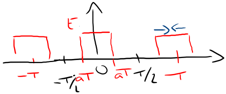
Exercice 1 TD1



(1) $$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-j\pi 2\frac{n}{T}t}\, dt.$$

$f(t)$ nulle sur $[-T/2 ; -aT]$ et $[aT ; T/2]$.

$$\hookrightarrow c_n = \frac{1}{T}\int_{-aT}^{aT} E\, e^{-2j\frac{n}{T}t\pi}\, dt = \frac{E}{T}\left[-\frac{e^{-2\pi j\frac{n}{T}t}}{2\pi j\frac{n}{T}}\right]_{-aT}^{aT}$$

$$= \frac{-E\cancel{T}}{2\pi j n\cancel{T}}\left(e^{-2\pi j\frac{n}{\cancel{T}}a\cancel{T}} - e^{2\pi j\frac{n}{\cancel{T}}a\cancel{T}}\right).$$

$$\hookrightarrow \sin(x) = \frac{e^{ix}-e^{-ix}}{2i} \quad \text{donc} \quad c_n = \frac{E}{\pi n}\left(\frac{e^{2\pi j n a}-e^{-2\pi j n a}}{2j}\right)$$

$$\text{donc} \quad c_n = \frac{E}{\pi n}\sin(2\pi n a) = 2aE\,\frac{\sin(2\pi n a)}{2\pi n a} = 2aE\,\mathrm{sinc}(2\pi n a).$$

c_0 ?

$$c_n = 2aE\,\mathrm{sinc}(2\pi n a) \quad \text{on prend } n=0 \rightarrow c_0 = 2aE\,\mathrm{sinc}(0) = \underbrace{2at}$$

$$c_0 = \frac{1}{T}\int_{-aT}^{aT} E\,dt = \frac{E}{T}\left(aT-(-aT)\right) = \underbrace{2aE}$$

$$x(t) = \sum_{-\infty}^{+\infty} c_n e^{2j\pi\frac{n}{T}t} = \sum_{-\infty}^{+\infty} \frac{E}{\pi n}\sin(2\pi n a)\, e^{2j\pi\frac{n}{T}t}$$

$$= \sum_{-\infty}^{-1} \frac{E}{\pi n}\sin(2\pi n a)\, e^{2j\pi\frac{n}{T}t} + \underbrace{2aE}\,\underset{1}{\underset{\uparrow}{e^{0}}} + \sum_{1}^{+\infty}\frac{E}{\pi n}\sin(2\pi n a)\, e^{2j\pi\frac{n}{T}t}$$

On pose $n'=-n$, pour la 1ère somme

$$x(t) = \sum_{1}^{+\infty}\frac{E}{\pi(-n')}\sin(2\pi(-n')a)\, e^{-2j\pi\frac{n'}{T}t} + 2aE + \sum_{1}^{+\infty}\frac{E}{\pi n}\sin(2\pi n a)\, e^{2j\pi\frac{n}{T}t}$$

$\sin(-x) = -\sin(x)$.

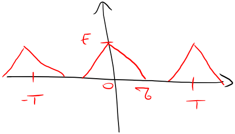
$$\hookrightarrow x(t) = \sum_{1}^{+\infty}\frac{E}{\pi n'}\sin(2\pi n' a)\, e^{-2j\pi\frac{n'}{T}t} + 2aE + \sum_{1}^{+\infty}\frac{E}{\pi n}\sin(2\pi n a)\, e^{2j\pi\frac{n}{T}t}$$

$n' = n$:

$$\hookrightarrow x(t) = 2aE + \sum_{1}^{+\infty}\frac{E}{\pi}\left(\frac{1}{n}\sin(2\pi n a)\left[e^{2\pi j\frac{n}{T}t} + e^{-2\pi j\frac{n}{T}t}\right]\right) \quad \Bigg|\; \cos(x) = \frac{(e^{ix}+e^{-ix})}{2}$$

$$\boxed{x(t) = 2aE + \frac{2E}{\pi}\sum_{1}^{+\infty}\frac{1}{n}\sin(2\pi n a)\cos\left(2\pi\frac{n}{T}t\right)}$$

Signal Triangulaire.



