

# Quantum Entanglement

Spooky and Hidden

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Avila, Herrera, Rodríguez

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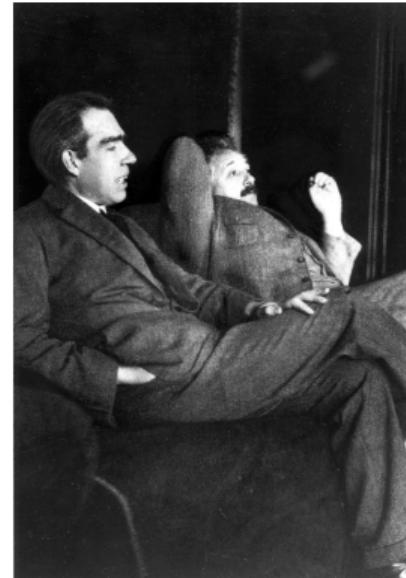
Universidad Distrital Francisco José de Caldas

## EPR Debate

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# The Clash of Titans: Einstein's Realism

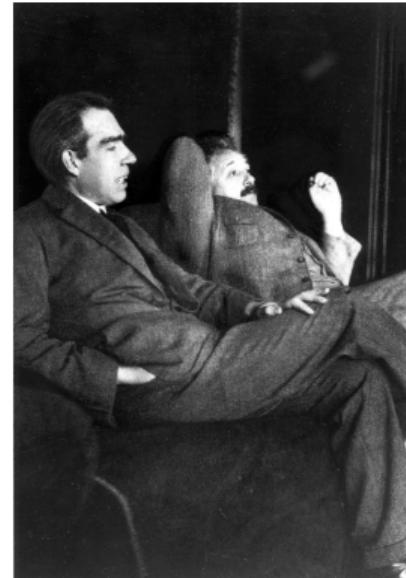
- **Deterministic View:** The universe is knowable and orderly.
- **Objective Reality:** Physical properties exist independently of measurement.
- **No Fundamental Randomness:** Probability is not intrinsic to nature.



**Figure 1:** Niels Bohr with Albert Einstein at Paul Ehrenfest's home in Leiden [12].

# The Clash of Titans: Einstein's Realism

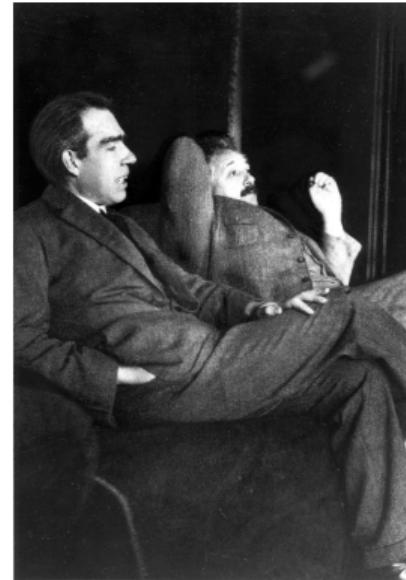
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# The EPR Paradox: A Challenge in 1935

## The Seminal Paper

- In 1935, Einstein, Podolsky, and Rosen published a key paper.
- A “frontal assault” on the conceptual foundations of quantum mechanics.
- Their goal: To question whether the quantum description of reality is *complete*.
- The title: “Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?”.



**Figure 2:** Paper: Can Quantum-Mechanical Description of Physical Reality Be Considered Complete? [7]

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In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the theory to be considered complete is that it contain no elements which do not correspond to an element of reality. If it is impossible, by recombining elements, the knowledge of one element of reality to be obtained. If this is possible, the description of reality given by the wave function is not complete.

A NY serious consideration of a physical theory must take into account the distinction between the objective reality, which is independent of any theory, and the physical concepts with which we theorize about it. These concepts are intended to correspond with the knowledge of the objective reality that we can acquire by picking up the reality to ourselves.

Any attempt to judge the success of a physical theory must therefore answer two questions: (1) "Is the theory correct?" and (2) "Is the theory complete?" It is only in the case in which positive answers may be given to both of these questions, that the concept of a theory can be regarded as being meaningful. The correctness of the theory is judged by the degree of agreement between the results of the theory and the results of observation. The correctness, which alone counts in making a theory acceptable, is not the same as its completeness. Completeness is only a necessary condition for the acceptability of a theory; it is not a sufficient condition.

The question, whether the quantum-mechanical description of physical reality is complete or not, can be reduced to the following two questions: (1) "Is the theory correct?" and (2) "Is the theory complete?" It is only in the case in which positive answers may be given to both of these questions, that the concept of a theory can be regarded as being meaningful. The correctness of the theory is judged by the degree of agreement between the results of the theory and the results of observation. The correctness, which alone counts in making a theory acceptable, is not the same as its completeness. Completeness is only a necessary condition for the acceptability of a theory; it is not a sufficient condition.

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In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the theory to be considered complete is that it be possible to associate with every element of reality an observable concept which can be realized in experience. These concepts are intended to correspond with the elements of reality in such a way that the theory may be regarded as giving a complete description of reality given by the wave function. It

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is appropriate to judge the success of a physical theory by two questions: (1) "Is the theory correct?" and (2) "Is the theory meaningful?" It is only in the case in which positive answers may be given to both of these questions, that the concept of a theory may be regarded as being meaningful. The correctness of the theory is judged by the degree of agreement between the concepts of the theory and the results of observation. The experiences, which alone enable us to make this comparison, are the results of experiment and measurement. It is the second question that we wish to consider here, in relation to quantum mechanics.

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# The EPR Criterion of Reality

## The “rule” to define an element of reality

To formalize their attack, EPR introduced an explicit and, seemingly, irrefutable criterion:

*“If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.”*

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## Violation of the Criterion: Position and Momentum

Example with a state  $\psi = e^{(2\pi i/\hbar)p_0 x}$

- When applying the momentum operator ( $p$ ), we get an exact value:

$$p\psi = \left( \frac{\hbar}{2\pi i} \frac{\partial}{\partial x} \right) \psi = p_0 \psi$$

It is concluded that the momentum,  $p_0$ , is an **element of reality**.

- However, when applying the position operator ( $x$ ), an exact value is not obtained:

$$x\psi \neq \text{constant} \cdot \psi$$

We can only calculate the **probability** of finding the particle in an interval  $[a, b]$ , which turns out to be  $P(a, b) = b - a$ .

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## The Fundamental Dichotomy

From the previous example, it follows that there are only two possibilities:

1. Either the quantum-mechanical description of reality given by the wave function **is not complete**.
2. Or when the operators for two physical quantities do not commute, the two quantities **cannot have simultaneous reality**.

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# The Principle of Locality

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## A fundamental idea in physics

- An object can only be directly influenced by its **immediate surroundings**.
- Any influence at a distance cannot be instantaneous; it must propagate at a finite speed, not exceeding the speed of light ( $c$ ).
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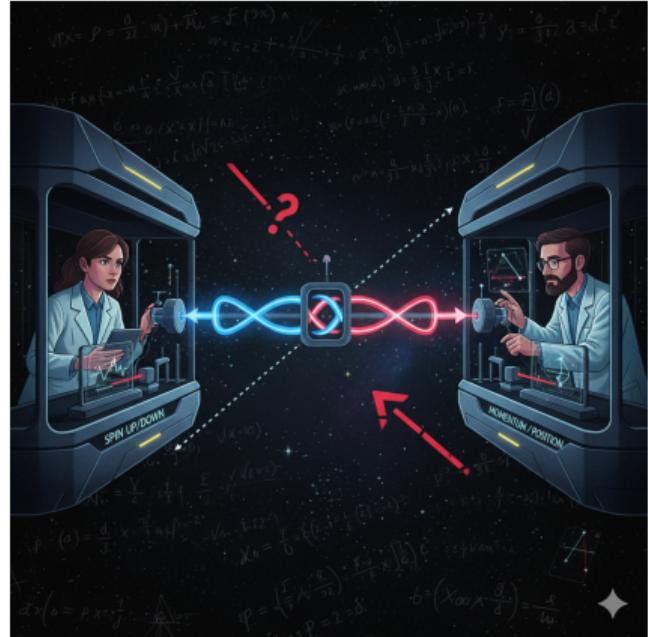
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# The EPR Thought Experiment

## The two-particle system

- A system of two particles is prepared in an entangled state and then separated by a large distance.
- Option 1: If Alice measures the momentum of her particle, she can predict with certainty the momentum of Bob's particle.
- Option 2: If Alice measures the position, she can predict with certainty Bob's position.
- Partial Conclusion: Both the position and momentum of Bob's particle must be “elements of reality”.

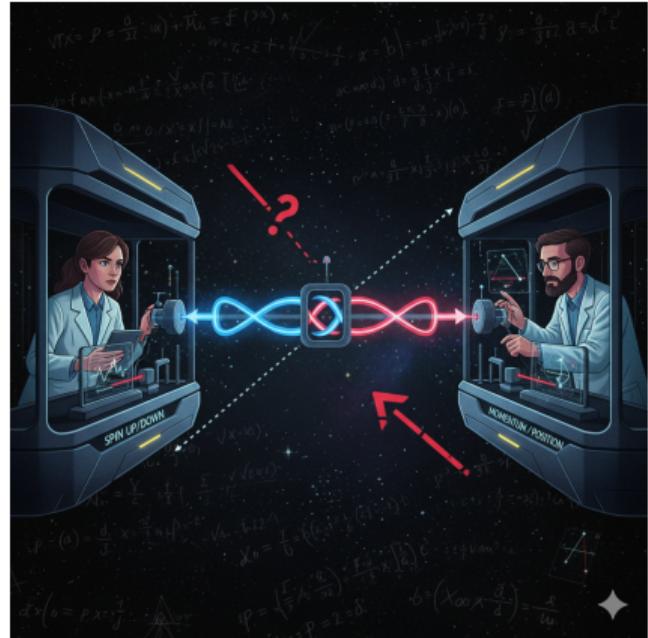


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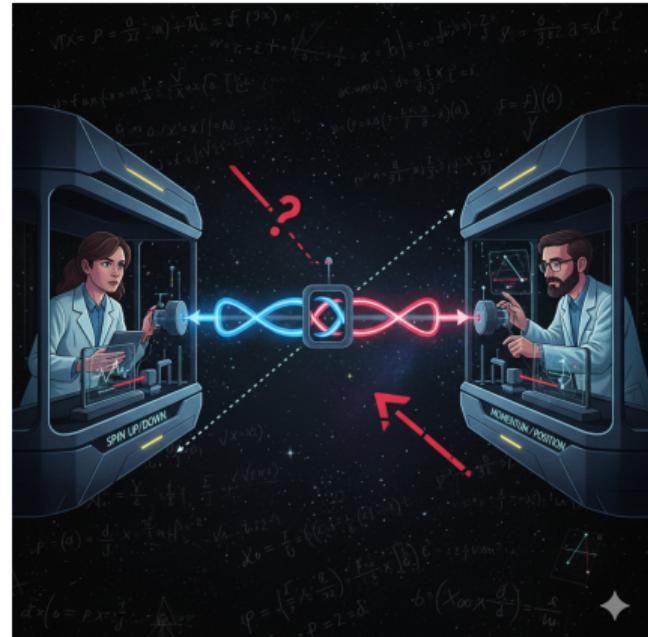


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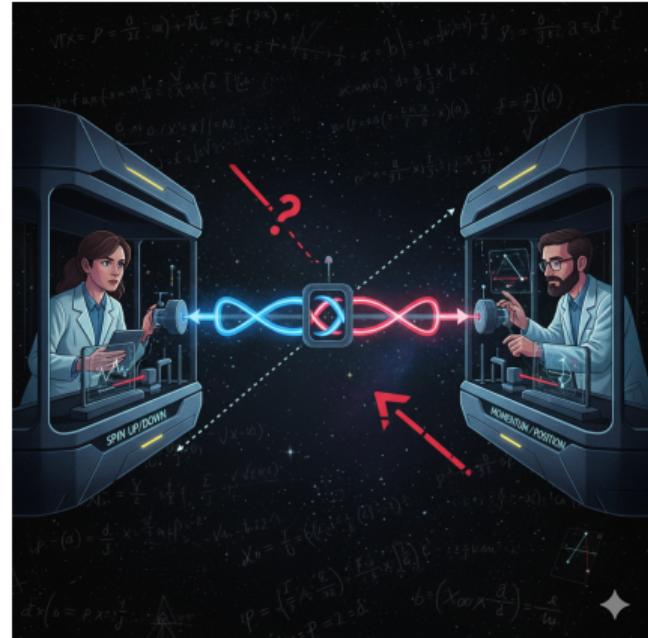


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# The Contradiction and EPR's Conclusion

## The Conflict with Quantum Mechanics

- The conclusion that Bob's particle has a definite position and momentum **simultaneously...**
- ...is in direct conflict with the Heisenberg Uncertainty Principle, which forbids it.

## The EPR Conclusion

Faced with this dilemma, if local realism is correct, the only possible conclusion is that:

- The description of reality provided by the wave function must be incomplete.
- There must be an underlying theory ("hidden variables") that provides a complete description.

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# Bohr's Response: The Principle of Complementarity

## Mutually Exclusive, Yet Necessary Concepts

- Complementarity posits that pairs of classical concepts are necessary for a complete description.
- But they are *mutually exclusive* in any single experiment.
- They are not contradictory properties of the object, but complementary aspects of a **phenomenon** revealed by incompatible experimental arrangements.



Figure 4: Niels Bohr [14].

