Multiple scattering of waves

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Abstract

Here we show and deduce the T-matrix and a general multiple scattering formulation which can be adapted to acoustics, electromagnetism, and elasticity. For details on each specific physical medium see the other documents.

Keywords: Multiple scattering, T-matrix, Scattering matrix

1 Using a T-matrix

A T-matrix denotes how one single particle scatters waves [4, 3].

For convenience and generality we denote:

$$\begin{aligned} \mathrm{u}_n(k\boldsymbol{r}) &= \text{outgoing spherical waves}, \\ \mathrm{v}_n(k\boldsymbol{r}) &= \text{regular spherical waves}, \end{aligned} \tag{1}$$

where n denotes a multi index which depends on the dimension and if the waves are scalar or vector fields.

Any incident wave and scattered wave*, centred at the same coordinate axis, can be written as

$$u_{\text{inc}} = \sum_n g_n \mathrm{v}_n(k\boldsymbol{r}), \tag{2}$$

$$u_{\text{sc}} = \sum_n f_n \mathrm{u}_n(k\boldsymbol{r}). \tag{3}$$

*For the scattered wave we need only use outgoing spherical waves when measuring the field outside of a sphere which completely encompasses the particle.

The T-matrix is an infinite matrix such that

$$f_n = \sum_{n'} T_{nn'} g_{n'}. \tag{4}$$

Such a matrix T exists when scattering is a linear operation (elastic scattering).

We can also estimate the field inside the particle by assuming that the field is smooth and continuous. This approximation is exact for homogeneous spheres and cylinders, but not for a Circular cylindrical capsule.

Assume the field inside the particle can be described by a regular spherical series:

$$v_{\text{in}} = \sum_n b_n \mathrm{v}_n(k_o \boldsymbol{r}), \tag{5}$$

where k_o if the particles wavenumber. Now if we assume that the total field is continuous everywhere so that $u_{\text{inc}} + u_{\text{sc}} = v_{\text{in}}$ on the boundary of the particle. If the field was smooth enough, we could analytically extend the field v_{in} to a spherical boundary, with radius a, which contains the particle. Let's take this as an assumption and equate $u_{\text{inc}} + u_{\text{sc}} = v_{\text{in}}$ for $r = a$. Due to orthogonality of the angular components of the basis functions this will result in

$$g_n \mathrm{v}_n(k\boldsymbol{r}) + f_n \mathrm{u}_n(k\boldsymbol{r}) = b_n \mathrm{v}_n(k_o \boldsymbol{r}), \quad \text{for } \; |\boldsymbol{r}| = a \tag{6}$$

using the T-matrix we can then write $g_n = T^{-1}_{nm} f_m$, which substituted above leads to

$$b_n = \frac{1}{\mathrm{v}_n(k_o \boldsymbol{r})} [\mathrm{v}_n(k\boldsymbol{r}) T^{-1}_{nm} f_m + \mathrm{u}_n(k\boldsymbol{r}) f_n], \quad \text{for } \; |\boldsymbol{r}| = a. \tag{7}$$

2 General multiple scattering

For multiple scattering in higher dimensions and for vector wave equations we use the notation given in [5].

For a point $\boldsymbol{r}$, outside of the circumscribed spheres of all particles, we can write the total field $u(\boldsymbol{r})$ as a sum of the incident wave $u_{\text{inc}}(\boldsymbol{r})$ and all scattered waves in the form [6, 7, 8]

$$u(\boldsymbol{r}) = u_{\text{inc}}(\boldsymbol{r}) + u_{\text{sc}}(\boldsymbol{r}), \quad u_{\text{sc}}(\boldsymbol{r}) = \sum_{i=1}^{N} \sum_n f_n^i \mathrm{u}_n(k\boldsymbol{r} - k\boldsymbol{r}_i), \tag{8}$$

where we assumed $|\boldsymbol{r} - \boldsymbol{r}_i| > a_i$ for $i = 1, 2, \dots N$, the f_n^i are coefficients we need to determine, where again:

$$\begin{cases} \mathrm{u}_n(k\boldsymbol{r}) = \text{outgoing spherical waves}, \\ \mathrm{v}_n(k\boldsymbol{r}) = \text{regular spherical waves}, \end{cases} \tag{9}$$

where n denotes a multi index which depends on the dimension and if the waves are scalar or vector fields.

In general, we can write the multiple scattering system in the form:

$$\alpha_n^i = g_n^i + \sum_{\substack{j=1 \\ j\neq i}}^{N} \sum_{n'n''} \mathcal{U}_{n''n}(k\boldsymbol{r}_i - k\boldsymbol{r}_j) T^j_{n''n'} \alpha^j_{n'}, \tag{10}$$

for $i = 1, 2, \ldots, N$, where $f_n^i = \sum_{n'} T^i_{nn'} \alpha^i_{n'}$ and $\mathcal{U}_{nn'}$ is a translation matrix [1, 2]. Let $\boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{d}$, then the translation matrices for a translation $\boldsymbol{d}$ can be defined by the property [1]

$$\begin{cases} \mathrm{v}_n(k\boldsymbol{r}') = \sum_{n'} \mathcal{V}_{nn'}(k\boldsymbol{d}) \mathrm{v}_{n'}(k\boldsymbol{r}), & \text{for all } \boldsymbol{d} \\ \mathrm{u}_n(k\boldsymbol{r}') = \sum_{n'} \mathcal{V}_{nn'}(k\boldsymbol{d}) \mathrm{u}_{n'}(k\boldsymbol{r}), & |\boldsymbol{r}| > |\boldsymbol{d}| \\ \mathrm{u}_n(k\boldsymbol{r}') = \sum_{n'} \mathcal{U}_{nn'}(k\boldsymbol{d}) \mathrm{v}_{n'}(k\boldsymbol{r}), & |\boldsymbol{r}| < |\boldsymbol{d}| \end{cases} \tag{11}$$

