Probability (homework 2).

Problem 1

What does it mean for one event C to cause another event E — for example, smoking (C) to cause cancer (E)? There is a long history in philosophy, statistics, and the sciences of trying to clearly analyze the concept of a cause. One tradition says that causes raise the probability of their effects; we may write this symbolically is

$$P(E|C) > P(E). \qquad (1)$$

Part a)

Does equation (1) imply that $P(C|E) > P(C)$? If so, prove it. If not, give a counter example.

Answer:

In fact, we shall prove that both expressions are equivalents. This means that:

$$\boxed{P(E|C) > P(E) \longleftrightarrow P(C|E) > P(C)}$$

and the proof of this claim is given below:

$$P(E|C) > P(E) \longleftrightarrow \frac{P(E \cap C)}{P(C)} > P(E) \longleftrightarrow$$

$$\longleftrightarrow P(E \cap C) > P(E) \cdot P(C) \longleftrightarrow \frac{P(E \cap C)}{P(E)} > P(C)$$

$\longleftrightarrow \quad p(C|E) > p(C).$

Therefore, the previous argument proves that both expressions are equivalents.

$\blacksquare$

Note (previous proof).

It is possible that somebody thinks that in this process $p(E)=0$ or $p(C)=0$, but if any of the previous equations is true, is easy to prove that $p(E)\neq 0$ and $p(C)\neq 0$.

Part b) ¶

Another way to formulate a probabilistic theory of causation is to say that

$$P(E|C) > P(E|C^C). \tag{2}$$

Show that equation (1) implies equation (2).

Answer:

Similarly as the literal (a), we shall prove that both expressions are equivalents. This means that

$$\boxed{p(E|C) > p(E|C^c) \longleftrightarrow p(E|C) > p(E)}$$

and the proof of this claim is given below.

$$p(E|C) > p(E|C^c) \longleftrightarrow \frac{p(E\cap C)}{p(C)} > \frac{p(E\cap C^c)}{p(C^c)}$$

$$\longleftrightarrow \frac{p(C^c)\cdot p(E\cap C)}{p(C)} > p(E\cap C^c) = p(E) - p(E\cap C)$$

$$\longleftrightarrow \left(\frac{1-p(C)}{p(C)}\right)\cdot p(E\cap C) + p(E\cap C) > p(E)$$

$$\longleftrightarrow \left(\frac{1-p(C)+p(C)}{p(C)}\right)\cdot p(E\cap C) > p(E)$$

$$\longleftrightarrow \frac{p(E\cap C)}{p(C)} > p(E) \longleftrightarrow p(E|C) > p(E).$$

Thus, the previous argument proves that both expressions are equivalents.

Note (previous proof).

It is possible that somebody thinks that in this process $p(c)=0$ or $p(c^c)=0$, but if any of the previous equations is true, is easy to prove that $0<p(c), p(c^c)<1$.

Part c)

Let C be the drop in the level of mercury in a barometer and let E be a storm. Briefly describe why this leads to a problem with using equation (1) (or equation (2)) as a theory of causation.

Answer:

This is because there exist a relationship between storms and the barometric pressure, and this relationship is given below:

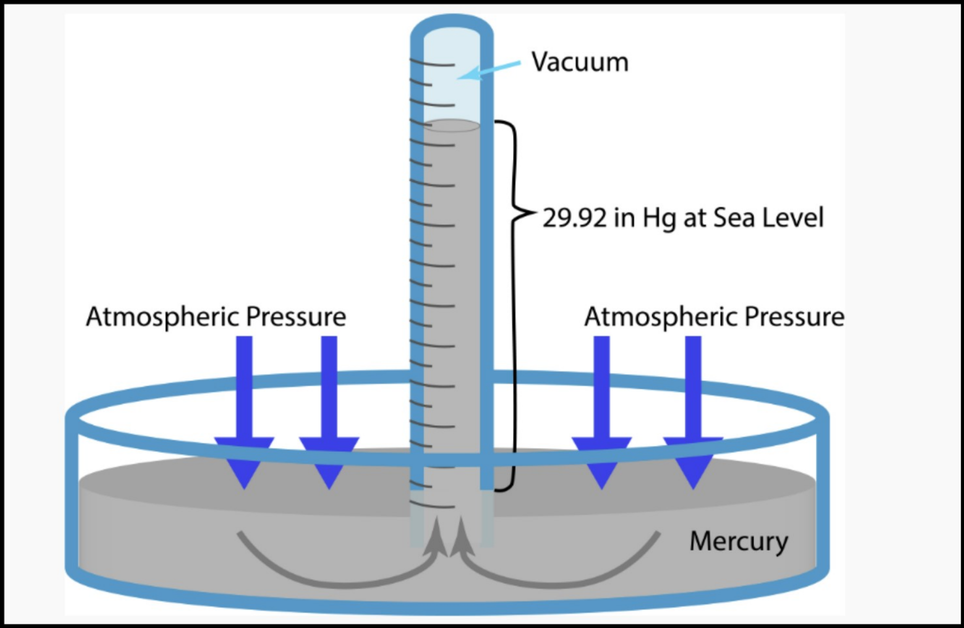
Less air pressure imply more intensity in a storm.

