Systematic Construction of a Complete Set of Commuting Observables on a 4D Hilbert Space

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Abstract

This document elucidates a rigorous framework for constructing a Complete Set of Commuting Observables (CSCO) for a quantum system defined on a four-dimensional Hilbert space, $\mathcal{H}$. Commencing with a self-adjoint operator A possessing a degenerate spectrum, we methodically introduce subsequent commuting observables, B and C, to resolve all degeneracies. The analysis leverages the principles of spectral theory and invariant subspaces. We provide a detailed treatment of the measurement process, including the derivation of state transition probabilities and the resulting state vector collapse. A central feature of this work is the explicit algebraic and geometric construction of the orthogonal projection operators associated with the eigenspaces of each observable, culminating in the projectors onto the unique simultaneous eigenbasis of the CSCO. This version includes a dedicated section with explicit probability calculations for a sequential measurement on a general state vector and a discussion of the Gram-Schmidt orthonormalization procedure.

1 Problem Statement and First Observable A

We consider a quantum system whose states are represented by vectors in a complex Hilbert space $\mathcal{H}$. The observables of the system are self-adjoint operators acting on $\mathcal{H}$. Our objective is to construct a CSCO, which is a set of mutually commuting observables $\{O_1, O_2, \ldots, O_k\}$ whose common eigenspaces are all one-dimensional. The set of eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ for a given simultaneous eigenvector then provides a unique label for that state.

1.1 Construction of the Primary Observable A

We begin by defining a primary observable $A \in \mathcal{B}(\mathcal{H})$ (the algebra of bounded linear operators on $\mathcal{H}$) with the following properties:

- $A = A^\dagger$ (A is self-adjoint).
- The spectrum of A, denoted $\sigma(A)$, consists of exactly two distinct real eigenvalues: $\sigma(A) = \{\alpha, \beta\}$, with $\alpha \neq \beta$.
- The eigenspace corresponding to α, denoted $V_\alpha = \ker(A - \alpha I)$, is three-dimensional: $\dim(V_\alpha) = 3$.
- The Hilbert space $\mathcal{H}$ has the minimal dimension consistent with these constraints, not to exceed 4.

According to the spectral theorem for self-adjoint operators on a finite-dimensional Hilbert space, $\mathcal{H}$ admits an orthogonal direct sum decomposition into the eigenspaces of A: $\mathcal{H} = V_\alpha \oplus V_\beta$. This implies $\dim(\mathcal{H}) = \dim(V_\alpha) + \dim(V_\beta)$. Given $\dim(V_\alpha) = 3$ and the necessary condition

$\dim(V_\beta) \geq 1$, the minimality constraint forces $\dim(V_\beta) = 1$ and thus $\dim(\mathcal{H}) = 4$. Our space is isomorphic to $\mathbb{C}^4$.

Let $\{|j\rangle\}_{j=1}^4$ be an orthonormal basis for $\mathcal{H}$, which we will refer to as the computational basis. To construct A in its simplest form, we choose this basis to be an eigenbasis of A. We assign the basis vectors to the eigenspaces as follows:

$$V_\alpha = \text{span}\{|1\rangle, |2\rangle, |3\rangle\} \quad \text{and} \quad V_\beta = \text{span}\{|4\rangle\}$$

In this basis, the operator A has the diagonal matrix representation:

$$A \mapsto \mathbf{A} = \text{diag}(\alpha, \alpha, \alpha, \beta) = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}$$

The 3-fold degeneracy of the eigenvalue α signifies that a measurement of A alone cannot uniquely determine the state of the system if the outcome is α.

1.2 Orthonormalization of Degenerate Eigenspaces

An essential property of self-adjoint operators is that eigenvectors corresponding to distinct eigenvalues are automatically orthogonal. However, within a degenerate eigenspace, such as our 3D space V_α, any linear combination of eigenvectors is also an eigenvector for the same eigenvalue. When solving the eigenvalue problem for a given operator, one typically first finds a set of linearly independent eigenvectors spanning the degenerate eigenspace; this set is not guaranteed to be orthogonal.

To construct the orthonormal basis required by the postulates of quantum mechanics, one must apply an orthonormalization procedure, most commonly the **Gram-Schmidt process**. Given a set of linearly independent eigenvectors $\{|u_1\rangle, |u_2\rangle, \ldots, |u_k\rangle\}$ for a k-dimensional eigenspace, the algorithm constructs an orthonormal basis $\{|e_1\rangle, |e_2\rangle, \ldots, |e_k\rangle\}$ as follows:

