TD 2 :

$\rightarrow TF(g) = \int_{-\infty}^{+\infty} g(t)\, e^{-2\pi j f t}\, dt.$

$TF^{-1}(G) = \int_{-\infty}^{+\infty} G(f)\, e^{2\pi j f t}\, df.$

Exercice 1 :

① $g(t-t_0)$? $G(f-f_0)$?

$$TF(g(t-t_0)) = \int_{-\infty}^{+\infty} g(t-t_0)\, e^{-2\pi j f t}\, dt$$

on pose $\begin{cases} u = t - t_0 \rightarrow du = dt. \text{ on a } t = u + t_0. \\ u \in]-\infty ; +\infty[\end{cases}$

$$= \int_{-\infty}^{+\infty} g(u)\, e^{-2\pi f j (u+t_0)}\, du.$$

$$= e^{-2\pi j f t_0} \underbrace{\int_{-\infty}^{+\infty} g(u)\, e^{-2\pi f j u}\, du}_{TF(g).} \rightarrow TF(g(t-t_0)) = e^{-2\pi j f t_0}\, TF(g).$$

$$TF^{-1}(G(f-f_0)) = \int_{-\infty}^{+\infty} G(f-f_0)\, e^{2\pi j f t}\, df$$

on pose $u = f - f_0$.

$$= \int_{-\infty}^{+\infty} G(u)\, e^{2\pi j t (u+f_0)}\, du \Rightarrow TF^{-1}(G(f-f_0)) = e^{2\pi j f_0 t}\, TF^{-1}(G(f)).$$

2) $g^{(n)}(t)$? $G^{(n)}(f)$?

$$TF(g^{(n)}) = \int_{-\infty}^{+\infty} g^{(n)}(t) e^{-2\pi j f t} dt.$$

Intégration par partie : $\int uv' = [uv] - \int u'v$.

$u = e^{-2\pi j f t}$ $\rightarrow$ $u' = (-2\pi j f) e^{-2\pi j f t}$.

$v' = g^{(n)}(t)$.

$$TF(g^{(n)}(t)) = \underline{\left[e^{-2\pi j f t} g^{(n-1)}(t) \right]_{-\infty}^{+\infty}} - \underbrace{\int_{-\infty}^{+\infty} g^{(n-1)}(t)(-2\pi j f) e^{-2\pi j f t} dt}_{2\pi j f \, TF(g^{(n-1)}(t))}.$$

g a un domaine temporel fini.
donc ses dérivées également
$\hookrightarrow g(\pm\infty) = 0$ et $g^{(n)}(\pm\infty) = 0$.

$$TF(g^{(n)}(t)) = 2\pi j f \, TF(g^{(n-1)}(t)) = \dots = (2\pi f j)^n TF(g(t)).$$

$$\underbrace{2\pi j f}_{A} \int_{-\infty}^{+\infty} g^{n-1}(t) e^{-2\pi j f t} dt. \rightarrow u = e, \; v' = g^{(n-1)}.$$

$$A \underbrace{\left[e^{-2\pi j f t} g^{(n-2)} \right]}_{0} - A \int_{-\infty}^{+\infty} g^{n-2} (A) e^{\cdots} dt.$$

$$A^2 \underbrace{\int_{-\infty}^{+\infty} g^{n-2} e^{\cdots} dt}_{TF(g^{(n-2)})} ..$$

$$TF^{-1}(G^{(n)}(f)) = (-2\pi t j)^n TF^{-1}(G(f))$$

③ $\delta(t)$?

$$TF(\delta(t)) = \int_{-\infty}^{+\infty} \delta(t)\, e^{-2\pi j f t}\, dt$$ on sait que $\underline{\delta(t-t_0)\,x(t)} = \delta(t-t_0)\,x(t_0)$.

on pose $t_0 = 0$ et $x(t) = e^{-2\pi j f t}$ et $x(t_0) = 1$.

$$TF(\delta(t)) = \int_{-\infty}^{+\infty} \delta(t)\, dt$$ par définition $\int_{-\infty}^{+\infty} \delta(t)\, dt = 1$.

