Space time

Relativity makes us work in a four-dimensional vector space - the use of complex numbers its to include electric charge, however we can remove them and work only in a real algebra. We will continue with the complex algebra to follow the article - endowed with a non-degenerate symmetric bilinear form, the metric $g : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ which defines its geometry.

We can construct a basis $\{\gamma_\mu\}$ for $\mathcal{M}$, called an observer's frame, allowing any event x to be represented by $x = x^\mu \gamma_\mu = (x^0, x^1, x^2, x^3) = (t, \mathbf{x})$, $x^\mu \in \mathbb{R}$ are the coordinates relative to $\{\gamma_\mu\}$

For two events $u = u^\mu \gamma_\mu$, $v = v^\mu \gamma_\mu$, their inner product is given by:

$$g(u,v) = u \cdot v = u^0 v^0 - u^1 v^1 - u^2 v^2 - u^3 v^3$$

we choose $\{\gamma_\mu\}$ such that $g(\gamma_\mu, \gamma_\nu) = \eta_{\mu\nu}$

Poincare Group

By the principles of relativity, a transform that relates any two frames (physically compatible) must be

$u' = R(u) + a$, $R(\cdot)$ is a linear transform and a a constant vector

Relativity needs us to leave the interval between two events invarient under said transformations

$$x = u - v \quad \rightarrow \quad x' = u' - v'$$

$$\begin{aligned} x' &= (R(u) + a) - (R(v) + a) \\ &= R(u) - R(v) \\ &= R(u - v) \\ &= R(x) \end{aligned}$$

the invariance of the interval means

$$g(x', x') = g(x, x) \quad \rightarrow \quad g(x', x') = g(R(x), R(x))$$

lets define $\bar{R}$ as the adjoint of R with respect to g, such that

$$g(x, R(y)) = g(\bar{R}(x), y) \qquad \forall x, y \in M$$

For $g(x', x') = g(x, x)$

$$\Rightarrow \quad g(x', x') = g(\bar{R}R(x), x) = g(x, x)$$

$$\Rightarrow \quad \bar{R}R(x) = x \quad \forall x \quad \Rightarrow \quad \bar{R}R \text{ is the identity map}$$

The set of transform $R(x)$ are the Lorentz transforms and form the group $O(1,3)$ - or $O(3,1)$ depending on $\eta_{\mu\nu}$ - and they leave the origin fixed, they are both rotations (on a spatial plane) or boosts (rotation on a space time plane).

The transform $x' = x + a$ is a pure translation and forms an Abelian group $T(1,3)$ - or $T(3,1)$ -.

The set of isometries of $\mathcal{M}$ combining both is the Poincare group $IO(1,3) \cong T(1,3) \rtimes O(1,3)$ - or 3,1 -.

This as $P_i = (R_i, a_i) \in IO(1,3)$

$$\begin{aligned} P_2 P_1 (x) &= P_2 (R_1(x) + a_1) \\ &= R_2 R_1 (x) + R_2(a_1) + a_2 \\ &= R_3 (x) + a_3 \end{aligned}$$

$$\Rightarrow \quad P_2 \circ P_1 = (R_2, a_2) \circ (R_1, a_1) = (R_2 R_1, R_2(a_1) + a_2)$$

Clifford Algebra

A vector space $\mathcal{M}$ with a bilinear non-degenerate form g is the natural foundation for a Clifford Algebra $Cl(\mathcal{M}, g)$. The algebra is generated by the Clifford product (indicated by juxtaposition) and satisfy

$$uv + vu = 2g(u,v) \qquad \forall u, v \in \mathcal{M}$$

$Cl_{p,q}$ denotes an algebra with p basis vectors that square to 1 and q square to -1.

$$\dim(\mathcal{M}) = p + q \quad \text{and} \quad \dim(Cl_{p,q}) = 2^{p+q}$$

On this document we will use $Cl_{1,3}(\mathbb{C})$ meaning that the basis vectors γ_μ obey

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} = 2\,\mathrm{diag}(1,-1,-1,-1)$$

$$\gamma_0^2 = 1 \quad , \quad \gamma_i^2 = -1 \quad , \quad \gamma_{\mu\nu} = -\gamma_{\nu\mu} \quad \forall_{\mu \neq \nu}$$

the full algebra is

1 scalar, $\{\gamma_\mu\}$ 4 vectors
$\{\gamma_{\mu\nu}\}$ 6 bivetors, $\{\gamma_{\mu\nu\lambda}\}$ 4 trivectors,

