Construction of a Complete Set of Commuting Observables for a 4D Hilbert Space with Degeneracies

Julian Avila   Laura Herrera   Sebastian Rodriguez

September 26, 2025

Abstract

This document details the systematic construction of a Complete Set of Commuting Observables (CSCO) for a quantum system in a four-dimensional Hilbert space, $\mathcal{H}$. Starting from a deliberately constructed degenerate observable A, we introduce subsequent commuting observables B and C to sequentially lift the degeneracies. We analyse the process of measurement, including the calculation of probabilities, state collapse, and the algebraic construction of projection operators using both spectral decomposition and the minimal polynomial method.

1 Problem Statement and First Observable A

We consider a quantum system described by a state vector in a Hilbert space $\mathcal{H}$. The initial information about the system is provided by an observable A which possesses a degenerate spectrum. Our goal is to find a set of additional observables that commute with A and each other, such that their collective set of eigenvalues uniquely specifies any basis vector of $\mathcal{H}$.

1.1 Constraints and Construction of Operator A

The primary observable A is a self-adjoint operator, $A : \mathcal{H} \to \mathcal{H}$, subject to the following constraints:

- A is diagonal in the computational basis $\{|1\rangle, |2\rangle, |3\rangle, \dots\}$.
- The spectrum of A, $\sigma(A)$, contains exactly two distinct real eigenvalues, $\alpha \neq \beta$.
- One eigenspace, V_α, corresponding to eigenvalue α, is three-dimensional: $\dim(V_\alpha) = 3$.
- The total dimension of the Hilbert space is minimal, but no greater than 4.

The spectral theorem dictates that $\mathcal{H}$ is the orthogonal direct sum of the eigenspaces of A, so $\dim(\mathcal{H}) = \dim(V_\alpha) + \dim(V_\beta)$. Given $\dim(V_\alpha) = 3$ and the requirement that $\dim(V_\beta) \geq 1$, the constraint $\dim(\mathcal{H}) \leq 4$ uniquely determines that $\dim(V_\beta) = 1$ and $\dim(\mathcal{H}) = 4$. Our Hilbert space is thus isomorphic to $\mathbb{C}^4$.

We adopt the standard orthonormal computational basis $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$. Since A must be diagonal in this basis, these basis vectors are its eigenvectors. To satisfy the dimensionality constraints, we assign three of them to the eigenvalue α and one to β. A canonical matrix representation for A is:

$$A \mapsto \mathbf{A} = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}$$

The eigenspaces are therefore $V_\alpha = \mathrm{span}\{|1\rangle, |2\rangle, |3\rangle\}$ and $V_\beta = \mathrm{span}\{|4\rangle\}$.

2 Measurement Formalism for Operator A

To analyse a measurement, we consider a general state $|\psi\rangle \in \mathcal{H}$ prepared in a basis that is incompatible with the eigenbasis of A.

2.1 A General State in a Superposition Basis

Let us define a new orthonormal basis, the V-basis, denoted by $\{|v_i\rangle\}_{i=1}^{4}$:

$$\begin{aligned} |v_1\rangle &= \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) & |v_3\rangle &= \frac{1}{\sqrt{2}}(|3\rangle + |4\rangle) \\ |v_2\rangle &= \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle) & |v_4\rangle &= \frac{1}{\sqrt{2}}(|3\rangle - |4\rangle) \end{aligned}$$

An arbitrary normalized state $|\psi\rangle$ can be expressed as a superposition in this basis:

$$|\psi\rangle = \sum_{i=1}^{4} c_i |v_i\rangle, \quad \text{with} \quad \sum_{i=1}^{4} |c_i|^2 = 1$$

To analyse the measurement of A, we must express $|\psi\rangle$ in the computational basis (the eigenbasis of A). This is a change of basis:

$$\begin{aligned} |\psi\rangle &= c_1\left(\frac{|1\rangle + |2\rangle}{\sqrt{2}}\right) + c_2\left(\frac{|1\rangle - |2\rangle}{\sqrt{2}}\right) + c_3\left(\frac{|3\rangle + |4\rangle}{\sqrt{2}}\right) + c_4\left(\frac{|3\rangle - |4\rangle}{\sqrt{2}}\right) \\ &= \frac{1}{\sqrt{2}}[(c_1 + c_2)|1\rangle + (c_1 - c_2)|2\rangle + (c_3 + c_4)|3\rangle + (c_3 - c_4)|4\rangle] \end{aligned}$$

The column vector representation of $|\psi\rangle$ in the computational basis is:

$$|\psi\rangle \mapsto \boldsymbol{\psi}_A = \frac{1}{\sqrt{2}}\begin{pmatrix} c_1 + c_2 \\ c_1 - c_2 \\ c_3 + c_4 \\ c_3 - c_4 \end{pmatrix}$$

This transformation is mediated by a unitary matrix U whose columns are the basis vectors $|v_i\rangle$ expressed in the computational basis, such that $\boldsymbol{\psi}_A = U\boldsymbol{\psi}_V$.

