Average reflection from a random particulate material

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Abstract

Does a halfspace filled with randomly placed cylinders behave, on average, like a homogeneous halfspace? To answer this, we compare the reflection from a homogeneous halfspace with the average reflection from a halfspace filled with cylinders. In the end we reach an absurd result for cylinders with Dirichlet boundary condition. An explanation for this absurd result would be great.

Keywords: blue sky thinking

1 Reflection from a halfspace

We consider an incident plane wave

$$u_{\mathrm{in}}(x,y) = \mathrm{e}^{\mathrm{i}(\alpha x + \beta y)}, \quad \text{with } (\alpha, \beta) = k(\cos\theta_{\mathrm{in}}, \sin\theta_{\mathrm{in}}),$$

and assume time-harmonic dependence of the form $\mathrm{e}^{-\mathrm{i}\omega t}$. The incident wave $u_{\mathrm{in}}(x,y)$ is heading towards the interface $x = 0$, which divides two homogeneous materials. The material on the left (right) has wavenumber and density k and ρ (k_* and ρ_*). The reflected and transmitted wave will be of the form

$$u_R = R\mathrm{e}^{\mathrm{i}(-x\alpha + y\beta)} \quad \text{and} \quad u_T = T\mathrm{e}^{\mathrm{i}(x\alpha_* + y\beta_*)},$$

where $k_*(\cos\theta_*, \sin\theta_*) = (\alpha_*, \beta_*)$.

The boundary conditions for the acoustic pressure are

$$u_{\text{in}} + u_R = u_T \quad \text{and} \quad \frac{1}{\rho}\frac{\partial u_{\text{in}}}{\partial x} + \frac{1}{\rho}\frac{\partial u_R}{\partial x} = \frac{1}{\rho_*}\frac{\partial u_T}{\partial x}, \quad \text{for} \quad x = 0,$$

from which we get Snell's law

$$k \sin\theta_{\text{in}} = k_* \sin\theta_*, \tag{1}$$

and

$$R = \frac{q_* \cos\theta_{\text{in}} - \cos\theta_*}{q_* \cos\theta_{\text{in}} + \cos\theta_*}, \quad T = \frac{2q_* \cos\theta_{\text{in}}}{q_* \cos\theta_{\text{in}} + \cos\theta_*}, \quad \text{with} \quad q_* = \frac{k\rho_*}{k_*\rho}. \tag{2}$$

Note that $1 + R = T$.

From this we can establish bounds such as $|R| \leq 1$, can you prove this? What happens when k_* is a complex number? Later, we will see that the reflection coefficient from a random mix of cylinders (with Dirichlet boundary condition), is unbounded! And the problem is in the limit for small k. This is likely wrong, and we are not sure why.

2 Reflection from multiple random cylinders

2.1 Multipole method for cylinders

Here we give the exact theory for scalar multiple wave scattering from a finite number N of circular cylinders. The pressure u outside all the cylinders satisfies the scalar Helmholtz equation $\nabla^2 u + k^2 u = 0, and inside the j th cylinder the pressure \mathrm{u}_j$ satisfies $\nabla^2 u_j + k_o^2 u_j = 0, \quad$ for $j = 1, 2, \ldots, N,$

where ∇^2 is the two-dimensional Laplacian and k $= \omega/c \quad$ and $\quad k_o = \omega/c_o.$