$\{\gamma_{0123}\}$ 1 pseudoscalar

where it is common to use $\gamma^5 \equiv i\gamma^{0123} \equiv \gamma_5$

As $\gamma_0^2 = 1 \Rightarrow \gamma^0 = \gamma_0$ but $\gamma^i = -\gamma_i$

$$(\gamma^0)^2 = 1 \; , \quad (\gamma^i)^2 = \gamma_i^2 = -1$$

A element of the algebra is a linear combination of the basis (scalar + vector + bivector + trivector + pseudoscalar)

The pseudo scalar commutes with even grade elements (scalar, bivector, pseudosaler) and

$$\begin{aligned}
(\gamma_{0123})^2 &= \gamma_{01230123} = -\gamma_{01203123} = \gamma_{01023123} = -\gamma_{00123123} \\
&= -\gamma_{123123} = \gamma_{121323} = -\gamma_{112323} \\
&= \gamma_{2323} = -\gamma_{2233} \\
&= \gamma_{33} \\
&= -1
\end{aligned}$$

$$(\gamma^5)^2 = (i\,\gamma^{0123})^2 = i^2(-1) = 1$$

For commutation

$$\gamma^0\gamma^5 = i\gamma^{00123} = i\gamma^{01203} = -i\gamma^{01230} = -\gamma^5\gamma^0$$

$$\Rightarrow \quad [\gamma^0, \gamma^5] \neq 0 \quad , \quad \{\gamma^0, \gamma^5\} = 0$$

$$\gamma^i\gamma^5 = i\gamma^{i0123} = -i\,\gamma^{0i123} = -i\,\gamma^{0123i} = -\gamma_5\gamma_i$$

With γ_{i123}, γ_i has to commute with one and anticommute with two
so either way $\gamma_{i123} = \gamma_{123i}$

$$\Rightarrow \underline{\{\gamma^\mu, \gamma^5\} = 0}$$

For Bivectors

- $$[\gamma^{\mu\nu}, \gamma^5] = \gamma^{\mu\nu}\gamma^5 - \gamma^5\gamma^{\mu\nu}$$

$$= \gamma^{\mu\nu}\gamma^5 + \gamma^{\mu5}\gamma^\nu - \gamma^{\mu5}\gamma^\nu - \gamma^5\gamma^{\mu\nu}$$

$$= \gamma^\mu(\gamma^\nu\gamma^5 + \gamma^5\gamma^\nu) - (\gamma^\mu\gamma^5 + \gamma^5\gamma^\mu)\gamma^\nu$$

$$= \gamma^\mu\{\gamma^\nu, \gamma^5\} - \{\gamma^\mu, \gamma^5\}\gamma^\nu$$

$$\therefore \underline{[\gamma^{\mu\nu}, \gamma^5] = 0}$$

For trivectors

$$\{\gamma^{\mu\nu\lambda}, \gamma^5\} = \gamma^{\mu\nu\lambda}\gamma^5 + \gamma^5\gamma^{\mu\nu\lambda}$$

$$= \gamma^{\mu\nu\lambda}\gamma^5 - \gamma^\mu\gamma^{5\nu\lambda} + \gamma^\mu\gamma^{5\nu\lambda} + \gamma^5\gamma^{\mu\nu\lambda}$$

$$= \gamma^\mu(\gamma^{\nu\lambda}\gamma^5 - \gamma^5\gamma^{\nu\lambda}) + (\gamma^\mu\gamma^5 + \gamma^5\gamma^\mu)\gamma^{\nu\lambda}$$

$$= \gamma^\mu[\gamma^{\nu\lambda}, \gamma^5] + \{\gamma^\mu, \gamma^5\}\gamma^{\nu\lambda}$$

$$\therefore \{\gamma^{\mu\nu\lambda}, \gamma^5\} = 0$$

Exponential function

if $\gamma^2 = 1$ $\quad e^{x\gamma} = \cosh x + \gamma \sinh x \rightarrow$ hyperbolic rotation

if $\gamma^2 = -1$ $\quad e^{x\gamma} = \cos x + \gamma \sin x \rightarrow$ circular rotation

