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A Method for Outer Interval Solution of Systems of Linear Equations Depending Linearly on Interval Parameters

IWONA SKALNA
Department of Applied Computer Science, University of Science and Technology, ul. Gramatyka 10, 30–067 Cracow (Kraków), Poland, e-mail: skalna@agh.edu.pl

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Abstract. Consider the systems of linear interval equations whose coefficients are linear functions of interval parameters. Such systems, called parametrized systems of linear interval equations, are encountered in many practical problems, e.g in electrical engineering and structure mechanics. A direct method for computing a tight enclosure for the solution set is proposed in this paper. It is proved that for systems with real matrix and interval right-hand vector the method generates the hull of the solution set. For such systems an explicit formula for the hull is also given. Finally some numerical examples are provided to demonstrate the usefulness of the method in structure mechanics.

1. Introduction

A system of linear interval equations

$$\boldsymbol{A}x = \boldsymbol{b} \tag{1.1}$$

with coefficient matrix $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ and right-hand vector $\boldsymbol{b} \in \mathbb{IR}^n$ is defined as a family of linear equations

$$Ax = b, \qquad (A \in \boldsymbol{A},\ b \in \boldsymbol{b}). \tag{1.2}$$

The solution set of (1.1) is given by

$$\sum(\boldsymbol{A}, \boldsymbol{b}) = \{x \mid Ax = b,\ A \in \boldsymbol{A},\ b \in \boldsymbol{b}\}. \tag{1.3}$$

When computing inner and outer bounds for the solution set (1.3) it is implicitly assumed A and b to vary independently within $\boldsymbol{A}$ and $\boldsymbol{b}$. In practice there might be further constraints on matrices within $\boldsymbol{A}$ and $\boldsymbol{b}$. Taking into account these constraints leads to the parametrized systems of linear interval equations. Consider the family of linear algebraic systems of the following type

$$A(p)x = b(p), \tag{1.4}$$

with

$$A_{ij}(p) = \omega(i,j)^{\mathrm{T}} p, \tag{1.5a}$$

$$b_j(p) = \omega(0,j)^{\mathrm{T}} p, \tag{1.5b}$$

and $p \in \boldsymbol{p} \in \mathbb{IR}^k$ [11]. Such systems are encountered in many practical applications, e.g. in electrical engineering [3], [5] and structure mechanics [6], [12].

The family of systems (1.4) is usually written in the form

$$A(\boldsymbol{p})x = b(\boldsymbol{p}), \tag{1.6}$$

and called parametrized system of linear interval equations.

The (united) solution set of the system (1.6) is defined as

$$\sum \big(A(\boldsymbol{p}), b(\boldsymbol{p})\big) = \{x \mid A(p)x = b(p),\ p \in \boldsymbol{p}\}. \tag{1.7}$$

In order to guarantee that the solution set is bounded the matrix $A(\boldsymbol{p})$ must be regular ($A(p)$ must be regular for all $p \in \boldsymbol{p}$). In practice it is usually required that matrix $A(\boldsymbol{p})$ is an H-matrix [7]. H-matrices are defined in Section 2.

Most of the methods for enclosing the solution set of parametrized systems of equations are iterative [1], [2], [8], [11], an exception is a method proposed in [4]. In this paper an alternative, simple and efficient direct method for computing a tight enclosure for (1.7) is proposed. The method is based on the following inclusion

$$\diamondsuit\Big(\sum \big(A(\boldsymbol{p}), b(\boldsymbol{p})\big)\Big) \subseteq \tilde{x} + \langle \boldsymbol{D} \rangle |\boldsymbol{z}| [-1, 1], \tag{1.8}$$

where

$$\boldsymbol{z}_i = \sum_{j=1}^{n} R_{ij} \left(\omega(0,j) - \sum_{\nu=1}^{n} \tilde{x}_\nu \omega(j,\nu) \right)^{\mathrm{T}} \boldsymbol{p}, \tag{1.9}$$

$$\boldsymbol{D}_{ij} = \left(\sum_{\nu=1}^{n} R_{i\nu} \omega(\nu,j) \right)^{\mathrm{T}} \boldsymbol{p}, \tag{1.10}$$

$R = \mathrm{mid}(A(\boldsymbol{p}))^{-1}$, $\tilde{x} = R \cdot \mathrm{mid}(b(\boldsymbol{p}))$, $\langle \boldsymbol{D} \rangle$ is an Ostrowsky matrix related to $\boldsymbol{D}$, and symbol $\diamondsuit$ denotes an interval hull. Ostrowsky matrix and interval hull are introduced in Section 2.

It is proved that for systems with $\mathrm{rad}(A(\boldsymbol{p})) = 0$ the inclusion in (1.8) is an equality. For such systems an explicit formula for the hull of the solution set (1.7) is also given.

Finally some numerical examples of truss structures are provided to demonstrate the usefulness of the proposed direct method in structure mechanics and its superiority over the classical methods.

