Analysis of a Degenerate Observable in a 4D Hilbert Space

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1 Problem Statement

We consider a quantum system whose state vectors belong to a Hilbert space of dimension four ($\dim(\mathcal{H}) = 4$). We are tasked with defining a Complete Set of Commuting Observables (CSCO) for this system. Subsequently, we will construct a diagonal observable, $\mathcal{A}$, that possesses a degenerate spectrum. The primary objective is to perform a complete spectral analysis of $\mathcal{A}$, determining its eigenvalues (*Eigenwerte*) and the corresponding eigenspaces and eigenvectors (*Eigenvektoren*).

2 System Definition and Hilbert Space

The canonical physical realization of a four-dimensional Hilbert space is the system of two non-interacting spin-1/2 particles. Each particle is described by a state in a two-dimensional Hilbert space, $\mathcal{H}_i \cong \mathbb{C}^2$. The composite system is thus described by the tensor product space:

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \tag{1}$$

The dimension of $\mathcal{H}$ is $\dim(\mathcal{H}) = 2^2 = 4$, which satisfies the problem's constraint.

We choose as our standard basis the simultaneous eigenbasis of the individual z-spin operators, S_{1z} and S_{2z}. This basis, often called the computational basis, consists of four orthonormal vectors:

$$|s_1 s_2\rangle \equiv |s_1\rangle_1 \otimes |s_2\rangle_2 \quad \text{where } s_i \in \{\uparrow, \downarrow\} \tag{2}$$

The four basis kets are explicitly:

$$\begin{aligned}
|\psi_1\rangle &= |\uparrow\uparrow\rangle \\
|\psi_2\rangle &= |\uparrow\downarrow\rangle \\
|\psi_3\rangle &= |\downarrow\uparrow\rangle \\
|\psi_4\rangle &= |\downarrow\downarrow\rangle
\end{aligned}$$

3 Construction of the CSCO

For this two-particle system, a natural CSCO is the set of the individual z-spin projection operators:

$$\text{CSCO} = \{S_{1z}, S_{2z}\} \tag{3}$$

These operators are Hermitian and, because they act on independent subspaces of the tensor product space, they mutually commute: $[S_{1z}, S_{2z}] = 0$. Their set of simultaneous eigenvalues, (m_1, m_2), where $m_i = \pm\hbar/2$, provides a unique label for each of the four basis vectors, thus forming a complete, non-degenerate basis for $\mathcal{H}$.

4 The Degenerate Observable $\mathcal{A}$

We define our observable $\mathcal{A}$ as the total spin projection along the z-axis. This is a common and physically significant observable.

$$\mathcal{A} \equiv S_z^{\text{total}} = S_{1z} + S_{2z} \tag{4}$$

Since $\mathcal{A}$ is a sum of operators that are diagonal in the chosen basis, $\mathcal{A}$ is also diagonal in this basis. We will now show that its spectrum is degenerate.

5 Eigenvalue and Eigenvector Analysis

We find the spectrum of $\mathcal{A}$ by applying it to each basis vector. The eigenvalue equation is $\mathcal{A}\,|s_1 s_2\rangle = \lambda\,|s_1 s_2\rangle$, where $\lambda = m_{s_1} + m_{s_2}$.

5.1 Eigenvalue $\lambda_1 = +\hbar$

This result is obtained from the state where both spins are aligned up.

- **Eigenvektor:** $|\uparrow\uparrow\rangle$
- **Degeneracy** $g_1 = 1$. The eigenspace $\mathcal{E}_\hbar$ is one-dimensional.

$$\mathcal{A}\,|\uparrow\uparrow\rangle = (S_{1z} + S_{2z})\,|\uparrow\uparrow\rangle = (\frac{\hbar}{2} + \frac{\hbar}{2})\,|\uparrow\uparrow\rangle = \hbar\,|\uparrow\uparrow\rangle \tag{5}$$

5.2 Eigenvalue $\lambda_2 = 0$

This result is obtained from states where the two spins are anti-aligned. This is the degenerate case.