The clifford product is bilinear and for two vectors

$$a \cdot b = (ab + ba)2^{-1}, \qquad a \wedge b = (ab - ba)2^{-1}$$

We also define the

- Reverse $\quad \widetilde{ab} = \tilde{b}\tilde{a}, \quad \widetilde{\gamma^\mu} = \gamma^\mu$
- Grade involution $\quad a(A_n) = (-1)^n A_n$
- Clifford conjugate $\quad \hat{A} = a(\tilde{A}_n)$
- Hermitian conjugate $\quad (zA)^\dagger = z^* A^\dagger, \quad (\gamma^0)^\dagger = \gamma^0, \quad (\gamma^i)^\dagger = -\gamma^i$
- Dirac adjoint $\quad \bar{A} = \gamma^0 A^\dagger \gamma^0$

Inverse

For vectors $v^2 \in \mathbb{C} \Rightarrow v^{-1} = v^{-2} v$

For a general element $A^{-1} = (A\tilde{A})^{-1}\tilde{A}$

Spinors

$Spin(1,3)$ is the even subalgebra $Cl_{1,3}(\mathbb{C})$

they form a minimal left ideal if $S \in Spin$ and $A \in Cl$
$\Rightarrow SA \in Spin \quad \forall A$

As result they transform as $\psi' = S\psi$

Lorentz transforms

The Lorentz transforms have 6 generators (3 rotations and 3 boost) they are identify with the 6 bivectors and a general Lorentz transform is expresed as

$$\Lambda = e^{-\frac{\phi}{2}K - \frac{\theta}{2}J} \qquad K = K^{0i}\gamma_{0i} \quad , \quad J = J^{ij}\gamma_{ij}$$

ϕ is the rapidity and θ the angle of rotation.
B is the plane of the boost
K is the plane of rotation

// The article defines the transform with $e^{iJ\theta + iK\phi}$, as both J and K have a $-i$ in its definiton and the θ and ϕ without ½ is because it is for spinors and not vectors \\

Article

Wigner's work

Wigner recognised that elemental particles must transform as irreducible representation of the Poincare group.

let's start with $T(1,3)$.

Translation

A translation is specified by a vector $a = a^\mu \gamma_\mu$, the unitary representation of the group is $U(a) = e^{-i P \cdot a}$ P being the generators (momentum)

$$\Rightarrow \quad U(a) = e^{-iP\cdot a} = e^{-i\hat{x}_\mu a^\mu}$$

$$\Rightarrow \quad \psi' = U(a)\psi = e^{-i\hat{x}_\mu a^\mu}\psi$$

$U(a)$ represents a translation, as such if we consider $\psi(x)$ a small change δa is ψ and U is by Taylor

$$\Rightarrow \quad \psi'(x) = \psi(x - \delta a) \approx \psi(x) - \delta a_\mu \eta^{\mu\nu}\partial_\nu \psi(x)$$

$$\Rightarrow \quad U(\delta a) = 1 - i\delta a_\mu P^\mu$$

$$\Rightarrow \quad \underbrace{(1 - \delta a_\mu \eta^{\mu\nu}\partial_\nu)\psi(x)}_{\psi'} = \underbrace{(1 - i\delta a_\mu P^\mu)}_{U(\delta a)}\psi(x)$$

$$\Rightarrow \quad \boxed{P^\mu = i\partial_\mu}$$

Lorentz transform

After a Lorentz transform, the particles momentum change but we are describing the same particle and as

$(\Lambda x)^2 = \Lambda x x \tilde{\Lambda} = x^2 \Lambda \tilde{\Lambda} = x^2 \Rightarrow x^2$ is an invariant quantity we call it mass or really $x_\mu x^\mu = x^2 = m^2$

- $m^2 > 0$ massive particle
- $m^2 = 0$ massless particle
- $m^2 < 0$ Tachyons

Now consider a massive particle, its simplest possible momentum is $p = m\gamma_0$

Any spatial rotation leaves p unchanged

let $\Theta = e^{-i\frac{\theta}{2} J^{ij}\gamma_{ij}} = \cos(\theta/2) - J^{ij}\gamma_{ij}\sin(\theta/2)$

$$\Theta(p) = \Theta p \Theta^{-1} = e^{-i\frac{\theta}{2} J^{ij}\gamma_{ij}}\, p\, e^{i\frac{\theta}{2} J^{ij}\gamma_{ij}}$$

$$= (\cos(\theta/2) - J^{ij}\gamma_{ij}\sin\theta/2)\, m\gamma_0\, e^{i\frac{\theta}{2} J^{ij}\gamma_{ij}}$$

$$= (m\gamma_0\cos(\theta/2) - m J^{ij}\gamma_{ij0}\sin\theta/2)\, e^{i\frac{\theta}{2} J^{ij}\gamma_{ij}}$$

