A relation similar to (2.16) holds in the continuous-time case. Here, for large t

$$\ln \det \mathsf{P} = \lim_{t\to\infty} \ln \left[[\det \mathsf{H}(t)]^{1/t} \right] = \lim_{t\to\infty} \frac{1}{t} \ln \det \exp \left(\int_0^t dt' \mathsf{K}(\mathbf{U}(t'), t') \right).$$

Taking into account the linear-algebra relation $\det \exp \mathsf{A} = \exp \mathrm{tr}\mathsf{A}$, we obtain

$$\mathcal{S}_N = \sum_{k=1}^{N} \lambda_k = \ln \det \mathsf{P} = \lim_{t\to\infty} \frac{1}{t} \int_0^t dt' \mathrm{tr}\mathsf{J}(\mathbf{U}(t'), t') = \langle \mathrm{tr}\, \mathsf{K} \rangle .$$

In the so-called dissipative systems, where $\det \mathsf{J} < 1$ (in the discrete case) and $\mathrm{tr}\, \mathsf{K} < 0$ (in the continuous case), the sum $\mathcal{S}_N$ of the Lyapunov exponents is negative, meaning that volumes around generic trajectories shrink exponentially to zero. In volume-preserving (e.g. Hamiltonian) systems, since $\det \mathsf{J} = 1$ for discrete time and $\mathrm{tr}\, \mathsf{K} = 0$ for continuous time, the sum of the exponents vanishes, $\mathcal{S}_N = 0$. This property can be used to test the numerical precision in the calculation of Lyapunov exponents.

These relations can be extended to partial volumes as well. Let us assume, for the sake of simplicity, that the Oseledets splitting (Section 2.3.2) is valid and consider two generic initial vectors $\mathbf{u}(0)$ and $\mathbf{v}(0)$. The area spanned by their iterates is given by the vector product $\mathbf{u}(t) \times \mathbf{v}(t)$. The two initial vectors typically have components along all directions $\mathbf{E}^k(0)$. As a result of the evolution over a long time interval t, each component will be roughly multiplied by a factor $\exp(t\lambda_k)$. Accordingly, in the vector product, as the term $\mathbf{E}^1(t) \times \mathbf{E}^1(t)$, involving the largest component, vanishes, the leading contribution is provided by the cross terms $\mathbf{E}^1(t) \times \mathbf{E}^2(t)$, which is proportional to $\exp[t(\lambda_1 + \lambda_2)]$. Therefore, the area V_2 spanned by two generic initial vectors grows as the sum of the two largest Lyapunov exponents:

$$\lambda_1 + \lambda_2 = \lim_{n\to\infty} \frac{1}{t} \log \frac{V_2(t)}{V_2(0)}. \tag{2.17}$$

Similarly, for a typical M-dimensional volume V_M spanned by M generic different initial vectors,

$$\mathcal{S}_M = \sum_{k=1}^{M} \lambda_k = \lim_{n\to\infty} \frac{1}{t} \log \frac{V_M(t)}{V_M(0)}. \tag{2.18}$$

This relation, which holds also for infinite-dimensional systems, lies at the heart of the numerical methods for determining Lyapunov exponents discussed in Chapter 3.

2.5.5 Time parametrisation

In the Lyapunov analysis, time plays a special role. One can often change variables to find the most appropriate ones (this "game" will be often played throughout the book) for either numerical or analytical computations. There exists, however, only one *time* and usually one can at most think of rescaling it according to some characteristic scale of the problem. In relativistic dynamics, this is not so, as there is no absolute time axis: the Lyapunov exponents depend on the observer (Francisco and Matsas, 1988) and the use of them as chaos indicators has even been challenged. Occasionally, it may be useful to perform a non-trivial change of the time variable also in classical contexts.

Let us start by introducing a generic scalar variable:

$$\phi = \phi(\mathbf{U}).$$

This is a proper time-like (phase-like) variable if its time derivative

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial \mathbf{U}} \cdot \mathbf{F}(\mathbf{U}) \tag{2.19}$$

is strictly positive everywhere (if not in the entire phase space, at least on the invariant set). If this is true, one can use ϕ to order all of the events along the true time axis. Furthermore, under this assumption, one can switch the independent variable from t to ϕ, arriving at the evolution equation

$$\frac{d\mathbf{U}}{d\phi} = \frac{\mathbf{F}(\mathbf{U})}{\mathbf{F}(\mathbf{U}) \cdot \partial\phi/\partial\mathbf{U}}. \tag{2.20}$$

