OTH OSTBAYERISCHE
TECHNISCHE HOCHSCHULE
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Formelsammlung Felder, Wellen und Leitungen

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Name:	Ayham Alhalaibi
Matrikelnummer:	MATNR
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Lizenz:	GPLv3

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1 Grundlagen

1.1 mathematische

Divergenz/Rotation/Gradient

div: macht aus einem Vektor ein Skalar.
rot: bildet ein Vektor auf Vektorfeld ab.
grad: bildet ein Skalar-/Gradientenfeld in ein Vektorfeld ab.
Zeigt Richtung stärkster Zunahme des Feldes.

$$\operatorname{div} \vec{G} = \nabla \cdot \vec{G} = \frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z}$$

$$= 0 \quad \Rightarrow \text{Volumen}$$
$$> 0 \quad \Rightarrow \text{Quelle}$$
$$< 0 \quad \Rightarrow \text{Senke}$$

$$\operatorname{rot} \vec{G} = \nabla \times \vec{G} = \begin{pmatrix} \frac{\partial G_z}{\partial y} - \frac{\partial G_y}{\partial z} \\ \frac{\partial G_x}{\partial z} - \frac{\partial G_z}{\partial x} \\ \frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y} \end{pmatrix}$$

$$\operatorname{grad} G = \nabla \cdot G = \begin{pmatrix} \frac{\partial G}{\partial x} \\ \frac{\partial G}{\partial y} \\ \frac{\partial G}{\partial z} \end{pmatrix}$$

Nabla Operator

$$\nabla = \vec{\nabla} = \left(\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}, \frac{\partial G}{\partial z} \right)$$

Feldänderung bei Bewegung

$$\Delta G = \frac{\partial G}{\partial x} \Delta x + \frac{\partial G}{\partial y} \Delta y + \frac{\partial G}{\partial z} \Delta z$$
$$= dG = \operatorname{grad} G \cdot d\vec{s}$$

1.2 Randbedingung

Dirichlet-RB	Funktion nimmt an den Rändern einen bestimmten Wert an (Bsp.: $\rho_r = 5V$)
Neumann-RB	Die Normalableitung der Fkt. nimmt an den Rändern einen bestimmten Wert an

1.3 Begriffe

	Begriff	Beschreibung
ρ	Raumladungsdichte	

1.4 Kartesische Koordinaten

Einheitsvektoren: $\vec{e_x}, \vec{e_y}, \vec{e_z}$
Rechtssystem: $\vec{e_x} \times \vec{e_y} = \vec{e_z}$
Linienelemente: $ds = \sqrt{dx^2 + dy^2 + dz^2}$
Nabla Operator: $\nabla = \frac{\partial}{\partial x} \vec{e_x} + \frac{\partial}{\partial y} \vec{e_y} + \frac{\partial}{\partial z} \vec{e_z}$
Gradient: $\operatorname{grad} \varphi \equiv \nabla \varphi = \frac{\partial \varphi}{\partial x} \vec{e_x} + \frac{\partial \varphi}{\partial y} \vec{e_y} + \frac{\partial \varphi}{\partial z} \vec{e_z}$
Divergenz: $\operatorname{div} \vec{D} \equiv \nabla \vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$
Rotation: $\operatorname{rot} \vec{E} \equiv \nabla \times \vec{E} =$
$\left[\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right] \vec{e_x} + \left[\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right] \vec{e_y} + \left[\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right] \vec{e_z}$

Laplace Operator: $\Delta = \frac{\partial^2 \cdots}{\partial x^2} + \frac{\partial^2 \cdots}{\partial y^2} + \frac{\partial^2 \cdots}{\partial z^2}$

$$\Delta \vec{E} = \operatorname{grad} \operatorname{div} \vec{E} - \operatorname{rot} \operatorname{rot} \vec{E} = \Delta E_x \vec{e_x} + \Delta E_y \vec{e_y} + \Delta E_z \vec{e_z} =$$
$$= \left[\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} \right] \vec{e_x} + \left[\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} \right] \vec{e_y}$$
$$+ \left[\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} \right] \vec{e_z}$$