2. Preliminaries

By $\mathbb{IR}$, $\mathbb{IR}^n$, $\mathbb{IR}^{n\times n}$ denote the set of real compact intervals, respectively interval vectors with n components and the set of interval $n \times n$ matrices. Ordinary letters denote real quantities, and bold face letters stand for interval values.

For interval $\boldsymbol{a} = [\underline{a}, \overline{a}] = \{x \mid \underline{a} \leq x \leq \overline{a}\}$ define the midpoint

$$\check{a} = \mathrm{mid}(\boldsymbol{a}) = (\underline{a} + \overline{a}) / 2,$$

the radius

$$\mathrm{rad}(\boldsymbol{a}) = (\overline{a} - \underline{a}) / 2,$$

maximal absolute value (magnitude)

$$|\boldsymbol{a}| = \mathrm{mag}(\boldsymbol{a}) = \max\{|x| \mid x \in \boldsymbol{a}\},$$

and minimal absolute value (mignitude)

$$\langle \boldsymbol{a} \rangle = \mathrm{mig}(\boldsymbol{a}) = \min\{|x| \mid x \in \mathbf{a}\}.$$

An interval matrix $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ is interpreted as a set of real $n \times n$ matrices

$$\boldsymbol{A} = \{A \in \mathbb{R}^{n\times n} \mid A_{ij} \in \boldsymbol{A}_{ij},\ i,j = 1, \ldots, n\}.$$

An $n \times 1$ matrix is just an interval vector. In analogy to one-dimensional case certain real matrices are related to each interval matrix. Middle matrix mid($\boldsymbol{A}$) and the radius rad($\boldsymbol{A}$) are computed componentwise. For square interval matrices an Ostrowsky matrix [7] $\langle \boldsymbol{A} \rangle$ is defined with entries

$$\begin{aligned} \langle \boldsymbol{A} \rangle_{ii} &= \langle \boldsymbol{A}_{ii} \rangle, \\ \langle \boldsymbol{A} \rangle_{ij} &= -|\boldsymbol{A}_{ij}|, \qquad i \neq j. \end{aligned}$$

A square matrix $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ is called regular if all $A \in \boldsymbol{A}$ are nonsingular. If $\check{A}\boldsymbol{A}$ is regular then $\boldsymbol{A}$ is strongly regular [7].

An interval matrix $\boldsymbol{A}$ is an H-matrix iff there exist a vector $u > 0$ such that

$$\langle \boldsymbol{A} \rangle u > 0.$$

If S is a bounded set of real matrices then $\inf S$ and $\sup S$ exist, and the *hull* of S,

$$\diamondsuit(S) = [\inf S,\ \sup S] = \bigcap\{\boldsymbol{Y} \mid \boldsymbol{Y} \in \mathbb{IR},\ \boldsymbol{Y} \supseteq S\},$$

is the tightest interval matrix enclosing S.

3. Main Result

As it was mentioned before most of the methods for enclosing the solution set of parametized systems of linear equations are iterative. However, in each iteration step overestimation of the “exact” bounds increases because of the directed roundings made while performing subsequent arithmetic operations on interval values. The method based on the formula (1.8) has polynomial complexity and computes the enclosure of the solution set (1.7) in one step, hence has a great advantage over the iterative methods. In what follows the theoretical background for the method is presented.

THEOREM 3.1 (Neumaier [7, Chapter 4, p. 121]). *Let* $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$. *If* $\boldsymbol{A}$ *is an H-matrix then for all* $\boldsymbol{b} \in \mathbb{IR}^n$ *holds*

$$\Diamond\Big(\sum(\boldsymbol{A},\boldsymbol{b})\Big) \subseteq \langle \boldsymbol{A}\rangle^{-1}\boldsymbol{b}[-1,1].$$

THEOREM 3.2. *Let* $A(\boldsymbol{p})x = b(\boldsymbol{p})$ *with* $\boldsymbol{p} \in \mathbb{IR}^k$, $R \in \mathbb{R}^{n\times n}$, *and* $\tilde{x} \in \mathbb{R}^n$. *If* $\boldsymbol{D}$ *given by formula (1.10) is an H-matrix then*

$$\Diamond\Big(\sum\big(A(\boldsymbol{p}),b(\boldsymbol{p})\big)\Big) \subseteq \tilde{x} + \langle \boldsymbol{D}\rangle^{-1}|\boldsymbol{z}|[-1,1], \tag{3.1}$$

where $\boldsymbol{z}$ *is defined by formula (1.9).*

Proof. Vector $x \in \sum(A(\boldsymbol{p}),b(\boldsymbol{p}))$ iff there exists such $p \in \boldsymbol{p}$ that $A(p)x = b(p)$. Since $\boldsymbol{D}$ is an H-matrix then both sides of this equality can be multiplied by $A(p)^{-1}$. Hence

$$\begin{aligned} x = A(p)^{-1}b(p) &= \tilde{x} + A(p)^{-1}\big(b(p) - A(p)\tilde{x}\big) \\ &= \tilde{x} + \big(R\cdot A(p)\big)^{-1}\Big(R\big(b(p) - A(p)\tilde{x}\big)\Big). \end{aligned}$$

