A small note on a certain differential

```julia
using Manifolds, Manopt, Plots, Random
```

```
TaskLocalRNG()
```

Let $\mathcal{M}$ be a manifold, $p, q \in \mathcal{M}$ and $t \in \mathbb{R}$ be given.

We consider the function $f\colon T_p\mathcal{M} \to \mathbb{R}$

$$f(X) = \frac{1}{2} d^2_{\mathcal{M}}(q, \exp_p(tX)), \qquad X \in T_p\mathcal{M}$$

Hence the differential is a map from $T_p\mathcal{M}$ to $\mathbb{R}$ since both domain and codomain are their own tangent spaces.

If we use this within minimisation, we can even specify the minmiser $X^* = \frac{1}{t}\log_p q$ aince then the distance is zero.

We decompose f into the functions

$$\begin{aligned}
g(s) &= \frac{1}{2} d_{\mathcal{M}}(q, s), & D_s g(s)[X] &= \langle -\log_s q, X\rangle_s, \quad X \in T_s\mathcal{M}\\
h(Y) &= \exp_p(Y), & D_Y h(Y)[Z] &= D_Y \exp_p(Y)[Z]\\
i(X) &= tX, & D_X i(X)[Y] &= tY.
\end{aligned}$$

And we use the Chain rule on manifolds, i.e. for f, g the concatenation $f(g(p)) = (f \circ g)(p)$ then the chain rule reads $D(f \circ g)(p)[X] = Df(g(p))\bigl[Dg(p)[X]\bigr]$ (cf. AMS08 p. 195).

For our case this reads $f(X) = g(h(i(X)))$ so that the differential reads

$$Df(X)[Y] = D\,g\big(h(i(X))\big)\Big[Dh(i(X))\big[Di(X)[Y]\big]\Big]$$

where

- $Di(X)[Y] = tY$
- $h(i(X)) = \exp_p(tX)$
- hence $Dh(i(X))\big[Di(X)[Y]\big] = D_e xp_p(tX)[tY]$

so that we finally have with $s = \exp_p tX$ that

$$\begin{aligned} Df(X)[Y] &= D\frac{1}{2}d^2_{\mathcal{M}}\big(q, \exp_p(tX)\big)[Y] = D\frac{1}{2}d^2_{\mathcal{M}}(q, exp_p tX)\big[D_e xp_p(tX)[tY]\big] \\ &= \langle -\log_{\exp_p(tX)} q, D\exp_p(tX)[tY]\rangle_{\exp_p(tX)} \end{aligned}$$

Let's check this in code

```
M = Sphere(2, ℝ)
```

```
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```

```
p =  [0.222164, -0.799887, -0.557516]
```

```
p = random_point(M)
```

```
q =  [-0.321372, -0.206102, 0.924252]
```

```
t = 0.3
```

```
t = 0.3
```

```
Xstar =  [-1.67917, -4.00888, 5.08253]
```

```
Xstar = 1/t * log(M, p, q)
```

```
f (generic function with 1 method)
```

```
f(X) = 0.5 * distance(M, q, exp(M, p, t*X)).^2
```

```
0.0
```

```
f(Xstar)
```

```
Df (generic function with 1 method)
```

```julia
Df(X, Y) = inner(M,
    exp(M, p, t*X),
    -log(M, exp(M, p, t*X), q),
    differential_exp_argument(M, p, t*X, t*Y),
)
```

the following ready checks, that in X^* the differential is zero

```
1.4426566739515313e-16
```

```julia
Df(Xstar, random_tangent(M, p))
```

```
Enter cell code...
```