Spontaneous-Synchronized Compound Pendulums

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I. Description

A curious phenomenon occurs when multiple metronomes are released while attached to a rigid body. The metronomes may start out of sync, but eventually, they should all influence each other such that they will oscillate at the same frequency. This project aims to analyze the solution to the same problem when each pendulum is instead a double pendulum. Each pendulum could be represented by two compound rods of each of mass m_{ij} where i takes on all values through n pendulums and j takes on the value 1 or 2 to describe each link. These n double pendulums are rigidly attached to a communal rigid body of mass m_c which moves freely on two massless wheels. This problem will be analyzed and solved using the Euler-Lagrangian and DAE Methods.

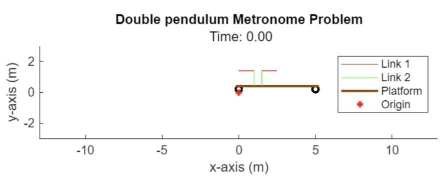

Fig. 1. A screenshot of a potential initial condition for the system.

II. Setup of Model

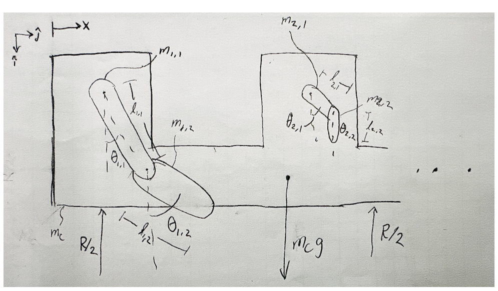

Fig. 2. An overall FBD for n pendulums

The above overall Free Body Diagram shows an example of 2 pendulums that could be extended n times. The coordinates that describe this system are the x coordinate of the platform, and $\theta_{1,1}, \theta_{1,2}, \theta_{2,1}, \theta_{2,2}, ..., \theta_{n,1}, \theta_{n,2}$. Each pendulum introduces two holonomic constraints that limit each link's center of mass to be a fixed length from the pin that it hangs from. There is an additional holonomic constraint that the cart cannot leave the ground or rotate, it can only move in the $\hat{j}$ direction.

III. Lagrangian Method

To start, n pendulums means the total number of generalized coordinates the system needs are $2n+1$. This is because we need two angles to describe each pendulum and some x value to describe the cart (and therefore the pendulum origins) motion.

A. Method

The Lagrangian solution stems from defining the kinetic and potential energy of the system:

$$\mathscr{L} = E_K - E_p$$

Once the Lagrangian is found the Equations of motion can be found by applying the equation:

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathscr{L}}{\partial q_i} = Q_i$$

In this case, Q is some $2n+1 \times 1$ sized zero vector due to the absence of non-conservative forces, external forces, and non-holonomic constraints. q will be the $2*n+1 \times 1$ generalized coordinate vector that represents each state.

$$q = \begin{bmatrix} x \\ \theta_{n,1} \\ \theta_{n,2} \\ \cdots \end{bmatrix}$$

B. Kinetic Energy

The kinetic energy is

$$E_k = \frac{1}{2}m_c\dot{x}^2 + \sum_{i=1}^{n}\frac{1}{2}m_{n,1}v_{n,1}^2 + \frac{1}{2}m_{n,2}v_{n,2}^2$$

for n pendulums. To find the $v_{n,1}^2$ (for the first link) and $v_{n,2}^2$ (for the second link) terms we need to define the position of each link as the following.

$$r_{n,1} = \frac{\ell_{n,1}}{2}\cos(\theta_{n,1})\hat{\imath} + \left(\frac{\ell_{n,1}}{2}\sin(\theta_{n,1}) + x\right)\hat{\jmath}$$

$$r_{n,2} = \left(\ell_{n,1}\cos(\theta_{n,1}) + \frac{\ell_{n,2}}{2}\cos(\theta_{n,2})\right)\hat{\imath} + \left(\ell_{n,1}\sin(\theta_{n,1}) + \frac{\ell_{n,2}}{2}\sin(\theta_{n,2}) + x\right)\hat{\jmath}$$

By defining this in matlab using an $n \times 4$ symvector dependent on time for theta and an x(t) symbol, we can let matlab differentiate and square the velocity:

$$v_{n,1}^2 = (\dot{x} + \frac{l_{n,1}}{2}\dot{\theta}_{n,1}\cos(\theta_{n,1})^2 + (\frac{1}{4}l_{n,1}^2\dot{\theta}_{n,1}\sin^2(\theta_{n,1}))^2$$

$$v_{n,2} = (\dot{x} + l_{n,1}\dot{\theta}_{n,1}\cos(\theta_{n,1}) + \frac{1}{2}l_{n,2}\dot{\theta}_{n,2}\cos(\theta n, 2))^2 + (\frac{1}{2}l_{n,2}\dot{\theta}_{n,2}\sin(\theta_{n,2}))^2$$

By then plugging that in and storing it into a $n \times 1$ E_k vector, I can save it to use in my Lagrangian.

