Quantum Entanglement

Spooky and Hidden

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Universidad Distrital Francisco José de Caldas

EPR Debate

The Clash of Titans: Einstein's Realism

- **Deterministic View:** The universe is knowable and orderly.
- **Objective Reality:** Physical properties exist independently of measurement.
- **No Fundamental Randomness:** Probability is not intrinsic to nature.

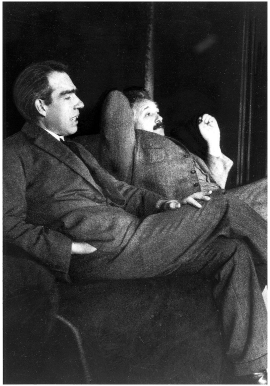

Figure 1: Niels Bohr with Albert Einstein at Paul Ehrenfest's home in Leiden [12].

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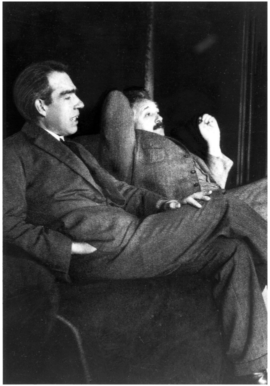

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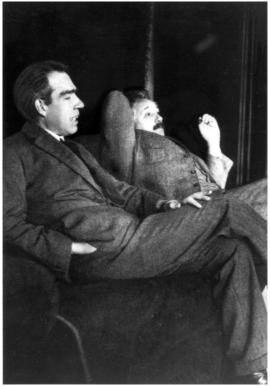

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- In 1935, Einstein, Podolsky, and Rosen published a key paper.
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- The title: “Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?”.

MAY 15, 1935 PHYSICAL REVIEW VOLUME 47

Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

A. Einstein, B. Podolsky and N. Rosen, *Institute for Advanced Study, Princeton, New Jersey*
(Received March 25, 1935)

In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete.

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ANY serious consideration of a physical theory must take into account the distinction between the objective reality, which is independent of any theory, and the physical concepts with which the theory operates. These concepts are intended to correspond with the objective reality, and by means of these concepts we picture this reality to ourselves.

In attempting to judge the success of a physical theory, we may ask ourselves two questions: (1) "Is the theory correct?" and (2) "Is the description given by the theory complete?" It is only in the case in which positive answers may be given to both of these questions, that the concepts of the theory may be said to be satisfactory. The correctness of the theory is judged by the degree of agreement between the conclusions of the theory and human experience. This experience, which alone enables us to make inferences about reality, in physics takes the form of experiment and measurement. It is the second question that we wish to consider here, as applied to quantum mechanics.

Whatever the meaning assigned to the term *complete*, the following requirement for a complete theory seems to be a necessary one: *every element of the physical reality must have a counterpart in the physical theory*. We shall call this the condition of completeness. The second question is thus easily answered, as soon as we are able to decide what are the elements of the physical reality.

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Figure 2: Paper: Can Quantum-Mechanical Description of Physical Reality Be Considered Complete? [7]

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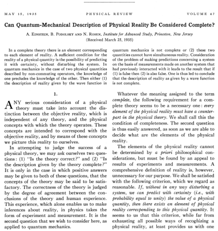
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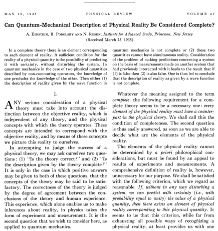
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The EPR Criterion of Reality

The “rule” to define an element of reality

To formalize their attack, EPR introduced an explicit and, seemingly, irrefutable criterion:

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Violation of the Criterion: Position and Momentum

Example with a state $\psi = e^{(2\pi i/h)p_0 x}$

- When applying the momentum operator (p), we get an exact value:

$$p\psi = \left(\frac{h}{2\pi i}\frac{\partial}{\partial x}\right)\psi = p_0\psi$$

  It is concluded that the momentum, p_0, is an **element of reality**.

- However, when applying the position operator (x), an exact value is not obtained:

$$x\psi \neq \text{constant} \cdot \psi$$

  We can only calculate the **probability** of finding the particle in an interval $[a, b]$, which turns out to be $P(a, b) = b - a$.

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The Fundamental Dichotomy

From the previous example, it follows that there are only two possibilities:

1. Either the quantum-mechanical description of reality given by the wave function **is not complete**.
2. Or when the operators for two physical quantities do not commute, the two quantities **cannot have simultaneous reality**.

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The Principle of Locality

A fundamental idea in physics

- An object can only be directly influenced by its **immediate surroundings**.
- Any influence at a distance cannot be instantaneous; it must propagate at a finite speed, not exceeding the speed of light (c).
- For Einstein, locality was a sacred axiom. Its violation would imply the disintegration of the universe's **causal structure** (cause-and-effect).

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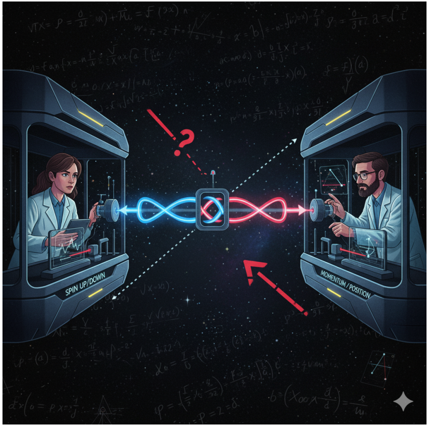
The EPR Thought Experiment

The two-particle system

- A system of two particles is prepared in an entangled state and then separated by a large distance.
- **Option 1:** If Alice measures the **momentum** of her particle, she can predict with certainty the momentum of Bob's particle.
- **Option 2:** If Alice measures the **position**, she can predict with certainty Bob's position.
- **Partial Conclusion:** Both the position and momentum of Bob's particle must be "elements of reality".



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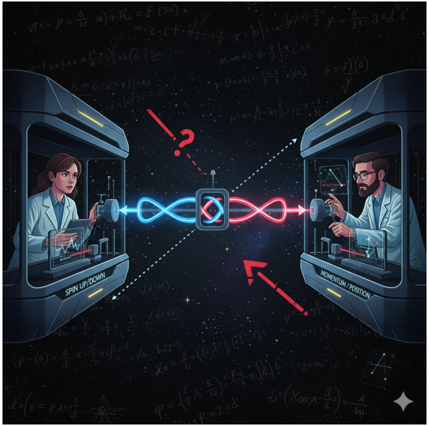

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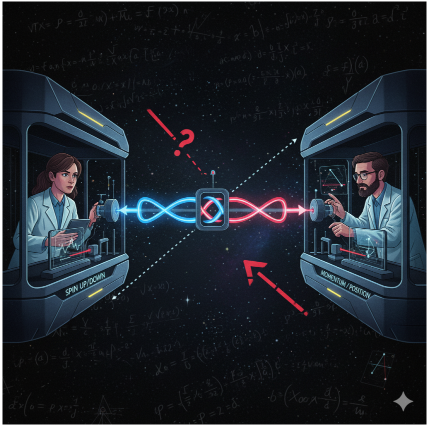

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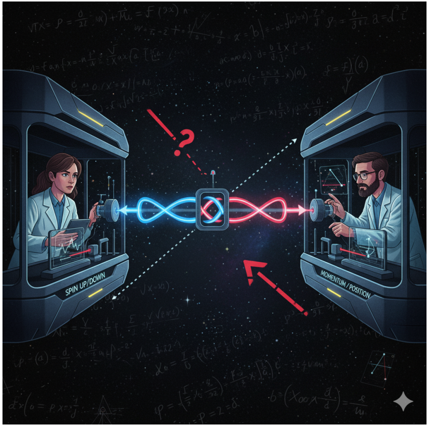

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The Contradiction and EPR's Conclusion

The Conflict with Quantum Mechanics

- The conclusion that Bob's particle has a definite position and momentum **simultaneously**...
- ...is in direct conflict with the **Heisenberg Uncertainty Principle**, which forbids it.

