Quantum Entanglement

Spooky and Hidden

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EPR Debate

The Clash of Titans: Einstein's Realism

- **Deterministic View:** The universe is knowable and orderly.
- **Objective Reality:** Physical properties exist independently of measurement.
- **No Fundamental Randomness:** Probability is not intrinsic to nature.

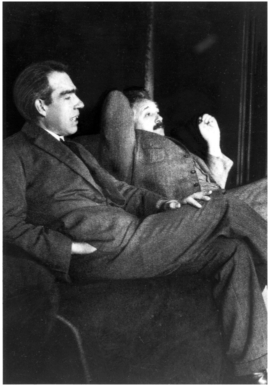

Figure 1: Niels Bohr with Albert Einstein at Paul Ehrenfest's home in Leiden [1].

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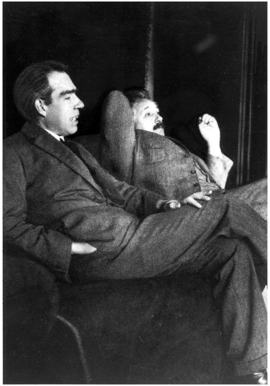

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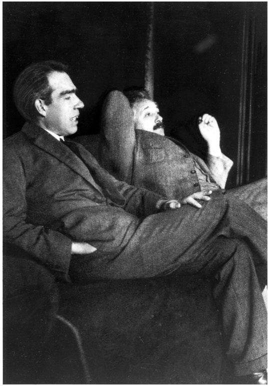

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The EPR Paradox: A Challenge in 1935

The Seminal Paper

- In 1935, Einstein, Podolsky, and Rosen published a key paper.
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MAY 15, 1935 PHYSICAL REVIEW VOLUME 47

Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

A. Einstein, B. Podolsky and N. Rosen, *Institute for Advanced Study, Princeton, New Jersey*
(Received March 25, 1935)

In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete.

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Any serious consideration of a physical theory must take into account the distinction between the objective reality, which is independent of any theory, and the physical concepts with which the theory operates. These concepts are intended to correspond with the objective reality, and by means of these concepts we picture this reality to ourselves.

In attempting to judge the success of a physical theory, we may ask ourselves two questions: (1) "Is the theory correct?" and (2) "Is the description given by the theory complete?" It is only in the case in which positive answers may be given to both of these questions, that the concepts of the theory may be said to be satisfactory. The correctness of the theory is judged by the degree of agreement between the conclusions of the theory and human experience. This experience, which alone enables us to make inferences about reality, in physics takes the form of experiment and measurement. It is the second question that we wish to consider here, as applied to quantum mechanics.

Whatever the meaning assigned to the term *complete*, the following requirement for a complete theory seems to be a necessary one: *every element of the physical reality must have a counterpart in the physical theory*. We shall call this the condition of completeness. The second question is thus easily answered, as soon as we are able to decide what are the elements of the physical reality.

The elements of the physical reality cannot be determined by *a priori* philosophical considerations, but must be found by an appeal to results of experiments and measurements. A comprehensive definition of reality is, however, unnecessary for our purpose. We shall be satisfied with the following criterion, which we regard as reasonable. *If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.* It seems to us that this criterion, while far from exhausting all possible ways of recognizing a physical reality, at least provides us with one

Figure 2: Paper: Can Quantum-Mechanical Description of Physical Reality Be Considered Complete? [2]

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The EPR Criterion of Reality

The "rule" to define an element of reality

To formalize their attack, EPR introduced an explicit and, seemingly, irrefutable criterion:

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Violation of the Criterion: Position and Momentum

Example with a state $\psi = e^{(2\pi i/h)p_0 x}$

- When applying the momentum operator (p), we get an exact value:

$$p\psi = \left(\frac{h}{2\pi i}\frac{\partial}{\partial x}\right)\psi = p_0\psi$$

  It is concluded that the momentum, p_0, is an **element of reality**.

- However, when applying the position operator (x), an exact value is not obtained:

$$x\psi \neq \text{constant} \cdot \psi$$

  We can only calculate the **probability** of finding the particle in an interval $[a, b]$, which turns out to be $P(a, b) = b - a$.

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The Fundamental Dichotomy

From the previous example, it follows that there are only two possibilities:

1. Either the quantum-mechanical description of reality given by the wave function **is not complete**.
2. Or when the operators for two physical quantities do not commute, the two quantities **cannot have simultaneous reality**.

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- An object can only be directly influenced by its **immediate surroundings**.
- Any influence at a distance cannot be instantaneous; it must propagate at a finite speed, not exceeding the speed of light (c).
- For Einstein, locality was a sacred axiom. Its violation would imply the disintegration of the universe's **causal structure** (cause-and-effect).

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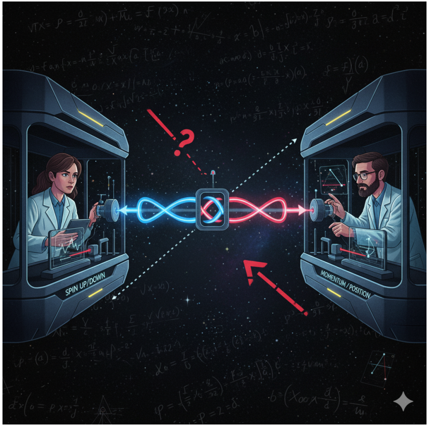
The two-particle system

- A system of two particles is prepared in an entangled state and then separated by a large distance.
- **Option 1:** If Alice measures the **momentum** of her particle, she can predict with certainty the momentum of Bob's particle.
- **Option 2:** If Alice measures the **position**, she can predict with certainty Bob's position.
- **Partial Conclusion:** Both the position and momentum of Bob's particle must be "elements of reality".



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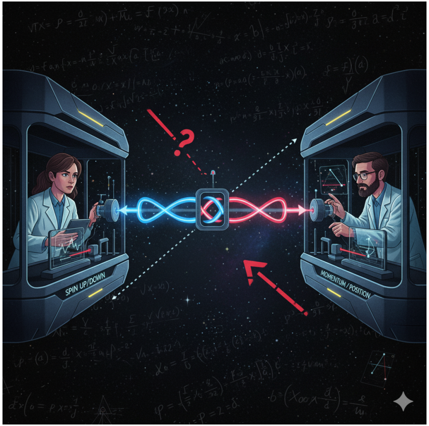

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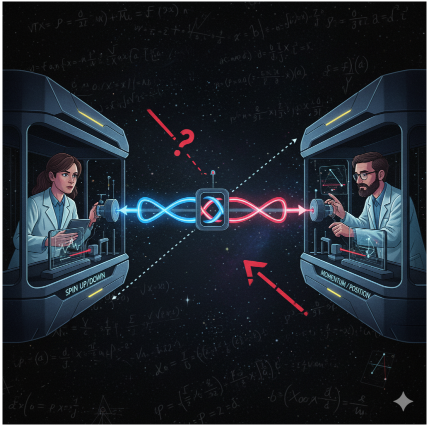

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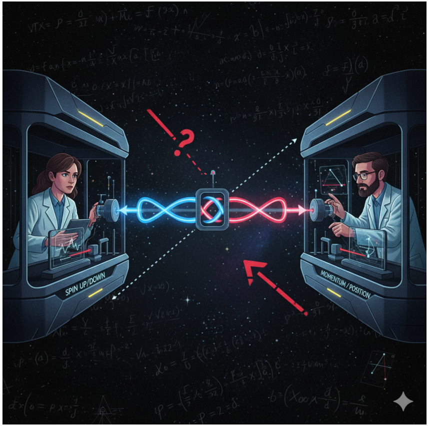

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The Contradiction and EPR's Conclusion

The Conflict with Quantum Mechanics

- The conclusion that Bob's particle has a definite position and momentum **simultaneously**...
- ...is in direct conflict with the **Heisenberg Uncertainty Principle**, which forbids it.