C. Potential Energy

The potential energy of the cart is always 0 due to the holonomic constraint limiting it to the x axis. This means the potential energy is

$$E_p = \sum_{i=1}^{n} m_{n,1} g \frac{\ell_{n,1}}{2}\cos(\theta_{n,1}) + m_{n,1} g(\ell_{n,1}\cos(\theta_{n,1}) + \frac{\ell_{n,2}}{2}\cos(\theta_{n,2})).$$

This can similarly be placed in an $n \times 1$ vector E_p.

D. The Lagrangian

By iterating through the stored vectors (adding $E_k(n)$ and subtracting $E_p(n)$) we get the following Lagrangian for n pendulums:

$$\mathscr{L} = \frac{1}{2}m_c\dot{x}^2 + \sum_{i=1}^{n}(\frac{1}{2}m_{n,1}v_{n,1}^2 + \frac{1}{2}m_{n,2}v_{n,2}^2) - \sum_{i=1}^{n}(m_{n,1} g \frac{\ell_{n,1}}{2}\cos(\theta_{n,1}) + m_{n,1} g(\ell_{n,1}\cos(\theta_{n,1}) + \frac{\ell_{n,2}}{2}\cos(\theta_{n,2})))$$

I then used the Jacobian method to solve this using the following lines of matlab code:

```
EoM = J(J(L,q_dot),q_dot)*q_ddot +...
J(J(L,q_dot),q)*q_dot - J(L,q).' == Q

[A,F] = equationsToMatrix(EoM,q_ddot)
fsymbolic = [q_dot ; A\F]
```

Solving this way becomes increasingly complex and time intensive fairly quickly. To show the robustness of my system I solved the system for $n = \{1, 2, 3\}$. When solving the initial conditions can be defined as follows:

$$\begin{aligned} m_{n,j} &= 1\text{kg for } j = 1, 2 \\ g &= 9.81\frac{m}{s^2} \\ m_c &= 100\text{kg} \\ x &= 0\text{m} \\ \dot{x} &= 0\text{m/s} \\ \theta_{n,1} &= 90^\circ \text{ for odd n} \\ \theta_{n,1} &= -90^\circ \text{ for even n} \\ \dot{\theta}_n &= 0 \end{aligned}$$

Which will yield the results as scene in Figure 3

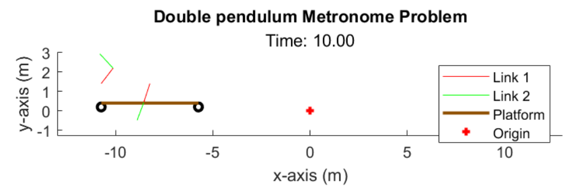

Fig. 3. The end of the n=2 animation. This is the end of the animation from the still frame in Figure1.

IV. DAE Method

Similarly to the lagrangian method, I'll start by defining the coordinates needed to define the solution. For n double pendulums I would need to solve for the following states:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{x}_{n,1} \\ \ddot{y}_{n,1} \\ \ddot{\theta}_{n,1} \\ \ddot{x}_{n,2} \\ \ddot{y}_{n,2} \\ \ddot{\theta}_{n,2} \\ R \\ R_{Bxn} \\ R_{Byn} \\ R_{Pxn} \\ R_{Pyn} \end{bmatrix}$$

This means my state matrix would be $10n + 3$ states long for n pendulums.

A. Free Body Diagrams

The following Free Body Diagrams can be used to describe the system and their LMB and AMB balances are under them:

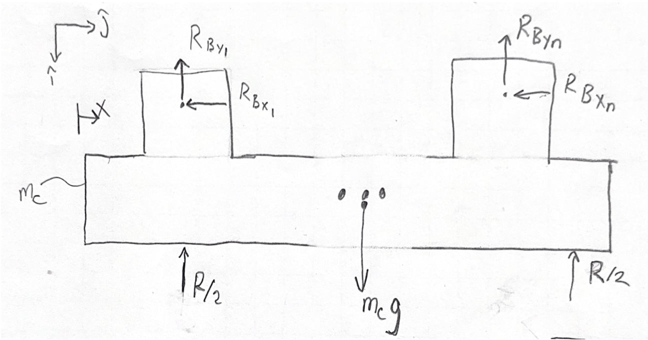

Fig. 4. The platform free body diagram (without pendulums).

LMB about inertial frame

$$\sum F = (m_c g - R - \sum_{n=i}^{n} R_{Byn})\hat{i} + (\sum_{n=1}^{n} -R_{Bxn})\hat{j} = m_c(\ddot{x}\hat{j} + \ddot{y}\hat{i})$$

