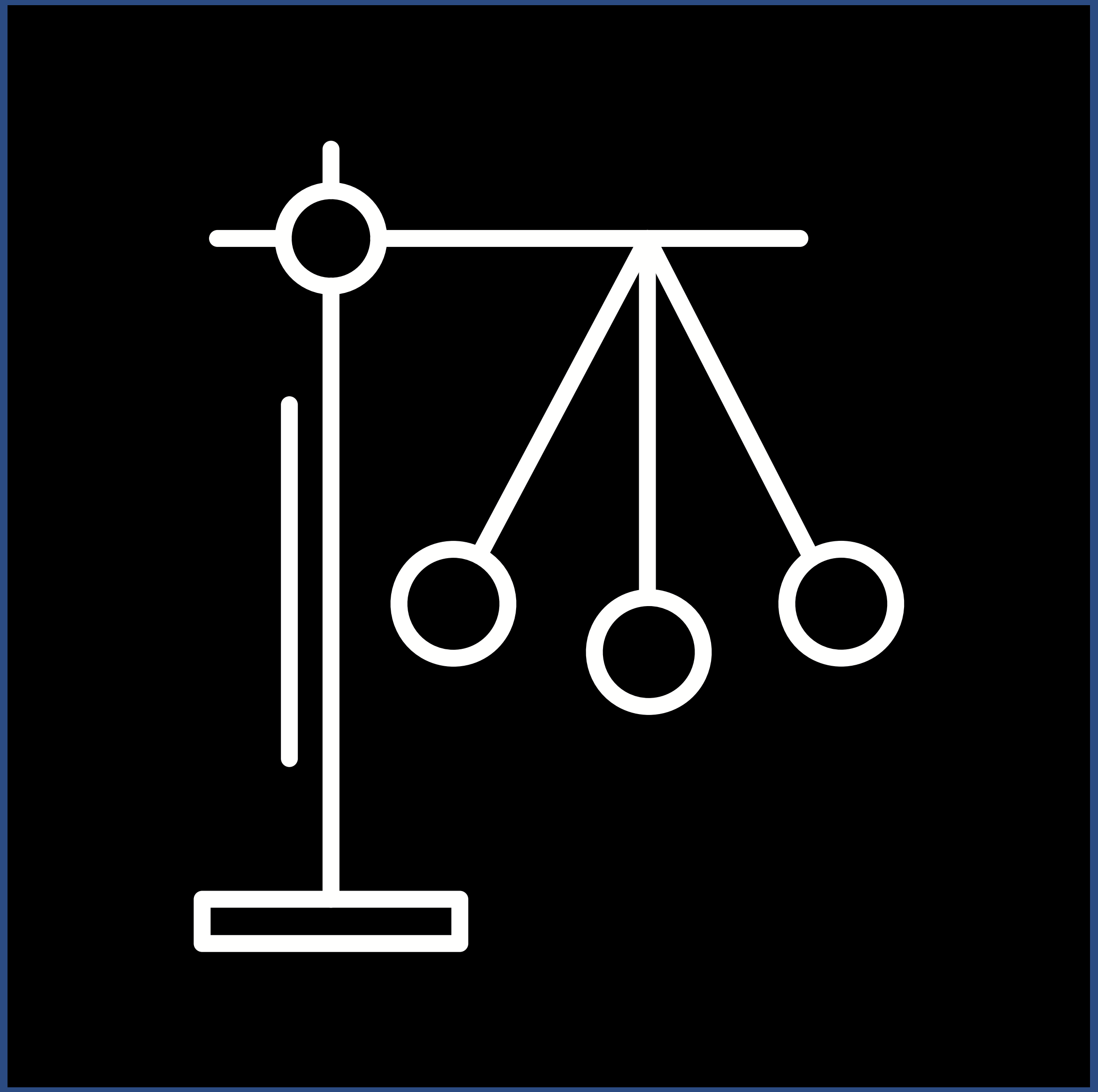
El péndulo



-Julian Leonardo Avila Martinez
