetric positive definite matrix are given in [15].

In this note we are interested in bounds for all eigenpairs of a general real or complex matrix. The first and seemingly only paper to that is Miyajima's [14], which is based on nonlinear matrix equations derived from the eigenproblem with special emphasis on multiple eigenvalues including the defective case. The nonlinear system and Newton correction are similar to [21]. His paper gives two methods. The first method uses numerical spectral decomposition and computes eigenvalue inclusions based on Gershgorin circles of the preconditioned matrix. Then $A - \lambda I$ is singular for λ in an eigenvalue inclusion, so that an eigenvector inclusion follows by solving a linear system with one row and column of $A - \lambda I$ deleted. The linear system is solved using a transformation of variables. For clusters, a basis of an invariant subspace is enclosed by solving the nonlinear matrix equation. The second method uses numerical block diagonalization analogous to [3], in which a numerical Jordan decomposition of A is included. The decomposition is known to be ill-posed, occasionally leading to computational problems. The method encloses all eigenvalues and eigenvectors (bases of invariant subspaces in the cluster case) by solving the nonlinear matrix equation. Miyajima's methods are also suitable for the generalized algebraic eigenproblem. That is also true for our method, but for simplicity we refrain from presenting that.

The approach presented in this note is based on [21]. As for Miyajima's approach, inclusions for some or all eigenspaces may be computed. In contrast, we do not rely on a numerical Jordan decomposition. In the next section the method will be presented, and in the final section on computational results we show the new method to be faster and more stable than Miyajima's.

2. Main result. Denote by $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ the field of real or complex numbers, and by $\mathbb{K}^n$ and $\mathbb{K}^{n,k}$ the set of n-vectors and $n \times k$ matrices over $\mathbb{K}$, respectively. We also use the short notation $M_{n,k}$ to denote a (real or complex) $n \times k$ matrix, and M_n if $k = n$.

Denote by $\mathbb{I}$ a set of real or complex intervals, then similarly we write $\mathbb{I}^n$ for interval n-vectors and $\mathbb{I}^{n,k}$ for $n \times k$ interval matrices. We will use boldface letters for interval quantities, and denote by $int(\cdot)$ the topological interior. The $n \times n$ identity matrix is denoted by I_n, where the subindex is omitted if clear from the context, and its j-th column by e_j. Entrywise matrix multiplication (the Hadamard product) is denoted by $\circ$.

The principles of verification methods and interval arithmetic can be found in [17, 22, 18]. However, we only use that interval operations $op \in \{+, -, \cdot, /\}$ are defined such that for compatible interval quantities $\mathbf{A}, \mathbf{B}$ the inclusion property

(2.1) $$\forall A \in \mathbf{A} \; \forall B \in \mathbf{B}: \qquad A \; op \; B \in \mathbf{A} \; op \; \mathbf{B}$$

is satisfied. Intervals may be represented by infimum-supremum or midpoint-radius, where the former is often used for real and the latter for complex intervals. The representation may influence the quality of the bounds, however, the only important property for the mathematical correctness is (2.1).

Concerning notation, we write $\mathrm{mig}(\mathbf{A}) := \min\{|a|: a \in \mathbf{A}\}$ and $\mathrm{mag}(\mathbf{A}) := \max\{|a|: a \in \mathbf{A}\}$ for a scalar interval $\mathbf{A}$, and the definition extends to interval vectors and matrices entrywise. Properties like, for example, being non-singular extend to an interval matrix $\mathbf{A}$ by requesting that all $A \in \mathbf{A}$ are non-singular.

We will use Matlab notation [13] and INTLAB [20], the Matlab-Octave toolbox for reliable computing. In INTLAB, real interval quantities are represented by infimum-supremum, complex interval quantities by midpoint-radius, but the following applies, *mutatis mutandis*, to other representations as well.

In [21, Theorem 3.2] we proved the following.

THEOREM 2.1. *Let* $A, R \in M_n$, $\widetilde{X} \in M_{n,k}$, $\tilde{\lambda} \in \mathbb{K}$ *and* $\mathbf{X} \in \mathbb{I}^{n,k}$ *be given. Define* $[n] := \{1, \ldots, n\}$, *and for non-empty* $\mu \subseteq [n]$ *with* $k := |\mu|$ *denote a partition of the* $n \times n$ *identity matrix* I *by*

(2.2) $$V := I(:, \mu) \in M_{n,k} \quad and \quad U = I(:, [n]\backslash\mu) \in M_{n,n-k}.$$

It follows that $UU^T + VV^T = I$. *Define*

(2.3) $$f(\mathbf{X}) := -R(A\widetilde{X} - \tilde{\lambda}\widetilde{X}) + \{I - R\big((A - \tilde{\lambda}I)UU^T - (\widetilde{X} + UU^T \cdot \mathbf{X})V^T\big)\} \cdot \mathbf{X}.$$

Suppose

(2.4) $$f(\mathbf{X}) \subseteq int(\mathbf{X}).$$

Then there exists $\widehat{M} \in M_k$ *with* $\widehat{M} \in \tilde{\lambda}I_k + V^T\mathbf{X}$ *such that the Jordan canonical form of* $\widehat{M}$ *is identical to a* $k \times k$ *principal submatrix of the Jordan canonical form of* A, *and there exists* $\widehat{Y} \in M_{n,k}$ *with* $\widehat{Y} \in \widetilde{X} + UU^T\mathbf{X}$ *such that* $\widehat{Y}$ *spans the corresponding invariant subspace of* A, *i.e.,* $A\widehat{Y} = \widehat{Y}\widehat{M}$.

The $k \times k$ submatrix of $\mathbf{X}$ with rows in μ is the inclusion of a Jordan block of A shifted by $\tilde{\lambda}$, where for the invariant subspace that $k \times k$ submatrix is replaced by the corresponding submatrix of $\widetilde{X}$ for normalization.

Theorem 2.1 gives a sufficient criterion for $\mathbf{X} \in \mathbb{I}^{n,k}$ to contain $\widehat{M}$ and $\widehat{Y}$ with the described properties. The better the approximations $\tilde{\lambda}, \widetilde{X}$ for the eigenspace, the preconditioner R and the choice of $\mathbf{X}$, the more likely condition (2.4) is satisfied. If (2.3) is not satisfied for the initial $\mathbf{X}$, then an interval iteration with epsilon-inflation

starting with $\mathbf{X} := f(\mathbf{X})$ is applied, see [21]. The matrix R is an approximate inverse of the Jacobian of the underlying Newton iteration for $\mathbf{X}$. Note that the assertions are true regardless of the quality of $\tilde{\lambda}$, $\widetilde{X}$ and R. The computational effort is $\mathcal{O}(n^3)$.

The quality of the inclusions depend on the Jordan structure of the matrix. There is a huge literature on the sensitivity of eigendecompositions, among them [23, 24, 1, 7, 11]. As a rule of thumb the sensitivity of an eigenvalue λ is of the order $\mathbf{u}^{1/k}$ for $\mathbf{u}$ denoting the relative rounding error unit and k the size its largest Jordan block [27, Section 2.23]. As a consequence, this is the minimum width of an inclusion of λ when using a floating-point arithmetic with relative rounding error unit $\mathbf{u}$.

Let $B \in M_n$. In order to derive an $\mathcal{O}(n^3)$ method for computing inclusions of some or all eigenpairs, the purpose of this note, we compute an approximate eigende-composition [V,D]=eig(B) of B. Suppose V is nonsingular and apply Theorem 2.1 to $V^{-1}BV$, then inclusions of the spectrum and Jordan structure of B are obtained.

If B is not defective and V its exact eigenbasis, then $V^{-1}BV$ is diagonal. But even if B has multiple eigenvalues of geometric multiplicity 1, we may expect that computationally $V^{-1}BV$ is almost diagonal because of the following. For a $k \times k$ Jordan block to an eigenvalue λ, a numerical algorithm like Matlab's eig tends to compute a basis of the corresponding invariant subspace with all columns close to a multiple of the corresponding eigenvector. The sensitivity of λ and the condition number of the corresponding columns of V is about $\mathbf{u}^{1/k}$. A numerically singular eigenapproximation matrix V may occur, but to our experience only in special cases or when searching for it. For numerical evidence see the beginning of Section 3.

Therefore a reasonable approximation to the eigenvectors of $V^{-1}BV$ is the iden-tity matrix. For each eigenvalue or cluster of eigenvalues, the corresponding matrix R in Theorem 2.1 is close to diagonal. Therefore we can combine the application of Theorem 2.1 to a set of eigenvalues or clusters into one matrix to achieve a total computing time of $\mathcal{O}(n^3)$.

The matrix $V^{-1}BV$ is usually not representable in floating-point, so that we have to work with an inclusion $\mathbf{A}$ of it. In verification methods it is a standard procedure to replace A in Theorem 2.1 by an interval matrix $\mathbf{A}$ and to conclude that, due to the inclusion property (2.1), all assertions are true for all $A \in \mathbf{A}$, in particular for $V^{-1}BV$. The eigenvalues and Jordan structure of B and $V^{-1}BV$ coincide, and the eigenvector inclusions transform by V. An inclusion $\mathbf{A}$ of $V^{-1}BV$ is computed by standard verification methods [22] including the proof that V is nonsingular. A corresponding INTLAB command is A = verifylss(V,intval(A)*V). Here verifylss(A,b) is a verified inclusion of the linear system $Ax = b$, where the type cast intval(A) assures that the right hand side intval(A)*V is an inclusion of AV. For details see [20, 22].

Before we can state our main result, we need some notation. For simplicity we state our theorem for the computation of inclusions of all eigenvalues and correspond-ing invariant subspaces. If only partially successful, i.e., for some eigenvalues the verification failed, it will be clear how to state and apply the theorem only to these remaining eigenvalues. We refrain to state the result for such a partial set because the notation would be even more involved.