The EPR Conclusion

Faced with this dilemma, if local realism is correct, the only possible conclusion is that:

- The description of reality provided by the wave function must be **incomplete**.
- There must be an underlying theory ("hidden variables") that provides a complete description.

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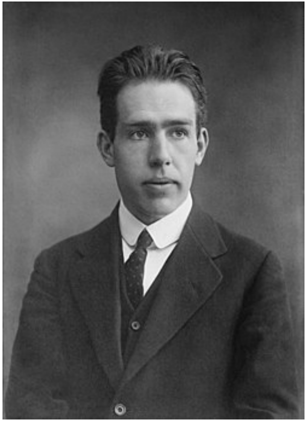
Bohr's Response: The Principle of Complementarity

Mutually Exclusive, Yet Necessary Concepts

- Complementarity posits that pairs of classical concepts are necessary for a complete description.
- But they are **mutually exclusive** in any single experiment.
- They are not contradictory properties of the object, but complementary aspects of a **phenomenon** revealed by incompatible experimental arrangements.



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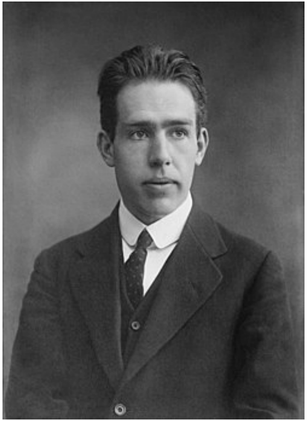

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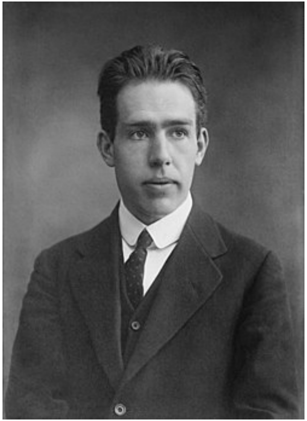

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Is Quantum Mechanics Complete? Bohr's View

Radical Revision of Physical Reality

- The experimenter's "freedom of choice" does not reveal pre-existing realities, but rather **creates different experimental conditions** and phenomena.
- Quantum mechanics is **complete** because it correctly and exhaustively describes the results of *every possible, well-defined experiment*.
- Bohr compares this revision of physical reality to the modification of ideas about the absolute character of phenomena introduced by the **Theory of Relativity**.
- This philosophical debate was crucial and led to **Bell's Theorem** and subsequent experiments, turning EPR into a testable question.

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Schrödinger's Response: The Birth of "Entanglement"

Context and the Essence of the New Physics

- A few months after EPR, Schrödinger published his response: "Discussion of Probability Relations between Separated Systems".
- He demonstrated a deep understanding and took the EPR argument to its most extreme conclusions.
- He understood that the phenomenon was not an anomaly, but **the characteristic trait of quantum mechanics**.
- He coined the term **"Entanglement" (Verschränkung)**.

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“Steering” and the Schmidt Decomposition

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- This is achieved through the local choice of measurement.
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- Rigorously demonstrated with a mathematical formalism: the **Schmidt decomposition**.

“Steering” and the Schmidt Decomposition

The most unsettling aspect of Entanglement

- Schrödinger identified an experimenter’s ability to **“steer” (steuern)** the state of a distant system.
- This is achieved through the local choice of measurement.
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Bohr vs. Schrödinger: Two Approaches to the Paradox

Fundamental Differences in the Response to EPR

- **Bohr:** Offered a philosophical framework (complementarity) to *dissolve the paradox*, declaring that EPR's questions were ill-posed.
- **Schrödinger:** Accepted the validity of EPR's questions and delved into the unsettling answers the theory provided, seeking to *intensify the paradox*.

Schrödinger's Key Contributions

- **Named and defined the central concept:** He coined the term "entanglement".
- **Identified its most powerful manifestation:** The concept of "steering," which captured the essence of non-locality.
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Bohm-Aharonov

Bohm & Aharonov: Simplifying EPR

A clearer, more experimental approach

- In 1951, **David Bohm** reformulated the EPR paradox in a conceptually clearer way.
- Together with **Yakir Aharonov** (in 1957), they simplified the example from continuous variables to the **discrete variable of spin**.
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Figure 6: David Bohm [5].

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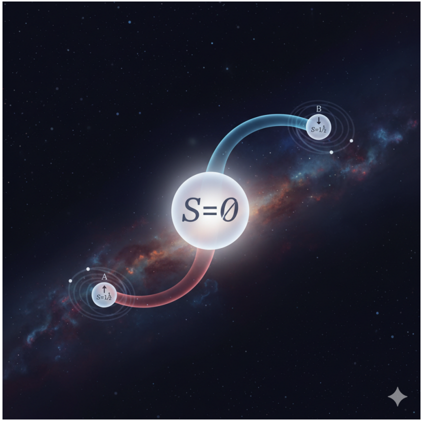
The Spin Singlet State

An example with spin-1/2 particles

- Consider a molecule with total spin zero that decays into two atoms (A and B) with spin 1/2.
- It is an **entangled** state where the total spin is zero, regardless of the measurement axis.

The entangled wave function:

$$\Psi = \frac{1}{\sqrt{2}} \left(\Psi_+(1)\Psi_-(2) - \Psi_-(1)\Psi_+(2) \right)$$



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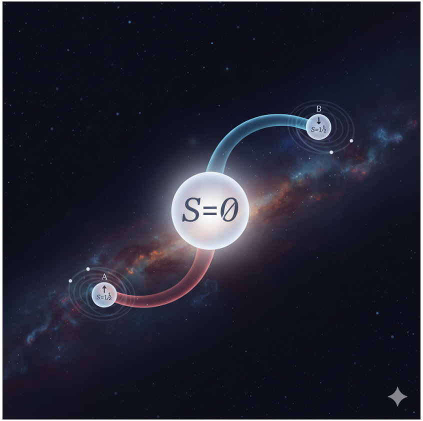

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The Core of the Paradox: Classical vs. Quantum

Classical Interpretation (Local Realism)

- The atoms have **pre-existing** and perfectly anti-correlated spin vectors.
- The measurement on A only **reveals** an already existing property.
- There is no instantaneous influence at a distance; the correlation was established locally at the source.

Quantum Interpretation

- A particle's spin state is **not defined until it is measured**.
- The choice to measure A's spin **realizes** its value, and due to entanglement, simultaneously realizes the opposite value for B.
- This happens **instantaneously**, despite the distance, seemingly violating locality.

Key conclusion: Quantum uncertainty in entangled systems is an intrinsic property of the **non-separable correlations** that define the state of the system as a whole.

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Entanglement that "decays" with distance

- The idea is raised that the quantum many-body formulation **might not be valid** for widely separated particles.
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- An alternative **"classical" state** is proposed for the system after separation.

Alternative state (statistical mixture):

$$\Psi = \Psi_{+\theta,\phi}(1)\Psi_{-\theta,\phi}(2)$$

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- **Avoidance of the EPR Paradox:** It restores local realism; the measurement on A only reveals a pre-existing spin direction.
- **Violation of Angular Momentum Conservation:** In individual events, total angular momentum would not be conserved (but it would on average).
- This pre-existing spin direction acts as a **“hidden variable”**.