$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-2j\pi \frac{n}{T}t}\, dt.$$

② $[-T/2\,;\,-\tau] \to 0.$ $\quad\circ\ [-\tau\,;\,0] \to \frac{E}{\tau}t + E$

$[\tau\,;\,T/2] \to 0$ $\quad\circ\ [0\,;\,\tau] \to -\frac{E}{\tau}t + E.$

$$C_n = \underbrace{\frac{1}{T}\int_{-\tau}^{0}\frac{Et}{\tau}e^{-2j\pi\frac{n}{T}t}\,dt}_{A} + \frac{1}{T}\int_{-\tau}^{0} E\, e^{-2j\pi \frac{n}{T}t}\,dt - \underbrace{\frac{1}{T}\int_{0}^{\tau} \frac{E}{\tau}\, t\, e^{-2\pi j \frac{n}{T}t}\,dt}_{C} + \frac{1}{T}\int_{0}^{\tau} E\, e^{-2j\pi\frac{n}{T}t}\,dt$$

B.

B] $$\frac{E}{T}\int_{-\tau}^{\tau} e^{-2j\pi\frac{n}{T}t}\,dt = \frac{E}{T}\left(\frac{e^{-2\pi j \frac{n}{T}t}}{-2\pi j\frac{n}{T}}\right)_{-\tau}^{\tau} = -\frac{E}{2\pi j n}\left(e^{-2\pi j\frac{n}{T}\tau} - e^{2\pi j \frac{n}{T}\tau}\right)$$

$$= \frac{E}{\pi n}\sin\left(2\pi\frac{n}{T}\tau\right).$$

A + C)

$$\frac{E}{T}\int_{-\tau}^{0}\frac{t}{\tau}e^{-2\pi j\frac{N}{T}t}dt - \frac{E}{T}\int_{0}^{\tau}\frac{t}{\tau}e^{-2\pi j\frac{N}{T}t}dt$$ ON pose $t' = -t \to t' \in [0;\tau]$.

$\hookrightarrow t = -t'$

$dt = -dt'$ 1ère intégrale

$$\frac{E}{T}\int_{\tau}^{0}\frac{-t'}{\tau}e^{-2\pi j\frac{N}{T}(-t')}(-dt') - \frac{E}{T}\int_{0}^{\tau}\frac{t}{\tau}e^{-2\pi j\frac{N}{T}t}dt$$

$$\frac{E}{T}\int_{\tau}^{0}\frac{t'}{\tau}e^{2\pi j\frac{N}{T}t'}dt' - \frac{E}{T}\int_{0}^{\tau}\frac{t}{\tau}e^{-2\pi j\frac{N}{T}t}dt \to \int_a^b = -\int_b^a$$

primitive

dérivée

$$-\frac{E}{T}\int_{0}^{\tau}\frac{t'}{\tau}e^{2\pi\frac{N}{T}t'}dt' - \frac{E}{T}\int_{0}^{\tau}\frac{t}{\tau}e^{-2\pi j\frac{N}{T}t}dt$$

$$2\cos(x) = e^{ix} + e^{-ix}$$

$$\hookrightarrow -\frac{E}{T}\int_{0}^{\tau}\frac{t}{\tau}\left(e^{2\pi j\frac{N}{T}t} + e^{-2\pi j\frac{N}{T}t}\right)dt$$

$U = t \to U' = 1$

$$-\frac{2E}{T}\int_{0}^{\tau}\frac{t}{\tau}\cos\left(2\pi\frac{N}{T}t\right)dt \to \int UV' = [UV] - \int U'V$$

$V' = \cos(2\pi\frac{N}{T}t)$

$V = \dfrac{\sin\left(2\pi\frac{N}{T}t\right)}{2\pi\frac{N}{T}}$

$$\frac{-2E}{T\tau}\left(\left[t\,\frac{\sin(2\pi\frac{N}{T}t)}{2\pi\frac{N}{T}}\right]_0^{\tau} - \int_0^{\tau}\frac{\sin\left(2\pi\frac{N}{T}t\right)}{2\pi\frac{N}{T}}dt\right)$$