As before denote $[n] := \{1, \ldots, n\}$. For $1 \leq m \leq n$, let $\mu_1 \cup \ldots \cup \mu_m = [n]$ be a partition of $[n]$, i.e., $\mu_i \cap \mu_j = \emptyset$ for $i \neq j$. Then, for given $i \in [m]$ and $k := |\mu_i|$, the splitting of I into columns within and outside μ_i is denoted by $V_i := I(:, \mu_i) \in M_{n,k}$ and $U_i = I(:, [n] \backslash \mu_i) \in M_{n,n-k}$. Multiplying a matrix from the left by $V_i^T V_i$ sets all rows outside μ_i to zero, from the right the columns outside μ_i. Furthermore, $V_i^T V_j$ is

185   the zero matrix for $i \neq j$ and

186   (2.5) $U_i U_i^T + V_i V_i^T = I_n$ and $V_i V_i^T V_i = V_i$ and $U_i^T V_i = 0$ for $i \in [m]$.

187   For a matrix $C \in M_n$ we define $C_{\mathcal{D}} \in M_n$ to be the block diagonal matrix composed
188   of the μ_i-blocks of C, i.e., $C_{\mathcal{D}} := \sum_{i=1}^m V_i V_i^T C V_i V_i^T$. The subindex "$\mathcal{D}$" serves as
189   an operator on C and applies to interval matrices as well; it means to extract the
190   elements in the diagonal μ_i-blocks. Similarly we define $C_{\mathcal{O}} := C - C_{\mathcal{D}}$ to be the
191   matrix of complement indices of C. Then (2.5) and $\sum_{i=1}^m V_i V_i^T = I_n$ yield

192   (2.6) $$C_{\mathcal{O}} := \sum_{i=1}^m U_i U_i^T C V_i V_i^T.$$

193   Again, the subindex $\mathcal{O}$ serves as an operator to C and applies also to interval matrices.

194   THEOREM 2.2. *We use the notation just given. Let $A \in M_n$, let mutually distinct*
195    $\tilde{\lambda}_i \in \mathbb{K}$ *for $i \in [m]$ be given, and let $D \in M_n$ be a diagonal matrix with $D_{jj} = \tilde{\lambda}_i$ for*
196   *all $i \in [m]$ and for all $j \in \mu_i$. Then $D_{jj} \neq \tilde{\lambda}_i$ for all $j \notin \mu_i$ and*

197   (2.7) $$DV_i = \tilde{\lambda}_i V_i \quad \text{for } i \in [m].$$

198   *Furthermore, let diagonal $R_i \in M_n$ be such that*

199   (2.8) $$R_i(D - \tilde{\lambda}_i I)U_i = U_i \quad \text{and} \quad R_i V_i = -V_i \quad \text{for } i \in [m].$$

200   *Note that R_i is well defined because the $\tilde{\lambda}_i$ are mutually distinct. Let $\widetilde{R} \in M_n$ be such*
201   *that*

202   (2.9) $$\forall i \in [m]: \quad j \in \mu_i \quad \Rightarrow \quad \widetilde{R}e_j = \text{diag}(R_i).$$

203   *Let $\mathbf{X} \in \mathbb{I}^{n,n}$ be given, define $E := A - D$ and set $\mathbf{Y} := \mathbf{X}_{\mathcal{O}}\mathbf{X}_{\mathcal{D}} - E - E\mathbf{X}_{\mathcal{O}}$, where*
204   *$\mathbf{X}_{\mathcal{O}}$ and $\mathbf{X}_{\mathcal{D}}$ are the matrices extracted from $\mathbf{X}$ according to the index sets μ_i.*
205   *Then for each $i \in [m]$ the following is true. If*

206   (2.10) $$\mathbf{Z}_i := (\widetilde{R} \circ \mathbf{Y})V_i \subseteq int(\mathbf{X}V_i) \quad \text{for some } i \in [m],$$

207   *then there exists a Jordan block $\widehat{M}_i \in \tilde{\lambda}_i I + V_i^T \mathbf{Z}_i$ with corresponding invariant sub-*
208   *space $\widehat{Y}_i \in V_i + U_i U_i^T \mathbf{Z}_i$ of A, i.e., $A\widehat{Y}_i = \widehat{Y}_i \widehat{M}_i$.*
209   *Denote the set of $i \in [m]$ satisfying (2.10) by Φ, define $\mathcal{J} := \bigcup\{\mu_i : i \in \Phi\}$*
210   *and $k := \sum_{i \in \Phi}|\mu_i| = |\mathcal{J}|$. Denote by $\mathbf{Z} \in \mathbb{I}^{k,k}$ the matrix collected of rows and*
211   *columns $i \in \mathcal{J}$ of $\widetilde{R} \circ \mathbf{Y}$, and suppose $\max\{\varrho(Z) : Z \in \mathbf{Z}\} < 1$. Then for each*
212   *$i \in \Phi$ the matrix $\widehat{M}_i$ corresponds to an individual Jordan block of A. If $k = n$, then*
213   *$\bigcup\{spec(\widehat{M}_i) : i \in \Phi\} = spec(A)$, where $spec(A)$ denotes the spectrum of the matrix A.*

214   Remark 2.3. The main point is that the $\mathbf{Z}_i$ in (2.10) are computed in one matrix
215    $\mathbf{Z} := \widetilde{R} \circ \mathbf{Y}$ and $\mathbf{Z} \subseteq int(\mathbf{X})$ is checked. That requires $\mathcal{O}(n^2)$ operations, only the
216   transformation $V^{-1}BV$ at the beginning costs $\mathcal{O}(n^3)$ operations. The eigenvalue and
217   eigenvector inclusions correspond to the columns μ_i in $\mathbf{Z}$, where $|\mu_i| > 1$ for a cluster
218   or multiple eigenvalue. Inclusions for a subset of k eigenvalues or clusters are obtained
219   by computing only the corresponding columns of $\widetilde{R} \circ \mathbf{Y}$ in $\mathcal{O}(nk)$ operations.

220   Remark 2.4. The assumption $\max\{\varrho(Z) : Z \in \mathbf{Z}\} < 1$ may be certified by Perron-
221   Frobenius theory and $\varrho(\text{mag}(\mathbf{Z})) < 1$, or by $\max\{\|\mathbf{Z}\|_\infty, \|\mathbf{Z}\|_1\} < 1$. Fortunately, that
222   test needs only $\mathcal{O}(n^2)$ operations.

223 *Remark* 2.5. The first part of Theorem 2.2 assures that for $i_1, i_2 \in \Phi$ there are
224 matrices $\widehat{M}_{i_1}$ and $\widehat{M}_{i_2}$ corresponding to a Jordan block of the matrix A, respectively,
225 and the last assertion certifies that these are different Jordan blocks. We think that
226 this is always true, also without the additional condition $\max\{\varrho(Z) : Z \in \mathbf{Z}\} < 1$, but
227 could not prove it.

228 *Proof.* Let $i \in [m]$ be fixed but arbitrary. Then (2.9) implies for every $T \in M_n$

229 (2.11)
$$(\widetilde{R} \circ T)V_i = R_i T V_i.$$

230 Let $X \in \mathbf{X}$ be fixed but arbitrary, and abbreviate $X_i := XV_i$. Then (2.5) and the
231 definition of $X_{\mathcal{D}}$ and $X_{\mathcal{O}}$ imply

232 (2.12)
$$X_{\mathcal{D}} V_i = V_i V_i^T X_i \ \text{ and } \ X_{\mathcal{O}} V_i = X_i - V_i V_i^T X_i = U_i U_i^T X_i.$$

233 Moreover,

234 (2.13)
$$X_{\mathcal{O}} X_{\mathcal{D}} V_i = X_{\mathcal{O}} V_i V_i^T X_i = U_i U_i^T X_i V_i^T X_i.$$

235 Using (2.7) and $A = D + E$ give

236 (2.14)
$$R_i(A - \tilde{\lambda}_i I)V_i = R_i(D - \tilde{\lambda}_i I)V_i + R_i E V_i = R_i E V_i \ ,$$

237 so that (2.8) and (2.5) imply

238 (2.15)
$$\begin{aligned}
\{I - R_i((A - \tilde{\lambda}_i I)U_i U_i^T - (V_i + U_i U_i^T X_i)V_i^T)\}X_i &= \\
X_i - R_i((D - \tilde{\lambda}_i I)U_i U_i^T + E U_i U_i^T - V_i V_i^T - U_i U_i^T X_i V_i^T)X_i &= \\
X_i - (U_i U_i^T + R_i E U_i U_i^T + V_i V_i^T - R_i U_i U_i^T X_i V_i^T)X_i &= \\
-R_i(E U_i U_i^T - U_i U_i^T X_i V_i^T)X_i \ .
\end{aligned}$$

239 Setting $Y := X_{\mathcal{O}} X_{\mathcal{D}} - E - E X_{\mathcal{O}} \in \mathbf{Y}$ and using (2.11), (2.13), (2.12), (2.14) and
240 (2.15) proves

241
$$(\widetilde{R} \circ Y)V_i = R_i(X_{\mathcal{O}} X_{\mathcal{D}} - E - E X_{\mathcal{O}})V_i = R_i(U_i U_i^T X_i V_i^T X_i - E V_i - E U_i U_i^T X_i)$$
242
243
$$= -R_i(A - \tilde{\lambda}_i I)V_i + \{I - R_i((A - \tilde{\lambda}_i I)U_i U_i^T - (V_i + U_i U_i^T X_i)V_i^T)\}X_i.$$

244 Since this is true for every $X \in \mathbf{X}$, it follows that the assumptions of Theorem 2.1
245 with $\widetilde{X} := V_i$ and $\mathbf{X}$ replaced by $\mathbf{X}V_i$ are satisfied. This proves the first part of the
246 theorem.

