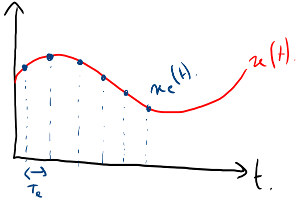
TD 5

Exercice 1 :

$$x_e(t) = \sum_{-\infty}^{+\infty} x(nT_e)\,\delta(t - nT_e)$$



TFD $\hookrightarrow$ $$X_e(f) = T_e \sum_{-\infty}^{+\infty} x(nT_e)\, e^{-j2\pi f n T_e}$$

$$= T_e \sum_{-\infty}^{+\infty} x(nT_e) \left(e^{j2\pi f T_e}\right)^{-n}$$

$$TZ(z) = \sum_{-\infty}^{+\infty} x(n)\, z^{-n}$$

$$\boxed{X_e(f) = T_e\, X(z)\Big|_{z = e^{j2\pi f T_e}}}$$

Exercice 2 :

① $U_k = \begin{cases} 1 & \text{si } k \geq 0. \\ 0 & \text{sinon.} \end{cases}$

$$TZ(u_k) = \sum_{-\infty}^{+\infty} u_n z^{-n} = \cancel{\sum_{-\infty}^{-1} u_n z^{-n}}^{\,0} + \sum_{0}^{+\infty} u_n z^{-n}.$$

$$TZ(u_k) = \sum_{0}^{+\infty} z^{-n}$$

$$\left.\begin{array}{ll} (1) & TZ(u_k) = 1 + z^{-1} + z^{-2} \cdots + z^{-n} \\ (2) & z^{-1} TZ(u_k) = z^{-1} + z^{-2} + \cdots + z^{-n} + z^{-(n+1)} \end{array}\right\} \begin{array}{l} (1)-(2) \\ (1 - z^{-1})\, TZ(u_k) = 1 - z^{-(n+1)}. \end{array}$$

$$\hookrightarrow TZ(u_k) = \lim_{n \to +\infty} \frac{1 - z^{-(n+1)}}{1 - z^{-1}}.$$

$$\boxed{TZ(u_k) = \frac{1}{1 - z^{-1}}}$$

⚠ dénominateur s'annule pour $z = 0$.

(2) $y_k = a^k x_k$.

$$TZ(y_k) = Y(z) = \sum_{-\infty}^{+\infty} a^n x_n z^{-n}$$

$$= \sum_{-\infty}^{+\infty} \left(\frac{1}{a}\right)^{-n} x_n z^{-n}$$

$$= \sum_{-\infty}^{+\infty} x_n \left(\frac{z}{a}\right)^{-n}$$

Formule

$$TZ(x) = X(z) = \sum_{-\infty}^{+\infty} x_n z^{-n}$$

$$\boxed{Y(z) = X\left(\frac{z}{a}\right)}$$ ou $TZ(a^k x_k) = X\left(\frac{z}{a}\right)$.

(3) $h_{1k} = U_k - U_{k-3}$.

Application linéaire $TZ(x+y) = TZ(x) + TZ(y)$.

$TZ \hookrightarrow H_1(z) = U(z) - TZ(U_{k-3})$.

Théorème du retard $TZ(x_{k-n_0}) = z^{-n_0} TZ(x_k)$.

On démontre le Théorème du retard.

$$TZ(x_{k-N_0}) = \sum_{-\infty}^{+\infty} x(N-N_0)\, z^{-N}$$

on pose $N' = N - N_0$.

$N = N' + N_0$.

$$= \sum_{-\infty}^{+\infty} x(N')\, z^{-(N'+N_0)} = \underbrace{\sum_{-\infty}^{+\infty} x(N')\, z^{-N'}}_{X(z)} z^{-N_0}$$

$$\boxed{\hookrightarrow TZ(x_{k-N_0}) = z^{-N_0} X(z)}$$ (Théorème du retard)

On revient à $H_1(z)$.

$$H_1(z) = U(z) - z^{-3} U(z) = \frac{1 - z^{-3}}{1 - z^{-1}}.$$

$$h_{2k} = a^k (u_k - u_{k-3}) = a^k h_{1k}$$

$$\rightarrow TZ(a^k x_k) = X\left(\frac{z}{a}\right).$$

$$TZ \hookrightarrow H_2(z) = H_1\left(\frac{z}{a}\right) = \frac{1 - (z/a)^{-3}}{1 - (z/a)^{-1}}.$$

