Computing Reachable Sets of Semi-Discrete Solid Dynamics Equations with ReachabilityAnalysis.jl

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From continuum to semi-discrete equations. In the context of linear solid dynamics, the spatial discretization of the governing partial differential equations (PDEs) using the Finite-Element Method (FEM) [3], results in

$$\mathbf{M}\mathbf{x}''(t) + \mathbf{C}\mathbf{x}'(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{F}(t), \quad (1)$$

a system of second-order differential equations in time, where $\mathbf{x}(0) \in \mathcal{X}_0$ and $\mathbf{x}'(0) \in \mathcal{V}_0$ are the sets of initial displacements and velocities, $\mathbf{x} \in \mathbb{R}^n$ is the state vector, and $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices, respectively. Depending on the problem and the mesh size, n is typically between 10^2 and 10^5.

Representing solutions with sets. In [7] we present a novel approach for time integration of solid heat transfer and structural dynamics equations based on reachability analysis techniques [1]. These methods are implemented in ReachabilityAnalysis.jl, a core package of JuliaReach, while FEM assembly is done using the Octave tool ONSAS [5]. Reachability is a modern computational approach where solutions to differential equations are represented using sets. The set-based conservative time integration of Eq. (1) returns solution sets (*flowpipes*) that include all exact trajectories, with convergence to the true reachable states as the time step decreases.

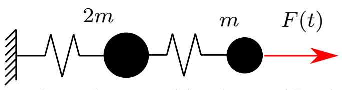

Fig. 1: Diagram of two degrees of freedom and Rayleigh damping.

Minimal example. We solve the system in Fig. 1 loaded with a Heaviside step function. Given the FEM assembled matrices (Line 3), the range of variation of the external loads (Line 4) is 10% around the nominal value 1. Initial displacements and velocities for both masses belong to the interval $[-0.5, 0.5]$ (Line 5). The initial-value problem is instantiated and homogenized as described in [7].

```
1 using ReachabilityAnalysis
2 m = 0.25; k = 2.0
3 M = [2m 0; 0 m]; K = [2k -k; -k k]; C = (M+K)/20
4 F = [0.0, 1.0]; ΔF0 = Interval(0.9, 1.1)
5 U0 = BallInf(zeros(4), 0.5)
6 sys = SecondOrderLinearContinuousSystem(M, C, K, F)
7 prob = InitialValueProblem(homogenize(sys), U0 × ΔF0)
8 solA = solve(prob, 50, LGG09(δ=5e-2, dirs=:box, dim=5))
9 solB = solve(prob, 50, LGG09(δ=5e-2, dirs=:oct, dim=5))
```

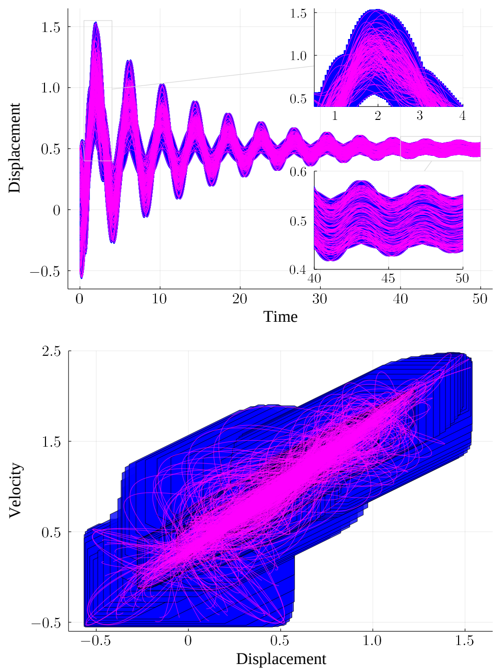

Fig. 2: Flowpipe using canonical directions projected on time (top), and using octagonal directions at node 1 (bottom). We additionally plot random simulations.

Solution method. To illustrate the flexibility of our approach, two algorithm choices are considered, both relying on support functions [10] (`LGG09` algorithm in Lines 8-9). `solA` contains the flowpipe efficiently computed along box directions $\pm e_1 = [\pm 1, 0, 0, 0]^T$, while `solB` contains the projection of the flowpipe for node 1 coordinates. To improve the accuracy, the latter method uses octagonal template directions.

Perspectives. We envision to model variations in mass and stiffness parameters using interval methods [6, 8]. Probabilistic reachability, and modeling nonlinear behaviors using state-space abstraction methods, are also planned. Julia has a thriving ecosystem of open source FEM projects [2, 4, 9], and we think that integrating set propagation into some of those tools is the next key step for solving real world problems using reachability.

References

[1] Matthias Althoff, Goran Frehse, and Antoine Girard. Set propagation techniques for reachability analysis. *Annual Review of Control, Robotics, and Autonomous Systems*, 4, 2020.

[2] Santiago Badia and Francesc Verdugo. Gridap: An extensible finite element toolbox in julia. *Journal of Open Source Software*, 5(52):2520, 2020. doi:10.21105/joss.02520.

[3] Klaus-Jürgen Bathe. *Finite Element Procedures*. Watertown, USA, 2 edition, 2014.

[4] Kristoffer Carlsson, Fredrik Ekre, and Contributors. Ferrite.jl, 3 2021.

[5] Jorge M. Pérez Zerpa et al. Open Nonlinear Structural Analysis Solver ONSAS. `https://github.com/ONSAS/ONSAS.m/`, 2021.

[6] Luca Feranti, Marcelo Forets, and David P. Sanders. IntervalLinearAlgebra.jl: linear algebra done rigorously, 9 2021. doi:10.5281/zenodo.5363563.

[7] Marcelo Forets, Daniel Freire Caporale, and Jorge M Pérez Zerpa. Combining set propagation with finite element methods for time integration in transient solid mechanics problems. *arXiv preprint arXiv:2105.05841. Accepted in Computers & Structures*, 2021.

[8] Marcelo Forets, Christian Schilling, and Luca Feranti. Intervalmatrices.jl: Matrices with interval coefficients in julia, 9 2021. doi:10.5281/zenodo.5516249.

[9] Petr Krysl and Contributors. FinEtools.jl, 10 2021.

[10] Colas Le Guernic and Antoine Girard. Reachability analysis of linear systems using support functions. *Nonlinear Analysis: Hybrid Systems*, 4(2):250 – 262, 2010. IFAC World Congress 2008.