2.1 Turning equations into code

For easy implementation we need the functions:

$$\psi_{\mathrm{inc}} \mapsto g_n^j \quad \text{and} \quad \text{particle} \mapsto T^j_{nn'}.$$

For efficient implementation we rewrite (10) as a matrix equation. Let

$$(\boldsymbol{\alpha}_j)_n = \alpha_n^j, \quad (\boldsymbol{g}_j)_n = g_n^j, \tag{12}$$

$$(\boldsymbol{T}_j)_{nn'} = T^j_{nn'}, \quad (\boldsymbol{\mathcal{U}}_{j\ell})_{n'n} = \mathcal{U}_{n'n}(k\boldsymbol{r}_j - k\boldsymbol{r}_\ell), \tag{13}$$

Then

$$\sum_{\ell} (\delta_{j\ell} + (\delta_{j\ell} - 1) \boldsymbol{\mathcal{U}}^{\mathrm{T}}_{j\ell} \boldsymbol{T}_\ell) \boldsymbol{\alpha}_\ell = \boldsymbol{g}_j, \tag{14}$$

where $\cdot^{\mathrm{T}}$ is the transpose operation. The above then leads to a block matrix equation:

$$\begin{bmatrix} \boldsymbol{I} & -\boldsymbol{\mathcal{U}}^{\mathrm{T}}_{12}\boldsymbol{T}_2 & \cdots & -\boldsymbol{\mathcal{U}}^{\mathrm{T}}_{1(N-1)}\boldsymbol{T}_{N-1} & -\boldsymbol{\mathcal{U}}^{\mathrm{T}}_{1N}\boldsymbol{T}_N \\ -\boldsymbol{\mathcal{U}}^{\mathrm{T}}_{21}\boldsymbol{T}_1 & \boldsymbol{I} & -\boldsymbol{\mathcal{U}}^{\mathrm{T}}_{23}\boldsymbol{T}_3 & \cdots & -\boldsymbol{\mathcal{U}}^{\mathrm{T}}_{2N}\boldsymbol{T}_N \\ & \vdots & & & \vdots \\ -\boldsymbol{\mathcal{U}}^{\mathrm{T}}_{N1}\boldsymbol{T}_1 & \cdots & \cdots & -\boldsymbol{\mathcal{U}}^{\mathrm{T}}_{N(N-1)}\boldsymbol{T}_{N-1} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \vdots \\ \boldsymbol{\alpha}_N \end{bmatrix} = \begin{bmatrix} \boldsymbol{g}_1 \\ \vdots \\ \boldsymbol{g}_N \end{bmatrix} \tag{15}$$

3 Periodic multiple scattering

Here we consider a unit cell filled with particles that is repeated periodically. The particles can take any positions within the cell.

Let us start with the simplest case of just one particle centred at $\boldsymbol{r}_1$. We assume there are identical particles centered at the positions $\boldsymbol{r}_1 \in \mathcal{P}$.

The total field is again given by (8), with $\boldsymbol{r}_i \in \mathcal{P}$. However, if we assume the source is periodic with

$$u_{\text{inc}}(\boldsymbol{r}) = u_{\text{inc}}(\boldsymbol{r} + \boldsymbol{r}_1), \quad \text{for every } \ \boldsymbol{r}_1 \in \mathcal{P}, \tag{16}$$

then, due to symmetry, the scattering coefficients are the same $f_n := f_n^i$, and as a result the total field is given by

$$u(\boldsymbol{r}) = u_{\text{inc}}(\boldsymbol{r}) + \sum_n f_n \sum_{mp} \mathrm{u}_n(k\boldsymbol{r} - k\boldsymbol{r}_1 - km\boldsymbol{v}_p).$$

Taking $\boldsymbol{r} = \boldsymbol{v} + \boldsymbol{r}_1 + m_1\boldsymbol{v}_{p_1}$, we can then write the wave arriving at (or exciting) the particle at $\boldsymbol{r}_1 + m_1\boldsymbol{v}_{p_1}$ in the form

$$u_{\text{ex}}^{m_1 p_1}(\boldsymbol{v}) = u_{\text{inc}}(\boldsymbol{v} + \boldsymbol{r}_1) + \sum_n f_n \sum_{m \neq m_1, p \neq p_1} \mathrm{u}_n(k\boldsymbol{v} + km_1\boldsymbol{v}_{p_1} - km\boldsymbol{v}_p),$$

where we used (16). Writing the above as a series of regular spherical waves cented at then leads to

$$u_{\text{ex}}^{m_1 p_1}(\boldsymbol{v}) = \sum_{n_1} g_{n_1} \mathrm{v}_{n_1}(\boldsymbol{r}) + \sum_n f_n \sum_{m \neq m_1, p \neq p_1} \sum_{n_1} \mathcal{U}_{nn_1}() \mathrm{u}_n(k\boldsymbol{v} + km_1\boldsymbol{v}_{p_1} - km\boldsymbol{v}_p),$$

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