$\Rightarrow TF(\delta(t)) = 1 \rightarrow TF^{-1}(1) = \delta(t) = \int_{-\infty}^{+\infty} e^{2\pi j f t}\, df = \int_{-\infty}^{+\infty} e^{-2\pi j f t}\, df$.

$\hookrightarrow \underline{TF^{-1}(TF(\delta(t)))} = TF^{-1}(1)$.

car fonction paire.

Il en découle

$$TF(1) = \delta(f) = \int_{-\infty}^{+\infty} e^{2\pi j f t}\, dt = \int_{-\infty}^{+\infty} e^{-2\pi j f t}\, dt.$$

$$\Rightarrow \delta(f - f_0) = \int_{-\infty}^{+\infty} e^{2\pi j (f-f_0) t}\, dt = \int_{-\infty}^{+\infty} e^{-2\pi (f-f_0) t}\, dt.$$

sin ? cos ?

$$TF(\cos(2\pi f_0 t)) = \int_{-\infty}^{+\infty} \cos(2\pi f_0 t) e^{-2\pi j f t} dt$$

$$= \int_{-\infty}^{+\infty} \left(\frac{e^{2\pi j f_0 t} + e^{-2\pi j f_0 t}}{2} \right) e^{-2\pi f j t} dt$$

$$= \frac{1}{2} \int_{-\infty}^{+\infty} e^{2\pi j t(f_0 - f)} dt + \underbrace{\frac{1}{2} \int_{-\infty}^{+\infty} e^{-2\pi j t(f + f_0)} dt}$$

$$= \underbrace{\frac{1}{2} \int_{-\infty}^{+\infty} e^{-2\pi j t(f - f_0)} dt} + \frac{1}{2} \delta(f + f_0)$$

$$TF = \frac{1}{2} \delta(f - f_0) + \frac{1}{2} \delta(f + f_0).$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$$\updownarrow$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}.$$

$$TF(\sin) = \frac{1}{2j} \delta(f - f_0) - \frac{1}{2j} \delta(f + f_0).$$

$x * y$?

$$x * y = \int_{-\infty}^{+\infty} x(u)\, y(t-u)\, du$$

$$TF(x * y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x(u)\, y(t-u)\, e^{-2\pi j f t}\, du\, dt$$

$$= \int_{-\infty}^{+\infty} x(u) \underbrace{\int_{-\infty}^{+\infty} y(t-u)\, e^{-2\pi j f t}\, dt}_{e^{-2\pi j u t}\, TF(y)}\, du$$

$$= \underbrace{\int_{-\infty}^{+\infty} x(u)\, e^{-2\pi j u t}\, du}_{TF(x)}\; TF(y)$$

$$TF(x * y) = TF(x)\, TF(y)$$

$\underline{x\,y}$?

$$TF(x\,y) = \underbrace{TF(x) * TF(y)}\ ?$$

$$X(u) = \int_{t\,-\infty}^{+\infty} x(t)\, e^{-j2\pi u t}\, dt$$

$$Y(f-u) = \int_{t'\,-\infty}^{+\infty} y(t')\, e^{-j2\pi(f-u)t'}\, dt'$$

$$TF(x) * TF(y) = \int_{u\,-\infty}^{+\infty} X(u)\, Y(f-u)\, du$$

$$= \int_{u\,-\infty}^{+\infty} \left(\int_{t\,-\infty}^{+\infty} x(t)\, e^{-j2\pi u t}\, dt \right) \int_{t'\,-\infty}^{+\infty} y(t')\, e^{-j2\pi(f-u)t'}\, dt'\, du$$

$$= \int_{t\,-\infty}^{+\infty} \int_{t'\,-\infty}^{+\infty} \int_{u\,-\infty}^{+\infty} x(t)\, y(t')\, e^{-j2\pi u t}\, e^{-j2\pi(f-u)t'}\, du\, dt'\, dt$$

$$\rightarrow = \int_{t\,-\infty}^{+\infty} \int_{t'\,-\infty}^{+\infty} \int_{u\,-\infty}^{+\infty} x(t)\, y(t')\, e^{+j2\pi u(t'-t)}\, e^{-j2\pi f t'}\, du\, dt'\, dt \longleftarrow \delta(t-t')$$