Crucial Difference: Quantum Superposition vs. Statistical Mixture

- **Quantum Superposition (Entanglement):** Maintains perfect correlations in **any** measurement basis.
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A Reinterpretation of Existing Data

- Bohm and Aharonov did not propose a new experiment, but reinterpreted data already published by **Chien-Shiung Wu and Irving Shaknov** in 1950.
- Their contribution was theoretical: to unveil the fundamental implications of a previously known experimental result.

Measurement Process and Setups

- Polarization is measured indirectly via **Compton Scattering**.
- The rate of **coincidences** (simultaneous detection of both photons) is measured in two geometries:
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The Crucial Ratio and the Predictions

The ratio R is defined as the quotient of the coincidence rates between the two geometries:

$$R = \frac{\text{Coincidence Rate(parallel)}}{\text{Coincidence Rate(perpendicular)}}$$

- **Quantum Mechanics Prediction (Entangled State):**
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- Local Realism Predictions (Statistical Mixture):
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Over 80 Years Apart

Albert Einstein

Physics Nobel Prize

Formulations vs Interpretations

Formulations of Quantum Mechanics

- Heisenberg's Matrix Mechanics
- Schrödinger's Wave Mechanics
- Hilbert Space Formalism (von Neumann)
- Bohmian Mechanics (Wave function decomposition: R and S)

Interpretations of Quantum Mechanics

- Copenhagen Interpretation (Statistical, Collapse of the wavefunction)
- EPR Argument and Reality Criterion
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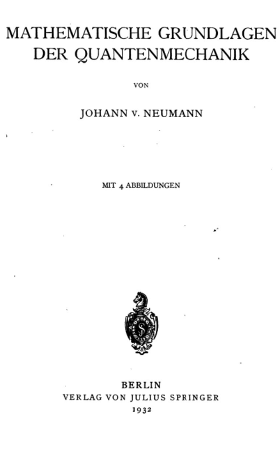
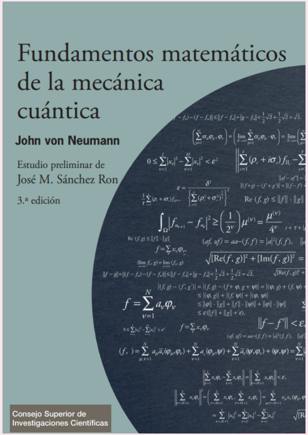
Mathematische Grundlagen der Quantenmechanik



First German edition (1932)



Spanish translation (CSIC, 3rd edition)

Von Neumann's Axioms (Section IV.1)

Von Neumann does not start from the quantum formalism itself, but rather establishes general axioms for the mean value of any physical magnitude R in a statistical ensemble, denoted as $V_m(R)$.

A'. **Positivity:** If a magnitude R is intrinsically non-negative (e.g. the square of another quantity), then

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B'. **Linearity:** For arbitrary magnitudes $R, S, \ldots$ and real numbers $a, b, \ldots$:

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2. **Dispersion-Free Condition:** The variance of any magnitude R must vanish, i.e.

$$V_m(R^2) = [V_m(R)]^2$$

Conditions for a Hidden-Variable System

In the hidden-variable hypothesis, any physical magnitude R must satisfy the following conditions:

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$$V_m(R) \in \{\text{eigenvalues of } R\}$$

2. **Dispersion-Free Condition:** The variance of any magnitude R must vanish, i.e.

$$V_m(R^2) = [V_m(R)]^2$$

Von Neumann's Impossibility Proof

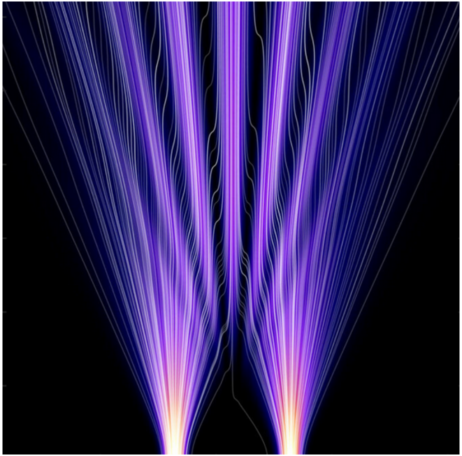
The core of von Neumann's argument is to show that there is no statistical operator U (and therefore no expectation-value function $V_m(R)$) that can simultaneously satisfy the linearity hypothesis (B') and the condition of absence of dispersion for all observables.

$$V_m(RS) + V_m(SR) = 2\,V_m(R)\,V_m(S)$$

For non-commuting observables ($[R, S] \neq 0$), there is in general no assignment of values $V_m(\cdot)$ that can satisfy this for all operators.

Bhom And Hidden Variables

Albert Einstein



Bhom Trayectories

Bhom Hidden Variables Papers

PHYSICAL REVIEW VOLUME 85, NUMBER 2 JANUARY 15, 1952

A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables. I

DAVID BOHM*
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey
(Received July 5, 1951)

The usual interpretation of the quantum theory is self-consistent, but it involves an assumption that cannot be tested experimentally, *viz.*, that the most complete possible specification of an individual system is in terms of a wave function that determines only probable results of actual measurement processes. The only way of investigating the truth of this assumption is by trying to find some other interpretation of the quantum theory in terms of at present "hidden" variables, which in principle determine the precise behavior of an individual system, but which are in practice averaged over in measurements of the types that can now be carried out. In this paper and in a subsequent paper, an interpretation of the quantum theory in terms of just such "hidden" variables is suggested. It is shown that as long as the mathematical theory retains its present general form, this suggested interpretation leads to precisely the same results for all physical processes as does the usual interpretation. Nevertheless, the suggested interpretation provides a broader conceptual framework than the usual interpretation, because it makes possible a precise and continuous description of all processes, even at the quantum level. This broader conceptual framework allows more general mathematical formulations of the theory than those allowed by the usual interpretation. Now, the usual mathematical formulation seems to lead to insoluble difficulties when it is extrapolated into the domain of distances of the order of 10^{-13} cm or less. It is therefore entirely possible that the interpretation suggested here may be needed for the resolution of these difficulties. In any case, the mere possibility of such an interpretation proves that it is not necessary for us to give up a precise, rational, and objective description of individual systems at a quantum level of accuracy.

1. INTRODUCTION

THE usual interpretation of the quantum theory is based on an assumption having very far-reaching implications, *viz.*, that the physical state of an individual system is completely specified by a wave function that determines only the probabilities of actual results that can be obtained in a statistical ensemble of similar experiments. This assumption has been the object of severe criticisms, notably on the part of Einstein, who has always believed that, even at the quantum level, there must exist precisely definable elements or dynamical variables determining (as in classical physics) the actual behavior of each individual system, and not merely its probable behavior. Since these elements or variables are not now included in the quantum theory and have not yet been detected experimentally, Einstein has always regarded the present form of the quantum theory as incomplete, although he admits its internal consistency.[1–5]

