Notes on the T-matrix in 2D

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Abstract

Here we show and deduce the T-matrix and multiple scattering for acoustics. The general multiple scattering formulation shown can be adapted for electromagnetism and elasticity.

Keywords: Multiple scattering, T-matrix, Scattering matrix

1 Using a T-matrix

A T-matrix denotes how one single particle scatters waves [4, 3].

For convenience and generality we denote:

$$\begin{aligned} \mathrm{u}_n(k\boldsymbol{r}) &= \text{outgoing spherical waves}, \\ \mathrm{v}_n(k\boldsymbol{r}) &= \text{regular spherical waves}, \end{aligned} \tag{1}$$

where n denotes a multi index which depends on the dimension and if the waves are scalar or vector fields.

Any incident wave and scattered wave*, centred at the same coordinate axis, can be written as

$$u_{\text{in}} = \sum_n g_n \mathrm{v}_n(k\boldsymbol{r}), \tag{2}$$

$$u_{\text{sc}} = \sum_{n=-\infty}^{\infty} f_n \mathrm{u}_n(k\boldsymbol{r}). \tag{3}$$

*For the scattered wave we need only use outgoing spherical waves when measuring the field outside of a sphere which completely encompasses the particle.

The T-matrix is an infinite matrix such that

$$f_n = \sum_{n'} T_{nn'} g_{n'}. \tag{4}$$

Such a matrix T exists when scattering is a linear operation (elastic scattering).

2 T-matrix 2D acoustics

We will be following a similar notation as used in A T-Matrix Reduced Order Model Software [4, 3].

For 2D acoustics we have that

$$\mathrm{u}_n(k\boldsymbol{r}) = J_n(kr)\mathrm{e}^{\mathrm{i}n\theta}, \tag{5}$$
$$\mathrm{v}_n(k\boldsymbol{r}) = H_n(kr)\mathrm{e}^{\mathrm{i}n\theta}. \tag{6}$$

When truncating up to some order N we would sum over $n = -N, -N+1, \ldots, N-1, N$.

For instance, if ρ and c are the background density and wavespeed, then for a circular scatterer with density ρ_j, soundspeed c_j and radius a_j, we have that

$$T_{nm} = -\delta_{nm} \frac{q_j J_m'(ka_j) J_m(k_j a_j) - J_m(ka_j) J_m'(k_j a_j)}{q_j H_m'(ka_j) J_m(k_j a_j) - H_m(ka_j) J_m'(k_j a_j)}, \tag{7}$$

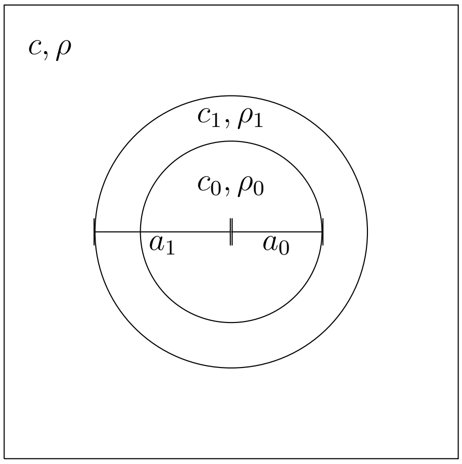
where $q_j = (\rho_j c_j)/(\rho c)$ and $k_j = \omega/c_j$.