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# Is Quantum Mechanics Complete? Bohr's View

## Radical Revision of Physical Reality

- The experimenter's "freedom of choice" does not reveal pre-existing realities, but rather **creates different experimental conditions** and phenomena.
- Quantum mechanics is **complete** because it correctly and exhaustively describes the results of *every possible, well-defined experiment*.
- Bohr compares this revision of physical reality to the modification of ideas about the absolute character of phenomena introduced by the **Theory of Relativity**.
- This philosophical debate was crucial and led to **Bell's Theorem** and subsequent experiments, turning EPR into a testable question.

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# Schrödinger's Response: The Birth of “Entanglement”

## Context and the Essence of the New Physics

- A few months after EPR, Schrödinger published his response: “Discussion of Probability Relations between Separated Systems”.
- He demonstrated a deep understanding and took the EPR argument to its most extreme conclusions.
- He understood that the phenomenon was not an anomaly, but the characteristic trait of quantum mechanics.
- He coined the term “Entanglement” (Verschränkung).



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# “Steering” and the Schmidt Decomposition

## The most unsettling aspect of Entanglement

- Schrödinger identified an experimenter's ability to “steer” (**steuern**) the state of a distant system.
- This is achieved through the local choice of measurement.
- He described it as “rather discomforting that the theory allows a system to be steered... at the whim of the experimenter, even though they have no access to it”.
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# Bohr vs. Schrödinger: Two Approaches to the Paradox

## Fundamental Differences in the Response to EPR

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- **Schrödinger:** Accepted the validity of EPR's questions and delved into the unsettling answers the theory provided, seeking to *intensify the paradox*.

## Schrödinger's Key Contributions

- Named and defined the central concept: He coined the term “entanglement”.
- Identified its most powerful manifestation: The concept of “steering,” which captured the essence of non-locality.
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## Bohm-Aharonov

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## A clearer, more experimental approach

- In 1951, **David Bohm** reformulated the EPR paradox in a conceptually clearer way.
- Together with Yakir Aharonov (in 1957), they simplified the example from continuous variables to the discrete variable of spin.
- This made the thought experiment much more amenable to experimental verification.



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# The Spin Singlet State

## An example with spin-1/2 particles

- Consider a molecule with total spin zero that decays into two atoms (A and B) with spin 1/2.
- It is an entangled state where the total spin is zero, regardless of the measurement axis.

The entangled wave function:

$$\Psi = \frac{1}{\sqrt{2}} (\Psi_+(1)\Psi_-(2) - \Psi_-(1)\Psi_+(2))$$

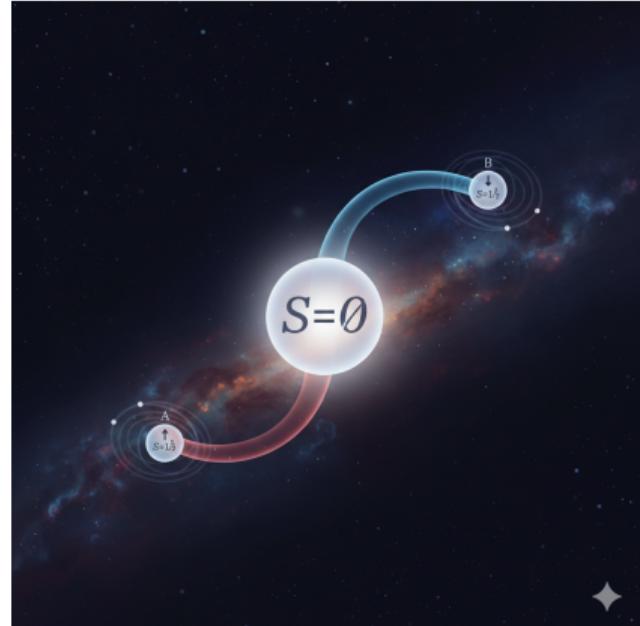


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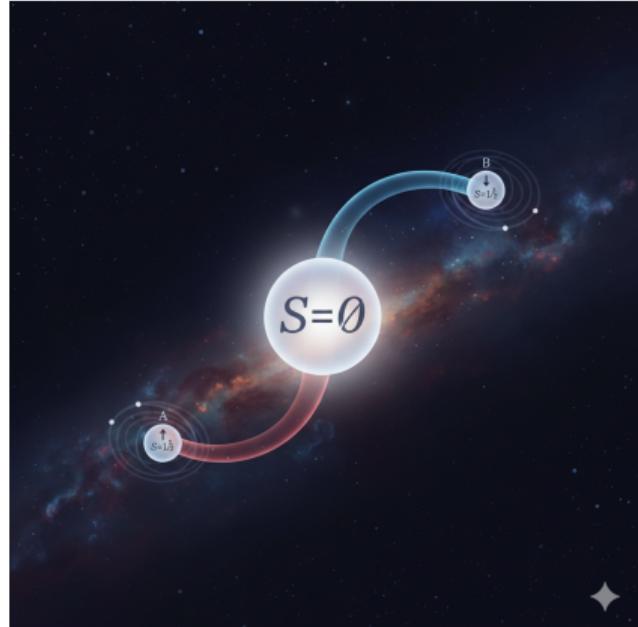


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# The Core of the Paradox: Classical vs. Quantum

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## Classical Interpretation (Local Realism)

- The atoms have **pre-existing** and perfectly anti-correlated spin vectors.
- The measurement on A only reveals an already existing property.
- There is no instantaneous influence at a distance; the correlation was established locally at the source.

## Quantum Interpretation

- A particle's spin state is **not defined** until it is measured.
- The choice to measure A's spin **realizes** its value, and due to entanglement, simultaneously realizes the opposite value for B.
- This happens **instantaneously**, despite the distance, seemingly violating locality.

**Key conclusion:** Quantum uncertainty in entangled systems is an intrinsic property of the **non-separable correlations** that define the state of the system as a whole.

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- The atoms have **pre-existing** and perfectly anti-correlated spin vectors.
- The measurement on A only **reveals** an already existing property.
- There is no instantaneous influence at a distance; the correlation was established locally at the source.

## Quantum Interpretation

- A particle's spin state is **not defined** until it is measured.
- The choice to measure A's spin realizes its value, and due to entanglement, simultaneously realizes the opposite value for B.
- This happens **instantaneously**, despite the distance, seemingly violating locality.

Key conclusion: Quantum uncertainty in entangled systems is an intrinsic property of the **non-separable correlations** that define the state of the system as a whole.

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# The Alternative Hypothesis: Entanglement Breaking

## Entanglement that “decays” with distance

- The idea is raised that the quantum many-body formulation **might not be valid** for widely separated particles.
- The proposal suggests that entanglement is a phenomenon that decreases with distance.
- An alternative “classical” state is proposed for the system after separation.

Alternative state (statistical mixture):

$$\Psi = \Psi_{+\theta,\phi}(1)\Psi_{-\theta,\phi}(2)$$

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## Consequences: Quantum Superposition vs. Classical Mixture

### Implications of the Breaking Hypothesis

- **Avoidance of the EPR Paradox:** It restores local realism; the measurement on A only reveals a pre-existing spin direction.
- **Violation of Angular Momentum Conservation:** In individual events, total angular momentum would not be conserved (but it would on average).
- This pre-existing spin direction acts as a “hidden variable”.

### Crucial Difference: Quantum Superposition vs. Statistical Mixture

- **Quantum Superposition (Entanglement):** Maintains perfect correlations in any measurement basis.
- **Classical Statistical Mixture:** Only exhibits perfect correlations in the basis defined by the hidden variable.

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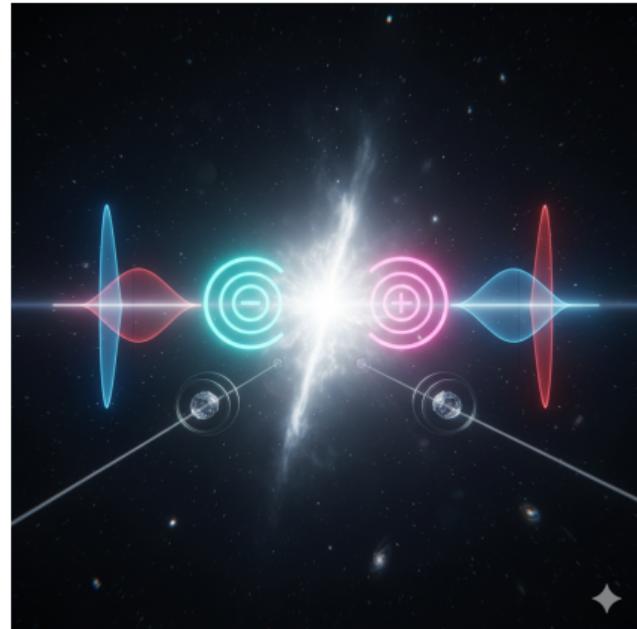
# An Experimental Analogue: Photon Polarization

## From Spin to Polarization

- A more viable analogue is proposed: the **polarization** of photons from **positron-electron annihilation**.
- Conservation Laws: Due to conservation of angular momentum and parity, the two emitted photons must have mutually perpendicular polarizations.

Wave function of the system:

$$\Phi = \frac{1}{\sqrt{2}} (C_1^x C_2^y - C_1^y C_2^x) \Psi_0$$



**Figure 8:** Scheme of the experiment: a source emits two entangled photons

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**Figure 8:** Scheme of the experiment: a source emits two entangled photons

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### A Reinterpretation of Existing Data

- Bohm and Aharonov did not propose a new experiment, but reinterpreted data already published by **Chien-Shiung Wu and Irving Shaknov** in 1950.
- Their contribution was theoretical: to unveil the fundamental implications of a previously known experimental result.

### Measurement Process and Setups

- Polarization is measured indirectly via Compton Scattering.
- The rate of coincidences (simultaneous detection of both photons) is measured in two geometries:
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# The Experimental Verdict: The Ratio R

## The Crucial Ratio and the Predictions

The ratio  $R$  is defined as the quotient of the coincidence rates between the two geometries:

$$R = \frac{\text{Coincidence Rate(parallel)}}{\text{Coincidence Rate(perpendicular)}}$$

- Quantum Mechanics Prediction (Entangled State):
  - $R = 2.00$
- Local Realism Predictions (Statistical Mixture):
  - Circular Polarization (B1):  $R = 1.00$
  - Linear Polarization (B2):  $R < 2$  (approx. 1.5)

The Experimental Result (Wu, 1950):  $R = 2.04 \pm 0.08$

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John Bell

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## Over 80 Years Apart



Albert Einstein



Physics Nobel Prize

# Formulations vs Interpretations

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## Formulations of Quantum Mechanics

- Heisenberg's Matrix Mechanics
- Schrödinger's Wave Mechanics
- Hilbert Space Formalism (von Neumann)
- Bohmian Mechanics (Wave function decomposition:  $R$  and  $S$ )

## Interpretations of Quantum Mechanics

- Copenhagen Interpretation (Statistical, Collapse of the wavefunction)
- EPR Argument and Reality Criterion
- de Broglie-Bohm Interpretation (Hidden Variables, Deterministic)
- Quantum Potential and Nonlocality
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# Mathematische Grundlagen der Quantenmechanik