$$= m\gamma_0 (\cos(\theta/2) - J^{ij}\gamma_{ij}\sin\theta/2) e^{i\frac{\theta}{2} J^{ij}\gamma_{ij}}$$

$$= m\gamma_0\, e^{-i\frac{\theta}{2} J^{ij}\gamma_{ij}}\, e^{i\frac{\theta}{2} J^{ij}\gamma_{ij}}$$

$$= m\gamma_0 = p$$

The behaviour under SO(3) determines if the particle has

- spin 0 $\rightarrow$ $e^{i\frac{\theta}{2}J}\psi = \psi$ any rotation
- Spin 1/2 $\rightarrow$ $e^{i\pi J}\psi = -\psi$ 2π rotation
- Spin 1 $\rightarrow$ $e^{i\pi J}\psi = \psi$ 2π rotation

2. Eigenvalue Equation for Bispinors

As shown before $P_\mu = i\partial_\mu$, then for a scalar field

$$P_\mu \phi = k_\mu \phi \quad \Rightarrow \quad i\partial_\mu \phi = k_\mu \phi \qquad (1)$$

We can consider a constant element A

$$A^\mu \psi \rightarrow i\partial_\mu (A^\mu \psi) = iA^\mu \partial_\mu \psi$$

$$iA^\mu \partial_\mu \psi = A^\mu k_\mu \psi \qquad (2)$$

let $X^\mu = -iA^\mu$, $Y = A^\mu k_\mu$

$$\Rightarrow \left(-iA^\mu \partial_\mu + A^\mu k_\mu\right)\psi = 0$$

$$\left(X^\mu \partial_\mu + Y\right)\psi = 0 \qquad (3)$$

As said, this makes ψ to transform as a irreps of $T(1,3)$

We can define a parity operator γ^5 to determine its chirality

- If $\gamma^5 \psi = \psi$ is Right-handed
- If $\gamma^5 \psi = -\psi$ is Left-handed

$\Rightarrow \quad \gamma^5 \psi_R = \psi_R \quad \Rightarrow \quad (1-\gamma^5)\psi_R = 0$

$\gamma^5 \psi_L = -\psi_L \quad \Rightarrow \quad (1+\gamma^5)\psi_L = 0$

we define $P_L = 2^{-1}(1-\gamma^5)$ and $P_R = 2^{-1}(1+\gamma^5)$
this way

- $P_L^2 = P_L$ as $2^{-2}(1-\gamma^5)(1-\gamma^5) = 2^{-2}(1-\gamma^5-\gamma^5+1)$
$= 2^{-1}(1-\gamma^5)$
- $P_R^2 = P_R$ as $2^{-2}(1+\gamma^5)(1+\gamma^5) = 2^{-2}(1+\gamma^5+\gamma^5+1)$
$= 2^{-1}(1+\gamma^5)$

- $P_L + P_R = 1$ as $2^{-1}(1-\gamma^5+1+\gamma^5) = 1$

For a general ψ we can have

$\psi = \chi_L + \chi_R = P_L\psi + P_R\psi$

- $\chi_L = P_L\psi, \quad \chi_R = P_R\psi$

$\psi = \chi_L + \chi_R$ 4

Given that L,R are different representations of ψ
we asked for

$\psi_L \to \psi_L' = R\psi_L$ As rotations $\bar{R} = R^{-1}$

$\psi_R \to \psi_R' = \bar{R}^{-1}\psi_R$ As boost $\bar{K} = K$

$$\Lambda = \Lambda_L + \Lambda_R = e^{i J\theta}\left(e^{-\phi K} + e^{\phi K}\right) \quad (5)$$

$$J = i J^{ij}\gamma_{ij}, \quad K = K^{0i}\gamma_{0i}$$

3. $\partial_\mu \to \partial_\nu = \Lambda^\mu_\nu \partial_\mu$

I have doubts on the process but it is correct

$$\Rightarrow \left(\left(\Lambda^{-1} X^\mu \Lambda\right)\Lambda^\nu_\mu \partial_\nu + \Lambda^{-1} Y \Lambda\right)\psi = 0 \quad (6)$$

$$\underbrace{\Lambda^\nu_\mu X^\mu = \Lambda X^\mu \Lambda^{-1}}_{\text{multivector}}, \quad \underbrace{Y = \Lambda Y \Lambda^{-1}}_{\text{Lorentz - scalar}} \quad (7)$$

