Systematic Construction of a Complete Set of Commuting Observables on a 4D Hilbert Space

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Abstract

This document elucidates a rigorous framework for constructing a Complete Set of Commuting Observables (CSCO) for a quantum system defined on a four-dimensional Hilbert space, $\mathcal{H}$. Commencing with a self-adjoint operator A possessing a degenerate spectrum, we methodically introduce subsequent commuting observables, B and C, to resolve all degeneracies. The analysis leverages the principles of spectral theory and invariant subspaces. We provide a detailed treatment of the measurement process, including the derivation of state transition probabilities and the resulting state vector collapse. A central feature of this work is the explicit algebraic and geometric construction of the orthogonal projection operators associated with the eigenspaces of each observable, culminating in the projectors onto the unique simultaneous eigenbasis of the CSCO. This version includes a dedicated section with explicit probability calculations for a sequential measurement on a general state vector and a discussion of the Gram-Schmidt orthonormalization procedure.

1 Problem Statement and First Observable A

We consider a quantum system whose states are represented by vectors in a complex Hilbert space $\mathcal{H}$. The observables of the system are self-adjoint operators acting on $\mathcal{H}$. Our objective is to construct a CSCO, which is a set of mutually commuting observables $\{O_1, O_2, \ldots, O_k\}$ whose common eigenspaces are all one-dimensional. The set of eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ for a given simultaneous eigenvector then provides a unique label for that state.

1.1 Construction of the Primary Observable A

We begin by defining a primary observable $A \in \mathcal{B}(\mathcal{H})$ (the algebra of bounded linear operators on $\mathcal{H}$) with the following properties:

- $A = A^\dagger$ (A is self-adjoint).
- The spectrum of A, denoted $\sigma(A)$, consists of exactly two distinct real eigenvalues: $\sigma(A) = \{\alpha, \beta\}$, with $\alpha \neq \beta$.
- The eigenspace corresponding to α, denoted $V_\alpha = \ker(A - \alpha I)$, is three-dimensional: $\dim(V_\alpha) = 3$.
- The Hilbert space $\mathcal{H}$ has the minimal dimension consistent with these constraints, not to exceed 4.

According to the spectral theorem for self-adjoint operators on a finite-dimensional Hilbert space, $\mathcal{H}$ admits an orthogonal direct sum decomposition into the eigenspaces of A: $\mathcal{H} = V_\alpha \oplus V_\beta$. This implies $\dim(\mathcal{H}) = \dim(V_\alpha) + \dim(V_\beta)$. Given $\dim(V_\alpha) = 3$ and the necessary condition

$\dim(V_\beta) \geq 1$, the minimality constraint forces $\dim(V_\beta) = 1$ and thus $\dim(\mathcal{H}) = 4$. Our space is isomorphic to $\mathbb{C}^4$.

Let $\{|j\rangle\}_{j=1}^4$ be an orthonormal basis for $\mathcal{H}$, which we will refer to as the computational basis. To construct A in its simplest form, we choose this basis to be an eigenbasis of A. We assign the basis vectors to the eigenspaces as follows:

$$V_\alpha = \text{span}\{|1\rangle, |2\rangle, |3\rangle\} \quad \text{and} \quad V_\beta = \text{span}\{|4\rangle\}$$

In this basis, the operator A has the diagonal matrix representation:

$$A \mapsto \mathbf{A} = \text{diag}(\alpha, \alpha, \alpha, \beta) = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}$$

The 3-fold degeneracy of the eigenvalue α signifies that a measurement of A alone cannot uniquely determine the state of the system if the outcome is α.

1.2 Orthonormalization of Degenerate Eigenspaces

An essential property of self-adjoint operators is that eigenvectors corresponding to distinct eigenvalues are automatically orthogonal. However, within a degenerate eigenspace, such as our 3D space V_α, any linear combination of eigenvectors is also an eigenvector for the same eigenvalue. When solving the eigenvalue problem for a given operator, one typically first finds a set of linearly independent eigenvectors spanning the degenerate eigenspace; this set is not guaranteed to be orthogonal.