2.2 Projection Operators and Probabilities

The measurement outcomes are the eigenvalues of A. The probability of measuring a particular eigenvalue is found using projection operators. The projectors onto the eigenspaces V_α and V_β are:

$$P_\alpha = |1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| \quad \text{and} \quad P_\beta = |4\rangle\langle 4|$$

The probability $P(\lambda)$ of measuring eigenvalue λ for a system in state $|\psi\rangle$ is $P(\lambda) = \| P_\lambda |\psi\rangle \|^2 = \langle\psi| P_\lambda |\psi\rangle$.

Probability of measuring α: The projection of $|\psi\rangle$ onto V_α is $P_\alpha|\psi\rangle = \frac{1}{\sqrt{2}}[(c_1 + c_2)|1\rangle + (c_1 - c_2)|2\rangle + (c_3 + c_4)|3\rangle]$.

$$\begin{aligned} P(\alpha) = \| P_\alpha |\psi\rangle \|^2 &= \frac{1}{2}\left(|c_1 + c_2|^2 + |c_1 - c_2|^2 + |c_3 + c_4|^2\right) \\ &= \frac{1}{2}\left[2(|c_1|^2 + |c_2|^2) + |c_3 + c_4|^2\right] = |c_1|^2 + |c_2|^2 + \frac{1}{2}|c_3 + c_4|^2 \end{aligned}$$

Probability of measuring β: The projection of $|\psi\rangle$ onto V_β is $P_\beta |\psi\rangle = \frac{1}{\sqrt{2}}(c_3 - c_4)|4\rangle$.

$$P(\beta) = \|P_\beta |\psi\rangle\|^2 = \frac{1}{2}|c_3 - c_4|^2$$

As required, $P(\alpha) + P(\beta) = \sum |c_i|^2 = 1$.

2.3 Post-Measurement States and Expectation Value

Upon measurement, the state collapses to the normalized projection onto the corresponding eigenspace.

- If the outcome is α, the state becomes $|\psi'\rangle_\alpha = \frac{P_\alpha|\psi\rangle}{\|P_\alpha|\psi\rangle\|}$. The state is now confined to the 3D subspace V_α, but is not fully determined.
- If the outcome is β, the state becomes $|\psi'\rangle_\beta = \frac{P_\beta|\psi\rangle}{\|P_\beta|\psi\rangle\|} = e^{i\phi}|4\rangle$, which is a uniquely determined state (up to a global phase).

The expectation value of A is given by $\langle\psi|A|\psi\rangle = \alpha P(\alpha) + \beta P(\beta)$.

2.4 Algebraic Construction of Projectors

An elegant and powerful method to construct projectors utilizes the functional calculus of operators. Since A is diagonalizable, its minimal polynomial is $m(\lambda) = (\lambda - \alpha)(\lambda - \beta)$. The projector P_α can be expressed as a polynomial in A, namely the Lagrange interpolating polynomial $q_\alpha(\lambda)$ that satisfies $q_\alpha(\alpha) = 1$ and $q_\alpha(\beta) = 0$.

$$q_\alpha(\lambda) = \frac{\lambda - \beta}{\alpha - \beta} \implies P_\alpha = \frac{A - \beta I}{\alpha - \beta}$$

This algebraic formula yields the exact same matrix operator as the geometric construction via sum of dyads, providing a crucial consistency check.

3 Lifting the Degeneracy: Operator B

The 3-fold degeneracy of eigenvalue α implies A is not a complete observable. We introduce a second observable B to partially lift this degeneracy. For A and B to be simultaneously measurable, they must commute: $[A, B] = 0$. This condition implies that B must be block-diagonal with respect to the eigenspace decomposition of A.

3.1 Construction of Operator B

We design B to act non-trivially on V_α and to possess its own degenerate spectrum. Within V_α, let B have two eigenvalues: γ (2-fold degenerate) and δ (non-degenerate), with $\gamma \neq \delta$. The eigenspaces of B within V_α are:

- $W_\gamma = \text{span}\{|b_1\rangle, |b_2\rangle\}$ for eigenvalue γ.
- $W_\delta = \text{span}\{|b_3\rangle\}$ for eigenvalue δ.