The EPR Conclusion

Faced with this dilemma, if local realism is correct, the only possible conclusion is that:

- The description of reality provided by the wave function must be **incomplete**.
- There must be an underlying theory ("hidden variables") that provides a complete description.

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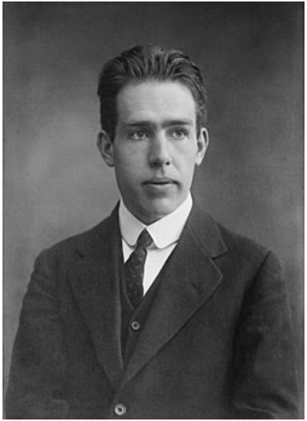
Mutually Exclusive, Yet Necessary Concepts

- Complementarity posits that pairs of classical concepts are necessary for a complete description.
- But they are **mutually exclusive** in any single experiment.
- They are not contradictory properties of the object, but complementary aspects of a **phenomenon** revealed by incompatible experimental arrangements.



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Bohr's Response: The Principle of Complementarity

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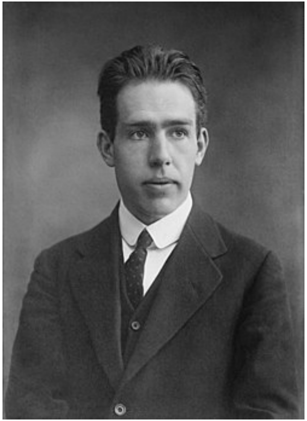

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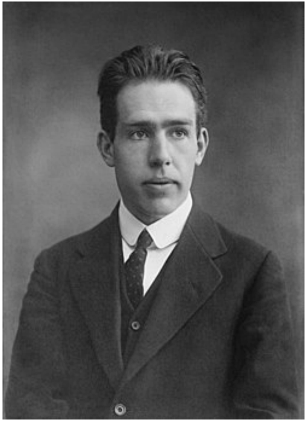

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Is Quantum Mechanics Complete? Bohr's View

Radical Revision of Physical Reality

- The experimenter's "freedom of choice" does not reveal pre-existing realities, but rather **creates different experimental conditions** and phenomena.
- Quantum mechanics is **complete** because it correctly and exhaustively describes the results of *every possible, well-defined experiment*.
- Bohr compares this revision of physical reality to the modification of ideas about the absolute character of phenomena introduced by the **Theory of Relativity**.
- This philosophical debate was crucial and led to **Bell's Theorem** and subsequent experiments, turning EPR into a testable question.

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Schrödinger's Response: The Birth of "Entanglement"

Context and the Essence of the New Physics

- A few months after EPR, Schrödinger published his response: "Discussion of Probability Relations between Separated Systems".
- He demonstrated a deep understanding and took the EPR argument to its most extreme conclusions.
- He understood that the phenomenon was not an anomaly, but **the characteristic trait of quantum mechanics**.
- He coined the term **"Entanglement"** **(Verschränkung)**.

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“Steering” and the Schmidt Decomposition

The most unsettling aspect of Entanglement

- Schrödinger identified an experimenter’s ability to **“steer” (steuern)** the state of a distant system.
- This is achieved through the local choice of measurement.
- He described it as “rather discomforting that the theory allows a system to be steered... at the whim of the experimenter, even though they have no access to it”.
- Rigorously demonstrated with a mathematical formalism: the **Schmidt decomposition**.

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Fundamental Differences in the Response to EPR

- **Bohr:** Offered a philosophical framework (complementarity) to *dissolve the paradox*, declaring that EPR's questions were ill-posed.
- **Schrödinger:** Accepted the validity of EPR's questions and delved into the unsettling answers the theory provided, seeking to *intensify the paradox*.

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- **Named and defined the central concept:** He coined the term "entanglement".
- **Identified its most powerful manifestation:** The concept of "steering," which captured the essence of non-locality.
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Bohm-Aharonov

Bohm & Aharonov: Simplifying EPR

A clearer, more experimental approach

- In 1951, **David Bohm** reformulated the EPR paradox in a conceptually clearer way.
- Together with **Yakir Aharonov** (in 1957), they simplified the example from continuous variables to the **discrete variable of spin**.
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Figure 6: David Bohm [16].

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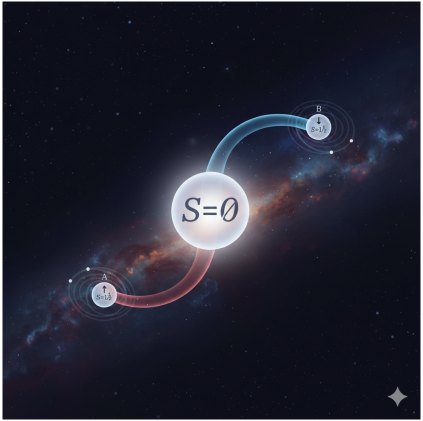
An example with spin-1/2 particles

- Consider a molecule with total spin zero that decays into two atoms (A and B) with spin 1/2.
- It is an **entangled** state where the total spin is zero, regardless of the measurement axis.

The entangled wave function:

$$\Psi = \frac{1}{\sqrt{2}}\left(\Psi_+(1)\Psi_-(2) - \Psi_-(1)\Psi_+(2)\right)$$



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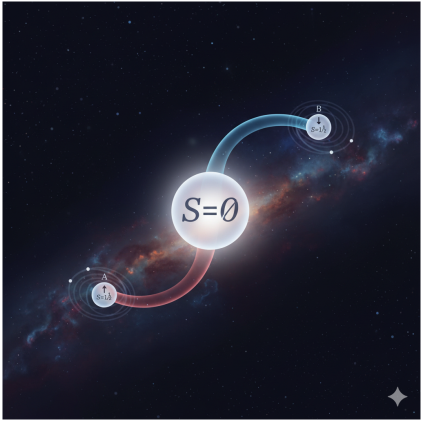

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The Core of the Paradox: Classical vs. Quantum

Classical Interpretation (Local Realism)

- The atoms have **pre-existing** and perfectly anti-correlated spin vectors.
- The measurement on A only **reveals** an already existing property.
- There is no instantaneous influence at a distance; the correlation was established locally at the source.

Quantum Interpretation

- A particle's spin state is **not defined until it is measured**.
- The choice to measure A's spin **realizes** its value, and due to entanglement, simultaneously realizes the opposite value for B.
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Key conclusion: Quantum uncertainty in entangled systems is an intrinsic property of the **non-separable correlations** that define the state of the system as a whole.

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The Alternative Hypothesis: Entanglement Breaking

Entanglement that “decays” with distance

- The idea is raised that the quantum many-body formulation **might not be valid** for widely separated particles.
- The proposal suggests that entanglement is a phenomenon that **decreases with distance**.
- An alternative **“classical” state** is proposed for the system after separation.