To construct the orthonormal basis required by the postulates of quantum mechanics, one must apply an orthonormalization procedure, most commonly the **Gram-Schmidt process**. Given a set of linearly independent eigenvectors $\{|u_1\rangle, |u_2\rangle, \dots, |u_k\rangle\}$ for a k-dimensional eigenspace, the algorithm constructs an orthonormal basis $\{|e_1\rangle, |e_2\rangle, \dots, |e_k\rangle\}$ as follows:

1. Normalize the first vector: $|e_1\rangle = \frac{|u_1\rangle}{\||u_1\rangle\|}$.
2. For the second vector, subtract its component parallel to $|e_1\rangle$ and then normalize the resulting orthogonal vector:
$$|v_2\rangle = |u_2\rangle - \langle e_1|u_2\rangle |e_1\rangle ; \quad |e_2\rangle = \frac{|v_2\rangle}{\||v_2\rangle\|}$$
3. This process is iterated. For the j-th vector:
$$|v_j\rangle = |u_j\rangle - \sum_{i=1}^{j-1} \langle e_i|u_j\rangle |e_i\rangle ; \quad |e_j\rangle = \frac{|v_j\rangle}{\||v_j\rangle\|}$$

In the context of this document, we employ a "top-down" construction. We begin by *postulating* an orthonormal basis for the Hilbert space (the computational basis $\{|j\rangle\}$) and then define the operators A, B, and C in terms of their desired properties with respect to this basis and its successors. For instance, we started by asserting that V_α is spanned by the mutually orthogonal vectors $\{|1\rangle, |2\rangle, |3\rangle\}$. This constructive approach bypasses the need to explicitly perform the Gram-Schmidt process, as orthonormality is imposed by definition from the outset.

2 Measurement of Observable A

We now formalize the measurement process for a system prepared in an arbitrary state $|\psi\rangle \in \mathcal{H}$.

2.1 Projection Operators for A

The postulates of quantum mechanics state that the probability of measuring an eigenvalue λ is determined by the orthogonal projector onto the corresponding eigenspace V_λ.

Geometric Construction. The projectors P_α and P_β onto V_α and V_β are constructed from the basis vectors spanning these subspaces:

$$P_\alpha = \sum_{j=1}^{3} |j\rangle\langle j| \quad \text{and} \quad P_\beta = |4\rangle\langle 4|$$

In the computational basis, their matrix representations are:

$$\mathbf{P}_\alpha = \mathrm{diag}(1,1,1,0), \quad \mathbf{P}_\beta = \mathrm{diag}(0,0,0,1)$$

These operators are self-adjoint ($P_\lambda = P_\lambda^\dagger$) and idempotent ($P_\lambda^2 = P_\lambda$), and they form a resolution of identity, $P_\alpha + P_\beta = I$, as expected.

Algebraic Construction (Functional Calculus). A more powerful method for finding projectors arises from the functional calculus of operators. The minimal polynomial of A is $m_A(\lambda) = (\lambda - \alpha)(\lambda - \beta)$. The projectors can be expressed as polynomials in A:

$$P_\alpha = \frac{A - \beta I}{\alpha - \beta} \quad \text{and} \quad P_\beta = \frac{A - \alpha I}{\beta - \alpha}$$

This algebraic formulation is demonstrably equivalent to the geometric one and will be used extensively.

2.2 Probabilities and State Collapse

Let the system be in a normalized state $|\psi\rangle \in \mathcal{H}$. The probability of measuring eigenvalue $\lambda \in \sigma(A)$ is given by Born's rule:

$$\mathcal{P}(\lambda) = \|P_\lambda |\psi\rangle\|^2 = \langle\psi| P_\lambda |\psi\rangle$$

If the measurement yields λ, the state of the system collapses to the normalized projection onto the corresponding eigenspace:

$$|\psi\rangle \xrightarrow{\text{measure } A \to \lambda} |\psi'\rangle_\lambda = \frac{P_\lambda |\psi\rangle}{\|P_\lambda |\psi\rangle\|}$$

3 Second Observable B: Resolving Degeneracy

To resolve the degeneracy in V_α, we introduce a second observable B that commutes with A, i.e., $[A, B] = 0$. This condition implies that B leaves the eigenspaces of A invariant, making its matrix representation block-diagonal in the eigenbasis of A.