Exercice 3 :

• $y_k = u_{k-N}$.

$$U(z) = \frac{1}{1 - z^{-1}}.$$

Théorème du retard

$$Y(z) = z^{-N} U(z) \rightarrow Y(z) = \frac{z^{-N}}{1 - z^{-1}}.$$

• $y_k = \left(\frac{1}{2}\right)^k u_k$.

$$\xrightarrow{TZ} Y(z) = U\left(\frac{z}{1/2}\right) = U(2z).$$

$$Y(z) = \frac{1}{1 - (2z)^{-1}}.$$

• $y_k = \left(\frac{1}{3}\right)^k u_{k-N}$. $\rightarrow y_k = \left(\frac{1}{3}\right)^N \underbrace{\left(\frac{1}{3}\right)^{k-N} u_{k-N}}_{v_{k-N}}$. on définit $v_k = \left(\frac{1}{3}\right)^k u_k \xrightarrow{TZ} V(z) = \frac{1}{1 - (3z)^{-1}}$.

$$Y(z) = \left(\frac{1}{3}\right)^N z^{-N} V(z).$$

$$Y(z) = \left(\frac{1}{3}\right)^N \frac{z^{-N}}{1-(3z)^{-1}} = \left(\frac{1}{3}\right) \frac{z^{-N}}{1-\frac{1}{3}z^{-1}}.$$

- $y_k = e^{-ak} u_k = \left(e^{-a}\right)^k u_k.$

TZ $\hookrightarrow$ $Y(z) = U\left(\frac{z}{e^{-a}}\right)$

$$= \frac{1}{1-\left(z/e^{-a}\right)^{-1}} \Longrightarrow Y(z) = \frac{1}{1-e^{-a}z^{-1}}.$$

- $y_k = -a_1 y_{k-1} - a_2 y_{k-2} + b u_{k-1}.$

TZ $\hookrightarrow$ $Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) + b z^{-1} U(z).$

$$Y(z)\left(1+a_1 z^{-1}+a_2 z^{-2}\right)=\frac{b z^{-1}}{1-z^{-1}}.$$

$$\boxed{Y(z)=\frac{b z^{-1}}{\left(1-z^{-1}\right)\left(1+a_1 z^{-1}+a_2 z^{-2}\right)}}$$

Exercice 4:

- $$X_1(z)=\frac{1}{z^3-\frac{1}{2}z^2}=\frac{1}{z^3\left(1-\frac{1}{2}z^{-1}\right)}=\frac{z^{-3}}{1-\frac{1}{2}z^{-1}}.$$

$$TZ^{-1} \hookrightarrow x_{1k}=\left(\frac{1}{2}\right)^{k-3} u_{k-3}$$

$\cdot\ X_2(z) = \dfrac{3z^2 - 6z + 2}{(z-1)(z-2)}$.

Décomposition en éléments simples :

$$\frac{3z^2 - 6z + 2}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)} + C.$$

$$\frac{3z^2 - 6z + 2}{(z-1)(z-2)} = \frac{Az - 2A + Bz - B + C(z-1)(z-2)}{(z-1)(z-2)}$$

$$\frac{3z^2 - 6z + 2}{(z-1)(z-2)} = \frac{(A+B)z - 2A - B + C(z^2 - 3z + 2)}{(z-1)(z-2)}$$

$$\begin{cases} C = 3 \\ A + B - 3C = -6. \\ -2A - B + 2C = 2 \end{cases} \rightarrow \begin{cases} C = 3 \\ A = 3 - B. \\ -6 + 2B - B + 6 = 2 \end{cases} \rightarrow \begin{cases} C = 3. \\ A = 1. \\ B = 2. \end{cases} \Rightarrow X_2(z) = \frac{1}{(z-1)} + \frac{2}{z-2} + 3.$$

$$X_2(z) = \frac{1}{z-1} + \frac{2}{z-2} + 3.$$

$$X_2(z) = \underbrace{\frac{z^{-1}}{1-z^{-1}}}_{u_{k-1}} + 2\,\frac{z^{-1}}{\underbrace{1-2z^{-1}}} + 3 \rightarrow TZ(\delta(k)) = 1.$$