lets suppose $X^\mu = x_1\gamma^\mu + x_2\gamma^\mu\gamma^5$ and $Y = y_1 + y_2\gamma^5$

$$\Rightarrow X^\mu = (x_1 - x_2\gamma^5)\gamma^\mu = 2^{-1}\left((x_1+x_2)(1-\gamma^5) + (x_1-x_2)(1+\gamma^5)\right)\gamma^\mu$$

$$= 2^{-1}\left(x_R(1-\gamma^5) + x_L(1+\gamma^5)\right)\gamma^\mu$$
$$= (x_R P_L + x_L P_R)\gamma^\mu$$

$$\therefore \boxed{X^\mu = (x_R P_L + x_L P_R)\gamma^\mu} \quad (8)$$

$$Y = y_1 + y_2\gamma^5 = 2^{-1}\left((y_1+y_2)(1+\gamma^5) + (y_1-y_2)(1-\gamma^5)\right)$$

$$= 2^{-1}\left(y_R(1+\gamma^5) + y_L(1-\gamma^5)\right)$$

$$= y_R P_R + y_L P_L$$

$$\therefore \boxed{Y = y_R P_R + y_L P_L} \quad (9)$$

$$\Rightarrow \quad (\chi^\mu \partial_\mu + Y)\psi = 0$$

$$\Rightarrow ((\chi_L P_R + \chi_R P_L)\gamma^\mu \partial_\mu + (\gamma_L P_L + \gamma_R P_R))\psi = 0 \qquad (12)$$

Given that $\quad P_R P_L = P_R P_L = 0 \quad$ and $\quad P_R^2 = P_R \;,\; P_L^2 = P_L$

$$P_R P_L = 2^{-2}(1+\gamma^5)(1-\gamma^5) = 2^{-2}(1-\gamma^5+\gamma^5-1) = 0$$
$$P_L P_R = 2^{-2}(1-\gamma^5)(1+\gamma^5) = 2^{-2}(1+\gamma^5-\gamma^5-1) = 0$$

$$\cdot \; (a_1 P_L + a_2 P_R)(b_1 P_L + b_2 P_R) = a_1 b_1 P_L + a_2 b_2 P_R$$

if $\quad a_1 b_1 = a_2 b_2 \quad \Rightarrow$

$$(a_1 P_L + a_2 P_R)(b_1 P_L + b_2 P_R) = a_1 b_1 (P_L + P_R) = a_1 b_1$$

$$\Rightarrow \quad (a_1 P_L + a_2 P_R)(a_1^{-1} P_L + b_2 (a_1 b_1)^{-1} P_R) = 1 \qquad b_2 = a_1 b_1 a_2^{-1}$$

$$(a_1 P_L + a_2 P_R)(a_1^{-1} P_L + a_2^{-1} P_R) = 1$$

$$\Rightarrow \quad \underline{(a_1 P_L + a_2 P_R)^{-1} = a_1^{-1} P_L + a_2^{-1} P_R}$$

$$\Rightarrow i(\chi_R^{-1} P_L + \chi_L^{-1} P_R)((\chi_L P_R + \chi_R P_L)\gamma^\mu \partial_\mu + (\gamma_L P_L + \gamma_R P_R))\psi = 0$$

$$\Rightarrow \quad i(\gamma^\mu \partial_\mu + (\gamma_L \chi_R^{-1} P_L + \gamma_R \chi_L P_R))\psi = 0 \qquad (13)$$

$$P_\pm \gamma^0 = 2^{-1}(\gamma^0 \pm \gamma^5 \gamma^0) = \gamma^0 2^{-1}(1 \mp \gamma^5) = \gamma^0 P_\mp$$

$$\Rightarrow \quad i\gamma^0 \partial_0 \psi + i(\gamma^x \partial_x + (m_1 P_L + m_2 P_R))\psi = 0$$

$$\Rightarrow \mathcal{H}\psi = i\partial_0\psi = -i\gamma^0\left(\gamma^x\partial_x + (m_1 P_L + m_2 P_R)\right)\psi$$

$$\mathcal{H}^2\psi = -\partial_0^2\psi = -\gamma^0\left(\gamma^x\partial_x + (m_1P_L + m_2P_R)\right)\gamma^0\left(\gamma^j\partial_j + (m_1P_L + m_2P_R)\right)\psi$$

$$= -(\gamma^0)^2\left(-\gamma^x\partial_x + (m_1P_R + m_2P_L)\right)\left(\gamma^j\partial_j + (m_1P_L + m_2P_R)\right)\psi$$