Alternative state (statistical mixture):

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- **Avoidance of the EPR Paradox:** It restores local realism; the measurement on A only reveals a pre-existing spin direction.
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Measurement Process and Setups

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The Crucial Ratio and the Predictions

The ratio R is defined as the quotient of the coincidence rates between the two geometries:

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- Local Realism Predictions (Statistical Mixture):
    - Circular Polarization (B1): R = 1.00
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Over 80 Years Apart

Albert Einstein

Physics Nobel Prize

Formulations vs Interpretations

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- Heisenberg's Matrix Mechanics
- Schrödinger's Wave Mechanics
- Hilbert Space Formalism (von Neumann)
- Bohmian Mechanics (Wave function decomposition: R and S)

Interpretations of Quantum Mechanics

- Copenhagen Interpretation (Statistical, Collapse of the wavefunction)
- EPR Argument and Reality Criterion
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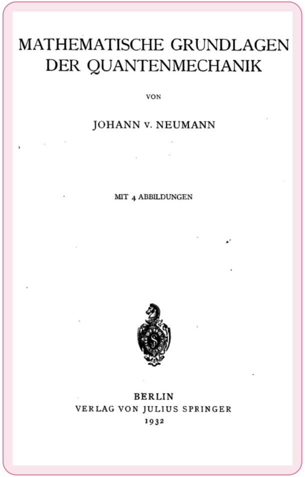
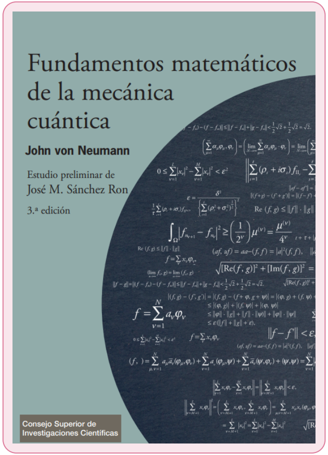
Mathematische Grundlagen der Quantenmechanik



First German edition (1932)



Spanish translation (CSIC, 3rd edition)

Von Neumann's Axioms (Section IV.1)

Von Neumann does not start from the quantum formalism itself, but rather establishes general axioms for the mean value of any physical magnitude R in a statistical ensemble, denoted as $V_m(R)$.

A'. **Positivity:** If a magnitude R is intrinsically non-negative (e.g. the square of another quantity), then

$$V_m(R) \geq 0$$

B'. **Linearity:** For arbitrary magnitudes $R, S, \ldots$ and real numbers $a, b, \ldots$:

$$V_m(aR + bS + \ldots) = aV_m(R) + bV_m(S) + \ldots$$

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Conditions for a Hidden-Variable System

In the hidden-variable hypothesis, any physical magnitude R must satisfy the following conditions:

1. **Eigenvalue Condition:** The expectation value of any magnitude R must coincide with one of its possible values (eigenvalues).

$$V_m(R) \in \{\text{eigenvalues of } R\}$$

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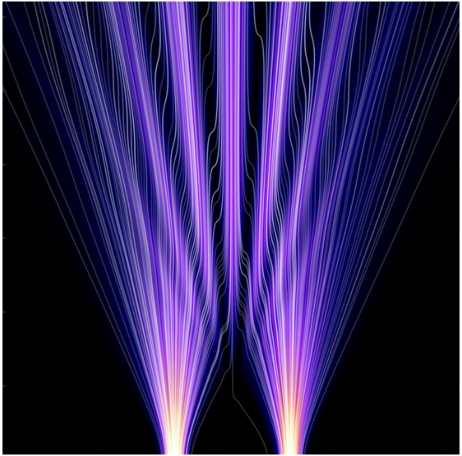
The core of von Neumann's argument is to show that there is no statistical operator U (and therefore no expectation-value function $V_m(R)$) that can simultaneously satisfy the linearity hypothesis (B') and the condition of absence of dispersion for all observables.

$$V_m(RS) + V_m(SR) = 2\,V_m(R)\,V_m(S)$$

For non-commuting observables ($[R, S] \neq 0$), there is in general no assignment of values $V_m(\cdot)$ that can satisfy this for all operators.

Bhom And Hidden Variables

Albert Einstein



Bhom Trayectories

Bhom Hidden Variables Papers

PHYSICAL REVIEW VOLUME 85, NUMBER 2 JANUARY 15, 1952

A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables. I

DAVID BOHM*

Palmer Physical Laboratory, Princeton University, Princeton, New Jersey

(Received July 5, 1951)

1. INTRODUCTION

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Hidden Variable I (1951)

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A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables. II

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1. INTRODUCTION

2. QUANTUM THEORY OF MEASUREMENTS

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Hidden Variables II (1951)

Bohmian Mechanics: Wave Function in Polar Form

Starting Point: Time-Dependent Schrödinger Equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\mathbf{x}, t)\psi$$

Polar Decomposition of the Wave Function

$$\psi(\mathbf{x}, t) = R(\mathbf{x}, t)\, e^{\frac{i}{\hbar}S(\mathbf{x}, t)}, \quad R \geq 0,\ S \in \mathbb{R}$$

This representation will split Schrödinger's equation into two coupled *real* equations: a continuity equation and a Hamilton–Jacobi–like equation.

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Bohmian Mechanics: Continuity and Quantum Potential

Continuity Equation (Probability Conservation)

$$\frac{\partial P}{\partial t} + \nabla \cdot \left(P \frac{\nabla S}{m}\right) = 0, \qquad P = R^2$$

Modified Hamilton–Jacobi Equation

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V(\mathbf{x}, t) + U(\mathbf{x}, t) = 0$$

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$$U(\mathbf{x}, t) = -\frac{\hbar^2}{2m} \frac{\nabla^2 R}{R}$$

Guidance Equation (Deterministic Trajectories)

$$\mathbf{p} = \nabla S(\mathbf{x}, t), \qquad \mathbf{v} = \frac{\nabla S}{m}$$

Equation of Motion (Newtonian Form with Quantum Force)

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Determinism vs. Statistics

Although Bohm's theory is deterministic for single systems, it reproduces all statistical predictions of standard quantum mechanics under three key assumptions:

1. The field ψ satisfies the Schrödinger equation.
2. The particle's momentum is always given by $p = \nabla S(\mathbf{x})$.
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Statistical Predictions in Bohmian Mechanics

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Continuity Equation Ensures Born's Rule Preservation

Starting from the polar decomposition, the continuity equation is:

$$\frac{\partial P}{\partial t} + \nabla \cdot (P\mathbf{v}) = 0, \qquad \mathbf{v} = \frac{\nabla S}{m}$$

This guarantees that if $P(\mathbf{x}, t_0) = |\psi(\mathbf{x}, t_0)|^2$ at one instant, then $P(\mathbf{x}, t) = |\psi(\mathbf{x}, t)|^2$ for all later times.

Interpretation: Probability is not fundamental but reflects ignorance of initial conditions, just as in classical statistical mechanics.

Consistency of the Statistical Postulate

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Extension to n particles

For a system of n particles, the wavefunction is:

$$\psi(x_1, x_2, \ldots, x_n, t) = R(x_1, \ldots, x_n, t)\, e^{\frac{i}{\hbar} S(x_1, \ldots, x_n, t)}$$

The quantum potential generalizes to:

$$U(x_1, \ldots, x_n) = -\frac{\hbar^2}{2m} \frac{\sum_i \nabla_i^2 R}{R}$$

Key feature: U depends on the coordinates of *all* particles simultaneously.

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Quantum Potential in Many-Body Systems

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Implications of the many-body quantum potential:

- The force on particle i,

$$\mathbf{F}_i = -\nabla_i U,$$

  depends instantaneously on the positions of all other particles.
- This represents an effective "many-body force" that is inherently **nonlocal**.
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Bell 1964 Paper

ON THE EINSTEIN PODOLSKY ROSEN PARADOX*

J. S. BELL†
Department of Physics, University of Wisconsin, Madison, Wisconsin

(*Received* 4 *November* 1964)

I. Introduction

THE paradox of Einstein, Podolsky and Rosen [1] was advanced as an argument that quantum mechanics could not be a complete theory but should be supplemented by additional variables. These additional variables were to restore to the theory causality and locality [2]. In this note that idea will be formulated mathematically and shown to be incompatible with the statistical predictions of quantum mechanics. It is the requirement of locality, or more precisely that the result of a measurement on one system be unaffected by operations on a distant system with which it has interacted in the past, that creates the essential difficulty. There have been attempts [3] to show that even without such a separability or locality requirement no "hidden variable" interpretation of quantum mechanics is possible. These attempts have been examined elsewhere [4] and found wanting. Moreover, a hidden variable interpretation of elementary quantum theory [5] has been explicitly constructed. That particular interpretation has indeed a grossly non-local structure. This is characteristic, according to the result to be proved here, of any such theory which reproduces exactly the quantum mechanical predictions.