Since $R\cdot A(p) \in \boldsymbol{D}$, $R(b(p) - A(p)\tilde{x}) \in \boldsymbol{z}$, then the following relation holds

$$\big(R\cdot A(p)\big)^{-1}\Big(R\big(b(p) - A(p)\tilde{x}\big)\Big) \in \Diamond\Big(\sum(\boldsymbol{D},\boldsymbol{z})\Big),$$

and hence

$$x \in \tilde{x} + \Diamond\Big(\sum(\boldsymbol{D},\boldsymbol{z})\Big). \tag{3.2}$$

Since matrix $\boldsymbol{D}$ is an H-matrix then by Theorem 3.1

$$\Diamond\Big(\sum(\boldsymbol{D},\boldsymbol{z})\Big) \subseteq \langle \boldsymbol{D}\rangle^{-1}|\boldsymbol{z}|[-1,1]. \tag{3.3}$$

Equations (3.2) and (3.3) give the thesis of the theorem. □

It is recommended to choose $R = \mathrm{mid}(A(\boldsymbol{p}))^{-1}$ and $\tilde{x} = R\cdot\mathrm{mid}(b(\boldsymbol{p}))$ so that $\boldsymbol{D}$ and $\boldsymbol{z}$ are of small norms (see [7, Theorem 4.1.10]).

In case of parametrized systems with real matrices the hull of the solution set (1.7) is given by an explicit formula.

THEOREM 3.3. *Let* $A(\boldsymbol{p})x = b(\boldsymbol{p})$, $\boldsymbol{p} \in \mathbb{IR}^k$. *If* $\mathrm{rad}(A(\boldsymbol{p})) = 0$ *then*

$$\Diamond\Big(\sum\big(A(\boldsymbol{p}),b(\boldsymbol{p})\big)\Big) = \tilde{x} + \boldsymbol{z}',$$

where

$$z_i' = \sum_{j=1}^{n} \left(R_{ij} \cdot \omega(0,j)\right)^{\mathrm{T}} [-\mathrm{rad}(\boldsymbol{p}), \mathrm{rad}(\boldsymbol{p})]. \tag{3.4}$$

Proof. Obvious. □

THEOREM 3.4. *Let* $A(\boldsymbol{p})x = b(\boldsymbol{p})$, $\boldsymbol{p} \in \mathbb{IR}^k$. *If* $\mathrm{rad}(A(\boldsymbol{p})) = 0$ *then*

$$\diamondsuit\Big(\sum \left(A(\boldsymbol{p}), b(\boldsymbol{p})\right)\Big) = \tilde{x} + \langle \boldsymbol{D} \rangle^{-1} |\boldsymbol{z}| [-1, 1].$$

where $\boldsymbol{z}$ *and* $\boldsymbol{D}$ *are given respectively by formulas (1.9) and (1.10).*

Proof. To prove the thesis of this theorem it suffices to show that

$$\tilde{x} + \langle \boldsymbol{D} \rangle^{-1} |\boldsymbol{z}| [-1, 1] = \tilde{x} + \boldsymbol{z}',$$

where $\boldsymbol{z}'$ is given by (3.4).

Since $\mathrm{rad}(A(\boldsymbol{p})) = 0$, then $R = A^{-1}$, $\boldsymbol{D} = I$, and

$$z_i = \diamondsuit\{R\left(b(p) - A\tilde{x}\right), \, p \in \boldsymbol{p}\}_i = \diamondsuit\{R\left(b(p) - \check{b}\right), \, p \in \boldsymbol{p}\}_i = z_i'.$$

Hence

$$\tilde{x} + \langle [D] \rangle^{-1} |\boldsymbol{z}| [-1, 1] = \tilde{x} + |\boldsymbol{z}'| [-1, 1]. \tag{3.5}$$

Let now $\alpha = \sum_{j=1}^{n} (R_{ij} \cdot \omega(0,j))$. Then by symmetry of the interval $[-\mathrm{rad}(\boldsymbol{p}), \mathrm{rad}(\boldsymbol{p})]$ holds

$$\begin{aligned}
\{|z|'[-1,1]\}_i &= \left|\sum_{\nu=1}^{n} \alpha_\nu [-\mathrm{rad}(\boldsymbol{p}_\nu), \ \mathrm{rad}(\boldsymbol{p}_\nu)]\right| [-1, 1] \\
&= \left|\left[-\sum_{\nu=1}^{n} \alpha_\nu \mathrm{rad}(\boldsymbol{p}_\nu), \ \sum_{\nu=1}^{n} \alpha_\nu \mathrm{rad}(\boldsymbol{p}_\nu)\right]\right| [-1, 1] \\
&= \left|\sum_{\nu=1}^{n} \alpha_\nu \mathrm{rad}(\boldsymbol{p}_\nu)\right| [-1, 1] \left[-\sum_{\nu=1}^{n} \alpha_\nu \mathrm{rad}(\boldsymbol{p}_\nu), \ \sum_{\nu=1}^{n} \alpha_\nu \mathrm{rad}(\boldsymbol{p}_\nu)\right] \\
&= \sum_{\nu=1}^{n} \alpha_\nu [-\mathrm{rad}(\boldsymbol{p}_\nu), \ \mathrm{rad}(\boldsymbol{p}_\nu)] = z_i'.
\end{aligned}$$

Theorem 3.3 and equation (3.5) give the thesis. □

Table 1 presents an algorithm of the proposed direct method. The algorithm was implemented in Smalltalk (an object-oriented programming language). The computation was performed under Windows 98.