1. Normalize the first vector: $|e_1\rangle = \frac{|u_1\rangle}{\||u_1\rangle\|}$.
2. For the second vector, subtract its component parallel to $|e_1\rangle$ and then normalize the resulting orthogonal vector:
$$|v_2\rangle = |u_2\rangle - \langle e_1|u_2\rangle |e_1\rangle; \quad |e_2\rangle = \frac{|v_2\rangle}{\||v_2\rangle\|}$$
3. This process is iterated. For the j-th vector:
$$|v_j\rangle = |u_j\rangle - \sum_{i=1}^{j-1} \langle e_i|u_j\rangle |e_i\rangle; \quad |e_j\rangle = \frac{|v_j\rangle}{\||v_j\rangle\|}$$

In the context of this document, we employ a "top-down" construction. We begin by *postulating* an orthonormal basis for the Hilbert space (the computational basis $\{|j\rangle\}$) and then define the operators A, B, and C in terms of their desired properties with respect to this basis and its successors. For instance, we started by asserting that V_α is spanned by the mutually orthogonal vectors $\{|1\rangle, |2\rangle, |3\rangle\}$. This constructive approach bypasses the need to explicitly perform the Gram-Schmidt process, as orthonormality is imposed by definition from the outset.

2 Measurement of Observable A

We now formalize the measurement process for a system prepared in an arbitrary state $|\psi\rangle \in \mathcal{H}$.

2.1 Projection Operators for A

The postulates of quantum mechanics state that the probability of measuring an eigenvalue λ is determined by the orthogonal projector onto the corresponding eigenspace V_λ.

Geometric Construction. The projectors P_α and P_β onto V_α and V_β are constructed from the basis vectors spanning these subspaces:

$$P_\alpha = \sum_{j=1}^{3} |j\rangle\langle j| \quad \text{and} \quad P_\beta = |4\rangle\langle 4|$$

In the computational basis, their matrix representations are:

$$\mathbf{P}_\alpha = \mathrm{diag}(1,1,1,0), \quad \mathbf{P}_\beta = \mathrm{diag}(0,0,0,1)$$

These operators are self-adjoint ($P_\lambda = P_\lambda^\dagger$) and idempotent ($P_\lambda^2 = P_\lambda$), and they form a resolution of identity, $P_\alpha + P_\beta = I$, as expected.

Algebraic Construction (Functional Calculus). A more powerful method for finding projectors arises from the functional calculus of operators. The minimal polynomial of A is $m_A(\lambda) = (\lambda - \alpha)(\lambda - \beta)$. The projectors can be expressed as polynomials in A:

$$P_\alpha = \frac{A - \beta I}{\alpha - \beta} \quad \text{and} \quad P_\beta = \frac{A - \alpha I}{\beta - \alpha}$$

This algebraic formulation is demonstrably equivalent to the geometric one and will be used extensively.

2.2 Probabilities and State Collapse

Let the system be in a normalized state $|\psi\rangle \in \mathcal{H}$. The probability of measuring eigenvalue $\lambda \in \sigma(A)$ is given by Born's rule:

$$\mathcal{P}(\lambda) = \|P_\lambda |\psi\rangle\|^2 = \langle\psi| P_\lambda |\psi\rangle$$

If the measurement yields λ, the state of the system collapses to the normalized projection onto the corresponding eigenspace:

$$|\psi\rangle \xrightarrow{\text{measure } A\to\lambda} |\psi'\rangle_\lambda = \frac{P_\lambda |\psi\rangle}{\|P_\lambda |\psi\rangle\|}$$

3 Second Observable B: Resolving Degeneracy

To resolve the degeneracy in V_α, we introduce a second observable B that commutes with A, i.e., $[A, B] = 0$. This condition implies that B leaves the eigenspaces of A invariant, making its matrix representation block-diagonal in the eigenbasis of A.

3.1 Construction from Invariant Subspaces

We design B to have eigenvalues γ (2-fold degenerate) and δ (non-degenerate) on V_α. We choose an orthonormal basis for V_α that is not aligned with the computational basis:

$$|b_1\rangle = |3\rangle, \quad |b_2\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle), \quad |b_3\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$$

We define the action of B on this basis: $B\,|b_1\rangle = \gamma\,|b_1\rangle$, $B\,|b_2\rangle = \gamma\,|b_2\rangle$, and $B\,|b_3\rangle = \delta\,|b_3\rangle$. We also set $B\,|4\rangle = \delta\,|4\rangle$. The matrix representation of B in the computational basis is:

$$\mathbf{B} = \begin{pmatrix} \frac{\gamma+\delta}{2} & \frac{\gamma-\delta}{2} & 0 & 0 \\ \frac{\gamma-\delta}{2} & \frac{\gamma+\delta}{2} & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix}$$

The projectors for B are $P_\gamma = |b_1\rangle\langle b_1| + |b_2\rangle\langle b_2|$ and $P_\delta = |b_3\rangle\langle b_3| + |4\rangle\langle 4|$.

4 Third Observable C: Completing the Set

A degeneracy remains in the simultaneous eigenspace $V_{\alpha,\gamma} = \text{span}\{|b_1\rangle, |b_2\rangle\}$. We introduce a third observable C commuting with both A and B.