247 Denote by $\widehat{Y} \in M_{n,k}$ the matrix collected of block columns $\widehat{Y}_i$ for $i \in \Phi$. For
248 $V \in M_{n,k}$ denoting the matrix of corresponding columns of the identity matrix, it
249 follows $\widehat{Y} = V + Z$ for some $Z \in \mathbf{Z}$, so that $\varrho(Z) < 1$ implies that $\widehat{Y}$ has full rank.
250 Therefore, $\widehat{Y}$ is a basis of an invariant subspace of A, and all assertions follow. $\square$

251 The interval matrix $\mathbf{Z}$ contains inclusions of the error of the approximate eigen-
252 values and -vectors. Thus, the intervals can be expected to be narrow, one of the three
253 main principles of verification methods mentioned in the introduction. Therefore it
254 is most likely that the final condition $\max\{\varrho(Z) : Z \in \mathbf{Z}\} < 1$ is satisfied.

255 Following we give some implementation details. Let a matrix $B \in M_n$ be given,
256 then Theorem 2.2 is applied to a similarity transformation of B as follows.

257
```
function [A, W] = transform(B)
   [W, X] = eig(B);
   X = X + W\prodK(B, W, −W, X);
   [Res, E] = prodK(B, W, −W, X);
   A = X + W\midrad(Res, E);
```

258 The approximate eigenvalue matrix X is improved by one Newton iteration. Here
259 `prodK` in line 2 with one output parameter computes an accurate approximation of
260 the residual $BW - WX$ using error-free transformations, and in line 3 with two output
261 parameters an inclusion with midpoint `Res` and radius `E`, wherefore `midrad(Res,E)`
262 is an inclusion of $BW - WX$.

263 As always in numerical analysis the residual should be calculated accurately. In
264 Algorithm `transform` we use error-free transformations which are accurate but costly.
265 Although using BLAS3 routines, as always in INTLAB, that takes about a third of the
266 total computing time of the inclusion Algorithm `verifyeigall` to be presented. In the
267 final line `W\midrad(Res,E)` computes an inclusion of $\{W^{-1}R \colon R \in \mathtt{midrad(Res,E)}\}$,
268 so that $\mathbf{A}$ is an inclusion of $X + W^{-1}(BW - WX) = W^{-1}BW =: A$. Hence, the
269 spectra of B and A are identical, and the invariant subspaces transform by W.

270 Note that the input matrix B in the function `transform` may be an interval
271 matrix $\mathbf{B}$ as well with assertions being true for all $B \in \mathbf{B}$. In that case use `[W,X]`
272 `= eig(B.mid)` . But not much cancellation is expected in the computation of the
273 residuals, so that `prodK(B,W,-W,X)` can be replaced by `B*W-W*X`. Multiple eigenvalues
274 with small geometric multiplicity become sensitive to perturbations, and even narrow
275 interval components of $\mathbf{B}$ may widen the computed inclusions of eigenvalues and
276 eigenvectors significantly.

277 We then apply Theorem 2.2 to the interval matrix $\mathbf{A}$. By the inclusion principle
278 (2.1) it follows that the assertions are true for all matrices within $\mathbf{A}$, in particular
279 for $A = W^{-1}BW$. Hence (2.10) for some $i \in [m]$ implies that $B\widehat{Y}_i = \widehat{Y}_i\widehat{M}_i$ for
280 $\widehat{M}_i \in \tilde{\lambda}_i I + V_i^T\mathbf{Z}_i$ and $\widehat{Y}_i \in (V_i + U_iU_i^T\mathbf{Z}_i)W$. By Perron-Frobenius theory, $\varrho(M) \le$
281 $\varrho(|M|)$ for every matrix M. Thus, the eigenvalues of $\widehat{M}_i$ are included in $\tilde{\lambda}_i \pm \varrho(|V_i^T\mathbf{Z}_i|)$,
282 where the spectral radius is bounded as in [21] by a few power iterations and Collatz's
283 inclusion [8].

284 Based on that our algorithm is as follows, partly using Matlab and INTLAB
285 notation.

function [L,X,mu] = verifyeigall(B)
1) Calculate A and W by Algorithm `transform`
2) $\mathrm{normA} = \mathtt{norm(A.mid, inf)}$; $\mathrm{d} = \mathtt{diag(A)}$;
   $\mathrm{dist} = (\mathtt{mig}(\mathrm{d} - \mathrm{d.'}) <= 1\mathrm{e} - 14 * \mathrm{normA})$;
   $[\mathtt{mu,binsizes}] = \mathtt{conncomp(graph(dist),'OutputForm','cell')}$;
   $\mathrm{J} = \mathtt{find(binsizes} > 1)$;
3) $\mathrm{D} = \mathrm{d.mid}$;
   $\mathrm{E} = \mathrm{A} - \mathtt{diag}(\mathrm{D})$;
4) $\mathrm{RR} = 1./(\mathrm{D} - \mathrm{D.'})$;
   $\mathrm{RR}(1:\mathrm{n}+1:\mathrm{n}^2) = -1$;
   for $j = J$, $\mathrm{RR(mu\{j\},mu\{j\})} = -1$; end
5) $\mathrm{Y} = -\mathrm{RR}.*\mathrm{E}$; $\mathrm{cols} = 0$;
6) repeat
      cols_old = cols;
      Compute an epsilon-inflation X of Y
      Compute $X_{\mathcal{D}}$ and $X_{\mathcal{O}}$ according to Theorem 2.2
      cols = # of columns satisfying (2.10)
      If cols=n then `[L,X] = final(D,Y,W)`, return
      If cols < cols_old, then apply verifyeigall recursively

286

287 In step 1) the input matrix B is transformed as described before. Step 2) computes

288   connected components of the graph of the matrix of distances of diagonal elements
289   of A which is a guess of the Jordan structure. In step 3) the interval matrix $\mathbf{A}$ is
290   splitted into $D + \mathbf{E}$ with $D \in M_n$ and $\mathbf{E} \in \mathbb{I}^{n,n}$. Thus for $A \in \mathbf{A}$ there exists $E \in \mathbf{E}$
291   with $A = D + E$. Note that the diagonal elements of $\mathbf{E}$ are, in general, nonzero.

292   Step 4) computes the matrix $\widetilde{R}$ as in Theorem 2.2. The main loop is in step 6).
293   The loop stops with success if (2.10) is satisfied for all columns, where Algorithm
294   `final` computes bounds for the eigenvalues of $\widehat{M_i}$ and transforms the invariant sub-
295   spaces using W. If the number of successful columns does not increase, the function
296   `verifyeigall` is applied recursively to the columns with no inclusion. The algorithm
297   terminates if that recursion does not increase the number of successful columns.

298   The output `mu` identifies the clusters, i.e., L_j for $j \in \mu_i$ and $X(:,\mu_i)$ form an
299   inclusion of an eigenvalue and corresponding invariant subspace. If the union of the
300   indices in `mu` is equal to $\{1,\ldots,n\}$, then inclusions for all eigenvalues and invariant
301   subspaces have been computed.

302   **3. Numerical results.** All computational results are produced using Matlab
303   and INTLAB and double precision (binary64) with a relative rounding error unit
304   $\mathbf{u} = 2^{-53} \approx 10^{-16}$ on a standard laptop. The relative error of an interval is the
305   maximum relative distance between two members of the interval, so that a value of
306   order 10^{-16} means that an inclusion is almost maximally accurate.

307   We start with some general remarks on the construction of test examples. Con-
308   sider the set $\mathcal{D}_n \subseteq M_n$ of matrices with double eigenvalue, which is of measure zero
309   within M_n. The set of diagonalizable matrices within $\mathcal{D}_n$ is again of measure zero. As
310   a consequence, we may expect that if there are non-trivial Jordan blocks, they belong
311   to mutually different eigenvalues.

312   Let `J = diag(randn(n,1))` and replace a $k \times k$ block with all diagonal elements
313   equal to one random number λ and 1's on the superdiagonal. Let V be a nonsingular
314   matrix, then $V^{-1}JV$ has a $k \times k$ Jordan block to the eigenvalue λ.

315   When computing $V^{-1}JV$ in floating-point arithmetic, likely the resulting matrix
316   A has a cluster of eigenvalues with center not far from λ. In fact, it needs some
317   effort to construct a matrix within $\mathbb{F}_n$ with multiple eigenvalues, see Subsection 3.2.
318   The radius of the cluster is usually close to the sensitivity of the multiple eigenvalue,
319   which is $\mathbf{u}^{1/k}$. For example, this attempt to construct a matrix with 3-fold eigenvalue
320   generates a matrix with a cluster of radius 10^{-5}.

321   We therefore split this section of computational results into a first part with
322   matrices the eigenvalues of which are generated as described above, and a second
323   part using special methods to generate matrices with truly multiple eigenvalues and
324   specified Jordan blocks. In the last subsection we compare our new algorithm with
325   Miyajima's methods in [14].

326   **3.1. Numerical results for eigenvalue clusters.** Suppose the input matrix B
327   has multiple eigenvalues. Then `[W,D] = eig(A)` produces W with almost linearly de-
328   pendent columns for each Jordan block, and one may expect W to be ill-conditioned.
329   However, those columns are only linear dependent up to the sensitivity of the clusters:
330   a $k \times k$ Jordan block produces eigenvector approximations becoming linearly depen-
331   dent for a perturbation of order $\mathbf{u}^{1/k}$. In that sense floating-point arithmetic has a
332   regularizing effect, the condition number of the eigenvector approximation matrix is
333   of the order $\mathbf{u}^{-1/k}$.