On the Einstein Podolsky Rosen Paradox (1964)

Fundamental Assumptions of Bell

- **Hidden variables:** Each particle pair has a set of parameters λ that determines measurement outcomes locally.
- **Measurement results:** $A(a,\lambda), B(b,\lambda) \in \{+1,-1\}$ represent outcomes along directions a and b.
- **Perfect anticorrelation:** If $a = b$, then $A(a,\lambda) = -B(a,\lambda)$, consistent with quantum prediction $P(a,a) = -1$.
- **Local correlation:**

$$P(a,b) = \int d\lambda\, \rho(\lambda)\, A(a,\lambda)B(b,\lambda)$$

  where $\rho(\lambda)$ is the probability distribution over the hidden variables.
- *Interpretation: correlations are determined solely by local properties of the particles and their hidden variables.*

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Bell Derivation: Key Intermediate Steps

1. Rewrite using anticorrelation:

$$P(a,b) = -\int d\lambda\, \rho(\lambda) A(a,\lambda) A(b,\lambda)$$

2. Comparing three directions:

$$P(a,b) - P(a,c) = -\int d\lambda\, \rho(\lambda) [A(a)A(b) - A(a)A(c)]$$

3. Regroup using $A(b)^2 = 1$:

$$P(a,b) - P(a,c) = -\int d\lambda\, \rho(\lambda) A(a)A(b) [1 - A(b)A(c)]$$

The difference in correlations depends on how particle 2's "instructions" change between b and c.

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Bell Inequality and Physical Meaning

$$|P(a,b) - P(a,c)| \leq 1 + P(b,c)$$

- $P(a,b)$: correlation measured between results along a and b.
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- **Physical interpretation:** No local realistic theory can produce correlations stronger than this bound.
- Violation of this inequality by quantum predictions implies:
  - Realism and locality cannot both hold simultaneously.
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Quantum mechanics prediction for entangled particles:

$$P_{\text{QM}}(\mathbf{a}, \mathbf{b}) = -\mathbf{a} \cdot \mathbf{b}$$

Bell inequality :

$$|P(\mathbf{a}, \mathbf{b}) - P(\mathbf{a}, \mathbf{c})| \leq 1 + P(\mathbf{b}, \mathbf{c})$$

Specific setup:

- Take a vector **c** "halfway" between **a** and **b**.
- Define angles: $\theta = \angle(\mathbf{a}, \mathbf{c}) = \angle(\mathbf{c}, -\mathbf{b})$

Substituting the quantum prediction:

$$|-\mathbf{a} \cdot \mathbf{b} - (-\mathbf{a} \cdot \mathbf{c})| \leq 1 - \mathbf{c} \cdot \mathbf{b} \quad \Rightarrow \quad |\cos(2\theta) - \cos(\theta)| \leq 1 - \cos(\theta)$$

Violation of Bell Inequality: Quantum Prediction

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Explicit Violation for Small Angles

Approximations for small angles $\theta \ll 1$:

$$\cos(\theta) \approx 1 - \frac{\theta^2}{2}, \quad \cos(2\theta) \approx 1 - 2\theta^2$$

Bell inequality becomes:

$$|\cos(2\theta) - \cos(\theta)| \leq 1 - \cos(\theta) \quad \Rightarrow \quad |(1 - 2\theta^2) - (1 - \frac{\theta^2}{2})| \leq \frac{\theta^2}{2}$$

Simplifying:

$$\frac{3}{2}\theta^2 \leq \frac{1}{2}\theta^2$$

Conclusion: This inequality is **false** for any $\theta \neq 0$, demonstrating that **quantum mechanics predictions violate Bell's inequality**.

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$$|\cos(2\theta) - \cos(\theta)| \leq 1 - \cos(\theta) \quad \Rightarrow \quad |(1 - 2\theta^2) - (1 - \frac{\theta^2}{2})| \leq \frac{\theta^2}{2}$$

Simplifying:

$$\frac{3}{2}\theta^2 \leq \frac{1}{2}\theta^2$$

Conclusion: This inequality is **false** for any $\theta \neq 0$, demonstrating that **quantum mechanics predictions violate Bell's inequality**.

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Whether this question is indeed interesting has been the subject of debate.[1,2] The present paper does not contribute to that debate. It is addressed to those who do find the question interesting, and more particularly to those among them who believe that[3] "the question concerning the existence of such hidden variables received an early and rather decisive answer in the form of von Neumann's proof on the mathematical impossibility of such variables in quantum theory." An attempt will be made to clarify what von Neumann and his successors actually demonstrated. This will cover, as well as von Neumann's treatment, the recent version of the argument by Jauch and Piron,[3] and the stronger result consequent on the work of Gleason.[4] It will be urged that these analyses leave the real question untouched. In fact it will be seen that these demonstrations require from the hypothetical dispersion free states, not only that appropriate ensembles thereof should have all measurable properties of quantum mechanical states, but certain other properties as well. These additional demands appear reasonable when results of measurement are loosely identified with properties of isolated systems. They are seen to be quite unreasonable when one remembers with Bohr[5] "the impossibility of any sharp distinction between the behavior of atomic objects and the interaction with the measuring instruments which serve to define the conditions under which the phenomena appear."

The realization that von Neumann's proof is of limited relevance has been gaining ground since the 1952 work of Bohm.[6] However, it is far from universal. Moreover, the writer has not found in the literature any adequate analysis of what went wrong.[7] Like all authors of noncommissioned reviews, he thinks that he can restate the position with such clarity and simplicity that all previous discussions will be eclipsed.

II. ASSUMPTIONS, AND A SIMPLE EXAMPLE

The authors of the demonstrations to be reviewed were concerned to assume as little as possible about quantum mechanics. This is valuable for some purposes, but not for ours. We are interested only in the possibility of hidden variables in ordinary quantum me-

* Work supported by U.S. Atomic Energy Commission.
† Permanent address: CERN, Geneva.
[1] The following works contain discussions of and references on the hidden variable problem: L. de Broglie, *Physicien et Penseur* (Albin Michel, Paris, 1953); W. Heisenberg, in *Niels Bohr and the Development of Physics*, W. Pauli, Ed. (McGraw-Hill Book Co., Inc., New York, and Pergamon Press, Ltd., London, 1955); *Observation and Interpretation*, S. Körner, Ed. (Academic Press Inc., New York, and Butterworths Scientific Publ., Ltd., London, 1957); N. R. Hansen, *The Concept of the Positron* (Cambridge University Press, Cambridge, England, 1963). See also the various works by D. Bohm cited later, and Bell and Nauenberg.[8] For the view that the possibility of hidden variables has little interest, see especially the contributions of Rosenfeld to the first and third of these references, of Pauli to the first, the article of Heisenberg, and many passages in Hansen.
[2] A. Einstein, *Philosopher Scientist*, P. A. Schilp, Ed. (Library of Living Philosophers, Evanston, Ill., 1949). Einstein's "Autobiographical Notes" and "Reply to Critics" suggest that the hidden variable problem has some interest.
[3] J. M. Jauch and C. Piron, Helv. Phys. Acta **36**, 827 (1963).
[4] A. M. Gleason, J. Math. & Mech. **6**, 885 (1957). I am much indebted to Professor Jauch for drawing my attention to this work.
[5] N. Bohr, in Ref. 2.
[6] D. Bohm, Phys. Rev. **85**, 166, 180 (1952).
[7] In particular the analysis of Bohm[6] seems to lack clarity, or else accuracy. He fully emphasizes the role of the experimental arrangement. However, it seems to be implied (Ref. 6, p. 187) that the circumvention of the theorem *requires* the association of *hidden* variables with the apparatus as well as with the system observed. The scheme of Sec. II is a counter example to this. Moreover, it will be seen in Sec. III that if the essential additivity assumption of von Neumann were granted, hidden variables wherever located would not avail. Bohm's further remarks in Ref. 16 (p. 95) and Ref. 17 (p. 358) are also unconvincing. Other critiques of the theorem are cited, and some of them rebutted, by Albertson [J. Albertson, Am. J. Phys. **29**, 478 (1961)].