4.1 Construction and Constraints

To resolve the final degeneracy, we define C to have distinct eigenvalues on $|b_1\rangle$ and $|b_2\rangle$:

$$C\,|b_1\rangle = \kappa\,|b_1\rangle \quad \text{and} \quad C\,|b_2\rangle = \zeta\,|b_2\rangle \quad (\kappa \neq \zeta)$$

To ensure neither $\{A, C\}$ nor $\{B, C\}$ are CSCOs, we strategically set $C\,|b_3\rangle = \kappa\,|b_3\rangle$ and $C\,|4\rangle = \kappa\,|4\rangle$. The resulting matrix in the computational basis is:

$$\mathbf{C} = \begin{pmatrix} \frac{\zeta+\kappa}{2} & \frac{\zeta-\kappa}{2} & 0 & 0 \\ \frac{\zeta-\kappa}{2} & \frac{\zeta+\kappa}{2} & 0 & 0 \\ 0 & 0 & \kappa & 0 \\ 0 & 0 & 0 & \kappa \end{pmatrix}$$

The projectors for C are $P_\kappa = |b_1\rangle\langle b_1| + |b_3\rangle\langle b_3| + |4\rangle\langle 4|$ and $P_\zeta = |b_2\rangle\langle b_2|$.

5 Explicit Probability Calculation for a General State

We now provide a concrete example of the measurement process. It is crucial to note that measurement probabilities are calculated using **projection operators**, not by applying the observables themselves to the state vector. The probability of obtaining a sequence of outcomes $(\lambda_A, \lambda_B, \dots)$ is found by successively projecting the state onto the corresponding eigenspaces.

5.1 Initial State Preparation

Let's prepare the system in a general normalized state $|\psi\rangle$ expressed in a basis that is different from the computational basis, for instance the V-basis:

$$|v_1\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) \qquad |v_3\rangle = \frac{1}{\sqrt{2}}(|3\rangle + |4\rangle)$$
$$|v_2\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle) \qquad |v_4\rangle = \frac{1}{\sqrt{2}}(|3\rangle - |4\rangle)$$

The general state is $|\psi\rangle = \sum_{i=1}^{4} c_i\,|v_i\rangle$, with $\sum_{i=1}^{4} |c_i|^2 = 1$.

5.2 Change to the Computational (A-eigen) Basis

To analyze the measurement of A, we express $|\psi\rangle$ in the computational basis:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\big[(c_1 + c_2)\,|1\rangle + (c_1 - c_2)\,|2\rangle + (c_3 + c_4)\,|3\rangle + (c_3 - c_4)\,|4\rangle\,\big]$$

5.3 Measurement Probabilities

We calculate the joint probability $\mathcal{P}(\lambda_A, \lambda_B, \lambda_C) = \|P_{\lambda_C} P_{\lambda_B} P_{\lambda_A} |\psi\rangle\|^2$.

1. Probabilities for Measurement of A. The probability of measuring α is $\mathcal{P}(\alpha) = \|P_\alpha |\psi\rangle\|^2$. The projected (unnormalized) state is:

$$P_\alpha |\psi\rangle = \frac{1}{\sqrt{2}}\big[(c_1 + c_2)|1\rangle + (c_1 - c_2)|2\rangle + (c_3 + c_4)|3\rangle\big]$$

The probability is the squared norm of this vector:

$$\begin{aligned}\mathcal{P}(\alpha) &= \frac{1}{2}\left(|c_1 + c_2|^2 + |c_1 - c_2|^2 + |c_3 + c_4|^2\right) \\ &= \frac{1}{2}\left(2|c_1|^2 + 2|c_2|^2 + |c_3 + c_4|^2\right) = |c_1|^2 + |c_2|^2 + \frac{1}{2}|c_3 + c_4|^2\end{aligned}$$

Similarly, for eigenvalue β, the projected state is $P_\beta |\psi\rangle = \frac{1}{\sqrt{2}}(c_3 - c_4)|4\rangle$, so:

$$\mathcal{P}(\beta) = \|P_\beta |\psi\rangle\|^2 = \frac{1}{2}|c_3 - c_4|^2$$

As required, $\mathcal{P}(\alpha) + \mathcal{P}(\beta) = |c_1|^2 + |c_2|^2 + |c_3|^2 + |c_4|^2 = 1$.

2. Joint Probabilities for A and B. We now calculate the probability of measuring an eigenvalue of B subsequent to measuring an eigenvalue of A.