## MATHEMATISCHE GRUNDLAGEN DER QUANTENMECHANIK

VON

JOHANN V. NEUMANN

MIT 4 ABBILDUNGEN



BERLIN  
VERLAG VON JULIUS SPRINGER  
1932

*First German edition (1932)*

## Fundamentos matemáticos de la mecánica cuántica

John von Neumann

Estudio preliminar de  
José M. Sánchez Ron

3.ª edición

$$\begin{aligned}
 & -\|f\| - \|f' - f_n\| \leq \|f\| - \|f_n\| + \|g - f_n\| \leq \frac{1}{2} \sqrt{2} + \frac{1}{2} \sqrt{2} = \sqrt{2}, \\
 & \left( \sum_{v=1}^V \alpha_v \varphi_v, \varphi_i \right) = \left( \lim_{n \rightarrow \infty} \sum_{v=1}^V \alpha_v \varphi_v, \varphi_i \right) = \lim_{n \rightarrow \infty} \left( \sum_{v=1}^V \alpha_v \varphi_v, \varphi_i \right) = \sum_{v=1}^V \alpha_v \varphi_v, \varphi_i, \\
 & 0 \leq \sum_{v=1}^V |\alpha_v|^2 - \sum_{v=1}^V |\beta_v|^2 < \varepsilon^2 \quad \left\| \sum_{v=1}^V (\rho_v + i\sigma_v) f_n \right\| \leq \sum_{v=1}^V \left\| (\rho_v + i\sigma_v) f_n \right\|, \\
 & E = \frac{\delta^2}{\left( \sum_{v=1}^V (\rho_v + \sigma_v)^2 \right)^{\frac{1}{2}}} \quad \frac{\|f\| - \|f' - f_n\|}{\|f\| - \|f' - f_n\| + \|f - g\|} = \frac{\|f\| - \|f'\|}{\|f\| - \|f'\| + \|f - g\|} = \frac{\|f\|}{\|f\| + \|g\|}, \\
 & \text{Re } (f, g) \leq \|f\| \cdot \|g\| \\
 & \int_Q |f_{n+1} - f_n|^2 \geq \left( \frac{1}{2^r} \right) \mu^{(r)} = \frac{\mu^{(r)}}{4^r} \quad r = r+1, \\
 & \text{Re } (f, g) \leq \|f\| \cdot \|g\| \quad (af', gf') = a\text{Re}(f, f) = |a|^2 \cdot \langle f, f \rangle, \quad \|a\| = \sqrt{|\text{Re}(f, f)| + |\text{Im}(f, f)|^2} = \sqrt{|\text{Re}(f, f)|^2 + |\text{Im}(f, f)|^2}, \\
 & f = \sum_v x_v \varphi_v, \quad \text{Re}(f, g) = \text{Re}(f, f) = \|f\|^2, \\
 & \lim_{n \rightarrow \infty} \langle f_n, f_n \rangle = \lim_{n \rightarrow \infty} \langle f_n, f_n \rangle, \quad \text{Im}(f, g) = \text{Im}(f, f) = \|f\|^2, \\
 & \|f\|^2 = \|f\| - \|f' - f_n\| \leq \|f\| - \|f_n\| + \frac{1}{2} \sqrt{2} + \frac{1}{2} \sqrt{2} = \sqrt{2}, \\
 & \langle f, f \rangle - \langle f', f \rangle = \langle f, f \rangle - \langle f', f \rangle = \langle f, f \rangle - \langle f', f \rangle = \langle g, f \rangle + \langle f, g \rangle, \\
 & \leq \|g\| \cdot \|f\| + \|f\| \cdot \|g\| = \|g\| \cdot \|f\| + \|f\| \cdot \|g\|, \\
 & \leq \|g\| \cdot \|f\| + \|g\| \cdot \|f\| = 2\|g\| \cdot \|f\|, \\
 & f = \sum_{v=1}^V a_v \varphi_v, \quad \|f - f'\| < \varepsilon, \\
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 & (f, f) = \sum_{v=1}^V a_v \bar{a}_v \langle \varphi_v, \varphi_v \rangle + \sum_{v=1}^V x_v \langle \varphi_v, \psi \rangle + \sum_{v=1}^V \bar{x}_v \langle \psi, \varphi_v \rangle + \langle \psi, \psi \rangle, \\
 & \left| \sum_{v=1}^V a_v \bar{a}_v \langle \varphi_v, \varphi_v \rangle \right| = \left| \sum_{v=1}^V a_v \bar{a}_v \right| \leq \varepsilon, \\
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 & = \sum_{v=1}^V |a_v|^2 = \sum_{v=1}^V |x_v|^2 = \sum_{v=1}^V |\bar{x}_v|^2 = \sqrt{|\mu|}.
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Consejo Superior de  
Investigaciones Científicas

*Spanish translation (CSIC, 3rd edition)*

## Von Neumann's Axioms (Section IV.1)

Von Neumann does not start from the quantum formalism itself, but rather establishes general axioms for the mean value of any physical magnitude  $R$  in a statistical ensemble, denoted as  $V_m(R)$ .

A'. **Positivity:** If a magnitude  $R$  is intrinsically non-negative (e.g. the square of another quantity), then

$$V_m(R) \geq 0$$

B'. **Linearity:** For arbitrary magnitudes  $R, S, \dots$  and real numbers  $a, b, \dots$ :

$$V_m(aR + bS + \dots) = aV_m(R) + bV_m(S) + \dots$$

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## Statistical Operator and Impossibility Result (Section IV.2)

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Von Neumann proves a fundamental theorem:

Any expectation value function  $V_m(R)$  that satisfies axioms A' and B' can be uniquely represented by a Hermitian, positive operator  $U$ , called the *statistical operator* (today: density matrix), via the trace formula:

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## Conditions for a Hidden-Variable System

In the hidden-variable hypothesis, any physical magnitude  $R$  must satisfy the following conditions:

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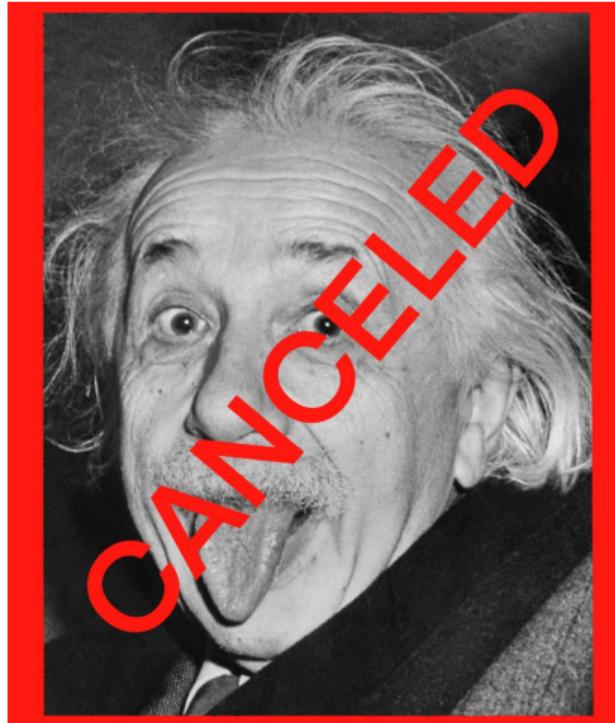
## Von Neumann's Impossibility Proof

The core of von Neumann's argument is to show that there is no statistical operator  $U$  (and therefore no expectation-value function  $V_m(R)$ ) that can simultaneously satisfy the linearity hypothesis (B') and the condition of absence of dispersion for all observables.

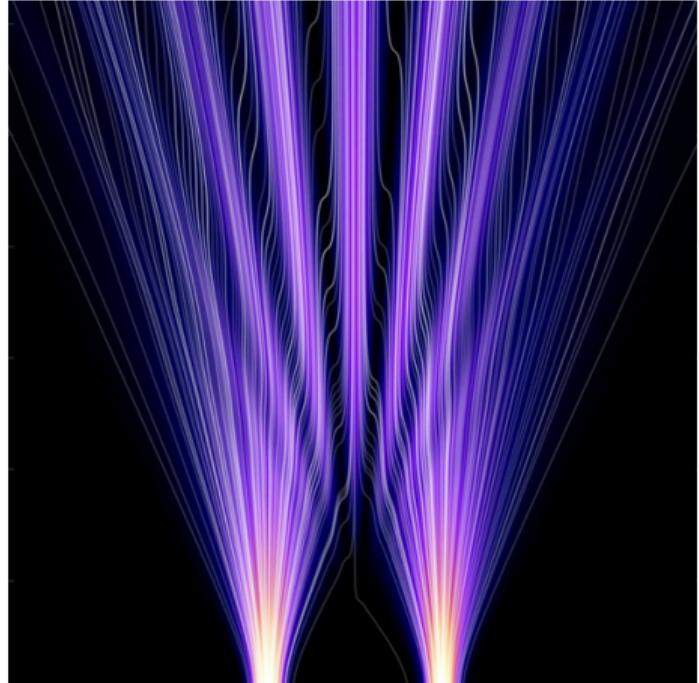
$$V_m(RS) + V_m(SR) = 2 V_m(R) V_m(S)$$

For non-commuting observables ( $[R, S] \neq 0$ ), there is in general no assignment of values  $V_m(\cdot)$  that can satisfy this for all operators.

# Bhom And Hidden Variables



Albert Einstein



Bhom Trajectories

# Bhom Hidden Variables Papers

PHYSICAL REVIEW

VOLUME 85, NUMBER 2

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## A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables. I

David Bohm\*

Princeton Physical Laboratory, Princeton University, Princeton, New Jersey  
(Received July 5, 1951)

The usual interpretation of the quantum theory is self-consistent, but it involves an assumption that cannot be tested experimentally, viz., that the most complete possible specification of an individual system is in terms of a wave function that determines only probable results of actual measurement processes. The author suggests an alternative interpretation, which is based on trying to find some other interpretation of the quantum theory in terms of at present "hidden" variables, which in principle determine all the details of the behavior of the system, but which in practice averaged over measurements of the types that can now be carried out. In this paper and in a subsequent paper, an interpretation is suggested in which the hidden variables are "local," i.e., "hidden" variables are suggested. It is shown that so long as the mathematical theory retains its present general form, this suggested interpretation leads to precisely the same results for all

physical processes as does the usual interpretation. Nevertheless, the suggested interpretation provides a broader conceptual framework than the usual interpretation, because it makes possible a precise and continuous description of all processes, even at the quantum level. This broader conceptual framework allows more realistic physical models to be developed, and it also allows those allowed by the usual interpretation. Now, the usual mathematical formalism seems to lead to insoluble difficulties when it is extended to cover the suggested interpretation, unless one uses a theory of averages over measurements of the types that can now be carried out. In this paper and in a subsequent paper, an interpretation is suggested in which the hidden variables are "non-local," i.e., "hidden" variables are suggested. It is shown that so long as the mathematical theory retains its present general form, this suggested interpretation leads to precisely the same results for all

### 1. INTRODUCTION

THE usual interpretation of the quantum theory is based on an assumption having very far-reaching implications, viz., that the physical state of an individual system is completely specified by a wave function that determines only the probabilities of actual results that can be obtained in a statistical averaging of many individual states, which changes with time, are determined by definite laws, analogous to (but not identical with) the classical equations of motion. Quantum-mechanical probabilities are regarded (like their counterparts in classical mechanics) as being only an indication of incomplete knowledge, and not as a manifestation of an inherent lack of complete determination in the properties of matter at the quantum level. As long as the present general form of Schrödinger's equation is retained, the physical results obtained with our suggested alternative interpretation are precisely the same as those obtained with the usual interpretation. We add, we see, however, that this alternative interpretation permits modification of the mathematical formulation which could not even be described in terms of the usual interpretation. Moreover, the modifications can quite easily be formulated in such a way that their effects are insignificant in the usual domain of applications. The present quantum theory is in fair good agreement with experiment, but of crucial importance in the domains of dimensions of the order of  $10^{-3}$  cm, where, as we have seen, the present theory is totally inadequate. It is thus entirely possible that some of the modifications describable in terms of our suggested alternative interpretation, but

tions have as yet been suggested. The purpose of this paper (and of a subsequent paper hereafter denoted by II) is, however, to suggest just such an alternative interpretation. In contrast to the usual interpretation, this alternative interpretation permits us to conceive of each individual system as being in a definite definable state, whose changes with time are determined by definite laws, analogous to (but not identical with) the classical equations of motion. Quantum-mechanical probabilities are regarded (like their counterparts in classical mechanics) as being only an indication of incomplete knowledge, and not as a manifestation of an inherent lack of complete determination in the properties of matter at the quantum level. As long as the present general form of Schrödinger's equation is retained, the physical results obtained with our suggested alternative interpretation are precisely the same as those obtained with the usual interpretation. We add, we see, however, that this alternative interpretation permits modification of the mathematical formulation which could not even be described in terms of the usual interpretation. Moreover, the modifications can quite easily be formulated in such a way that their effects are insignificant in the usual domain of applications. The present quantum theory is in fair good agreement with experiment, but of crucial importance in the domains of dimensions of the order of  $10^{-3}$  cm, where, as we have seen, the present theory is totally inadequate. It is thus entirely possible that some of the modifications describable in terms of our suggested alternative interpretation, but

\* Now at Universidade de São Paulo, Faculdade de Filosofia, Ciências e Letras, São Paulo, Brazil.  
† Einst. Poldy, and Rausch, Phys. Rev. 47, 777 (1935).  
‡ Pauli, *Relativistic Quantum Theory* (Princeton-Hall, Inc., New York, 1951), Chap. 10.

§ Pauli, Phys. Rev. 80, 763 (1950).  
¶ Pauli, Phys. Rev. 80, 770 (1950).

\*\* Pauli, *Relativistic Quantum Theory* (Princeton-Hall, Inc., New York, 1951), Chap. 10.

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‡‡ Pauli, Phys. Rev. 84, 166 (1951).

in a very crude sense. Thus, it is generally expected that in connection with phenomena associated with this so-called "standard interpretation" of the quantum theory, a problem may arise. It is believed that the theory could not agree precisely with such processes as meson production and scattering of elementary particles, or with the theory of the motion of charged particles, mass, charge, spin, etc., of the large number of so-called "elementary" particles that have already been found, as well as those of new particles which might be found in the future.

## Hidden Variable I (1951)

PHYSICAL REVIEW

VOLUME 85, NUMBER 2

JANUARY 15, 1952

## A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables. II

David Bohm\*

Princeton Physical Laboratory, Princeton University  
(Received July 5, 1951)

In this paper, we shall show how the theory of measurements is to be understood from the point of view of a physical interpretation of the quantum theory in terms of "hidden" variables, developed in Paper I. This interpretation provides a broader conceptual framework than the usual interpretation, because it makes possible a precise and continuous description of all processes, even at the quantum level. This broader conceptual framework allows more realistic physical models to be developed, and it also allows those allowed by the usual interpretation.

The author suggests an alternative interpretation, which is based on trying to find some other interpretation of the quantum theory in terms of at present "hidden" variables, which in principle determine all the details of the behavior of the system, but which in practice averaged over measurements of the types that can now be carried out. In this paper and in a subsequent paper, an interpretation is suggested in which the hidden variables are "local," i.e., "hidden" variables are suggested. It is shown that so long as the mathematical theory retains its present general form, this suggested interpretation leads to precisely the same results for all

ways that are consistent with our interpretation but not with the usual interpretation.

We give a simple explanation of the results of quantum-mechanical measurements in terms of the local hidden variables. These variables are the basic objects in the hypothetical approach of Einstein, Podolsky, and Rosen, which was suggested by these authors as a criticism of the usual interpretation.