Most physicists have felt that objections such as those raised by Einstein are not relevant, first, because the present form of the quantum theory with its usual probability interpretation is in excellent agreement with an extremely wide range of experiments, at least in the domain of distances[6] larger than 10^{-13} cm, and, secondly, because no consistent alternative interpretations have as yet been suggested. The purpose of this paper (and of a subsequent paper hereafter denoted by II) is, however, to suggest just such an alternative interpretation. In contrast to the usual interpretation, this alternative interpretation permits us to conceive of each individual system as being in a precisely definable state, whose changes with time are determined by definite laws, analogous to (but not identical with) the classical equations of motion. Quantum-mechanical probabilities are regarded (like their counterparts in classical statistical mechanics) as only a practical necessity and not as a manifestation of an inherent lack of complete determination in the properties of matter at the quantum level. As long as the present general form of Schroedinger's equation is retained, the physical results obtained with our suggested alternative interpretation are precisely the same as those obtained with the usual interpretation. We shall see, however, that our alternative interpretation permits modifications of the mathematical formulation which could not even be described in terms of the usual interpretation. Moreover, the modifications can quite easily be formulated in such a way that their effects are insignificant in the atomic domain, where the present quantum theory is in such good agreement with experiment, but of crucial importance in the domain of dimensions of the order of 10^{-13} cm, where, as we have seen, the present theory is totally inadequate. It is thus entirely possible that some of the modifications describable in terms of our suggested alternative interpretation, but

* Now at Universidade de São Paulo, Faculdade de Filosofia, Ciencias, e Letras, São Paulo, Brasil.
[1] Einstein, Podolsky, and Rosen, Phys. Rev. **47**, 777 (1933).
[2] D. Bohm, *Quantum Theory* (Prentice-Hall, Inc., New York, 1951), see p. 611.
[3] N. Bohr, Phys. Rev. **48**, 696 (1935).
[4] W. Furry, Phys. Rev. **49**, 393, 476 (1936).
[5] Paul Arthur Schilp, editor, *Albert Einstein, Philosopher-Scientist* (Library of Living Philosophers, Evanston, Illinois, 1949). This book contains a thorough summary of the entire controversy.
[6] At distances of the order of 10^{-13} cm or smaller and for times of the order of this distance divided by the velocity of light or smaller, present theories become so inadequate that it is generally believed that they are probably not applicable, except perhaps in a very crude sense. Thus, it is generally expected that in connection with phenomena associated with this so-called "fundamental length," a totally new theory will probably be needed. It is hoped that this theory could not only deal precisely with such processes as meson production and scattering of elementary particles, but that it would also systematically predict the masses, charges, spins, etc., of the large number of so-called "elementary" particles that have already been found, as well as those of new particles which might be found in the future.

PHYSICAL REVIEW VOLUME 85, NUMBER 2 JANUARY 15, 1952

A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables. II

DAVID BOHM*
Palmer Physical Laboratory, Princeton University
(Received July 5, 1951)

In this paper, we shall show how the theory of measurements is to be understood from the point of view of a physical interpretation of the quantum theory in terms of "hidden" variables, developed in a previous paper. We find that in principle, these "hidden" variables determine the precise results of each individual measurement process. In practice, however, in measurements that we now know how to carry out, the observing apparatus disturbs the observed system in an unpredictable and uncontrollable way, so that the uncertainty principle is obtained as a practical limitation on the possible precision of measurements. This limitation is not, however, inherent in the conceptual structure of our interpretation. We shall see, for example, that simultaneous measurements of position and momentum having unlimited precision would in principle be possible if, as suggested in the previous paper, the mathematical formulation of the quantum theory needs to be modified at very short distances in certain ways that are consistent with our interpretation but not with the usual interpretation.

We give a simple explanation of the origin of quantum-mechanical correlations of distant objects in the hypothetical experiment of Einstein, Podolsky, and Rosen, which was suggested by these authors as a criticism of the usual interpretation.

Finally, we show that von Neumann's proof that quantum theory is not consistent with hidden variables does not apply to our interpretation, because the hidden variables contemplated here depend both on the state of the measuring apparatus and the observed system and therefore go beyond certain of von Neumann's assumptions.

In two appendixes, we treat the problem of the electromagnetic field in our interpretation and answer certain additional objections which have arisen in the attempt to give a precise description for an individual system at the quantum level.

1. INTRODUCTION

IN a previous paper,[1] to which we shall hereafter refer as I, we have suggested an interpretation of the quantum theory in terms of "hidden" variables. We have shown that although this interpretation provides a conceptual framework that is broader than that of the usual interpretation, it permits of a set of three mutually consistent special assumptions, which lead to the same physical results as are obtained from the usual interpretation of the quantum theory. These three special assumptions are: (1) The ψ-field satisfies Schroedinger's equation. (2) If we write $\psi = R\exp(is/\hbar)$, then the particle momentum is restricted to $\mathbf{p} = \nabla S(\mathbf{x})$. (3) We have a statistical ensemble of particle positions, with a probability density, $P = |\psi(\mathbf{x})|^2$. If the above three special assumptions are not made, then one obtains a more general theory that cannot be made consistent with the usual interpretation. It was suggested in Paper I that such generalizations may actually be needed for an understanding of phenomena associated with distances of the order of 10^{-13} cm or less, but may produce changes of negligible importance in the atomic domain.

In this paper, we shall apply the interpretation of the quantum theory suggested in Paper I to the development of a theory of measurements in order to show that as long as one makes the special assumptions indicated above, one is led to the same predictions for all measurements as are obtained from the usual interpretation. In our interpretation, however, the uncertainty principle is regarded, not as an inherent limitation on the precision with which we can correctly conceive of the simultaneous definition of momentum and position, but rather as a practical limitation on the precision with which these quantities can simultaneously be measured, arising from unpredictable and uncontrollable disturbances of the observed system by the measuring apparatus. If the theory needs to be generalized in the ways suggested in Paper I, Secs. 4 and 9, however, then these disturbances could in principle either be eliminated, or else be made subject to prediction and control, so that their effects could be corrected for. Our interpretation therefore demonstrates that measurements violating the uncertainty principle are at least conceivable.

2. QUANTUM THEORY OF MEASUREMENTS

We shall now show how the quantum theory of measurements is to be expressed in terms of our suggested interpretation of the quantum theory.[2]

In general, a measurement of any variable must always be carried out by means of an interaction of the system of interest with a suitable piece of measuring apparatus. The apparatus must be so constructed that any given state of the system of interest will lead to a certain range of states of the apparatus. Thus, the interaction introduces correlations between the state of the observed system and the state of the apparatus. The range of indefiniteness in this correlation may be called the uncertainty, or the error, in the measurement.

Let us now consider an observation designed to measure an arbitrary (hermitian) "observable" Q, associated with an electron. Let $\mathbf{x}$ represent the position of the electron, y that of the significant apparatus coordinate (or coordinates if there are more than one). Now, one can show[2] that it is enough to consider an impulsive measurement, i.e., a measurement utilizing a very strong interaction between apparatus and system

* Now at Universidade de São Paulo, Faculdade de Filosofia, Ciencias e Letras, São Paulo, Brasil.
[1] D. Bohm, Phys. Rev. **84**, 166 (1951).
[2] For a treatment of how the theory of measurements can be carried out with the usual interpretation, see D. Bohm, *Quantum Theory* (Prentice-Hall, Inc., New York, 1951), Chapter 22.

Hidden Variable I (1951)

Hidden Variables II (1951)

Bohmian Mechanics: Wave Function in Polar Form

Starting Point: Time-Dependent Schrödinger Equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 \psi + V(\mathbf{x}, t)\psi$$

Polar Decomposition of the Wave Function

$$\psi(\mathbf{x}, t) = R(\mathbf{x}, t)\, e^{\frac{i}{\hbar} S(\mathbf{x}, t)}, \quad R \geq 0,\ S \in \mathbb{R}$$

This representation will split Schrödinger's equation into two coupled *real* equations: a continuity equation and a Hamilton–Jacobi–like equation.