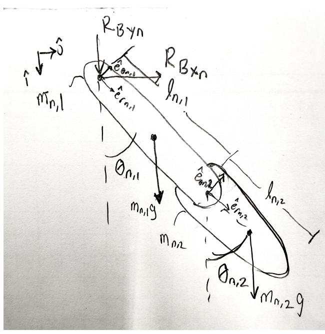

Fig. 5. A Free Body Diagram of a Pendulum attached to a rigid body

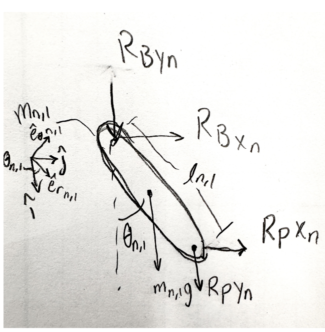

Fig. 6. A Free Body Diagram of the first Pendulum link

LMB about inertial frame for a single pendulum

$$\sum F = (R_{Pyn} + R_{Byn} + m_{n,1}g)\hat{i} + (R_{Bxn} + R_{Pxn})\hat{j} = m_{n,1}(\ddot{x}\hat{j} + \ddot{y}\hat{i})$$

AMB about center of mass

$$(-\frac{1}{2}\ell_{n,1}\cos(\theta_{n,1})R_{Bxn} + \frac{1}{2}\ell_{n,1}\sin(\theta_{n,1})R_{Byn} - \frac{1}{2}\ell_{n,1}\sin(\theta_{n,1})R_{Pyn} + \frac{1}{2}\ell_{n,1}\cos(\theta_{n,1})R_{Pxn})\hat{k} = (\frac{1}{3}m_{n,1}(\frac{\ell_{n,1}}{2})^2\ddot{\theta}_{n,1})\hat{k}$$

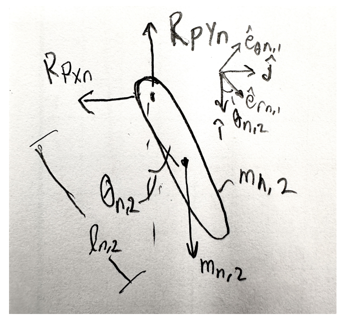

Fig. 7. A Free Body Diagram of the second Pendulum link

LMB about inertial frame for a single pendulum

$$\sum F = (-R_{Pyn} + m_{n,2}g)\hat{i} + (-R_{Pxn})\hat{j} = m_{n,2}(\ddot{x}\hat{j} + \ddot{y}\hat{i})$$

AMB about the center of mass

$$\sum M_{G_2} = (-\frac{1}{2}\ell_{n,2}\sin(\theta_{n,2})R_{Pyn} + \frac{1}{2}\ell_{n,2}\cos(\theta_{n,2})R_{Pxn})\hat{k} = (\frac{1}{3}m_{n,2}(\frac{\ell_{n,2}}{2})^2\ddot{\theta}_{n,2})\hat{k}$$

B. *Constraint Equations*

So far there are only $6n+2$ equations. The solution to DAE would not be complete without the constraint equations that complete the $10n+3$ equations needed to solve the problem.

Cart Constraints

$$\ddot{y} = 0$$

Pendulum Constraints

$$\ddot{y}_{n,1} = \frac{1}{2}\ell_{n,1}(\ddot{\theta}_{n,1}\cos(\theta_{n,1}) - \dot{\theta}_{n,1}^2\sin(\theta_{n,1}))$$

$$\ddot{x}_{n,1} = -\frac{1}{2}\ell_{n,1}(\ddot{\theta}_{n,1}\sin(\theta_{n,1}) + \dot{\theta}_{n,1}^2\cos(\theta_{n,1}))$$

$$\ddot{x}_{n,2} = -\frac{1}{2}\ell_{n,2}\ddot{\theta}_{n,2}\sin(\theta_{n,2}) - \frac{1}{2}\ell_2\dot{\theta}_{n,2}^2\cos(\theta_{n,2}) - \ell_{n,1}(\ddot{\theta}_{n,1}\sin(\theta_{n,1}) + \dot{\theta}_{n,1}^2\cos(\theta_{n,1}))$$

$$\ddot{y}_{n,2} = \frac{1}{2}\ell_{n,2}\ddot{\theta}_{n,2}\cos(\theta_{n,2}) - \frac{1}{2}\ell_2\dot{\theta}_{n,2}^2\sin(\theta_{n,2}) + \ell_{n,1}(\ddot{\theta}_{n,1}\cos(\theta_{n,1}) - \dot{\theta}_{n,1}^2\sin(\theta_{n,1}))$$

With these additions, we reach the requisite $10n+3$ equations and can put this into a matrix to solve. By writing a matrix A such that,

$$F = Au$$

where u is the second derivative of the maximal state matrix and F holds the remaining terms, we can perform the operation

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u = F\A
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in order to solve the equations of motion.

V. Discussion

The $n = 1$ pendulum is relatively uninteresting. Mostly it proved the code I was running worked correctly. When analyzing $n = 2$, however, I can analyze how the total energy of each object changes over time. If the metronomes match up over time, these should gradually equalize as time goes on. The simulations I ran couldn't simulate for long enough to see this sort of steady state behavior (my simulations ran for about 10 seconds). The most evident fact about the simulated dynamics of this system is that the system still presents chaotic motion just like a normal double pendulum and the coupling of the pendulums seems to bring a little more predictability to its motion. When the cart swings one way, we know that the pendulum most move the other to balance out the energy of the system.

VI. Comparison

The DAE method has many more states than the lagrangian solution and potentially takes longer to run given the massive matrices involved