334   We first compare the accuracy of the eigenvalue inclusions by `verifyeigall` with
335   those of Gershgorin circles. The latter provide verified inclusions of the eigenvalues,
336   but not of eigenvectors. In that respect the comparison is not fair. However, as an

337 advantage, the Gershgorin approach cannot fail.

338   The computational results displayed in Table 1 are as follows. For different dimen-
339 sions n, we generate a random matrix being diagonal except one Jordan block of size
340 k (which is just a random matrix for $k = 1$), and perform a similarity transformation
341 by some random matrix. That is one test matrix B. Then inclusions of all eigen-
342 values are first computed by Gershgorin circles applied to `transform(B)`, and second
343 by `verifyeigall(B)`. The results displayed in Table 1 are the mean of all means,
344 the median of all medians and the maximum of all maxima of the relative errors of
345 the inclusions, calculated over 100 samples. In this test set `verifyeigall` could not
346 compute inclusions of all eigenvalues in 1 out of the 100 test cases for $n = 1000$ and
347 $k = 3$.

		relerr Gershgorin			relerr new		
n	k	mean	median	max	mean	median	max
10	1	$7.4 \cdot 10^{-13}$	$1.8 \cdot 10^{-14}$	$9.6 \cdot 10^{-11}$	$2.3 \cdot 10^{-16}$	$2.3 \cdot 10^{-16}$	$3.3 \cdot 10^{-16}$
	2	$7.5 \cdot 10^{-7}$	$3.0 \cdot 10^{-14}$	$1.9 \cdot 10^{-4}$	$6.4 \cdot 10^{-8}$	$3.0 \cdot 10^{-16}$	$7.1 \cdot 10^{-6}$
	3	$3.9 \cdot 10^{-5}$	$3.9 \cdot 10^{-14}$	$1.0 \cdot 10^{-3}$	$1.5 \cdot 10^{-5}$	$3.6 \cdot 10^{-16}$	$1.1 \cdot 10^{-3}$
100	1	$2.5 \cdot 10^{-10}$	$1.9 \cdot 10^{-12}$	$5.2 \cdot 10^{-7}$	$2.1 \cdot 10^{-16}$	$2.1 \cdot 10^{-16}$	$3.3 \cdot 10^{-16}$
	2	$8.6 \cdot 10^{-7}$	$2.2 \cdot 10^{-12}$	$1.5 \cdot 10^{-3}$	$1.5 \cdot 10^{-8}$	$3.0 \cdot 10^{-16}$	$1.8 \cdot 10^{-5}$
	3	$2.9 \cdot 10^{-4}$	$2.7 \cdot 10^{-12}$	$2.3 \cdot 10^{-1}$	$6.5 \cdot 10^{-6}$	$3.2 \cdot 10^{-16}$	$2.4 \cdot 10^{-3}$
1000	1	$2.0 \cdot 10^{-7}$	$7.3 \cdot 10^{-11}$	$4.3 \cdot 10^{-3}$	$1.9 \cdot 10^{-16}$	$1.9 \cdot 10^{-16}$	$3.3 \cdot 10^{-16}$
	2	$3.3 \cdot 10^{-6}$	$7.3 \cdot 10^{-11}$	$8.8 \cdot 10^{-2}$	$8.3 \cdot 10^{-9}$	$3.1 \cdot 10^{-16}$	$2.0 \cdot 10^{-4}$
	3	$2.4 \cdot 10^{-4}$	$7.9 \cdot 10^{-11}$	$1.3 \cdot 10^{0}$	$1.2 \cdot 10^{-6}$	$3.1 \cdot 10^{-16}$	$5.5 \cdot 10^{-3}$

TABLE 1
Eigenvalue bounds by Gershgorin circles and the new method `verifyeigall`

348   As can be seen, the accuracy of eigenvalue inclusions of both methods decrease
349 with dimension and size of cluster, and the new method is generally more accurate
350 than inclusions by Gershgorin circles. The medians of the medians is better than the
351 mean because the size of the cluster is small compared to the dimension. For $n = 1000$
352 and $k = 3$ some inclusions by Gershgorin circles are very wide.

353   Next we test the performance of `verifyeigall`, first for real and complex clusters
354 of size k, respectively. Again, for $k = 1$ this is just a random matrix. For one test
355 matrix we compute the mean and the median of the relative error of all eigenvalue
356 and of all eigenvector inclusions. Then, for different dimensions and 100 test cases
357 each, the median of those numbers is displayed in columns 3 to 6 in Tables 2 and 3.
358 The number of test cases where inclusions could not be computed for all eigenpairs is
359 listed in column 'fail'.

360   The mean of the ratio of computing times between `verifyeigall` and Matlab's
361 `eig` is displayed in the last column. That compares apples and oranges because
362 `verifyeigall` computes verified inclusions of all results, whereas Matlab's `eig` cal-
363 culates approximations without error bound. For random matrices the error bounds
364 computed by `verifyeigall` are pretty accurate; for a cluster of size 3 the mean
365 relative error of the approximations by Matlab's `eig` against a multiple precision cal-
366 culation is about 10^{-8} and the maximum about 10^{-6}. In any case, the comparison
367 gives an impression on the necessary effort for verified inclusions.

368   The accuracy of the inclusions corresponds to the sensitivity of the cluster $\mathbf{u}^{-1/k}$.
369 That is true for the eigenvalues and for the invariant subspaces. The ratio of comput-
370 ing time increases with the cluster size because likely a recursive call of `verifyeigall`

n	k	relerr L		relerr X		fail	$t_{\mathrm{new}}/t_{\mathrm{eig}}$
100	1	$3.2 \cdot 10^{-16}$	$3.2 \cdot 10^{-16}$	$1.2 \cdot 10^{-15}$	$1.1 \cdot 10^{-15}$	-	6.8
200	1	$3.2 \cdot 10^{-16}$	$3.2 \cdot 10^{-16}$	$1.7 \cdot 10^{-15}$	$1.9 \cdot 10^{-15}$	-	5.0
500	1	$3.3 \cdot 10^{-16}$	$3.3 \cdot 10^{-16}$	$2.6 \cdot 10^{-15}$	$2.6 \cdot 10^{-15}$	-	9.2
1000	1	$3.1 \cdot 10^{-16}$	$3.1 \cdot 10^{-16}$	$3.4 \cdot 10^{-15}$	$3.8 \cdot 10^{-15}$	-	10.9
100	2	$6.0 \cdot 10^{-9}$	$4.0 \cdot 10^{-14}$	$1.8 \cdot 10^{-11}$	$3.0 \cdot 10^{-12}$	-	12.1
200	2	$3.1 \cdot 10^{-9}$	$4.5 \cdot 10^{-14}$	$4.7 \cdot 10^{-11}$	$6.9 \cdot 10^{-12}$	-	8.3
500	2	$1.0 \cdot 10^{-9}$	$4.7 \cdot 10^{-14}$	$1.4 \cdot 10^{-10}$	$1.9 \cdot 10^{-11}$	-	9.2
1000	2	$7.5 \cdot 10^{-10}$	$5.1 \cdot 10^{-14}$	$3.4 \cdot 10^{-10}$	$3.9 \cdot 10^{-11}$	-	19.6

TABLE 2
Random matrix with real cluster of size k

n	k	relerr L		relerr X		fail	$t_{\mathrm{new}}/t_{\mathrm{eig}}$
100	1	$3.1 \cdot 10^{-16}$	$3.1 \cdot 10^{-16}$	$1.0 \cdot 10^{-15}$	$1.1 \cdot 10^{-15}$	-	3.7
200	1	$3.3 \cdot 10^{-16}$	$3.3 \cdot 10^{-16}$	$1.4 \cdot 10^{-15}$	$1.4 \cdot 10^{-15}$	-	3.2
500	1	$3.2 \cdot 10^{-16}$	$3.1 \cdot 10^{-16}$	$2.1 \cdot 10^{-15}$	$2.1 \cdot 10^{-15}$	-	4.5
1000	1	$3.3 \cdot 10^{-16}$	$3.3 \cdot 10^{-16}$	$3.0 \cdot 10^{-15}$	$3.7 \cdot 10^{-15}$	-	5.0
100	2	$7.4 \cdot 10^{-9}$	$4.3 \cdot 10^{-14}$	$1.7 \cdot 10^{-11}$	$3.0 \cdot 10^{-12}$	-	17.1
200	2	$5.1 \cdot 10^{-9}$	$4.5 \cdot 10^{-14}$	$4.4 \cdot 10^{-11}$	$6.7 \cdot 10^{-12}$	-	11.7
500	2	$3.6 \cdot 10^{-9}$	$4.9 \cdot 10^{-14}$	$1.4 \cdot 10^{-10}$	$1.9 \cdot 10^{-11}$	-	18.9
1000	2	$1.9 \cdot 10^{-9}$	$5.2 \cdot 10^{-14}$	$3.3 \cdot 10^{-10}$	$4.0 \cdot 10^{-11}$	-	20.6

TABLE 3
Random matrix with complex cluster of size k

is necessary. The ratio is a little better for complex clusters, seemingly because `eig` slows down. There is no failure, i.e., verified inclusions have been computed for all eigenpairs in all test cases.