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-Juan Sebastian Acuña Tellez
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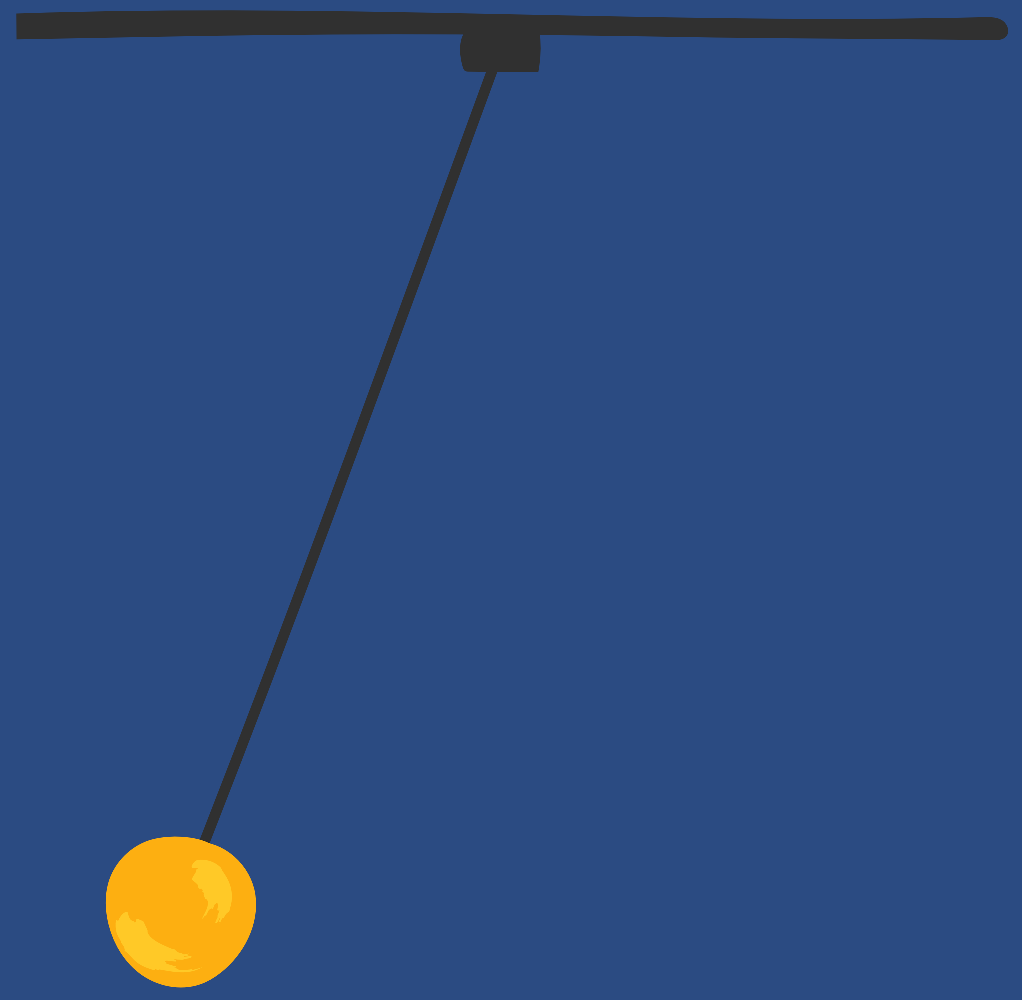
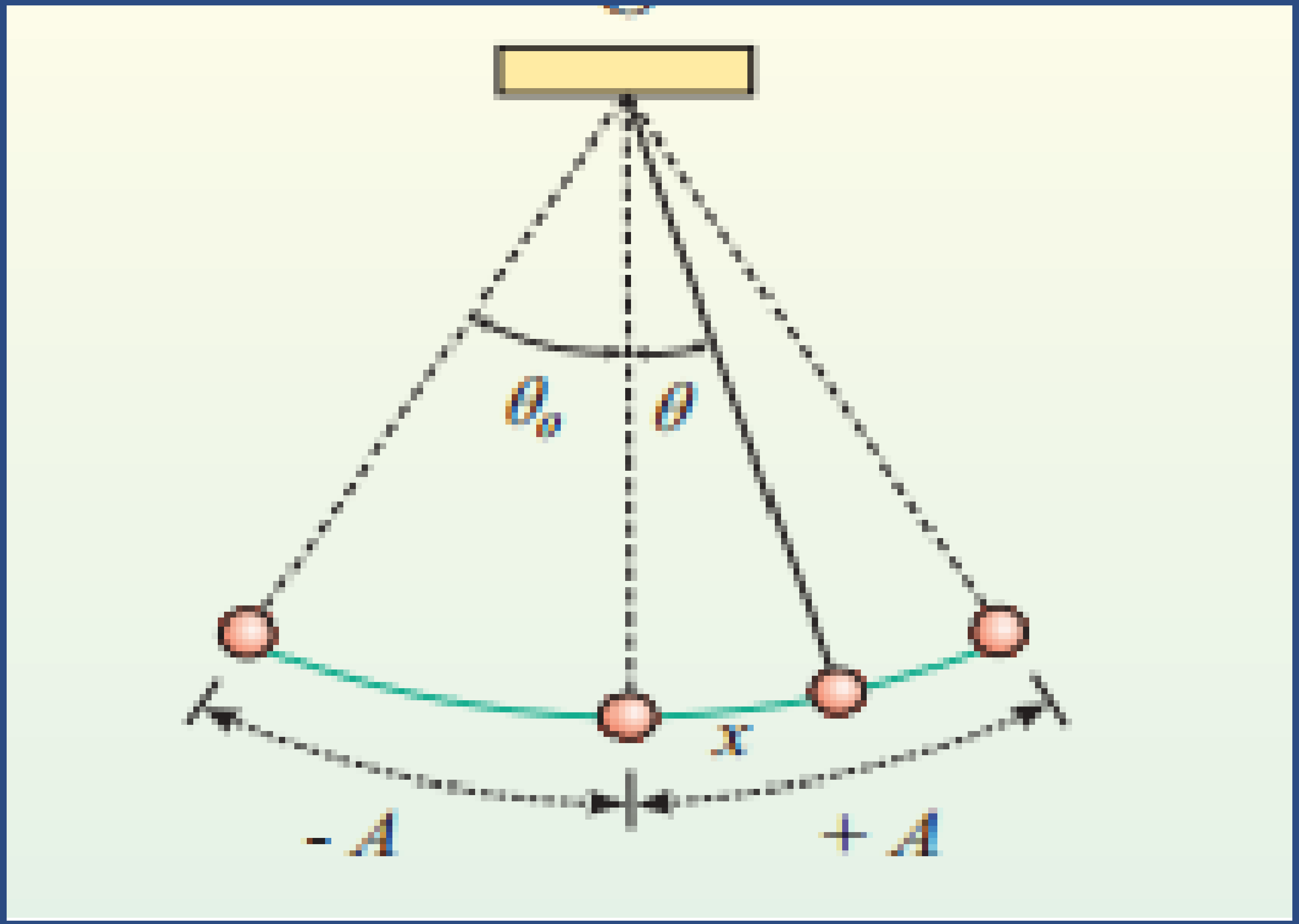
Péndulo simple