We use for each cylinder the polar coordinates $\mathrm{R}_j = \|\mathbf{x} - \mathbf{x}_j\|, \quad \Theta_j = \arctan\left(\frac{y-y_j}{x-x_j}\right), where \mathrm{x}_j$ is the centre of the j-th cylinder and $\mathbf{x} = (x, y)$ is an arbitrary point with origin O. See Figure **??** for a schematic of the material properties and coordinate systems. Then we can define u_j as the scattered pressure field from the j-th cylinder, $\mathrm{u}_j(R_j, \Theta_j) = \sum_{m=-\infty}^{\infty} A_j^m Z^m H_m(kR_j) \mathrm{e}^{\mathrm{i}m\Theta_j}, \quad$ for $\ R_j > a_j, where \mathrm{H}_m$ are Hankel functions of the first kind, A_j^m are arbitrary coefficients and Z^m characterises the type of scatterer: $\mathrm{Z}^m = \frac{qJ_m'(ka)J_m(k_o a) - J_m(ka)J_m'(k_o a)}{qH_m'(ka)J_m(k_o a) - H_m(ka)J_m'(k_o a)} = Z^{-m}, with \mathrm{q} = (\rho_o k)/(\rho k_o).$ In the limits $q \to 0$ or $q \to \infty$, the coefficients for Dirichlet or Neumann boundary conditions are recovered, respectively.

The pressure outside all cylinders is the sum of the incident wave u_{in} and all scattered waves, u(x,y) $= \mathrm{u}_{\mathrm{in}}(x, y) + \sum_{j=1}^{N} u_j(R_j, \Theta_j). and the total field inside the \mathrm{j}-th cylinder is u_j^{\mathrm{I}}(R_j, \Theta_j) = \sum_{m=-\infty}^{\infty} B_j^m J_m(k_j R_j) \mathrm{e}^{\mathrm{i}m\Theta_j}, \quad$ for $\ R_j < a_j.$

The unknown coefficients are determined through the boundary conditions of continuity of pressure and normal velocity on the cylinder boundaries: u $= \mathrm{u}_j^{\mathrm{I}} \quad$ and $\quad \frac{1}{\rho}\frac{\partial u}{\partial R_j} = \frac{1}{\rho_o}\frac{\partial u_j^{\mathrm{I}}}{\partial R_j}, \quad$ on $\ R_j = a\ $ for $\ j = 1, \ldots, N.$

When the cylinders are far apart, the solution for the A_j^m are similar to the solution

Figure 1: represents a multi-species material comprising different species of cylinders to the right of the origin $O = (0, 0)$. The vector $\mathbf{x}_j$ points to the centre of the j-th cylinder, with a local polar coordinate system (R_j, Θ_j). Each cylinder has a radius a_j, density ρ_j, and wave speed c_j, while the background has density ρ and wave speed c. The vector $\mathbf{k}$ is the direction of the incident plane wave.

for one lone cylinder scattering the incident wave u_{in}, which is

$$A_j^m = -\mathrm{i}^m \mathrm{e}^{-\mathrm{i}m\theta_{\text{in}}} \mathrm{e}^{\mathrm{i}\mathbf{x}_j \cdot \mathbf{k}}. \tag{3}$$

Using the above and assuming the cylinders are far apart, the scattered field far away from the cylinder (**??**) becomes

$$\lim_{R_j \to \infty} u_j(R_j, \Theta_j) \sim \sqrt{\frac{2}{\pi k R_j}} f_\circ(\Theta_j - \theta_{\text{in}}) \mathrm{e}^{\mathrm{i}kR_j - \mathrm{i}\pi/4}, \tag{4}$$

where

$$f_\circ(\theta) = -\sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m\theta} Z^m. \tag{5}$$

2.2 Ensemble average

For an introduction to ensemble-averaging of multiple scattering see Foldy (1945).

Consider a configuration of N circular cylinders centred at $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$. Each $\mathbf{x}_j$ is in the region $\mathcal{R}_N$, where $\mathfrak{n} = N/|\mathcal{R}_N|$ is the total number density and $|\mathcal{R}_N|$ is the area of $\mathcal{R}_N$. The probability of the cylinders being in a specific configuration is given by the probability density function $p(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$, so that $\int p(\mathbf{x}_1) d\mathbf{x}_1 = \int \int p(\mathbf{x}_1, \mathbf{x}_2) d\mathbf{x}_1 d\mathbf{x}_2 = \ldots = 1. Andasthecylindersareindistinguishable$:p(x$_1, \mathbf{x}_2) = p(\mathbf{x}_2, \mathbf{x}_1)$.