Bell, J.S.

On the problem of hidden variables in QM (1966)

The CHSH Inequality

Defining the correlation function:

$$E(a,b) = \langle A(a)\,B(b)\rangle, \quad A,B = \pm 1$$

The CHSH combination:

$$S = E(a,b) - E(a,b') + E(a',b) + E(a',b')$$

Local realism constraint:

$$-2 \leq S \leq 2$$

Quantum mechanics prediction:

$$|S| \leq 2\sqrt{2} \quad \Rightarrow \quad \text{Violation of local realism}$$

Impact:

- The CHSH form was **experimentally testable**.
- Enabled landmark tests (e.g. Aspect, 1982).

Why correlations matter:

- The values $E(a,b)$ come from **statistical counts** of coincident detections.
- Individual outcomes are random, but correlations reveal hidden structure.
- The strength of violation depends entirely on these **count-based correlations**.

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Conclusions on Quantum Entanglement

- EPR raised the dilemma about the **completeness** of quantum mechanics.
- Bell turned the **philosophical** debate into **mathematical and experimental predictions**.
- Entanglement reveals **nonlocal correlations**.
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Mathematical Framework for Entanglement

Motivation: From Feature to Formalism

The Origin of Entanglement

- Emerged as a mathematical feature of early quantum mechanics.
- Now rigorously described via experiment and theory.
- Core Idea: How to represent states of *composite quantum systems*.

A Note on Interpretation

While the math is well-established, its physical interpretation remains a subject of active debate—a common theme in quantum theory.

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Classical Composite Systems

For a two-particle system, the total state space is the Cartesian product of individual phase spaces:

$$\Gamma_{AB} = \Gamma_A \times \Gamma_B$$

The state (x_A, x_B) of the parts completely defines the whole.

A Naive Quantum Extrapolation

A simple guess might be $\mathcal{H}_{AB} = \mathcal{H}_A \times \mathcal{H}_B$. **This is incorrect.**

- It fails to incorporate the *superposition principle* for combined states.
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The Correct Postulate

The state space for a composite quantum system is the *tensor product* of the individual Hilbert spaces:

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \tag{1}$$

General Pure State

A general pure state $|\Psi\rangle \in \mathcal{H}_{AB}$ is a superposition:

$$|\Psi\rangle = \sum_{i,j} c_{ij}(|a_i\rangle \otimes |b_j\rangle) \tag{2}$$

where $\{|a_i\rangle\}$ and $\{|b_j\rangle\}$ are orthonormal bases for $\mathcal{H}_A$ and $\mathcal{H}_B$.

Dimension: $\mathbf{dim}(\mathcal{H}_{AB}) = \mathbf{dim}(\mathcal{H}_A) \cdot \mathbf{dim}(\mathcal{H}_B)$

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Two Fundamental Classes of States

Separable (Product) States

A state is *separable* if it can be written as a single tensor product:

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Subsystems have definite, independent properties (classical analogue).

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A state is *entangled* if it *cannot* be written in the separable form.

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The Symmetrization Postulate

The formalism $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ implicitly assumes particles are distinguishable.

For identical particles, quantum mechanics imposes the symmetrization postulate:

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Consequence for Fermions

A Restricted State Space

- Physically allowed states live in the *antisymmetric subspace* $\mathcal{H} \wedge \mathcal{H}$, not the full tensor product space $\mathcal{H} \otimes \mathcal{H}$.
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A New Paradigm: Mode Entanglement

Entanglement is reformulated based on correlations between modes (e.g., spatial vs. spin), not between labeled particles.

- The role of a separable state is now played by a single Slater determinant.

For two fermions in single-particle states $|\phi_1\rangle, |\phi_2\rangle$:

$$|\Psi\rangle_{\text{Slater}} = \frac{1}{\sqrt{2}}\left(|\phi_1\rangle_1 \otimes |\phi_2\rangle_2 - |\phi_2\rangle_1 \otimes |\phi_1\rangle_2\right)$$

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Defining Fermionic Entanglement: Slater Rank

Condition for Entanglement

A pure fermionic state is considered **entangled if and only if its Slater rank is greater than one**.

- This means it requires a superposition of multiple Slater determinants to be described.
- Slater Rank 1 $\iff$ "Separable" (unentangled)
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Example: Pauli Exclusion *Forces* Entanglement

Two Electrons in the Same Spatial Orbital

The total wave function $|\Psi\rangle_{\text{total}} = |\psi\rangle_{\text{spatial}} \otimes |\chi\rangle_{\text{spin}}$ must be antisymmetric.

1. **Spatial State:** $|\psi\rangle_{\text{spatial}}$ is *symmetric* (same orbital).
2. **Spin State:** To ensure total antisymmetry, $|\chi\rangle_{\text{spin}}$ *must be antisymmetric*.
3. **Result:** The unique antisymmetric spin state for two spin-1/2 particles is the maximally entangled spin-singlet:

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 - |\downarrow\rangle_1 \otimes |\uparrow\rangle_2)$$

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An operator O_A on subsystem A is represented on $\mathcal{H}_{AB}$ as a local operator:

$$O_A \to O_A \otimes I_B$$

Inner Product

The inner product is defined by linear extension:

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Formalism: Operators on Composite Systems

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Why We Need the Density Matrix (ρ)

Handling Incomplete Information

- When we look at only one part of an entangled pair, we have incomplete information about it.
- The state of such a subsystem cannot be described by a state vector.
- The density matrix formalism is essential for describing these subsystems.

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The Density Matrix: Pure vs. Mixed States

Pure States

For a pure state $|\Psi\rangle$, the density matrix is a projection operator:

$$\rho_{\text{pure}} = |\Psi\rangle\langle\Psi| \tag{4}$$

It is a projector ($\rho^2 = \rho$), so purity can be tested:

$$\text{Tr}(\rho^2) = 1$$

Mixed States

For a statistical ensemble of pure states $\{p_i, |\psi_i\rangle\}$:

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$$S(\rho) = -\mathrm{Tr}(\rho \ln \rho) = -\sum_i \lambda_i \ln \lambda_i \tag{6}$$

where λ_i are the eigenvalues of ρ.

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The state of subsystem A is found via the *reduced density matrix*:

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This quantity, $S(\rho_A)$, is the *entropy of entanglement*.

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Entropy as a Signature of Entanglement

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Quantifying and Detecting Entanglement

The Schmidt Decomposition Theorem

Any pure bipartite state $|\Psi\rangle_{AB}$ can be written in a special orthonormal basis:

$$|\Psi\rangle_{AB} = \sum_{k=1}^{r_S} \sqrt{\lambda_k}\, |k\rangle_A \otimes |k\rangle_B \tag{7}$$

- $\{|k\rangle_A\}$ and $\{|k\rangle_B\}$ are orthonormal bases (the Schmidt bases).
- The number of non-zero terms, r_S, is the *Schmidt rank*.

A Simple Criterion

A state is *separable if and only if its Schmidt rank is 1.*

For pure states, the degree of entanglement is uniquely quantified by the *entropy of entanglement*:

$$E(|\Psi\rangle) = S(\rho_A) = -\sum_{k=1}^{r_S} \lambda_k \ln \lambda_k \qquad (8)$$

(where λ_k are the squares of the Schmidt coefficients).

A Simple Criterion

A state is *separable if and only if its Schmidt rank is 1.*

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Mixed States: A Zoo of Measures

Quantifying mixed-state entanglement is complex; no single measure exists.

Entanglement of Formation (E_F)

Question: What is the minimum average pure-state entanglement needed to create ρ?

$$E_F(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle)$$

Relative Entropy of Entanglement (E_R)

Question: How “distant” is ρ from the set of separable states (SEP)?

$$E_R(\rho) = \min_{\sigma \in \mathrm{SEP}} S(\rho \| \sigma) = \min_{\sigma \in \mathrm{SEP}} \mathrm{Tr}(\rho \ln \rho - \rho \ln \sigma)$$

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Detecting Mixed-State Entanglement: The PPT Criterion

The Positive Partial Transpose (PPT) Test

A simple but powerful necessary condition for separability.