From the point of view of the usual interpretation, it is believed that quantum theory is not consistent with hidden variables does not apply to the present interpretation, because the hidden variables are not yet defined, nor is there any reason to suppose that they will ever be defined, because the state of the measuring apparatus and the observed system and therefore go beyond certain of von Neumann's axioms.

In two appendices, we treat the problems of the electromagnetic field in our interpretation and answer certain additional questions which have arisen in the attempt to give a precise description for the previous paper, the mathematical formulation of the quantum theory needs to be modified at a short distance in certain

### 1. INTRODUCTION

In a previous paper,<sup>1</sup> to which we shall hereafter refer as Paper I, we have suggested an interpretation of the quantum theory in terms of "hidden" variables. We have shown that although this interpretation provides a conceptual framework that is broader than that of the usual interpretation, it permits of all the standard mathematical and specific assumptions, which lead to the same physical results as are obtained from the usual interpretation of the quantum theory. These three special assumptions are: (1) The  $\psi$ -field satisfies Schrödinger's equation;<sup>2</sup> (2) it is weak enough to satisfy the uncertainty principle;<sup>3</sup> and (3) it is localized in space.<sup>4</sup>

(3) We have a statistical ensemble of particle positions, with a probability density,  $P = |\psi(x)|^2$ . If the above three special assumptions are not made, then one can obtain a theory that is not in accordance with the usual interpretation. We note, however, that in Paper I that each generalization may actually be needed for an understanding of phenomena associated with distances of the order of  $10^{-3}$  cm or less, but may have changes of negligible importance in the atomic domain.

In this paper, we shall apply the interpretation of the quantum theory suggested in Paper I to the development of a theory of measurements in order to show that it is possible to make the generalizations indicated above, one led to the same predictions for all measurements as are obtained from the usual interpretation. In our interpretation, however, the uncertainty principle is regarded, not as an inherent limitation on the precision of measurements, but rather as a more concise of the simultaneous definition of momentum and position, but rather as a practical limitation on the

precision with which these quantities can simultaneously be measured, arising from unpredictable and uncontrollable disturbances of the observed system by the measuring apparatus. If the theory needs to be generalized in the ways suggested in Paper I, Secs. 4 and 5, then one of them must be adopted. In either case either be eliminated, or else be made subject to prediction and control, so that their effects could be corrected for. Our interpretation therefore demonstrates that measurements violating the uncertainty principle are at least conceivable.

### 2. QUANTUM THEORY OF MEASUREMENTS

We shall now show how the quantum theory of measurements is to be expressed in terms of our suggested interpretation of the quantum theory.<sup>5</sup>

In our interpretation, the measurement procedure must always be carried out by means of an interaction of the system of interest with a suitable piece of measuring apparatus. The apparatus must be so constructed that any given state of the system of interest will lead to a unique state of the apparatus. The interaction between the interaction introduces correlations between the state of the observed system and the state of the apparatus. The range of indefiniteness in this correlation may be called the "uncertainty, or the error, of the measurement."

Let us consider some idealized device designed to measure an arbitrary (hermitian) "observable"  $Q$ , associated with an electron. Let  $x$  represent the position of the electron,  $y$  that of the significant apparatus coordinate (or coordinates if there are more than one). The electron is to be considered as being subject to an impulsive measurement, i.e., a measurement utilizing a very strong interaction between apparatus and system

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# Bohmian Mechanics: Wave Function in Polar Form

Starting Point: Time-Dependent Schrödinger Equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(\mathbf{x}, t) \psi$$

Polar Decomposition of the Wave Function

$$\psi(\mathbf{x}, t) = R(\mathbf{x}, t) e^{\frac{i}{\hbar} S(\mathbf{x}, t)}, \quad R \geq 0, S \in \mathbb{R}$$

This representation will split Schrödinger's equation into two coupled *real* equations: a continuity equation and a Hamilton–Jacobi–like equation.

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# Bohmian Mechanics: Continuity and Quantum Potential

Continuity Equation (Probability Conservation)

$$\frac{\partial P}{\partial t} + \nabla \cdot (P \frac{\nabla S}{m}) = 0, \quad P = R^2$$

Modified Hamilton–Jacobi Equation

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V(\mathbf{x}, t) + U(\mathbf{x}, t) = 0$$

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# Bohmian Mechanics: Continuity, Quantum Potential and Motion

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## Quantum Potential

$$U(\mathbf{x}, t) = -\frac{\hbar^2}{2m} \frac{\nabla^2 R}{R}$$

## Guidance Equation (Deterministic Trajectories)

$$\mathbf{p} = \nabla S(\mathbf{x}, t), \quad \mathbf{v} = \frac{\nabla S}{m}$$

## Equation of Motion (Newtonian Form with Quantum Force)

$$m \frac{d^2 \mathbf{x}}{dt^2} = -\nabla(V(\mathbf{x}) + U(\mathbf{x}, t))$$

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# Statistical Predictions in Bohmian Mechanics

## Determinism vs. Statistics

Although Bohm's theory is deterministic for single systems, it reproduces all statistical predictions of standard quantum mechanics under three key assumptions:

1. The field  $\psi$  satisfies the Schrödinger equation.
2. The particle's momentum is always given by  $p = \nabla S(\mathbf{x})$ .
3. In practice, we cannot predict or control the exact initial position, so we work with an ensemble with probability density:

$$P(\mathbf{x}) = R^2 = |\psi(\mathbf{x})|^2$$

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## Consistency of the Statistical Postulate

### Continuity Equation Ensures Born's Rule Preservation

Starting from the polar decomposition, the continuity equation is:

$$\frac{\partial P}{\partial t} + \nabla \cdot (P\mathbf{v}) = 0, \quad \mathbf{v} = \frac{\nabla S}{m}$$

This guarantees that if  $P(\mathbf{x}, t_0) = |\psi(\mathbf{x}, t_0)|^2$  at one instant, then  $P(\mathbf{x}, t) = |\psi(\mathbf{x}, t)|^2$  for all later times.

**Interpretation:** Probability is not fundamental but reflects ignorance of initial conditions, just as in classical statistical mechanics.

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# Quantum Potential in Many-Body Systems

## Extension to $n$ particles

For a system of  $n$  particles, the wavefunction is:

$$\psi(x_1, x_2, \dots, x_n, t) = R(x_1, \dots, x_n, t) e^{\frac{i}{\hbar} S(x_1, \dots, x_n, t)}$$

The quantum potential generalizes to:

$$U(x_1, \dots, x_n) = -\frac{\hbar^2}{2m} \frac{\sum_i \nabla_i^2 R}{R}$$

Key feature:  $U$  depends on the coordinates of *all* particles simultaneously.

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Key feature:  $U$  depends on the coordinates of *all* particles simultaneously.

# Quantum Potential in Many-Body Systems

## Extension to $n$ particles

For a system of  $n$  particles, the wavefunction is:

$$\psi(x_1, x_2, \dots, x_n, t) = R(x_1, \dots, x_n, t) e^{\frac{i}{\hbar} S(x_1, \dots, x_n, t)}$$

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# Nonlocality and Entanglement in Bohm's Theory

## Implications of the many-body quantum potential:

- The force on particle  $i$ ,

$$\mathbf{F}_i = -\nabla_i U,$$

depends instantaneously on the positions of all other particles.

- This represents an effective "many-body force" that is inherently nonlocal.
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- Thus, Bohm's theory is **causal but nonlocal**, consistent with the EPR scenario.

## ON THE EINSTEIN PODOLSKY ROSEN PARADOX\*

J. S. BELL<sup>†</sup>

*Department of Physics, University of Wisconsin, Madison, Wisconsin*

(Received 4 November 1964)

### I. Introduction

THE paradox of Einstein, Podolsky and Rosen [1] was advanced as an argument that quantum mechanics could not be a complete theory but should be supplemented by additional variables. These additional variables were to restore to the theory causality and locality [2]. In this note that idea will be formulated mathematically and shown to be incompatible with the statistical predictions of quantum mechanics. It is the requirement of locality, or more precisely that the result of a measurement on one system be unaffected by operations on a distant system with which it has interacted in the past, that creates the essential difficulty. There have been attempts [3] to show that even without such a separability or locality requirement no "hidden variable" interpretation of quantum mechanics is possible. These attempts have been examined elsewhere [4] and found wanting. Moreover, a hidden variable interpretation of elementary quantum theory [5] has been explicitly constructed. That particular interpretation has indeed a grossly non-local structure. This is characteristic, according to the result to be proved here, of any such theory which reproduces exactly the quantum mechanical predictions.

*On the Einstein Podolsky Rosen Paradox (1964)*

## Fundamental Assumptions of Bell

- **Hidden variables:** Each particle pair has a set of parameters  $\lambda$  that determines measurement outcomes locally.
- **Measurement results:**  $A(a, \lambda), B(b, \lambda) \in \{+1, -1\}$  represent outcomes along directions  $a$  and  $b$ .
- **Perfect anticorrelation:** If  $a = b$ , then  $A(a, \lambda) = -B(a, \lambda)$ , consistent with quantum prediction  $P(a, a) = -1$ .
- **Local correlation:**

$$P(a, b) = \int d\lambda \rho(\lambda) A(a, \lambda) B(b, \lambda)$$

where  $\rho(\lambda)$  is the probability distribution over the hidden variables.

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## Bell Derivation: Key Intermediate Steps

1. Rewrite using anticorrelation:

$$P(a, b) = - \int d\lambda \rho(\lambda) A(a, \lambda) A(b, \lambda)$$

2. Comparing three directions:

$$P(a, b) - P(a, c) = - \int d\lambda \rho(\lambda) [A(a) A(b) - A(a) A(c)]$$

3. Regroup using  $A(b)^2 = 1$ :

$$P(a, b) - P(a, c) = - \int d\lambda \rho(\lambda) A(a) A(b) [1 - A(b) A(c)]$$

*The difference in correlations depends on how particle 2's "instructions" change between  $b$  and  $c$ .*

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# Bell Inequality and Physical Meaning

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$$|P(a,b) - P(a,c)| \leq 1 + P(b,c)$$

- $P(a,b)$ : correlation measured between results along  $a$  and  $b$ .
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- $P(b,c)$ : correlation between directions  $b$  and  $c$ .
- **Physical interpretation:** No local realistic theory can produce correlations stronger than this bound.
- Violation of this inequality by quantum predictions implies:
  - Realism and locality cannot both hold simultaneously.
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## Violation of Bell Inequality: Quantum Prediction

Quantum mechanics prediction for entangled particles:

$$P_{QM}(\mathbf{a}, \mathbf{b}) = -\mathbf{a} \cdot \mathbf{b}$$

Bell inequality :

$$|P(\mathbf{a}, \mathbf{b}) - P(\mathbf{a}, \mathbf{c})| \leq 1 + P(\mathbf{b}, \mathbf{c})$$

Specific setup:

- Take a vector  $\mathbf{c}$  “halfway” between  $\mathbf{a}$  and  $\mathbf{b}$ .
- Define angles:  $\theta = \angle(\mathbf{a}, \mathbf{c}) = \angle(\mathbf{c}, -\mathbf{b})$

Substituting the quantum prediction:

$$|-\mathbf{a} \cdot \mathbf{b} - (-\mathbf{a} \cdot \mathbf{c})| \leq 1 - \mathbf{c} \cdot \mathbf{b} \quad \Rightarrow \quad |\cos(2\theta) - \cos(\theta)| \leq 1 - \cos(\theta)$$

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## Explicit Violation for Small Angles

Approximations for small angles  $\theta \ll 1$ :

$$\cos(\theta) \approx 1 - \frac{\theta^2}{2}, \quad \cos(2\theta) \approx 1 - 2\theta^2$$

Bell inequality becomes:

$$|\cos(2\theta) - \cos(\theta)| \leq 1 - \cos(\theta) \quad \Rightarrow \quad |(1 - 2\theta^2) - (1 - \frac{\theta^2}{2})| \leq \frac{\theta^2}{2}$$

Simplifying:

$$\frac{3}{2}\theta^2 \leq \frac{1}{2}\theta^2$$

Conclusion: This inequality is false for any  $\theta \neq 0$ , demonstrating that quantum mechanics predictions violate Bell's inequality.

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# Bell's Refutation of von Neumann (1966)

## Bell's Conceptual Criticism:

- Von Neumann's **linearity assumption**:

$$V_m(aR + bS) = aV_m(R) + bV_m(S)$$

- Imposed even on dispersion-free states.
- Bell argued this is unjustified for non-commuting observables.
- Example: measuring  $\sigma_x + \sigma_y$  is not the same as separately measuring  $\sigma_x$  and  $\sigma_y$ .
- Therefore, von Neumann's proof excludes only a very restrictive class of hidden-variable theories.
- Bohm's 1952 theory escapes because it is contextual.

REVIEWS OF MODERN PHYSICS

VOLUME 38, NUMBER 3

JULY 1966

### On the Problem of Hidden Variables in Quantum Mechanics\*

JOHN S. BELL†

Stanford Linear Accelerator Center, Stanford University, Stanford, California

The demonstrations of von Neumann and others, that quantum mechanics does not permit a hidden variable interpretation, are reconsidered. It is shown that their essential axioms are unreasonable. It is urged that in further examination of this problem an interesting axiom would be that mutually distant systems are independent of one another.

#### I. INTRODUCTION

To know the quantum mechanical state of a system implies, in general, only statistical restrictions on the results of measurements. It seems interesting to ask if that statistical element can be thought of as arising, as in classical mechanics, from chance. Because the states in question are averages over better defined states for which individually the results would be quite determined. These hypothetical "dispersion free" states would be specified not only by the quantum mechanical state  $|s\rangle$ , but also by the values of the variables of these variables could actually be prepared, quantum mechanics would be obviously inadequate.