To make B non-diagonal in the computational basis, we choose an orthonormal basis for these eigenspaces that mixes $\{|1\rangle, |2\rangle, |3\rangle\}$:

$$|b_1\rangle = |3\rangle, \quad |b_2\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle), \quad |b_3\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$$

The action of B restricted to V_α is $B_\alpha = \gamma(|b_1\rangle\langle b_1| + |b_2\rangle\langle b_2|) + \delta\,|b_3\rangle\langle b_3|$. To complete the definition of B on $\mathcal{H}$, we define its action on $V_\beta = \text{span}\{|4\rangle\}$, for instance by assigning it the eigenvalue δ. The full matrix for B in the computational basis is:

$$\mathbf{B} = \begin{pmatrix} \frac{\gamma+\delta}{2} & \frac{\gamma-\delta}{2} & 0 & 0 \\ \frac{\gamma-\delta}{2} & \frac{\gamma+\delta}{2} & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix}$$

By construction, $[A, B] = 0$.

3.2 Sequential Measurement of B

Suppose a measurement of A yields the outcome α. The system is now in the state $|\psi'\rangle_\alpha$. We now measure B. The possible outcomes are γ and δ. The conditional probability is calculated as:

$$P(\gamma|\alpha) = \left\|P_\gamma\,|\psi'\rangle_\alpha\right\|^2 = \langle\psi'|_\alpha\,P_\gamma\,|\psi'\rangle_\alpha$$

where P_γ is the projector onto the eigenspace W_γ. From its basis vectors, $P_\gamma = |b_1\rangle\langle b_1| + |b_2\rangle\langle b_2|$. It is often simpler to work with the unnormalized post-A-measurement state, $|\phi_A\rangle = P_\alpha\,|\psi\rangle$. The conditional probability is then:

$$P(\gamma|\alpha) = \frac{\|P_\gamma\,|\phi_A\rangle\|^2}{\||\phi_A\rangle\|^2} = \frac{|c_1|^2 + \frac{1}{2}|c_3 + c_4|^2}{|c_1|^2 + |c_2|^2 + \frac{1}{2}|c_3 + c_4|^2}$$

And similarly for the outcome δ, with projector $P_\delta = |b_3\rangle\langle b_3|$:

$$P(\delta|\alpha) = \frac{\|P_\delta\,|\phi_A\rangle\|^2}{\||\phi_A\rangle\|^2} = \frac{|c_2|^2}{|c_1|^2 + |c_2|^2 + \frac{1}{2}|c_3 + c_4|^2}$$

If the outcome is (α, δ), the state collapses to $|b_3\rangle$, which is uniquely specified. However, if the outcome is (α, γ), the state collapses into the 2D subspace W_γ, indicating a remaining degeneracy.

4 Completing the Set: Operator C

To lift the final degeneracy, we introduce a third operator C that commutes with both A and B. This requires that C respects the simultaneous eigenspace decomposition of $\{A, B\}$. The only remaining degenerate eigenspace is $W_{\alpha,\gamma} \equiv W_\gamma$.

4.1 Construction of Operator C

We define C to act non-trivially only within W_γ. Let its eigenvalues be $\lambda_1 \neq \lambda_2$. Its eigenvectors, $\{|c_1\rangle, |c_2\rangle\}$, must form an orthonormal basis for W_γ. We choose:

$$|c_1\rangle = \frac{1}{\sqrt{2}}(|b_1\rangle + |b_2\rangle) = \frac{1}{2}(|1\rangle + |2\rangle) + \frac{1}{\sqrt{2}}\,|3\rangle$$
$$|c_2\rangle = \frac{1}{\sqrt{2}}(|b_1\rangle - |b_2\rangle) = -\frac{1}{2}(|1\rangle + |2\rangle) + \frac{1}{\sqrt{2}}\,|3\rangle$$

We define the action of C to be zero on all other simultaneous eigenspaces. The spectral decomposition is $C = \lambda_1\,|c_1\rangle\langle c_1| + \lambda_2\,|c_2\rangle\langle c_2|$. This guarantees $[C, A] = [C, B] = 0$.