3.1 Construction from Invariant Subspaces

We design B to have eigenvalues γ (2-fold degenerate) and δ (non-degenerate) on V_α. We choose an orthonormal basis for V_α that is not aligned with the computational basis:

$$|b_1\rangle = |3\rangle, \quad |b_2\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle), \quad |b_3\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$$

We define the action of B on this basis: $B\,|b_1\rangle = \gamma\,|b_1\rangle$, $B\,|b_2\rangle = \gamma\,|b_2\rangle$, and $B\,|b_3\rangle = \delta\,|b_3\rangle$. We also set $B\,|4\rangle = \delta\,|4\rangle$. The matrix representation of B in the computational basis is:

$$\mathbf{B} = \begin{pmatrix} \frac{\gamma+\delta}{2} & \frac{\gamma-\delta}{2} & 0 & 0 \\ \frac{\gamma-\delta}{2} & \frac{\gamma+\delta}{2} & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix}$$

The projectors for B are $P_\gamma = |b_1\rangle\langle b_1| + |b_2\rangle\langle b_2|$ and $P_\delta = |b_3\rangle\langle b_3| + |4\rangle\langle 4|$.

4 Third Observable C: Completing the Set

A degeneracy remains in the simultaneous eigenspace $V_{\alpha,\gamma} = \mathrm{span}\{|b_1\rangle, |b_2\rangle\}$. We introduce a third observable C commuting with both A and B.

4.1 Construction and Constraints

To resolve the final degeneracy, we define C to have distinct eigenvalues on $|b_1\rangle$ and $|b_2\rangle$:

$$C\,|b_1\rangle = \kappa\,|b_1\rangle \quad \text{and} \quad C\,|b_2\rangle = \zeta\,|b_2\rangle \quad (\kappa \neq \zeta)$$

To ensure neither $\{A, C\}$ nor $\{B, C\}$ are CSCOs, we strategically set $C\,|b_3\rangle = \kappa\,|b_3\rangle$ and $C\,|4\rangle = \kappa\,|4\rangle$. The resulting matrix in the computational basis is:

$$\mathbf{C} = \begin{pmatrix} \frac{\zeta+\kappa}{2} & \frac{\zeta-\kappa}{2} & 0 & 0 \\ \frac{\zeta-\kappa}{2} & \frac{\zeta+\kappa}{2} & 0 & 0 \\ 0 & 0 & \kappa & 0 \\ 0 & 0 & 0 & \kappa \end{pmatrix}$$

The projectors for C are $P_\kappa = |b_1\rangle\langle b_1| + |b_3\rangle\langle b_3| + |4\rangle\langle 4|$ and $P_\zeta = |b_2\rangle\langle b_2|$.

5 Explicit Probability Calculation for a General State

We now provide a concrete example of the measurement process. It is crucial to note that measurement probabilities are calculated using **projection operators**, not by applying the observables themselves to the state vector. The probability of obtaining a sequence of outcomes $(\lambda_A, \lambda_B, \dots)$ is found by successively projecting the state onto the corresponding eigenspaces.

5.1 Initial State Preparation

Let's prepare the system in a general normalized state $|\psi\rangle$ expressed in a basis that is different from the computational basis, for instance the V-basis:

$$|v_1\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) \qquad |v_3\rangle = \frac{1}{\sqrt{2}}(|3\rangle + |4\rangle)$$
$$|v_2\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle) \qquad |v_4\rangle = \frac{1}{\sqrt{2}}(|3\rangle - |4\rangle)$$

The general state is $|\psi\rangle = \sum_{i=1}^4 c_i\,|v_i\rangle$, with $\sum_{i=1}^4 |c_i|^2 = 1$.

