Scattering from a spherical inhomogeneity

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Abstract

Some brief notes on calculating scattering from a sphere.

1 Introduction

Here we consider an elastic wave, of any type, scattered from an elastic sphere. The goal is to obtain the general T-matrix which is useful when calculating multiple scattering. There are many papers using vector spherical harmonics to solve this problem [1], while those that do no use vector spherical harmonics only solve the problem of a plane wave incident on a sphere [2, 4]. Here we make a slight extension by deducing in a simple, yet self contained way, how any elastic wave is scattered from a sphere without vector spherical harmonics.

If needed, I could add the thermal wave potential, as that would certainly add some novelty. The motivation is to better resolve wave scattering by pockets of air in visco-elastic solids. Although there is work on this, no one has done it elegantly, and I know how to do this. We could even use motivations like using ultrasound to heat up and burn different tissues, which could be enhanced with air bubbles.

2 Problem setup

Due to the Helmholtz decomposition, the total displacement $\boldsymbol{u}$ outside of the sphere can be written in the form

$$\boldsymbol{u} = \nabla\varphi + \nabla \times \boldsymbol{\psi}, \quad \text{with} \quad \nabla \cdot \boldsymbol{\psi} = 0. \tag{2.1}$$

Enforcing $\nabla \cdot \boldsymbol{\psi} = 0$ makes the representation unique, when both φ and $\boldsymbol{\psi}$ decay at infinity.

Both φ and $\boldsymbol{\psi}$ satisfy a Helmholtz equation

$$\nabla^2\varphi + k_P^2\varphi = 0 \quad \text{and} \quad \nabla^2\boldsymbol{\psi} + k_S^2\boldsymbol{\psi} = 0. \tag{2.2}$$

The governing equation for homogenous isotropic elasticity, when $\boldsymbol{u}$ is harmonic in time with the factor $\mathrm{e}^{\mathrm{i}\omega t}$ is

$$(\nabla^2 + k_S^2)\boldsymbol{u} - (1 - k_S^2/k_P^2)\nabla(\nabla \cdot \boldsymbol{u}) = 0, \tag{2.3}$$

where

$$k_P = \omega\sqrt{\frac{\rho}{\lambda + 2\mu}} \ \text{and} \ k_S = \omega\sqrt{\frac{\rho}{\mu}}. \tag{2.4}$$

In general $\boldsymbol{\psi}$ can be written in terms of Debye Potentials [7, 5, 2, 6]:

$$\boldsymbol{\psi} = \nabla \times (\Phi\boldsymbol{r}) + \frac{1}{k_S}\nabla \times (\nabla \times (\chi\boldsymbol{r})), \tag{2.5}$$

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where $\boldsymbol{r}$ is a vector from the origin of the coordinate system, which is also the centre of the sphere, to any position. By substituting the shear potentials into the governing equation (2.3) it can be shown [7, 4] that they satisfy the Helmholtz equations

$$\nabla^2\Phi + k_S^2\Phi = 0 \quad \text{and} \quad \nabla^2\chi + k_S^2\chi = 0. \tag{2.6}$$

Because Φ and χ are scalar potentials, it is more convenient to solve directly for these potentials instead of solving for $\boldsymbol{\psi}$. We note that there are other ways to write $\boldsymbol{\psi}$: the paper [4] does not enforce $\nabla \cdot \boldsymbol{\psi} = 0$, which simplfies the algebra, but then leads to non standard potentials.

