HW Problems for Assignment 2 - Part II
Due 6:00 PM Tuesday, October 4, 2016

Note : this is the second half of HW 2. It covers material from Lecture 4 on Tuesday, September 27, 2016. The first half of HW 2 was posted on Tuesday, September 20th, and covered material from Lecture 3 on Tuesday, September 20th. Both halves are to be submitted by 6 PM on Tuesday, October 4th. There are 50 points possible for the questions covering Lecture 3, and 50 points possible for the questions covering Lecture 4.

4. (25 Points) VaR and Time Aggregation. In this exercise you will replicate the results given at the end of Lecture 3 on the time aggregation of VaR, where we produced the 10 day VaR for a hedged call option in the Black-Scholes model.

Recall, we have sold a call option with strike K and maturity T, and we are hedging the position by being long the underlying stock in accordance with the classical delta hedging formula from the Black-Scholes model. Specifically, the asset evolves according to

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t,$$

where μ and $\sigma > 0$ are given constants, and W is a Brownian motion under the physical measure $\mathbb{P}$. The risk free rate is a constant $r > 0$. Now, consider a time period $[t, t+\Delta]$ where Δ is one day. At time t we compute our position in the stock as

$$h_t = C_S^{BS}(t, S_t; r, \sigma, K, T),$$

where C^{BS} is the classical Black-Scholes formula and the subscript S above refers to differentiation with respect to S. Our portfolio value is thus

$$V_t = h_t S_t - C^{BS}(t, S_t; r, \sigma, K, T).$$

We keep our position in S constant over $[t, t+\Delta]$ so that at the end of the day our portfolio value is

$$V_{t+\Delta} = h_t S_{t+\Delta} - C^{BS}(t+\Delta, S_{t+\Delta}; r, \sigma, K, T).$$

The one period loss $L_{t+\Delta}$ is thus given by $V_t - V_{t+\Delta}$, and we can think of this as a function of the log return $X_{t+\Delta} = \log(S_{t+\Delta}) - \log(S_t)$ since $S_{t+\Delta} = S_t e^{X_{t+\Delta}}$. From here we can produce an estimate of the one time period VaR_α conditional on time t by sampling $X_{t+\Delta}$, computing the loss $L_{t+\Delta}$ and taking averages. This is an easy simulation since, given $\mathcal{F}_t$, $X_{t+\Delta} \sim (\mu - \sigma^2/2)\Delta + \sigma\sqrt{\Delta}Z$, $Z \sim N(0,1)$ under $\mathbb{P}$.

We are now interested in estimating the 10 day VaR: i.e. over the period $[t, t+10\Delta]$. To do this, we have to make some kind of decision on how we will re-balance our position in the stock. Since we are delta-hedging, we can automatically build in a re-balancing rule into our simulation.

Specifically, we re-balance as follows:

- Over $[t, t+\Delta]$ create the portfolio as above.
- Over the period $[\tau, \tau+\Delta]$, $\tau = t + (k-1)\Delta, k = 2, ..., 10$ we
  - · Re-balance our position in the stock at τ via $h_\tau = C_S^{BS}(\tau, S_\tau; r, \sigma, K, T)$.
  - · Put the residual money in the bank account. Here, the residual money is the difference between the portfolio value at the end of $[\tau - \Delta, \tau]$:
  $$V_\tau = h_{\tau-\Delta} S_\tau - C^{BS}(\tau, S_\tau; r, \sigma, K, T),$$
  and the amount after re-balancing at τ, which is
  $$\hat{V}_\tau = h_\tau S_\tau - C^{BS}(\tau, S_\tau; r, \sigma, K, T).$$
  The difference between these two values is thus
  $$Y_\tau = V_\tau - \hat{V}_\tau = (h_{\tau-\Delta} - h_\tau) S_\tau.$$
  We put Y_τ in the bank account at the beginning of $[\tau, \tau+\Delta]$.
  - · At the end of the period $\tau + \Delta$ our portfolio value is given by
  $$V_{\tau+\Delta} = h_\tau S_{\tau+\Delta} - C^{BS}(\tau+\Delta, S_{\tau+\Delta}; r, \sigma, K, T) + Y_\tau e^{r\Delta}.$$
  and hence our loss is
  $$L_{\tau+\Delta} = V_\tau - V_{\tau+\Delta}.$$

The above methodology will produce losses over $[t, t+10\Delta]$ which we can simulate by sampling the log returns $X_{t+k\Delta}, k = 1, ..., 10$.

Using the above methodology, use simulation to

(1) Estimate the one day VaR over $[t, t+\Delta]$.

(2) Estimate the ten day VaR over $[t, t+10\Delta]$ using the above re-balancing and re-investing rule.

For parameters, use $t = 0$, $\Delta = 1/250$, $T = .376$, $S_t = 56.47$, $K = 55$, $r = .84\%$, $\sigma = 20.66\%$ and $\mu = 16.89\%$. For your simulations, use $10,000$ runs. Compare the full 10 day VaR with the square root of time rule produced $\sqrt{10}\times$ the 1 day VaR. How close are they?

For error checking purposes, I am seeing a 10 day VaR at the 95% confidence of approximately .1107 and a $\sqrt{10}\times$ one day VaR of .1286.

5. (25 Points) Backtesting VaR. In this exercise, you will replicate the results given in Lecture 4 on the backtesting of VaR. Specifically, we will back-test three different methods for computing VaR for a portfolio which keeps a constant $1 in the S&P 500 index throughout time.

The historical S&P 500 index data is given in the file

"S&P_Daily_Log_Return_Data.csv".

Here, you will find 5 columns: the date, index value, log index value, log return, and the loss of the hypothetical portfolio. For this latter value note that for all t:

$$L_{t+\Delta} = l_{[t]}(X_{t+\Delta}) = -\left(e^{X_{t+\Delta}} - 1\right),$$

where $X_{t+\Delta}$ is the log return.

For the above data, produce a series of one-day VaRand exceedance estimates using

(a) The GARCH(1,1) method with EWMA means.
(b) The empirical distribution method.
(c) The normal variance/covariance method using EWMA to update the mean and variance.

Use the methodology described in class: see slides 13-19 in

"Value_at_Risk_II.pdf".

Note that for the GARCH and EWMA methods, you will need an initial estimate for $\hat{\mu}, \hat{\sigma}$ to perform the first update (which takes place at the 1000^{th} data point). To obtain these, it is fine to take the sample mean and sample variance for the first 999 data points.

Obtain the total number of exceedances for each method and compare them with what was obtained in class (see slide 19). Additionally, produce the $1 - \beta$ confidence interval and see if the number of exceedances falls within this range. Use parameter values $\alpha = .95$, $\beta = .05$, $m = 1000$ days of historical data, and $\lambda = .97$ EWMA value, to fit the models. **Note:** you might have to massage the data file in order to import it properly into whatever program you use to run the simulation. For example, to import into MatLab, I had to eliminate all the non-numeric fields (i.e. the header rows), and change the date column to display the numeric value for the date. Be mindful of this.