$$-\frac{2E}{T\tau}\left(\frac{\tau T}{2\pi N}\sin\left(2\pi\frac{N}{T}\tau\right)+\left[\frac{\cos\left(2\pi\frac{N}{T}t\right)}{4\pi^2\frac{N^2}{T^2}}\right]_0^{\tau}\right)$$

$$-\frac{2E}{\cancel{T}\cancel{\tau}}\frac{\cancel{\tau}\cancel{T}}{\cancel{2}\pi N}\sin\left(2\pi\frac{N}{T}\tau\right)-\frac{\cancel{2}E}{\cancel{T}\tau}\frac{1}{\underset{2}{\cancel{4}}\pi^2\frac{N^2}{T^{\cancel{2}}}}\left(\cos\left(2\pi\frac{N}{T}\tau\right)-1\right)$$

$$-\frac{E}{\pi N}\sin\left(2\pi\frac{N}{T}\tau\right)+\frac{ET}{2\pi^2N^2\tau}\left(1-\cos\left(2\pi\frac{N}{T}\tau\right)\right)$$

$$\hookrightarrow c_N=\cancel{\frac{E}{\pi N}\sin\left(2\pi\frac{N}{T}\tau\right)}-\cancel{\frac{E}{\pi N}\sin\left(2\pi\frac{N}{T}\tau\right)}+\frac{ET}{2\pi^2N^2\tau}\left(1-\cos\left(2\pi\frac{N}{T}\tau\right)\right)$$

$$\boxed{c_N=\frac{ET}{2\pi^2N^2\tau}\left(1-\cos\left(2\pi\frac{N}{T}\tau\right)\right)}\qquad 1-\cos(2x)=2\sin^2(x)$$

$$c_N=\frac{ET}{\cancel{2}\pi^2N^2\tau}\cancel{2}\sin^2\left(\pi\frac{N}{T}\tau\right)\qquad \sin(x)=x\,\mathrm{sinc}(x)$$

$$c_N=\frac{E\cancel{T}}{\cancel{\pi^2}\cancel{N^2}\cancel{\tau}}\frac{\cancel{\pi^2}\cancel{N^2}\tau^{\cancel{2}}}{T^{\cancel{2}}}\mathrm{sinc}^2\left(\pi\frac{N}{T}\tau\right)\quad \text{donc}\quad \boxed{c_N=\frac{E\tau}{T}\mathrm{sinc}^2\left(\pi\frac{N}{T}\tau\right)}$$

C_0 ?

$C_0 \to \mathrm{sinc}(0) = 1 \to C_0 = \frac{E\tau}{T}$.

$$C_N = \frac{E\tau}{T}\,\mathrm{sinc}^2\left(\pi \frac{N}{T}\tau\right)$$

$$x(t) = \sum_{-\infty}^{+\infty} c_N e^{2j\pi\frac{N}{T}t} = \sum_{-\infty}^{-1} C_N e^{2j\pi\frac{N}{T}t} + C_0 + \sum_{1}^{+\infty} C_N e^{2j\pi\frac{N}{T}t}$$

$N' = -N$

$$= C_0 + \sum_{+\infty}^{1} C_{-N'} e^{-2j\pi\frac{N'}{T}t} + \sum_{1}^{+\infty} C_N e^{2j\pi\frac{N}{T}t}$$

donc $C_N = C_{-N}$.

$$= C_0 + \sum_{1}^{+\infty} C_N \Big(\underbrace{e^{2j\pi\frac{N}{T}t} + e^{-2j\pi\frac{N}{T}t}}_{2\cos\left(2\pi\frac{N}{T}t\right)}\Big)$$

$$\boxed{\begin{gathered} \hookrightarrow x(t) = \frac{E\tau}{T} + \frac{2ET}{\cancel{2}\pi^2\tau}\sum_{1}^{+\infty}\frac{1}{N^2}\left(1-\cos\left(2\pi\frac{N}{T}t\right)\right)\cos\left(2\pi\frac{N}{T}t\right). \\ \text{ou bien} \\ x(t) = \frac{E\tau}{T} + 2\frac{E\tau}{T}\sum_{1}^{+\infty}\mathrm{sinc}^2\left(\pi\frac{N}{T}t\right)\cos\left(2\pi\frac{N}{T}t\right). \end{gathered}}$$

Exercice 2 :

signal carré avec $a = \frac{1}{4}$ et $E = 1$.

$$c_n = \frac{E}{\pi n} \sin(2\pi n a) \text{ donc ici } c_n = \frac{1}{\pi n} \sin\left(\frac{\pi n}{2}\right)$$

c_0 ?