Table 1. Algorithm of the proposed direct method.

Compute $R \approx \mathrm{mid}(A(\boldsymbol{p}))^{-1}$ using favorite algorithm;

$\tilde{x} := R \cdot \mathrm{mid}(b(\boldsymbol{p}))$;

$\boldsymbol{z}_i = \sum_{j=1}^{n} R_{ij}\left(\omega(0,j) - \sum_{v=1}^{n} \tilde{x}_k \omega(j,v)\right)^{\mathrm{T}} \boldsymbol{p}, \quad i = 1, \dots, n;$

$\boldsymbol{D}_{ij} := \left(\sum_{v=1}^{n} R_{iv}\omega(v,j)\right)^{\mathrm{T}} \boldsymbol{p}, \quad i,j = 1, \dots, n;$

Check if $\boldsymbol{D}$ is an H-matrix using favorite algorithm;

If $\boldsymbol{D}$ is an H-matrix Then outer $:= \tilde{x} + \langle \boldsymbol{D} \rangle^{-1} |\boldsymbol{z}| [-1, 1]$;

4. Examples

EXAMPLE 4.1. The 7-bar truss structure shown in Figure 1 is subjected to downward force of 10 kN at node 2. Young's modulus $E = 2.0 \times 10^{11}$ Pa and cross-section area $\sigma = 0.005\ m^2$ for each of the truss members. The lengths of the bars are shown in the figure (unit equals 1 m). Then the stiffness ($s = \frac{E\sigma}{l}$) of bar (2–3) is assumed to be uncertain by $\pm 10\%$. To compute the displacements of the nodes, the system

$$\boldsymbol{K} d = \boldsymbol{q}$$

has to be solved, where $\boldsymbol{K}$ is a stiffness matrix, d is unknown vector of node displacements, and $\boldsymbol{q}$ is vector of forces.

Since the stiffness of one bar is uncertain, then the vector of interval parameters $\boldsymbol{p} = \{1, \boldsymbol{s}_{23}\} = \{1, [636396103.068, 777817459.305]\}$. The stiffness matrix

$$K(\boldsymbol{p}) = \{K_1(\boldsymbol{p})\ K_2(\boldsymbol{p})\},$$

where

$$K_1(\boldsymbol{p}) = \begin{pmatrix} \frac{s_{12}}{2} + s_{13} & -\frac{s_{12}}{2} & -\frac{s_{12}}{2} & -s_{13} \\ -\frac{s_{21}}{2} & \frac{s_{21} + \boldsymbol{s}_{23}}{2} + s_{24} & \frac{s_{21} - \boldsymbol{s}_{23}}{2} & -\frac{\boldsymbol{s}_{23}}{2} \\ -\frac{s_{21}}{2} & \frac{s_{21} - \boldsymbol{s}_{23}}{2} & \frac{s_{21} + \boldsymbol{s}_{23}}{2} & \frac{\boldsymbol{s}_{23}}{2} \\ -s_{31} & -\frac{\boldsymbol{s}_{32}}{2} & \frac{\boldsymbol{s}_{32}}{2} & s_{31} + \frac{\boldsymbol{s}_{32} + s_{34}}{2} + s_{35} \\ 0 & \frac{\boldsymbol{s}_{32}}{2} & -\frac{\boldsymbol{s}_{32}}{2} & \frac{s_{34} - \boldsymbol{s}_{32}}{2} \\ 0 & -s_{42} & 0 & -\frac{s_{43}}{2} \\ 0 & 0 & 0 & -\frac{s_{43}}{2} \end{pmatrix},$$

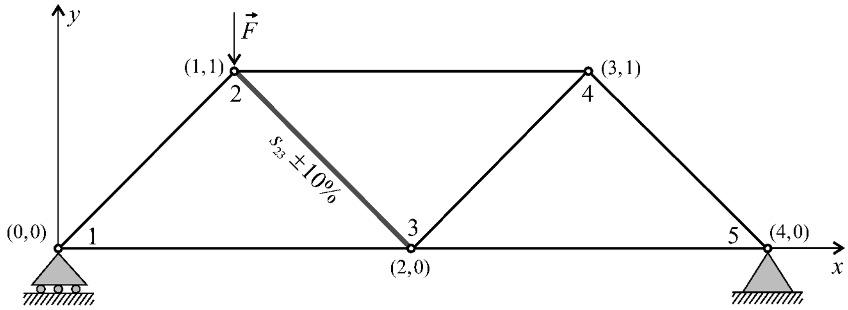

Figure 1. 7-bar plane truss structure.