For this reason, taking $\underline{\underline{C}}$ and $\underline{\underline{E}}$ as:

$C =$ " the drop in the level of mercury in a barometer "

$E =$ " the storm is intense "

then is natural to think that $p(E|C) > p(E)$.



Part d)

Let A, C, and E be events. If $P(E|A \cap C) = P(E|C)$, then C is said to screen A off from E. Suppose that $P(E \cap C) > 0$. Show that screening off is equivalent to saying that $P(A \cap E|C) = P(A|C)P(E|C)$. What does this latter equation say in terms of independence?

Answer:

we need to prove that the following expressions are equivalents.

$$\begin{cases} p(E|A\cap C) = p(E|C) \\ p(A\cap E|C) = p(A|C)\cdot p(E|C) \end{cases}$$

For this purpose, we note that:

$$(\bigstar)\quad p(E|A\cap C) = p(E|C) \longleftrightarrow p(A|C)\cdot p(E|A\cap C) = p(A|C)\cdot p(E|C).$$

This equivalence is true because $p(A\cap C) \neq 0$ and $p(C) \neq 0$ in both sides of the equivalence.

$$(\blacklozenge)\quad p(A|C)\cdot p(E|A\cap C) = \frac{p(A\cap C)\cdot p(A\cap C\cap E)}{p(C)\cdot p(A\cap C)} =$$

$$= \frac{p(A\cap C\cap E)}{p(C)} = p(A\cap E|C).$$

(●) $p(E|A\cap C) = p(E|C) \longleftrightarrow p(A\cap E|C) = p(A|C)\cdot p(E|C).$

This equivalence is true for (★) and (♦).

Thus, we have shown that $p(E|A\cap C) = p(E|C)$ and $p(A\cap E|C) = p(A|C)\cdot p(E|C)$ are equivalents expressions.

¿What is the meaning of the equation $p(A\cap E|C) = p(A|C)\cdot p(E|C)$?

To begin with, let us note the following things:

(√) $p(A\cap E|C) = p((A\cap C)\cap(E\cap C)|C).$

(✓) $\begin{cases} p(A|C) = p(A\cap C|C) \\ p(E|C) = p(E\cap C|C) \end{cases}$.

(✓)

$$p(A\cap E|C) = p(A|C)\cdot p(E|C)$$

$$\Updownarrow$$

$$p((A\cap C)\cap(E\cap C)|C) = p(A\cap C|C)\cdot p(E\cap C|C)$$

$$\Updownarrow$$

$$p((A\cap C)\cap(E\cap C)) = p(A\cap C)\cdot p(E\cap C)$$

therefore $p(A\cap E|C) = p(A|C)\cdot p(E|C)$ is equivalent to the equation $p((A\cap C)\cap(E\cap C)) = p(A\cap C)\cdot p(E\cap C)$, and this equation tells us particularly that $A\cap C$ and $E\cap C$ are independent events.

Part e)

Now let A be a the drop in the level of mercury in a barometer, E be a storm, and C be a drop in atmospheric pressure. Does the result from part (d) help fix the problem suggested in part (c)?

Answer.