The boundary conditions, for both visco-elastic solids and fluids, are continuity of displacement and traction, when disconsidering temperature effects [3]. The displacement vector $\boldsymbol{u}$ is a combination of the displacement from each potential, which by using (2.1) and (2.5) becomes

$$\boldsymbol{u} = \nabla\varphi + \boldsymbol{u}^\Phi + \boldsymbol{u}^\chi, \quad \text{with} \tag{2.7}$$

$$\boldsymbol{u}^\Phi = \nabla\frac{\partial(r\Phi)}{\partial r} + k_S^2\boldsymbol{r}\Phi, \quad \boldsymbol{u}^\chi = k_S\nabla\chi \times \boldsymbol{r}, \tag{2.8}$$

where we used (2.6) together with the identities

$$\boldsymbol{u}^\Phi = \nabla \times (\nabla \times (\Phi\boldsymbol{r})) = \nabla\frac{\partial(r\Phi)}{\partial r} - \boldsymbol{r}\nabla^2\Phi \tag{2.9}$$

$$k_S\boldsymbol{u}^\chi = \nabla \times (\nabla \times (\nabla \times (\boldsymbol{r}\chi))) = -\nabla(\nabla^2\chi) \times \boldsymbol{r}, \tag{2.10}$$

and $\boldsymbol{r}$ is the position vector.

The traction on the surface of a sphere, centred at the origin, is given by $\boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}}$, where $\hat{\boldsymbol{r}}$ is a unit vector pointing in the radial direction and $\boldsymbol{\sigma}$ is the Cauchy stress tensor given by

$$\boldsymbol{\sigma} = \lambda(\nabla \cdot \boldsymbol{u})\boldsymbol{I} + \mu(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T), \tag{2.11}$$

The components of $\boldsymbol{\tau}$ are given expliclity in Appendix A.

2.1 Boundary conditions

We now consider a sphere at the origin of our coordinate system with radius a. To solve the boundary conditions we consider the fields inside the sphere: $\boldsymbol{u}_1$ and $\boldsymbol{\tau}_1$, and the fields outside the sphere $\boldsymbol{u}_2$ and $\boldsymbol{\tau}_2$, and our aim is to solve

$$\boldsymbol{u}_1 = \boldsymbol{u}_2 \;\; \text{and} \;\; \boldsymbol{\tau}_1 = \boldsymbol{\tau}_2 \;\; \text{for} \;\; r = a, \tag{2.12}$$

and every $0 \leq \theta \leq \pi$ and $-\pi \leq \phi \leq \pi$ for the spherical coordinates (r, θ, ϕ), which are the radial distance r, polar angle θ, and azimuthal angle ϕ.

Let φ_1, Φ_1, and χ_1 be the potentials inside the sphere. As we assume the sphere is homogeneous, the field inside the sphere needs to be smooth, which implies that each of the potentials can be expanded in a series of regular spherical waves

$$\varphi_1 = \sum_{n=-\infty}^{\infty} b_n^\varphi \mathrm{j}_\ell(k_{P_1}r)\mathrm{Y}_n(\hat{\boldsymbol{r}}), \tag{2.13}$$

$$\Phi_1 = \sum_{n=-\infty}^{\infty} b_n^\Phi \mathrm{j}_\ell(k_{S_1}r)\mathrm{Y}_n(\hat{\boldsymbol{r}}), \tag{2.14}$$

$$\chi_1 = \sum_{n=-\infty}^{\infty} b_n^\chi \mathrm{j}_\ell(k_{S_1}r)\mathrm{Y}_n(\hat{\boldsymbol{r}}), \tag{2.15}$$

where j_ℓ is the spherical Bessel function, n denotes a multi index $n = \{\ell, m\}$, with summation being over $\ell = 0, 1, 2, 3\ldots$ and $m = -\ell, -\ell+1, \ldots, -1, 0, 1, \ldots, \ell$, and b_n^υ for $\upsilon = \varphi, \Phi, \chi$, are coefficients that need to be determined from boundary conditions.