- **Outcome** (α, γ): We project $P_\alpha |\psi\rangle$ with P_γ.

  $$P_\gamma P_\alpha |\psi\rangle = P_\gamma \left(\frac{1}{\sqrt{2}}\big[(c_1 + c_2)|1\rangle + (c_1 - c_2)|2\rangle + (c_3 + c_4)|3\rangle\big]\right)$$

  Using $P_\gamma = |b_1\rangle\langle b_1| + |b_2\rangle\langle b_2| = |3\rangle\langle 3| + \frac{1}{2}(|1\rangle\langle 1| + |1\rangle\langle 2| + |2\rangle\langle 1| + |2\rangle\langle 2|)$, we find:

  $$P_\gamma P_\alpha |\psi\rangle = \frac{1}{\sqrt{2}}\big[c_1(|1\rangle + |2\rangle) + (c_3 + c_4)|3\rangle\big]$$

  The joint probability is $\mathcal{P}(\alpha, \gamma) = \|P_\gamma P_\alpha |\psi\rangle\|^2 = \frac{1}{2}\left(|c_1|^2 \||1\rangle + |2\rangle\|^2 + |c_3 + c_4|^2\right) = |c_1|^2 + \frac{1}{2}|c_3 + c_4|^2$.

- **Outcome** (α, δ): We project $P_\alpha |\psi\rangle$ with P_δ. Using $P_\delta = I - P_\gamma$, we get:

  $$P_\delta P_\alpha |\psi\rangle = P_\alpha |\psi\rangle - P_\gamma P_\alpha |\psi\rangle = \frac{1}{\sqrt{2}} c_2(|1\rangle - |2\rangle)$$

  The joint probability is $\mathcal{P}(\alpha, \delta) = \|P_\delta P_\alpha |\psi\rangle\|^2 = \frac{|c_2|^2}{2}\||1\rangle - |2\rangle\|^2 = |c_2|^2$.

- **Outcome** (β, δ): After measuring $A = \beta$, the state is in the span of $|4\rangle$. Since $B|4\rangle = \delta|4\rangle$, a measurement of B must yield δ. Thus, $\mathcal{P}(\beta, \gamma) = 0$ and $\mathcal{P}(\beta, \delta) = \mathcal{P}(\beta) = \frac{1}{2}|c_3 - c_4|^2$.

3. Joint Probabilities for the Full CSCO. Finally, we project the results from the A and B measurements with the projectors of C.

- **Outcome** (α, γ, ζ): We project the state $P_\gamma P_\alpha |\psi\rangle$ with $P_\zeta = |b_2\rangle\langle b_2|$.

  $$P_\zeta P_\gamma P_\alpha |\psi\rangle = P_\zeta \left(\frac{1}{\sqrt{2}}\big[c_1(|1\rangle + |2\rangle) + (c_3 + c_4)|3\rangle\big]\right) = c_1 \frac{|1\rangle + |2\rangle}{\sqrt{2}} = c_1 |b_2\rangle$$

  The joint probability is $\mathcal{P}(\alpha, \gamma, \zeta) = \|c_1 |b_2\rangle\|^2 = |c_1|^2$.

- **Outcome** (α, γ, κ): We project with $P_\kappa = I - P_\zeta$.

$$P_\kappa P_\gamma P_\alpha |\psi\rangle = (P_\gamma P_\alpha |\psi\rangle) - (P_\zeta P_\gamma P_\alpha |\psi\rangle) = \frac{c_3 + c_4}{\sqrt{2}} |3\rangle$$

  The joint probability is $\mathcal{P}(\alpha, \gamma, \kappa) = \left\| \frac{c_3+c_4}{\sqrt{2}} |3\rangle \right\|^2 = \frac{1}{2}|c_3 + c_4|^2$.

- **Outcome** (α, δ, κ): After measuring $(A, B) = (\alpha, \delta)$, the state is proportional to $|1\rangle - |2\rangle$, which is $\sqrt{2}|b_3\rangle$. Since $C|b_3\rangle = \kappa|b_3\rangle$, the outcome must be κ. Thus, $\mathcal{P}(\alpha, \delta, \zeta) = 0$ and $\mathcal{P}(\alpha, \delta, \kappa) = \mathcal{P}(\alpha, \delta) = |c_2|^2$.

- **Outcome** (β, δ, κ): After measuring $(A, B) = (\beta, \delta)$, the state is $|4\rangle$. Since $C|4\rangle = \kappa|4\rangle$, the outcome must be κ. Thus, $\mathcal{P}(\beta, \delta, \zeta) = 0$ and $\mathcal{P}(\beta, \delta, \kappa) = \mathcal{P}(\beta, \delta) = \frac{1}{2}|c_3 - c_4|^2$.