Lets start showing the descriptions of the events:

(✓) $A =$ "the drop in the level of mercury in a barometer"

(✓) $E =$ "storm is intense"

(✓) $C =$ "the drop in the atmospheric pressure"

(✓) $A \cap C =$ "the drop in the level of mercury in a barometer and the drop in the atmospheric pressure".

(r) $E \cap C =$ "the storm is intense and the drop in the atmospheric pressure"

And if A, C and E satisfy the result of the literal (d), then

$$p((A \cap C) \cap (E \cap C)) = p(A \cap C) \cdot p(E \cap C),$$

and this equation say to us that the events $A \cap C$ and $E \cap C$ are independents events, but this result doesn't help to solve the problem suggested in the part (c), because in the part (c) we have that $p(E|A) > p(E)$ and this implies that the events A and E are not independent events, and we could think that $A \cap C \approx A$ and $E \cap C \approx E$.

Problem 2

Suppose you have two bags of marbles that are in a box. Bag 1 contains 7 white marbles, 6 black marbles, and 3 gold marbles. Bag 2 contains 4 white marbles, 5 black marbles, and 15 gold marbles. The probability of grabbing the Bag 1 from the box is twice the probability of grabbing the Bag 2.

If you close your eyes, grab a bag from the box, and then grab a marble from that bag, what is the probability that it is gold?

Part a)

Solve this problem by hand. This should give us a theoretical value for pulling a gold marble.

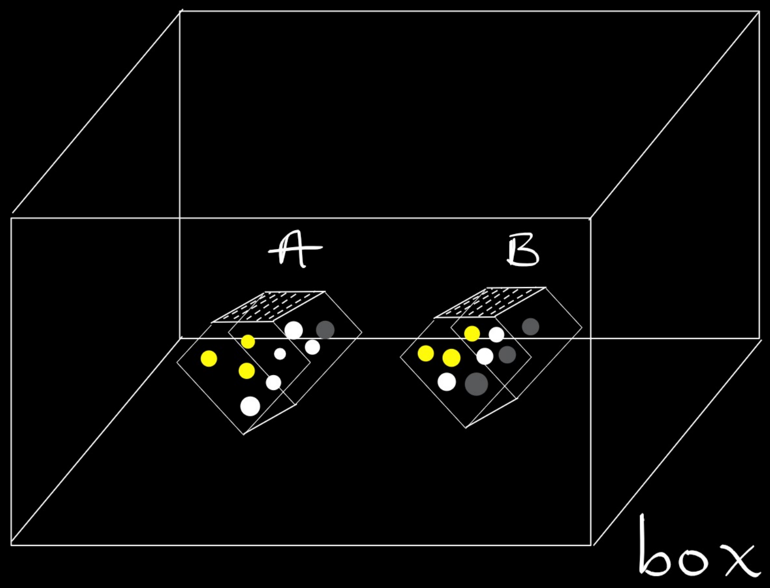
Answer:

Let's suposse that A and B are the bags of marbles such that:

(r) A contains
→ 7 white marbles
→ 6 black marbles
→ 3 gold marbles

(r) B contains
→ 4 white marbles
→ 5 black marbles
→ 15 gold marbles

(r) A and B are in a box.



(r) The probability of grabbing the bag A from the box is twice the probability of grabbing the bag B. That means that:

$$\boxed{p(A) = 2\,p(B)}.$$

(r) $p(A) = \frac{2}{3}$ and $p(B) = \frac{1}{3}$.

this is because the equations $p(A)+p(B)=1$ and $p(A)=2\,p(B)$.

(✓) $P(\text{"Marble is gold"} \mid A) = \dfrac{3}{7+6+3} = \dfrac{3}{16}$.