Outside of the sphere, i.e. $r > a$, the displacement $\boldsymbol{u}_2 = \boldsymbol{u}_\circ + \boldsymbol{u}_{\text{scat}}$, where $\boldsymbol{u}_\circ$ is an incident displacement, while $\boldsymbol{u}_{\text{scat}}$ is composed of scattered fields. Both displacements $\boldsymbol{u}_\circ$ and $\boldsymbol{u}_{\text{scat}}$ can again be written in terms of three potentials, as shown by (2.7). As we assume the source of the incident wave is some distance away from the sphere, all the potentials of the incident field $\varphi_\circ$, $\Phi_\circ$, and $\chi_\circ$ should be smooth fields when close enough to the boundary of the sphere. Which implies that they can be written in terms of a series of regular spherical waves. On the other hand, the potentials of the scattered field φ_{scat}, Φ_{scat}, and χ_{scat} need to be composed of outgoing waves, moving away from the origin, to ensure that $\boldsymbol{u}_{\text{scat}}$ is a scattered wave, which leads to the representation

$$\varphi_{\text{scat}} = \sum_{n=-\infty}^{\infty} f_n^\varphi \mathrm{h}_\ell(k_{P_2} r) \mathrm{Y}_n(\hat{\boldsymbol{r}}), \quad \varphi_\circ = \sum_{n=-\infty}^{\infty} g_n^\varphi \mathrm{j}_\ell(k_{P_2} r) \mathrm{Y}_n(\hat{\boldsymbol{r}}), \tag{2.16}$$

$$\Phi_{\text{scat}} = \sum_{n=-\infty}^{\infty} f_n^\Phi \mathrm{h}_\ell(k_{S_2} r) \mathrm{Y}_n(\hat{\boldsymbol{r}}), \quad \Phi_\circ = \sum_{n=-\infty}^{\infty} g_n^\Phi \mathrm{j}_\ell(k_{S_2} r) \mathrm{Y}_n(\hat{\boldsymbol{r}}), \tag{2.17}$$

$$\chi_{\text{scat}} = \sum_{n=-\infty}^{\infty} f_n^\chi \mathrm{h}_\ell(k_{S_2} r) \mathrm{Y}_n(\hat{\boldsymbol{r}}), \quad \chi_\circ = \sum_{n=-\infty}^{\infty} g_n^\chi \mathrm{j}_\ell(k_{S_2} r) \mathrm{Y}_n(\hat{\boldsymbol{r}}), \tag{2.18}$$

where h_ℓ is the spherical Hankel function of the first kind, and f_n^υ for $\upsilon = \varphi, \Phi, \chi$, are also coefficients that need to be determined from boundary conditions.

Substituting the above series into the boundary condition (2.12) leads to the equations

$$\sum_n \mathbf{M}_n \boldsymbol{F}_n = \sum_n \mathbf{B}_n \boldsymbol{G}_n, \tag{2.19}$$

where

$$\mathbf{M}_n = \begin{bmatrix} \mathrm{Y}_n & \mathrm{Y}_n & \mathrm{Y}_n & \mathrm{Y}_n & 0 & 0 \\ \mathrm{Y}_n & \mathrm{Y}_n & \mathrm{Y}_n & \mathrm{Y}_n & 0 & 0 \\ \mathrm{Z}_n & \mathrm{Z}_n & \mathrm{Z}_n & \mathrm{Z}_n & \mathrm{Y}_n & \mathrm{Y}_n \\ \mathrm{Z}_n & \mathrm{Z}_n & \mathrm{Z}_n & \mathrm{Z}_n & \mathrm{Y}_n & \mathrm{Y}_n \\ \mathrm{Y}_n & \mathrm{Y}_n & \mathrm{Y}_n & \mathrm{Y}_n & \mathrm{Z}_n & \mathrm{Z}_n \\ \mathrm{Y}_n & \mathrm{Y}_n & \mathrm{Y}_n & \mathrm{Y}_n & \mathrm{Z}_n & \mathrm{Z}_n \end{bmatrix}, \quad \boldsymbol{F}_n = \begin{bmatrix} f_n^\varphi \\ b_n^\varphi \\ f_n^\Phi \\ b_n^\Phi \\ f_n^\chi \\ b_n^\chi \end{bmatrix}, \quad \mathbf{B}_n = \begin{bmatrix} \mathrm{Y}_n & \mathrm{Y}_n & 0 \\ \mathrm{Y}_n & \mathrm{Y}_n & 0 \\ \mathrm{Z}_n & \mathrm{Z}_n & \mathrm{Y}_n \\ \mathrm{Z}_n & \mathrm{Z}_n & \mathrm{Y}_n \\ \mathrm{Y}_n & \mathrm{Y}_n & \mathrm{Z}_n \\ \mathrm{Y}_n & \mathrm{Y}_n & \mathrm{Z}_n \end{bmatrix}, \quad \boldsymbol{G}_n = \begin{bmatrix} g_n^\varphi \\ g_n^\Phi \\ g_n^\chi \end{bmatrix},$$