- **Eigenvektoren:** $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$
- **Degeneracy** $g_2 = 2$. The eigenspace $\mathcal{E}_0$ is a two-dimensional subspace spanned by these two vectors.

$$\mathcal{E}_0 = \text{span}\{|\uparrow\downarrow\rangle\,, |\downarrow\uparrow\rangle\} \tag{6}$$

$$\mathcal{A}\,|\uparrow\downarrow\rangle = (S_{1z} + S_{2z})\,|\uparrow\downarrow\rangle = (\frac{\hbar}{2} - \frac{\hbar}{2})\,|\uparrow\downarrow\rangle = 0 \tag{7}$$

$$\mathcal{A}\,|\downarrow\uparrow\rangle = (S_{1z} + S_{2z})\,|\downarrow\uparrow\rangle = (-\frac{\hbar}{2} + \frac{\hbar}{2})\,|\downarrow\uparrow\rangle = 0 \tag{8}$$

5.3 Eigenvalue $\lambda_3 = -\hbar$

This result is obtained from the state where both spins are aligned down.

- **Eigenvektor:** $|\downarrow\downarrow\rangle$
- **Degeneracy** $g_3 = 1$. The eigenspace $\mathcal{E}_{-\hbar}$ is one-dimensional.

$$\mathcal{A}\,|\downarrow\downarrow\rangle = (S_{1z} + S_{2z})\,|\downarrow\downarrow\rangle = (-\frac{\hbar}{2} - \frac{\hbar}{2})\,|\downarrow\downarrow\rangle = -\hbar\,|\downarrow\downarrow\rangle \tag{9}$$

6 Summary and Matrix Representation

The spectrum of $\mathcal{A}$ is $\text{Spec}(\mathcal{A}) = \{+\hbar, 0, -\hbar\}$. The eigenvalue $\lambda = 0$ is two-fold degenerate, while the eigenvalues $\lambda = \pm\hbar$ are non-degenerate. The sum of degeneracies $1 + 2 + 1 = 4$ correctly matches the dimension of the Hilbert space.

In the ordered basis $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$, the matrix representation of $\mathcal{A}$ is a 4x4 diagonal matrix:

$$[\mathcal{A}] = \hbar \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

This matrix explicitly demonstrates the degeneracy associated with the eigenvalue 0. Within the degenerate eigenspace $\mathcal{E}_0$, the two vectors $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ are not distinguished by $\mathcal{A}$. However, they are distinguished by the individual operators S_{1z} and S_{2z} from our CSCO, which lift the degeneracy and provide a complete description of the state.

7 Transformation of an Initial State

We now introduce an arbitrary initial state of the system, $|\psi(0)\rangle$, defined in a basis that is different from the eigenbasis of our observable $\mathcal{A} = S_z^{\text{total}}$. A standard choice for an alternative basis is the eigenbasis of the spin-x operator, S_x.

7.1 Defining the State in the S_x Basis

For a single spin-1/2 particle, the eigenvectors of S_x (with eigenvalues $\pm\hbar/2$) are given in terms of the S_z eigenvectors as:

$$|+\rangle_x = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \tag{10}$$

$$|-\rangle_x = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle) \tag{11}$$

For our two-particle system, we can form a complete orthonormal basis by taking the tensor products of these states. Let's define our arbitrary initial state $|\psi\rangle$ to be the state where the first particle is spin-up along the x-axis and the second particle is spin-down along the x-axis.

$$|\psi\rangle = |+\rangle_{x,1} \otimes |-\rangle_{x,2} \equiv |+-\rangle_x \tag{12}$$

This state is, by construction, normalized and defined in a basis other than the eigenbasis of $\mathcal{A}$.

7.2 Change of Basis to the Eigenbasis of $\mathcal{A}$

Our goal is to express $|\psi\rangle$ as a linear combination of the eigenvectors of $\mathcal{A}$, which are the vectors of the z-basis $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$. This is achieved by substituting the definitions of the S_x eigenvectors into our expression for $|\psi\rangle$.

$$\begin{aligned} |\psi\rangle &= \left(\frac{1}{\sqrt{2}}(|\uparrow\rangle_1 + |\downarrow\rangle_1)\right) \otimes \left(\frac{1}{\sqrt{2}}(|\uparrow\rangle_2 - |\downarrow\rangle_2)\right) \\ &= \frac{1}{2}(|\uparrow\rangle_1 \otimes (|\uparrow\rangle_2 - |\downarrow\rangle_2) + |\downarrow\rangle_1 \otimes (|\uparrow\rangle_2 - |\downarrow\rangle_2)) \\ &= \frac{1}{2}(|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle) \end{aligned} \tag{13}$$

This is the desired expression of $|\psi\rangle$ in the eigenbasis of $\mathcal{A}$.

