Computing Reachable Sets of Semi-Discrete Solid Dynamics Equations with ReachabilityAnalysis.jl

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The Set-Based Approach. Many real-world problems require the resolution of ODEs with uncertainties in initial conditions or in applied loads. Obtaining solutions considering these uncertainties is a challenging task, particularly in large scale systems. The set-based approach consists in the construction of sets that contain all the feasible solutions of the ODEs [1].

Solid Dynamics ODEs. In problems such as wave propagation or structural vibrations, solid dynamics problems are formulated. In these cases, large systems of ODEs of the form:

$$\mathbf{M}\mathbf{x}''(t) + \mathbf{C}\mathbf{x}'(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{F}(t), \tag{1}$$

are formulated using the Finite-Element Method (FEM) [3], where $\mathbf{x} \in \mathbb{R}^n$ is the displacements (or state) vector, and $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices, respectively. Depending on the problem, n is typically between 10^2 and 10^5.

Set-Based Solid Dynamics. When uncertainty is present, the initial displacements $\mathbf{x}(0)$ and the initial velocities velocities $\mathbf{x}'(0)$ can be considered as part of corresponding feasible states sets $\mathcal{X}_0$ and $\mathcal{V}_0$, respectively. In [6] a novel approach for time integration of solid dynamics equations based on set-based techniques was presented. The approach allows to compute, in a single integration, the solution sets (or *flowpipes*) that include all exact trajectories under uncertainties in the initial conditions and applied loads. Such solution sets cannot be obtained using standard numerical integrators, since they are designed to propagate initial points, not sets.

Set-Based FEM Implementation and Application. We have extended the package ReachabilityAnalysis.jl[7] to support the set-based approach for solid dynamics. Moreover, through a simple interface, it can be integrated with FEM tools such as ONSAS [10]. The implementations developed can be used to solve large systems, however, a minimal problem was chosen to illustrate its use. The spring-mass system shown in Fig. 1, loaded with a Heaviside step function with uncertainties in load and initial conditions, can be easily modeled and solved using the code shown below. The resulting solutions sets are shown in Fig. 2 together with random simulations results.

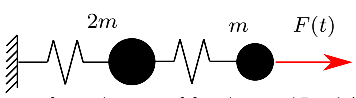

Fig. 1: Diagram of two degrees of freedom and Rayleigh damping.

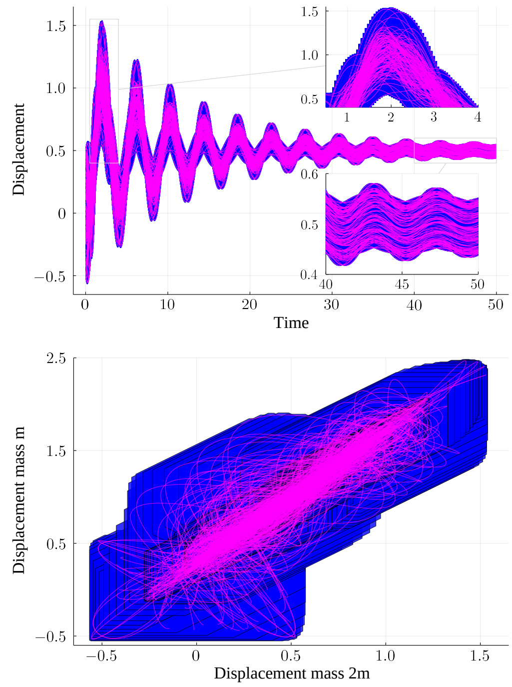

Fig. 2: Solution sets of mass $2m$ vs time (top) and displacement of both masses (bottom).

```julia
using ReachabilityAnalysis
m = 0.25; k = 2.0
M = [2m 0; 0 m]; K = [2k -k; -k k]; C = (M+K)/20
F = [0.0, 1.0]; ΔF0 = Interval(0.9, 1.1)
U0 = BallInf(zeros(4), 0.5)
sys = SecondOrderLinearContinuousSystem(M, C, K, F)
prob = InitialValueProblem(homogenize(sys), U0 × ΔF0)
solA = solve(prob, 50, LGG09(δ=5e-2, dirs=:box, dim=5))
solB = solve(prob, 50, LGG09(δ=5e-2, dirs=:oct, dim=5))
```

Perspectives. We envision modeling uncertainties in density or stiffness using interval methods [5, 8], as well as integrating our work with Julia's FEM projects [2, 4, 9].

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