2.1 Single circular capsule

$$\psi^0 = \sum_{n=-\infty}^{\infty} g_n^0 J_n(k_0 r)\mathrm{e}^{\mathrm{i}n\theta}, \tag{8}$$
$$\psi^1 = \sum_{n=-\infty}^{\infty} \left[g_n^1 J_n(k_1 r) + f_n^1 H_n(k_1 r)\right] \mathrm{e}^{\mathrm{i}n\theta}. \tag{9}$$

Applying the boundary conditions,

$$\psi^0 = \psi^1 \quad \text{and} \quad \frac{1}{\rho_0}\frac{\partial \psi^0}{\partial r} = \frac{1}{\rho_1}\frac{\partial \psi^1}{\partial r}, \quad \text{on } r = a_0, \tag{10}$$
$$\psi^1 = \psi^{\mathrm{s}} + \psi^{\mathrm{inc}} \quad \text{and} \quad \frac{1}{\rho_1}\frac{\partial \psi^1}{\partial r} = \frac{1}{\rho}\frac{\partial(\psi^{\mathrm{s}} + \psi^{\mathrm{inc}})}{\partial r}, \quad \text{on } r = a_1. \tag{11}$$



Solving these boundary conditions (see capsule-boundary-conditions.nb) leads to

$$
\begin{aligned}
T_{nn} = -\frac{J_n(ka_1)}{H_n(ka_1)} - \frac{Y_{\prime}^{n}(ka_1, ka_1)}{H_n(ka_1)} &\left[Y^n(k_1a_1, k_1a_0)J_n'(k_0a_0) - q_0 J_n(k_0a_0) Y_{\prime}^{n}(k_1a_1, k_1a_0)\right] \\
\times \big[J_n'(k_0a_0)(qH_n(ka_1)Y_{\prime}^{n}(k_1a_0, k_1a_1) &+ H_n'(ka_1)Y^n(k_1a_1, k_1a_0)) \\
+ q_0 J_n(k_0a_0)(qH_n(ka_1)Y_{\prime\prime}^{n}(k_1a_1, k_1a_0) &- H_n'(ka_1)Y_{\prime}^{n}(k_1a_1, k_1a_0))\big]^{-1}. \qquad (12)
\end{aligned}
$$

where $q = \rho c/(\rho_1 c_1)$, $q_0 = \rho_0 c_0/(\rho_1 c_1)$, and

$$
\begin{aligned}
Y^n(x, y) &= H_n(x)J_n(y) - H_n(y)J_n(x), && (13) \\
Y_{\prime}^{n}(x, y) &= H_n(x)J_n'(y) - H_n'(y)J_n(x), && (14) \\
Y_{\prime\prime}^{n}(x, y) &= H_n'(x)J_n'(y) - H_n'(y)J_n'(x). && (15)
\end{aligned}
$$

3 Multiple scattering in 2D

Graf's addition theorem in two spatial dimensions:

$$
H_n(kR_\ell)\mathrm{e}^{\mathrm{i}n\Theta_\ell} = \sum_{m=-\infty}^{\infty} H_{n-m}(kR_{\ell j})\mathrm{e}^{\mathrm{i}(n-m)\Theta_{\ell j}} J_m(kR_j)\mathrm{e}^{\mathrm{i}m\Theta_j}, \quad \text{for} \;\; R_j < R_{\ell j}, \tag{16}
$$

where $(R_{\ell j}, \Theta_{\ell j})$ are the polar coordinates of $\boldsymbol{r}_j - \boldsymbol{r}_\ell$. The above is also valid if we swap H_n for J_n, and swap H_{n-m} for J_{n-m}.

Particle-j scatters a field

$$
u_j = \sum_n f_n^j \mathrm{u}_n(k\boldsymbol{r} - k\boldsymbol{r}_j), \quad \text{for} \;\; |\boldsymbol{r} - \boldsymbol{r}_j| > a_j, \tag{17}
$$

where $\boldsymbol{r}_j$ is the centre of particle j.