and Y_n stands for $\mathrm{Y}_n(\theta, \phi)$ multipied by some term which does not depend on θ and ϕ. The term can be different for every appearance of Y_n. Likewise, Z_n stands for $\sin\theta \partial_\theta Y_n(\theta, \phi)$ multipied by some term which does not depend on θ and ϕ. The order of the equations (2.19) are for the boundary conditions for $u_r, \tau_r, u_\theta, \tau_\theta, u_\phi, \tau_\phi$.

The Z_n terms can be written in terms of spherical harmonics by using the identity:

$$\sin\theta \frac{\partial Y_{(\ell,m)}(\theta, \phi)}{\partial \theta} = \ell c_{\ell+1,m} Y_{(\ell+1,m)}(\theta, \phi) - (\ell+1) c_{\ell,m} Y_{(\ell-1,m)}(\theta, \phi), \tag{2.20}$$

with $c_{\ell,m} := \frac{\sqrt{\ell^2 - m^2}}{4\ell^2 - 1}$.

To solve (2.19) it is conceptually easier to consider that the incident wave has only one mode, for example

$$\chi_\circ = g_{n_1}^\chi \mathrm{j}_{\ell_1}(k_{S_2} r) \mathrm{Y}_{n_1}(\hat{\boldsymbol{r}}),$$

which would remove the sum on the right side of (2.19). One systematic way to then solve the resulting equations is to multiply both sides by $Y^*_{(\ell_2, m_1)}(\theta, \phi)$, for varios different choices of ℓ_2, integrate over θ and ϕ, and then use the orthogonality of the spherical harmonics. However, this is not straightforward because $F_{(\ell_2-1,m)}$ and $F_{(\ell_2+1,m)}$ will appear, which introduces more unknowns. In fact, by varying ℓ_2 and repeating this process we would obtain an infinite number of equations for an infinite number of unknowns. Instead of tackling this infinite system we guess a solution, which is the only valid solution as it is typical to assume uniqueness.

Art: The only dependance on m (from $n = (\ell, m)$) in the system is through Y_n and Z_n. This can explain how to solve the system. That is, we can calculate the solution for $m = 0$ and then show it is the solution for every m...

The solution for the coefficients $\boldsymbol{F}_n$ in (2.19) is given by solving the two seperate systems:

$$\mathbf{M}_n \begin{bmatrix} f_n^{\varphi} \\ b_n^{\varphi} \\ f_n^{\Phi} \\ b_n^{\Phi} \\ 0 \\ 0 \end{bmatrix} = \mathbf{B}_N \begin{bmatrix} g_n^{\varphi} \\ g_n^{\Phi} \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_n \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ f_n^{\chi} \\ b_n^{\chi} \end{bmatrix} = \mathbf{B}_N \begin{bmatrix} 0 \\ 0 \\ g_n^{\chi} \end{bmatrix}, \tag{2.21}$$

where both systems have a unique solution, as in both cases the number of independent equations is equal to the number of unknowns. Both systems are written in full, ignoring repeated equations, in Appendix B. The solution to these equations leads to the T-matrix which relate the incident coefficients to the scattering coefficients in the form:

$$\begin{bmatrix} f_n^{\varphi} \\ f_n^{\Phi} \end{bmatrix} = \begin{bmatrix} T_\ell^{\varphi\varphi} & T_\ell^{\varphi\Phi} \\ T_\ell^{\Phi\Phi} & T_\ell^{\Phi\varphi} \end{bmatrix} \begin{bmatrix} g_n^{\varphi} \\ g_n^{\Phi} \end{bmatrix} \quad \text{and} \quad f_n^{\chi} = T_\ell^{\chi\chi} g_n^{\chi}. \tag{2.22}$$