$$K_2(\boldsymbol{p}) = \begin{pmatrix} 0 & 0 & 0 \\ \frac{\boldsymbol{s}_{23}}{2} & -s_{24} & 0 \\ -\frac{\boldsymbol{s}_{23}}{2} & 0 & 0 \\ \frac{s_{34}-\boldsymbol{s}_{32}}{2} & -\frac{s_{34}}{2} & -\frac{s_{34}}{2} \\ \frac{s_{34}+\boldsymbol{s}_{32}}{2} & -\frac{s_{34}}{2} & -\frac{s_{34}}{2} \\ -\frac{s_{43}}{2} & s_{42}+\frac{s_{43}+s_{45}}{2} & 0 \\ -\frac{s_{43}}{2} & 0 & \frac{s_{43}+s_{45}}{2} \end{pmatrix}.$$

The vector of forces

$$\boldsymbol{q} = (0, 0, -10^4, 0, 0, 0, 0)^{\mathrm{T}}.$$

Then matrix $\omega \in (\mathbb{R}^2)^{8\times 7}$ is of the form

$$\omega = \{\omega_1 \;\; \omega_2\},$$

where

$$\omega_1 = \begin{pmatrix} (0,0) & (0,0) & (0,0) & (-10,0) \\ \left(\frac{s_{12}}{2}+s_{13}, 0\right) & \left(-\frac{s_{12}}{2}, 0\right) & \left(-\frac{s_{12}}{2}, 0\right) & (-s_{13}, 0) \\ \left(-\frac{s_{21}}{2}, 0\right) & \left(\frac{s_{21}}{2}+s_{24}, \frac{1}{2}\right) & \left(\frac{s_{21}}{2}, -\frac{1}{2}\right) & \left(0, -\frac{1}{2}\right) \\ \left(-\frac{s_{21}}{2}, 0\right) & \left(\frac{s_{21}}{2}, -\frac{1}{2}\right) & \left(\frac{s_{21}}{2}, \frac{1}{2}\right) & \left(0, \frac{1}{2}\right) \\ (-s_{31}, 0) & \left(0, -\frac{1}{2}\right) & \left(0, \frac{1}{2}\right) & \left(s_{31}+\frac{s_{34}}{2}+s_{35}, \frac{1}{2}\right) \\ (0,0) & \left(0, \frac{1}{2}\right) & \left(0, -\frac{1}{2}\right) & \left(\frac{s_{34}}{2}, -\frac{1}{2}\right) \\ (0,0) & (-s_{42}, 0) & (0,0) & \left(-\frac{s_{43}}{2}, 0\right) \\ (0,0) & (0,0) & (0,0) & \left(-\frac{s_{43}}{2}, 0\right) \end{pmatrix},$$

Table 2. Example 4.1.

	d_0 [× 10^{-6}]	RSA [× 10^{-6}]	r[∗] / d_0 [%]	DM [× 10^{-6}]	r[∗] / d_0 [%]
d_1^x	−20	[−824.57, −6]	2046.4	−20	0
d_2^x	−2.5	[−425.61, <u>28.57</u>]	9083.6	[−2.7, −2.3]	8
d_2^y	−38.71	[−846.86, −24.65]	1062	[−38.91, −38.52]	0.5
d_3^x	−5	[−507.86, <u>11.76</u>]	5196.2	−5	0
d_3^y	−34.14	[−1064.26, −16.22]	1534.9	[−34.53, −33.75]	1.2
d_4^x	−12.5	[−636.76, −1.64]	2540.5	[−12.7, −12.3]	1.6
d_4^y	−19.57	[−786.06, −6.24]	1992.4	[−19.77, −19.37]	1

$$\omega_2 = \begin{pmatrix} (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) \\ \left(0,\frac{1}{2}\right) & (-s_{24},0) & (0,0) \\ \left(0,-\frac{1}{2}\right) & (0,0) & (0,0) \\ \left(\frac{s_{34}}{2},-\frac{1}{2}\right) & \left(-\frac{s_{34}}{2},0\right) & \left(-\frac{s_{34}}{2},0\right) \\ \left(\frac{s_{34}}{2},\frac{1}{2}\right) & \left(-\frac{s_{34}}{2},0\right) & \left(-\frac{s_{34}}{2},0\right) \\ \left(-\frac{s_{43}}{2},0\right) & \left(s_{42}+\frac{s_{43}+s_{45}}{2},0\right) & (0,0) \\ \left(-\frac{s_{43}}{2},0\right) & (0,0) & \left(\frac{s_{43}+s_{45}}{2},0\right) \end{pmatrix}.$$

The results produced by direct method and Rohn-Signed Algorithm (RSA) [9], [10] are presented in Table 2. Rohn-Signed Algorithm gives the tightest interval enclosure for (1.3) that can be obtained using the classical methods.

EXAMPLE 4.2 Baltimore bridge (1870).

For the plane truss structure (all bars, loads and displacements are in the same x-y plane) shown in Figure 2 subjected to downward forces of 80 *kN* at node 11, 120 *kN* at node 12 and 80 *kN* at node 15, the displacements of the nodes are computed. Young's modulus E = 2.1×10^{11} *Pa* and cross-section area $\sigma = 0.004$ m^2. The lengths of the bar elements are shown in the figure (unit equals 1 *m*). Assume the stiffness of some of the bar elements (denoted in the figure with thick lines) to be uncertain by ±5%. To compute the displacement, the parametrized system of linear interval equations must be solved. The results are in Tables 3 and 4.