Péndulo simple-lineal



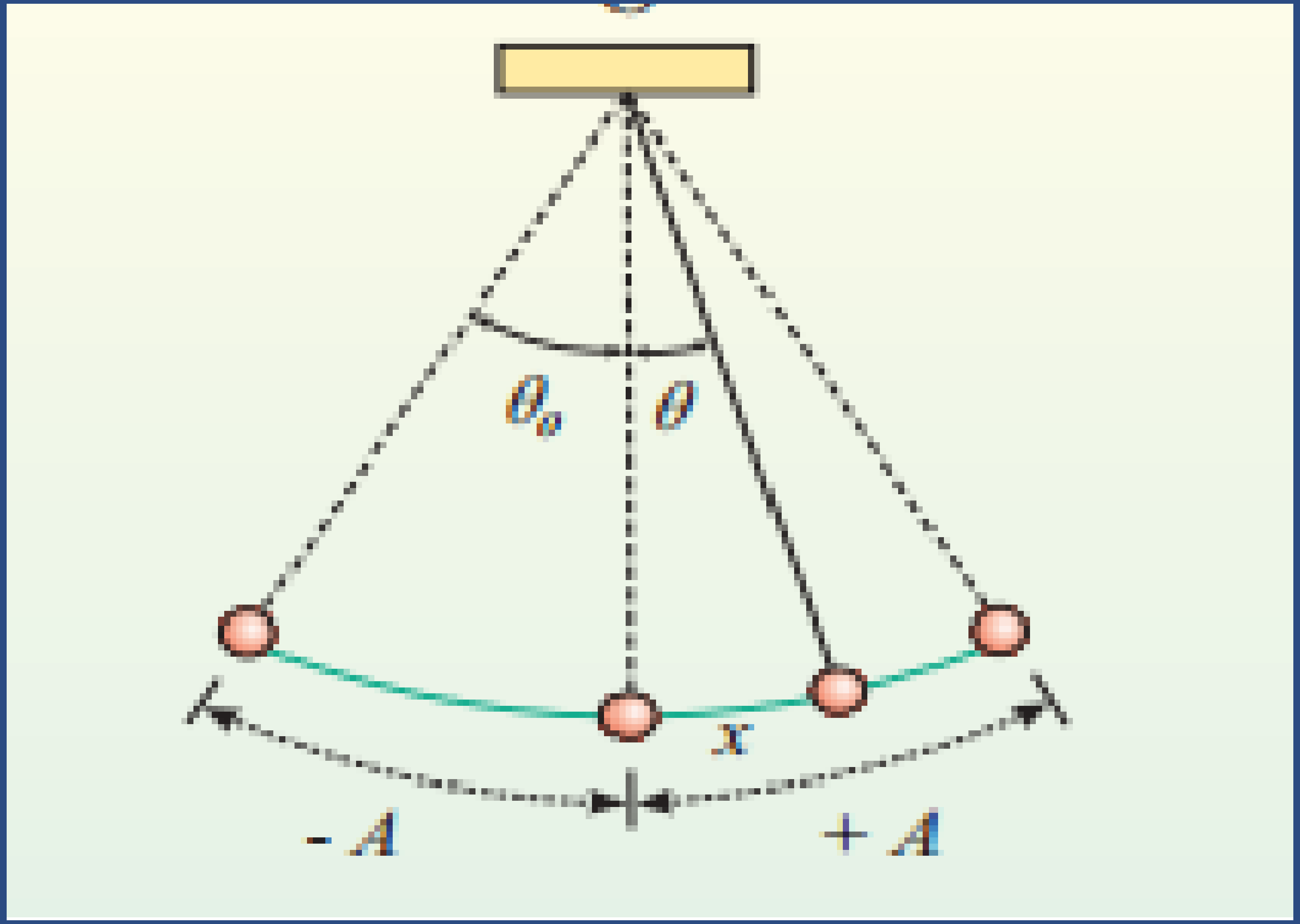
Ecuación

$$\ddot{\theta} + \omega_n^2 \theta = 0$$

Solución

$$\boxed{\theta(t) = \theta_0 \cos \omega_n t + \frac{\omega_0}{\omega_n} \sin \omega_n t}$$