(✓) $P(\text{"Marble is gold"} \mid B) = \dfrac{15}{4+5+15} = \dfrac{15}{24}$.

therefore, the probability to get a marble gold is given by:

$$P(\text{"Marble is gold"}) =$$

$$= P(\text{"Marble is gold"} \cap A) + P(\text{"Marble is gold"} \cap B)$$

$$= P(\text{"Marble is gold"} \mid A) \cdot p(A) + P(\text{"Marble is gold"} \mid B) \cdot p(B)$$

$$= \frac{3}{16} \cdot \frac{2}{3} + \frac{15}{24} \cdot \frac{1}{3} = \frac{1}{8} + \frac{5}{24} = \frac{8}{24} = \frac{1}{3}.$$

thus

$$\boxed{P(\text{"Marble is gold"}) = \frac{1}{3} \approx 0{,}333}$$

Also is easy to calculate

$P(\text{"Marble is white"})$ and $P(\text{"Marble is black"})$.

$$P(\text{"Marble is white"}) =$$

$$= P(\text{"Marble is white"} \cap A) + P(\text{"Marble is white"} \cap B)$$

$$= P(\text{"Marble is white"} | A) \cdot P(A) + P(\text{"Marble is white"} | B) \cdot P(B)$$

$$= \frac{7}{16} \cdot \frac{2}{3} + \frac{4}{25} \cdot \frac{1}{3} = \frac{7}{24} + \frac{4}{75} = 0{,}345$$

$$P(\text{"Marble is black"}) =$$

$$= P(\text{"Marble is black"} \cap A) + P(\text{"Marble is black"} \cap B)$$

$$= P(\text{"Marble is black"} | A) \cdot P(A) + P(\text{"Marble is black"} | B) \cdot P(B)$$

$$= \frac{6}{16} \cdot \frac{2}{3} + \frac{5}{24} \cdot \frac{1}{3} = \frac{1}{4} + \frac{5}{72} = 0{,}3194$$

$$\begin{cases} P(\text{"Marble is white"}) = 0{,}345 \\ P(\text{"Marble is black"}) = 0{,}319 \\ P(\text{"Marble is gold"}) = 0{,}333 \end{cases}$$

Part b)

Create a simulation to estimate the probability of pulling a gold marble. Assume you put the marble back in the bag each time you pull one out. Make sure to run the simulation enough times to be confident in your final result.

Note: To generate n random values between [0,1], use the `runif(n)` function. This function generates n random variables from the Uniform(0,1) distribution, which we will learn more about later in this course!

Answer:

To begin with, we are going to describe A and B as:

$$(\text{r})\ A = \{[\text{"white"}, i, 1]\}_{i=1}^{7} \cup \{[\text{"black"}, i, 1]\}_{i=1}^{6} \cup \{[\text{"gold"}, i, 1]\}_{i=1}^{3}.$$

$$(\text{r})\ B = \{[\text{"white"}, i, 2]\}_{i=1}^{4} \cup \{[\text{"black"}, i, 2]\}_{i=1}^{5} \cup \{[\text{"gold"}, i, 2]\}_{i=1}^{15}.$$

And we define a probability

$P : \{T \,/\, T \subseteq A \cup B\} \longrightarrow \mathbb{R}$ as:

- $P(\{[\text{"white"}, i, 1]\}) = \frac{1}{16} \cdot P(A) = \frac{1}{16} \cdot \frac{2}{3} = \frac{1}{24}$

for $i \in \{1,2,3,4,5,6,7\}$.

- $P(\{["white", i, 2]\}) = \frac{1}{24} \cdot P(B) = \frac{1}{24} \cdot \frac{1}{3} = \frac{1}{72}$

for $i \in \{1,2,3,4\}$.

- $P(\{["black", i, 1]\}) = \frac{1}{16} \cdot P(A) = \frac{1}{16} \cdot \frac{2}{3} = \frac{1}{24}$

for $i \in \{1,2,3,4,5,6\}$.

- $P(\{["black", i, 2]\}) = \frac{1}{24} \cdot P(B) = \frac{1}{24} \cdot \frac{1}{3} = \frac{1}{72}$

for $i \in \{1,2,3,4,5\}$.

- $P(\{["gold", i, 1]\}) = \frac{1}{16} \cdot P(A) = \frac{1}{16} \cdot \frac{2}{3} = \frac{1}{24}$

for $i \in \{1,2,3\}$.

- $P(\{["gold", i, 2]\}) = \frac{1}{24} \cdot P(B) = \frac{1}{24} \cdot \frac{1}{3} = \frac{1}{72}$

for $i \in \{1,2,3,\ldots,14,15\}$.