4.2 Matrix Representation of C

To find the matrix $\mathbf{C}$ in the computational basis, we compute the dyads:

$$|c_1\rangle\langle c_1| = \frac{1}{4}(|1\rangle\langle 1| + |1\rangle\langle 2| + |2\rangle\langle 1| + |2\rangle\langle 2|) + \frac{1}{2}|3\rangle\langle 3| + \frac{1}{2\sqrt{2}}(|1\rangle\langle 3| + |3\rangle\langle 1| + |2\rangle\langle 3| + |3\rangle\langle 2|)$$
$$|c_2\rangle\langle c_2| = \frac{1}{4}(|1\rangle\langle 1| + |1\rangle\langle 2| + |2\rangle\langle 1| + |2\rangle\langle 2|) + \frac{1}{2}|3\rangle\langle 3| - \frac{1}{2\sqrt{2}}(|1\rangle\langle 3| + |3\rangle\langle 1| + |2\rangle\langle 3| + |3\rangle\langle 2|)$$

Combining these gives the matrix for C:

$$\mathbf{C} = \frac{1}{4}\begin{pmatrix} \lambda_1+\lambda_2 & \lambda_1+\lambda_2 & \sqrt{2}(\lambda_1-\lambda_2) & 0 \\ \lambda_1+\lambda_2 & \lambda_1+\lambda_2 & \sqrt{2}(\lambda_1-\lambda_2) & 0 \\ \sqrt{2}(\lambda_1-\lambda_2) & \sqrt{2}(\lambda_1-\lambda_2) & 2(\lambda_1+\lambda_2) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

4.3 Final Measurement Probabilities

Suppose the measurement sequence has yielded outcomes (α, γ). The unnormalized state is $|\phi_{AB}\rangle = P_\gamma P_\alpha |\psi\rangle = c_1 |b_2\rangle + \frac{c_3+c_4}{\sqrt{2}} |b_1\rangle$. We now measure C. The probability of obtaining λ_1 is $P(\lambda_1|\alpha,\gamma) = \frac{|\langle c_1|\phi_{AB}\rangle|^2}{\||\phi_{AB}\rangle\|^2}$. The required inner product is $\langle c_1|\phi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(c_1 + \frac{c_3+c_4}{\sqrt{2}}\right)$. The probability is:

$$P(\lambda_1|\alpha,\gamma) = \frac{\frac{1}{2}\left|c_1 + \frac{c_3+c_4}{\sqrt{2}}\right|^2}{|c_1|^2 + \frac{1}{2}|c_3+c_4|^2}$$

And for λ_2:

$$P(\lambda_2|\alpha,\gamma) = \frac{\frac{1}{2}\left|-c_1 + \frac{c_3+c_4}{\sqrt{2}}\right|^2}{|c_1|^2 + \frac{1}{2}|c_3+c_4|^2}$$

These probabilities sum to 1. After this final measurement, the state collapses to either $|c_1\rangle$ or $|c_2\rangle$, and the state of the system is now uniquely determined.

5 Conclusion: The Complete Set of Commuting Observables

The set of operators $\{A, B, C\}$ forms a CSCO. Their simultaneous eigenvectors form a complete orthonormal basis for the Hilbert space $\mathcal{H}$. Each basis vector is uniquely specified by a triplet of eigenvalues (a, b, c). The complete basis and corresponding eigenvalues are summarized below:

Basis Vector	**Definition in Computational Basis**	**Eigenvalues**		
		A	B	C
$\|c_1\rangle$	$\frac{1}{2}(\|1\rangle + \|2\rangle) + \frac{1}{\sqrt{2}}\|3\rangle$	α	γ	λ_1
$\|c_2\rangle$	$-\frac{1}{2}(\|1\rangle + \|2\rangle) + \frac{1}{\sqrt{2}}\|3\rangle$	α	γ	λ_2
$\|b_3\rangle$	$\frac{1}{\sqrt{2}}(\|1\rangle - \|2\rangle)$	α	δ	0
$\|4\rangle$	$\|4\rangle$	β	δ	0

Table 1: The CSCO basis and their corresponding eigenvalues.

A sequential measurement of A, then B, then C will project an arbitrary initial state $|\psi\rangle$ onto one of these four basis vectors, with probabilities calculable at each stage. The process demonstrates how introducing compatible observables resolves spectral degeneracies and allows for the complete determination of a quantum state.

6 Time Evolution of the System

Having fully specified the state of our system through a sequence of measurements, we now consider its dynamics, governed by the Time-Dependent Schrödinger Equation (TDSE). For a discrete spectrum and a time-independent Hamiltonian H, the equation and its formal solution are:

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle \quad \Longrightarrow \quad |\psi(t)\rangle = \sum_n e^{-iE_n t/\hbar} |E_n\rangle \langle E_n|\psi(0)\rangle$$

where $\{|E_n\rangle\}$ is the orthonormal basis of energy eigenvectors (autokets) with corresponding energy eigenvalues $\{E_n\}$.