$$= -\left(\partial^x\partial_x + m_1m_2(P_R + P_L)\right)\psi$$

inconsistent with paper

$$\therefore \quad \mathcal{H}^2\psi = -\left(\partial^x\partial_x + \gamma_L\gamma_R(x_Lx_R)^{-1}\right)\psi \qquad (14)$$

$$\text{But} \quad \left(-\partial_0^2 + \partial^x\partial_x + \gamma_L\gamma_R(x_Lx_R)^{-1}\right)\psi = 0$$

$$\left(\Box - \gamma_L\gamma_R(x_Lx_R)^{-1}\right)\psi = 0$$

$$\Rightarrow \quad m^2 = -\gamma_L\gamma_R(x_Lx_R)^{-1} \qquad (15) \qquad \Rightarrow \quad m = \pm i\left(\gamma_L\gamma_R(x_Lx_R)^{-1}\right)^{1/2}$$

$$i\left(\gamma^\mu\partial_\mu + \left(\gamma_L x_R^{-1}P_L + \gamma_R x_L P_R\right)\right)\psi = 0$$

$$\Rightarrow \left(i\gamma^\mu\partial_\mu + i(m_1P_L + m_2P_R)\right)\psi = 0$$

$$\Rightarrow \quad i(m_1P_L + m_2P_R) = 2^{-1}i\left((m_1+m_2) - (m_1-m_2)\gamma^5\right)$$

$$\text{let} \quad m\cos 2\alpha = -2^{-1}i(m_1+m_2) \equiv \mathcal{M} \qquad (18)$$

$$-im\sin 2\alpha = -2^{-1}i(m_1-m_2) \equiv \widetilde{\mathcal{M}} \qquad (19)$$

$$\Rightarrow \quad i(m_1P_L + m_2P_R) = -m\left(\cos 2\alpha - i\sin 2\alpha\,\gamma^5\right) = \underline{-me^{-i2\alpha\gamma^5}} \qquad (17)$$

$$m(\cos 2\alpha + i\sin 2\alpha) = me^{i2\alpha} = -im_2$$

$$\Rightarrow \quad e^{i2\alpha} = -im_2m^{-1} = \mp\gamma_Lx_R^{-1}\left(\gamma_L\gamma_R(x_Lx_R)^{-1}\right)^{-1/2}$$

$$\Rightarrow \quad e^{i2\alpha} = \mp \left(y_L x_L (x_R y_R)^{-1} \right)^{1/2}$$

$$i2\alpha = \ln\left(\mp \left(y_L x_L (x_R y_R)^{-1} \right)^{1/2} \right)$$

$$\Rightarrow \quad \underline{\alpha = -i 2^{-1} \ln\left(\mp \left(y_L x_2 (x_R y_R)^{-1} \right)^{1/2} \right)} \qquad 16$$

$$\Rightarrow \left(i\gamma^\mu \partial_\mu - m e^{-2i\alpha\gamma^5} \right)\psi = 0 \qquad 17$$

$$\Rightarrow \left(i\gamma^\mu \partial_\mu - (M + \tilde{M}\gamma^5) \right)\psi = 0 \qquad 19$$

Consider a chiral transform

$$\psi' = e^{i\beta\gamma^5}\psi \qquad 20$$

$$\bar{\psi}' = \overline{\left(e^{i\beta\gamma^5}\psi \right)} = \bar{\psi}\,\overline{\left(e^{i\beta\gamma^5} \right)} = \bar{\psi}\, e^{i\beta\gamma^5} \quad , \quad \overline{(i\gamma^5)} = (-i)(-\gamma^5)$$

The Legrian

$$\mathcal{L} = \underbrace{i\bar{\psi}\gamma^\mu\partial_\mu\psi}_{\text{Kinnetic}} - \underbrace{\bar{\psi}(M + \tilde{M}\gamma^5)\psi}_{\text{Mass terms}}$$

$$\mathcal{L}' = i\bar{\psi}e^{i\beta\gamma^5}\gamma^\mu\partial_\mu\left(e^{i\beta\gamma^5}\psi \right) - \bar{\psi}e^{i\beta\gamma^5}(M + \tilde{M}\gamma^5)e^{i\beta\gamma^5}\psi$$

