Computing Reachable Sets of Semi-Discrete Solid Dynamics Equations with ReachabilityAnalysis.jl

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From continuum to semi-discrete equations. In the context of linear solid dynamics, the spatial discretization of the governing partial differential equations (PDEs) using the Finite-Element Method (FEM) [3], results in

$$\mathbf{M}\mathbf{x}''(t) + \mathbf{C}\mathbf{x}'(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{F}(t), \qquad (1)$$

a system of second-order differential equations in time, where $\mathbf{x}(0) \in \mathcal{X}_0$ and $\mathbf{x}'(0) \in \mathcal{V}_0$ are the sets of initial displacements and velocities, $\mathbf{x} \in \mathbb{R}^n$ is the state vector, and $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices, respectively. Depending on the problem and the mesh size, n is typically between 10^2 and 10^5.

Representing solutions with sets. In [7] we present a novel approach for time integration of heat transfer and structural dynamics equations based on reachability analysis techniques [1]. These methods are implemented in ReachabilityAnalysis.jl, a core package of JuliaReach, while FEM matrices assembly is done using the Octave tool ONSAS [5]. Reachability is a modern computational approach where differential equations solutions are represented using sets. The set-based conservative time integration of Eq. (1) returns solution sets (*flowpipes*) that include all exact trajectories, with convergence to the true reachable states as the time step decreases.

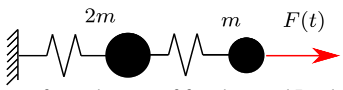

Fig. 1: Diagram of two degrees of freedom and Rayleigh damping.

Minimal example. We solve the system in Fig. 1 loaded with a Heaviside step function. Given the FEM assembled matrices (Line 3), the range of variation of the external loads (Line 4) is 10% around the nominal value 1. Initial displacements and velocities for both masses belong to the interval $[-0.5, 0.5]$ (Line 5). The initial-value problem is instantiated and homogenized as described in [7].

```
1 using ReachabilityAnalysis
2 m = 0.25; k = 2.0
3 M = [2m 0; 0 m]; K = [2k -k; -k k]; C = (M+K)/20
4 F = [0.0, 1.0]; △F0 = Interval(0.9, 1.1)
5 U0 = BallInf(zeros(4), 0.5)
6 sys = SecondOrderLinearContinuousSystem(M, C, K, F)
7 prob = InitialValueProblem(homogenize(sys), U0 × △F0)
8 solA = solve(prob, 50, LGG09(δ=5e-2, dirs=:box, dim=5))
9 solB = solve(prob, 50, LGG09(δ=5e-2, dirs=:oct, dim=5))
```

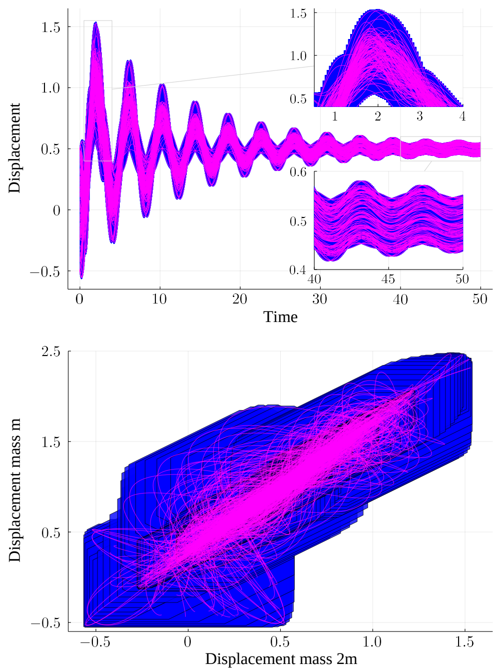

Fig. 2: Flowpipe using canonical directions projected on time (top), and using octagonal directions at displacement coordinates (bottom). We additionally plot random simulations.

Solution method. To illustrate the flexibility of our approach, two algorithm choices are considered, both relying on support functions [10] (`LGG09` algorithm in Lines 8-9). `solA` contains the flowpipe efficiently computed along box directions $\pm e_1 = [\pm 1, 0, 0, 0]^T$, while `solB` contains the projection of the flowpipe for displacement coordinates of each mass. To improve the accuracy, the latter method uses octagonal template directions.

Perspectives. We envision to model variations in mass and stiffness parameters using interval methods [6, 8]. Probabilistic reachability, and modeling nonlinear behaviors using state-space abstraction methods, are also planned. Julia has a thriving ecosystem of open source FEM projects [2, 4, 9], and we think that integrating set propagation into such tools is the next key step for solving real world problems using reachability.

References

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