5.4 Summary of Results

The four possible unique outcomes of the CSCO measurement have the following joint probabilities:

- $\mathcal{P}(\alpha, \gamma, \zeta) = |c_1|^2$ (State collapses to $|\psi_2\rangle$)
- $\mathcal{P}(\alpha, \gamma, \kappa) = \frac{1}{2}|c_3 + c_4|^2$ (State collapses to $|\psi_1\rangle$)
- $\mathcal{P}(\alpha, \delta, \kappa) = |c_2|^2$ (State collapses to $|\psi_3\rangle$)
- $\mathcal{P}(\beta, \delta, \kappa) = \frac{1}{2}|c_3 - c_4|^2$ (State collapses to $|\psi_4\rangle$)

The sum of these probabilities is $|c_1|^2+|c_2|^2+\frac{1}{2}(|c_3+c_4|^2+|c_3-c_4|^2) = |c_1|^2+|c_2|^2+|c_3|^2+|c_4|^2 = 1$, confirming the consistency of the calculation.

6 Sequential Measurement and CSCO Summary

A sequential measurement of A, then B, then C will uniquely determine the final state of the system. Suppose the system starts in a normalized state $|\psi\rangle$.

1. **Measure** A: The probability of obtaining α is $\mathcal{P}(\alpha) = \|P_\alpha |\psi\rangle\|^2$. The state collapses to $|\psi'\rangle_\alpha = P_\alpha |\psi\rangle / \sqrt{\mathcal{P}(\alpha)}$.

2. **Measure** B **on** $|\psi'\rangle_\alpha$: The conditional probability of obtaining γ given α is

$$\mathcal{P}(\gamma|\alpha) = \frac{\mathcal{P}(\alpha, \gamma)}{\mathcal{P}(\alpha)} = \frac{\|P_\gamma P_\alpha |\psi\rangle\|^2}{\|P_\alpha |\psi\rangle\|^2}$$

   The state then collapses to $|\psi''\rangle_{\alpha,\gamma} = P_\gamma |\psi'\rangle_\alpha / \sqrt{\mathcal{P}(\gamma|\alpha)}$.

3. **Measure** C **on** $|\psi''\rangle_{\alpha,\gamma}$: The conditional probability of obtaining κ given (α, γ) is

$$\mathcal{P}(\kappa|\alpha, \gamma) = \frac{\mathcal{P}(\alpha, \gamma, \kappa)}{\mathcal{P}(\alpha, \gamma)} = \frac{\|P_\kappa P_\gamma P_\alpha |\psi\rangle\|^2}{\|P_\gamma P_\alpha |\psi\rangle\|^2}$$

   The state collapses to the final, unique state $|\psi'''\rangle_{\alpha,\gamma,\kappa}$.

6.1 The Simultaneous Eigenbasis

The set $\{A, B, C\}$ forms a CSCO. Their simultaneous eigenbasis, which we denote $\{|\psi_i\rangle\}_{i=1}^4$, consists of the vectors that are uniquely specified by a triplet of eigenvalues $(\lambda_A, \lambda_B, \lambda_C)$. This basis is precisely $\{|b_1\rangle, |b_2\rangle, |b_3\rangle, |4\rangle\}$.

Simultaneous Eigenvector	Eigenvalue of A	Eigenvalue of B	Eigenvalue of C
$\|\psi_1\rangle = \|b_1\rangle = \|3\rangle$	α	γ	κ
$\|\psi_2\rangle = \|b_2\rangle = \frac{1}{\sqrt{2}}(\|1\rangle + \|2\rangle)$	α	γ	ζ
$\|\psi_3\rangle = \|b_3\rangle = \frac{1}{\sqrt{2}}(\|1\rangle - \|2\rangle)$	α	δ	κ
$\|\psi_4\rangle = \|4\rangle$	β	δ	κ

Table 1: The simultaneous eigenbasis of the CSCO $\{A, B, C\}$. Each row corresponds to a unique state vector identified by a distinct triplet of eigenvalues. The set is complete provided $\alpha \neq \beta$, $\gamma \neq \delta$, and $\kappa \neq \zeta$.

6.2 Projectors onto the CSCO Basis

The most fundamental projectors are those onto the one-dimensional subspaces spanned by the simultaneous eigenvectors. These can be constructed by multiplying the projectors for the corresponding eigenvalues. For instance, the projector onto the state $|\psi_1\rangle$, specified by the eigenvalues (α, γ, κ), is:

$$P_{\psi_1} = P_{\alpha,\gamma,\kappa} = P_\alpha P_\gamma P_\kappa = |\psi_1\rangle\langle\psi_1|$$

Since the operators commute, their projectors also commute, so the order of multiplication is irrelevant. The measurement of the CSCO is thus equivalent to projecting the initial state vector onto this unique, physically distinguished basis. The joint probabilities calculated in the previous section are precisely $\mathcal{P}(\lambda_A, \lambda_B, \lambda_C) = \|P_{\lambda_A,\lambda_B,\lambda_C}\,|\psi\rangle\|^2$.

7 Time Evolution of the System

Having fully specified the state of our system through a sequence of measurements that define the CSCO, we now consider its dynamics. The evolution is governed by the Time-Dependent Schrödinger Equation (TDSE). For a discrete spectrum and a time-independent Hamiltonian H, the equation and its formal solution are:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle \quad \Longrightarrow \quad |\psi(t)\rangle = \sum_n e^{-iE_nt/\hbar}|E_n\rangle\langle E_n|\psi(0)\rangle$$

where $\{|E_n\rangle\}$ is the orthonormal basis of energy eigenvectors with corresponding energy eigenvalues $\{E_n\}$.