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Bohmian Mechanics: Continuity and Quantum Potential

Continuity Equation (Probability Conservation)

$$\frac{\partial P}{\partial t} + \nabla \cdot (P \, \tfrac{\nabla S}{m}) = 0, \qquad P = R^2$$

Modified Hamilton–Jacobi Equation

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V(\mathbf{x}, t) + U(\mathbf{x}, t) = 0$$

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Quantum Potential

$$U(\mathbf{x}, t) = -\frac{\hbar^2}{2m} \frac{\nabla^2 R}{R}$$

Guidance Equation (Deterministic Trajectories)

$$\mathbf{p} = \nabla S(\mathbf{x}, t), \qquad \mathbf{v} = \frac{\nabla S}{m}$$

Equation of Motion (Newtonian Form with Quantum Force)

$$m\frac{d^2\mathbf{x}}{dt^2} = -\nabla\Big(V(\mathbf{x}) + U(\mathbf{x}, t)\Big)$$

Bohmian Mechanics: Continuity, Quantum Potential and Motion

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Determinism vs. Statistics

Although Bohm's theory is deterministic for single systems, it reproduces all statistical predictions of standard quantum mechanics under three key assumptions:

1. The field ψ satisfies the Schrödinger equation.
2. The particle's momentum is always given by $p = \nabla S(\mathbf{x})$.
3. In practice, we cannot predict or control the exact initial position, so we work with an ensemble with probability density:

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Continuity Equation Ensures Born's Rule Preservation

Starting from the polar decomposition, the continuity equation is:

$$\frac{\partial P}{\partial t} + \nabla \cdot (P\mathbf{v}) = 0, \qquad \mathbf{v} = \frac{\nabla S}{m}$$

This guarantees that if $P(\mathbf{x}, t_0) = |\psi(\mathbf{x}, t_0)|^2$ at one instant, then $P(\mathbf{x}, t) = |\psi(\mathbf{x}, t)|^2$ for all later times.

Interpretation: Probability is not fundamental but reflects ignorance of initial conditions, just as in classical statistical mechanics.

Consistency of the Statistical Postulate

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Extension to n particles

For a system of n particles, the wavefunction is:

$$\psi(x_1, x_2, \ldots, x_n, t) = R(x_1, \ldots, x_n, t)\, e^{\frac{i}{\hbar} S(x_1, \ldots, x_n, t)}$$

The quantum potential generalizes to:

$$U(x_1, \ldots, x_n) = -\frac{\hbar^2}{2m} \frac{\sum_i \nabla_i^2 R}{R}$$

Key feature: U depends on the coordinates of *all* particles simultaneously.

Quantum Potential in Many-Body Systems

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Implications of the many-body quantum potential:

- The force on particle i,

$$\mathbf{F}_i = -\nabla_i U,$$

  depends instantaneously on the positions of all other particles.
- This represents an effective “many-body force” that is inherently **nonlocal**.
- In entangled states, the correlations predicted by quantum mechanics emerge naturally from this nonlocal potential.
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Nonlocality and Entanglement in Bohm's Theory

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Bell 1964 Paper

ON THE EINSTEIN PODOLSKY ROSEN PARADOX*

J. S. BELL†
Department of Physics, University of Wisconsin, Madison, Wisconsin

(*Received* 4 *November* 1964)

I. Introduction

THE paradox of Einstein, Podolsky and Rosen [1] was advanced as an argument that quantum mechanics could not be a complete theory but should be supplemented by additional variables. These additional variables were to restore to the theory causality and locality [2]. In this note that idea will be formulated mathematically and shown to be incompatible with the statistical predictions of quantum mechanics. It is the requirement of locality, or more precisely that the result of a measurement on one system be unaffected by operations on a distant system with which it has interacted in the past, that creates the essential difficulty. There have been attempts [3] to show that even without such a separability or locality requirement no "hidden variable" interpretation of quantum mechanics is possible. These attempts have been examined elsewhere [4] and found wanting. Moreover, a hidden variable interpretation of elementary quantum theory [5] has been explicitly constructed. That particular interpretation has indeed a grossly non-local structure. This is characteristic, according to the result to be proved here, of any such theory which reproduces exactly the quantum mechanical predictions.

On the Einstein Podolsky Rosen Paradox (1964)

Fundamental Assumptions of Bell

- **Hidden variables:** Each particle pair has a set of parameters λ that determines measurement outcomes locally.
- **Measurement results:** $A(a,\lambda), B(b,\lambda) \in \{+1, -1\}$ represent outcomes along directions a and b.
- **Perfect anticorrelation:** If $a = b$, then $A(a,\lambda) = -B(a,\lambda)$, consistent with quantum prediction $P(a,a) = -1$.
- **Local correlation:**

$$P(a,b) = \int d\lambda\, \rho(\lambda)\, A(a,\lambda)B(b,\lambda)$$

  where $\rho(\lambda)$ is the probability distribution over the hidden variables.
- *Interpretation: correlations are determined solely by local properties of the particles and their hidden variables.*

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Bell Derivation: Key Intermediate Steps

1. Rewrite using anticorrelation:

$$P(a,b) = -\int d\lambda\, \rho(\lambda) A(a,\lambda) A(b,\lambda)$$

2. Comparing three directions:

$$P(a,b) - P(a,c) = -\int d\lambda\, \rho(\lambda)[A(a)A(b) - A(a)A(c)]$$

3. Regroup using $A(b)^2 = 1$:

$$P(a,b) - P(a,c) = -\int d\lambda\, \rho(\lambda) A(a)A(b)[1 - A(b)A(c)]$$

The difference in correlations depends on how particle 2's "instructions" change between b and c.

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Bell Inequality and Physical Meaning

$$|P(a,b) - P(a,c)| \leq 1 + P(b,c)$$

- $P(a,b)$: correlation measured between results along a and b.
- $P(a,c)$: correlation measured between a and c.
- $P(b,c)$: correlation between directions b and c.
- **Physical interpretation:** No local realistic theory can produce correlations stronger than this bound.
- Violation of this inequality by quantum predictions implies:
  - Realism and locality cannot both hold simultaneously.
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- Violation of this inequality by quantum predictions implies:
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Violation of Bell Inequality: Quantum Prediction

Quantum mechanics prediction for entangled particles:

$$P_{\mathrm{QM}}(\mathbf{a}, \mathbf{b}) = -\mathbf{a} \cdot \mathbf{b}$$

Bell inequality :

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Specific setup:

- Take a vector **c** “halfway” between **a** and **b**.
- Define angles: $\theta = \angle(\mathbf{a}, \mathbf{c}) = \angle(\mathbf{c}, -\mathbf{b})$

Substituting the quantum prediction:

$$|-\mathbf{a} \cdot \mathbf{b} - (-\mathbf{a} \cdot \mathbf{c})| \leq 1 - \mathbf{c} \cdot \mathbf{b} \quad \Rightarrow \quad |\cos(2\theta) - \cos(\theta)| \leq 1 - \cos(\theta)$$

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Approximations for small angles $\theta \ll 1$:

$$\cos(\theta) \approx 1 - \frac{\theta^2}{2}, \quad \cos(2\theta) \approx 1 - 2\theta^2$$

Bell inequality becomes:

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Simplifying:

$$\frac{3}{2}\theta^2 \leq \frac{1}{2}\theta^2$$

Conclusion: This inequality is **false** for any $\theta \neq 0$, demonstrating that **quantum mechanics predictions violate Bell's inequality**.

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Bell's Conceptual Criticism:

- Von Neumann's **linearity assumption**:

$$V_m(aR + bS) = aV_m(R) + bV_m(S)$$

- Imposed even on dispersion-free states.
- Bell argued this is **unjustified for non-commuting observables**.
- Example: measuring $\sigma_x + \sigma_y$ is not the same as separately measuring σ_x and σ_y.
- Therefore, von Neumann's proof excludes only a very restrictive class of hidden-variable theories.
- Bohm's 1952 theory escapes because it is **contextual**.

