

# Notes on the T-matrix in 2D

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May 16, 2018

## Abstract

short explanation

*Keywords:* Multiple scattering, fresh fish

## 1 Single scatterer

We will be following a similar notation as used in A T-Matrix Reduced Order Model Software [2, 1].

Any incident and scattered wave in 2D, centred at the same polar coordinate axis, can be written as

$$\psi^{\text{inc}} = \sum_{n=-\infty}^{\infty} f_n J_n(kr) e^{in\theta}, \quad (1)$$

$$\psi^{\text{s}} = \sum_{n=-\infty}^{\infty} a_n H_n(kr) e^{in\theta}. \quad (2)$$

The T-matrix is an infinite matrix such that

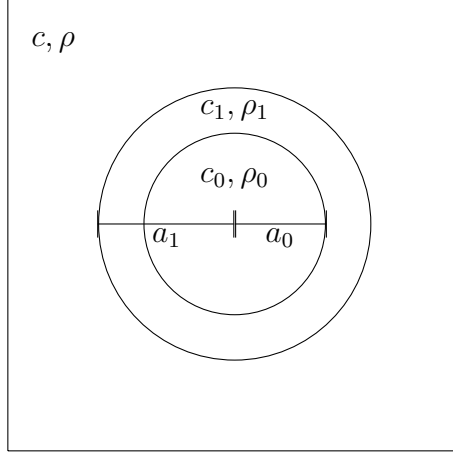
$$a_n = \sum_{m=-\infty}^{\infty} T_{nm} f_m. \quad (3)$$

Such a matrix  $T$  exists when scattering is a linear operation (elastic scattering).

For instance, if  $\rho$  and  $c$  are the background density and wavespeed, then for a circular scatterer with density  $\rho_j$ , soundspeed  $c_j$  and radius  $a_j$ , we have that

$$T_{nm} = -\delta_{nm} Z_j^m, \quad \text{with} \quad Z_j^m = \frac{q_j J'_m(ka_j) J_m(k_j a_j) - J_m(ka_j) J'_m(k_j a_j)}{q_j H'_m(ka_j) J_m(k_j a_j) - H_m(ka_j) J'_m(k_j a_j)}, \quad (4)$$

where  $q_j = (\rho_j c_j)/(\rho c)$  and  $k_j = \omega/c_j$ .



## 1.1 Single circular capsule

$$\psi^0 = \sum_{n=-\infty}^{\infty} f_n^0 J_n(k_0 r) e^{in\theta}, \quad (5)$$

$$\psi^1 = \sum_{n=-\infty}^{\infty} [f_n^1 J_n(k_1 r) + a_n^1 H_n(k_1 r)] e^{in\theta}. \quad (6)$$

Applying the boundary conditions,

$$\psi^0 = \psi^1 \quad \text{and} \quad \frac{1}{\rho_0} \frac{\partial \psi^0}{\partial r} = \frac{1}{\rho_1} \frac{\partial \psi^1}{\partial r}, \quad \text{on } r = r_0, \quad (7)$$

$$\psi^1 = \psi^s + \psi^{\text{inc}} \quad \text{and} \quad \frac{1}{\rho_1} \frac{\partial \psi^1}{\partial r} = \frac{1}{\rho} \frac{\partial (\psi^s + \psi^{\text{inc}})}{\partial r}, \quad \text{on } r = r_1. \quad (8)$$

Solving these boundary conditions (see capsule-boundary-conditions.nb) leads to

$$\begin{aligned} T_{nn} = & -\frac{J_n(ka_1)}{H_n(ka_1)} - \frac{Y_n'(ka_1, ka_1)}{H_n(ka_1)} [Y^n(k_1 a_1, k_1 a_0) J_n'(k_0 a_0) - q_0 J_n(k_0 a_0) Y_n'(k_1 a_1, k_1 a_0)] \\ & \times [J_n'(k_0 a_0) (q H_n(ka_1) Y_n^n(k_1 a_0, k_1 a_1) + H_n'(ka_1) Y^n(k_1 a_1, k_1 a_0)) \\ & + q_0 J_n(k_0 a_0) (q H_n(ka_1) Y_n''(k_1 a_1, k_1 a_0) - H_n'(ka_1) Y_n'(k_1 a_1, k_1 a_0))]^{-1}. \end{aligned} \quad (9)$$