7.1 Defining a Hamiltonian from the CSCO

To proceed, we must define a Hamiltonian for the system. A physically and algebraically motivated choice is to construct H from our set of commuting observables. Since A, B, and C all commute with each other, any function of them also commutes. Let us define H as a linear combination of our CSCO:

$$H = k_A A + k_B B + k_C C$$

where $k_A, k_B, k_C \in \mathbb{R}$ are constants that define the energy scale associated with each observable. This construction guarantees that H is compatible with A, B, and C, meaning they share a mutual eigenbasis.

7.2 Energy Spectrum and Eigenstates

A direct consequence of our construction is that the simultaneous eigenbasis of the CSCO is also the eigenbasis of our Hamiltonian H. Let's rename our basis vectors from Table 1 of the source document to reflect that they are now energy eigenstates:

- $|E_1\rangle := |\psi_1\rangle = |b_1\rangle$
- $|E_2\rangle := |\psi_2\rangle = |b_2\rangle$
- $|E_3\rangle := |\psi_3\rangle = |b_3\rangle$
- $|E_4\rangle := |\psi_4\rangle = |4\rangle$

The energy eigenvalue for each state is found by applying H and using the known eigenvalues of A, B, and C:

$$\begin{aligned} H|E_1\rangle &= (k_A A + k_B B + k_C C)|\psi_1\rangle = (k_A\alpha + k_B\gamma + k_C\kappa)|\psi_1\rangle \\ &\implies E_1 = k_A\alpha + k_B\gamma + k_C\kappa \\ H|E_2\rangle &= (k_A A + k_B B + k_C C)|\psi_2\rangle = (k_A\alpha + k_B\gamma + k_C\zeta)|\psi_2\rangle \\ &\implies E_2 = k_A\alpha + k_B\gamma + k_C\zeta \\ H|E_3\rangle &= (k_A A + k_B B + k_C C)|\psi_3\rangle = (k_A\alpha + k_B\delta + k_C\kappa)|\psi_3\rangle \\ &\implies E_3 = k_A\alpha + k_B\delta + k_C\kappa \\ H|E_4\rangle &= (k_A A + k_B B + k_C C)|\psi_4\rangle = (k_A\beta + k_B\delta + k_C\kappa)|\psi_4\rangle \\ &\implies E_4 = k_A\beta + k_B\delta + k_C\kappa \end{aligned}$$

By choosing the constants k_A, k_B, k_C appropriately, one can ensure that the energy spectrum $\{E_1, E_2, E_3, E_4\}$ is non-degenerate, with each energy level corresponding to a unique state vector.

7.3 General Solution for $|\psi(t)\rangle$

The time evolution of an arbitrary initial state, $|\psi(0)\rangle$, is now determined. First, we project the initial state onto the energy eigenbasis to find the expansion coefficients $d_n = \langle E_n|\psi(0)\rangle$:

$$|\psi(0)\rangle = d_1|E_1\rangle + d_2|E_2\rangle + d_3|E_3\rangle + d_4|E_4\rangle$$

The state at any subsequent time t is then given by evolving each component with its characteristic complex phase:

$$|\psi(t)\rangle = d_1 e^{-iE_1 t/\hbar}|E_1\rangle + d_2 e^{-iE_2 t/\hbar}|E_2\rangle + d_3 e^{-iE_3 t/\hbar}|E_3\rangle + d_4 e^{-iE_4 t/\hbar}|E_4\rangle$$

This expression is the complete solution to the dynamics of the system. It demonstrates how an initial superposition state evolves as a coherent "rotation" in Hilbert space, with each energy eigenstate component acquiring phase at a rate determined by its energy. This concludes our construction and analysis of a complete quantum mechanical problem.

8 Uncertainty Product and the CSCO

8.1 Expectation Values

The expectation value of an observable O in state $|\psi\rangle$ is defined as

$$\langle O\rangle = \langle\psi|O|\psi\rangle = \sum_i o_i P(o_i),$$

where o_i are the eigenvalues of O and $P(o_i) = |\langle o_i|\psi\rangle|^2$ the corresponding probabilities. Explicitly

- $\langle A\rangle = \alpha P(\alpha) + \beta P(\beta)$
- $\langle B\rangle = \gamma P(\gamma) + \delta P(\delta)$
- $\langle C\rangle = \kappa P(\kappa) + \zeta P(\zeta)$

8.2 Uncertainties

The variance of an observable O, denoted $(\Delta O)^2$, measures the spread of measurements around the expectation value:

$$(\Delta O)^2 = \langle O^2\rangle - \langle O\rangle^2$$

where the expectation value of O^2 is $\langle O^2\rangle = \sum_i o_i^2 P(o_i)$. Thus, the variance for observable A is:

$$(\Delta A)^2 = \alpha^2 P(\alpha) + \beta^2 P(\beta) - \langle A\rangle^2$$

The expressions for $(\Delta B)^2$ and $(\Delta C)^2$ follow the same structure.