EXAMPLE 4.3. Plane truss with uncertain stiffness of 8 bar elements.

The plane truss shown in Figure 3 is subjected to downward forces of 30 *kN* at nodes 2, 3, and 4. All bar elements have the same Young's modulus E = $7 \cdot 10^{10}$ *Pa* and cross-section area $\sigma = 0.003$ m^2. Assume the stiffness of 8 bar elements to

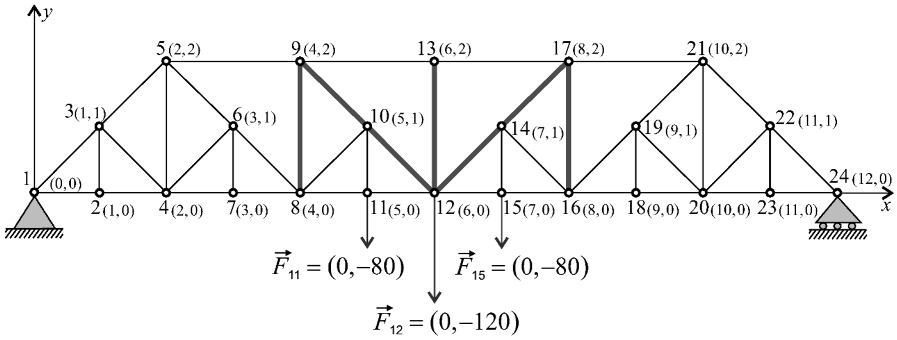

Figure 2. Scheme of the Baltimore bridge.

Table 3. Example 4.2 (x-coords).

	d_0 [$\times 10^{-5}$]	RSPI [$\times 10^{-5}$]	r[∗] / d_0 [%]	DM [$\times 10^{-5}$]	r[∗] / d_0 [%]
d_2^x	16.67	16.67	0	16.67	0
d_3^x	190.24	[189.72, 190.76]	0.3	[189.34, 191.13]	0.5
d_4^x	33.33	33.33	0	33.33	0
d_5^x	300	[298.96, 301.05]	0.3	[298.21, 301.79]	0.6
d_6^x	190.24	[189.72, 190.76]	0.3	[189.34, 191.13]	0.5
d_7^x	50	50	0	50	0
d_8^x	66.67	66.67	0	66.67	0
d_9^x	233.33	[232.29, 234.38]	0.4	[231.54, 235.12]	0.8
d_{10}^x	172.01	[170.6, 173.38]	0.8	[169.84, 174.17]	1.3
d_{11}^x	104.76	104.76	0	104.76	0
d_{12}^x	142.86	142.86	0	142.86	0
d_{13}^x	142.86	[141.81, 143.9]	0.7	[141.07, 144.65]	1.3
d_{14}^x	113.71	[112.38, 114.93]	1.1	[111.54, 115.88]	1.9
d_{15}^x	180.95	180.95	0	180.95	0
d_{16}^x	219.05	219.05	0	219.05	0
d_{17}^x	52.38	[51.34, 53.43]	2	[50.59, 54.17]	3.4
d_{18}^x	235.71	235.71	0	235.71	0
d_{19}^x	95.48	[94.96, 96]	0.5	[94.58, 96.37]	0.9
d_{20}^x	252.38	252.38	0	252.38	0
d_{21}^x	−14.29	[−15.33, −13.24]	7.3	[−16.08, −12.5]	12.5
d_{22}^x	95.48	[94.96, 96]	0.5	[94.58, 96.37]	0.9
d_{23}^x	269.05	269.05	0	269.05	0
d_{24}^x	285.71	285.71	0	285.71	0

be uncertain by $\pm 5\%$. The resulting intervals vectors are presented is Tables 5 and 6.

Table 4. Example 4.2 (y-coords).