Furthermore, we have

$$p(\mathbf{x}_1, \ldots, \mathbf{x}_N) = p(\mathbf{x}_j) p(\mathbf{x}_1, \ldots, \mathbf{x}_N | \mathbf{x}_j), \tag{6}$$

$$p(\mathbf{x}_1, \ldots, \mathbf{x}_N | \mathbf{x}_j) = p(\mathbf{x}_\ell | \mathbf{x}_j) p(\mathbf{x}_1, \ldots, \mathbf{x}_N | \mathbf{x}_\ell, \mathbf{x}_j), \tag{7}$$

where $p(\mathbf{x}_1, \ldots, \mathbf{x}_N | \mathbf{x}_j)$ is the conditional probability of having cylinders centred at $\mathbf{x}_1, \ldots, \mathbf{x}_N$ (not including $\mathbf{x}_j$), given that the j-th cylinder is fixed at $\mathbf{x}_j$. Likewise, $p(\mathbf{x}_1, \ldots, \mathbf{x}_N | \mathbf{x}_\ell, \mathbf{x}_j)$ is the conditional probability of having cylinders centred at $\mathbf{x}_1, \ldots, \mathbf{x}_N$ (not including $\mathbf{x}_\ell$ and $\mathbf{x}_j$) given that there are already two cylinders centred at $\mathbf{x}_\ell$ and $\mathbf{x}_j$.

Given some function $F(\mathbf{x}_1, \ldots, \mathbf{x}_N)$, we denote its average, or *expected value*, by $\langle F \rangle = \int \ldots \int F(\mathbf{x}_1, \ldots, \mathbf{x}_N) p(\mathbf{x}_1, \ldots, \mathbf{x}_N) d\mathbf{x}_1 \ldots d\mathbf{x}_N. Ifwefixthelocationandpropertiesofthe$j$-thcylinder,x_j$

and average over all the properties of the other cylinders, we obtain a *conditional average* of F given by $\langle F\rangle_{\mathbf{x}_j} = \int \ldots \int F(\mathbf{x}_1, \ldots, \mathbf{x}_N) p(\mathbf{x}_1, \ldots, \mathbf{x}_N | \mathbf{x}_j) d\mathbf{x}_1 \ldots \mathbf{x}_N, wherewedonotintegrateover \mathrm{x}_j.$ The average and conditional averages are related by $\langle F\rangle = \int \langle F\rangle_{\mathbf{x}_j} p(\mathbf{x}_j)\, d\mathbf{x}_j \quad \text{and} \quad \langle F\rangle_{\mathbf{x}_j} = \int \langle F\rangle_{\mathbf{x}_j\mathbf{x}_\ell} p(\mathbf{x}_\ell)\, d\mathbf{x}_\ell, where \langle F\rangle_{\mathbf{x}_\ell\mathbf{x}_j}$ is the conditional average when fixing both $\mathbf{x}_j$ and $\mathbf{x}_\ell$, and $\langle F\rangle_{\mathbf{x}_\ell\mathbf{x}_j} = \langle F\rangle_{\mathbf{x}_j\mathbf{x}_\ell}$.

We can now calculate the average total pressure (incident plus scattered), measured at some position $\mathbf{x}$ outside of $\mathcal{R}_N$, by averaging (**??**) to obtain $\langle u(x,y)\rangle = u_{\text{in}}(x,y) + \sum_{j=1}^N \int \ldots \int u_j(R_j, \Theta_j) p(\mathbf{x}_1, \ldots, \mathbf{x}_N) d\mathbf{x}_1 \ldots d\mathbf{x}_N, where \langle u_{\text{in}}(x,y)\rangle = u_{\text{in}}(x,y)$, because the incident field is independent of the scattering configuration. We can then rewrite the average outgoing wave u_j by fixing the properties of the j-th cylinder $\mathbf{x}_j$ and using equation (**??**) to reach

$$\langle u(x,y)\rangle - u_{\text{in}}(x,y) = \sum_{j=1}^{N} \int \langle u_j(R_j, \Theta_j)\rangle_{\mathbf{x}_j} p(\mathbf{x}_j) d\mathbf{x}_j = N \int \langle u_1(R_1, \Theta_1)\rangle_{\mathbf{x}_1} p(\mathbf{x}_1) d\mathbf{x}_1. \quad (8)$$