$$= \int_{t\,-\infty}^{+\infty} \int_{t'\,-\infty}^{+\infty} x(t)\, y(t')\, e^{-j2\pi f t'} \int_{u\,-\infty}^{+\infty} e^{+j2\pi u(t'-t)}\, du\, dt'\, dt$$

$$= \int_{-\infty}^{+\infty}\!\!\!\!\!\!\!_{t}\quad \int_{-\infty}^{+\infty}\!\!\!\!\!\!\!_{t'}\quad x(t)\, y(t')\, \delta(t-t')\, e^{-2\pi j f t'}\, dt'\, dt$$

$$\delta(t-t_0)\, x(t) = \delta(t-t_0)\, x(t_0)$$

$$\hookrightarrow \delta(t-t')\, x(t) = \delta(t-t')\, x(t')$$

$$= \int_{-\infty}^{+\infty}\!\!\!\!\!\!\!_{t}\quad \int_{-\infty}^{+\infty}\!\!\!\!\!\!\!_{t'}\quad x(t')\, y(t')\, \delta(t-t')\, e^{-2\pi j f t'}\, dt'\, dt$$

$$= \underbrace{\left(\int_{-\infty}^{+\infty}\!\!\!\!\!\!\!_{t}\quad \delta(t-t')\, dt\right)}_{1} \int_{-\infty}^{+\infty}\!\!\!\!\!\!\!_{t'}\quad x(t')\, y(t')\, e^{-2\pi j f t'}\, dt'$$

$$TF(x) * TF(y) = \int_{-\infty}^{+\infty} \underbrace{x(t')\, y(t')}_{g \,=\, x y} e^{-2\pi j f t'}\, dt'$$

$$TF(x) * TF(y) = TF(x y)$$

$\Pi_{\frac{T}{2}}$?

$$\Pi_{\frac{T}{2}} = \begin{cases} 0 \text{ si } t < -\frac{T}{2} \text{ ou } t > T/2 \\ 1 \text{ sinon} \end{cases}$$

$$TF(\Pi_{\frac{T}{2}}) = \int_{-\infty}^{+\infty} \Pi_{T/2}\, e^{-2\pi j f t}\, dt$$

$$= \int_{-T/2}^{T/2} e^{-2\pi j f t}\, dt$$

$$= \left[\frac{-e^{-2\pi j f t}}{2\pi j f} \right]_{-T/2}^{T/2} = \frac{-1}{2\pi j f} \left(e^{-\pi j f T} - e^{\pi j f T} \right)$$

$$= \frac{1}{\pi f} \frac{e^{\pi j f T} - e^{-\pi j f T}}{2j}$$

$$TF(\Pi_{T/2}) = \frac{1}{\pi f} \sin(\pi f T)$$

$Ш_T(t)$?

On exprime le peigne de dirac comme une somme infinie de dirac.

$$Ш_T(t) = \sum_{-\infty}^{+\infty} \delta(t - nT).$$

On décompose en série de Fourier.

$$Ш_T = \sum_{-\infty}^{+\infty} C_n(f) e^{j2\pi f_n t} \quad \text{avec} \quad C_n(f) = \frac{1}{T} \int_{-T/2}^{T/2} \sum_{-\infty}^{+\infty} \delta(t - nT) e^{-j2\pi f_n t} dt$$

$$C_n(f) = \frac{1}{T} \sum_{-\infty}^{+\infty} \int_{-T/2}^{T/2} \delta(t - nT) e^{-j2\pi f_n t} dt.$$

Sur $\left[\frac{-T}{2} ; \frac{T}{2}\right]$, seul $\delta(t)$ est non nul, pour tout les autres n, on aura 0.

$\hookrightarrow C_n(f) = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t - 0 \times T) e^{-j2\pi f_n t} dt$ on applique $\delta(t - t_0) x(t) = \delta(t - t_0) x(t_0)$.

$$= \frac{1}{T} \int_{-T/2}^{T/2} \delta(t - 0) e^{-j2\pi f_n \times 0} dt = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) dt = \frac{1}{T} \quad \text{donc } C_n(Ш) = \frac{1}{T}$$

On utilise la série de Fourier pour calculer la TF.