n	k	relerr L		relerr X		fail	$t_{\mathrm{new}}/t_{\mathrm{eig}}$
100	3	$1.4 \cdot 10^{-6}$	$4.1 \cdot 10^{-14}$	$1.8 \cdot 10^{-11}$	$3.0 \cdot 10^{-12}$	-	19.7
200	3	$7.2 \cdot 10^{-7}$	$4.4 \cdot 10^{-14}$	$4.7 \cdot 10^{-11}$	$6.7 \cdot 10^{-12}$	-	12.6
500	3	$3.8 \cdot 10^{-7}$	$4.7 \cdot 10^{-14}$	$1.5 \cdot 10^{-10}$	$1.9 \cdot 10^{-11}$	-	22.1
1000	3	$2.9 \cdot 10^{-7}$	$5.2 \cdot 10^{-14}$	$3.7 \cdot 10^{-10}$	$4.1 \cdot 10^{-11}$	2	26.0
100	5	$1.9 \cdot 10^{-4}$	$4.2 \cdot 10^{-14}$	$5.8 \cdot 10^{-11}$	$3.2 \cdot 10^{-12}$	-	23.4
200	5	$1.1 \cdot 10^{-4}$	$4.8 \cdot 10^{-14}$	$9.0 \cdot 10^{-11}$	$6.9 \cdot 10^{-12}$	4	15.5
500	5	$3.3 \cdot 10^{-5}$	$4.9 \cdot 10^{-14}$	$9.1 \cdot 10^{-10}$	$1.9 \cdot 10^{-11}$	13	26.8
1000	5	$2.1 \cdot 10^{-5}$	$5.4 \cdot 10^{-14}$	$2.3 \cdot 10^{-9}$	$4.3 \cdot 10^{-11}$	34	33.3
100	10	$9.0 \cdot 10^{-4}$	$4.3 \cdot 10^{-14}$	$2.9 \cdot 10^{-7}$	$3.2 \cdot 10^{-12}$	66	45.0
200	10	$6.1 \cdot 10^{-5}$	$6.2 \cdot 10^{-14}$	$7.8 \cdot 10^{-6}$	$9.1 \cdot 10^{-12}$	72	28.8
500	10	$9.5 \cdot 10^{-7}$	$8.2 \cdot 10^{-14}$	$7.0 \cdot 10^{-6}$	$3.4 \cdot 10^{-11}$	78	42.3
1000	10	$3.2 \cdot 10^{-7}$	$1.0 \cdot 10^{-13}$	$9.3 \cdot 10^{-6}$	$8.1 \cdot 10^{-11}$	65	41.9

TABLE 4
Random matrix with real cluster of size k

Results for clusters of size up to 10 are displayed in Table 4. Now we observe failures. That means, that not for all eigenpairs inclusions could be computed, usually

376 for those close to the cluster. Again the relative accuracy corresponds to the sensitivity
377 of the clusters.
378   For separated eigenvalues usually one repeat-loop in step 6) of `verifyeigall`
379 suffices. For clustered eigenvalues it may be executed a number of times in order
380 to identify and separate the clusters. Therefore, for larger clusters we observe a
381 significant increase of computing time relative to Matlab's `eig`. That is mainly due
382 to Matlab's interpretation overhead and in particular the use of the operator concept.
383 That can be improved significantly by using function calls and/or calculating left and
384 right bounds individually using directed rounding as Florian Bünger did for the Taylor
385 model and AWA toolbox in INTLAB [6]. We refrained from doing this for the sake
of better readability of the code.

n	k	relerr L		relerr X		fail	$t_{\text{new}}/t_{\text{eig}}$
100	2	$9.4 \cdot 10^{-9}$	$4.1 \cdot 10^{-14}$	$2.4 \cdot 10^{-11}$	$3.0 \cdot 10^{-12}$	-	18.9
200	2	$6.1 \cdot 10^{-9}$	$4.5 \cdot 10^{-14}$	$7.1 \cdot 10^{-11}$	$7.0 \cdot 10^{-12}$	-	12.2
500	2	$2.1 \cdot 10^{-9}$	$4.8 \cdot 10^{-14}$	$1.8 \cdot 10^{-10}$	$1.9 \cdot 10^{-11}$	-	20.2
1000	2	$2.0 \cdot 10^{-9}$	$5.2 \cdot 10^{-14}$	$4.0 \cdot 10^{-10}$	$4.1 \cdot 10^{-11}$	-	22.1
100	5	$5.2 \cdot 10^{-4}$	$4.9 \cdot 10^{-14}$	$5.0 \cdot 10^{-7}$	$3.3 \cdot 10^{-12}$	5	29.1
200	5	$3.4 \cdot 10^{-4}$	$5.1 \cdot 10^{-14}$	$5.7 \cdot 10^{-7}$	$7.7 \cdot 10^{-12}$	12	18.3
500	5	$7.1 \cdot 10^{-5}$	$5.2 \cdot 10^{-14}$	$2.5 \cdot 10^{-7}$	$2.1 \cdot 10^{-11}$	34	30.5
1000	5	$5.6 \cdot 10^{-5}$	$8.9 \cdot 10^{-14}$	$3.9 \cdot 10^{-6}$	$7.1 \cdot 10^{-11}$	65	37.7

TABLE 5
Random matrix with two real clusters of size k to different eigenvalues

386
387 In Table 5 results for two clusters of size k to different eigenvalues are reported. Again,
388 for clusters of size 5 sometimes inclusions could not be computed for all eigenpairs,
389 otherwise the results correspond to the previous ones.

n	k	relerr L		relerr X		fail	$t_{\text{new}}/t_{\text{eig}}$
100	1	$5.2 \cdot 10^{-16}$	$3.1 \cdot 10^{-16}$	$1.5 \cdot 10^{-16}$	$1.4 \cdot 10^{-16}$	-	5.2
200	1	$1.1 \cdot 10^{-15}$	$3.1 \cdot 10^{-16}$	$2.8 \cdot 10^{-16}$	$2.6 \cdot 10^{-16}$	-	5.1
500	1	$1.5 \cdot 10^{-13}$	$4.2 \cdot 10^{-14}$	$1.2 \cdot 10^{-10}$	$1.7 \cdot 10^{-11}$	-	7.4
1000	1	$1.9 \cdot 10^{-13}$	$4.8 \cdot 10^{-14}$	$3.1 \cdot 10^{-10}$	$3.8 \cdot 10^{-11}$	-	7.6
100	2	$1.4 \cdot 10^{-8}$	$4.3 \cdot 10^{-14}$	$1.8 \cdot 10^{-11}$	$3.0 \cdot 10^{-12}$	2	19.2
200	2	$9.6 \cdot 10^{-9}$	$4.5 \cdot 10^{-14}$	$4.7 \cdot 10^{-11}$	$6.8 \cdot 10^{-12}$	4	13.5
500	2	$4.0 \cdot 10^{-9}$	$4.8 \cdot 10^{-14}$	$1.5 \cdot 10^{-10}$	$1.9 \cdot 10^{-11}$	1	19.8
1000	2	$3.5 \cdot 10^{-9}$	$5.2 \cdot 10^{-14}$	$3.9 \cdot 10^{-10}$	$4.1 \cdot 10^{-11}$	-	24.8

TABLE 6
Random matrix with two real clusters of size k to the same eigenvalue

390   Finally we generate matrices with two clusters of size k to the same eigenvalue.
391 The sensitivity is about $\mathbf{u}^{-1/k}$ for k denoting the largest Jordan block. So again the
392 accuracy corresponds to the sensitivity, for cluster size 2 not always inclusions for all
393 eigenpairs could be computed, and the computing time increases due to a recursive
394 call of `verifyeigall`. The results are shown in Table 6.

395   **3.2. Numerical results for truly multiple eigenvalues.** Next we perform
396 similar tests but knowing the true Jordan structure of the test matrices. The con-
397 struction of the test matrices is as follows. First a diagonal matrix is generated with

small integer entries divided by a small power of 2, and with a $k \times k$ block with equal
diagonal entries and superdiagonal set to 1. Call that matrix J. Next sparse lower
and upper unit triangular matrices L and U with integer entries are generated, so
that their inverses have integer entries. Finally it is tested that $A = U^{-1}L^{-1}JLU$ is
computed without rounding errors.

The results displayed in Table 7 are structured as those in the previous section.
Here k is the size of the Jordan block, where 1/1 refers to a double eigenvalue of
(algebraic and) geometric multiplicity 2.