Likewise, for the conditionally averaged scattered field (**??**) measured at $\mathbf{x}$ we obtain $\langle u_1(R_1, \Theta_1)\rangle_{\mathbf{x}_1} = \sum_{m=-\infty}^{\infty} \langle A_1^m\rangle_{\mathbf{x}_1} Z^m H_m^{(1)}(kR_1) \mathrm{e}^{\mathrm{i}m\Theta_1}$.

We will use the simplest approximations possible, which are a random uniform distribution $\mathrm{p}(\mathrm{x}_1) = \frac{1}{|\mathcal{R}_N|}, whichcombinedwith$ (**??**)and(**??**)$, andtakingthelimit \mathrm{N} \to \infty$ with $\mathcal{R}_N$ turning into a halfspace $x_1 > 0$, leads to

$$\langle u(x,y)\rangle = u_{\text{in}}(x,y) + \mathfrak{n} \sum_{m=-\infty}^{\infty} Z^m \int_{x_1>0} \langle A_1^m\rangle_{\mathbf{x}_1} H_m^{(1)}(kR_1) \mathrm{e}^{\mathrm{i}m\Theta_1} d\mathbf{x}_1. \quad (9)$$

When $x < 0$, the above turns into the incident wave plus the average reflected field from the halfspace $x > 0$.

2.3 Effective medium approach

The simplest approach is to assume that, on average, the wave exciting a scatterer is a plane wave. That is, for $x_1 > 0$, we assume

$$\langle A_1^m \rangle_{\mathbf{x}_1} = \mathrm{i}^m \mathrm{e}^{-\mathrm{i}m\theta_*} \mathcal{A}_{m*} \mathrm{e}^{\mathrm{i}\mathbf{x}\cdot\mathbf{k}_*}, \quad \text{for} \quad x > 0, \tag{10}$$

where the constant factor $\mathrm{i}^m \mathrm{e}^{-\mathrm{i}m\theta_*}$ is just for later convenience, $\mathcal{A}_{m*}$ is an unknown constant (for now), and we define

$$\mathbf{k}_* = (\alpha_*, \beta) := k_*(\cos\theta_*, \sin\theta_*), \tag{11}$$

and from Snell's law

$$k_* \sin\theta_* = k \sin\theta_{\text{in}}, \tag{12}$$

noting that both θ_* and k_* are complex numbers.

$$\mathcal{A}_{m*}(\mathbf{s}_1) + 2\pi\mathfrak{n} \sum_{n=-\infty}^{\infty} \int_{\mathcal{S}} \mathcal{A}_{n*}(\mathbf{s}_2) \left[\frac{\mathcal{N}_{n-m}(ka_{12}, k_* a_{12})}{k^2 - k_*^2} \right] d\mathbf{s}_2^n = 0, \tag{13}$$

$$\sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}n(\theta_{\text{in}} - \theta_*)} \int_{\mathcal{S}} \mathcal{A}_{n*}(\mathbf{s}_2) d\mathbf{s}_2^n = (\alpha_* - \alpha) \frac{\alpha \mathrm{i}}{2\mathfrak{n}}, \tag{14}$$

where

$$d\mathbf{s}_2^n = Z^n(\mathbf{s}_2) p(\mathbf{s}_2) d\mathbf{s}_2, \tag{15}$$

we used whole-correction and ignored the boundary layer (which disappears in the low-frequency limit anyway). The above equations are sufficient to completely determine k_* and $\mathcal{A}_{n*}$.