	d_0 [$\times 10^{-5}$]	RSPI [$\times 10^{-5}$]	r[$*$] / d_0 [%]	DM [$\times 10^{-5}$]	r[$*$] / d_0 [%]
d_2^y	−237.38	[−237.9, −236.86]	0.2	[−238.27, −236.48]	0.4
d_3^y	−237.38	[−237.9, −236.86]	0.2	[−238.27, −236.48]	0.4
d_4^y	−394.28	[−395.33, −393.24]	0.3	[−396.07, −392.49]	0.5
d_5^y	−394.28	[−395.33, −393.24]	0.3	[−396.07, −392.49]	0.5
d_6^y	−551.18	[−552.75, −549.62]	0.3	[−553.87, −548.5]	0.5
d_7^y	−551.18	[−552.75, −549.62]	0.3	[−553.87, −548.5]	0.5
d_8^y	−721.9	[−723.99, −719.81]	0.3	[−725.47, −718.32]	0.5
d_9^y	−745.7	[−747.96, −743.69]	0.3	[−749.81, −741.6]	0.5
d_{10}^y	−840.7	[−842.68, −838.87]	0.2	[−843.9, −837.5]	0.4
d_{11}^y	−850.23	[−852.2, −848.39]	0.2	[−853.43, −847.03]	0.4
d_{12}^y	−890.06	[−893.29, −887.27]	0.3	[−895.42, −884.69]	0.6
d_{13}^y	−890.06	[−893.29, −887.27]	0.3	[−895.99, −884.12]	0.7
d_{14}^y	−840.7	[−842.64, −838.75]	0.2	[−843.9, −837.5]	0.4
d_{15}^y	−850.23	[−852.17, −848.27]	0.2	[−853.43, −847.03]	0.4
d_{16}^y	−721.9	[−723.98, −719.8]	0.3	[−725.47, −718.32]	0.5
d_{17}^y	−745.7	[−748.02, −743.3]	0.3	[−749.81, −741.6]	0.6
d_{18}^y	−551.18	[−552.75, −549.61]	0.3	[−553.87, −548.5]	0.5
d_{19}^y	−551.18	[−552.75, −549.61]	0.3	[−553.87, −548.5]	0.5
d_{20}^y	−394.28	[−395.32, −393.23]	0.3	[−396.07, −392.49]	0.5
d_{21}^y	−394.28	[−395.32, −393.23]	0.3	[−396.07, −392.49]	0.5
d_{22}^y	−237.38	[−237.9, −236.85]	0.2	[−238.27, −236.48]	0.4
d_{23}^y	−237.38	[−237.9, −236.85]	0.2	[−238.27, −236.48]	0.4

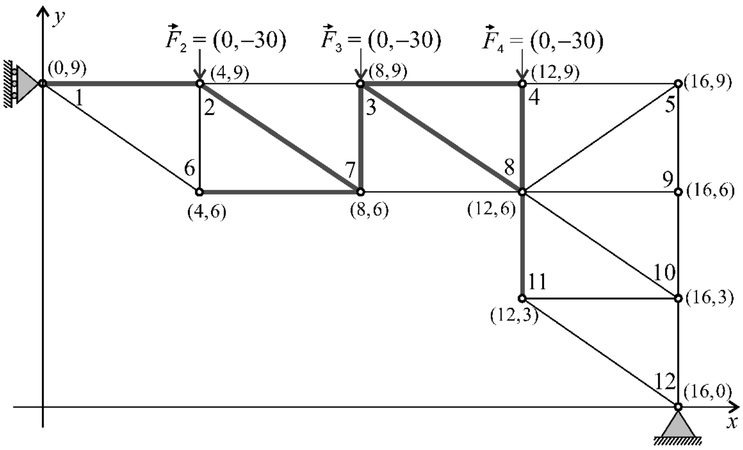

Figure 3. Truss structure with uncertain stiffness of 8 bar elements.

Table 5. Example 4.3 (x-coords).

	d_0 [$\times 10^{-5}$]	RSPI [$\times 10^{-5}$]	r[∗] / d_0 [%]	DM [$\times 10^{-5}$]	r[∗] / d_0 [%]
d_2^x	−152.38	[−160.4, −145.13]	5 5	[−160.4, −144.36]	5.3
d_3^x	−228.57	[−236.59, −221.32]	3.3	[−236.59, −220.55]	3.5
d_4^x	−152.38	[−163.76, −141.34]	7.4	[−165.26, −139.51]	8.5
d_5^x	−76.19	[−87.56, −65.15]	14.7	[−89.06, −63.32]	16.9
d_6^x	427.38	[419, 435.56]	1.9	[416.48, 438.28]	2.5
d_7^x	427.38	[419, 435.56]	1.9	[417.52, 437.24]	2.3
d_8^x	351.19	[342.81, 359.37]	2.4	[341.33, 361.05]	2.8
d_9^x	351.19	[342.81, 359.37]	2.4	[341.33, 361.05]	2.8
d_{10}^x	267.86	[262.19, 273.57]	2.1	[261, 274.7]	2.5
d_{11}^x	115.48	[109.81, 121.19]	4.9	[108.63, 122.32]	5.9

Table 6. Example 4.3 (y-coords).

	d_0 [$\times 10^{-4}$]	RSPI [$\times 10^{-4}$]	r[∗] / d_0 [%]	DM [$\times 10^{-4}$]	r[∗] / d_0 [%]
d_1^y	−308.25	[−312.45, −304.18]	1.3	[−315.09, −301.41]	2.2
d_2^y	−251.27	[−255.54, −247.22]	1.6	[−258.08, −244.46]	2.7
d_3^y	−149.84	[−152.8, −147.16]	1.9	[−153.43, −146.25]	2.4
d_4^y	−37.14	[−37.97, −36.34]	2.2	[−38.19, −36.1]	2.8
d_5^y	4.29	4.29	0	4.29	0
d_6^y	−251.27	[−255.53, −247.22]	1.7	[−258.08, −244.46]	2.7
d_7^y	−154.13	[−157.19, −151.38]	1.9	[−158.32, −149.93]	2.7
d_8^y	−32.86	[−33.48, −32.24]	1.9	[−33.6, −32.11]	2.3
d_9^y	0	0	0	0	0
d_{10}^y	−4.29	−4.29	0	−4.29	0
d_{11}^y	24.29	[−25.04, −23.52]	3.1	[−25.2, −23.37]	3.8

EXAMPLE 4.4. Plane truss with uncertain displacements of the supports.

