Clamped-Free Bar Model for ARCH-COMP 2021 AFF category

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1 Model description

We consider a uni-dimensional wave propagation problem from [1]. The problem is governed by the partial differential equation

$$EA\frac{\partial^2 u}{\partial x^2}(x,t) - \rho A \frac{\partial^2 u}{\partial t^2}(x,t) = 0, \tag{1}$$

where $u(x,t)$ is the displacement of the point in position x at time t, considering the axis shown in Figure 1. The model consists of a bar of length $L = 200$ and cross-section area $A = 1$. The bar is formed by a linear elastic material with Young modulus $E = 30 \times 10^6$ and density $\rho = 7.3 \times 10^{-4}$.

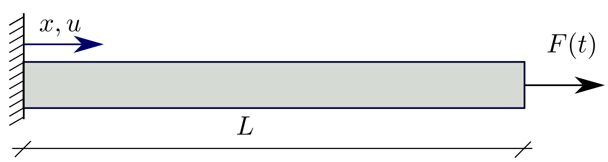

Figure 1: Example 2: diagram of the clamped-free bar excited by end load

The bar is considered to be initially at rest, with $u(x,0) = 0$ and $\frac{\partial u}{\partial t}(x,0) = 0$ for all $x \in [0, L]$. The boundary conditions are $u(0,t) = 0$, corresponding to the fixed end, and $\sigma(L)A = F(t)$, for the free end, where $\sigma(x,t) = E\frac{\partial u}{\partial x}(x,t)$. The free end is submitted to a step force $F(t) = 10000H(t)$, where $H(t)$ is the heaviside function.

The analytical solution of this problem in the continuum can be obtained using mode superposition [2] and it is given by:

$$u(x,t) = \frac{8FL}{\pi^2 EA}\sum_{s=1}^{\infty}\left\{\frac{(-1)^{s-1}}{(2s-1)^2}\sin\frac{(2s-1)\pi x}{2L}\left(1 - \cos\frac{(2s-1)\pi\mu t}{2L}\right)\right\}, \tag{2}$$

where $\mu = \sqrt{E/\rho}$.

Considering N two-node finite elements with linear interpolation, the following system of linear ODEs is obtained:

$$M\frac{\partial^2 u(t)}{\partial t^2} + Ku(t) = F(t), \qquad t \in [0, T], \tag{3}$$

where the $N \times N$ stiffness and mass matrices are respectively:

$$K = \frac{EA}{\ell}\begin{bmatrix} 2 & -1 & & & & 0 \\ -1 & 2 & -1 & & & \\ & -1 & 2 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 2 & -1 \\ 0 & & & & -1 & 1 \end{bmatrix}, \quad M = \frac{\rho A \ell}{2}\begin{bmatrix} 2 & & & & & 0 \\ & 2 & & & & \\ & & 2 & & & \\ & & & \ddots & & \\ & & & & 2 & \\ 0 & & & & & 1 \end{bmatrix},$$

where $\ell = L/N$ is the length of each element.

2 Transformation to first order

Since M is invertible, Eq. (3) can be rewritten as a system of first order ODEs. We introduce the auxiliary variables $x = [u, u']$, then multiply the equation $Mu''(t) + Ku(t) = F(t)$ by M^{-1}:

$$u'' = -M^{-1}Ku(t) + M^{-1}F(t) \tag{4}$$

The resulting system is:

$$x'(t) = \frac{d}{dt}\begin{bmatrix} u \\ u' \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & 0 \end{bmatrix}\begin{bmatrix} u \\ u' \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}F(t) \end{bmatrix} = Ax(t) + f(t) \tag{5}$$

The matrices A and b are provided in MAT format for problem sizes $N = 100$, $N = 500$ and $N = 1000$.

3 Reachability settings

There are two versions of this benchmark:

- CB21C: constant input $F(t)$.
- CB21F: the inputs $F(t)$ can change arbitrarily over time.

The bar is considered to start at rest: $x_i = 0$ for all $i = 1, \ldots, 2N$. In both CB21C and CB21F, the forcing term is $F(t) \in [9900, 10100]$. Three scenarios are considered with an increasing number of dimensions: $N = 100$, $N = 500$, $N = 1000$.

4 Requirements

The tools should report the computation time for the time horizon $T = 0.01s$. Discrete-time analysis should use a step size 9.88×10^{-7}. The accuracy of the results is measured by reporting the maximum value of the position (resp. velocity) at nodes 70, 350 and 700 for the scenarios with $N = 100$, $N = 500$ and $N = 1000$ respectively.

Tools should plot the velocity at node 700 as a function of time for the $N = 1000$ scenario (or the corresponding node for a smaller instance) for the time interval $[8.15 \times 10^{-3}, 8.40 \times 10^{-3}]$.

5 Damped case

To model a realistic beam, we consider adding a damping matrix to the system.

$$C = aK + bM, \tag{6}$$

with $a = b = 10^{-6}$. The system matrix then becomes

$$x'(t) = \frac{d}{dt}\begin{bmatrix} u \\ u' \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}\begin{bmatrix} u \\ u' \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}F(t) \end{bmatrix} = Ax(t) + f(t). \tag{7}$$

References

[1] Mohammad Mahdi Malakiyeh, Saeed Shojaee, and Klaus-Jürgen Bathe. The bathe time integration method revisited for prescribing desired numerical dissipation. *Computers & Structures*, 212:289–298, 2019.

[2] Michel Géradin and Daniel J Rixen. *Mechanical vibrations: theory and application to structural dynamics*. John Wiley & Sons, 2014.