First using $k_* = ck/c_*$:

$$\mathcal{N}_n(ka_{12}, k_* a_{12}) \sim \frac{2\mathrm{i} c^{|n|}}{\pi c_*^{|n|}} + \mathcal{O}(k^2),$$

because this does not depend on the species, we can move it outside the integral in (??), multiple $Z^m(\mathbf{s}_1)p(\mathbf{s}_1)$ on both sides of the equation and then integrate in $\mathbf{s}_1$ to reach,

$$\langle \mathcal{A}_{m*} \rangle^m + \frac{4\mathrm{i}\mathfrak{n}}{k^2} \frac{c_*^2}{c_*^2 - c^2} \sum_{n=-1}^{1} \frac{c^{|n-m|}}{c_*^{|n-m|}} \langle \mathcal{A}_{n*} \rangle^n \langle Z^m \rangle = 0, \tag{16}$$

where

$$\langle \mathcal{A}_{m*} \rangle^m = \int_{\mathcal{S}} \mathcal{A}_{m*}(\mathbf{s}_o) d\mathbf{s}_o^m, \quad \langle Z^n \rangle = \int_{\mathcal{S}} Z^n(\mathbf{s}_o) p(\mathbf{s}_o) d\mathbf{s}_o, \tag{17}$$

$$\langle Z^0 \rangle = \frac{\mathrm{i}k^2\pi}{4} \langle a_o \frac{\beta_o - \beta}{\beta_o} \rangle, \quad \langle Z^1 \rangle = \langle Z^{-1} \rangle = \frac{\mathrm{i}k^2\pi}{4} \langle a_o^2 \frac{\rho - \rho_o}{\rho + \rho_o} \rangle, \tag{18}$$

a_o is the radius* of the species $\mathbf{s}_o$, and we define $\langle f \rangle^m = \langle f Z^m \rangle$.

Equation (??) is now in the same form as the single species equation. By evaluating (??) for $m = -1, 0, 1$, we reach three equations with unknowns $\langle \mathcal{A}_{-1*} \rangle^{-1}$, $\langle \mathcal{A}_{0*} \rangle^0$, $\langle \mathcal{A}_{1*} \rangle^1$, and c_*. By forming a matrix equation for the $\langle \mathcal{A}_{m*} \rangle^m$, then setting the determinant of this matrix to zero, and solving for c_*, we reach

$$c_*^2 = \frac{\beta_*}{\rho_*}, \quad \text{with} \quad \frac{1}{\beta_*} = \frac{1 - \mathfrak{n}\pi \langle a_o^2 \rangle}{\beta} + \mathfrak{n}\pi \langle \frac{a_o^2}{\beta_o} \rangle, \quad \rho_* = \rho \frac{1 - \mathfrak{n}\pi \langle a_o^2 \frac{\rho - \rho_o}{\rho + \rho_o} \rangle}{1 + \mathfrak{n}\pi \langle a_o^2 \frac{\rho - \rho_o}{\rho + \rho_o} \rangle}. \tag{19}$$

Using the above in (??), we can reach

$$\langle \mathcal{A}_{0*} \rangle^0 = 2 \frac{\beta - \beta_*}{\rho - \rho_*} \sqrt{\frac{\rho \rho_*}{\beta \beta_*}} \langle \mathcal{A}_{1*} \rangle^1 \quad \text{and} \quad \langle \mathcal{A}_{-1*} \rangle^{-1} = \langle \mathcal{A}_{1*} \rangle^1. \tag{20}$$

*If you find the appearance of the radius a_o strange, have a look at the next section.

To determine $\langle \mathcal{A}_{1*} \rangle$ we use (??), which leads to

$$\langle \mathcal{A}_{1*} \rangle^1 = (\rho - \rho_*) \cos\theta_{\text{in}} \frac{\mathrm{i}a^2 k^2 \pi}{4\phi} \frac{\cos\theta_{\text{in}} - \sqrt{\frac{\rho_* \beta}{\rho \beta_*}} \cos\theta_*}{\sqrt{\frac{\beta_* \rho \rho_*}{\beta}} \left(\frac{\beta}{\beta_*} - 1 \right) - (\rho - \rho_*) \cos(\theta_{\text{in}} - \theta_*)}. \tag{21}$$

2.4 A discrete number of species

Here we show what are the effective properties (??) when there are a discrete number of species.

