Notes on the T-matrix in 2D

Artur L. Gower[a],

[a] School of Mathematics, University of Manchester, Oxford Road, Manchester, M13 9PL,UK

April 3, 2018

Abstract

short explanation

Keywords: Multiple scattering, fresh fish

1 Single scatterer

We will be following the notation used in A T-Matrix Reduced Order Model Software [**ganesh˙far-field˙2010**, **ganesh˙algorithm˙2017**].

Any incident and scattered wave in 2D, centred at the same polar coordinate axis, can be written as

$$\psi^{\mathrm{inc}} = \sum_n f_n J_{|n|}(kr)\mathrm{e}^{in\theta}, \tag{1}$$

$$\psi^{\mathrm{s}} = \sum_n a_n H_{|n|}(kr)\mathrm{e}^{in\theta}. \tag{2}$$

The T-matrix is an infinite matrix such that

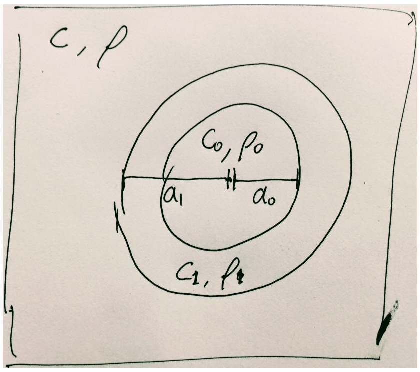
$$a_n = \sum_m T_{nm} f_m. \tag{3}$$

Such a matrix T exists when scattering is a linear (elastic scattering).

For instance, if ρ and c are the background density and wavespeed, then for a circular scatterer with density ρ_j, soundspeed c_j and radius a_j, we have that

$$T_{nm} = -\delta_{nm} Z_j^m, \;\; \text{with} \;\; Z_j^m = \frac{q_j J_m'(ka_j) J_m(k_j a_j) - J_m(ka_j) J_m'(k_j a_j)}{q_j H_m'(ka_j) J_m(k_j a_j) - H_m(ka_j) J_m'(k_j a_j)},$$

where $q_j = (\rho_j c_j)/(\rho c)$ and $k_j = \omega/c_j$.



1.1 Single circular capsule

$$\psi^0 = \sum_n f_n^0 J_{|n|}(k_0 r) \mathrm{e}^{\mathrm{i}n\theta}, \tag{4}$$

$$\psi^1 = \sum_n \left[f_n^1 J_{|n|}(k_1 r) + a_n^1 H_{|n|}(k_1 r) \right] \mathrm{e}^{\mathrm{i}n\theta}. \tag{5}$$

Applying the boundary conditions,

$$\psi^0 = \psi^1 \quad \text{and} \quad \frac{1}{\rho_0}\frac{\partial \psi^0}{\partial r} = \frac{1}{\rho_1}\frac{\partial \psi^1}{\partial r}, \quad \text{on} \;\; r = r_0, \tag{6}$$

$$\psi^1 = \psi^{\mathrm{s}} + \psi^{\mathrm{inc}} \quad \text{and} \quad \frac{1}{\rho_1}\frac{\partial \psi^1}{\partial r} = \frac{1}{\rho}\frac{\partial (\psi^{\mathrm{s}} + \psi^{\mathrm{inc}})}{\partial r}, \quad \text{on} \;\; r = r_1. \tag{7}$$

Solving these boundary conditions (see capsule-boundary-conditions.nb) leads to

$$\begin{aligned} T_{nn} = &-\frac{J_n(ka_1)}{H_n(ka_1)} - \frac{Y_{\prime}^{n}(ka_1, ka_1)}{H_n(ka_1)} \left[Y^n(k_1 a_1, k_1 a_0) J_n'(k_0 a_0) - q_0 J_n(k_0 a_0) Y_{\prime}^{n}(k_1 a_1, k_1 a_0) \right] \\ &\times \big[J_n'(k_0 a_0)(q H_n(ka_1) Y_{\prime}^{n}(k_1 a_0, k_1 a_1) + H_n'(ka_1) Y^n(k_1 a_1, k_1 a_0)) \\ &+ q_0 J_n(k_0 a_0)(q H_n(ka_1) Y_{\prime\prime}^{n}(k_1 a_1, k_1 a_0) - H_n'(ka_1) Y_{\prime}^{n}(k_1 a_1, k_1 a_0)) \big]^{-1}. \end{aligned} \tag{8}$$

where $q = \rho c/(\rho_1 c_1)$, $q_0 = \rho_0 c_0/(\rho_1 c_1)$, and

$$Y^n(x, y) = H_n(x) J_n(y) - H_n(y) J_n(x), \tag{9}$$

$$Y_{\prime}^{n}(x, y) = H_n(x) J_n'(y) - H_n'(y) J_n(x), \tag{10}$$

$$Y_{\prime\prime}^{n}(x, y) = H_n'(x) J_n'(y) - H_n'(y) J_n'(x). \tag{11}$$

2 Multiple scattering

Graf's addition theorem

$$H_n(kR_\ell)\mathrm{e}^{\mathrm{i}n\Theta_\ell} = \sum_{m=-\infty}^{\infty} H_{n-m}(kR_{\ell j})\mathrm{e}^{\mathrm{i}(n-m)\Theta_{\ell j}} J_m(kR_j)\mathrm{e}^{\mathrm{i}m\Theta_j}, \quad \text{for} \ \ R_j < R_{\ell j}, \tag{12}$$

where we can also swap H_n and H_{n-m} for J_n and J_{n-m}.

Each particle scatters a field Then we can define u_j as the scattered pressure field from the j-th cylinder,

$$\psi_j(R_j, \Theta_j) = \sum_{m=-\infty}^{\infty} A_j^m H_m(kR_j)\mathrm{e}^{\mathrm{i}m\Theta_j}, \quad \text{for} \ \ R_j > a_j, \tag{13}$$

where (R_j, Θ_j) are the polar coordinates of $\boldsymbol{x} - \boldsymbol{x}_j$, where $\boldsymbol{x}_j$ is the centre of particle j. Let the incident wave, with coordinate system centred at $\boldsymbol{x}_j$, be

$$\psi_{\mathrm{inc}}(R_j, \Theta_j) = \sum_{m=-\infty}^{\infty} f^m J_m(kR_j)\mathrm{e}^{\mathrm{i}m\Theta_j} \tag{14}$$