where  $q = \rho c / (\rho_1 c_1)$ ,  $q_0 = \rho_0 c_0 / (\rho_1 c_1)$ , and

$$Y^n(x, y) = H_n(x) J_n(y) - H_n(y) J_n(x), \quad (10)$$

$$Y_n'(x, y) = H_n(x) J_n'(y) - H_n'(y) J_n(x), \quad (11)$$

$$Y_n''(x, y) = H_n'(x) J_n'(y) - H_n'(y) J_n'(x). \quad (12)$$

## 2 Multiple scattering

Graf's addition theorem

$$H_n(kR_\ell)e^{in\Theta_\ell} = \sum_{m=-\infty}^{\infty} H_{n-m}(kR_{\ell j})e^{i(n-m)\Theta_{\ell j}} J_m(kR_j)e^{im\Theta_j}, \quad \text{for } R_j < R_{\ell j}, \quad (13)$$

where  $(R_{\ell j}, \Theta_{\ell j})$  are the polar coordinates of  $\mathbf{x}_j - \mathbf{x}_\ell$ . The above is also valid if we swap  $H_n$  for  $J_n$ , and swap  $H_{n-m}$  for  $J_{n-m}$ .

Particle- $j$  scatters a field

$$\psi_j = \sum_{m=-\infty}^{\infty} A_j^m H_m(kR_j)e^{im\Theta_j}, \quad \text{for } R_j > a_j, \quad (14)$$

where  $(R_j, \Theta_j)$  are the polar coordinates of  $\mathbf{x} - \mathbf{x}_j$ , where  $\mathbf{x}_j$  is the centre of particle  $j$ .

Let the incident wave, with coordinate system centred at  $\mathbf{x}_j$ , be

$$\psi_{\text{inc}} = \sum_{m=-\infty}^{\infty} f_j^m J_m(kR_j)e^{im\Theta_j}, \quad (15)$$

then the wave exciting particle- $j$  is

$$\psi_j^E = \sum_{m=-\infty}^{\infty} F_j^m J_m(kR_j)e^{im\Theta_j}, \quad (16)$$

where

$$F_j^m = f_j^m + \sum_{\ell \neq j} \sum_{p=-\infty}^{\infty} A_\ell^p H_{p-m}(kR_{\ell j})e^{i(p-m)\Theta_{\ell j}}. \quad (17)$$

Using the T-matrix of particle- $j$  we reach  $A_j^n = \sum_m T_j^{nm} F_j^m$ , which leads to

$$A_j^n = \sum_{m=-\infty}^{\infty} T_j^{nm} f_j^m + \sum_{\ell \neq j} \sum_{m,p=-\infty}^{\infty} A_\ell^p T_j^{nm} H_{p-m}(kR_{\ell j})e^{i(p-m)\Theta_{\ell j}}. \quad (18)$$

As a check, if we use (4) and swap  $A_j^n \rightarrow A_j^n Z_j^n$ , then we arrive at equation (2.11) in [3].

For easy implementation we need the functions:

$$\psi_{\text{inc}} \mapsto f_j^m \quad \text{and} \quad \text{particle} \mapsto T_j^{nm}.$$

For efficient implementation we need to rewrite (18) as a matrix equation. Let

$$(\mathbf{A}_j)_n = A_j^n, \quad (\mathbf{f}_j)_n = f_j^n, \quad (19)$$

$$(\mathbf{T}_j)_{nm} = T_j^{nm}, \quad (\mathbf{\Psi}_{j\ell})_{mp} = H_{p-m}(kR_{\ell j})e^{i(p-m)\Theta_{\ell j}}, \quad (20)$$

and note that  $\Theta_{\ell j} = \Theta_{j\ell} + \pi$ . Then

$$\sum_{\ell} (\delta_{\ell j} + (\delta_{\ell j} - 1)\mathbf{T}_j \mathbf{\Psi}_{j\ell}) \mathbf{A}_{\ell} = \mathbf{T}_j \mathbf{f}_j, \quad (21)$$

which leads to one massive square matrix:

$$\begin{bmatrix} \mathbf{I} & -\mathbf{T}_1 \mathbf{\Psi}_{12} & \cdots & -\mathbf{T}_1 \mathbf{\Psi}_{1(N-1)} & -\mathbf{T}_1 \mathbf{\Psi}_{1N} \\ -\mathbf{T}_2 \mathbf{\Psi}_{21} & \mathbf{I} & -\mathbf{T}_2 \mathbf{\Psi}_{23} & \cdots & -\mathbf{T}_2 \mathbf{\Psi}_{2N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\mathbf{T}_N \mathbf{\Psi}_{N1} & \cdots & \cdots & -\mathbf{T}_N \mathbf{\Psi}_{N(N-1)} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_N \end{bmatrix} = \begin{bmatrix} \mathbf{T}_1 \mathbf{f}_1 \\ \vdots \\ \mathbf{T}_N \mathbf{f}_N \end{bmatrix} \quad (22)$$

## References

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