REVIEWS OF MODERN PHYSICS VOLUME 38, NUMBER 3 JULY 1966

On the Problem of Hidden Variables in Quantum Mechanics*

JOHN S. BELL†

Stanford Linear Accelerator Center, Stanford University, Stanford, California

The demonstrations of von Neumann and others, that quantum mechanics does not permit a hidden variable interpretation, are reconsidered. It is shown that their essential axioms are unreasonable. It is urged that in further examination of this problem an interesting axiom would be that mutually distant systems are independent of one another.

I. INTRODUCTION

To know the quantum mechanical state of a system implies, in general, only statistical restrictions on the results of measurements. It seems interesting to ask if this statistical element be thought of as arising, as in classical statistical mechanics, because the states in question are averages over better defined states for which individually the results would be quite determined. These hypothetical "dispersion free" states would be specified not only by the quantum mechanical state vector but also by additional "hidden variables"—"hidden" because if states with prescribed values of these variables could actually be prepared, quantum mechanics would be observably inadequate.

Whether this question is indeed interesting has been the subject of debate.[1,2] The present paper does not contribute to that debate. It is addressed to those who do find the question interesting, and more particularly to those among them who believe that[3] "the question concerning the existence of such hidden variables received an early and rather decisive answer in the form of von Neumann's proof on the mathematical impossibility of such variables in quantum theory." An attempt will be made to clarify what von Neumann and his successors actually demonstrated. This will cover, as well as von Neumann's treatment, the recent version of the argument by Jauch and Piron,[3] and the stronger result consequent on the work of Gleason.[4] It will be urged that these analyses leave the real question untouched. In fact it will be seen that these demonstrations require from the hypothetical dispersion free states, not only that appropriate ensembles thereof should have all measurable properties of quantum mechanical states, but certain other properties as well. These additional demands appear reasonable when results of measurement are loosely identified with properties of isolated systems. They are seen to be quite unreasonable when one remembers with Bohr[5] "the impossibility of any sharp distinction between the behavior of atomic objects and the interaction with the measuring instruments which serve to define the conditions under which the phenomena appear."

The realization that von Neumann's proof is of limited relevance has been gaining ground since the 1952 work of Bohm.[6] However, it is far from universal. Moreover, the writer has not found in the literature any adequate analysis of what went wrong.[7] Like all authors of noncommissioned reviews, he thinks that he can restate the position with such clarity and simplicity that all previous discussions will be eclipsed.

II. ASSUMPTIONS, AND A SIMPLE EXAMPLE

The authors of the demonstrations to be reviewed were concerned to assume as little as possible about quantum mechanics. This is valuable for some purposes, but not for ours. We are interested only in the possibility of hidden variables in ordinary quantum me-

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The CHSH Inequality

Defining the correlation function:

$$E(a,b) = \langle A(a)\, B(b) \rangle, \quad A, B = \pm 1$$

The CHSH combination:

$$S = E(a,b) - E(a,b') + E(a',b) + E(a',b')$$

Local realism constraint:

$$-2 \leq S \leq 2$$

Quantum mechanics prediction:

$$|S| \leq 2\sqrt{2} \quad \Rightarrow \quad \text{Violation of local realism}$$

Impact:

- The CHSH form was **experimentally testable**.
- Enabled landmark tests (e.g. Aspect, 1982).

Why correlations matter:

- The values $E(a,b)$ come from **statistical counts** of coincident detections.
- Individual outcomes are random, but correlations reveal hidden structure.
- The strength of violation depends entirely on these **count-based correlations**.

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Conclusions on Quantum Entanglement

- EPR raised the dilemma about the **completeness** of quantum mechanics.
- Bell turned the **philosophical** debate into **mathematical and experimental predictions**.
- Entanglement reveals **nonlocal correlations**.
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Mathematical Framework for Entanglement

The Origin of Entanglement

- Emerged as a mathematical feature of early quantum mechanics.
- Now rigorously described via experiment and theory.
- Core Idea: How to represent states of *composite quantum systems*.

A Note on Interpretation

While the math is well-established, its physical interpretation remains a subject of active debate—a common theme in quantum theory.

Motivation: From Feature to Formalism

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Classical Composite Systems

For a two-particle system, the total state space is the Cartesian product of individual phase spaces:

$$\Gamma_{AB} = \Gamma_A \times \Gamma_B$$

The state (x_A, x_B) of the parts completely defines the whole.

A Naive Quantum Extrapolation

A simple guess might be $\mathcal{H}_{AB} = \mathcal{H}_A \times \mathcal{H}_B$. **This is incorrect.**

- It fails to incorporate the *superposition principle* for combined states.
- An ordered pair $(|\psi_A\rangle, |\phi_B\rangle)$ cannot represent linear combinations.

Departure from Classical State Spaces

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The Correct Postulate

The state space for a composite quantum system is the *tensor product* of the individual Hilbert spaces:

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \tag{1}$$

General Pure State

A general pure state $|\Psi\rangle \in \mathcal{H}_{AB}$ is a superposition:

$$|\Psi\rangle = \sum_{i,j} c_{ij}(|a_i\rangle \otimes |b_j\rangle) \tag{2}$$

where $\{|a_i\rangle\}$ and $\{|b_j\rangle\}$ are orthonormal bases for $\mathcal{H}_A$ and $\mathcal{H}_B$.

Dimension: $\dim(\mathcal{H}_{AB}) = \dim(\mathcal{H}_A) \cdot \dim(\mathcal{H}_B)$

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Two Fundamental Classes of States

Separable (Product) States

A state is *separable* if it can be written as a single tensor product:

$$|\Psi\rangle_{\text{sep}} = |\psi_A\rangle \otimes |\psi_B\rangle \tag{3}$$

Subsystems have definite, independent properties (classical analogue).

Entangled States

A state is *entangled* if it *cannot* be written in the separable form.

- Uniquely quantum-mechanical correlation.
- Measurement on A instantly affects B.

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A Complication: Identical Particles

The Symmetrization Postulate

The formalism $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ implicitly assumes particles are distinguishable.

For identical particles, quantum mechanics imposes the symmetrization postulate:

- Total wave function must be **symmetric** under particle exchange for **bosons**.
- Total wave function must be **antisymmetric** under particle exchange for **fermions**.

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Consequence for Fermions

A Restricted State Space

- Physically allowed states live in the *antisymmetric subspace* $\mathcal{H} \wedge \mathcal{H}$, not the full tensor product space $\mathcal{H} \otimes \mathcal{H}$.
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A New Paradigm: Mode Entanglement

Entanglement is reformulated based on correlations between modes (e.g., spatial vs. spin), not between labeled particles.

- The role of a separable state is now played by a single Slater determinant.

For two fermions in single-particle states $|\phi_1\rangle, |\phi_2\rangle$:

$$|\Psi\rangle_{\text{Slater}} = \frac{1}{\sqrt{2}}\left(|\phi_1\rangle_1 \otimes |\phi_2\rangle_2 - |\phi_2\rangle_1 \otimes |\phi_1\rangle_2\right)$$

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- Slater Rank 1 $\iff$ "Separable" (unentangled)
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Example: Pauli Exclusion *Forces* Entanglement

Two Electrons in the Same Spatial Orbital

The total wave function $|\Psi\rangle_{\text{total}} = |\psi\rangle_{\text{spatial}} \otimes |\chi\rangle_{\text{spin}}$ must be antisymmetric.