Whether such questions are indeed interesting has been the subject of debate.<sup>1,2</sup> It is addressed to those who do find the question interesting, and more particularly to those among them who believe that "the question concerning the existence of such hidden variables received an early and rather definitive answer in the form of von Neumann's proof that quantum mechanics is unable to support such variables in quantum theory."<sup>3</sup> An attempt will be made to clarify what von Neumann and his successors actually demonstrated. This will cover, as well as von Neumann's treatment, the recent version of the argument by Jauch and Piron,<sup>4</sup> the stronger

result consequent on the work of Gleason,<sup>5</sup> It will be urged that these analyses leave the real question unanswered, namely, whether it can be shown that determinations require from the hypothesis dispersion free states, not only that appropriate ensembles thereof should have all measurable properties of quantum mechanical states, but retain other properties as well. The following considerations apply to both the results of measurement as locally identified with properties of isolated systems. They are seen to be quite unreasonable when one remembers with Bohr<sup>6</sup> "the impossibility of any sharp distinction between the outer behavior of an object and its interaction with the measuring instruments which serve to define the conditions under which the phenomena appear."

The realization that von Neumann's proof is of limited relevance has been gaining ground since the 1952 work of Bell.<sup>7</sup> However, it is far from general. Moreover, the writer has not found in the literature any adequate analysis of what went wrong. Like all authors of noncommissioned reviews, he thinks that he can restore the position with such clarity and simplicity that all previous discussions will be eclipsed.

#### II. ASSUMPTIONS, AND A SIMPLE EXAMPLE

The authors of the demonstrations to be reviewed were concerned to assume as little as possible about quantum mechanics. This is valuable for some purposes, but not for ours. We are interested only in the possibility of hidden variables in ordinary quantum mechanics.

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<sup>3</sup>In particular the analysis of Bohr<sup>6</sup> seems to lack clarity, or else accuracy. He fully emphasizes the role of experimental fact in the derivation of the theory (see p. 187) that the circumvention of the theorem requires the association of a definite value with each observable that cannot be observed. The scheme of Sec. II is a counter example to this. Moreover, it will be seen in Sec. III that the theory advanced here is not at all like the one given by Bohr, which wherever located would not avoid Bohr's further remarks in the same section.

<sup>4</sup>J. P. Gaasen, *Philosophical Science*, F. A. Schilpp, Ed. (Library of Living Philosophers, Inc., New York, 1963), pp. 333–352. In the biographical Notes" and "Reply to Critics" suggest that the hidden-variable position has some interest.

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# Bell's Refutation of von Neumann (1966)

REVIEWS OF MODERN PHYSICS

VOLUME 38, NUMBER 3

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## Bell's Conceptual Criticism:

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$$V_m(aR + bS) = aV_m(R) + bV_m(S)$$

- Imposed even on dispersion-free states.
- Bell argued this is **unjustified for non-commuting observables**.
- Example: measuring  $\sigma_x + \sigma_y$  is not the same as separately measuring  $\sigma_x$  and  $\sigma_y$ .
- Therefore, von Neumann's proof excludes only a very restrictive class of hidden-variable theories.
- Bohm's 1952 theory escapes because it is contextual.

## On the Problem of Hidden Variables in Quantum Mechanics\*

JOHN S. BELL

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The demonstrations of von Neumann and others, that quantum mechanics does not permit a hidden variable interpretation, are reconsidered. It is shown that their essential axioms are unreasonable. It is urged that in further examination of this problem an interesting axiom would be that mutually distant systems are independent of one another.

### I. INTRODUCTION

To know the quantum mechanical state of a system implies, in general, only statistical restrictions on the results of measurements. This seems interesting to ask if that statistical element be thought of as arising, as in classical mechanics, from chance. Because the states in question are averages over better defined states for which individually the results would be quite determined. These hypothetical "dispersion free" states would be specified not only by the quantum mechanical state, but also by other properties. These are the "hidden" variables. If states with given values of these variables could actually be prepared, quantum mechanics would be obviously inadequate.

Whether such questions are indeed interesting has been the subject of debate.<sup>1,2</sup> The present paper does not contribute to that debate. It is addressed to those who do find the question interesting, and more particularly to those among them who believe that "the question concerning the existence of such hidden variables received an early and rather definitive answer in the form of von Neumann's proof that quantum mechanics is unable of such variables in quantum theory."<sup>3</sup> An attempt will be made to clarify what von Neumann and his successors actually demonstrated. This will cover, as well as von Neumann's treatment, the recent version of the argument by Jauch and Piron,<sup>4</sup> the stronger

result consequent on the work of Gleason,<sup>5</sup> and the result that these analyses leave the real question unanswered. It will be shown that the demonstrations required from the hypothesis "dispersion free states," not only that appropriate ensembles thereof should have all measurable properties of quantum mechanical states, but also obtain other properties as well. The following considerations apply equally to the results of measurement or to those identified with properties of isolated systems. They are seen to be quite unreasonable when one remembers with Bohr<sup>6</sup> "the impossibility of any sharp distinction between the objectified experiment and the measuring instruments which serve to define the conditions under which the phenomena appear."

The realization that von Neumann's proof is of limited relevance has been gaining ground since the 1952 work of Bell.<sup>7</sup> However, it is far from general. Moreover, the writer has not found in the literature any adequate analysis of what went wrong. Like all authors of noncommissioned reviews, he thinks that he can restore the position with such clarity and simplicity that all previous discussions will be eclipsed.

### II. ASSUMPTIONS, AND A SIMPLE EXAMPLE

The authors of the demonstrations to be reviewed were concerned to assume as little as possible about quantum mechanics. This is valuable for some purposes, but not for ours. We are interested only in the possibility of hidden variables in ordinary quantum mechanics.

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<sup>4</sup>J. P. Gaasen, *Fortschr. Phys.*, **14**, 125 (1966); J. P. Gaasen and F. A. de Vos, *Fortschr. Phys.*, **15**, 125 (1967).

<sup>5</sup>A. Gleason, *Philosophical Science*, P. A. Schilpp, Ed. (Library of Living Philosophers, Inc., New York, 1963), Vol. 1, pp. 75–154.

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<sup>15</sup>J. S. Bell, *Rev. Mod. Phys.*, **38**, 442 (1966).

<sup>16</sup>J. S. Bell, *Rev. Mod. Phys.*, **38**, 442 (1966).

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<sup>67</sup>J. S. Bell, *Rev. Mod. Phys.*, **38**, 442 (1966).

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<sup>71</sup>J. S. Bell, *Rev. Mod. Phys.*, **38**, 442 (1966).

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- Imposed even on dispersion-free states.
- Bell argued this is **unjustified for non-commuting observables**.
- Example: measuring  $\sigma_x + \sigma_y$  is not the same as separately measuring  $\sigma_x$  and  $\sigma_y$ .
- Therefore, von Neumann's proof excludes only a very restrictive class of hidden-variable theories.
- Bohm's 1952 theory escapes because it is contextual.

## On the Problem of Hidden Variables in Quantum Mechanics\*

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The demonstrations of von Neumann and others, that quantum mechanics does not permit a hidden variable interpretation, are reconsidered. It is shown that their essential axioms are unreasonable. It is urged that in further examination of this problem an interesting axiom would be that mutually distant systems are independent of one another.

### I. INTRODUCTION

To know the quantum mechanical state of a system implies, in general, only statistical restrictions on the results of measurements. It seems interesting to ask if that statistical element can be thought of as arising, as in classical mechanics, from chance. Because the states in question are averages over better defined states for which individually the results would be quite determined. These hypothetical "dispersion free" states would be specified not only by the quantum mechanical state, but also by some additional information. These "hidden" variables would presumably have values of these variables could actually be prepared, quantum mechanics would be obviously inadequate.

Whether such questions are indeed interesting has been the subject of debate.<sup>1,2</sup> This paper does not contribute to that debate. It is addressed to those who do find the question interesting, and more particularly to those among them who believe that "the question concerning the existence of such hidden variables received an early and rather definitive answer in the form of von Neumann's proof that quantum mechanics is unable of such variables in quantum theory."<sup>3</sup> An attempt will be made to clarify what von Neumann and his successors actually demonstrated. This will cover, as well as von Neumann's treatment, the recent version of the argument by Jauch and Piron,<sup>4</sup> the stronger

result consequent on the work of Gleason,<sup>5</sup> it will be urged that these analyses leave the real question unanswered. It will be shown that the demonstration requires from the hypothesis "dispersion free states, not only that appropriate ensembles thereof should have all measurable properties of quantum mechanical states, but retain other properties as well. The latter are found to depend on the way the results of measurement are locally identified with properties of isolated systems. They are seen to be quite unreasonable when one remembers with Bohr<sup>6</sup> "the impossibility of any sharp distinction between the outer behavior of an object and its interaction with the measuring instruments which serve to define the conditions under which the phenomena appear."

The realization that von Neumann's proof is of limited relevance has been growing ground since the 1952 work of Bell.<sup>7</sup> However, it is far from final. Moreover, the writer has not found in the literature any adequate analysis of what went wrong. Like all authors of noncommissioned reviews, he thinks that he can restore the position with such clarity and simplicity that all previous discussions will be eclipsed.

### II. ASSUMPTIONS, AND A SIMPLE EXAMPLE

The authors of the demonstrations to be reviewed were concerned to assume as little as possible about quantum mechanics. This is valuable for some purposes, but not for ours. We are interested only in the possibility of hidden variables in ordinary quantum mechanics.

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<sup>3</sup>In particular the analysis of Bohr<sup>6</sup> seems to lack clarity, or else accuracy. He fully emphasizes the role of the experimental arrangement in the proof, but fails to note (p. 187) that the circumvention of the theorem requires the association of different hidden-variable assignments with different observed. The scheme of Sec. II is a counter example to this. Moreover, it will be seen in Sec. III that the proof is ad hoc.

<sup>4</sup>Jauch, J. M., and Piron, C., *Comm. Math. Phys.*, 1, 50 (1965). In this paper, the reader will find a detailed account of the various works by D. Bohm cited later, and Bell and Neumann<sup>7</sup> as well. The present paper is a greatly simplified version, and little interest, we hope, is taken of the contributions of Rosenfeld to the field. In view of these references, of Pauli to the left, the article of Heisenberg, *Z. Phys.*, 133, 252 (1953), is omitted.

<sup>5</sup>A. Gleason, *Philosophical Science*, F. A. Schafft, Ed. (Library of Living Philosophy, Vol. 12), Philosophical Library, New York, 1957.

<sup>6</sup>Bohr, N., *Philosophical Papers*, Vol. 1, E. Schilpp, Ed. (Biographical Notes" and "Reply to Critics" suggest that the hidden-variable position has since changed.)

<sup>7</sup>J. S. Bell, and C. Fuchs, *Helv. Phys. Acta*, 39, 827 (1966).

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result consequent on the work of Gleason,<sup>5</sup> it will be urged that these analyses leave the real question unanswered. It is believed that the demonstration of the unreality of hidden variables requires the hypothesis "dispersion free states," not only that appropriate ensembles thereof should have all measurable properties of quantum mechanics, but also other properties as well. The following considerations apply to the results of measurement as loosely identified with properties of isolated systems. They are seen to be quite unreasonable when one remembers with Bohm<sup>6</sup> "the impossibility of any sharp distinction between the various elements which enter into what we call 'the measuring instruments' which serve to define the conditions under which the phenomena appear."

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<sup>3</sup>In particular the analysis of Bohm<sup>6</sup> seems to lack clarity, or else accuracy. He fully emphasizes the role of experimental data in the derivation of his theorem (see Fig. 1, p. 487) that the circumference of the hexagon requires the association of two hidden variables with each vertex, whereas only one was observed. The scheme of Sec. II is a counter example to this. Moreover, it will be seen in Sec. III that the theorem is not violated if the hidden variables are not associated with vertices, but with edges located would not avoid Bohm's further remarks in Sec. IV. The reader may note that the name "hidden variables" is misleading. The term "hidden variables" is used here to mean "variables whose values are not known." In this sense, the hidden variables in the Bohm theory are not hidden.

<sup>4</sup>J. J. Jauch and C. Piron, Helv. Phys. Acta 36, 827 (1963).

<sup>5</sup>A. Gleason, Philosophical Science, P. A. Schilpp, Ed. (Library of Living Philosophers, Inc., New York, 1963), Vol. 1, pp. 727-760.

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- Bell argued this is **unjustified for non-commuting observables**.
- Example: measuring  $\sigma_x + \sigma_y$  is not the same as separately measuring  $\sigma_x$  and  $\sigma_y$ .
- Therefore, von Neumann's proof excludes only a very restrictive class of hidden-variable theories.
- Bohm's 1952 theory escapes because it is **contextual**.

REVIEWS OF MODERN PHYSICS

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### On the Problem of Hidden Variables in Quantum Mechanics\*

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The demonstrations of von Neumann and others, that quantum mechanics does not permit a hidden variable interpretation, are reconsidered. It is shown that their essential axioms are unreasonable. It is urged that in further examination of this problem an interesting axiom would be that mutually distant systems are independent of one another.

#### I. INTRODUCTION

To know the quantum mechanical state of a system implies, in general, only statistical restrictions on the results of measurements. It seems interesting to ask if that statistical element can be thought of as arising, as in classical mechanics, from chance. Because the states in question are averages over better defined states for which individually the results would be quite determined. These hypothetical "dispersion free" states would be specified not only by the quantum mechanical state, but also by some additional information about "hidden" variables. If states with given values of these variables could actually be prepared, quantum mechanics would be obviously inadequate.