3 Incident plane shear wave

Let us consider an incident shear wave on a sphere of the form:

$$\boldsymbol{\psi}_\circ = \hat{\boldsymbol{y}} \Psi_\circ \frac{\mathrm{i} \mathrm{e}^{\mathrm{i} k_{S_2} z}}{k_{S_2}}, \tag{3.1}$$

which implies that the displacement field is given by $\boldsymbol{u}_\circ = \nabla \times \boldsymbol{\psi}_\circ = \hat{\boldsymbol{x}} \mathrm{e}^{\mathrm{i} k_{S_2} z}$. To calculate how this plane wave scatters from a sphere we need to expand the above in a spherical coordinate system by using

$$\hat{\boldsymbol{y}} = \hat{\boldsymbol{r}} \sin\theta \sin\phi + \hat{\boldsymbol{\theta}} \cos\theta \sin\phi + \hat{\boldsymbol{\phi}} \cos\phi,$$

and the plane wave expansion

$$\mathrm{e}^{\mathrm{i} k_{S_2} z} = \mathrm{e}^{\mathrm{i} k_{S_2} r \cos\theta} = \sum_{\ell=0} 2 \mathrm{i}^\ell \sqrt{\pi} \sqrt{2\ell+1} \mathrm{j}_\ell(k_{S_2} r) \mathrm{Y}_{(\ell,0)}(\hat{\boldsymbol{r}}).$$

With the above we can write (3.1) in terms of series of regular waves, and in a spherical coordinate system. This form of (3.1) can be written in terms of the Debye potentials by equating (3.1) in spherical coordinates with $\boldsymbol{\psi}_\circ = \nabla \times (\Phi_\circ \boldsymbol{r}) + k_S^{-1} \nabla \times \nabla \times (\chi_\circ \boldsymbol{r})$, then substituting the series representation $(2.17)_2$ and $(2.18)_2$, and solving for the coefficients g_n^{Φ} and g_n^{χ}. The result is given by

$$\Phi_\circ = \mathrm{i}^\ell \sqrt{\pi} \sqrt{\frac{2\ell+1}{\ell(\ell+1)}} \mathrm{j}_\ell(k_S r) [\mathrm{Y}_{(\ell,1)}(\hat{\boldsymbol{r}}) - \mathrm{Y}_{(\ell,-1)}(\hat{\boldsymbol{r}})], \tag{3.2}$$

$$\chi_\circ = \mathrm{i}^\ell \sqrt{\pi} \sqrt{\frac{2\ell+1}{\ell(\ell+1)}} \mathrm{j}_\ell(k_S r) [\mathrm{Y}_{(\ell,1)}(\hat{\boldsymbol{r}}) + \mathrm{Y}_{(\ell,-1)}(\hat{\boldsymbol{r}})]. \tag{3.3}$$

The above solution was first was derived in [2], though with some apparent sign mistakes.

A Displacement and Traction components

Here we write the traction in terms of the spherical coordinates (r, θ, ϕ), which are the radial distance r, polar angle θ, and azimuthal angle ϕ. We consider that the basis of our spherical coordinate system is orthonormal, which is the most standard choice.

Substituting (2.7) into (2.11) we obtain

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\varphi} + \boldsymbol{\sigma}^{\Phi} + \boldsymbol{\sigma}^{\chi}, \tag{A.1}$$

then by using $\nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{u}^\circ + \nabla^2\varphi = \nabla\cdot\boldsymbol{u}^\circ - k_P^2\varphi$ we can write the traction $\boldsymbol{\tau} = \boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}}$ expliclity in terms of its spherical coordinates (r,θ,Φ):