Péndulo simple- no lineal



Ecuación

$$\ddot{\theta} + \omega_n^2 \sin\theta = 0$$

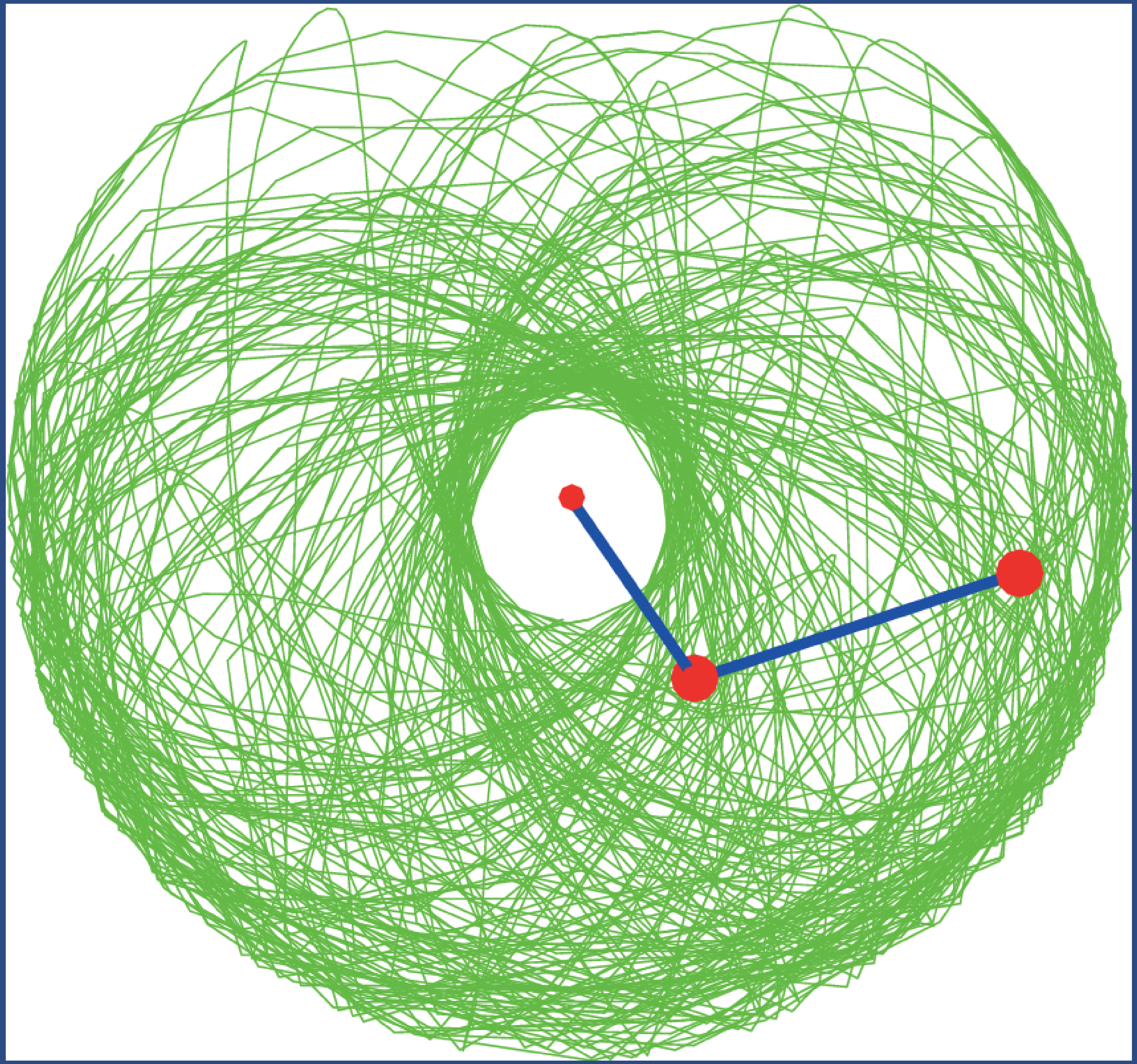
Solución

$$\theta(t) = 2\arcsin\left(\sin\frac{\theta_0}{2} sn\left[K\left(\sin^2\frac{\theta_0}{2}\right) - \omega_n t, \sin^2\frac{\theta_0}{2}\right]\right)$$

$$\dot{\theta}(t) = -2\omega_n \sec\left(\frac{\theta}{2}\right) cn[K(k) - \omega_n t, k] dn[K(k) - \omega_n t, k]$$

$$\boxed{T = \frac{\tau}{\omega_0}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 k^{2n}}$$