$$TF(Ш) = \int_{-\infty}^{+\infty} \left(\sum_{-\infty}^{+\infty} C_N e^{j2\pi f_N t} \right) e^{-2\pi j f t} \, dt$$

$$= \int_{-\infty}^{+\infty} \sum_{-\infty}^{+\infty} \frac{1}{T} e^{j2\pi f_N t} e^{-2\pi j f t} \, dt. \quad \text{avec } f_N = \frac{N}{T}$$

$$= \frac{1}{T} \sum_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{2\pi j t \left(f - \frac{N}{T}\right)} \, dt.$$

On reconnait $\displaystyle\int_{-\infty}^{+\infty} e^{2\pi j t \left(f - \frac{N}{T}\right)} \, dt = \delta\left(f - \frac{N}{T}\right).$

donc $\displaystyle TF(Ш_T) = \frac{1}{T} \sum_{-\infty}^{+\infty} \delta\left(f - \frac{N}{T}\right).$ or $\displaystyle\sum_{-\infty}^{+\infty} \delta\left(f - \frac{N}{T}\right) = Ш_{1/T}$ peigne de période $\frac{1}{T}$.

$$TF(Ш_T) = \frac{1}{T} Ш_{1/T}.$$

Λ_T ? fonction triangle

$$\Lambda_T = \begin{cases} 0 \text{ si } t < -T \text{ et si } t > T. \\ 1 + \frac{t}{T} \text{ si } -T < t < 0. \\ 1 - \frac{t}{T} \text{ si } 0 < t < T. \end{cases}$$

$$TF(\Lambda_T) = \int_{-\infty}^{+\infty} \Lambda_T e^{-2\pi j t f} dt = \int_{-T}^{0} \left(1 + \frac{t}{T}\right) e^{-2\pi j f t} dt + \int_{0}^{T} \left(1 - \frac{t}{T}\right) e^{-2\pi j f t} dt.$$

2 integrations par partie.

$$= \left[\left(1 + \frac{t}{T}\right) \frac{e^{-2\pi j f t}}{-2\pi j f}\right]_{-T}^{0} - \int_{-T}^{0} \frac{1}{T} \frac{e^{-2\pi j f t}}{(-2\pi j f)} dt + \left[\left(1 - \frac{t}{T}\right) \frac{e^{-2\pi j f t}}{-2\pi j f}\right]_{0}^{T} - \int_{0}^{T} \frac{1}{T} \frac{e^{-2\pi j f t}}{(-2\pi j f)} dt.$$

$\left(1 + \frac{t}{T}\right)$ et $\left(1 - \frac{t}{T}\right)$ non nulles uniquement si $t = 0$.

$$TF(\Lambda_T) = \underbrace{\left(\frac{1}{-2\pi j f} - 0\right) + \left(0 - \frac{1}{-2\pi j f}\right)} - \frac{1}{T} \left[\frac{e^{-2\pi j f t}}{(2\pi j f)^2}\right]_{-T}^{0} - \frac{1}{T} \left[\frac{e^{-2\pi j f t}}{(2\pi f j)^2}\right]_{0}^{T}$$

$$= \quad 0 \quad - \frac{1}{T} \left(\frac{1 - e^{+2\pi j f T}}{(2\pi j f)^2}\right) - \frac{1}{T} \left(\frac{e^{-2\pi j f T} - 1}{(2\pi j f)^2}\right)^2$$

$$TF(\Lambda_T) = \frac{1}{T}\,\frac{1}{(2j\pi f)^2}\left(-2 + \underbrace{e^{j2\pi fT} + e^{-j2\pi fT}}_{2\cos(2\pi fT)}\right).$$

$$= \frac{1}{T}\,\frac{1}{(2\pi j f)^2}\left(-2 + 2\cos(2\pi fT)\right)$$

$$= \frac{2}{T}\,\frac{1}{(2\pi j f)^2}\left(-1 + \cos(2\pi fT)\right)$$ or $1 - \cos(2x) = 2\sin^2(x)$ et $j^2 = -1$.

donc

$$TF(\Lambda_T) = \frac{2}{T}\,\frac{1}{(2\pi f)^2}\left(1 - \cos(2\pi fT)\right)$$

$$= \frac{4}{T}\,\frac{1}{(2\pi f)^2}\sin^2(\pi fT).$$ On veut faire apparaître un sinus cardinal.

$$TF(\Lambda_T) = \frac{2}{T\,4\pi^2 f^2}\sin^2(\pi fT)$$

$$= \frac{T}{T^2\,2\pi^2 f^2}\sin^2(\pi fT)$$ donc $TF(\Lambda_T) = T\,\mathrm{sinc}^2(\pi fT)$.

