

Novel Algebras for Advanced Analytics in Julia

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Abstract—A linear algebraic approach to graph algorithms that exploits the sparse adjacency matrix representation of graphs can provide a variety of benefits. These benefits include syntactic simplicity, easier implementation, and higher performance. One way to employ linear algebra techniques for graph algorithms is to use a broader definition of matrix and vector multiplication. We demonstrate through the use of the Julia language system how easy it is to explore semirings using linear algebraic methodologies.

I. INTRODUCTION

A. Semiring algebra

The duality between the canonical representation of graphs as abstract collections of vertices and edges and a sparse adjacency matrix representation has been a part of graph theory since its inception [?], [?]. Matrix algebra has been recognized as a useful tool in graph theory for nearly as long (see [?] and references therein). A linear algebraic approach to graph algorithms that exploits the sparse adjacency matrix representation of graphs can provide a variety of benefits. These benefits include syntactic simplicity, easier implementation, and higher performance. One way to employ linear algebra techniques for graph algorithms is to use a broader definition of matrix and vector multiplication. One such broader definition is that of a semiring. For example, in semiring notation we write matrix-matrix multiply as:

$$C = A + . * B$$

where

$$+.*$$

denotes standard matrix multiply. In such notation, a semiring requires that addition and multiplication are associative, addition is commutative, and multiplication distributes over addition from both left and right. In graph algorithms, a fundamental operation is matrix-matrix multiply where both matrices are sparse. This operation represents multi source 1-hop breadth first search (BFS) and combine, which is the foundation of many graph algorithms [?]. In addition, it is often the case that operations other than standard matrix multiply are desired, for example:

1)

$$C = A \max . + B$$

2)

$$C = A \min . \max B$$

3)

$$C = A |. \& B \text{ (or.and)}$$

4)

$$C = A f . g B$$

With this more general case of sparse matrix multiply, a wide range of graphs algorithms can be implemented [?].

II. APPLICATION EXAMPLE

A classic example of the utility of the semiring approach is in finding the minimum path between all vertices in graph (see [?] in [?]. Given a weighted adjacency matrix for a graph where $A(i, j) = w_{ij}$ is the weight of a directed edge from vertex i to vertex j . Let $C(i, j)_2$ be the minimum 2-hop cost from vertex i to vertex j . C_2 can be computed via the semiring matrix product:

$$C_2 = A \min . + A$$

Likewise, C_3 can be computed via

$$C_3 = A \min . + A \min . + A$$

and more generally

$$C_k = A^k$$

III. JULIA

It has become clear that programmers and scientists prefer high-level, interactive, dynamic environments for algorithm development and data analysis. Systems such as Matlab [?], Octave [?], R [?], SciPy [?], and SciLab [?] provide greatly increased convenience and productivity, yet C and Fortran remain the gold standard for computationally-intensive problems because these high-level dynamic systems still lag significantly in performance. As a result, the most challenging areas of technical computing have benefited the least from the increased abstraction and productivity offered by higher level languages. Julia [?] is a high-level, dynamic language, designed from the ground up to take advantage of modern techniques for executing dynamic languages efficiently. As a result, Julia has the performance of a statically compiled language while providing the interactive, dynamic experience and productivity that scientists have come to expect.

Julia also introduces many powerful computer science tools to scientific computing, including a sophisticated type system, multiple dispatch, coroutines, Lisp-style metaprogramming (including real macros), and built-in support for distributed computation. Although a powerful type system is made available to the programmer, it remains unobtrusive in the sense that one is never required to specify types, nor is performance dependent upon doing so: unless the programmer wants to take advantage of Julia’s dispatch system or create C-compatible types, their code will work just as well without ever mentioning types. Likewise, coroutines, macros and distributed programming primitives are right there, should the programmer ever need them, but are not required for day-to-day programming.

We recommend trying the code examples in this paper. Just a few moments may convince the reader that Julia is simple yet powerful. All of the code may be found on GitHub: github.com/ViralBShah/SemiringAlgebra.jl

IV. SEMIRING ALGEBRA IN JULIA

Julia’s ordinary matrix multiplication is recognizable to users of many high level languages:

```
A=rand(m,n)
B=rand(n,p)
C=A*B
or one can equally well use the prefix notation
C = *(A,B)
instead of the infix notation.
```

We believe that users of matrix-multiply who are used to such compact expressions would prefer not to have overly complicated syntax to express the more general semirings. Julia offers two approaches that are readily available for exploration:

1) *Star overloading*: This method is recommended for interactive exploration of many semiring operators. Without introducing any types, define $*(f, g)(A, B)$ to perform the semiring operation very generally, with no a-priori restriction on the binary functions f or g . Upon overloading the star operator, and defining the `ringmatmul`, the following immediately work in Julia:

```
*(max,+) (A,B)
*(min,max) (A,B)
*(|,&) (A,B)
*(+,*) (A,B)
```

The last example computes the usual matrix product.

2) *Creation of semiring objects*: This method is recommended for users who are working exclusively in one semiring and wish to optimize notation and performance. In this method, a semiring type is created, and one overloads scalar $+$ and scalar $*$ only. Julia’s generic definitions for matrix multiplication, which depends only on having appropriate definitions for $+$ and $*$ does the rest, allowing one to immediately compute matrix products in the newly defined semiring, using the usual notation for matrix products.