n	k	relerrL		relerrX		fail	$t_{\text{new}}/t_{\text{eig}}$
100	1/1	$3.2 \cdot 10^{-16}$	$3.0 \cdot 10^{-16}$	$2.7 \cdot 10^{-9}$	$4.4 \cdot 10^{-10}$	-	15.2
200	1/1	$1.3 \cdot 10^{-15}$	$3.2 \cdot 10^{-16}$	$5.9 \cdot 10^{-9}$	$1.1 \cdot 10^{-9}$	-	13.9
500	1/1	$5.0 \cdot 10^{-15}$	$3.6 \cdot 10^{-16}$	$1.2 \cdot 10^{-8}$	$4.2 \cdot 10^{-9}$	1	20.2
1000	1/1	$1.9 \cdot 10^{-14}$	$4.1 \cdot 10^{-16}$	$3.6 \cdot 10^{-8}$	$1.4 \cdot 10^{-8}$	3	20.7
100	2	$1.2 \cdot 10^{-11}$	$1.1 \cdot 10^{-12}$	$9.2 \cdot 10^{-2}$	$7.7 \cdot 10^{-3}$	-	33.2
200	2	$2.9 \cdot 10^{-11}$	$2.2 \cdot 10^{-12}$	$9.3 \cdot 10^{-2}$	$3.8 \cdot 10^{-3}$	-	31.0
500	2	$5.5 \cdot 10^{-11}$	$5.5 \cdot 10^{-12}$	$9.8 \cdot 10^{-2}$	$8.4 \cdot 10^{-9}$	-	49.3
1000	2	$8.4 \cdot 10^{-11}$	$9.1 \cdot 10^{-12}$	$1.0 \cdot 10^{-1}$	$2.8 \cdot 10^{-9}$	2	48.5
100	3	$1.7 \cdot 10^{-11}$	$1.6 \cdot 10^{-12}$	$9.6 \cdot 10^{-2}$	$1.8 \cdot 10^{-3}$	-	41.0
200	3	$2.2 \cdot 10^{-11}$	$2.7 \cdot 10^{-12}$	$9.6 \cdot 10^{-2}$	$8.6 \cdot 10^{-4}$	1	33.4
500	3	$5.0 \cdot 10^{-11}$	$5.4 \cdot 10^{-12}$	$9.9 \cdot 10^{-2}$	$3.3 \cdot 10^{-4}$	-	50.1
1000	3	$8.2 \cdot 10^{-11}$	$9.4 \cdot 10^{-12}$	$1.0 \cdot 10^{-1}$	$2.7 \cdot 10^{-9}$	4	49.0
100	5	$2.4 \cdot 10^{-11}$	$1.9 \cdot 10^{-12}$	$9.3 \cdot 10^{-2}$	$1.8 \cdot 10^{-3}$	11	42.2
200	5	$2.0 \cdot 10^{-11}$	$2.6 \cdot 10^{-12}$	$9.7 \cdot 10^{-2}$	$3.2 \cdot 10^{-3}$	22	26.9
500	5	$4.2 \cdot 10^{-11}$	$5.1 \cdot 10^{-12}$	$1.0 \cdot 10^{-1}$	$2.7 \cdot 10^{-3}$	29	51.1
1000	5	$8.2 \cdot 10^{-11}$	$7.5 \cdot 10^{-12}$	$1.0 \cdot 10^{-1}$	$6.4 \cdot 10^{-4}$	31	49.9

TABLE 7
Random matrix with true Jordan blocks of size k

For larger block size, the eigenvalue inclusions are more accurate, where the eigen-
vector inclusions are less accurate than expected. As before the number of cases where
not all eigenpairs are included increases with the the block size and dimension, and
the computing time as well. For larger block size and larger dimension the relative
accuracy of some eigenvector and/or invariant subspace inclusions is poor, sometimes
only one digit can be verified.

3.3. Comparison to Miyajima's methods. Finally we compare our algo-
rithm `verifyeigall` with the, to our knowledge, only competitor published in [14].
Miyajima presents two algorithms `VAE_NSD` and `VAE_NJD`. The source code of both
algorithms was kindly provided by the author. He also uses error-free transforma-
tions to improve the accuracy of the inclusions, and he uses NAClab [29], a publically
available Matlab toolbox which, in particular, offers algorithms to compute the Jor-
dan canonical form of a matrix. That is an ill-posed problem, thus the true Jordan
structure may not be determined correctly in floating-point arithmetic.

We mention that there are also block versions of the algorithms in [14], however,
the results shown in [14] are similar to the unblocked version so we refrain from
comparisons to save space.

We encountered hard Matlab errors or infinite loops when testing `VAE_NSD` and

424 VAE_NJD, and were advised by Miyajima to use the older 2013 version of NAClab
425 rather than the newer 2018 version. That improved the situation, but still infinite
426 loops occur and sometimes the routines stopped with a Matlab error. Moreover, the
routines VAE_NSD and VAE_NJD are time consuming. We therefore had to reduce the

dimension n	50	100	200	500	1000
# test cases	100	100	50	15	10

TABLE 8
Number of test cases for Tables 9, 10 and 11

427
428 number of test cases according to Table 8, and for larger sizes of clusters the dimension
429 had to be reduced.
430 We generate several test matrices, both with eigenvalue clusters and true Jordan
431 blocks as described in Subsections 3.1 and 3.2, respectively. For one test matrix
432 we compute the mean of the relative errors of all eigenvalue and of all eigenvector
433 inclusions, and for different dimensions the median of those numbers is displayed in the
434 blocks "relerr L" for the eigenvalues and in the blocks "relerr X" for the eigenvectors
435 or invariant subspaces. As before, the case $k = 1$ is added in which case the test
436 matrix is just a random matrix.
437 The number of test cases where no inclusion could be computed for one eigenpair
438 is listed in the column "failure". More precisely, if, as before, only for one eigenpair
439 no inclusion could be computed it is considered as failure. If an algorithm fails for all
440 test cases, the relative errors for L and X are noted as NaN.
441 For some test cases three numbers are displayed for VAE_NSD and VAE_NJD in the
442 block "failure". In that case the first number is the total number of failures as just
443 described, the second the number of cases where the algorithm runs into an infinite
444 loop, and the third where Matlab stops with an error. For example, for $n = 200$ and
445 $k = 5$, VAE_NSD failed in 17 of 50 cases, no infinite loop was encountered and 1 Matlab
446 error. Similarly, again for $n = 200$ and $k = 5$, VAE_NJD failed in 12 out of 50 test
447 cases, in 11 cases the algorithm ran into an infinite loop, and 1 test case ended with
448 a Matlab error.
449 Finally, in the last two columns, the mean time ratio of VAE_NSD and VAE_NJD to
450 our new routine is displayed. For example, for $n = 500$ and $k = 3$, VAE_NSD required
451 on the average 10.2 times the computing time of verifyeigall, and VAE_NJD was on
452 the average 29.3 times slower.
453 The results are shown in Table 9, where the numbers above the horizontal line
454 refer to clusters of eigenvalues as in Subsection 3.1, and below to true Jordan blocks
455 as in Subsection 3.2. For increasing dimension and size of clusters or Jordan blocks,
456 the number of failures of all algorithms increase, more moderately for verifyeigall.
457 Algorithm VAE_NSD failed for cluster size 10 completely, where VAE_NJD failed for all
458 true Jordan blocks with no inclusion, infinite loop and/or Matlab error. If successful,
459 the quality of the eigenvalue inclusions of VAE_NJD and verifyeigall are comparable,
460 those of VAE_NSD are a little weaker. The quality of the eigenvector inclusions of
461 both VAE_NSD and VAE_NJD are weaker than those of verifyeigall in our test cases.
462 Algorithm VAE_NSD is generally slower than verifyeigall, whereas VAE_NJD is much
463 slower in our test cases.
464 In the previous tables we computed the mean of the relative errors of all inclusion
465 components and took the median over all test cases. We think that gives a general
466 impression of the performance of the methods. Since the size of the clusters or Jordan
467 blocks is relatively small compared to the dimension, those numbers favor the majority