1. Start with a state ρ_{AB}.
2. Compute the *partial transpose* on one subsystem, e.g., B: ρ^{T_B}.
3. Check if ρ^{T_B} is positive semidefinite (all eigenvalues ≥ 0).

The Punchline

- If ρ_{AB} is separable $\implies$ ρ^{T_B} is positive.
- Therefore, if ρ^{T_B} has any negative eigenvalues, the state ρ_{AB} is certified as **entangled**.

(This condition is also sufficient only for 2×2 and 2×3 systems.)

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Worked Example: The Werner State

Definition

A mixture of a Bell state and a maximally mixed state ($p \in [0, 1]$):

$$\rho_W = p \left|\Psi^-\right\rangle \left\langle\Psi^-\right| + \frac{1-p}{4}\mathbb{I}_4$$

where $\left|\Psi^-\right\rangle = \frac{1}{\sqrt{2}}(\left|01\right\rangle - \left|10\right\rangle)$.

In the computational basis $\{\left|00\right\rangle, \left|01\right\rangle, \left|10\right\rangle, \left|11\right\rangle\}$:

$$\rho_W = \frac{1}{4}\begin{pmatrix} 1-p & 0 & 0 & 0 \\ 0 & 1+p & -2p & 0 \\ 0 & -2p & 1+p & 0 \\ 0 & 0 & 0 & 1-p \end{pmatrix}$$

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The Partial Transpose

We apply the transpose operation only to the second qubit's subspace. This swaps the $|01\rangle\langle 10|$ and $|10\rangle\langle 01|$ matrix elements.

$$\rho_W^{T_B} = \frac{1}{4}\begin{pmatrix} 1-p & 0 & 0 & -2p \\ 0 & 1+p & 0 & 0 \\ 0 & 0 & 1+p & 0 \\ -2p & 0 & 0 & 1-p \end{pmatrix}$$

Worked Example: Werner State (Conclusion)

Eigenvalues of the Partial Transpose

The eigenvalues of $\rho_W^{T_B}$ are:

$$\lambda_{1,2,3} = \frac{1+p}{4} \quad \lambda_4 = \frac{1-3p}{4} \tag{9}$$

Result

The eigenvalue λ_4 becomes negative if $1-3p<0$:

$$p > 1/3$$

Therefore, the Werner state is **entangled for** $p > 1/3$.

The magnitude of the negative eigenvalue is a measure of entanglement (a "negativity"): $\mathcal{N}(\rho_W) = \max(0, -\lambda_4)$

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Detecting Entanglement: Witnesses

Entanglement Witness

An entanglement witness is a Hermitian operator W designed such that:

- $\text{Tr}(W\rho_{\text{sep}}) \geq 0$ for *all* separable states.
- There exists at least one entangled state ρ_{ent} with $\text{Tr}(W\rho_{\text{ent}}) < 0$.

Experimental Implication

If an experiment measures an expectation value $\langle W \rangle = \text{Tr}(W\rho) < 0$, the state ρ is **certified as entangled**.

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Dynamics of Entanglement

Why Study Entanglement Dynamics?

We must understand how entanglement evolves under the influence of:

- The system's Hamiltonian (internal dynamics).
- Interaction with an environment (open system dynamics).

This is crucial for quantum information, where we must **generate**, **manipulate**, and **protect** entangled states.

Propagation in Closed Systems

In many-body systems with local interactions, entanglement propagates. The **Lieb-Robinson bounds** establish a finite maximum speed for information, creating a linear "light cone" for correlations.

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Non-Equilibrium Dynamics: Quantum Quenches

Quantum Quench

A sudden change in a system's Hamiltonian $(H_0 \to H_1)$ drives the system out of equilibrium, revealing universal entanglement dynamics.

The Quasiparticle Picture

For a subsystem of length ℓ, the entropy $S_A(t)$ shows a universal pattern:

1. Initial linear growth: $S_A(t) \propto t$.
2. Saturation to a volume-law: $S_A(t) \to \text{const} \cdot \ell$.

Explanation: The quench creates entangled quasiparticle pairs. Entanglement grows as pairs are split by the subsystem boundary.

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Entanglement in Open Systems: Fragility

Decoherence

Interaction with an external environment degrades a system's coherence and entanglement over time.

Entanglement is typically far more fragile than the coherence of individual subsystems.

Entanglement Sudden Death (ESD)

A striking feature of entanglement decay.

- Unlike local coherence (which decays asymptotically), entanglement can **vanish completely at a finite time**.
- After this time, the global state is separable, even if subsystems remain coherent.

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Alternative Perspectives

Beyond Hilbert Space: Geometric Algebra

An Alternative View

Alternative mathematical frameworks can offer different physical insights beyond the standard Hilbert space formalism.

Entanglement in Geometric Algebra

- Entanglement is not an intrinsic property of a state, but a relationship between reference frames.
- It is described by a superposition of *relative rotations*.
- The focus shifts from abstract states to the transformation operators themselves.

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The Algebra of Physical Space (APS)

- The language is Geometric Algebra, a powerful tool for describing geometric relationships.
- In this framework, spin states are represented as *rotors* (operators that perform rotations).

Entanglement as a Transformation

Central Thesis: Entanglement is encoded within a transformation operator, the entanglor, not in the state itself.

- Entangled State = (Entanglor) acting on a (Separable State).
- The entanglor contains the geometric info of the superposition of relative rotations between particle reference frames.

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Entangling Eigenspinors

A key advantage of APS is its natural compatibility with special relativity. A Lorentz boost (relative velocity) can be combined with an entanglor (relative rotation) into a single composite operator: the entangling eigenspinor Λ_{AB}.

Shift in Perspective

This reinterpretation has profound conceptual implications:

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Relativistic Context and Conceptual Implications

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Experimental Proofs

The Bridge to Reality: The CHSH Inequality

From Theory to Practice

- The CHSH inequality was the key theoretical tool that allowed for the design of the Freedman and Clauser experiment.

The Decisive Inequality (δ)

For the angles of predicted maximum violation ($\phi = 22.5^\circ$ and $3\phi = 67.5^\circ$), the inequality was simplified to a single expression:

$$\delta = \frac{R(22.5^\circ)}{R_0} - \frac{R(67.5^\circ)}{R_0} - \frac{1}{4}$$

- **Local Realism** requires: $\delta \leq 0$
- **Quantum Mechanics** predicts: $\delta > 0$

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Photon Source: Calcium Atomic Cascade

Generating Entangled Pairs

- An atomic cascade in calcium atoms was used to generate photon pairs.
- The **J=0 → J=1 → J=0** transition emits two photons.
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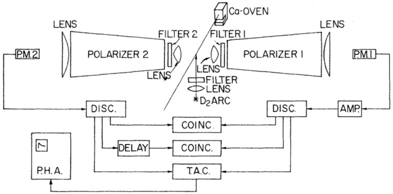
Key Measurement Components

- The optical system used lenses, filters, and **“pile-of-plates” polarizers**.
- The detectors were high-sensitivity **photomultiplier tubes (PMTs)**.
- The overall detection efficiency was extremely low ($\approx 1.6 \times 10^{-3}$), requiring long hours of measurement.



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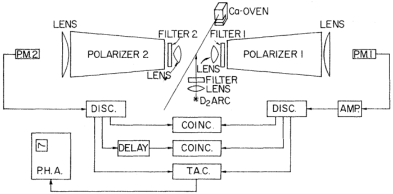

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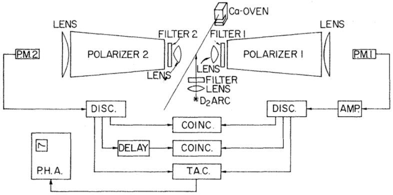

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Protocol and Result

- The experiment lasted approximately **200 hours** due to the low coincidence rate.
- A robust real-time normalization system was used to cancel out equipment drift.