As it is global $\partial_\mu e^{i\beta\gamma^5} = 0$

$$\Rightarrow \underline{i\bar{\psi}e^{i\beta\gamma^5}\gamma^\mu e^{i\beta\gamma^5}\partial_\mu\psi = i\bar{\psi}\gamma^\mu\partial_\mu\psi}$$

$$e^{i\beta\gamma^5}\gamma^\mu = (\cos\beta + i\sin\beta\gamma^5)\gamma^\mu = \gamma^\mu(\cos\beta - i\sin\beta\gamma^5)$$
$$= \gamma^\mu e^{-i\beta\gamma^5}$$

$$\gamma^5 = i\gamma^{0123}$$

$$\Rightarrow \bar{\Psi} e^{i\beta\gamma^5}(M + \tilde{M}\gamma^5) e^{i\beta\gamma^5}\psi = \bar{\psi}(M + \tilde{M}\gamma^5) e^{2i\beta\gamma^5}\psi$$

$$e^{i\beta\gamma^5}\gamma^5 = (\cos\beta + i\sin\beta\gamma^5)\gamma^5 = \gamma^5(\cos\beta + i\sin\beta\gamma^5) = \gamma^5 e^{i\beta\gamma^5}$$

$$\Rightarrow \mathcal{L}' = i\bar{\Psi}\gamma^\mu\partial_\mu\psi - \bar{\psi}(M + \tilde{M}\gamma^5) e^{2i\beta\gamma^5}\psi \qquad (22)$$

Here I belive the paper had a typo on (20)

$\psi' = e^{-i\beta\gamma^5}\psi$ should be the correct

$$\Rightarrow (M + \tilde{M}\gamma^5) e^{-2i\beta\gamma^5} = (M + \tilde{M}\gamma^5)(\cos 2\beta - i\sin 2\beta\gamma^5)$$

$$= (M\cos 2\beta - i\tilde{M}\sin 2\beta) + (\tilde{M}\cos 2\beta - i M\sin 2\beta)\gamma^5$$

$$= M' + i M'\gamma^5$$

$$\Rightarrow \begin{aligned} M' &= M\cos 2\beta - i\tilde{M}\sin 2\beta \\ i\tilde{M}' &= M\sin 2\beta + i\tilde{M}\cos 2\beta \end{aligned} \qquad (23)$$

5) Yukawa

Normaly we consider some scalar fields that couple minimaly

$$\mathcal{L}_Y = -\underbrace{\lambda_1\bar{\Psi}\phi_1\psi}_{\text{scalar}} - \underbrace{\lambda_2\bar{\Psi}\phi_2\gamma^5\psi}_{\text{Pseudo-scalar}} \qquad (24)$$

λ_i coupling constant, ϕ_i scalar field

We can consider that the field ϕ_i is its expected value plus some fluctuactions

$$\phi_i(x) = \langle \phi_i \rangle + h_i(x) \equiv \underbrace{2^{-1/2}}_{\text{Normalization}} v_i + h_i(x)$$

$$\Rightarrow \mathcal{L}_Y = \underbrace{-2^{-1/2}\left(\lambda_1 v_1 \bar{\psi}\psi + \lambda_2 v_2 \bar{\psi}\gamma^5\psi\right)}_{\text{Effective mass}} - \underbrace{\left(\lambda_1 h_1(x)\bar{\psi}\psi + \lambda_2 h_2(x)\bar{\psi}\gamma^5\psi\right)}_{\text{Interactions}}$$

$$\Rightarrow \mathcal{L}_Y \approx -2^{-1/2}\left(\lambda_1 v_1 \bar{\psi}\psi + \lambda_2 v_2 \bar{\psi}\gamma^5\psi\right) \quad (24)$$

$$\Rightarrow \mathcal{M} = 2^{-1/2}\lambda_1 v_1 \quad , \quad \tilde{\mathcal{M}} = 2^{-1/2}\lambda_2 v_2$$

$$\Rightarrow m\cos 2\alpha = 2^{-1/2}\lambda_1 v_1 \quad , \quad -im\sin(2\alpha) = 2^{-1/2}\lambda_2 v_2$$

$$\Rightarrow m^2(\cos^2 2\alpha + \sin^2 2\alpha) = 2^{-1}(\lambda_1^2 v_1^2 - \lambda_2^2 v_2^2)$$

$$\Rightarrow \underline{m^2 = 2^{-1}(\lambda_1^2 v_1^2 - \lambda_2^2 v_2^2)} \quad (25)$$