8.3 The Uncertainty Principle

For any two observables O_1 and O_2, the Robertson uncertainty relation provides a lower bound on the product of their uncertainties:

$$\Delta O_1 \Delta O_2 \geq \frac{1}{2}\,|\langle [O_1, O_2]\rangle|$$

This principle states that it is impossible to prepare a quantum state where two non-commuting observables can both be measured with arbitrary precision.

8.4 Application to the CSCO

A defining feature of a Complete Set of Commuting Observables is that all operators in the set mutually commute. By our construction, we ensured that:

$$[A, B] = [A, C] = [B, C] = 0$$

Because the commutators are zero, the right-hand side of the uncertainty relation vanishes:

$$\Delta O_i \Delta O_j \geq \frac{1}{2}\,|\langle 0\rangle| = 0$$

This result, $\Delta O_i \Delta O_j \geq 0$, is mathematically trivial, but it confirms the physical principle that there is no fundamental limit to the precision with which compatible (commuting) observables can be simultaneously known.

8.5 Effect of Degeneracy

The power of the CSCO lies in resolving degeneracy. The eigenvalue α of observable A is three-fold degenerate. This means that if a measurement of A yields α, the state collapses not to a unique vector, but to the 3D eigenspace V_α.

Consider a state $|\phi\rangle$ that is a superposition of vectors entirely within V_α (for example, $|\phi\rangle = c_2|E_2\rangle + c_3|E_3\rangle$). For this state:

- A measurement of A is guaranteed to yield α, so $\Delta A = 0$.

- However, $|\phi\rangle$ is a superposition of states with different eigenvalues for B (γ and δ) and C (ζ and κ). Therefore, $\Delta B \neq 0$ and $\Delta C \neq 0$.

This demonstrates that even when one observable is known with perfect certainty ($\Delta A = 0$), the uncertainties of other commuting observables can remain non-zero if the state lies within a degenerate subspace of the first observable.

8.6 Choice of Basis within Eigenspaces

Inside a degenerate eigenspace like V_α, one is free to choose any orthonormal basis. The specific basis we constructed, $\{|b_1\rangle, |b_2\rangle, |b_3\rangle\}$, was deliberately chosen to be the eigenbasis of the subsequent observables B and C within that subspace. This "top-down" approach is what resolves the degeneracy and minimizes the uncertainties. If we had chosen a different basis (like the computational basis $\{|1\rangle, |2\rangle, |3\rangle\}$), a state like $|1\rangle$ would still have $\Delta A = 0$, but it would not be an eigenstate of B or C, leading to non-zero uncertainties for both.

9 Verification of Postulates

The system constructed and analysed in this document serves as a concrete model to illustrate several of the fundamental postulates of quantum mechanics.

9.1 The Measurement Postulate and Collapse to Subspaces

The measurement postulate states that a measurement of an observable collapses the system's state to an eigenstate of the corresponding operator. Our system demonstrates a crucial facet of this postulate: the collapse to a degenerate eigenspace.

When the observable A is measured on a general state $|\psi\rangle$ and the result α is obtained, the state does not collapse to a single vector, since the eigenspace V_α is three-dimensional. Instead, the state of the system collapses to the normalized projection onto this subspace:

$$|\psi'\rangle_\alpha = \frac{P_\alpha|\psi\rangle}{\|P_\alpha|\psi\rangle\|}$$

This state $|\psi'\rangle_\alpha$ still lives in a 3D space, and therefore, the information about the system is not complete. Uncertainty still exists with respect to other observables like B and C. Only the subsequent measurements of B and C force the system to finally collapse to a unique vector (a simultaneous eigenstate), removing all degeneracy and uniquely determining the state.

9.2 The Necessity of Orthonormal Bases (Gram-Schmidt)

The postulates and formalism of quantum mechanics critically depend on the use of orthonormal bases. The expansion of a state vector, the construction of projectors like $P = \sum_i |e_i\rangle\langle e_i|$, and the Born rule for probabilities all presuppose that the basis vectors are mutually orthogonal and normalized.

The document explicitly acknowledges this necessity when discussing the Gram-Schmidt process. It is noted that, when solving the eigenvalue problem for an operator with a degenerate eigenspace (like V_α), one initially obtains a set of eigenvectors that is only guaranteed to be linearly independent, not necessarily orthogonal. The Gram-Schmidt algorithm is the standard procedure for converting such a basis into an orthonormal one.