5.2 Change to the Computational (A-eigen) Basis

To analyze the measurement of A, we express $|\psi\rangle$ in the computational basis:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\big[(c_1 + c_2)\,|1\rangle + (c_1 - c_2)\,|2\rangle + (c_3 + c_4)\,|3\rangle + (c_3 - c_4)\,|4\rangle\big]$$

5.3 Measurement Probabilities

We calculate the joint probability $\mathcal{P}(\lambda_A, \lambda_B, \lambda_C) = \|P_{\lambda_C} P_{\lambda_B} P_{\lambda_A} |\psi\rangle\|^2$.

1. Probabilities for Measurement of A. The probability of measuring α is $\mathcal{P}(\alpha) = \|P_\alpha |\psi\rangle\|^2$. The projected (unnormalized) state is:

$$P_\alpha |\psi\rangle = \frac{1}{\sqrt{2}}\big[(c_1 + c_2)|1\rangle + (c_1 - c_2)|2\rangle + (c_3 + c_4)|3\rangle\big]$$

The probability is the squared norm of this vector:

$$\begin{aligned}\mathcal{P}(\alpha) &= \frac{1}{2}\left(|c_1 + c_2|^2 + |c_1 - c_2|^2 + |c_3 + c_4|^2\right) \\ &= \frac{1}{2}\left(2|c_1|^2 + 2|c_2|^2 + |c_3 + c_4|^2\right) = |c_1|^2 + |c_2|^2 + \frac{1}{2}|c_3 + c_4|^2\end{aligned}$$

Similarly, for eigenvalue β, the projected state is $P_\beta |\psi\rangle = \frac{1}{\sqrt{2}}(c_3 - c_4)|4\rangle$, so:

$$\mathcal{P}(\beta) = \|P_\beta |\psi\rangle\|^2 = \frac{1}{2}|c_3 - c_4|^2$$

As required, $\mathcal{P}(\alpha) + \mathcal{P}(\beta) = |c_1|^2 + |c_2|^2 + |c_3|^2 + |c_4|^2 = 1$.

2. Joint Probabilities for A and B. We now calculate the probability of measuring an eigenvalue of B subsequent to measuring an eigenvalue of A.

- **Outcome** (α, γ): We project $P_\alpha |\psi\rangle$ with P_γ.

$$P_\gamma P_\alpha |\psi\rangle = P_\gamma \left(\frac{1}{\sqrt{2}}\big[(c_1 + c_2)|1\rangle + (c_1 - c_2)|2\rangle + (c_3 + c_4)|3\rangle\big]\right)$$

  Using $P_\gamma = |b_1\rangle\langle b_1| + |b_2\rangle\langle b_2| = |3\rangle\langle 3| + \frac{1}{2}(|1\rangle\langle 1| + |1\rangle\langle 2| + |2\rangle\langle 1| + |2\rangle\langle 2|)$, we find:

$$P_\gamma P_\alpha |\psi\rangle = \frac{1}{\sqrt{2}}\big[c_1(|1\rangle + |2\rangle) + (c_3 + c_4)|3\rangle\big]$$

  The joint probability is $\mathcal{P}(\alpha, \gamma) = \|P_\gamma P_\alpha |\psi\rangle\|^2 = \frac{1}{2}\left(|c_1|^2 \||1\rangle + |2\rangle\|^2 + |c_3 + c_4|^2\right) = |c_1|^2 + \frac{1}{2}|c_3 + c_4|^2$.

- **Outcome** (α, δ): We project $P_\alpha |\psi\rangle$ with P_δ. Using $P_\delta = I - P_\gamma$, we get:

$$P_\delta P_\alpha |\psi\rangle = P_\alpha |\psi\rangle - P_\gamma P_\alpha |\psi\rangle = \frac{1}{\sqrt{2}} c_2(|1\rangle - |2\rangle)$$

  The joint probability is $\mathcal{P}(\alpha, \delta) = \|P_\delta P_\alpha |\psi\rangle\|^2 = \frac{|c_2|^2}{2}\||1\rangle - |2\rangle\|^2 = |c_2|^2$.