The Decisive Experimental Result

The measured value was: $\delta = 0.050 \pm 0.008$

Required Improvements: The Loopholes

- Despite its success, the experiment was not airtight.

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Aspect and team

Clauser–Horne 1974 Paper

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Experimental consequences of objective local theories*

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A broad class of theories, called "objective local theories," is defined, motivation for considering these theories is given, and experimental consequences of the class are investigated. An extension of previous analyses by Bell and by Clauser *et al.* shows that predictions of objective local theories and of quantum mechanics differ, and that an experimental test of the entire family of objective local theories can be performed. The experimental requirements are given. Objective local theories satisfying a plausible but experimentally untestable supplementary assumption are shown to be incompatible with existing experimental data.

Clauser & Horne: Experimental Consequences of Objective Local Theories (1974)

Clauser–Horne Formulation (1974)

- Defined **Objective Local Theories (OLT)**: outcomes depend only on hidden variables λ and local settings, not on distant settings.
- Introduced the **No-Enhancement Assumption**: detection with a polarizer cannot exceed detection without it.
- Showed that this weaker assumption was still enough to make Freedman–Clauser results incompatible with all OLT satisfying it.
- Proved that some **supplementary assumption is necessary**: without no-enhancement, one can build an OLT reproducing quantum predictions.
- Revealed the **detection loophole**: experiments of that era only excluded OLT within the no-enhancement class.

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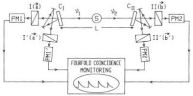
Aspect’s Experimental Setups (1980s)

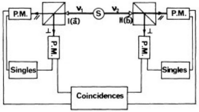
Experimental Program:

- Systematic tests of Bell inequalities using entangled photons from a calcium cascade.
- Progressively designed to close detection and locality loopholes.

Key Features:

- 1981: High-efficiency source, single-channel polarizers.
- 1982 (Two-channel): Polarizing beam splitters, closed the detection loophole.
- 1982 (Time-varying analyzers): Fast switches, addressed the locality loophole.





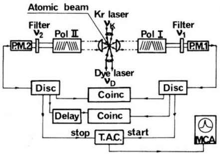


FIG. 2. Schematic diagram of apparatus and electronics. The laser beams are focused onto the atomic beam perpendicular to the figure. Feedback loops from the fluorescence signal control the krypton laser power and the dye-laser wavelength. The output of discriminators feed counters (not shown) and coincidence circuits. The multichannel analyzer (MCA) displays the time-delay spectrum.

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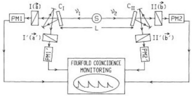

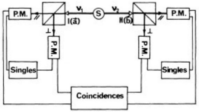

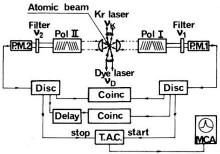

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Comparing Aspect's Three Experiments

Progression of Experimental Design

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Applications

Applications of Quantum Entanglement (I)

1. Quantum Computing

- Entangled **qubits** are the foundation of quantum computers.
- They allow for computations impossible for classical computers.
- Advances in cryptography, drug discovery, and materials science.

2. Quantum Cryptography (QKD)

- Uses pairs of entangled photons to generate cryptographic keys.
- Offers **unconditional security** (guaranteed by the laws of physics).
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Applications of Quantum Entanglement (II)

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    - Foundation for a future **quantum internet**.
    - Enables hyper-secure communications and the interconnection of distributed quantum computers.
    - Development of quantum repeaters for long distances.

4. Quantum Metrology and Sensing
    - Creates **sensors much more sensitive** than classical physics allows.
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    - **Quantum Teleportation:** Transferring quantum states (information) between distant particles.

Applications of Quantum Entanglement (III)

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References

References

[1] Ryszard Horodecki et al. **"Quantum entanglement".** In: *Reviews of Modern Physics* 81.2 (June 17, 2009), pp. 865–942. ISSN: 0034-6861, 1539-0756. DOI: 10.1103/RevModPhys.81.865. URL: https://link.aps.org/doi/10.1103/RevModPhys.81.865 (visited on 09/03/2025).

[2] Luigi Amico et al. **“Entanglement in many-body systems”.** In: *Reviews of Modern Physics* 80.2 (May 6, 2008), pp. 517–576. ISSN: 0034-6861, 1539-0756. DOI: `10.1103/RevModPhys.80.517`. URL: `https://link.aps.org/doi/10.1103/RevModPhys.80.517` (visited on 09/03/2025).

[3] Alain Aspect, Philippe Grangier, and Gérard Roger. **“Experimental Tests of Realistic Local Theories via Bell’s Theorem”.** In: *Physical Review Letters* 47.7 (Aug. 17, 1981), pp. 460–463. ISSN: 0031-9007. DOI: `10.1103/PhysRevLett.47.460`. URL: `https://link.aps.org/doi/10.1103/PhysRevLett.47.460` (visited on 09/05/2025).

[4] Alain Aspect, Philippe Grangier, and Gérard Roger. **“Experimental Realization of Einstein-Podolsky-Rosen-Bohm *Gedankenexperiment* : A New Violation of Bell’s Inequalities”.** In: *Physical Review Letters* 49.2 (July 12, 1982), pp. 91–94. ISSN: 0031-9007. DOI: `10.1103/PhysRevLett.49.91`. URL: `https://link.aps.org/doi/10.1103/PhysRevLett.49.91` (visited on 09/05/2025).

[5] Alain Aspect, Jean Dalibard, and Gérard Roger. **“Experimental Test of Bell’s Inequalities Using Time- Varying Analyzers”.** In: *Physical Review Letters* 49.25 (Dec. 20, 1982), pp. 1804–1807. ISSN: 0031-9007. DOI: `10.1103/PhysRevLett.49.1804`. URL: `https://link.aps.org/doi/10.1103/PhysRevLett.49.1804` (visited on 09/07/2025).

[6] Reinhold A. Bertlmann and Anton Zeilinger. ***Quantum [Un]speakables: From Bell to Quantum Information.*** Berlin, Heidelberg s.l: Springer Berlin Heidelberg, 2002. 485 pp. ISBN: 978-3-642-07664-0 978-3-662-05032-3. DOI: 10.1007/978-3-662-05032-3.

[7] A. Einstein, B. Podolsky, and N. Rosen. **"Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?"** In: *Physical Review* 47.10 (May 15, 1935), pp. 777–780. ISSN: 0031-899X. DOI: 10.1103/PhysRev.47.777. URL: https://link.aps.org/doi/10.1103/PhysRev.47.777 (visited on 09/07/2025).

[8] J S Bell. **"On the Einstein Podolsky Rosen paradox*".** In: (). DOI: https://doi.org/10.1103/PhysRev.47.777.

[9] Crystal-Ann McKenzie. **“An Interpretation of Relativistic Spin Entanglement Using Geometric Algebra”.** In: ().

[10] Timothy F. Havel and Chris J. L. Doran. **“Interaction and Entanglement in the Multiparticle Spacetime Algebra”.** In: *Applications of Geometric Algebra in Computer Science and Engineering*. Ed. by Leo Dorst, Chris Doran, and Joan Lasenby. Boston, MA: Birkhäuser Boston, 2002, pp. 227–247. ISBN: 978-1-4612-6606-8 978-1-4612-0089-5. DOI: 10.1007/978-1-4612-0089-5_21. URL: http://link.springer.com/10.1007/978-1-4612-0089-5_21 (visited on 09/17/2025).

[11] Yuri Alexandre Aoto and Márcio Fabiano Da Silva. **"Calculating the distance from an electronic wave function to the manifold of Slater determinants through the geometry of Grassmannians".** In: *Physical Review A* 102.5 (Nov. 6, 2020), p. 052803. ISSN: 2469-9926, 2469-9934. DOI: `10.1103/PhysRevA.102.052803`. URL: `https://link.aps.org/doi/10.1103/PhysRevA.102.052803` (visited on 10/01/2025).