- $$m(\cos 2\alpha - i\sin 2\alpha) = 2^{-1} m(e^{i2\alpha} + e^{-i2\alpha} - e^{i2\alpha} + e^{-i2\alpha}) = me^{-i2\alpha}$$

$$\Rightarrow me^{-i2\alpha} = 2^{-1/2}(\lambda_1 v_1 + \lambda_2 v_2)$$

- $$m(\cos 2\alpha + i\sin 2\alpha) = 2^{-1} m(e^{i2\alpha} + e^{-i2\alpha} + e^{i2\alpha} - e^{-i2\alpha}) = me^{i2\alpha}$$

$$\Rightarrow me^{i2\alpha} = 2^{-1/2}(\lambda_1 v_1 - \lambda_2 v_2)$$

$$\Rightarrow e^{-i4\alpha} = (\lambda_1 v_1 + \lambda_2 v_2)(\lambda_1 v_1 - \lambda_2 v_2)^{-1}$$

$$\Rightarrow \underline{\alpha = i\,4^{-1}\ln\left((\lambda_1 v_1 + \lambda_2 v_2)(\lambda_1 v_1 - \lambda_2 v_2)^{-1}\right)} \quad (24)$$

5.2 As we haven't seen right neutrinos

$$P_L \nu = \nu \Rightarrow 2^{-1}(1-\gamma^5)\nu = \nu \Rightarrow \nu = -\gamma^5 \nu$$

$$\Rightarrow \mathcal{L}_Y \approx -2^{-1/2}\left(\lambda_1 v_1 \bar{\nu}\nu + \lambda_2 v_2 \bar{\nu}\gamma^5 \nu\right)$$

$$\approx -2^{-1/2}\left(\lambda_1 v_1 \bar{\nu}\nu - \lambda_2 v_2 \bar{\nu}\nu\right)$$

$$\Rightarrow \mathcal{L}_Y \approx -2^{-1/2}\,\bar{\nu}\left(\lambda_1 v_1 - \lambda_2 v_2\right)\nu \quad (27)$$

2^{-1} the text adds a 2^{-1} I dont get

but there is a bigger problem

If $\nu = 2^{-1}(1-\gamma^5)\nu$

$$\bar{\nu} = 2^{-1}\bar{\nu}(1+\gamma^5)$$

$$\bar{\nu}\nu = 2^{-1}\bar{\nu}(1+\gamma^5)\nu = 0 \Rightarrow \mathcal{L}_Y = 0$$

Which destroys all the argument for massive neutrinos

Ignoring that $m = 2^{-1/2}\left(\lambda_1 v_1 - \lambda_2 v_2\right)$

missing 2^{-1}

Update

$$\psi \to \psi' = e^{i\gamma^5\frac{\alpha}{2}}\psi$$

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - \bar{\psi}(M + \tilde{M}\gamma^5)\psi$$

$$\mathcal{L}' = i\bar{\psi}e^{i\gamma^5\frac{\alpha}{2}}\gamma^\mu\partial_\mu e^{i\gamma^5\frac{\alpha}{2}}\psi - \bar{\psi}e^{i\gamma^5\frac{\alpha}{2}}(M + \tilde{M}\gamma^5)e^{i\gamma^5\frac{\alpha}{2}}\psi$$

$$\Rightarrow \quad i\bar{\psi}e^{i\gamma^5\frac{\alpha}{2}}\gamma^\mu e^{i\gamma^5\frac{\alpha}{2}}\partial_\mu\psi = i\bar{\psi}\gamma^\mu e^{-i\gamma^5\frac{\alpha}{2}}e^{i\gamma^5\frac{\alpha}{2}}\partial_\mu\psi = \underline{i\bar{\psi}\gamma^\mu\partial_\mu\psi}$$

$$\Rightarrow \quad \mathcal{L}' = i\bar{\psi}\gamma^\mu\partial_\mu\psi - \bar{\psi}(M + \tilde{M}\gamma^5)e^{i\gamma^5\alpha}\psi$$

$$(M + \tilde{M}\gamma^5)(\cos\alpha + i\gamma^5\sin\alpha) = (M\cos\alpha + i\tilde{M}\sin\alpha) + (\tilde{M}\cos\alpha + iM\sin\alpha)\gamma^5$$

$$M' = M\cos\alpha + i\tilde{M}\sin\alpha$$

$$\tilde{M}' = \tilde{M}\cos\alpha + iM\sin\alpha$$

$$i\tilde{M}' = (-M\sin\alpha + i\tilde{M}\cos\alpha)$$