$$\frac{\left(\frac{z}{2}\right)^{-1}}{1-\left(\frac{z}{2}\right)^{-1}} = U\left(\frac{z}{2}\right) \xrightarrow{TZ^{-1}} \underline{2^{k-1}\,u_{k-1}}$$

$$\xrightarrow{TZ^{-1}} \quad x_{2k} = u_{k-1} + 2^{k-1}u_{k-1} + 3\delta_k$$

$f(t)\cdot u(t)$	$F(p)$	$F(z)$
$\delta(t)$	1	1
$\delta(t-kT_e)$	$\exp(-kT_e p)$	z^{-k}
$u(t)$	$\frac{1}{p}$	$\frac{1}{1-z^{-1}}$
$t\cdot u(t)$	$\frac{1}{p^2}$	$\frac{T_e z^{-1}}{(1-z^{-1})^2}$

$f(t)\cdot\upsilon(t)$	$F(p)$	$F(z)$
$\delta(t)$	1	1
$\delta(t-kT_e)$	$\exp(-kT_e p)$	z^{-k}
$\upsilon(t)$	$\frac{1}{p}$	$\frac{1}{1-z^{-1}}$
$t\cdot\upsilon(t)$	$\frac{1}{p^2}$	$\frac{T_e z^{-1}}{\left(1-z^{-1}\right)^2}$
$\frac{t^2}{2!}\cdot\upsilon(t)$	$\frac{1}{p^3}$	$\frac{T_e^2 z^{-1}\left(1+z^{-1}\right)}{2\left(1-z^{-1}\right)^3}$
$e^{-at}\cdot\upsilon(t)$	$\frac{1}{p+a}$	$\frac{1}{1-e^{-aT_e}z^{-1}}$
$t\cdot e^{-at}\cdot\upsilon(t)$	$\frac{1}{(p+a)^2}$	$\frac{T_e\cdot e^{-aT_e}\cdot z^{-1}}{\left(1-e^{-aT_e}\cdot z^{-1}\right)^2}$
$\cos(\omega t)\cdot\upsilon(t)$	$\frac{p}{p^2+\omega^2}$	$\frac{1-\mathbf{cos}(\omega T_e)\cdot z^{-1}}{1-2\cdot\mathbf{cos}(\omega T_e)\cdot z^{-1}+z^{-2}}$
$\sin(\omega t)\cdot\upsilon(t)$	$\frac{\omega}{p^2+\omega^2}$	$\frac{\mathbf{sin}(\omega T_e)\cdot z^{-1}}{1-2\cdot\mathbf{cos}(\omega T_e)\cdot z^{-1}+z^{-2}}$
$e^{-at}\cdot\cos(\omega t)\cdot\upsilon(t)$	$\frac{p+a}{(p+a)^2+\omega^2}$	$\frac{z^2-e^{-aT_e}\cdot\cos(\omega T_e)\cdot z}{z^2-2\cdot e^{-aT_e}\cdot\cos(\omega T_e)\cdot z+e^{-2aT_e}}$
$e^{-at}\cdot\sin(\omega t)\cdot\upsilon(t)$	$\frac{\omega}{(p+a)^2+\omega^2}$	$\frac{e^{-aT_e}\cdot\sin(\omega T_e)\cdot z}{z^2-2\cdot e^{-aT_e}\cdot\cos(\omega T_e)\cdot z+e^{-2aT_e}}$
$e^{-at}\cdot f(t)\cdot\upsilon(t)$	$F(p+a)$	$F\left(e^{aT_e}z\right)$
$t^k\cdot\upsilon(t)$	$\frac{k!}{p^{k+1}}$	$-T_e z\cdot\frac{d}{dz}\mathbf{Z}\left(t^{k-1}\cdot\upsilon(t)\right)$
Théorème de la valeur finale	$\lim_{t\to\infty}\left(f(t)\right)=\lim_{p\to 0}\left(p\cdot F(p)\right)$	$\lim_{n\to\infty}\left(f(nT_e)\right)=\lim_{z\to 1}\left(\left(1-z^{-1}\right)\cdot F(z)\right)$