A. Star Overloading

First time users should readily be able to google, download, and install Julia. The star overloading functionality may be

explored by typing directly into a Julia session:

```
function ringmatmul(+,*,A::Matrix,B::Matrix)
    m, n, p = size(A,1), size(B,2), size(A,2)
    C = [A[i,1]*B[1,j] for i=1:m, j=1:n]
    for i=1:m, j=1:n, k=2:p
        C[i,j] += A[i,k]*B[k,j]
    end
    return C
end

*((+)::Function, (*)::Function) =
    (A,B)->ringmatmul(+,*,A,B)
```

The `ringmatmul` function is straightforward – it is a standard naïve triple loop matrix multiply. The function takes four arguments, the first two, $+$ and $*$, are functions that will serve as the local versions of addition and multiplication, while the last two are the matrices to multiply using those operations. This highlights a few relevant features of the Julia language. First, functions are first-class values that can be passed into other functions as arguments and then used with standard function call syntax. Second, the $+$ and $*$ operators are just regular functions with some special syntax. For example, the expression `C[i,j] += A[i,k]*B[k,j]` is translated into `C[i,j] = +(C[i,j],*(A[i,k],B[k,j]))` where the functions $+$ and $*$ are looked up in the current scope just like any other variables would be. The second function definition says that if “ $*$ ” is called with two arguments, both of which have the type `Function`, then the result is a function itself which takes two arrays as arguments, and calls `ringmatmul` with arguments f, g, A, B . This highlights a few more features of Julia: it uses multiple dispatch by proding a new behavior for $*$ when its arguments are functions; it returns a function as a value, which can then be used elsewhere, again with the standard function call syntax.

B. Semiring Objects

Here we create a “max-plus” semiring type. Other semirings can be implemented by changing the definitions of “ $+$ ” and “ $*$ ”, which below are set to use “max” and “plus”. The new type does not need to be taught matrix multiply; `matmul` will work with any underlying numeric type (e.g. `int`, `float`, ...), and will also work with both dense and sparse matrices. Arrays of the new semiring type will use the same amount of memory as “primitive” arrays of the underlying numeric type.

```
immutable MPNumber{T} <: Number
    val::T
end

+(a::MPNumber,b::MPNumber)
    = MPNumber(max(a.val,b.val))
*(a::MPNumber,b::MPNumber)
    = MPNumber(a.val+b.val)
show(io::IO, k::MPNumber)
    = print(io, k.val)

zero{T} (::Type{MPNumber{T}})
    = MPNumber(typemin{T})
one{T} (::Type{MPNumber{T}})
```

```

    = MPNumber(zero(T))
promote_rule{T<:Number} (::Type{MPNumber},
                        ::Type{T})
    = MPNumber

mparray(A::Array) = map(MPNumber, A)
array{T}(A::Array{MPNumber{T}})
    = map(x->x.val, A)
mpsparse(S::SparseMatrixCSC)
    = SparseMatrixCSC(S.m, S.n, S.colptr,
                      S.rowval, mparray(S.nzval))

```

The first definition describes the `MPNumber` type. Then plus and times are defined as `max` and `plus` respectively, followed by a routine `IO` function.

The zero and one in this semiring are defined as the identity elements for `max` and `plus`, and the `promote` rule is a Julia construction which allows semiring numbers and ordinary numbers to work together.

Finally the last three functions allow for the conversion between ordinary arrays and the semiring, with a special extra constructor for the important use case of sparse arrays.

Some examples of using the code:

```

A=mparray(rand(3,3)); B=mparray(rand(3,3));
A*B
A*A
A^2
C=mpsparse(sparse(eye(3,3)));
C*C

```

V. PERFORMANCE

Peak possible performance for an ordinary matrix multiply on floating point numbers would be achieved by a highly tuned BLAS routine. The goal of Julia is to have the right combination of reasonable performance for the machine and productivity for the human. So while we do not expect to match a tuned BLAS, performance will often be comparable to that of naive C code.

On a Macbook Pro, with dual-core 2.4 GHz Intel core i5, 8GB RAM, we compared a dense BLAS run of `matmul` with an `MPNumber{Float64}` run in the semiring. The BLAS time was 0.26msec (practically free!) for a 100×100 array. The semiring time was 313 msec. This is no surprise.

More interesting and more useful is the comparison for sparse matrices. For an $n \times n$ sparse random array with density $1/n$, where $n = 100,000$ the ordinary matrix multiply took 21 msec, and the semiring matrix multiply took a comparable 22 msec.

For a higher density ($5/n$) we found 461 msec for the `matmul` compared to 414 msec in the semiring.

VI. CONCLUSION

Julia facilitates the implementation and exploration of graph algorithms through the semiring notation of generalized matrix multiply. These algorithms appear in applications to data analysis and related fields. Exploitation of sparse

data structures provides easy implementations with reasonable performance. Sparse semiring performance is likely to be comparable to ordinary sparse `matmul` in the Julia language.

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