$c_0 = 2aE$ donc ici $c_0 = \frac{1}{2}$.

Pour n pair, ?

$$C_{2p} = \frac{1}{\pi 2p} \sin\left(\frac{\pi}{2} 2p\right) = \frac{1}{2\pi p} \sin(\pi p) = 0.$$

Pour n impair ?

$$C_{2p+1} = \frac{1}{\pi(2p+1)} \sin\left(\frac{\pi}{2}(2p+1)\right) = \frac{1}{\pi(2p+1)} \sin\left(p\pi + \frac{\pi}{2}\right) = \frac{\cos(p\pi)}{\pi(2p+1)} = \frac{(-1)^p}{\pi(2p+1)}.$$

Donc c_n ne comporte qu'une composante continue et des harmoniques impaires.

Puissance moyenne:

$$P_m = \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\,dt = \frac{1}{T}\int_{-T/4}^{T/4} E^2\,dt = \frac{1}{T}\Big[t\Big]_{-T/4}^{T/4} = \frac{1}{T}\left(\frac{T}{4} - \left(-\frac{T}{4}\right)\right) = \frac{1}{2}.$$

Puissance:

$$P_x = \sum_{-\infty}^{+\infty} |C_n^2| = C_0^2 + \sum_{-\infty}^{-1} |C_n^2| + \sum_{1}^{+\infty} |C_n^2|.$$ comme il ne reste que les harmoniques impaires $C_{-n} = C_n$.

$n' = -n$ $$P_x = C_0^2 + \sum_{1}^{\infty} |C_{-n'}^2| + \sum_{1}^{+\infty} |C_n^2|.$$

$$= C_0^2 + 2\sum_{1}^{+\infty} |C_n^2|$$ $\rightarrow p = \frac{n-1}{2} \rightarrow n = 1$ alors $p = 0$.

$$= C_0^2 + 2\sum_{0}^{+\infty} |C_{2p+1}^2|$$

$$\frac{1}{4}+2\sum_{0}^{+\infty}\frac{(-1)^{2p}}{(2p+1)^2\pi^2} = \frac{1}{4}+\frac{2}{\pi^2}\underbrace{\sum_{0}^{+\infty}\frac{1}{(2p+1)^2}}_{\frac{\pi^2}{8}}$$

$(-1)^{2p}=1 \quad \forall p.$

donc $\boxed{P_x = \frac{1}{4}+\frac{2}{\pi^2}\frac{\pi^2}{8} = \frac{1}{2}.}$

(2)

$P_y \geqslant \frac{98}{100}P_x.$ avec $P_y = \frac{1}{4}+\frac{2}{\pi^2}\sum_{0}^{k}\frac{1}{(2p+1)^2}$ avec k le nombre d'harmonique voulu.

$$\sum_{0}^{k}\frac{1}{(2p+1)^2} \geqslant \frac{\pi^2}{2}\left(\frac{98}{100}P_x-\frac{1}{4}\right) \geqslant$$

$$\geqslant \frac{\pi^2}{2}\left(\frac{98}{100}\frac{1}{2}-\frac{1}{4}\right)$$

$$\sum_{0}^{k}\frac{1}{(2p+1)^2} \geqslant 1{,}183$$

Par le calcul, il faut trouver K, tel que.

$$S_K = \sum_{0}^{K} \frac{1}{(2p+1)^2} \geq 1,183.$$

$$\left.\begin{array}{l} S_0 = 1 \\ S_1 = 1,11 \\ S_2 = 1,15 \\ S_3 = 1,17 \\ S_4 = 1,18 \\ S_5 = 1,188 \end{array}\right\}$$

Donc $K \geq 5$ et il faut laisser passer $2K + 1 = 11$ harmoniques.

La fréquence de coupure du filtre doit donc vérifier $\rightarrow f_c \geq \frac{11}{T}$