A Chiral Symmetric Dirac Equation
Watson & Musielak (Int. J. Mod. Phys. A, 2020)

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What is Chirality?

A Lorentz-invariant property distinguishing how spinor components transform under boosts Λ:

$$\psi = \psi_L + \psi_R \tag{1}$$
$$\psi_L \to \Lambda\psi_L \quad , \quad \psi_R \to \Lambda^{-1}\psi_R \tag{2}$$

The components transform under inequivalent Lorentz group representations:

- **Left-handed (ψ_L):** $(\frac{1}{2}, 0)$
- **Right-handed (ψ_R):** $(0, \frac{1}{2})$

Chiral Symmetric Dirac Equation

$$\left(i\gamma^\mu\partial_\mu - me^{-i2\alpha\gamma^5}\right)\psi = 0 \tag{3}$$

The properties that the DECS must satisfy

In the theory of irreducible representations of the Poincaré group, each irreducible representation is defined by how the objects it describes transform under $\mathcal{P}$.

- **Poincaré Principle:** Physics is unchanged under Lorentz transformations and translations.
- **Gauge Invariance:** Interactions emerge from requiring local symmetry of the Lagrangian.
- **Locality:** Locality: Fields interact only at the same space–time point.

DECS derivation

1. Start with the translation operator eigenvalue equation:

Given that $P_\mu = i\partial_\mu$:

$$i\,\partial_\mu\phi = k_\mu\phi \quad \longmapsto \quad X^\mu\partial_\mu\psi = -Y\psi \tag{4}$$

2. Impose Lorentz invariance:

This yields the most general first-order equation with two degrees of freedom (y_L/x_R, y_R/x_L):

$$\left[i\gamma^\mu\partial_\mu + i\left(\frac{y_L}{x_R}P_L + \frac{y_R}{x_L}P_R\right)\right]\psi = 0 \tag{5}$$

3. Squaring the Hamiltonian reveals the mass term:

The squared Hamiltonian derived from eq. (5) reads:

$$H^2\psi = \left(\partial_k\partial^k - \frac{y_L y_R}{x_L x_R}\right)\psi \tag{6}$$

4. Identify the mass m and chiral angle α:

The remaining degrees of freedom are identified as:

$$m \equiv \pm i\sqrt{\frac{y_L y_R}{x_L x_R}} \qquad \alpha \equiv -\frac{i}{2}\ln\left(\mp\sqrt{\frac{x_L y_L}{x_R y_R}}\right) \tag{7}$$

Substituting these into eq. (5) gives the DECS eq. (3).

Chiral Angle and Mass

What is the Chiral Angle?

Rewriting the DECS eq. (3) reveals scalar and pseudoscalar mass terms:

$$\left(i\gamma^\mu\partial_\mu - M - \tilde{M}\gamma^5\right)\psi = 0 \tag{8}$$

Where α mixes the **scalar mass** M and **pseudoscalar mass** $\tilde{M}$:

- $M = m\cos 2\alpha$ (Scalar)
- $\tilde{M} = -im\sin 2\alpha$ (Pseudoscalar)

Chiral Angle and Mass

Generating Mass: A Two-Field Higgs Model

These M and $\tilde{M}$ terms can arise from Yukawa couplings to two Higgs-like fields: a scalar ϕ_1 and a pseudoscalar ϕ_2.

$$\mathcal{L}_Y \approx -\frac{\lambda_1 v_1}{\sqrt{2}}\bar{\psi}\psi - \frac{\lambda_2 v_2}{\sqrt{2}}\bar{\psi}\gamma^5\psi \tag{9}$$

This *fixes* the parameters m and α:

$$m = \sqrt{\frac{\lambda_1^2 v_1^2 - \lambda_2^2 v_2^2}{2}} \tag{10}$$
$$\alpha = \frac{i}{4}\ln\left(\frac{\lambda_1 v_1 + \lambda_2 v_2}{\lambda_1 v_1 - \lambda_2 v_2}\right) \tag{11}$$

Neutrinos and Dark Matter

Neutrino Mass Proposal (and its flaw)

- Paper's Goal: Explain small ν mass by cancellation between M and $\tilde{M}$:

$$m_\nu = \frac{1}{2\sqrt{2}}(\lambda_1 v_1 - \lambda_2 v_2) \tag{12}$$

- Fundamental Contradiction:
- The paper **states** ν has no right-chiral component ($\frac{1}{2}(1+\gamma^5)\nu = 0$).
- *But...* Dirac mass terms (both M and $\tilde{M}$) **require** both left and right chiral components to be non-zero.

Dark Matter Candidate

Despite the neutrino flaw, the model offers a DM candidate:

- The **pseudoscalar Higgs** ϕ_2 is proposed as Dark Matter.
- It couples to the SM only via the pseudoscalar Yukawa (primarily to neutrinos).
- This makes it massive, long-lived, and "dark"—a viable WIMP-like particle.

Knowledge Gap

- Hermiticity of the Chiral Symmetric Hamiltonian
- A Chiral Equation for higher spins (Proca, Rarita-Schwinger-like equations)
- Detection and measurement of the free parameters that specify the model
- Validity of the proposed neutrino mass as a Dirac mass

Conclusions

- The DECS is a valid generalization of the Dirac equation, formally introducing scalar (M) and pseudoscalar ($\tilde{M}$) mass terms mixed by a chiral angle α.
- A two-field Higgs model (ϕ_1, ϕ_2) can generate these mass terms.
- **Key Flaw:** The paper's application of this model to neutrinos is contradictory, as Dirac mass terms cannot apply to a purely left-chiral field.
- The pseudoscalar Higgs (ϕ_2) remains a viable WIMP-like Dark Matter candidate.

References

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