Péndulo doble



Péndulo doble

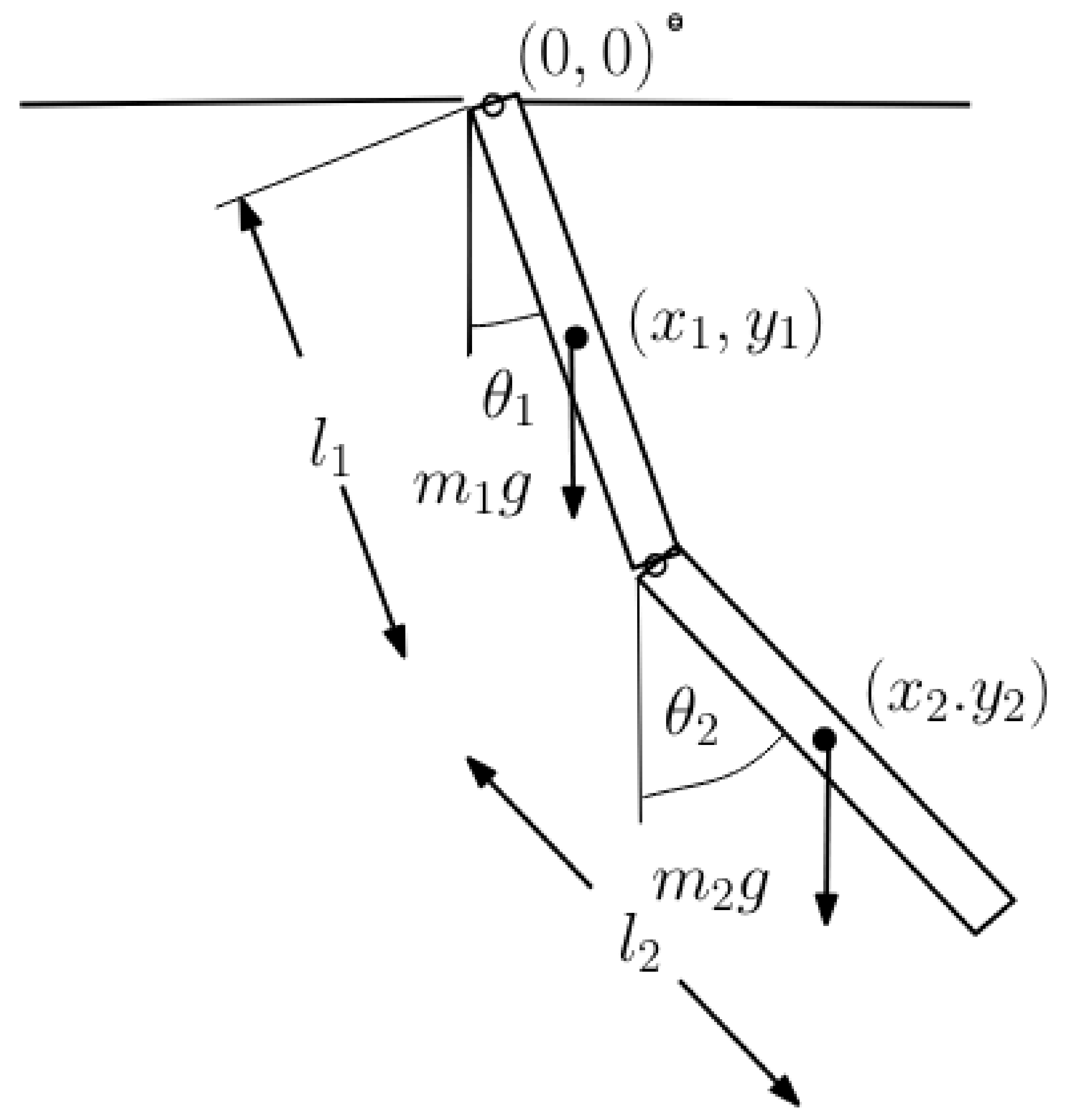


Figura 1: Péndulo doble

$$\mathbf{r_1} = \frac{l_1}{2}\left(\sin\theta_1, -\cos\theta_1\right)$$

$$\mathbf{r_2} = 2\mathbf{r_1} + \frac{l_2}{2}\left(\sin\theta_2, -\cos\theta_2\right)$$

