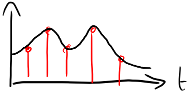
TD 3 : Echantillonnage.

Exercice 1 :

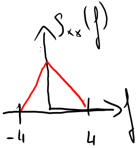
① signal périodique de période T $\xrightarrow{TF}$ raies à $f = \frac{N}{T}$ (harmoniques).

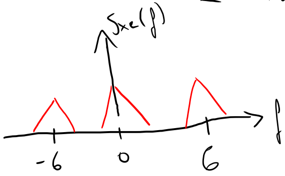
Spectre périodique $\xrightarrow{TF^{-1}}$ signal temporel avec des raies

ex : $x(t)$ échantillonné



②





③ F_e ?

$$x_e(t) = \sum_{-\infty}^{+\infty} x(t)\,\delta(t-nT_e) = \qquad \rightarrow \quad \delta(t-t_0)\,g(t) = \delta(t-t_0)\,g(t_0).$$

$$= \sum_{-\infty}^{+\infty} x(nT_e)\,\delta(t-nT_e) = x(t)\cdot Ш_{T_e}.$$

$x_e(t) = x(t) \cdot Ш_{Te}$

$\xrightarrow{TF}$ $X_e(f) = TF(x(t) \cdot Ш_{Te})$

$= TF(x(t)) * TF(Ш_{Te})$

$= X(f) * \frac{1}{T_e} Ш_{1/Te}(f)$

$= \frac{1}{Te} \sum_{-\infty}^{+\infty} X(f) * \delta\left(f - \frac{N}{Te}\right)$

$Xe(f) = \frac{1}{Te} \sum_{-\infty}^{+\infty} X\left(f - \frac{N}{Te}\right)$ donc $Fe = \frac{1}{Te}$.

Ici les raies de $X_e(f)$ sont à $-6, 0, 6 \ldots$ donc $Fe = 6$ kHz.

- Shannon impose $f_{max} \leq \frac{Fe}{2} \rightarrow f_{max} \leq 3$ kHz.

Or ici $f_{max} = 4$ kHz donc le signal est mal échantillonné.

Exercice 1 :

A. Echantillonnage Parfait

(1) $X^*(f) = \sum_{-\infty}^{+\infty} X(f - nF_e)$ $\rightarrow \delta(f-f_0) * G(f) = G(f-f_0)$.

$= \sum_{-\infty}^{+\infty} X(f) * \delta(f - nF_e)$.

$X^*(f) = X(f) * Ш_{F_e}$.

$\downarrow TF^{-1}$

$x^*(t) = TF^{-1}(X^*(f))$.

$= TF^{-1}(X(f) * Ш_{F_e})$.

$= TF^{-1}(X(f)) \, TF^{-1}(Ш_{F_e})$. et $TF^{-1}(Ш_{F_e}) = \frac{1}{F_e} Ш_{1/F_e}$

$= x(t) \, T_e \, Ш_{T_e}$. $= T_e Ш_{T_e}$.

x_n

$x^*(t) = T_e \sum_{-\infty}^{+\infty} x(t) \delta(t - nT_e) = T_e \sum_{-\infty}^{+\infty} x(nT_e) \delta(t - nT_e)$.

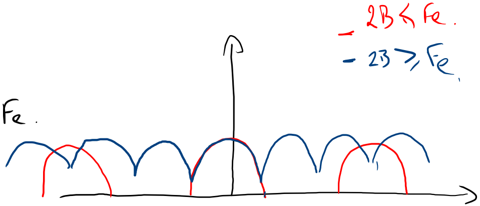
② $x_e(t) = x(t)\, Ш_{T_e}(t)$.

$$\xrightarrow{TF} X_e(f) = \frac{1}{T_e} \sum_{-\infty}^{+\infty} X(f - nF_e) \quad (\text{cf exo } 1).$$

$$= \frac{1}{T_e} X^*(f).$$

$|X(f)| = 0$ si $|f| > B \to f_{max} = B$.