n	k	relerr L			relerr X			failure			time/t_new	
		VAE_NSD	VAE_NJD	new	VAE_NSD	VAE_NJD	new	VAE_NSD	VAE_NJD	new	VAE_NSD	VAE_NJD
50	1	$1.7\cdot10^{-13}$	$2.2\cdot10^{-15}$	$3.2\cdot10^{-16}$	$7.5\cdot10^{-14}$	$7.6\cdot10^{-14}$	$1.0\cdot10^{-15}$	–	–	–	5.4	11.4
100	1	$2.9\cdot10^{-13}$	$2.7\cdot10^{-15}$	$3.2\cdot10^{-16}$	$1.7\cdot10^{-13}$	$1.8\cdot10^{-13}$	$1.1\cdot10^{-15}$	–	–	–	7.9	21.9
200	1	$5.4\cdot10^{-13}$	$3.6\cdot10^{-15}$	$3.2\cdot10^{-16}$	$4.3\cdot10^{-13}$	$4.3\cdot10^{-13}$	$1.9\cdot10^{-15}$	–	–	–	6.8	22.5
500	1	$1.6\cdot10^{-12}$	$4.7\cdot10^{-15}$	$3.3\cdot10^{-16}$	$1.3\cdot10^{-12}$	$1.3\cdot10^{-12}$	$2.6\cdot10^{-15}$	–	–	–	19.3	59.8
1000	1	$3.6\cdot10^{-12}$	$5.5\cdot10^{-15}$	$3.1\cdot10^{-16}$	$3.3\cdot10^{-12}$	$3.3\cdot10^{-12}$	$3.8\cdot10^{-15}$	–	4/–/4	–	29.5	90.3
50	2	$1.0\cdot10^{-6}$	$2.2\cdot10^{-8}$	$3.5\cdot10^{-8}$	$6.3\cdot10^{-8}$	$8.4\cdot10^{-13}$	$2.3\cdot10^{-16}$	–	3/2/1	–	3.0	5.3
100	2	$4.2\cdot10^{-6}$	$1.7\cdot10^{-8}$	$2.8\cdot10^{-8}$	$1.3\cdot10^{-7}$	$9.5\cdot10^{-12}$	$1.5\cdot10^{-16}$	–	–	–	5.0	12.1
200	2	$7.5\cdot10^{-7}$	$3.3\cdot10^{-9}$	$5.2\cdot10^{-9}$	$1.6\cdot10^{-7}$	$1.9\cdot10^{-11}$	$2.0\cdot10^{-16}$	–	–	–	6.3	17.8
500	2	$2.2\cdot10^{-6}$	$4.6\cdot10^{-9}$	$9.4\cdot10^{-9}$	$3.7\cdot10^{-7}$	$1.6\cdot10^{-10}$	$2.6\cdot10^{-16}$	–	–	–	14.6	36.8
1000	2	$4.5\cdot10^{-5}$	$1.9\cdot10^{-9}$	$6.2\cdot10^{-8}$	$2.4\cdot10^{-7}$	$3.6\cdot10^{-10}$	$2.7\cdot10^{-16}$	1	1/–/1	–	29.2	55.8
50	3	$2.6\cdot10^{-4}$	$1.0\cdot10^{-5}$	$1.4\cdot10^{-5}$	$1.8\cdot10^{-5}$	$1.2\cdot10^{-12}$	$2.7\cdot10^{-16}$	–	–	–	2.0	3.7
100	3	$7.5\cdot10^{-5}$	$1.3\cdot10^{-6}$	$2.5\cdot10^{-6}$	$2.4\cdot10^{-5}$	$6.6\cdot10^{-12}$	$2.7\cdot10^{-16}$	1	2/2/–	–	3.4	21.1
200	3	$2.6\cdot10^{-4}$	$1.2\cdot10^{-6}$	$3.6\cdot10^{-6}$	$1.8\cdot10^{-5}$	$3.3\cdot10^{-11}$	$2.7\cdot10^{-16}$	2	1/1/–	–	3.3	27.9
500	3	$4.3\cdot10^{-3}$	$4.9\cdot10^{-7}$	$7.4\cdot10^{-7}$	$1.8\cdot10^{-5}$	$8.3\cdot10^{-11}$	$2.7\cdot10^{-16}$	9	1/–/1	1	10.2	29.3
50	5	$5.8\cdot10^{-3}$	$5.9\cdot10^{-4}$	$1.2\cdot10^{-3}$	$9.0\cdot10^{-4}$	$1.5\cdot10^{-12}$	$3.4\cdot10^{-16}$	28	11/11/–	–	1.7	19.3
100	5	$6.3\cdot10^{-3}$	$2.7\cdot10^{-4}$	$6.3\cdot10^{-4}$	$6.9\cdot10^{-4}$	$9.2\cdot10^{-12}$	$3.8\cdot10^{-16}$	37	10/8/2	–	2.6	43.7
200	5	$1.2\cdot10^{-4}$	$6.3\cdot10^{-4}$	$2.5\cdot10^{-4}$	$4.5\cdot10^{-4}$	$4.6\cdot10^{-11}$	$4.6\cdot10^{-16}$	–	12/11/1	2	3.3	174.6
500	5	NaN	$3.5\cdot10^{-6}$	$1.9\cdot10^{-4}$	$1.6\cdot10^{-4}$	$4.2\cdot10^{-11}$	$4.2\cdot10^{-16}$	15	14/14/–	1	8.0	356.5
50	10	NaN	$8.1\cdot10^{-3}$	$2.0\cdot10^{-2}$	NaN	$3.4\cdot10^{-12}$	$2.9\cdot10^{-15}$	15	15/15/–	9	1.7	84.0
100	10	NaN	$5.3\cdot10^{-3}$	$9.3\cdot10^{-3}$	NaN	$3.8\cdot10^{-12}$	$5.0\cdot10^{-15}$	–	18/18/–	13	2.6	130.9
200	10	NaN	$1.0\cdot10^{-3}$	$6.8\cdot10^{-4}$	NaN	$5.0\cdot10^{-11}$	$1.2\cdot10^{-15}$	–	46/0/46	36	2.3	255.5
500	10	NaN	NaN	$2.3\cdot10^{-4}$	NaN	$1.2\cdot10^{-15}$	$4.5\cdot10^{-15}$	–	50/–/2	30	7.0	351.6
50	1/1	$1.2\cdot10^{-7}$	NaN	$5.1\cdot10^{-16}$	$4.2\cdot10^{-11}$	NaN	$1.4\cdot10^{-10}$	–	100/13/87	–	7.2	99.5
100	1/1	$2.5\cdot10^{-6}$	NaN	$1.4\cdot10^{-13}$	$1.4\cdot10^{-10}$	NaN	$4.3\cdot10^{-10}$	–	50/24/26	–	8.8	454.2
50	2	$1.5\cdot10^{-2}$	NaN	$2.6\cdot10^{-8}$	$7.2\cdot10^{-11}$	NaN	$1.9\cdot10^{-10}$	7	100/23/77	–	2.9	55.3
100	2	$1.6\cdot10^{-4}$	NaN	$1.9\cdot10^{-6}$	$4.3\cdot10^{-10}$	NaN	$4.8\cdot10^{-10}$	2	50/34/16	–	4.1	266.3
50	3	$3.3\cdot10^{-3}$	NaN	$1.6\cdot10^{-4}$	$2.8\cdot10^{-10}$	NaN	$3.1\cdot10^{-10}$	7	100/20/80	–	2.1	34.6
100	3	$2.8\cdot10^{-3}$	NaN	$1.9\cdot10^{-5}$	$3.8\cdot10^{-10}$	NaN	$6.5\cdot10^{-10}$	14	50/35/15	–	3.2	229.9
50	5	$4.6\cdot10^{-2}$	NaN	$3.9\cdot10^{-3}$	$3.4\cdot10^{-8}$	NaN	$5.4\cdot10^{-10}$	66	100/48/52	2	1.8	78.1
100	5	NaN	NaN	$1.7\cdot10^{-3}$	NaN	NaN	$1.2\cdot10^{-9}$	50	50/40/10	4	2.7	240.2

TABLE 9

Comparison with VAE_NSD and VAE_NSD for clusters and true Jordan blocks with number of test cases according to Table 8

n	k	relerr L			relerr X		
		VAE_NSD	VAE_NJD	new	VAE_NSD	VAE_NJD	new
50	1	$5.0 \cdot 10^{-11}$	$1.9 \cdot 10^{-13}$	$4.4 \cdot 10^{-16}$	$1.2 \cdot 10^{-9}$	$8.4 \cdot 10^{-10}$	$2.0 \cdot 10^{-15}$
100	1	$7.9 \cdot 10^{-11}$	$2.4 \cdot 10^{-13}$	$4.4 \cdot 10^{-16}$	$2.7 \cdot 10^{-8}$	$1.8 \cdot 10^{-8}$	$2.2 \cdot 10^{-15}$
200	1	$3.8 \cdot 10^{-11}$	$7.7 \cdot 10^{-14}$	$4.4 \cdot 10^{-16}$	$3.1 \cdot 10^{-8}$	$2.3 \cdot 10^{-8}$	$2.7 \cdot 10^{-15}$
500	1	$6.1 \cdot 10^{-10}$	$9.0 \cdot 10^{-13}$	$4.4 \cdot 10^{-16}$	$1.6 \cdot 10^{-7}$	$2.8 \cdot 10^{-7}$	$5.2 \cdot 10^{-15}$
1000	1	$6.9 \cdot 10^{-10}$	$3.5 \cdot 10^{-13}$	$4.4 \cdot 10^{-16}$	$8.3 \cdot 10^{-6}$	$8.4 \cdot 10^{-6}$	$7.7 \cdot 10^{-15}$
50	2	$2.3 \cdot 10^{-4}$	$5.1 \cdot 10^{-6}$	$1.2 \cdot 10^{-5}$	$2.7 \cdot 10^{-1}$	$1.7 \cdot 10^{-4}$	$3.0 \cdot 10^{-6}$
100	2	$3.6 \cdot 10^{-3}$	$9.8 \cdot 10^{-5}$	$2.0 \cdot 10^{-4}$	$4.2 \cdot 10^{-1}$	$3.1 \cdot 10^{-4}$	$3.3 \cdot 10^{-7}$
200	2	$1.3 \cdot 10^{-3}$	$5.1 \cdot 10^{-6}$	$5.8 \cdot 10^{-6}$	$9.5 \cdot 10^{-1}$	$1.5 \cdot 10^{-1}$	$4.4 \cdot 10^{-7}$
500	2	$9.7 \cdot 10^{-4}$	$1.2 \cdot 10^{-6}$	$3.0 \cdot 10^{-6}$	$6.7 \cdot 10^{-1}$	$1.1 \cdot 10^{-2}$	$3.4 \cdot 10^{-6}$
1000	2	$6.6 \cdot 10^{-3}$	$8.3 \cdot 10^{-6}$	$2.6 \cdot 10^{-5}$	$9.3 \cdot 10^{-1}$	$4.8 \cdot 10^{-2}$	$1.9 \cdot 10^{-7}$
50	3	$4.2 \cdot 10^{-2}$	$1.4 \cdot 10^{-3}$	$1.7 \cdot 10^{-3}$	$1.0 \cdot 10^{0}$	$7.9 \cdot 10^{-2}$	$6.8 \cdot 10^{-9}$
100	3	$3.0 \cdot 10^{-2}$	$3.1 \cdot 10^{-4}$	$6.7 \cdot 10^{-4}$	$9.5 \cdot 10^{-1}$	$2.4 \cdot 10^{-2}$	$3.6 \cdot 10^{-9}$
200	3	$2.5 \cdot 10^{-2}$	$1.4 \cdot 10^{-4}$	$2.8 \cdot 10^{-4}$	$9.8 \cdot 10^{-1}$	$7.4 \cdot 10^{-3}$	$2.0 \cdot 10^{-8}$
500	3	$3.5 \cdot 10^{-1}$	$5.7 \cdot 10^{-4}$	$2.2 \cdot 10^{-3}$	$1.0 \cdot 10^{0}$	$9.5 \cdot 10^{-1}$	$1.1 \cdot 10^{-7}$
1000	3	$2.9 \cdot 10^{-1}$	$2.7 \cdot 10^{-5}$	$1.6 \cdot 10^{-3}$	$1.0 \cdot 10^{0}$	$6.7 \cdot 10^{-1}$	$1.1 \cdot 10^{-7}$
50	5	$7.3 \cdot 10^{-1}$	$9.0 \cdot 10^{-2}$	$2.2 \cdot 10^{-1}$	$1.2 \cdot 10^{0}$	$7.8 \cdot 10^{-1}$	$1.3 \cdot 10^{-1}$
100	5	$7.2 \cdot 10^{-1}$	$5.1 \cdot 10^{-2}$	$9.8 \cdot 10^{-2}$	$1.0 \cdot 10^{0}$	$7.2 \cdot 10^{-1}$	$3.6 \cdot 10^{-1}$
200	5	$8.0 \cdot 10^{-1}$	$6.9 \cdot 10^{-3}$	$5.0 \cdot 10^{-2}$	$6.3 \cdot 10^{0}$	$8.6 \cdot 10^{-1}$	$9.2 \cdot 10^{-1}$
500	5	$2.1 \cdot 10^{0}$	$3.5 \cdot 10^{-4}$	$1.2 \cdot 10^{-1}$	$2.1 \cdot 10^{0}$	$3.8 \cdot 10^{-6}$	$9.9 \cdot 10^{-1}$

Table 10

Extract of Table 9 taking the maximum relative error over all samples

of eigenpairs which belong to simple eigenvalues. However, if in 51 out of 100 test cases narrow intervals are computed but not in the remaining, that may remain undetected by taking the median.