Whether such questions are indeed interesting has been the subject of debate.<sup>1,2</sup> This paper does not contribute to that debate. It is addressed to those who do find the question interesting, and more particularly to those among them who believe that "the question concerning the existence of such hidden variables received an early and rather definitive answer in the form of von Neumann's proof that quantum mechanics is unable of such variables in quantum theory."<sup>3</sup> An attempt will be made to clarify what von Neumann and his successors actually demonstrated. This will cover, as well as von Neumann's treatment, the recent version of the argument by Jauch and Piron,<sup>4</sup> the stronger

result consequent on the work of Gleason,<sup>5</sup> it will be urged that these analyses leave the real question unanswered. They must be considered to be demonstrations, requiring from the hypothesis "dispersion free states," not only that appropriate ensembles thereof should have all measurable properties of quantum mechanical states, but retain other properties as well. The following demonstration depends on the results of measurement being loosely identified with properties of isolated systems. They are seen to be quite unreasonable when one remembers with Bohm<sup>6</sup> "the impossibility of any sharp distinction between the objectified experiment and the measuring instruments which serve to define the conditions under which the phenomena appear."

The realization that von Neumann's proof is of little relevance has been gaining ground since the 1952 work of Bell.<sup>7</sup> However, it is far from general. Moreover, the writer has not found in the literature any adequate analysis of what went wrong. Like all authors of noncommissioned reviews, he thinks that he can restore the position with such clarity and simplicity that all previous discussions will be eclipsed.

#### II. ASSUMPTIONS, AND A SIMPLE EXAMPLE

The authors of the demonstrations to be reviewed were concerned to assume as little as possible about quantum mechanics. This is valuable for some purposes, but not for ours. We are interested only in the possibility of hidden variables in ordinary quantum mechanics.

\*A. M. Gleason, J. Math. & Mech. 6, 885 (1957). I am much indebted to Professor Jauch for drawing my attention to this.

<sup>1</sup>N. Bohr, in Ref. 2.

<sup>2</sup>A. Einstein, Philos. Natur., 1953, 16, 390 (1952).

<sup>3</sup>In particular the analysis of Bohm<sup>6</sup> seems to lack clarity, or else accuracy. He fully emphasizes the role of experimental data in the derivation of the theory (see, in particular, p. 147) that the circumvention of the theorem requires the association of a definite value with each of the two variables observed. The scheme of Sec. II is a counter example to this. Moreover, it will be seen in Sec. III that the theory is adequately described by the equations of the theory of the hidden variables, wherever located would not avoid Bohm's further remarks in the last part of his paper. In this connection, it may be mentioned that the "biographical Notes" and "Reply to Critics" suggest that the hidden variable position has some interest.

<sup>4</sup>J. S. Bell, J. Math. & Mech. 15, 212 (1964).

<sup>5</sup>A. Einstein, Philos. Natur., 1953, 16, 390 (1952).

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# The CHSH Inequality

Defining the correlation function:

$$E(a, b) = \langle A(a) B(b) \rangle, \quad A, B = \pm 1$$

The CHSH combination:

$$S = E(a, b) - E(a, b') + E(a', b) + E(a', b')$$

Local realism constraint:

$$-2 \leq S \leq 2$$

Quantum mechanics prediction:

$$|S| \leq 2\sqrt{2} \quad \Rightarrow \quad \text{Violation of local realism}$$

Impact:

- The CHSH form was experimentally testable.
- Enabled landmark tests (e.g. Aspect, 1982).

Why correlations matter:

- The values  $E(a, b)$  come from statistical counts of coincident detections.
- Individual outcomes are random, but correlations reveal hidden structure.
- The strength of violation depends entirely on these count-based correlations.

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## Conclusions on Quantum Entanglement

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- EPR raised the dilemma about the **completeness** of quantum mechanics.
- Bell turned the philosophical debate into mathematical and experimental predictions.
- Entanglement reveals **nonlocal correlations**.
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## Mathematical Framework for Entanglement

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## The Origin of Entanglement

- Emerged as a mathematical feature of early quantum mechanics.
- Now rigorously described via experiment and theory.
- Core Idea: How to represent states of *composite quantum systems*.

## A Note on Interpretation

While the math is well-established, its physical interpretation remains a subject of active debate—a common theme in quantum theory.

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# Departure from Classical State Spaces

## Classical Composite Systems

For a two-particle system, the total state space is the **Cartesian product** of individual phase spaces:

$$\Gamma_{AB} = \Gamma_A \times \Gamma_B$$

The state  $(x_A, x_B)$  of the parts completely defines the whole.

## A Naive Quantum Extrapolation

A simple guess might be  $\mathcal{H}_{AB} = \mathcal{H}_A \times \mathcal{H}_B$ . This is incorrect.

- It fails to incorporate the *superposition principle* for combined states.
- An ordered pair  $(|\psi_A\rangle, |\phi_B\rangle)$  cannot represent linear combinations.

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# The Tensor Product Structure

## The Correct Postulate

The state space for a composite quantum system is the *tensor product* of the individual Hilbert spaces:

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \quad (1)$$

## General Pure State

A general pure state  $|\Psi\rangle \in \mathcal{H}_{AB}$  is a superposition:

$$|\Psi\rangle = \sum_{i,j} c_{ij} (|a_i\rangle \otimes |b_j\rangle) \quad (2)$$

where  $\{|a_i\rangle\}$  and  $\{|b_j\rangle\}$  are orthonormal bases for  $\mathcal{H}_A$  and  $\mathcal{H}_B$ .

Dimension:  $\dim(\mathcal{H}_{AB}) = \dim(\mathcal{H}_A) \cdot \dim(\mathcal{H}_B)$

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# Two Fundamental Classes of States

## Separable (Product) States

A state is *separable* if it can be written as a single tensor product:

$$|\Psi\rangle_{\text{sep}} = |\psi_A\rangle \otimes |\psi_B\rangle \quad (3)$$

Subsystems have definite, independent properties (classical analogue).

## Entangled States

A state is *entangled* if it cannot be written in the separable form.

- Uniquely quantum-mechanical correlation.
- Measurement on A instantly affects B.

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## A Complication: Identical Particles

### The Symmetrization Postulate

The formalism  $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$  implicitly assumes particles are distinguishable.

For identical particles, quantum mechanics imposes the **symmetrization postulate**:

- Total wave function must be symmetric under particle exchange for bosons.
- Total wave function must be antisymmetric under particle exchange for fermions.

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## A Restricted State Space

- Physically allowed states live in the *antisymmetric subspace*  $\mathcal{H} \wedge \mathcal{H}$ , not the full tensor product space  $\mathcal{H} \otimes \mathcal{H}$ .
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# Entanglement for Identical Particles

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## A New Paradigm: Mode Entanglement

Entanglement is reformulated based on correlations between modes (e.g., spatial vs. spin), not between labeled particles.

- The role of a separable state is now played by a single **Slater determinant**.

For two fermions in single-particle states  $|\phi_1\rangle, |\phi_2\rangle$ :

$$|\Psi\rangle_{\text{Slater}} = \frac{1}{\sqrt{2}} (|\phi_1\rangle_1 \otimes |\phi_2\rangle_2 - |\phi_2\rangle_1 \otimes |\phi_1\rangle_2)$$

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# Defining Fermionic Entanglement: Slater Rank

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## Condition for Entanglement

A pure fermionic state is considered **entangled if and only if its Slater rank is greater than one.**

- This means it requires a superposition of multiple Slater determinants to be described.
- Slater Rank 1  $\iff$  “Separable” (unentangled)
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## Example: Pauli Exclusion Forces Entanglement

### Two Electrons in the Same Spatial Orbital

The total wave function  $|\Psi\rangle_{\text{total}} = |\psi\rangle_{\text{spatial}} \otimes |\chi\rangle_{\text{spin}}$  must be antisymmetric.

1. **Spatial State:**  $|\psi\rangle_{\text{spatial}}$  is *symmetric* (same orbital).
2. **Spin State:** To ensure total antisymmetry,  $|\chi\rangle_{\text{spin}}$  must be *antisymmetric*.
3. **Result:** The unique antisymmetric spin state for two spin-1/2 particles is the maximally entangled **spin-singlet**:

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 - |\downarrow\rangle_1 \otimes |\uparrow\rangle_2)$$

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### Conclusion

The Pauli exclusion principle *forces* the two electrons into a maximally entangled Bell state.

For indistinguishable particles, entanglement is not just a possibility but can be a necessity imposed by fundamental symmetries.

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# Formalism: Operators on Composite Systems

## Local Operators

An operator  $O_A$  on subsystem  $A$  is represented on  $\mathcal{H}_{AB}$  as a **local operator**:

$$O_A \rightarrow O_A \otimes I_B$$

## Inner Product

The inner product is defined by linear extension:

$$(\langle \phi_A | \otimes \langle \phi_B |)(|\psi_A\rangle \otimes |\psi_B\rangle) = \langle \phi_A | \psi_A \langle \phi_B | \psi_B$$

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# Why We Need the Density Matrix ( $\rho$ )

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## Handling Incomplete Information

- When we look at only one part of an entangled pair, we have incomplete information about it.
- The state of such a subsystem cannot be described by a state vector.
- The density matrix formalism is essential for describing these subsystems.

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# The Density Matrix: Pure vs. Mixed States

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## Pure States

For a pure state  $|\Psi\rangle$ , the density matrix is a projection operator:

$$\rho_{\text{pure}} = |\Psi\rangle \langle \Psi| \quad (4)$$

It is a projector ( $\rho^2 = \rho$ ), so purity can be tested:

$$\text{Tr}(\rho^2) = 1$$

## Mixed States

For a statistical ensemble of pure states  $\{p_i, |\psi_i\rangle\}$ :

$$\rho_{\text{mixed}} = \sum_i p_i |\psi_i\rangle \langle \psi_i| \quad (5)$$

For any mixed state, the purity is less than one:  $\text{Tr}(\rho_{\text{mixed}}^2) < 1$ .

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# Quantifying Uncertainty: Von Neumann Entropy

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## Definition

The degree of uncertainty or “mixedness” is quantified by the Von Neumann entropy:

$$S(\rho) = -\text{Tr}(\rho \ln \rho) = -\sum_i \lambda_i \ln \lambda_i \quad (6)$$

where  $\lambda_i$  are the eigenvalues of  $\rho$ .

- Pure state ( $\rho = |\psi\rangle\langle\psi|$ ):  $S(\rho) = 0$  (maximal knowledge).
- Maximally mixed state ( $\rho = I/d$ ):  $S(\rho) = \ln d$  (minimal knowledge).

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## The Entropy of Entanglement

Consider a composite system  $AB$  in a pure state  $|\Psi\rangle_{AB}$ , so  $S(\rho_{AB}) = 0$ .

The state of subsystem A is found via the *reduced density matrix*:

$$\rho_A = \text{Tr}_B(\rho_{AB})$$

If  $|\Psi\rangle_{AB}$  is entangled,  $\rho_A$  will be a mixed state, and its entropy  $S(\rho_A) > 0$ .

This quantity,  $S(\rho_A)$ , is the *entropy of entanglement*.

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# Entropy as a Signature of Entanglement

---

## The Entropy of Entanglement

Consider a composite system  $AB$  in a pure state  $|\Psi\rangle_{AB}$ , so  $S(\rho_{AB}) = 0$ .

The state of subsystem A is found via the *reduced density matrix*:

$$\rho_A = \text{Tr}_B(\rho_{AB})$$

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## Quantifying and Detecting Entanglement

---

### The Schmidt Decomposition Theorem

Any pure bipartite state  $|\Psi\rangle_{AB}$  can be written in a special orthonormal basis:

$$|\Psi\rangle_{AB} = \sum_{k=1}^{r_S} \sqrt{\lambda_k} |k\rangle_A \otimes |k\rangle_B \quad (7)$$

- $\{|k\rangle_A\}$  and  $\{|k\rangle_B\}$  are orthonormal bases (the Schmidt bases).
- The number of non-zero terms,  $r_S$ , is the *Schmidt rank*.

# Schmidt Rank and Entanglement

---

## A Simple Criterion

A state is *separable if and only if its Schmidt rank is 1.*

For pure states, the degree of entanglement is uniquely quantified by the *entropy of entanglement*:

$$E(|\Psi\rangle) = S(\rho_A) = - \sum_{k=1}^{r_S} \lambda_k \ln \lambda_k \quad (8)$$

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## Mixed States: A Zoo of Measures

Quantifying mixed-state entanglement is complex; no single measure exists.

### Entanglement of Formation ( $E_F$ )

*Question:* What is the minimum average pure-state entanglement needed to create  $\rho$ ?

$$E_F(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle)$$

### Relative Entropy of Entanglement ( $E_R$ )

*Question:* How “distant” is  $\rho$  from the set of separable states (SEP)?

$$E_R(\rho) = \min_{\sigma \in \text{SEP}} S(\rho||\sigma) = \min_{\sigma \in \text{SEP}} \text{Tr}(\rho \ln \rho - \rho \ln \sigma)$$

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# Detecting Mixed-State Entanglement: The PPT Criterion

## The Positive Partial Transpose (PPT) Test

A simple but powerful necessary condition for separability.

1. Start with a state  $\rho_{AB}$ .
2. Compute the *partial transpose* on one subsystem, e.g., B:  $\rho^{T_B}$ .
3. Check if  $\rho^{T_B}$  is positive semidefinite (all eigenvalues  $\geq 0$ ).