$$\mathbf{v_1} = \dot{\mathbf{r_1}} = \frac{l_1}{2}\dot{\theta_1}\left(\cos\theta_1, \sin\theta_1\right)$$

$$\mathbf{v_2} = \dot{\mathbf{r_2}} = 2\mathbf{v_1} + \frac{l_2}{2}\dot{\theta_2}\left(\cos\theta_2, -\sin\theta_2\right)$$

Péndulo doble

$$L = T - V$$

Enegía cinética

$$T = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \frac{1}{2}I_1 \dot{\theta_1}^2 + \frac{1}{2}I_2 \dot{\theta_2}^2$$

$$v_1^2 = \frac{l_1^2}{4}\dot{\theta_1}^2$$

$$v_2^2 = l_1^2\dot{\theta_1}^2 + l_1\dot{\theta}_1 l_2\dot{\theta}_2 \cos(\theta_1 - \theta_2) + \frac{l_2^2}{4}\dot{\theta}_2$$

$$I_1 = \frac{l_1^2 m_1}{12}$$

$$I_2 = \frac{l_2^2 m_2}{12}$$

$$T = \frac{1}{6}m_1 l_1^2\dot{\theta_1}^2 + \frac{1}{2}m_2\left(l_1^2\dot{\theta}_1^2 + \frac{l_2^2}{4}\dot{\theta}_2^2 + l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right)$$

Péndulo doble

Enegía potencial

$$
\begin{aligned}
V &= -m_1 g y_1 - m_2 g y_2 \\
V &= -m_1 g \frac{l_1}{2}\cos\theta_1 - m_2 g\left(l_1\cos\theta_1 + \frac{l_2}{2}\cos\theta_2\right) \\
V &= -\left(\frac{m_1}{2} + m_2\right) g l_1 \cos\theta_1 - \frac{m_2}{2} g l_2 \cos\theta_2
\end{aligned}
$$

Laplaciano

$$
\begin{aligned}
L &= T - V \\
&= \frac{1}{6} m_1 l_1^2 \dot{\theta_1}^2 + \frac{1}{2} m_2 \left(l_1^2 \dot{\theta}_1^2 + \frac{l_2^2}{4}\dot{\theta}_2^2 + l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)\right) \\
&+ \left(\frac{m_1}{2} + m_2\right) g l_1 \cos\theta_1 + \frac{m_2}{2} g l_2 \cos\theta_2
\end{aligned}
$$

Péndulo doble

Ecuación de Euler-Lagrange

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

Para θ1

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} = \left(\frac{m_1}{3} + m_2\right)\ddot{\theta}_1 l_1^2 + \frac{1}{2} m_2 l_1 l_2 \left(\ddot{\theta}_2 \cos(\theta_1 - \theta_2) - \dot{\theta}_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)\right)$$

$$\frac{\partial L}{\partial \theta_1} = -\frac{1}{2} m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - \left(\frac{m_1}{2} + m_2\right) g l_1 \sin\theta_1$$

$$\boxed{\left(\frac{m_1}{3} + m_2\right) l_1^2 \ddot{\theta}_1 + \frac{1}{2} m_2 l_1 l_2 \left(\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^{\,2} \sin(\theta_1 - \theta_2)\right) + \left(\frac{m_1}{2} + m_2\right) g l_1 \sin\theta_1 = 0}$$