The definition of the probability density $p(\mathbf{s}_o)$, is that given any point $\boldsymbol{x}$, $p(\mathbf{s}_o)$ is the probability of finding a particle of species $\mathbf{s}_o$ centred at $\boldsymbol{x}$. This means that if there are S species uniformly distributed we can use $p(\mathbf{s}_o) d\mathbf{s}_o = \frac{\mathfrak{n}_o}{\mathfrak{n}}$, where $\mathfrak{n}_o$ is the number density of the species $\mathbf{s}_o$. For example:

$$\mathfrak{n}\pi \langle f(\beta_o, \rho_o) a_o^2 \rangle = \mathfrak{n}\pi \sum_{j=1}^{S} a_j^2 f(\beta_j, \rho_j) \frac{\mathfrak{n}_j}{\mathfrak{n}} = \sum_{j=1}^{S} \phi_j f(\beta_j, \rho_j), \tag{22}$$

where $\phi_j = \pi a_j^2 \mathfrak{n}_j$ is the volume fraction of the j-th species.

This leads to the discrete version of the effective properties:

$$\frac{1}{\beta_*} = \frac{1-\phi}{\beta} + \sum_j \frac{\phi_j}{\beta_j}, \quad \rho_* = \rho \frac{1 - \sum_j \phi_j \frac{\rho - \rho_j}{\rho + \rho_j}}{1 + \sum_j \phi_j \frac{\rho - \rho_j}{\rho + \rho_j}}. \tag{23}$$

2.5 Average low-frequency reflection

To calculate the average reflected field (??), we use (??),

$$(\nabla^2 + k_*^2) \langle A_1^m \rangle_{\mathbf{x}_1} \quad \text{and} \quad (\nabla^2 + k_*^2) H_m^{(1)}(kR_1) \mathrm{e}^{\mathrm{i}m\Theta_1},$$

which allows us to use Green's second identity, or more specifically equation (88) from Gower et al. (2017), to calculate

$$\int_{x_1>0} \mathrm{e}^{\mathrm{i}\alpha_* x_1 + \mathrm{i}\beta y_1} H_m^{(1)}(kR_1) \mathrm{e}^{\mathrm{i}m\Theta_1} d\mathbf{x}_1 = \mathrm{e}^{-\mathrm{i}\alpha x + \mathrm{i}\beta y} \frac{2}{\alpha} \frac{(-\mathrm{i})^{-m}\mathrm{i}}{\alpha + \alpha_*} \mathrm{e}^{-\mathrm{i}m\theta_{\mathrm{in}}}. \tag{24}$$

Substituting the above into (**??**) we get

$$\langle u(x,y) \rangle = u_{\mathrm{in}}(x,y) + R_o \mathrm{e}^{-\mathrm{i}\alpha x + \mathrm{i}\beta y}, \quad \theta_{\mathrm{ref}} = \pi - \theta_* - \theta_{\mathrm{in}}, \tag{25}$$

$$R_o = \frac{1}{a^2 \pi k \cos\theta_{\mathrm{in}}} \frac{2\mathrm{i}\phi}{k\cos\theta_{\mathrm{in}} + k_* \cos\theta_*} \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m\theta_{\mathrm{ref}}} \langle \mathcal{A}_{m*} \rangle^m. \tag{26}$$

Substituting (**??**) and (**??**) we reach, after algebraic manipulation, that

$$R_o = R = \frac{q_* \cos\theta_{\mathrm{in}} - \cos\theta_*}{q_* \cos\theta_{\mathrm{in}} + \cos\theta_*}, \quad \text{with} \quad q_* = \sqrt{\frac{\rho_* \beta_*}{\rho\beta}}.$$

References

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