## The Punchline

- If  $\rho_{AB}$  is separable  $\Rightarrow \rho^{T_B}$  is positive.
- Therefore, if  $\rho^{T_B}$  has any **negative eigenvalues**, the state  $\rho_{AB}$  is certified as entangled.

(This condition is also sufficient only for  $2 \times 2$  and  $2 \times 3$  systems.)

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## Worked Example: The Werner State

### Definition

A mixture of a Bell state and a maximally mixed state ( $p \in [0, 1]$ ):

$$\rho_W = p |\Psi^-\rangle \langle \Psi^-| + \frac{1-p}{4} \mathbb{I}_4$$

where  $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ .

In the computational basis  $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ :

$$\rho_W = \frac{1}{4} \begin{pmatrix} 1-p & 0 & 0 & 0 \\ 0 & 1+p & -2p & 0 \\ 0 & -2p & 1+p & 0 \\ 0 & 0 & 0 & 1-p \end{pmatrix}$$

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## Worked Example: Applying the PPT Criterion

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### The Partial Transpose

We apply the transpose operation only to the second qubit's subspace. This swaps the  $|01\rangle\langle 10|$  and  $|10\rangle\langle 01|$  matrix elements.

$$\rho_W^{T_B} = \frac{1}{4} \begin{pmatrix} 1-p & 0 & 0 & -2p \\ 0 & 1+p & 0 & 0 \\ 0 & 0 & 1+p & 0 \\ -2p & 0 & 0 & 1-p \end{pmatrix}$$

## Worked Example: Werner State (Conclusion)

### Eigenvalues of the Partial Transpose

The eigenvalues of  $\rho_W^{T_B}$  are:

$$\lambda_{1,2,3} = \frac{1+p}{4} \quad \lambda_4 = \frac{1-3p}{4} \tag{9}$$

### Result

The eigenvalue  $\lambda_4$  becomes negative if  $1 - 3p < 0$ :

$$p > 1/3$$

Therefore, the Werner state is entangled for  $p > 1/3$ .

The magnitude of the negative eigenvalue is a measure of entanglement (a “negativity”):  $\mathcal{N}(\rho_W) = \max(0, -\lambda_4)$

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# Detecting Entanglement: Witnesses

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## Entanglement Witness

An entanglement witness is a Hermitian operator  $W$  designed such that:

- $\text{Tr}(W\rho_{\text{sep}}) \geq 0$  for all separable states.
- There exists at least one entangled state  $\rho_{\text{ent}}$  with  $\text{Tr}(W\rho_{\text{ent}}) < 0$ .

## Experimental Implication

If an experiment measures an expectation value  $\langle W \rangle = \text{Tr}(W\rho) < 0$ , the state  $\rho$  is certified as entangled.

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## Dynamics of Entanglement

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# Entanglement in Motion

## Why Study Entanglement Dynamics?

We must understand how entanglement evolves under the influence of:

- The system's Hamiltonian (internal dynamics).
- Interaction with an environment (open system dynamics).

This is crucial for quantum information, where we must generate, manipulate, and protect entangled states.

## Propagation in Closed Systems

In many-body systems with local interactions, entanglement propagates. The **Lieb-Robinson bounds** establish a finite maximum speed for information, creating a linear “light cone” for correlations.

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## Quantum Quench

A sudden change in a system's Hamiltonian ( $H_0 \rightarrow H_1$ ) drives the system out of equilibrium, revealing universal entanglement dynamics.

## The Quasiparticle Picture

For a subsystem of length  $\ell$ , the entropy  $S_A(t)$  shows a universal pattern:

1. Initial **linear growth**:  $S_A(t) \propto t$ .
2. **Saturation** to a volume-law:  $S_A(t) \rightarrow \text{const} \cdot \ell$ .

*Explanation:* The quench creates entangled quasiparticle pairs. Entanglement grows as pairs are split by the subsystem boundary.

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# Entanglement in Open Systems: Fragility

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## Decoherence

Interaction with an external environment degrades a system's coherence and entanglement over time.

Entanglement is typically far more fragile than the coherence of individual subsystems.

## Entanglement Sudden Death (ESD)

A striking feature of entanglement decay.

- Unlike local coherence (which decays asymptotically), entanglement can vanish completely at a finite time.
- After this time, the global state is separable, even if subsystems remain coherent.

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## Alternative Perspectives

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## An Alternative View

Alternative mathematical frameworks can offer different physical insights beyond the standard Hilbert space formalism.

## Entanglement in Geometric Algebra

- Entanglement is not an intrinsic property of a state, but a relationship between reference frames.
- It is described by a superposition of *relative rotations*.
- The focus shifts from abstract states to the transformation operators themselves.

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# The Entangler Operator

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## The Algebra of Physical Space (APS)

- The language is Geometric Algebra, a powerful tool for describing geometric relationships.
- In this framework, spin states are represented as *rotors* (operators that perform rotations).

## Entanglement as a Transformation

**Central Thesis:** Entanglement is encoded within a transformation operator, the **entangler**, not in the state itself.

- Entangled State = (Entangler) acting on a (Separable State).
- The entangler contains the geometric info of the superposition of relative rotations between particle reference frames.

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## Entangling Eigenspinors

A key advantage of APS is its natural compatibility with special relativity. A Lorentz boost (relative velocity) can be combined with an entangler (relative rotation) into a single composite operator: the **entangling eigenspinor**  $\Lambda_{AB}$ .

## Shift in Perspective

This reinterpretation has profound conceptual implications:

- The “weirdness” of QM is shifted from the state to the *transformation process*.
- Wave function collapse is not a physical change, but the **revelation of the specific geometric transformation** that was present all along.

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## Experimental Proofs

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# The Bridge to Reality: The CHSH Inequality

## From Theory to Practice

- The CHSH inequality was the key theoretical tool that allowed for the design of the Freedman and Clauser experiment.

### The Decisive Inequality ( $\delta$ )

For the angles of predicted maximum violation ( $\phi = 22.5^\circ$  and  $3\phi = 67.5^\circ$ ), the inequality was simplified to a single expression:

$$\delta = \frac{R(22.5^\circ)}{R_0} - \frac{R(67.5^\circ)}{R_0} - \frac{1}{4}$$

- Local Realism requires:  $\delta \leq 0$
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# Photon Source: Calcium Atomic Cascade

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## Generating Entangled Pairs

- An atomic cascade in calcium atoms was used to generate photon pairs.
- The  $J=0 \rightarrow J=1 \rightarrow J=0$  transition emits two photons.
- By conservation of angular momentum, the photons' polarizations are strongly **correlated** (parallel).

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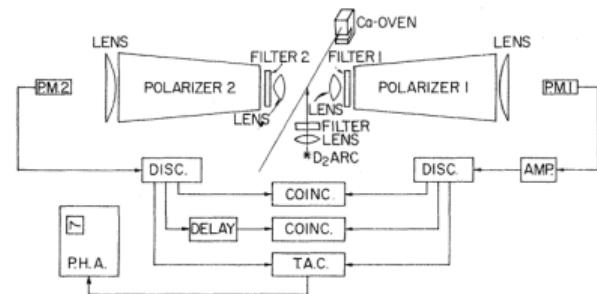
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# Optical and Detection Apparatus

## Key Measurement Components

- The optical system used lenses, filters, and “pile-of-plates” polarizers.
- The detectors were high-sensitivity photomultiplier tubes (PMTs).
- The overall detection efficiency was extremely low ( $\approx 1.6 \times 10^{-3}$ ), requiring long hours of measurement.

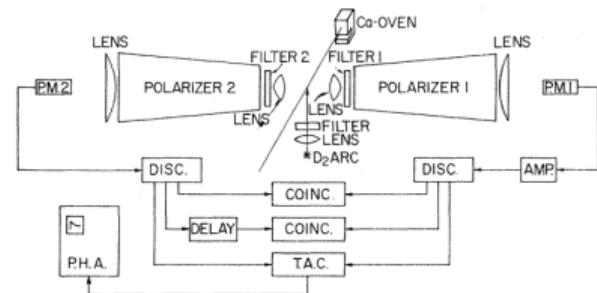


**Figure 9:** Scheme of the experimental setup [23].

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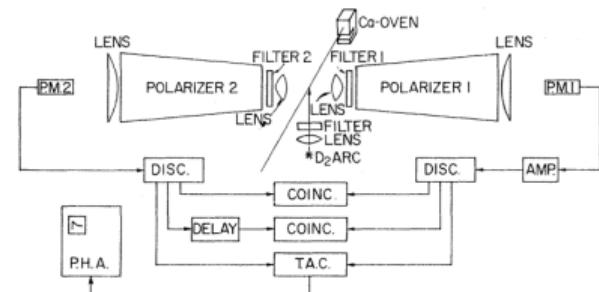


Figure 9: Scheme of the experimental setup [23].

# Results, Verdict, and Loopholes

---

## Protocol and Result

- The experiment lasted approximately **200 hours** due to the low coincidence rate.
- A robust real-time normalization system was used to cancel out equipment drift.

## The Decisive Experimental Result

The measured value was:  $\delta = 0.050 \pm 0.008$

## Required Improvements: The Loopholes

• Improve the signal-to-noise ratio by adding more detectors.

# Results, Verdict, and Loopholes

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## Protocol and Result

- The experiment lasted approximately **200 hours** due to the low coincidence rate.
- A robust real-time normalization system was used to cancel out equipment drift.

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## Required Improvements: The Loopholes

• Improve signal processing to reduce noise and increase efficiency.

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## Required Improvements: The Loopholes

• Improve signal-to-noise ratio by adding more detectors and better shielding.

• Implement a more sophisticated real-time normalization system.

• Increase the experiment duration to reduce statistical uncertainty.

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## Aspect and team

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PHYSICAL REVIEW D

VOLUME 10, NUMBER 2

15 JULY 1974

## Experimental consequences of objective local theories\*

John F. Clauser

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Michael A. Horne

*Department of Physics, Stonehill College, North Easton, Massachusetts 02356*

(Received 10 August 1973; revised manuscript received 8 April 1974)

A broad class of theories, called "objective local theories," is defined, motivation for considering these theories is given, and experimental consequences of the class are investigated. An extension of previous analyses by Bell and by Clauser *et al.* shows that predictions of objective local theories and of quantum mechanics differ, and that an experimental test of the entire family of objective local theories can be performed. The experimental requirements are given. Objective local theories satisfying a plausible but experimentally untestable supplementary assumption are shown to be incompatible with existing experimental data.

Clauser & Horne: Experimental Consequences of Objective Local Theories (1974)

## Clauser–Horne Formulation (1974)

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- Defined **Objective Local Theories (OLT)**: outcomes depend only on hidden variables  $\lambda$  and local settings, not on distant settings.
- Introduced the **No-Enhancement Assumption**: detection with a polarizer cannot exceed detection without it.
- Showed that this weaker assumption was still enough to make Freedman–Clauser results incompatible with all OLT satisfying it.
- Proved that some **supplementary assumption is necessary**: without no-enhancement, one can build an OLT reproducing quantum predictions.
- Revealed the **detection loophole**: experiments of that era only excluded OLT within the no-enhancement class.

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## Experimental Program:

- Systematic tests of Bell inequalities using entangled photons from a calcium cascade.
- Progressively designed to close detection and locality loopholes.

## Key Features:

- 1981: High-efficiency source, single-channel polarizers.
- 1982 (Two-channel): Polarizing beam splitters, closed the detection loophole.
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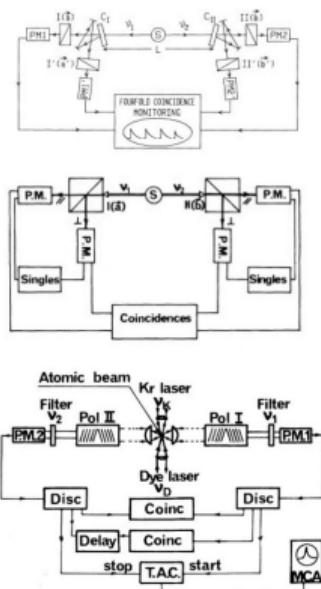


FIG. 2. Schematic diagram of apparatus and electronics. The laser beams are focused onto the atomic beam perpendicular to the figure. Feedback loops from the fluorescence signal control the krypton laser power and the dye-laser wavelength. The output of discriminators feed counters (not shown) and coincidence circuits. The multichannel analyzer (MCA) displays the time-delay spectrum.

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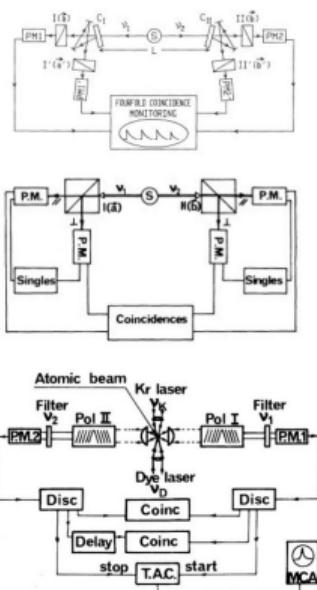


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# Comparing Aspect's Three Experiments

## Progression of Experimental Design

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**Impact:** Each step addressed a loophole, making the violation of Bell's inequalities increasingly robust and conclusive.

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## Applications

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# Applications of Quantum Entanglement (I)

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## 1. Quantum Computing

- Entangled **qubits** are the foundation of quantum computers.
- They allow for computations impossible for classical computers.
- Advances in cryptography, drug discovery, and materials science.