Péndulo doble

Para θ2

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} = \frac{m_2}{3}\ddot{\theta}_2 l_2^2 + \frac{1}{2}m_2 l_1 l_2 \left(\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1 \sin(\theta_1 - \theta_2)\left(\dot{\theta}_1 - \dot{\theta}_2\right)\right)$$

$$\frac{\partial L}{\partial \theta_2} = \frac{1}{2}m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_1 \sin(\theta_1 - \theta_2) - \frac{1}{2}m_2 g l_2 \sin\theta_2$$

$$\boxed{\frac{m_2}{3} l_2^2 \ddot{\theta}_2 + \frac{1}{2}m_2 l_1 l_2 \left(\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1^{\,2} \sin(\theta_1 - \theta_2)\right) + \frac{m_2}{2} g l_2 \sin\theta_2 = 0}$$

Péndulo doble

Sistema de ecuaciones

$$\begin{cases} 2\left(\frac{m_1}{3m_2}+1\right)\frac{l_1}{l_2}\ddot{\theta}_1 + \ddot{\theta}_2\cos(\theta_1-\theta_2) + \dot{\theta}_2^{\,2}\sin(\theta_1-\theta_2) + \left(\frac{m_1}{m_2}+2\right)g\frac{1}{l_2}\sin\theta_1 = 0 \\ \frac{2}{3}\frac{l_2}{l_1}\ddot{\theta}_2 + \ddot{\theta}_1\cos(\theta_1-\theta_2) - \dot{\theta}_1^{\,2}\sin(\theta_1-\theta_2) + \frac{1}{l_1}g\sin\theta_2 = 0 \end{cases}$$

$$\dot{\omega}_1 = \frac{18m_2\cos(\Delta\theta_{12})[g\sin\theta_2 - l_1\dot{\theta}_1^{\,2}\sin(\Delta\theta_{12})] - 12[m_2 l_2\dot{\theta}_2^2\sin(\Delta\theta_{12}) + (m_1+2m_2)g\sin\theta_1]}{l_1(15m_2+8m_1-9m_2\cos(2\Delta\theta_{12}))}$$

$$\dot{\omega}_2 = \frac{18\cos(\Delta\theta_{12})[m_2 l_2\dot{\theta}_2^2\sin(\Delta\theta_{12}) + (m_1+2m_2)g\sin\theta_1] - 12(m_1+3m_2)[g\sin\theta_2 - l_1\dot{\theta}_1^2\sin(\Delta\theta_{12})]}{l_1(15m_2+8m_1-9m_2\cos(2\Delta\theta_{12}))}$$

Método implementados

$$\mathbf{u}'(t) = \mathbf{F}(t, \mathbf{u}(t)), \quad \mathbf{u}(t_0) = \mathbf{u}_0.$$

Método de Euler

$$\mathbf{u} = \mathbf{u}_0 + h\mathbf{F}(t_0, \mathbf{u}_0)$$

Método de RK4

$$\begin{aligned}
\mathbf{k}_1 &= h\mathbf{F}(t_0, \mathbf{u}_0), \\
\mathbf{k}_1 &= h\mathbf{F}(t_0 + h/2, \mathbf{u}_0 + \mathbf{k}_1/2), \\
\mathbf{k}_1 &= h\mathbf{F}(t_0 + h/2, \mathbf{u}_0 + \mathbf{k}_2/2), \\
\mathbf{k}_1 &= h\mathbf{F}(t_0 + h, \mathbf{u}_0 + \mathbf{k}_3), \\
\mathbf{u} &= \mathbf{u}_0 + \frac{1}{6}\left(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4\right).
\end{aligned}$$

Método de Heun

$$\mathbf{w} = \mathbf{u}_0 + h\mathbf{F}(t_0, \mathbf{u}_0), \quad \mathbf{u} = \mathbf{u}_0 + \frac{h}{2}\left(\mathbf{F}(t_0, \mathbf{u}_0) + \mathbf{F}(t_0 + h, \mathbf{w})\right)$$

Implementación del código

Resultados