- **Outcome** (β, δ): After measuring $A = \beta$, the state is in the span of $|4\rangle$. Since $B|4\rangle = \delta|4\rangle$, a measurement of B must yield δ. Thus, $\mathcal{P}(\beta, \gamma) = 0$ and $\mathcal{P}(\beta, \delta) = \mathcal{P}(\beta) = \frac{1}{2}|c_3 - c_4|^2$.

3. Joint Probabilities for the Full CSCO. Finally, we project the results from the A and B measurements with the projectors of C.

- **Outcome** (α, γ, ζ): We project the state $P_\gamma P_\alpha |\psi\rangle$ with $P_\zeta = |b_2\rangle\langle b_2|$.

$$P_\zeta P_\gamma P_\alpha |\psi\rangle = P_\zeta \left(\frac{1}{\sqrt{2}}\big[c_1(|1\rangle + |2\rangle) + (c_3 + c_4)|3\rangle\big]\right) = c_1 \frac{|1\rangle + |2\rangle}{\sqrt{2}} = c_1 |b_2\rangle$$

  The joint probability is $\mathcal{P}(\alpha, \gamma, \zeta) = \|c_1 |b_2\rangle\|^2 = |c_1|^2$.

- **Outcome** (α, γ, κ): We project with $P_\kappa = I - P_\zeta$.

$$P_\kappa P_\gamma P_\alpha |\psi\rangle = (P_\gamma P_\alpha |\psi\rangle) - (P_\zeta P_\gamma P_\alpha |\psi\rangle) = \frac{c_3 + c_4}{\sqrt{2}} |3\rangle$$

  The joint probability is $\mathcal{P}(\alpha, \gamma, \kappa) = \left\| \frac{c_3+c_4}{\sqrt{2}} |3\rangle \right\|^2 = \frac{1}{2}|c_3 + c_4|^2$.

- **Outcome** (α, δ, κ): After measuring $(A, B) = (\alpha, \delta)$, the state is proportional to $|1\rangle - |2\rangle$, which is $\sqrt{2}\,|b_3\rangle$. Since $C\,|b_3\rangle = \kappa\,|b_3\rangle$, the outcome must be κ. Thus, $\mathcal{P}(\alpha, \delta, \zeta) = 0$ and $\mathcal{P}(\alpha, \delta, \kappa) = \mathcal{P}(\alpha, \delta) = |c_2|^2$.

- **Outcome** (β, δ, κ): After measuring $(A, B) = (\beta, \delta)$, the state is $|4\rangle$. Since $C\,|4\rangle = \kappa\,|4\rangle$, the outcome must be κ. Thus, $\mathcal{P}(\beta, \delta, \zeta) = 0$ and $\mathcal{P}(\beta, \delta, \kappa) = \mathcal{P}(\beta, \delta) = \frac{1}{2}|c_3 - c_4|^2$.

5.4 Summary of Results

The four possible unique outcomes of the CSCO measurement have the following joint probabilities:

- $\mathcal{P}(\alpha, \gamma, \zeta) = |c_1|^2$ (State collapses to $|\psi_2\rangle$)
- $\mathcal{P}(\alpha, \gamma, \kappa) = \frac{1}{2}|c_3 + c_4|^2$ (State collapses to $|\psi_1\rangle$)
- $\mathcal{P}(\alpha, \delta, \kappa) = |c_2|^2$ (State collapses to $|\psi_3\rangle$)
- $\mathcal{P}(\beta, \delta, \kappa) = \frac{1}{2}|c_3 - c_4|^2$ (State collapses to $|\psi_4\rangle$)

The sum of these probabilities is $|c_1|^2 + |c_2|^2 + \frac{1}{2}(|c_3 + c_4|^2 + |c_3 - c_4|^2) = |c_1|^2 + |c_2|^2 + |c_3|^2 + |c_4|^2 = 1$, confirming the consistency of the calculation.

6 Sequential Measurement and CSCO Summary

A sequential measurement of A, then B, then C will uniquely determine the final state of the system. Suppose the system starts in a normalized state $|\psi\rangle$.