1. **Spatial State:** $|\psi\rangle_{\text{spatial}}$ is *symmetric* (same orbital).
2. **Spin State:** To ensure total antisymmetry, $|\chi\rangle_{\text{spin}}$ *must be antisymmetric*.
3. **Result:** The unique antisymmetric spin state for two spin-1/2 particles is the maximally entangled spin-singlet:

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$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 - |\downarrow\rangle_1 \otimes |\uparrow\rangle_2)$$

Example: Pauli Exclusion (Conclusion)

Conclusion
The Pauli exclusion principle *forces* the two electrons into a maximally entangled Bell state.

For indistinguishable particles, entanglement is not just a possibility but can be a **necessity** imposed by fundamental symmetries.

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Formalism: Operators on Composite Systems

Local Operators

An operator O_A on subsystem A is represented on $\mathcal{H}_{AB}$ as a local operator:

$$O_A \to O_A \otimes I_B$$

Inner Product

The inner product is defined by linear extension:

$$(\langle\phi_A| \otimes \langle\phi_B|)(|\psi_A\rangle \otimes |\psi_B\rangle) = \langle\phi_A\rangle\, \psi_A \,\langle\phi_B\rangle\, \psi_B$$

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Why We Need the Density Matrix (ρ)

Handling Incomplete Information

- When we look at only one part of an entangled pair, we have incomplete information about it.
- The state of such a subsystem cannot be described by a state vector.
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The Density Matrix: Pure vs. Mixed States

Pure States

For a pure state $|\Psi\rangle$, the density matrix is a projection operator:

$$\rho_{\text{pure}} = |\Psi\rangle \langle\Psi| \tag{4}$$

It is a projector ($\rho^2 = \rho$), so purity can be tested:

$$\text{Tr}(\rho^2) = 1$$

Mixed States

For a statistical ensemble of pure states $\{p_i, |\psi_i\rangle\}$:

$$\rho_{\text{mixed}} = \sum_i p_i |\psi_i\rangle \langle\psi_i| \tag{5}$$

For any mixed state, the purity is less than one: $\text{Tr}(\rho^2_{\text{mixed}}) < 1$.

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Quantifying Uncertainty: Von Neumann Entropy

Definition

The degree of uncertainty or "mixedness" is quantified by the Von Neumann entropy:

$$S(\rho) = -\mathrm{Tr}(\rho \ln \rho) = -\sum_i \lambda_i \ln \lambda_i \tag{6}$$

where λ_i are the eigenvalues of ρ.

- Pure state ($\rho = |\psi\rangle\langle\psi|$): $S(\rho) = 0$ (maximal knowledge).
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Consider a composite system AB in a pure state $|\Psi\rangle_{AB}$, so $S(\rho_{AB}) = 0$.

The state of subsystem A is found via the *reduced density matrix*:

$$\rho_A = \mathrm{Tr}_B(\rho_{AB})$$

If $|\Psi\rangle_{AB}$ is entangled, ρ_A will be a mixed state, and its entropy $S(\rho_A) > 0$.

This quantity, $S(\rho_A)$, is the *entropy of entanglement*.

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Quantifying and Detecting Entanglement

The Schmidt Decomposition Theorem

Any pure bipartite state $|\Psi\rangle_{AB}$ can be written in a special orthonormal basis:

$$|\Psi\rangle_{AB} = \sum_{k=1}^{r_S} \sqrt{\lambda_k}\, |k\rangle_A \otimes |k\rangle_B \tag{7}$$

- $\{|k\rangle_A\}$ and $\{|k\rangle_B\}$ are orthonormal bases (the Schmidt bases).
- The number of non-zero terms, r_S, is the *Schmidt rank*.

Schmidt Rank and Entanglement

A Simple Criterion

A state is *separable if and only if its Schmidt rank is 1.*

For pure states, the degree of entanglement is uniquely quantified by the *entropy of entanglement*:

$$E(|\Psi\rangle) = S(\rho_A) = -\sum_{k=1}^{r_S} \lambda_k \ln \lambda_k \tag{8}$$

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Mixed States: A Zoo of Measures

Quantifying mixed-state entanglement is complex; no single measure exists.

Entanglement of Formation (E_F)

Question: What is the minimum average pure-state entanglement needed to create ρ?

$$E_F(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle)$$

Relative Entropy of Entanglement (E_R)

Question: How “distant” is ρ from the set of separable states (SEP)?

$$E_R(\rho) = \min_{\sigma \in \mathrm{SEP}} S(\rho || \sigma) = \min_{\sigma \in \mathrm{SEP}} \mathrm{Tr}(\rho \ln \rho - \rho \ln \sigma)$$

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Detecting Mixed-State Entanglement: The PPT Criterion

The Positive Partial Transpose (PPT) Test

A simple but powerful necessary condition for separability.

1. Start with a state ρ_{AB}.
2. Compute the *partial transpose* on one subsystem, e.g., B: ρ^{T_B}.
3. Check if ρ^{T_B} is positive semidefinite (all eigenvalues ≥ 0).

The Punchline

- If ρ_{AB} is separable $\implies$ ρ^{T_B} is positive.
- Therefore, if ρ^{T_B} has any negative eigenvalues, the state ρ_{AB} is certified as **entangled**.

(This condition is also sufficient only for 2×2 and 2×3 systems.)

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Worked Example: The Werner State

Definition

A mixture of a Bell state and a maximally mixed state ($p \in [0, 1]$):

$$\rho_W = p \left|\Psi^-\right\rangle \left\langle\Psi^-\right| + \frac{1-p}{4}\mathbb{I}_4$$

where $\left|\Psi^-\right\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.

In the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$:

$$\rho_W = \frac{1}{4}\begin{pmatrix} 1-p & 0 & 0 & 0 \\ 0 & 1+p & -2p & 0 \\ 0 & -2p & 1+p & 0 \\ 0 & 0 & 0 & 1-p \end{pmatrix}$$

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Worked Example: Applying the PPT Criterion

The Partial Transpose

We apply the transpose operation only to the second qubit's subspace. This swaps the $|01\rangle\langle 10|$ and $|10\rangle\langle 01|$ matrix elements.

$$\rho_W^{T_B} = \frac{1}{4}\begin{pmatrix} 1-p & 0 & 0 & -2p \\ 0 & 1+p & 0 & 0 \\ 0 & 0 & 1+p & 0 \\ -2p & 0 & 0 & 1-p \end{pmatrix}$$

Worked Example: Werner State (Conclusion)

Eigenvalues of the Partial Transpose

The eigenvalues of $\rho_W^{T_B}$ are:

$$\lambda_{1,2,3} = \frac{1+p}{4} \quad \lambda_4 = \frac{1-3p}{4} \tag{9}$$

Result

The eigenvalue λ_4 becomes negative if $1 - 3p < 0$:

$$p > 1/3$$

Therefore, the Werner state is **entangled for** $p > 1/3$.

The magnitude of the negative eigenvalue is a measure of entanglement (a "negativity"): $\mathcal{N}(\rho_W) = \max(0, -\lambda_4)$

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Detecting Entanglement: Witnesses

Entanglement Witness

An entanglement witness is a Hermitian operator W designed such that:

- $\mathrm{Tr}(W\rho_{\mathrm{sep}}) \geq 0$ for *all* separable states.
- There exists at least one entangled state ρ_{ent} with $\mathrm{Tr}(W\rho_{\mathrm{ent}}) < 0$.

Experimental Implication

If an experiment measures an expectation value $\langle W \rangle = \mathrm{Tr}(W\rho) < 0$, the state ρ is **certified as entangled**.

Geometrically, W is a hyperplane separating ρ_{ent} from the convex set of separable states.