6.1 Defining a Hamiltonian from the CSCO

To proceed, we must define a Hamiltonian for the system. A physically and algebraically motivated choice is to construct the Hamiltonian from our set of commuting observables. Since A, B, and C all commute with each other, any function of them also commutes. Let us define H as a linear combination of our CSCO:

$$H = k_A A + k_B B + k_C C$$

where $k_A, k_B, k_C \in \mathbb{R}$ are constants that define the energy scale associated with each observable. This construction guarantees that H is compatible with A, B, and C, meaning they can all be measured simultaneously without uncertainty.

6.2 Energy Spectrum and Eigenstates

A direct consequence of our construction is that the simultaneous eigenbasis of the CSCO, which we have painstakingly constructed, is also the eigenbasis of our Hamiltonian H. Let's rename our basis vectors to reflect that they are energy eigenstates:

$$\begin{aligned}
|E_1\rangle &:= |c_1\rangle \\
|E_2\rangle &:= |c_2\rangle \\
|E_3\rangle &:= |b_3\rangle \\
|E_4\rangle &:= |4\rangle
\end{aligned}$$

The energy eigenvalue for each state is found by applying H and using the known eigenvalues of A, B, and C:

$$\begin{aligned}
H |E_1\rangle &= (k_A\alpha + k_B\gamma + k_C\lambda_1) |E_1\rangle \implies E_1 = k_A\alpha + k_B\gamma + k_C\lambda_1 \\
H |E_2\rangle &= (k_A\alpha + k_B\gamma + k_C\lambda_2) |E_2\rangle \implies E_2 = k_A\alpha + k_B\gamma + k_C\lambda_2 \\
H |E_3\rangle &= (k_A\alpha + k_B\delta + k_C \cdot 0) |E_3\rangle \implies E_3 = k_A\alpha + k_B\delta \\
H |E_4\rangle &= (k_A\beta + k_B\delta + k_C \cdot 0) |E_4\rangle \implies E_4 = k_A\beta + k_B\delta
\end{aligned}$$

By choosing the constants k_A, k_B, k_C appropriately, one can ensure that the energy spectrum is non-degenerate, with each energy level corresponding to a unique state vector.

6.3 General Solution for $|\psi(t)\rangle$

The time evolution of our arbitrary initial state, $|\psi(0)\rangle$, is now determined. First, we project the initial state onto the energy eigenbasis to find the expansion coefficients $d_n = \langle E_n|\psi(0)\rangle$.

$$|\psi(0)\rangle = d_1 |E_1\rangle + d_2 |E_2\rangle + d_3 |E_3\rangle + d_4 |E_4\rangle$$

The state at any subsequent time t is then given by evolving each component with its characteristic complex phase:

$$|\psi(t)\rangle = d_1 e^{-iE_1 t/\hbar}|E_1\rangle + d_2 e^{-iE_2 t/\hbar}|E_2\rangle + d_3 e^{-iE_3 t/\hbar}|E_3\rangle + d_4 e^{-iE_4 t/\hbar}|E_4\rangle$$

This expression is the complete solution to the dynamics of the system. It demonstrates how an initial superposition state evolves as a coherent "rotation" in Hilbert space, with each energy eigenstate component acquiring phase at a rate determined by its energy. This concludes our construction and analysis of a complete quantum mechanical problem.

7 Uncertainty Product

1. **Expectation values of A, B, and C**
   The expectation value of an observable O in state $|\psi\rangle$ is defined as

   $$\langle O\rangle = \langle\psi|O|\psi\rangle = \sum_i o_i P(o_i),$$

   where o_i are the eigenvalues of O and $P(o_i) = |\langle o_i|\psi\rangle|^2$ the corresponding probabilities. Explicitly,

   $$\langle A\rangle = \alpha P(\alpha) + \beta P(\beta), \quad \langle B\rangle = \gamma P(\gamma) + \delta P(\delta), \quad \langle C\rangle = \lambda_1 P(\lambda_1) + \lambda_2 P(\lambda_2).$$

2. **Uncertainties**
   The variance of an observable O is

   $$\Delta O^2 = \langle O^2\rangle - \langle O\rangle^2,$$

   with $\langle O^2\rangle = \sum_i o_i^2 P(o_i)$. Thus,

   $$\Delta A^2 = \alpha^2 P(\alpha) + \beta^2 P(\beta) - \langle A\rangle^2,$$

   and similarly for B and C.