7.3 State Vector Representation in the $\mathcal{A}$-Basis

The transformation is completed by writing the state as a column vector whose components are the coefficients (or probability amplitudes) of the corresponding basis kets. Using the ordered basis $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$, the state vector representation of $|\psi\rangle$ is:

$$[\psi]_{\mathcal{A}} = \begin{pmatrix} \langle\uparrow\uparrow|\psi\rangle \\ \langle\uparrow\downarrow|\psi\rangle \\ \langle\downarrow\uparrow|\psi\rangle \\ \langle\downarrow\downarrow|\psi\rangle \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \tag{14}$$

We can verify that the state is normalized by calculating the sum of the squared moduli of the amplitudes:

$$\langle\psi|\psi\rangle = \left(\frac{1}{2}\right)^2 \left(|1|^2 + |-1|^2 + |1|^2 + |-1|^2\right) = \frac{1}{4}(1+1+1+1) = 1 \tag{15}$$

Thus, we have successfully transformed the arbitrarily chosen initial state $|\psi\rangle$ into the eigenbasis of the observable $\mathcal{A}$.

8 State Analysis and Projection onto the Degenerate Subspace

We now perform a deeper analysis of the state $|\psi\rangle$ with respect to the observable $\mathcal{A}$. This involves calculating the action of $\mathcal{A}$ on the state, and then focusing on the degenerate eigenspace $\mathcal{E}_0$ to determine measurement probabilities.

8.1 Action of the Operator $\mathcal{A}$ on $|\psi\rangle$

We apply the observable $\mathcal{A} = S_z^{\text{total}}$ to our state $|\psi\rangle$. Using the linear nature of the operator and the fact that the basis kets are its eigenvectors, we have:

$$\begin{aligned} \mathcal{A}|\psi\rangle &= \mathcal{A}\left(\frac{1}{2}(|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle)\right) \\ &= \frac{1}{2}\left(\mathcal{A}|\uparrow\uparrow\rangle - \mathcal{A}|\uparrow\downarrow\rangle + \mathcal{A}|\downarrow\uparrow\rangle - \mathcal{A}|\downarrow\downarrow\rangle\right) \\ &= \frac{1}{2}\left((\hbar)|\uparrow\uparrow\rangle - (0)|\uparrow\downarrow\rangle + (0)|\downarrow\uparrow\rangle - (-\hbar)|\downarrow\downarrow\rangle\right) \\ &= \frac{\hbar}{2}\left(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\right) \end{aligned} \tag{16}$$

This resulting state is the state $|\psi\rangle$ after the action of the operator $\mathcal{A}$. As expected, the components in the degenerate subspace with eigenvalue 0 have been annihilated.

8.2 A Non-Orthogonal Basis for the Degenerate Subspace

The degenerate subspace is the eigenspace $\mathcal{E}_0$ corresponding to the eigenvalue $\lambda = 0$, which is spanned by the orthogonal vectors $\{|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle\}$. To fulfill the request, we must construct a basis for this subspace from vectors that are linearly independent but **not orthogonal**.

Let the original orthonormal basis vectors be $|v_1\rangle = |\uparrow\downarrow\rangle$ and $|v_2\rangle = |\downarrow\uparrow\rangle$. We can construct a new basis $\{|u_1\rangle, |u_2\rangle\}$ via a simple linear transformation:

$$|u_1\rangle = |v_1\rangle = |\uparrow\downarrow\rangle \tag{17}$$

$$|u_2\rangle = |v_1\rangle + |v_2\rangle = |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \tag{18}$$

These vectors are linearly independent, as one is not a scalar multiple of the other. We check for orthogonality by computing their inner product:

$$\langle u_1|u_2\rangle = \langle\uparrow\downarrow|(\uparrow\downarrow + \downarrow\uparrow)\rangle = \langle\uparrow\downarrow\mid\uparrow\downarrow\rangle + \langle\uparrow\downarrow\mid\downarrow\uparrow\rangle = 1 + 0 = 1 \tag{19}$$

Since $\langle u_1|u_2\rangle \neq 0$, the vectors $\{|u_1\rangle, |u_2\rangle\}$ form a linearly independent, non-orthogonal basis for the degenerate subspace $\mathcal{E}_0$.

8.3 Orthonormalization via Gram-Schmidt Process

We now apply the Gram-Schmidt process to our non-orthogonal basis $\{|u_1\rangle, |u_2\rangle\}$ to recover an orthonormal basis $\{|w_1\rangle, |w_2\rangle\}$.