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Dynamics of Entanglement

Why Study Entanglement Dynamics?

We must understand how entanglement evolves under the influence of:

- The system's Hamiltonian (internal dynamics).
- Interaction with an environment (open system dynamics).

This is crucial for quantum information, where we must **generate**, **manipulate**, and **protect** entangled states.

Propagation in Closed Systems

In many-body systems with local interactions, entanglement propagates. The Lieb-Robinson bounds establish a finite maximum speed for information, creating a linear "light cone" for correlations.

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Non-Equilibrium Dynamics: Quantum Quenches

Quantum Quench

A sudden change in a system's Hamiltonian $(H_0 \to H_1)$ drives the system out of equilibrium, revealing universal entanglement dynamics.

The Quasiparticle Picture

For a subsystem of length ℓ, the entropy $S_A(t)$ shows a universal pattern:

1. Initial linear growth: $S_A(t) \propto t$.
2. Saturation to a volume-law: $S_A(t) \to \text{const} \cdot \ell$.

Explanation: The quench creates entangled quasiparticle pairs. Entanglement grows as pairs are split by the subsystem boundary.

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Entanglement in Open Systems: Fragility

Decoherence

Interaction with an external environment degrades a system's coherence and entanglement over time.

Entanglement is typically far more fragile than the coherence of individual subsystems.

Entanglement Sudden Death (ESD)

A striking feature of entanglement decay.

- Unlike local coherence (which decays asymptotically), entanglement can **vanish completely at a finite time**.
- After this time, the global state is separable, even if subsystems remain coherent.

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Alternative Perspectives

An Alternative View

Alternative mathematical frameworks can offer different physical insights beyond the standard Hilbert space formalism.

Entanglement in Geometric Algebra

- Entanglement is not an intrinsic property of a state, but a relationship between reference frames.
- It is described by a superposition of *relative rotations*.
- The focus shifts from abstract states to the transformation operators themselves.

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The Algebra of Physical Space (APS)

- The language is Geometric Algebra, a powerful tool for describing geometric relationships.
- In this framework, spin states are represented as *rotors* (operators that perform rotations).

Entanglement as a Transformation

Central Thesis: Entanglement is encoded within a transformation operator, the entanglor, not in the state itself.

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A key advantage of APS is its natural compatibility with special relativity. A Lorentz boost (relative velocity) can be combined with an entanglor (relative rotation) into a single composite operator: the entangling eigenspinor Λ_{AB}.

Shift in Perspective

This reinterpretation has profound conceptual implications:

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Experimental Proofs

The Bridge to Reality: The CHSH Inequality

From Theory to Practice

- The CHSH inequality was the key theoretical tool that allowed for the design of the Freedman and Clauser experiment.

The Decisive Inequality (δ)

For the angles of predicted maximum violation ($\phi = 22.5^\circ$ and $3\phi = 67.5^\circ$), the inequality was simplified to a single expression:

$$\delta = \frac{R(22.5^\circ)}{R_0} - \frac{R(67.5^\circ)}{R_0} - \frac{1}{4}$$

- **Local Realism** requires: $\delta \leq 0$
- **Quantum Mechanics** predicts: $\delta > 0$

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$$\delta = \frac{R(22.5^\circ)}{R_0} - \frac{R(67.5^\circ)}{R_0} - \frac{1}{4}$$

- **Local Realism** requires: $\delta \leq 0$
- **Quantum Mechanics** predicts: $\delta > 0$

The Bridge to Reality: The CHSH Inequality

From Theory to Practice

- The CHSH inequality was the key theoretical tool that allowed for the design of the Freedman and Clauser experiment.

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Generating Entangled Pairs

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- The **J=0** → **J=1** → **J=0** transition emits two photons.
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- The optical system used lenses, filters, and **“pile-of-plates” polarizers**.
- The detectors were high-sensitivity **photomultiplier tubes (PMTs)**.
- The overall detection efficiency was extremely low ($\approx 1.6 \times 10^{-3}$), requiring long hours of measurement.

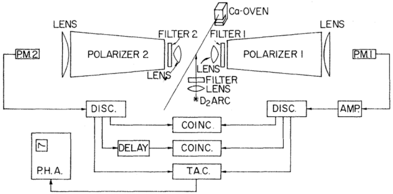

Figure 9: Scheme of the experimental setup [7].

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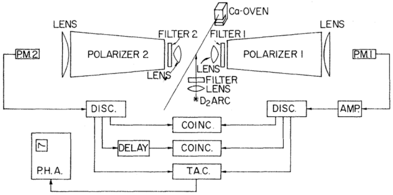

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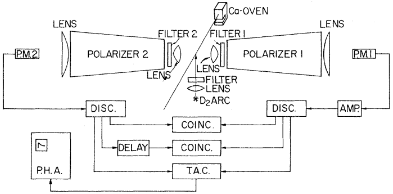

Figure 9: Scheme of the experimental setup [7].

Protocol and Result

- The experiment lasted approximately **200 hours** due to the low coincidence rate.
- A robust real-time normalization system was used to cancel out equipment drift.

The Decisive Experimental Result

The measured value was: $\delta = 0.050 \pm 0.008$

Required Improvements: The Loopholes

- Despite its success, the experiment was not airtight.

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Aspect and team

PHYSICAL REVIEW D VOLUME 10, NUMBER 2 15 JULY 1974

Experimental consequences of objective local theories*

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Department of Physics, Stonehill College, North Easton, Massachusetts 02356

(Received 10 August 1973; revised manuscript received 8 April 1974)

A broad class of theories, called "objective local theories," is defined, motivation for considering these theories is given, and experimental consequences of the class are investigated. An extension of previous analyses by Bell and by Clauser *et al.* shows that predictions of objective local theories and of quantum mechanics differ, and that an experimental test of the entire family of objective local theories can be performed. The experimental requirements are given. Objective local theories satisfying a plausible but experimentally untestable supplementary assumption are shown to be incompatible with existing experimental data.

Clauser & Horne: Experimental Consequences of Objective Local Theories (1974)

Clauser–Horne Formulation (1974)

- Defined **Objective Local Theories (OLT)**: outcomes depend only on hidden variables λ and local settings, not on distant settings.
- Introduced the **No-Enhancement Assumption**: detection with a polarizer cannot exceed detection without it.
- Showed that this weaker assumption was still enough to make Freedman–Clauser results incompatible with all OLT satisfying it.
- Proved that some **supplementary assumption is necessary**: without no-enhancement, one can build an OLT reproducing quantum predictions.
- Revealed the **detection loophole**: experiments of that era only excluded OLT within the no-enhancement class.

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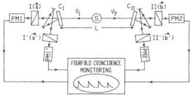

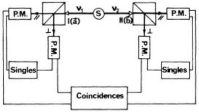

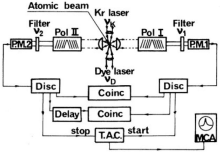

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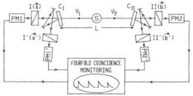

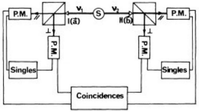

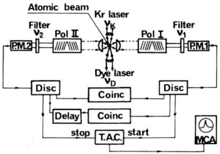

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Progression of Experimental Design

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1. Quantum Computing
   - Entangled **qubits** are the foundation of quantum computers.
   - They allow for computations impossible for classical computers.
   - Advances in cryptography, drug discovery, and materials science.

2. Quantum Cryptography (QKD)
   - Uses pairs of entangled photons to generate cryptographic keys.
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Thank you!

Questions or Comments?

Quantum Entanglement

Spooky and Hidden

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