For completeness we display therefore part of the results of Table 9, again but now taking the maximum relative error of all inclusions of one test case and then the maximum of that number over all test cases. That means, the displayed relative errors in Table 10 are the maximum over all inclusions and over all test cases.

Again we can see a correspondence between the sensitivity of the clusters to the accuracy of the inclusions. For the eigenvalue inclusions, verifyeigall is a little better than both VAE_NSD and VAE_NJD, the eigenvector inclusions are significantly better than both VAE_NSD and VAE_NJD for clusters up to size 3. For size 5 the inclusions of all methods are poor in the worst case.

For the same test cases as in Table 10 we calculate the time ratios

$$t_{\texttt{VAE\_NSD}}/t_{\texttt{verifyeigall}} \quad \text{and} \quad t_{\texttt{VAE\_NJD}}/t_{\texttt{verifyeigall}}$$

and display the minimum, mean, median and maximum over all test cases in Table 11. As before, verifyeigall is on the average significantly faster than VAE_NSD and VAE_NJD, in the best case the ratio is close to 1.

Next we show numerical evidence that an approximated eigenmatrix approximation is hardly ill-conditioned. The first candidate is an integer matrix with 3 Jordan blocks to eigenvalues $1, 2$ and 3, respectively, each of size 10, and no other eigenvalues. An integer similarity transformation is applied to produce a matrix without zero entries retaining the anticipated Jordan structure. For this matrix there are only 3 linear independent eigenvectors, the remaining 27 are corresponding principal vectors.

n	k	$t_{\texttt{VAE\_NSD}}/t_{\texttt{verifyeigall}}$				$t_{\texttt{VAE\_NSD}}/t_{\texttt{verifyeigall}}$			
---	---	min	mean	median	max	min	mean	median	max
50	1	4.4	5.3	5.4	6.1	9.2	11.4	11.6	12.6
100	1	7.1	7.9	7.8	8.5	19.9	21.8	21.7	23.0
200	1	6.0	6.8	6.8	7.4	19.5	22.4	22.5	24.3
500	1	15.2	19.5	19.4	23.5	48.6	60.4	60.7	69.5
1000	1	27.2	29.4	29.5	31.1	83.8	90.2	91.2	94.1
50	2	1.2	3.7	3.8	7.8	2.2	6.3	6.3	12.3
100	2	3.6	5.4	5.5	7.8	8.7	13.2	12.9	18.7
200	2	3.5	7.7	8.4	13.2	10.7	22.5	24.4	36.3
500	2	8.0	18.0	13.4	31.5	19.8	47.6	35.2	88.5
1000	2	14.8	36.3	40.8	57.6	36.9	94.3	103.2	147.9
50	3	1.5	2.6	2.7	3.7	3.3	4.7	4.7	16.5
100	3	2.3	3.6	3.6	4.7	6.0	9.6	9.5	21.9
200	3	3.2	4.2	4.3	4.9	10.6	57.9	14.0	458.8
500	3	7.2	9.9	9.6	12.5	25.0	105.4	28.5	680.3
1000	3	12.6	14.7	13.5	18.1	34.8	40.6	37.1	50.0
50	5	1.1	1.7	1.7	3.4	2.1	16.7	3.4	231.1
100	5	1.4	2.8	2.7	4.7	4.5	40.0	7.3	265.7
200	5	2.1	3.3	3.2	5.0	8.9	176.8	128.7	517.1
500	5	5.0	8.4	8.4	11.1	28.9	380.6	392.9	572.6

TABLE 11

Extract of Table 9: detailed timing

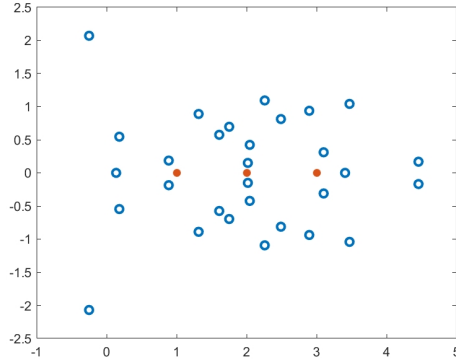The eigenvalue approximations computed by Matlab's `eig` are displayed in Figure 1.

FIG. 1. *Approximate eigenvalues 'o' of the matrix "Jordan" with 3 Jordan blocks with 10-fold eigenvalues 1, 2 and 3 depicited in red '*'.*

Recall that there should be only the 3 dots at $1, 2$ and 3 depicted in red. Given these approximations verified inclusions can hardly be computed - and they are not.

Since the Matlab function `eig` can only approximate eigenvectors, not principal vectors, we may expect that `[W,D] = eig(A)` produces a matrix W of numerical rank 3. However, as displayed in the first line of Table 12, $\text{cond}(W) \approx 3.5 \cdot 10^9$. That is because of the smoothening effect of floating-point arithmetic, some kind of regularization. We remark that in that specific case of an integer matrix a computer

algebra system such as the symbolic toolbox in Matlab computed easily the correct eigenvalues together with the Jordan structure.

In Table 12 the maximal errors of the inclusions of the eigenvalues and of the eigenvectors are displayed, each for Algorithms `VAE_NSD` and `VAE_NJD` in 3.2, and for the new algorithm, followed by the time in seconds. An entry "-" means that not for all eigenpairs inclusions could be computed.

Next we try some test matrices from the Matlab gallery which may cause problems. In a number of cases no inclusion could be computed, in some cases the inclusions are very wide. For example, the eigenvalue inclusions for the matrix `gallery/lesp` of `VAE_NSD` and the new algorithm are of reasonable quality, where `VAE_NJD` fails. However, the eigenvector inclusions are poor with a relative error close to 1.

For the "frank 0" and "frank 1" matrices we observe ill-conditioned W. However, these are integer Hessenberg matrices for which, because of the zeros below the first subdiagonal, the smoothing of floating-point arithmetic is not as effective.

The timing is self-explaining; in three cases, the "Jordan" matrix with 3 Jordan blocks of size 10, "frank 0" and "frank 1" matrices of dimension 30 Algorithm `VAE_NJD` was caught in an infinite loop, for the "lesp" matrix of dimension 30 Algorithm `VAE_NJD` caused a Matlab error.

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matrix	n	cond(W)	relerr L			relerr X			time [sec]		
			VAE_NSD	VAE_NJD	new	VAE_NSD	VAE_NJD	new	VAE_NSD	VAE_NJD	new
Jordan	30	$3.5 \cdot 10^9$	$2.5 \cdot 10^4$	-	-	0	-	-	0.04	∞	0.03
frank 0	10	$7.2 \cdot 10^5$	$6.1 \cdot 10^{-8}$	$4.7 \cdot 10^{-10}$	$3.3 \cdot 10^{-16}$	$1.3 \cdot 10^{-4}$	$6.6 \cdot 10^{-5}$	$1.9 \cdot 10^{-15}$	0.02	0.03	0.006
frank 0	20	$2.5 \cdot 10^5$	1.7	-	0.18	0.88	-	-	0.03	13	0.07
frank 0	30	$4.1 \cdot 10^{14}$	-	-	-	-	-	-	0.03	∞	0.09
frank 1	10	$8.4 \cdot 10^5$	$8.5 \cdot 10^{-8}$	$1.5 \cdot 10^{-9}$	$3.3 \cdot 10^{-16}$	$4.8 \cdot 10^{-5}$	$3.7 \cdot 10^{-5}$	$5.2 \cdot 10^{-14}$	0.02	0.04	0.007
frank 1	20	$2.8 \cdot 10^{13}$	0.99	-	-	1.8	-	-	0.03	21	0.07
frank 1	30	$1.3 \cdot 10^{14}$	-	-	-	-	-	-	0.03	∞	0.09
lesp	10	$4.4 \cdot 10^2$	$6.8 \cdot 10^{-14}$	$2.0 \cdot 10^{-15}$	$3.1 \cdot 10^{-16}$	$1.2 \cdot 10^{-11}$	$8.0 \cdot 10^{-5}$	$3.9 \cdot 10^{-12}$	0.02	0.03	0.007
lesp	20	$4.5 \cdot 10^5$	$8.3 \cdot 10^{-12}$	$2.5 \cdot 10^{-15}$	$3.2 \cdot 10^{-16}$	0.93	0.58	0.57	0.03	0.29	0.008
lesp	30	$4.4 \cdot 10^8$	$8.6 \cdot 10^{-10}$	-	$3.2 \cdot 10^{-16}$	0.51	-	0.88	0.04	-	0.007
wilk	21	1	$3.6 \cdot 10^{-14}$	$4.4 \cdot 10^{-10}$	$3.6 \cdot 10^{-15}$	$2.3 \cdot 10^{-2}$	$1.4 \cdot 10^{-5}$	$8.2 \cdot 10^{-9}$	0.03	0.09	0.01

TABLE 12
Selected matrices from the Matlab gallery

pp. 24–45.

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