[12] Paul Ehrenfest. ***Niels Bohr and Albert Einstein.*** Published: Photograph. Dec. 1925. URL: `http://www.dfi.dk/dfi/pressroom/kbhfortolkningen/`.

[13] Google. *Ilustración de Alice y Bob midiendo partículas entrelazadas separadas por una gran distancia.* Published: Imagen generada por IA. Oct. 2025.

[14] Bain News Service. *Prof. Bohr.* Published: Fotografía en negativo de vidrio. 1910. URL: https://www.loc.gov/pictures/item/2014715454/.

[15] Fotógrafo desconocido. *Retrato de Erwin Schrödinger.* Published: Fotografía. 1933. URL: http://search.bildarchiv.at/object/161477.

[16] Fotógrafo desconocido. *Fotografía de David Bohm.* Published: Fotografía. URL: http://www.cartage.org.lb/en/themes/Biographies/MainBiographies/B/Bohm/1.html (visited on 10/01/2025).

[17] Google. ***Una fuente emite dos partículas entrelazadas con espín opuesto.*** Published: Imagen generada por IA. Oct. 2025.

[18] Google. ***Esquema del experimento: una fuente emite dos fotones entrelazados.*** Published: Imagen generada por IA. Oct. 2025.

[19] N. Bohr. **"Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?"** In: *Physical Review* 48.8 (Oct. 1935). Publisher: American Physical Society (APS), pp. 696–702. DOI: `10.1103/PhysRev.48.696`.

[20] E. Schrödinger. **"Discussion of Probability Relations between Separated Systems".** In: *Mathematical Proceedings of the Cambridge Philosophical Society* 31.4 (Oct. 1935). Publisher: Cambridge University Press, pp. 555–563. DOI: `10.1017/S0305004100013554`.

[21] D. Bohm and Y. Aharonov. **"Discussion of Experimental Proof for the Paradox of Einstein, Rosen, and Podolsky".** In: *Physical Review* 108.5 (Dec. 1957). Publisher: American Physical Society (APS), pp. 1371–1373. DOI: 10.1103/PhysRev.108.1371.

[22] C. S. Wu and I. Shaknov. **"The Angular Correlation of Scattered Annihilation Radiation".** In: *Physical Review* 77.1 (Jan. 1950). Publisher: American Physical Society (APS), p. 136. DOI: 10.1103/PhysRev.77.136.

[23] Stuart J. Freedman and John F. Clauser. **"Experimental Test of Local Hidden-Variable Theories".** In: *Physical Review Letters* 28.14 (Apr. 1972). Publisher: American Physical Society (APS), pp. 938–941. DOI: 10.1103/PhysRevLett.28.938.

[24] Mohamed Hatifi. **“Relativistic Bohmian mechanics revisited: A covariant reformulation for spin-1/2 particles”.** In: *Physics Letters A* 518 (July 2024), p. 129680. DOI: `10.1016/j.physleta.2024.129680`. URL: `https://doi.org/10.1016/j.physleta.2024.129680`.

[25] colaboradores de Wikipedia. ***Albert Einstein.*** Oct. 2025. URL: `https://es.wikipedia.org/wiki/Albert_Einstein`.

[26] Wikipedia contributors. ***File:Nobel Prize.png - Wikipedia.*** URL: `https://en.wikipedia.org/wiki/File:Nobel_Prize.png`.

[27] *Mathematische Grundlagen der Quantenmechanik - GDZ - Göttinger Digitalisierungszentrum.* URL: https://gdz.sub.uni-goettingen.de/id/PPN379400774?tify=%7B%22pages%22%3A%5B2%5D%2C%22pan%22%3A%7B%22x%22%3A0.329%2C%22y%22%3A0.927%7D%2C%22view%22%3A%22info%22%2C%22zoom%22%3A0.31%7D.

[28] Anthony Duncan. **"Von Neumann's 1927 Trilogy on the Foundations of Quantum Mechanics. Annotated Translations".** In: *arXiv (Cornell University)* (June 2024). DOI: 10.48550/arxiv.2406.02149. URL: https://arxiv.org/abs/2406.02149.

[29] David Bohm. **“A Suggested Interpretation of the Quantum Theory in Terms of ”Hidden” Variables. I”.** In: *Physical Review* 85.2 (Jan. 1952), pp. 166–179. DOI: 10.1103/physrev.85.166. URL: https://doi.org/10.1103/physrev.85.166.

[30] David Bohm. **“A Suggested Interpretation of the Quantum Theory in Terms of ”Hidden” Variables. II”.** In: *Physical Review* 85.2 (Jan. 1952), pp. 180–193. DOI: 10.1103/physrev.85.180. URL: https://doi.org/10.1103/physrev.85.180.

[31] Lucien Hardy. **"Nonlocality for two particles without inequalities for almost all entangled states".** In: *Physical Review Letters* 71.11 (Sept. 1993), pp. 1665–1668. DOI: 10.1103/physrevlett.71.1665. URL: https://doi.org/10.1103/physrevlett.71.1665.

[32] John F. Clauser et al. **"Proposed Experiment to Test Local Hidden-Variable Theories".** In: *Physical Review Letters* 23.15 (Oct. 1969), pp. 880–884. DOI: 10.1103/physrevlett.23.880. URL: https://doi.org/10.1103/physrevlett.23.880.

[33] John Von Neumann. ***Fundamentos matemáticos de la mecánica cuántica.*** May 2018.

[34] Daniel F. Styer et al. **“Nine formulations of quantum mechanics”.** In: *American Journal of Physics* 70.3 (Mar. 2002), pp. 288–297. DOI: `10.1119/1.1445404`. URL: `https://doi.org/10.1119/1.1445404`.

[35] Ian J. R. Aitchison, David A. MacManus, and Thomas M. Snyder. **“Understanding Heisenberg’s “magical” paper of July 1925: A new look at the calculational details”.** In: *American Journal of Physics* 72.11 (Sept. 2004), pp. 1370–1379. DOI: `10.1119/1.1775243`. URL: `https://doi.org/10.1119/1.1775243`.

[36] John S. Bell. **“On the Problem of Hidden Variables in Quantum Mechanics”.** In: *Reviews of Modern Physics* 38.3 (July 1966), pp. 447–452. DOI: 10.1103/revmodphys.38.447. URL: https://doi.org/10.1103/revmodphys.38.447.

[37] D. Bohm and Y. Aharonov. **“Discussion of Experimental Proof for the Paradox of Einstein, Rosen, and Podolsky”.** In: *Physical Review* 108.4 (Nov. 1957), pp. 1070–1076. DOI: 10.1103/physrev.108.1070. URL: https://doi.org/10.1103/physrev.108.1070.

[38] John F. Clauser and Michael A. Horne. **“Experimental consequences of objective local theories”.** In: *Physical review. D. Particles, fields, gravitation, and cosmology/Physical review. D. Particles and fields* 10.2 (July 1974), pp. 526–535. DOI: `10.1103/physrevd.10.526`. URL: `https://doi.org/10.1103/physrevd.10.526`.

[39] Eugene P. Wigner. **“On Hidden Variables and Quantum Mechanical Probabilities”.** In: *American Journal of Physics* 38.8 (Aug. 1970), pp. 1005–1009. DOI: `10.1119/1.1976526`. URL: `https://doi.org/10.1119/1.1976526`.

[40] William A. Fedak and Jeffrey J. Prentis. **"The 1925 Born and Jordan paper "On quantum mechanics""**. In: *American Journal of Physics* 77.2 (Jan. 2009), pp. 128–139. DOI: 10.1119/1.3009634. URL: https://doi.org/10.1119/1.3009634.

Thank you!

Questions or Comments?

Quantum Entanglement

Spooky and Hidden

Avila, Herrera, Rodríguez

October 2, 2025

Universidad Distrital Francisco José de Caldas