1.5 Zylinderkoordinaten

Variablen: r, α, z
Einheitsvektoren: $\vec{e_r}, \vec{e_\alpha}, \vec{e_z}$
Rechtssystem: $\vec{e_r} \times \vec{e_\alpha} = \vec{e_z}$
Linienelemente: $ds = \sqrt{dr^2 + r^2 d\alpha^2 + dz^2}$
Volumenelemente: $dv = r \, dr \, d\alpha \, dz$
Nabla Operator: $\nabla = \frac{\partial}{\partial r} \vec{e_r} + \frac{1}{r} \frac{\partial}{\partial \alpha} \vec{e_\alpha} + \frac{\partial}{\partial z} \vec{e_z}$
Gradient: $\operatorname{grad} \varphi \equiv \nabla \varphi = \frac{\partial \varphi}{\partial r} \vec{e_r} + \frac{1}{r} \frac{\partial \varphi}{\partial \alpha} \vec{e_\alpha} + \frac{\partial \varphi}{\partial z} \vec{e_z}$
Divergenz: $\operatorname{div} \vec{D} \equiv \nabla \vec{D} = \frac{1}{r} \frac{\partial (r \vec{D_r})}{\partial r} + \frac{1}{r} \frac{\partial \vec{D_\alpha}}{\partial \alpha} + \frac{\partial \vec{D_z}}{\partial z}$
Rotation: $\operatorname{rot} \vec{E} \equiv \nabla \times \vec{E} = \left[\frac{1}{r} \frac{\partial E_z}{\partial \alpha} - \frac{\partial E_\alpha}{\partial z} \right] \vec{e_r} +$
$\left[\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} \right] \vec{e_\alpha} + \left[\frac{1}{r} \frac{\partial (r E_\alpha)}{\partial r} - \frac{1}{r} \frac{\partial E_r}{\partial \alpha} \right] \vec{e_z}$
Laplace Operator: $\Delta = \frac{1}{r} \frac{\partial \left(r \frac{\partial \cdots}{\partial r} \right)}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \cdots}{\partial \alpha^2} + \frac{\partial^2 \cdots}{\partial z^2}$

$$\vec{E} = \left[\Delta E_r - \frac{2}{r^2} \frac{\partial E_\alpha}{\partial \alpha} - \frac{E_r}{r^2} \right] \vec{e_r}$$
$$+ \left[\Delta E_\alpha + \frac{2}{r^2} \frac{\partial E_r}{\partial \alpha} - \frac{E_\alpha}{r^2} \right] \vec{e_\alpha} + \left[\Delta E_z \right] \vec{e_z}$$

1.6 Kugelkoordinaten

Variablen: r, ϑ, α
Einheitsvektoren: $\vec{e_r}, \vec{e_\vartheta}, \vec{e_\alpha}$
Rechtssystem: $\vec{e_r} \times \vec{e_\vartheta} = \vec{e_\alpha}$
Linienelement: $ds = \sqrt{dr^2 + r^2 \sin^2 \vartheta \, d\alpha^2 + r^2 d\vartheta^2}$
Volumenelement: $dv = r^2 \sin \vartheta \, dr \, d\vartheta \, d\alpha$
Nabla Operator: $\nabla = \frac{\partial}{\partial r} \vec{e_r} + \frac{1}{r} \frac{\partial}{\partial \vartheta} \vec{e_\vartheta} + \frac{1}{r \sin \vartheta} \frac{\partial}{\partial \alpha} \vec{e_\alpha}$
Gradient: $\operatorname{grad} \varphi \equiv \nabla \varphi = \frac{\partial \varphi}{\partial r} \vec{e_r} + \frac{1}{r} \frac{\partial \varphi}{\partial \vartheta} \vec{e_\vartheta} + \frac{1}{r \sin \vartheta} \frac{\partial \varphi}{\partial \alpha} \vec{e_\alpha}$
Divergenz: $\operatorname{div} \vec{D} \equiv \nabla \vec{D} = \frac{1}{r^2} \frac{\partial (r^2 D_r)}{\partial r} + \frac{1}{r \sin \vartheta} \frac{\partial (\sin \vartheta \cdot D_\vartheta)}{\partial \vartheta} + \frac{1}{r \sin \vartheta} \frac{\partial D_\alpha}{\partial \alpha}$
Rotation:

$$\operatorname{rot} \vec{E} \equiv \nabla \times \vec{E} = \frac{1}{r \sin \vartheta} \left[\frac{\partial (\sin \vartheta \cdot E_\alpha)}{\partial \vartheta} - \frac{\partial E_\vartheta}{\partial \alpha} \right] \vec{e_r}$$
$$+ \frac{1}{r} \left[\frac{1}{\sin \vartheta} \frac{\partial E_r}{\partial \alpha} - \frac{\partial r E_\alpha}{\partial r} \right] \vec{e_\vartheta} + \frac{1}{r} \left[\frac{\partial (r E_\vartheta)}{\partial r} - \frac{\partial E_r}{\partial \vartheta} \right] \vec{e_\alpha}$$

Laplace Operator: $\Delta = \frac{1}{r^2} \frac{\partial \left(r^2 \frac{\partial \cdots}{\partial r} \right)}{\partial r} + \frac{1}{r^2 \sin \vartheta} \frac{\partial \left(\sin \vartheta \frac{\partial \cdots}{\partial \vartheta} \right)}{\partial \vartheta} + \frac{1}{r^2 \sin^2 \vartheta} \frac{\partial^2 \cdots}{\partial \alpha^2}$