Formally, one of the variables is redundant because of the link between ϕ and the position in phase space, but one can nevertheless disregard this property, select a generic initial condition $\mathbf{U}$ and let it evolve. In practice, the system (2.20) is a reformulation of the initial problem, where the time variable has been adjusted. In order to establish a full link with the original equations, it is necessary to complete the mathematical model with the equation that allows the determination of the true time as a function of ϕ. This is nothing but the inverse of Eq. (2.19):

$$\frac{dt}{d\phi} = \frac{1}{\mathbf{F}(\mathbf{U}) \cdot \partial\phi/\partial\mathbf{U}}. \tag{2.21}$$

In practice, it is easy to convince oneself that the Lyapunov exponents λ_i^ϕ of the system (2.20) are equal to those of the original model up to a multiplicative constant

$$\lambda_i = \lambda_i^\phi/\tau, \tag{2.22}$$

where

$$\tau = \lim_{\phi\to\infty} \frac{t}{\phi}. \tag{2.23}$$

An example where a meanigful variable can be introduced is the Rössler attractor (A.8) for the standard parameter values, setting $\phi = \arctan(y/x)$ as the "phase" of the oscillations.

Within relativistic dynamics ϕ is not a time-like variable, but the true time measured by some other observer (Motter, 2003). Remarkably Eq. (2.22) still holds when a Lorentz tranformation turns a bounded trajectory into an unbounded one (or vice versa) – see Motter and Saa (2009) for a more detailed discussion. As a result, the sign of the Lyapunov exponent is preserved, and this assures that a positive value is a valid criterion to identify chaos across space-time transformations.

This change of variables proves rather useful whenever one has to determine a Poincaré surface of a section. Imagine that the section is defined by the condition $\phi = C$. It is much easier to integrate Eq. (2.20) until the "time" variable ϕ reaches the preassigned value C than to incorporate this condition into the original system. In general, one cannot expect that the variable defining the surface of section is a global time-like variable. One can nevertheless introduce the new variable ϕ locally, sufficiently close to the Poincaré surface, such that one can be sure it behaves monotonically. This is basically the trick suggested long ago by Hénon (1982).

This construction also allows one to establish a relationship between the continuous- and discrete-time representations of a given dynamical system. There are two ways to reduce a continuous-time system to a map: (i) by monitoring the continuous-time system stroboscopically, with a prescribed time interval T and (ii) by constructing a Poincaré map, sampling a trajectory only when it crosses some $(N-1)$-dimensional surface of section in the N-dimensional phase space. In the case of the stroboscopic map, the dimension of the phase space is unchanged, and the N Lyapunov exponents are related by the equation

$$T\lambda_k^{\text{cont}} = \lambda_k^{\text{discr}}.$$

When a Poincaré map construction is used (typically in autonomous continuous systems), the number of Lyapunov exponents reduces to $N-1$. The zero exponent, which corresponds to the invariance of the original trajectories under time shift (see Section 2.5.6), "disappears" because this symmetry is lifted in discrete time. In fact, as discussed, the Poincaré map can be interpreted as a stroboscopic map for a time-like variable ϕ with constant $C = 1$. In this latter case, Eq. (2.23) yields $\tau = \langle T_n \rangle$ where T_n is the Poincaré return time, so that all other exponents are related by

$$\langle T_n \rangle \, \lambda_k^{\text{cont}} = \lambda_k^{\text{Poin}}.$$

The zero exponent may be eventually recovered by including the evolution equation (2.21) for the true time.

2.5.6 Symmetries and zero Lyapunov exponents

Symmetries and conservation laws play an important role in determining the spectrum of Lyapunov exponents. Continuous symmetries, for instance, typically yield zero Lyapunov exponents. We start by showing that autonomous continuous-time dynamical systems have a zero Lyapunov exponent, provided the trajectory does not converge to a steady state. To see this, let us consider a trajectory $\mathbf{U}(t)$ of Eq. (2.4) and select the perturbation $\mathbf{u} = \mathbf{F}$. Since in autonomous system $\mathbf{F}$ does not explicitly depend on time, time differentiaton yields

$$\frac{d\mathbf{F}}{dt} = \frac{\partial \mathbf{F}}{\partial \mathbf{U}}\frac{d\mathbf{U}}{dt} = \frac{\partial \mathbf{F}}{\partial \mathbf{U}}\mathbf{F} = \mathsf{J}(\mathbf{U})\mathbf{F},$$

which tells us that $\mathbf{u} = \mathbf{F}(\mathbf{U})$ satisfies Eq. (2.5). Thus, one can use the norm of the "velocity vector" to determine the corresponding Lyapunov exponent from Eq. (2.12). As $\|\mathbf{F}\|$ remains bounded from zero for a trajectory that does not converge to a fixed point, we conclude that the corresponding Lyapunov exponent vanishes. This fact has a simple physical interpretation: the zero Lyapunov exponent is measured by perturbing a phase point along its trajectory. Because the system is autonomous, such a perturbation on average neither grows nor decays; it just fluctuates, depending on the local velocity.