Let the incident wave, with coordinate system centred at $\boldsymbol{r}_j$, be

$$u_{\text{in}} = \sum_n g_n^j \mathrm{v}_n(k\boldsymbol{r} - k\boldsymbol{r}_j), \tag{18}$$

then the wave exciting particle-j is

$$u_j^E = \sum_n F_j^n \mathrm{v}_n(k\boldsymbol{r} - k\boldsymbol{r}_j) \tag{19}$$

where

$$F_n^j = g_n^j + \sum_{\ell \neq j} \sum_{p=-\infty}^{\infty} f_p^\ell H_{p-m}(kR_{\ell j}) \mathrm{e}^{\mathrm{i}(p-m)\Theta_{\ell j}}. \tag{20}$$

Using the T-matrix of particle-j we reach $f_n^j = \sum_m T_{nm}^j F_m^j$, which leads to

$$f_q^j = \sum_m T_{qm}^j g_m^j + \sum_{\ell \neq j} \sum_{m,p=-\infty}^{\infty} f_p^\ell T_{qm}^j H_{p-m}(kR_{\ell j}) \mathrm{e}^{\mathrm{i}(p-m)\Theta_{\ell j}}. \tag{21}$$

The above simplifies if we substitute $f_q^j = T_{qd}^j \alpha_d^j$, and then multiple across by $\{T_{qn}^j\}^{-1}$ and sum over q to arrive at

$$\alpha_n^j = g_n^j + \sum_{\ell \neq j} \sum_{m,p=-\infty}^{\infty} H_{p-n}(kR_{\ell j}) \mathrm{e}^{\mathrm{i}(p-n)\Theta_{\ell j}} T_{pm}^\ell \alpha_m^\ell. \tag{22}$$

As a check, if we use (7), then we arrive at equation (2.11) in [5].

In the general formulation below we would have

$$\mathcal{U}_{n'n}(kR_{\ell j}) = H_{n'-n}(kR_{\ell j}) \mathrm{e}^{\mathrm{i}(n'-n)\Theta_{\ell j}}.$$

Note that swapping ℓ for j would result in $\Theta_{\ell j} = \Theta_{j\ell} + \pi$.

4 Multiple scattering in general

For multiple scattering in higher dimensions and for vector wave equations we use the notation given in [6].

For a point $\boldsymbol{r}$, outside of the circumscribed spheres of all particles, we can write the total field $u(\boldsymbol{r})$ as a sum of the incident wave $u_{\text{in}}(\boldsymbol{r})$ and all scattered waves in the form [7, 8, 9]

$$u(\boldsymbol{r}) = u_{\text{in}}(\boldsymbol{r}) + u_{\text{sc}}(\boldsymbol{r}), \quad u_{\text{sc}}(\boldsymbol{r}) = \sum_{i=1}^{N} \sum_n f_n^i \mathrm{u}_n(k\boldsymbol{r} - k\boldsymbol{r}_i), \tag{23}$$

where we assumed $|\boldsymbol{r} - \boldsymbol{r}_i| > a_i$ for $i = 1, 2, \dots N$, the f_n^i are coefficients we need to determine, where again:

$$\begin{cases} \mathrm{u}_n(k\boldsymbol{r}) = \text{outgoing spherical waves}, \\ \mathrm{v}_n(k\boldsymbol{r}) = \text{regular spherical waves}, \end{cases} \tag{24}$$

where n denotes a multi index which depends on the dimension and if the waves are scalar or vector fields.

In general, we can write the multiple scattering system in the form:

$$\alpha_n^i = g_n^i + \sum_{\substack{j=1 \\ j \neq i}}^{N} \sum_{n'n''} \mathcal{U}_{n''n}(k\boldsymbol{r}_i - k\boldsymbol{r}_j) T^j_{n''n'} \alpha^j_{n'}, \tag{25}$$

for $i = 1, 2, \dots, N$, where $f_n^i = \sum_{n'} T^i_{nn'} \alpha^i_{n'}$ and $\mathcal{U}_{nn'}$ is a translation matrix [1, 2]. Let $\boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{d}$, then the translation matrices for a translation $\boldsymbol{d}$ can be defined by the property [1]

