HW Problems for Assignment 2 - Part I
Due 6:00 PM Tuesday, October 4, 2016

Note : this is the first half of HW 2. It covers material from Lecture 3 on Tuesday, September 20, 2016. The second half of HW 2 will be posted on Tuesday, September 27th, and covers material from Lecture 4 on Tuesday, September 27th.. Both halves are to be submitted by 6 PM on Tuesday, October 4th. There are 50 points possible for the questions covering Lecture 3, and 50 points possible for the questions covering Lecture 4.

1. (15 Points) Forward Start Options, Risk Factors, and Loss Distributions.

Let S denote the price process for a stock. Assume we are at time $t \geq 0$ and let $t < T_S < T$ be given. A *forward-start option* is a contract entered at time t that pays $(S_T - S_{T_S})^+$ at time T. In other words, it is a European call except that the strike is set "at the money" at time T_S rather than being set at time t.

In this exercise, we work in the Black-Scholes model where S evolves according to

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t,$$

where $W = \{W_t\}_{t \leq T}$ is a Brownian motion under the physical measure $\mathbb{P}$. The interest rate $r > 0$ is constant, as well as $\mu, \sigma > 0$. According to the theory of risk neutral pricing, the price at time t of the forward-start option is

$$V_t = E^{\mathbb{Q}}\left[e^{-r(T-t)}\left(S_T - S_{T_S}\right)^+ \mid \mathcal{F}_t\right], \tag{0.1}$$

where $\mathbb{Q}$ is risk-neutral measure, under which S evolves as a Geometric Brownian motion with drift rate r and volatility σ.

Let $C^{BS}(t, x; r, \sigma, K, T)$ denote the price of a (regular) call option with strike K and expiry T, given that $S_t = x$ at time t. I.e.

$$C^{BS}(t, x; r, \sigma, K, T) = E^{\mathbb{Q}}\left[e^{-r(T-t)}\left(S_T - K\right)^+ \mid \mathcal{F}_t, S_t = x\right]. \tag{0.2}$$

(a) **(9 Points)** Show that

$$V_t = C^{BS}(T_S, S_t; r, \sigma, S_t, T).$$

In other words, the value at time t of the forward start option coincides with that of a regular call option which is at the money with strike S_t and time to expiry of $T - T_S$.

Hint: Starting with the generic formula for V_t in (0.1), use the method of conditioning and the properties of geometric Brownian motion to get the result. You may use without proof (but feel free to show it!) that for every positive numbers x, y and time $s < T$ we have

$$yC^{BS}(s, x; r, \sigma, K, T) = C^{BS}(s, yx; r, \sigma, yK, T). \tag{0.3}$$

(b) **(6 Points)** Given you result in part (a), we may view the value V_t as a function of the risk factor $X_t = \log(S_t)$. Given this, identify the first order approximation $L'_{t+\Delta}$ for the loss. You may assume that Δ is such that $t + \Delta$ is well-before T_S.

2. (10 Points) An Exact VaR Computation. The purpose of this exercise is to explicitly compute the VaR associated to holding a call option in the Black-Scholes model. As in Problem 1, we assume the stock price evolves according to

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t,$$

where W is a Brownian motion under the physical measure $\mathbb{P}$. The interest rate $r > 0$ is constant, as well as $\mu, \sigma > 0$. Also as in Problem 1, we denote by $C^{BS}(t, x; r, \sigma, K, T)$ the price for a call option expiring at time T with strike K, give that at the earlier time $t < T$, we have $S_t = x$.

Let Δ be such that $t + \Delta < T$. We are interested in the Value at Risk associated to the portfolio consisting of holding the call at time $t + \Delta$, conditional time t. I.e. we wish to compute

$$\mathrm{VaR}_\alpha\left(C^{BS}(t + \Delta, S_{t+\Delta}; r, \sigma, K, T)\right),$$

which, to be precise, is the smallest ℓ so that

$$\mathbb{P}\left[C^{BS}(t + \Delta, S_{t+\Delta}; r, \sigma, K, T) \leq \ell \;\middle|\; \mathcal{F}_t\right] \geq \alpha.$$

Here, the interest rate and volatility are constant so that at time $t + \Delta$ our portfolio value is the random quantity $C^{BS}(t+\Delta, S_{t+\Delta}; r, \sigma, K, T)$. Furthermore, we are interested in computing the value at risk under the physical, rather than risk neutral measure, as we live in the physical measure. Note lastly that as the above is a conditional expectation,

$$\mathrm{VaR}_\alpha\left(C^{BS}(t + \Delta, S_{t+\Delta}; r, \sigma, K, T)\right),$$

will be a random variable, however, one that is known to us at time t.

Provide an explicit formula for the value at risk in terms of the model parameter, the Black-Scholes call price and N^{-1}, the inverse of the standard normal cdf.

3. (25 Points) VaR for a Portfolio of Microsoft and Apple stocks. The data file "APPL_MSFT_Price_Data.xlsx" contains one year of historical prices for Microsoft (MSFT) and Apple (APPL). The data period is 8/31/15 – 8/31/16. Additionally, both the prices and market capitalizations are given for 9/1/16. In this exercise you will use this data to compute the Value at Risk for a market capitalization weighted portfolio of these two stocks. As in class, the risk factors are the log stock prices and the changes in risk factors are the log returns.

(a) **(10) Points** The first thing to do is estimate averages and covariances of the log returns. Do this two ways:
  (i) Take the sample mean vector and covariance matrix.
  (ii) Using an exponentially weighted moving average. Here, the procedure is as follows. Start with some initial value μ_0, Σ_0 for the mean vector and covariance matrix. You may use your results from part (*i*) to obtain these. Next, assume at time $t - \Delta$ we have our estimate for $\mu_{t-\Delta}, \Sigma_{t-\Delta}$. To get the new estimate use the recursive equations

$$\mu_t = \lambda \mu_{t-\Delta} + (1-\lambda) X_t$$
$$\Sigma_t = \lambda \Sigma_{t-\Delta} + (1-\lambda)(X_t - \mu_t)(X_t - \mu_t)^{\mathrm{T}}.$$

  Above, X_t is the log return over $[t-\Delta, t]$. Starting this updating procedure from you earliest date $t = \Delta = 9/1/15$ use the recursive equations to get an estimate for the mean vector and covariance matrix as of 9/1/16. For λ, use $\lambda = 0.97$.

(b) **(15 Points)** Now that we have our model parameters we may estimate the one day VaR of our market cap weighted portfolio. Perform the VaR estimation for a 95% confidence in three (actually five) ways
  (i) Using the empirical distribution of the log returns and the full loss operator $l_{[t])}$ as defined in class.
  (ii) Assuming the log returns are normally distributed, and using the full loss operator $l_{[t]}$. Here, the mean vector and covariance matrix are obtained from parts (i), (ii) above. Note: this will produce two VaR estimates. To obtain the VaR run a simulation sampling the normal r.v. N times, where $N = 10,000$.
  (iii) Assuming the log returns are normally distributed, and using the first order approximation to the loss operator $l'_{[t]}$. Again, produce answers for each of your μ, Σ values obtained above. Note also that here there is no need to run a simulation.

  How do the VaR estimates compare? In particular, is any one (or more than one) estimate different from the rest? If so, please explain why you think this is the case.