## 2. Quantum Cryptography (QKD)

- Uses pairs of entangled photons to generate cryptographic keys.
- Offers **unconditional security** (guaranteed by the laws of physics).
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- Foundation for a future **quantum internet**.
- Enables hyper-secure communications and the interconnection of distributed quantum computers.
- Development of quantum repeaters for long distances.

## 4. Quantum Metrology and Sensing

- Creates **sensors** much more sensitive than classical physics allows.
- Allows for measurements with precision beyond the “standard quantum limit”.
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- [1] Ryszard Horodecki et al. “Quantum entanglement”. In: *Reviews of Modern Physics* 81.2 (June 17, 2009), pp. 865–942. ISSN: 0034-6861, 1539-0756. DOI: 10.1103/RevModPhys.81.865. URL: <https://link.aps.org/doi/10.1103/RevModPhys.81.865> (visited on 09/03/2025).

- [2] Luigi Amico et al. “**Entanglement in many-body systems**”. In: *Reviews of Modern Physics* 80.2 (May 6, 2008), pp. 517–576. ISSN: 0034-6861, 1539-0756. DOI: 10.1103/RevModPhys.80.517. URL: <https://link.aps.org/doi/10.1103/RevModPhys.80.517> (visited on 09/03/2025).
- [3] Alain Aspect, Philippe Grangier, and Gérard Roger. “**Experimental Tests of Realistic Local Theories via Bell’s Theorem**”. In: *Physical Review Letters* 47.7 (Aug. 17, 1981), pp. 460–463. ISSN: 0031-9007. DOI: 10.1103/PhysRevLett.47.460. URL: <https://link.aps.org/doi/10.1103/PhysRevLett.47.460> (visited on 09/05/2025).

- [4] Alain Aspect, Philippe Grangier, and Gérard Roger. “**Experimental Realization of Einstein-Podolsky-Rosen-Bohm Gedankenexperiment : A New Violation of Bell’s Inequalities**”. In: *Physical Review Letters* 49.2 (July 12, 1982), pp. 91–94. ISSN: 0031-9007. DOI: [10.1103/PhysRevLett.49.91](https://doi.org/10.1103/PhysRevLett.49.91). URL: <https://link.aps.org/doi/10.1103/PhysRevLett.49.91> (visited on 09/05/2025).
- [5] Alain Aspect, Jean Dalibard, and Gérard Roger. “**Experimental Test of Bell’s Inequalities Using Time- Varying Analyzers**”. In: *Physical Review Letters* 49.25 (Dec. 20, 1982), pp. 1804–1807. ISSN: 0031-9007. DOI: [10.1103/PhysRevLett.49.1804](https://doi.org/10.1103/PhysRevLett.49.1804). URL: <https://link.aps.org/doi/10.1103/PhysRevLett.49.1804> (visited on 09/07/2025).

- [6] Reinhold A. Bertlmann and Anton Zeilinger. *Quantum [Un]speakables: From Bell to Quantum Information*. Berlin, Heidelberg s.l: Springer Berlin Heidelberg, 2002. 485 pp. ISBN: 978-3-642-07664-0 978-3-662-05032-3. DOI: [10.1007/978-3-662-05032-3](https://doi.org/10.1007/978-3-662-05032-3).
- [7] A. Einstein, B. Podolsky, and N. Rosen. “Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?” In: *Physical Review* 47.10 (May 15, 1935), pp. 777–780. ISSN: 0031-899X. DOI: [10.1103/PhysRev.47.777](https://doi.org/10.1103/PhysRev.47.777). URL: <https://link.aps.org/doi/10.1103/PhysRev.47.777> (visited on 09/07/2025).
- [8] J S Bell. “On the Einstein Podolsky Rosen paradox\*”. In: (). DOI: <https://doi.org/10.1103/PhysRev.47.777>.

- [9] Crystal-Ann McKenzie. “An Interpretation of Relativistic Spin Entanglement Using Geometric Algebra”. In: ().
- [10] Timothy F. Havel and Chris J. L. Doran. “Interaction and Entanglement in the Multiparticle Spacetime Algebra”. In: *Applications of Geometric Algebra in Computer Science and Engineering*. Ed. by Leo Dorst, Chris Doran, and Joan Lasenby. Boston, MA: Birkhäuser Boston, 2002, pp. 227–247. ISBN: 978-1-4612-6606-8 978-1-4612-0089-5. DOI: [10.1007/978-1-4612-0089-5\\_21](https://doi.org/10.1007/978-1-4612-0089-5_21). URL: [http://link.springer.com/10.1007/978-1-4612-0089-5\\_21](http://link.springer.com/10.1007/978-1-4612-0089-5_21) (visited on 09/17/2025).

- [11] Yuri Alexandre Aoto and Márcio Fabiano Da Silva. “Calculating the distance from an electronic wave function to the manifold of Slater determinants through the geometry of Grassmannians”. In: *Physical Review A* 102.5 (Nov. 6, 2020), p. 052803. ISSN: 2469-9926, 2469-9934. DOI: 10.1103/PhysRevA.102.052803. URL: <https://link.aps.org/doi/10.1103/PhysRevA.102.052803> (visited on 10/01/2025).
- [12] Paul Ehrenfest. *Niels Bohr and Albert Einstein*. Published: Photograph. Dec. 1925. URL: <http://www.dfi.dk/dfi/pressroom/kbhfortolkningen/>.

- [13] Google. *Ilustración de Alice y Bob midiendo partículas entrelazadas separadas por una gran distancia.* Published: Imagen generada por IA. Oct. 2025.
- [14] Bain News Service. *Prof. Bohr.* Published: Fotografía en negativo de vidrio. 1910. URL: <https://www.loc.gov/pictures/item/2014715454/>.
- [15] Fotógrafo desconocido. *Retrato de Erwin Schrödinger.* Published: Fotografía. 1933. URL: <http://search.bildarchiv.at/object/161477>.
- [16] Fotógrafo desconocido. *Fotografía de David Bohm.* Published: Fotografía. URL: <http://www.cartage.org.lb/en/themes/Biographies/MainBiographies/B/Bohm/1.html> (visited on 10/01/2025).

- [17] Google. *Una fuente emite dos partículas entrelazadas con espín opuesto.* Published: Imagen generada por IA. Oct. 2025.
- [18] Google. *Esquema del experimento: una fuente emite dos fotones entrelazados.* Published: Imagen generada por IA. Oct. 2025.
- [19] N. Bohr. “Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?” In: *Physical Review* 48.8 (Oct. 1935). Publisher: American Physical Society (APS), pp. 696–702. DOI: [10.1103/PhysRev.48.696](https://doi.org/10.1103/PhysRev.48.696).
- [20] E. Schrödinger. “Discussion of Probability Relations between Separated Systems”. In: *Mathematical Proceedings of the Cambridge Philosophical Society* 31.4 (Oct. 1935). Publisher: Cambridge University Press, pp. 555–563. DOI: [10.1017/S0305004100013554](https://doi.org/10.1017/S0305004100013554).

- [21] D. Bohm and Y. Aharonov. “**Discussion of Experimental Proof for the Paradox of Einstein, Rosen, and Podolsky**”. In: *Physical Review* 108.5 (Dec. 1957). Publisher: American Physical Society (APS), pp. 1371–1373. DOI: [10.1103/PhysRev.108.1371](https://doi.org/10.1103/PhysRev.108.1371).
- [22] C. S. Wu and I. Shaknov. “**The Angular Correlation of Scattered Annihilation Radiation**”. In: *Physical Review* 77.1 (Jan. 1950). Publisher: American Physical Society (APS), p. 136. DOI: [10.1103/PhysRev.77.136](https://doi.org/10.1103/PhysRev.77.136).
- [23] Stuart J. Freedman and John F. Clauser. “**Experimental Test of Local Hidden-Variable Theories**”. In: *Physical Review Letters* 28.14 (Apr. 1972). Publisher: American Physical Society (APS), pp. 938–941. DOI: [10.1103/PhysRevLett.28.938](https://doi.org/10.1103/PhysRevLett.28.938).

- [24] Mohamed Hatifi. “**Relativistic Bohmian mechanics revisited: A covariant reformulation for spin-1/2 particles**”. In: *Physics Letters A* 518 (July 2024), p. 129680. DOI: [10.1016/j.physleta.2024.129680](https://doi.org/10.1016/j.physleta.2024.129680). URL: <https://doi.org/10.1016/j.physleta.2024.129680>.
- [25] colaboradores de Wikipedia. **Albert Einstein**. Oct. 2025. URL: [https://es.wikipedia.org/wiki/Albert\\_Einstein](https://es.wikipedia.org/wiki/Albert_Einstein).
- [26] Wikipedia contributors. **File:Nobel Prize.png - Wikipedia**. URL: [https://en.wikipedia.org/wiki/File:Nobel\\_Prize.png](https://en.wikipedia.org/wiki/File:Nobel_Prize.png).

- [27] *Mathematische Grundlagen der Quantenmechanik - GDZ - Göttinger Digitalisierungszentrum.* URL: <https://gdz.sub.uni-goettingen.de/id/PPN379400774?ify=%7B%22pages%22%3A%5B2%5D%2C%22pan%22%3A%7B%22x%22%3A0.329%2C%22y%22%3A0.927%7D%2C%22view%22%3A%22info%22%2C%22zoom%22%3A0.31%7D>.
- [28] Anthony Duncan. “Von Neumann’s 1927 Trilogy on the Foundations of Quantum Mechanics. Annotated Translations”. In: arXiv (Cornell University) (June 2024). DOI: 10.48550/arxiv.2406.02149. URL: <https://arxiv.org/abs/2406.02149>.

- [29] David Bohm. “A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables. I”. In: *Physical Review* 85.2 (Jan. 1952), pp. 166–179. DOI: 10.1103/physrev.85.166. URL: <https://doi.org/10.1103/physrev.85.166>.
- [30] David Bohm. “A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables. II”. In: *Physical Review* 85.2 (Jan. 1952), pp. 180–193. DOI: 10.1103/physrev.85.180. URL: <https://doi.org/10.1103/physrev.85.180>.

- [31] Lucien Hardy. “**Nonlocality for two particles without inequalities for almost all entangled states**”. In: *Physical Review Letters* 71.11 (Sept. 1993), pp. 1665–1668. DOI: [10.1103/physrevlett.71.1665](https://doi.org/10.1103/physrevlett.71.1665). URL: <https://doi.org/10.1103/physrevlett.71.1665>.
- [32] John F. Clauser et al. “**Proposed Experiment to Test Local Hidden-Variable Theories**”. In: *Physical Review Letters* 23.15 (Oct. 1969), pp. 880–884. DOI: [10.1103/physrevlett.23.880](https://doi.org/10.1103/physrevlett.23.880). URL: <https://doi.org/10.1103/physrevlett.23.880>.
- [33] John Von Neumann. *Fundamentos matemáticos de la mecánica cuántica*. May 2018.

- [34] Daniel F. Styer et al. “Nine formulations of quantum mechanics”. In: *American Journal of Physics* 70.3 (Mar. 2002), pp. 288–297. DOI: [10.1119/1.1445404](https://doi.org/10.1119/1.1445404). URL: <https://doi.org/10.1119/1.1445404>.
- [35] Ian J. R. Aitchison, David A. MacManus, and Thomas M. Snyder. “Understanding Heisenberg’s “magical” paper of July 1925: A new look at the calculational details”. In: *American Journal of Physics* 72.11 (Sept. 2004), pp. 1370–1379. DOI: [10.1119/1.1775243](https://doi.org/10.1119/1.1775243). URL: <https://doi.org/10.1119/1.1775243>.

- [36] John S. Bell. “On the Problem of Hidden Variables in Quantum Mechanics”. In: *Reviews of Modern Physics* 38.3 (July 1966), pp. 447–452. DOI: [10.1103/revmodphys.38.447](https://doi.org/10.1103/revmodphys.38.447). URL: <https://doi.org/10.1103/revmodphys.38.447>.
- [37] D. Bohm and Y. Aharonov. “Discussion of Experimental Proof for the Paradox of Einstein, Rosen, and Podolsky”. In: *Physical Review* 108.4 (Nov. 1957), pp. 1070–1076. DOI: [10.1103/physrev.108.1070](https://doi.org/10.1103/physrev.108.1070). URL: <https://doi.org/10.1103/physrev.108.1070>.

- [38] John F. Clauser and Michael A. Horne. “**Experimental consequences of objective local theories**”. In: *Physical review. D. Particles, fields, gravitation, and cosmology/Physical review. D. Particles and fields* 10.2 (July 1974), pp. 526–535. DOI: 10.1103/physrevd.10.526. URL: <https://doi.org/10.1103/physrevd.10.526>.
- [39] Eugene P. Wigner. “**On Hidden Variables and Quantum Mechanical Probabilities**”. In: *American Journal of Physics* 38.8 (Aug. 1970), pp. 1005–1009. DOI: 10.1119/1.1976526. URL: <https://doi.org/10.1119/1.1976526>.

- [40] William A. Fedak and Jeffrey J. Prentis. “**The 1925 Born and Jordan paper “On quantum mechanics”**”. In: *American Journal of Physics* 77.2 (Jan. 2009), pp. 128–139. DOI: [10.1119/1.3009634](https://doi.org/10.1119/1.3009634). URL: <https://doi.org/10.1119/1.3009634>.

# Thank you!

Questions or Comments?

# Quantum Entanglement

Spooky and Hidden

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