In our case, a "top-down construction" was employed where orthonormality was imposed by definition from the outset (e.g., by postulating that the V-basis $\{|v_i\rangle\}$ was orthonormal). This approach bypasses the manual application of the process but underscores its conceptual importance as a fundamental requirement of the theoretical framework.

9.3 Consistency between Probabilities, Projectors, and Expectation Values

The analysis demonstrates the mathematical coherence between the different components of the quantum formalism:

1. Projectors as the Foundation: The projection operators ($P_\alpha, P_\beta, P_\gamma$, etc.) are the fundamental tools that represent the question "is the system in this subspace?". They are constructed directly from the eigenspaces of the observables.

2. Probabilities from Projectors: The probabilities of obtaining any measurement result are calculated directly from these projectors using the Born rule, $\mathcal{P}(\lambda) = \langle\psi|P_\lambda|\psi\rangle$. The explicit calculations in the document for the joint probabilities (e.g., $\mathcal{P}(\alpha, \gamma, \zeta)$) show that the total probability sums to 1, confirming the model's consistency.

3. Expectation Values from Probabilities: The expectation value, $\langle O\rangle$, is defined as the average of the eigenvalues weighted by their probabilities: $\langle O\rangle = \sum_i \lambda_i \mathcal{P}(\lambda_i)$. This directly connects the statistical outcomes (expectation values) with the probabilities calculated from the projectors. For instance, the value $\langle A\rangle = \alpha P(\alpha) + \beta P(\beta)$ is a direct manifestation of this consistency, where $P(\alpha)$ and $P(\beta)$ were determined using P_α and P_β.

This logical flow, from operators to probabilities and expectation values, illustrates the robust and self-consistent mathematical structure of quantum mechanics

10 The Role of Artificial Intelligence as a Computational Collaborator

The use of Artificial Intelligence (AI) in the analysis of quantum systems, such as the one explored here, redefines the dynamics of learning and problem-solving. The AI positions itself as a powerful "computational collaborator", handling intensive and error-prone algorithmic operations like matrix diagonalizations, basis changes, and the symbolic application of projectors. By delegating these tasks, the student is freed from a significant computational burden, allowing them to focus on the most fundamental aspect of the problem: the physical analysis and mathematical interpretation. For instance, while an AI can instantly construct the operators B and C, it is the student who must provide the physical guidance, ensuring the operators commute ($[A, B] = 0$, etc.) to properly resolve the system's degeneracy.

This highlights a crucial aspect of the human-AI collaboration: the need for explicit and well-defined instructions. Leaving a prompt open to interpretation can lead to undesired results. In preliminary stages of this work, vague prompts caused the AI to generate a redundant observable that, while mathematically valid, did not contribute to resolving the system's degeneracy, failing to illustrate the primary physical objective. Therefore, the student's role is not just to ask a question, but to frame it with precision, transforming a physical goal into a specific computational task.

This exercise provides a practical illustration of how successive measurements can define the state of a system. Starting with an observable A with a degenerate spectrum, a first measurement only allows us to confine the state to a subspace (in this case, the 3-dimensional V_α). It is through the application of additional compatible observables, B and C, that the degeneracy is sequentially broken or "lifted," until the system's state is uniquely determined by a set of eigenvalues (e.g., (α, γ, κ)), collapsing to a single eigenvector like $|b_1\rangle$.

Furthermore, the development highlights the deep relationship between projection operators and the calculation of probabilities. The probability of measuring a specific eigenvalue is calculated by applying the projector associated with its subspace onto the system's state, for example, $P(\alpha) = \|P_\alpha|\psi\rangle\|^2$. This connection is not merely computational; it is a cornerstone of the postulates of quantum mechanics that links the algebraic structure of observables (through

their projectors) with the statistical outcomes of experiments (the measurement probabilities). The AI can construct the projector and compute the norm, but interpreting this value as a probability is a conceptual task that falls entirely on the student.

This collaborative model extends seamlessly from the static measurement problem to the analysis of the system's dynamics, governed by the Schrödinger equation. The AI can efficiently compute the energy spectrum and determine the time evolution of any initial state by applying the corresponding phase factors $e^{-iE_nt/\hbar}$. In both the static case (resolving degeneracy) and the dynamic case (time evolution), the AI provides the final mathematical expression, but the student is responsible for interpreting its physical meaning: understanding it as a collapse to a subspace, a coherent rotation in Hilbert space, or recognizing that energy measurement probabilities remain constant.

In conclusion, the synergy between the student and the AI is powerful and multifaceted. The AI acts as a computational engine, flawlessly executing complex tasks. The student, in turn, acts as the project director and physical interpreter. This role involves translating physical objectives into precise, unambiguous instructions and, most importantly, interpreting the final mathematical results. The AI accelerates the journey to the solution, but it does not provide a shortcut to understanding; the conceptual grasp of quantum postulates and physical intuition remain the exclusive and essential domain of the human learner.