3. **Uncertainty product inequality**
   For any two observables O_1, O_2, the Robertson relation states:

   $$\Delta O_1\, \Delta O_2 \geq \frac{1}{2}\left|\langle[O_1, O_2]\rangle\right|.$$

4. **Application to the CSCO**
   In our construction $[A, B] = [A, C] = [B, C] = 0$. Hence,

   $$\Delta O_1\, \Delta O_2 \geq 0,$$

   which is always satisfied. The lower bound is trivial in this case.

5. **Effect of degeneracy**
   The degeneracy of A implies that measuring A may collapse the state only into a subspace, not into a unique vector. Consequently, ΔA can vanish while ΔB or ΔC remain non-zero, depending on the chosen state within the degenerate subspace.

6. **Choice of basis within eigenspaces**
   Inside a degenerate eigenspace one can freely choose an orthonormal basis. Selecting the basis to diagonalize a second observable (e.g. B inside V_α) minimizes uncertainties for that observable. Different basis choices lead to different distributions of probabilities and variances for B and C.

8 Verification of Postulates

1. **Postulate of measurement**
   The collapse of the state into an eigenspace of A (e.g. V_α) upon measurement illustrates the measurement postulate, including the case of degenerate spectra.

2. **Need for orthonormal bases (Gram–Schmidt)**
   Within degenerate subspaces, an orthonormal basis is required to define projectors and probabilities consistently. The Gram–Schmidt process guarantees the construction of such orthonormal sets.

3. **Consistency of probabilities, projectors, and expectation values**
   The relations
   $$P(o_i) = \langle\psi|P_{o_i}|\psi\rangle, \qquad \langle O\rangle = \sum_i o_i P(o_i),$$
   demonstrate the internal consistency of the formalism: probabilities, projectors, and expectation values are fully compatible and lead to identical results, regardless of the representation.

The Role of Artificial Intelligence as a Computational Collaborator in Quantum Mechanics

The use of Artificial Intelligence (AI) in the analysis of quantum systems, such as the one explored in this workshop, redefines the dynamics of learning and problem-solving, positioning itself as a powerful "computational collaborator". This tool handles the most intensive and error-prone algorithmic operations, such as matrix diagonalizations, the application of the Gram-Schmidt orthogonalization process, and the execution of basis changes. By delegating these tasks to the AI, the student is freed from a significant computational burden, allowing them to focus on the most fundamental and enriching aspect of the problem: the physical analysis and mathematical interpretation.

As observed in the exercise, AI is an extremely useful tool for performing extensive calculations. For instance, constructing the operators B and C in the computational basis from their eigenvectors is a dense algebraic process that an AI can execute instantly. However, it is crucial to always keep the purpose of these procedures in mind. The goal is not merely to obtain a matrix, but to construct an observable that commutes with the previous ones ($[A, B] = 0$, $[C, A] = [C, B] = 0$) in order to lift the degeneracy in a controlled manner. The AI performs the calculation, but the student must guide the process with a clear understanding of the physical constraints and mathematical objectives.

This exercise provides a practical illustration of how successive measurements can define the state of a system. Starting with an observable A with a degenerate spectrum, a first measurement only allows us to confine the state to a subspace (in this case, the 3-dimensional V_α). It is through the application of additional compatible observables, B and C, that the degeneracy is sequentially broken or "lifted", until the system's state is uniquely determined by a set of eigenvalues $(\alpha, \gamma, \lambda_1)$, collapsing to a single eigenvector like $|c_1\rangle$.

Furthermore, the development highlights the deep relationship between projection operators and the calculation of probabilities. The probability of measuring a specific eigenvalue is calculated by applying the projector associated with its subspace onto the system's state. For example, $P(\alpha) = ||P_\alpha|\psi\rangle||^2$. This connection is not merely computational; it is a cornerstone of the postulates of quantum mechanics that links the algebraic structure of observables (through their projectors) with the statistical outcomes of experiments (the measurement probabilities). The AI can construct the projector and compute the norm, but interpreting this value as a probability is a conceptual task that falls entirely on the student.

In conclusion, the synergy between the student and the AI is clear: the AI serves as the computational engine, while the student acts as the physical director. The tool accelerates results but does not replace the need for a deep understanding of quantum postulates, operator theory, and the physical meaning behind each mathematical step.