1. **Measure** A: The probability of obtaining α is $\mathcal{P}(\alpha) = \|P_\alpha |\psi\rangle\|^2$. The state collapses to $|\psi'\rangle_\alpha = P_\alpha |\psi\rangle / \sqrt{\mathcal{P}(\alpha)}$.
2. **Measure** B **on** $|\psi'\rangle_\alpha$: The conditional probability of obtaining γ given α is

$$\mathcal{P}(\gamma|\alpha) = \frac{\mathcal{P}(\alpha, \gamma)}{\mathcal{P}(\alpha)} = \frac{\|P_\gamma P_\alpha |\psi\rangle\|^2}{\|P_\alpha |\psi\rangle\|^2}$$

   The state then collapses to $|\psi''\rangle_{\alpha,\gamma} = P_\gamma |\psi'\rangle_\alpha / \sqrt{\mathcal{P}(\gamma|\alpha)}$.
3. **Measure** C **on** $|\psi''\rangle_{\alpha,\gamma}$: The conditional probability of obtaining κ given (α, γ) is

$$\mathcal{P}(\kappa|\alpha, \gamma) = \frac{\mathcal{P}(\alpha, \gamma, \kappa)}{\mathcal{P}(\alpha, \gamma)} = \frac{\|P_\kappa P_\gamma P_\alpha |\psi\rangle\|^2}{\|P_\gamma P_\alpha |\psi\rangle\|^2}$$

   The state collapses to the final, unique state $|\psi'''\rangle_{\alpha,\gamma,\kappa}$.

6.1 The Simultaneous Eigenbasis

The set $\{A, B, C\}$ forms a CSCO. Their simultaneous eigenbasis, which we denote $\{|\psi_i\rangle\}_{i=1}^4$, consists of the vectors that are uniquely specified by a triplet of eigenvalues $(\lambda_A, \lambda_B, \lambda_C)$. This basis is precisely $\{|b_1\rangle, |b_2\rangle, |b_3\rangle, |4\rangle\}$.

Simultaneous Eigenvector	**Eigenvalue of A**	**Eigenvalue of B**	**Eigenvalue of C**
$\lvert\psi_1\rangle = \lvert b_1\rangle = \lvert 3\rangle$	α	γ	κ
$\lvert\psi_2\rangle = \lvert b_2\rangle = \frac{1}{\sqrt{2}}(\lvert 1\rangle + \lvert 2\rangle)$	α	γ	ζ
$\lvert\psi_3\rangle = \lvert b_3\rangle = \frac{1}{\sqrt{2}}(\lvert 1\rangle - \lvert 2\rangle)$	α	δ	κ
$\lvert\psi_4\rangle = \lvert 4\rangle$	β	δ	κ

Table 1: The simultaneous eigenbasis of the CSCO $\{A, B, C\}$. Each row corresponds to a unique state vector identified by a distinct triplet of eigenvalues. The set is complete provided $\alpha \neq \beta$, $\gamma \neq \delta$, and $\kappa \neq \zeta$.

6.2 Projectors onto the CSCO Basis

The most fundamental projectors are those onto the one-dimensional subspaces spanned by the simultaneous eigenvectors. These can be constructed by multiplying the projectors for the corresponding eigenvalues. For instance, the projector onto the state $|\psi_1\rangle$, specified by the eigenvalues (α, γ, κ), is:

$$P_{\psi_1} = P_{\alpha,\gamma,\kappa} = P_\alpha P_\gamma P_\kappa = |\psi_1\rangle\langle\psi_1|$$

Since the operators commute, their projectors also commute, so the order of multiplication is irrelevant. The measurement of the CSCO is thus equivalent to projecting the initial state vector onto this unique, physically distinguished basis. The joint probabilities calculated in the previous section are precisely $\mathcal{P}(\lambda_A, \lambda_B, \lambda_C) = \|P_{\lambda_A,\lambda_B,\lambda_C} |\psi\rangle\|^2$.