$$\begin{cases} \mathrm{v}_n(k\boldsymbol{r}') = \sum_{n'} \mathcal{V}_{nn'}(k\boldsymbol{d}) \mathrm{v}_{n'}(k\boldsymbol{r}), & \text{for all } \boldsymbol{d} \\ \mathrm{u}_n(k\boldsymbol{r}') = \sum_{n'} \mathcal{V}_{nn'}(k\boldsymbol{d}) \mathrm{u}_{n'}(k\boldsymbol{r}), & |\boldsymbol{r}| > |\boldsymbol{d}| \\ \mathrm{u}_n(k\boldsymbol{r}') = \sum_{n'} \mathcal{U}_{nn'}(k\boldsymbol{d}) \mathrm{v}_{n'}(k\boldsymbol{r}), & |\boldsymbol{r}| < |\boldsymbol{d}| \end{cases} \tag{26}$$

4.1 3D acoustics

For all the details on acoustics in three spatial dimensions see [6]. Here we all only provide:

$$\begin{cases} \mathrm{u}_n(k\boldsymbol{r}) = \mathrm{h}_\ell^{(1)}(kr) \mathrm{Y}_n(\hat{\boldsymbol{r}}), \\ \mathrm{v}_n(k\boldsymbol{r}) = \mathrm{j}_\ell(kr) \mathrm{Y}_n(\hat{\boldsymbol{r}}), \end{cases} \tag{27}$$

where $r = |\boldsymbol{r}|$, $n = \{\ell, m\}$, with summation being over $\ell = 0, 1, 2, 3 \dots$ and $m = -\ell, -\ell + 1, \dots, -1, 0, 1, \dots, \ell$, and the spherical Hankel and Bessel functions are denoted $\mathrm{h}_\ell^{(1)}(z)$ and $\mathrm{j}_\ell(z)$, respectively.

4.2 Turing equations into code

For easy implementation we need the functions:

$$\psi_{\text{inc}} \mapsto g_j^m \quad \text{and} \quad \text{particle} \mapsto T_j^{nm}.$$

For efficient implementation we rewrite (22) as a matrix equation. Let

$$(\boldsymbol{\alpha}_j)_n = \alpha_n^j, \quad (\boldsymbol{g}_j)_n = g_n^j, \tag{28}$$

$$(\boldsymbol{T}_j)_{nn'} = T_{nn'}^j, \quad (\boldsymbol{\mathcal{U}}_{j\ell})_{nn'} = \mathcal{U}_{n'n}(k\boldsymbol{r}_j - k\boldsymbol{r}_\ell), \tag{29}$$

Then

$$\sum_\ell (\delta_{j\ell} + (\delta_{j\ell} - 1)\boldsymbol{\mathcal{U}}_{j\ell}\boldsymbol{T}_\ell)\boldsymbol{\alpha}_\ell = \boldsymbol{g}_j, \tag{30}$$

which leads to a block matrix equation:

$$\begin{bmatrix} \boldsymbol{I} & -\boldsymbol{\mathcal{U}}_{12}\boldsymbol{T}_2 & \cdots & -\boldsymbol{\mathcal{U}}_{1(N-1)}\boldsymbol{T}_{N-1} & -\boldsymbol{\mathcal{U}}_{1N}\boldsymbol{T}_N \\ -\boldsymbol{\mathcal{U}}_{21}\boldsymbol{T}_1 & \boldsymbol{I} & -\boldsymbol{\mathcal{U}}_{23}\boldsymbol{T}_3 & \cdots & -\boldsymbol{\mathcal{U}}_{2N}\boldsymbol{T}_N \\ & \vdots & & & \vdots \\ -\boldsymbol{\mathcal{U}}_{N1}\boldsymbol{T}_1 & \cdots & \cdots & -\boldsymbol{\mathcal{U}}_{N(N-1)}\boldsymbol{T}_{N-1} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \vdots \\ \boldsymbol{\alpha}_N \end{bmatrix} = \begin{bmatrix} \boldsymbol{g}_1 \\ \vdots \\ \boldsymbol{g}_N \end{bmatrix} \tag{31}$$

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