$$\sigma_{rr}^{\varphi} = 2\mu\partial_r^2\varphi - \lambda k_P^2\varphi, \quad \sigma_{r\theta}^{\varphi} = 2\mu\frac{\partial}{\partial r}\left[\frac{\partial_\theta\varphi}{r}\right], \quad \sigma_{r\phi}^{\varphi} = \frac{2\mu}{\sin\theta}\frac{\partial}{\partial r}\left[\frac{\partial_\phi\varphi}{r}\right], \tag{A.2}$$

$$\sigma_{rr}^{\Phi} = 2\mu k_S^2\partial_r[r\Phi] + 2\mu\partial_r^3[r\Phi], \quad \sigma_{r\theta}^{\Phi} = \mu k_S^2\partial_\theta\Phi + 2\mu\frac{\partial}{\partial r}\frac{\partial_{r\theta}[r\Phi]}{r}, \tag{A.3}$$

$$\sigma_{r\phi}^{\Phi} = \frac{\mu k_S^2}{\sin\theta}\partial_\phi\Phi + \frac{2\mu}{\sin\theta}\frac{\partial}{\partial r}\frac{\partial_{r\phi}[r\Phi]}{r}, \tag{A.4}$$

$$\sigma_{rr}^{\chi} = 0, \quad \sigma_{r\theta}^{\chi} = \frac{\mu k_S r}{\sin\theta}\frac{\partial}{\partial r}\left[\frac{\partial_\phi\chi}{r}\right], \quad \sigma_{r\phi}^{\chi} = -\mu k_S r\frac{\partial}{\partial r}\left[\frac{\partial_\theta\chi}{r}\right], \tag{A.5}$$

B T-matrix system

The solution for the coefficients $f_n^\varphi, b_n^\varphi, f_n^\Phi, b_n^\Phi, f_n^\chi, b_n^\chi$ in (2.19) is given by solving the two seperate systems:

$$\begin{bmatrix}\mathbf{M}_{11} & \mathbf{M}_{12}\\ \mathbf{M}_{21} & \mathbf{M}_{22}\end{bmatrix}\begin{bmatrix}f_n^\varphi\\ b_n^\varphi\\ f_n^\Phi\\ b_n^\Phi\end{bmatrix} = \mathbf{G}^\varphi g_n^\varphi + \mathbf{G}^\Phi g_n^\Phi, \quad \mathbf{M}^\chi\begin{bmatrix}f_n^\chi\\ b_n^\chi\end{bmatrix} = \mathbf{G}^\chi g_n^\chi, \tag{B.1}$$

where we use the shorthand

$$a_{P_j} := k_{P_j}a, \quad \mathrm{j}_{\ell P_j} := \mathrm{j}_\ell(k_{P_j}a), \quad \mathrm{h}_{\ell P_j} := \mathrm{h}_\ell(k_{P_j}a),$$

and likewise for the terms with k_{S_j}, and defined the matrices:

$$\mathbf{G}^\varphi = \begin{bmatrix} a_{P_2}\mathrm{j}'_{\ell P_2}\\ a_{P_2}^2\left(2\mu_2\mathrm{j}''_{\ell P_2} - \lambda_2\mathrm{j}_{\ell P_2}\right)\\ \mathrm{j}_{\ell P_2}\\ 2\mu_2\left(a_{P_2}\mathrm{j}'_{\ell P_2} - \mathrm{j}_{\ell P_2}\right)\end{bmatrix}, \quad \mathbf{G}^\Phi = \begin{bmatrix} a_{S_2}\left(2\mathrm{j}'_{\ell S_2} + a_{S_2}\mathrm{j}_{\ell S_2} + a_{S_2}\mathrm{j}''_{\ell S_2}\right)\\ 2a_{S_2}^2\mu_2\left(\mathrm{j}_{\ell S_2} + a_{S_2}\mathrm{j}'_{\ell S_2} + 3\mathrm{j}''_{\ell S_2} + a_{S_2}\mathrm{j}'''_{\ell S_2}\right)\\ \mathrm{j}_{\ell S_2} + a_{S_2}\mathrm{j}'_{\ell S_2}\\ \mu_2\left(\left(a_{S_2}^2 - 2\right)\mathrm{j}_{\ell S_2} + 2a_{S_2}\left(\mathrm{j}'_{\ell S_2} + a_{S_2}\mathrm{j}''_{\ell S_2}\right)\right)\end{bmatrix},$$