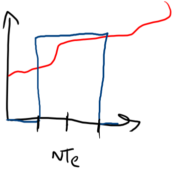
Shannon impose $B \leqslant \frac{F_e}{2} \to 2B \leqslant F_e$.



Echantillonnage réel :

① $x^*_\tau(nT_e) = \frac{1}{\tau} \int_{nT_e-\tau}^{nT_e+\tau} x(t)\,dt$.

$$= \frac{1}{\tau} \int_{-\infty}^{+\infty} x(t)\, h(t - nT_e)\, dt$$ avec $h(t)$ la fonction porte $\Pi_{\tau/2}(t)$.



$$x^*_\tau(t) = \sum_{-\infty}^{+\infty} x^*_\tau(nT_e)\,\delta(t-nT_e).$$

$$= \sum_{-\infty}^{+\infty} \left(\frac{1}{\tau}\int_{-\infty}^{+\infty} x(u)\,h(u-nT_e)\,du\right)\delta(t-nT_e).$$

$$= \frac{1}{\tau}\int_{-\infty}^{+\infty} x(u) \sum_{-\infty}^{+\infty} \underbrace{h(u-nT_e)\,\delta(t-nT_e)}_{h(u-t)\,\delta(t-nT_e)}\,du. \longrightarrow g(t)\,\delta(t-t_0) = g(t_0)\,\delta(t-t_0)$$

$$= \frac{1}{\tau}\int_{-\infty}^{+\infty} x(u)\,h(u-t) \underbrace{\sum_{-\infty}^{+\infty}\delta(t-nT_e)}_{Ш_{T_e}}\,du.$$

$$= \frac{1}{\tau}\int_{-\infty}^{+\infty} x(u)\,h(u-t)\,du\; Ш_{T_e}.$$

donc $x^*_\tau(t) = \underbrace{\frac{1}{\tau}x(t) * h(-t)}_{p(t)}\, Ш_{T_e}(t).$

Propriétés de $p(t)$ → Spectre.

$$P(f) = \frac{1}{\tau} TF\left(x(t) * h(-t)\right)$$

$$= \frac{1}{\tau} X(f)\, TF\left(h(-t)\right) \rightarrow \quad TF\left(h(t)\right) = \tau \operatorname{sinc}(\pi f \tau)$$

$$TF\left(h(-t)\right) = TF\left(h(t)\right)^*$$

$$= \tau \operatorname{sinc}(\pi f \tau)$$

$$\hookrightarrow P(f) = \frac{1}{\cancel{\tau}} X(f)\, \cancel{\tau} \operatorname{sinc}(\pi f \tau) = X(f) \operatorname{sinc}(\pi f \tau)$$

② Spectre $x^*_\tau(t)$?

$$X^*_\tau(f) = TF\left(p(t)\, Ш_{T_e}\right) = TF\left(p(t)\right) * TF\left(Ш_{T_e}\right)$$

$$= X(f) \operatorname{sinc}(\pi f \tau)\, \frac{1}{T_e} Ш_{1/T_e} = X(f) \operatorname{sinc}(\pi f \tau)\, \frac{1}{T_e} * \sum_{-\infty}^{+\infty} \delta(f - nF_e)$$

$$X^*_\tau(f) = \frac{1}{T_e} \sum_{-\infty}^{+\infty} X(f - nF_e) \operatorname{sinc}\left(\pi \tau (f - nF_e)\right)$$

③ $\tau \to 0 \rightarrow$ fonction porte tend vers un Dirac $\delta(t)$.

$$X^*_{\tau \to 0}(f) = \frac{1}{T_e} \sum_{-\infty}^{+\infty} X(f - nF_e)\, \mathrm{sinc}(\pi \tau (f - nF_e)).$$

$$\mathrm{sinc}(\pi \tau (f - nF_e)) \xrightarrow[\tau \to 0]{} \mathrm{sinc}(0) = 1.$$

$\hookrightarrow X^*_{\tau \to 0}(f) = \frac{1}{T_e} \sum_{-\infty}^{+\infty} X(f - nF_e)$. (cf question échantillonnage parfait).