The plane truss shown in Figure 4 has two supports: partial (sliding) support along y axis at node 1 and full support at node 12. Allow the movements of the supports at node 1 by $\Delta_{1-2} = 0.2\ m$ along y axis, at node 12 by $\Delta_{11-12} = 0.3\ m$ along x axis and by $\Delta_{10-12} = 0.4\ m$ along y axis. Now assume all movements to be uncertain by ±5%. This cause interval parameters appear only in the right-hand vector. In this case the direct method should give the hull. The results in Tables 7 and 8 prove this to be true.

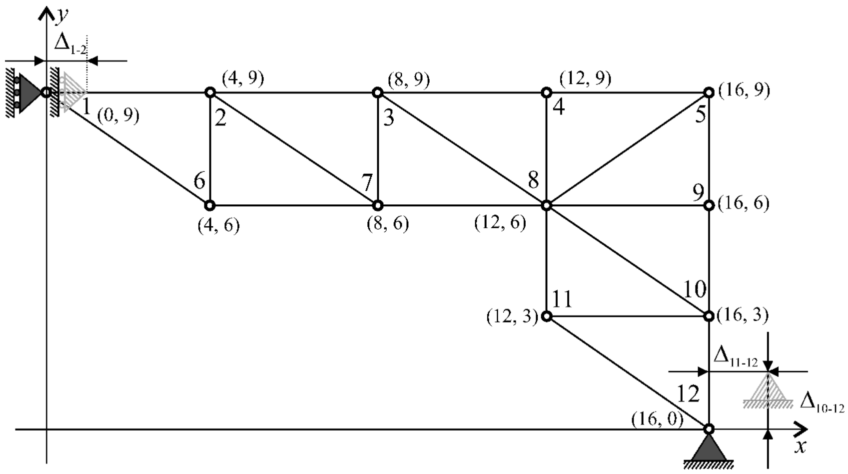

Figure 4. Truss structure with uncertain displacements of the supports.

5. Results

The results produced by the direct method described in Section 3 are presented in Tables 2–8. Column 2 contains the solution of non-interval system, column 5 contains the results of the proposed direct method (DM), columns 4 and 6 contain the relative error of the resulting intervals (in percent). Column 3 of Table 2 contains the results produced by Rohn-Signed Algorithm (RSA). Same column of remaining tables contains the inner estimation obtained using the method of random sampling of parameter intervals (RSPI) which is a modification of the method of sampling of parameter intervals (SPI) described in [6]. The endpoints of the resulting intervals with sign different then corresponding solution of non-interval system are indicated by underlining.

6. Conclusions

The problem of solving the parametrized systems of linear interval equations is very important in practical applications. Well known classical methods, such as interval version of Gauss Elimination, Preconditioned Interval Gauss-Seidel iteration or Rohn-Signed Algorithm fail since they compute enclosure for the solution set (1.3) which is generally much larger then solution set (1.7). A direct method for solving paratmerized systems of linear interval equations based on the inclusion (1.8) was proposed and checked to be useful in structure mechanics. The method produced tight enclosure for the solutions set of parametrized systems for all exemplary truss structures.

Table 7. Example 4.4 (x-coords).

	d_0 [$\times 10^{-3}$]	DM (hull) [$\times 10^{-3}$]	r[$*$] / d_0 [%]
d_2^x	20	$[19, 21]$	5
d_3^x	20	$[19, 21]$	5
d_4^x	20	$[19, 21]$	5
d_5^x	20	$[19, 21]$	5
d_6^x	13.33	$[11.67, 15]$	12.5
d_7^x	13.33	$[11.67, 15]$	12.5
d_8^x	13.33	$[11.67, 15]$	12.5
d_9^x	13.33	$[11.67, 15]$	12.5
d_{10}^x	6.67	$[4.33, 9]$	35
d_{11}^x	6.67	$[4.33, 9]$	35

Table 8. Example 4.4 (y-coords).

	d_0 [$\times 10^{-3}$]	DM (hull) [$\times 10^{-3}$]	r[$*$] / d_0 [%]
d_1^y	35.56	$[28.44, 42.67]$	20
d_2^y	26.67	$[21.33, 32]$	20
d_3^y	17.78	$[14.22, 21.33]$	20
d_4^y	8.89	$[7.11, 10.67]$	20
d_5^y	0	0	–
d_6^y	26.67	$[21.33, 32]$	20
d_7^y	17.78	$[14.22, 21.33]$	20
d_8^y	8.89	$[7.11, 10.67]$	20
d_9^y	0	0	–
d_{10}^y	0	0	–
d_{11}^y	8.89	$[7.11, 10.67]$	20

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