$$\mathbf{M}_{11} = \begin{bmatrix} -a_{P_2}\mathrm{h}'_{\ell P_2} & a_{P_1}\mathrm{j}'_{\ell P_1}\\ a_{P_2}^2\left(\lambda_2\mathrm{h}_{\ell P_2} - 2\mu_2\mathrm{h}''_{\ell P_2}\right) & a_{P_1}^2\left(2\mu_1\mathrm{j}''_\ell(k_{P_1}a) - \lambda_1\mathrm{j}_{\ell P_1}\right)\end{bmatrix},$$

$$\mathbf{M}_{21} = \begin{bmatrix} -\mathrm{h}_{\ell P_2} & \mathrm{j}_{\ell P_1}\\ 2\mu_2\mathrm{h}_{\ell P_2} - 2\mu_2 a_{P_2}\mathrm{h}'_{\ell P_2} & 2\mu_1 a_{P_1}\mathrm{j}'_{\ell P_1} - 2\mu_1\mathrm{j}_{\ell P_1}\end{bmatrix},$$

$$\mathbf{M}_{12} = \begin{bmatrix} -a_{S_2}\left(a_{S_2}\mathrm{h}_{\ell S_2} + 2\mathrm{h}'_{\ell S_2} + a_{S_2}\mathrm{h}''_{\ell S_2}\right) & a_{S_1}\left(a_{S_1}\mathrm{j}_{\ell S_1} + 2\mathrm{j}'_{\ell S_1} + a_{S_1}\mathrm{j}''_{\ell S_1}\right)\\ -2a_{S_2}^2\mu_2\left(\mathrm{h}_{\ell S_2} + a_{S_2}\mathrm{h}'_{\ell S_2} + 3\mathrm{h}''_{\ell S_2} + a_{S_2}\mathrm{h}'''_{\ell S_2}\right) & 2a_{S_1}^2\mu_1\left(\mathrm{j}_{\ell S_1} + a_{S_1}\mathrm{j}'_{\ell S_1} + 3\mathrm{j}''_{\ell S_1} + a_{S_1}\mathrm{j}'''_{\ell S_1}\right)\end{bmatrix},$$

$$\mathbf{M}_{22} = \begin{bmatrix} -\mathrm{h}_{\ell S_2} - a_{S_2}\mathrm{h}''_{\ell S_2} & \mathrm{j}_{\ell S_1} + a_{S_1}\mathrm{j}'_{\ell S_1}\\ \mu_2(2 - a_{S_2}^2)\mathrm{h}_{\ell S_2} - 2\mu_2 a_{S_2}\left(\mathrm{h}'_{\ell S_2} + a_{S_2}\mathrm{h}''_{\ell S_2}\right) & \mu_1(a_{S_1}^2 - 2)\mathrm{j}_{\ell S_1} + 2\mu_1 a_{S_1}\left(\mathrm{j}'_{\ell S_1} + a_{S_1}\mathrm{j}''_{\ell S_1}\right)\end{bmatrix},$$

$$\mathbf{M}^\chi = \begin{bmatrix} a_{S_2}\mathrm{h}_{\ell S_2} & -a_{S_1}\mathrm{j}_{\ell S_1}\\ a_{S_2}\mu_2\left(a_{S_2}\mathrm{h}'_{\ell S_2} - \mathrm{h}_{\ell S_2}\right) & a_{S_1}\mu_1\left(\mathrm{j}_{\ell S_1} - a_{S_1}\mathrm{j}'_{\ell S_1}\right)\end{bmatrix}, \quad \mathbf{G}^\chi = \begin{bmatrix} -a_{S_2}\mathrm{j}_{\ell S_2}\\ \mu_2 a_{S_2}(\mathrm{j}_{\ell S_2} - a_{S_2}\mathrm{j}'_{\ell S_2})\end{bmatrix}.$$

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