Laplace Operator in Kugelkoordinaten, angewandt auf einen Vektor:

$$\Delta \vec{E} = \left[\Delta E_r - \frac{2}{r^2} E_r - \frac{2}{r^2 \sin \vartheta} \frac{\partial (\sin \vartheta \cdot E_\vartheta)}{\partial \vartheta} - \frac{2}{r^2 \sin \vartheta} \frac{\partial E_\alpha}{\partial \alpha} \right] \vec{e_r}$$
$$+ \left[\Delta E_\vartheta - \frac{E_\vartheta}{r^2 \sin^2 \vartheta} + \frac{2}{r^2} \frac{\partial E_r}{\partial \vartheta} - \frac{2 \cot \vartheta}{r^2 \sin \vartheta} \frac{\partial E_\alpha}{\partial \alpha} \right] \vec{e_\vartheta}$$
$$+ \left[\Delta E_\alpha - \frac{E_\alpha}{r^2 \sin^2 \vartheta} + \frac{2}{r^2 \sin \vartheta} \frac{\partial E_r}{\partial \alpha} + \frac{2 \cot \vartheta}{r^2 \sin \vartheta} \frac{\partial E_\vartheta}{\partial \alpha} \right] \vec{e_\alpha}$$

1.7 Vergleich/Umrechnung

Kart.	Zyl.	Kug.
x	$r\cos\alpha$	$r\sin\vartheta\cos\alpha$
y	$r\sin\alpha$	$r\sin\vartheta\sin\alpha$
z	z	$r\cos\vartheta$
$\sqrt{x^2+y^2}$	r	
$\arctan\frac{y}{x}$	α	
z	z	
$dx\cos\alpha+dy\sin\alpha$	dr	
$dy\cos\alpha-dx\sin\alpha$	$rd\alpha$	
dz	dz	
$\sqrt{x^2+y^2+z^2}$		r
$\arctan\frac{y}{x}$		α
$\arctan\frac{\sqrt{x^2+y^2}}{z}$		ϑ
$dx\sin\vartheta\cos\alpha\quad+$ $dy\sin\vartheta\sin\alpha\quad+$ $dz\cos\vartheta$		dr
$dy\cos\alpha-dx\sin\alpha$		$r\sin\vartheta d\alpha$
$dx\cos\vartheta\cos\alpha\quad+$ $dy\cos\vartheta\sin\alpha\quad-$ $dz\sin\vartheta$		$rd\vartheta$

2 Maxwell'schen Gleichungen

2.1 Intergralform I, II

Gauß'sches Gesetz
Induktionsgesetz
Durchflutungsgesetz
Quellenfreiheit B-Feld
Zusammenhang

$$\oiint_{A_{\text{Hülle}}} \vec{D}d\vec{A} = Q_{\text{eing.}}$$

$$\oint_{\text{Rand}} \vec{E}d\vec{s} = -\frac{d\Phi_{\text{eing.}}}{dt}$$

$$\oint_{\text{Rand}} \vec{H}d\vec{s} = I_{\text{eing.}} + I_{\text{versch.}}$$

$$\oiint_{A_{\text{Hülle}}} \vec{B}d\vec{A} = 0$$

$$\vec{D} = \varepsilon \cdot \vec{E} \qquad \vec{B} = \mu \cdot \vec{H}$$

Bei isotropen Stoffen sind ε u. μ Skalare:

$$\varepsilon = \varepsilon_0 \cdot \varepsilon_r \qquad \mu = \mu_0 \cdot \mu_r$$

2.2 Differentialform I, II

$$\text{div}\,\vec{D} = \rho$$

$$\text{rot}\,\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\text{rot}\,\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$$

$$\text{div}\,\vec{B} = 0$$

2.3 stationäre Felder

$$\nabla \cdot \vec{D} = \rho \qquad \vec{D} = \varepsilon \cdot \vec{E}$$

$$\nabla \times \vec{E} = 0$$

$$\nabla \times \vec{H} = \vec{J} \qquad \vec{B} = \mu \cdot \vec{H}$$

$$\nabla \cdot \vec{B} = 0$$

$$\vec{J} = \kappa\vec{E}$$

3 Felder

3.1 Elektrostatik

ist ein wirbelfreies Feld. Elek. Ladungen sind Quellen des Feldes.

$$\text{div}\,\vec{D} = \nabla \cdot \vec{D} = \rho \qquad \vec{D} = \varepsilon\vec{E}$$

$$\text{rot}\,\vec{E} = \nabla \times \vec{E} = 0 \quad = \text{rot}\,\text{grad}\,E$$

$$\vec{E} = -\,\text{grad}\,\varphi$$

3.1.1 Potetialgleichung

$$\text{div}\,\text{grad} = -\frac{\rho}{\varepsilon}$$

$\Rightarrow$ **Poisson-Gleichung** mit $\rho = 0 \Rightarrow$ **Laplace-Gleichung**

$$\Delta\varphi + \underbrace{\frac{\text{grad}\,\varepsilon \cdot \text{grad}\,\varphi}{\varepsilon}}_{=0,\ wenn\ homogen} = -\frac{\rho(x,y,z)}{\varepsilon}$$

$$\frac{d^2\varphi}{dx^2} + \frac{d^2\varphi}{dy^2} + \frac{d^2\varphi}{dz^2} = -\frac{\rho(x,y,z)}{\varepsilon}$$