Exercice 2 :

$\Pi(f)$ bornée sur $[-f_0 ; f_0]$.

$$x_1(t) = A_0 m(t) \cos(2\pi F_0 t).$$

$$TF(x_1) = TF\left(A_0 \cos(2\pi F_0 t)\, m(t)\right) = TF\left(A_0 \cos(2\pi F_0 t)\right) * TF(m(t)).$$

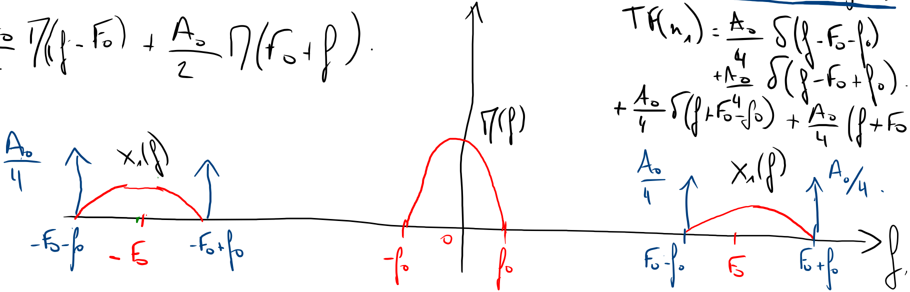
$$TF(x_1) = A_0 \int_{-\infty}^{+\infty} m(t) \cos(2\pi F_0 t)\, e^{-2\pi f t j}\, dt.$$

$$= \frac{A_0}{2} \int_{-\infty}^{+\infty} m(t)\, e^{-2\pi j (f - F_0) t}\, dt + \frac{A_0}{2} \int_{-\infty}^{+\infty} m(t)\, e^{-2\pi j (f + F_0) t}\, dt.$$

$$= \frac{A_0}{2} \Pi(f - F_0) + \frac{A_0}{2} \Pi(F_0 + f).$$

Si $m(t) = \cos(2\pi f_0 t)$.

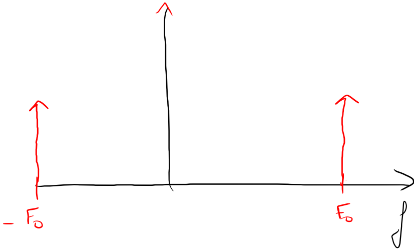
$$TF(x_1) = \frac{A_0}{4} \delta(f - F_0 - f_0) + \frac{A_0}{4} \delta(f - F_0 + f_0) + \frac{A_0}{4} \delta(f + F_0 - f_0) + \frac{A_0}{4} (f + F_0 + f_0)$$



$$x_2(t) = A_0(1+m)\cos(2\pi F_0 t).$$

$$TF(x_2) = \underbrace{A_0 \int_{-\infty}^{+\infty} \cos(2\pi F_0 t)\, e^{-2\pi j f t}\, dt} + \underbrace{A_0 \int_{-\infty}^{+\infty} m(t) \cos(2\pi F_0 t)\, e^{-2\pi j f t}\, dt}.$$

$$TF(x_2) = \frac{A_0}{2}\delta(f - F_0) + \frac{A_0}{2}\delta(f - F) \quad + \quad TF(x_1).$$



$$x_2(t) = A_0(1+m)\cos(2\pi F_0 t).$$

$$TF(x_2) = \underbrace{A_0\int_{-\infty}^{+\infty}\cos(2\pi F_0 t)e^{-2\pi j f t}\,dt} + \underbrace{A_0\int_{-\infty}^{+\infty} m(t)\cos(2\pi F_0 t)e^{-2\pi j f t}\,dt}.$$

$$TF(x_2) = \underline{\frac{A_0}{2}\delta(f-F_0) + \frac{A_0}{2}\delta(f-F)} + \underline{TF(x_1)}.$$

si $\underline{m(t) = \cos(2\pi f_0 t)}$.