1. **First vector:** We normalize $|u_1\rangle$.

$$|w_1\rangle = \frac{|u_1\rangle}{\sqrt{\langle u_1|u_1\rangle}} = \frac{|\uparrow\downarrow\rangle}{\sqrt{1}} = |\uparrow\downarrow\rangle \tag{20}$$

2. **Second vector:** We project $|u_2\rangle$ onto $|w_1\rangle$ and subtract this projection from $|u_2\rangle$ to find an orthogonal vector $|\tilde{w}_2\rangle$.

$$\begin{aligned}|\tilde{w}_2\rangle &= |u_2\rangle - |w_1\rangle\langle w_1|u_2\rangle \\ &= (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) - |\uparrow\downarrow\rangle\langle\uparrow\downarrow|(\uparrow\downarrow + \downarrow\uparrow)\rangle \\ &= (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) - |\uparrow\downarrow\rangle(1) = |\downarrow\uparrow\rangle\end{aligned} \tag{21}$$

3. **Normalize the second vector:**

$$|w_2\rangle = \frac{|\tilde{w}_2\rangle}{\sqrt{\langle\tilde{w}_2|\tilde{w}_2\rangle}} = \frac{|\downarrow\uparrow\rangle}{\sqrt{1}} = |\downarrow\uparrow\rangle \tag{22}$$

The process correctly yields the original orthonormal basis for the subspace $\mathcal{E}_0$, which is $\{|w_1\rangle, |w_2\rangle\} = \{|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle\}$.

8.4 Projection onto the Degenerate Subspace

The projection operator P_0 onto the eigenspace $\mathcal{E}_0$ is constructed from its orthonormal basis:

$$P_0 = \sum_{i=1}^{2} |w_i\rangle\langle w_i| = |\uparrow\downarrow\rangle\langle\uparrow\downarrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow| \tag{23}$$

We now apply this projector to our state $|\psi\rangle$ to find the component of the state that lies within this subspace.

$$\begin{aligned}P_0|\psi\rangle &= (|\uparrow\downarrow\rangle\langle\uparrow\downarrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow|)\left[\frac{1}{2}(|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle)\right] \\ &= \frac{1}{2}(|\uparrow\downarrow\rangle\langle\uparrow\downarrow\mid\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle\langle\uparrow\downarrow\mid\uparrow\downarrow\rangle + \dots) \\ &= \frac{1}{2}(|\uparrow\downarrow\rangle(0) - |\uparrow\downarrow\rangle(1) + |\downarrow\uparrow\rangle(1) - |\downarrow\uparrow\rangle(0)) \\ &= \frac{1}{2}(|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle)\end{aligned}$$

This is the projection of $|\psi\rangle$ onto the degenerate subspace $\mathcal{E}_0$.

8.5 Probabilities of Degenerate Eigenvalues

The probability of measuring an eigenvalue λ_i upon measurement of the observable $\mathcal{A}$ on a system in state $|\psi\rangle$ is given by the squared norm of the projection of $|\psi\rangle$ onto the corresponding eigenspace $\mathcal{E}_i$.

$$\text{Prob}(\lambda_i) = \|P_i\,|\psi\rangle\,\|^2 = \langle\psi|P_i^\dagger P_i|\psi\rangle = \langle\psi|P_i|\psi\rangle \tag{24}$$

The only degenerate eigenvalue is $\lambda_2 = 0$. The probability of measuring this value is:

$$\begin{aligned}
\text{Prob}(\lambda = 0) = \|P_0\,|\psi\rangle\,\|^2 &= \left\|\frac{1}{2}(|{\downarrow\uparrow}\rangle - |{\uparrow\downarrow}\rangle)\right\|^2 \\
&= \left(\frac{1}{2}\right)^2 \langle({\downarrow\uparrow} - {\uparrow\downarrow})|({\downarrow\uparrow} - {\uparrow\downarrow})\rangle \\
&= \frac{1}{4}\left(\langle{\downarrow\uparrow} \mid {\downarrow\uparrow}\rangle - \langle{\downarrow\uparrow} \mid {\uparrow\downarrow}\rangle - \langle{\uparrow\downarrow} \mid {\downarrow\uparrow}\rangle + \langle{\uparrow\downarrow} \mid {\uparrow\downarrow}\rangle\right) \\
&= \frac{1}{4}(1 - 0 - 0 + 1) = \frac{2}{4} = \frac{1}{2}
\end{aligned} \tag{25}$$

Therefore, the probability of measuring the energy eigenvalue $\lambda = 0$ is exactly 1/2 or 50%.