Thus, with the previous information, we can built a simulation of this situation as follows:

```r
# Your Code Here
S <- c("white", "white", "white", "white", "white", "white", "white", "black", "black", "black", "black", "black", "black","gold'
probabilities <- c(1/24, 1/24, 1/24, 1/24, 1/24, 1/24, 1/24, 1/24, 1/24, 1/24, 1/24, 1/24, 1/24, 1/24, 1/24, 1/24, 1/72, 1/72, 1/

sample_marbles=sample(x=S, 1000, replace=TRUE, probabilities)
sample_marbles
```

```
'white' · 'gold' · 'white' · 'gold' · 'gold' · 'white' · 'gold' · 'black' · 'gold' · 'gold' · 'white' · 'white' · 'gold' · 'gold' · 'black' · 'white' · 'white' ·
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'gold' · 'white' · 'white' · 'gold' · 'white' · 'white' · 'white' · 'gold' · 'white' · 'black' · 'gold' · 'white' · 'gold' · 'gold' · 'black' · 'white' · 'black' ·
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'gold' · 'gold' · 'white' · 'gold' · 'gold' · 'black' · 'white' · 'gold' · 'white' · 'black' · 'black' · 'black' · 'white' · 'black' · 'gold' · 'black' · 'black' ·
'black' · 'black' · 'gold'
```

and we have the following results

```
{'white': 0.377, 'gold': 0.328, 'black': 0.295}
```

that are so close to the theorical values.

Problem 3

Suppose you roll a fair die two times. Let A be the event "the sum of the throws equals 5" and B be the event "at least one of the throws is a 4".

Part a)

By hand, solve for the probability that the sum of the throws equals 5, given that at least one of the throws is a 4. That is, solve $P(A|B)$.

Answer:

Let $A = \{(x,y) / x+y=5 \text{ and } 1 \leq x,y \leq 6\}$ and $B = \{(x,y)/(x=4 \text{ or } y=4) \text{ and } 1 \leq x,y \leq 6\}$, then:

(✓) $A \cap B = \{(4,1),(1,4)\}$.

(✓) $B = \{(4,i)\}_{i=1}^{6} \cup \{(j,4)\}_{j=1}^{6}$.

(✓) $|A \cap B| = 2$ and $|B| = 11$.

(✓) $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{2}{11}$.

Thus the probabitity of A given B is $P(A|B) = \dfrac{2}{11} \approx 0{,}182$.

Part b)

Write a simple simulation to confirm our result. Make sure you run your simulation enough times to be confident in your result.

Hint: Think about the definition of conditional probability.

Answer:

To begin with, remember that $A \cap B$ and B are described as:

(r) $A \cap B = \{(4,1),(1,4)\}$.

(r) $B = \{(4,i)\}_{i=1}^{6} \cup \{(j,4)\}_{j=1}^{6} =$

$$= \left\{ \begin{array}{l} (4,1),(4,2),(4,3),(4,4),(4,5),(4,6),(1,4) \\ (2,4),(3,4),(5,4),(6,4) \end{array} \right\}.$$

Thus, we have the following simulation:

```
# Your Code Here
B= c("(4,1)","(4,2)","(4,3)","(4,4)","(4,5)","(4,6)","(1,4)","(2,4)","(3,4)","(5,4)","(6,4)")

probabilities_elements_of_B = c(1/11, 1/11, 1/11, 1/11, 1/11, 1/11, 1/11, 1/11, 1/11, 1/11, 1/11)

sample_B=sample(x=B, 10000, replace=TRUE, probabilities_elements_of_B)

sample_B
```

```
'(6,4)' · '(2,4)' · '(1,4)' · '(4,4)' · '(4,6)' · '(6,4)' · '(4,2)' · '(6,4)' · '(2,4)' · '(4,2)' · '(4,2)' · '(4,5)' · '(3,4)' · '(5,4)' · '(4,2)' · '(3,4)' · '(4,4)' ·
'(4,4)' · '(4,1)' · '(4,1)' · '(4,2)' · '(4,5)' · '(1,4)' · '(2,4)' · '(5,4)' · '(2,4)' · '(6,4)' · '(4,2)' · '(1,4)' · '(2,4)' · '(4,2)' · '(4,4)' · '(5,4)' · '(4,2)' ·
'(1,4)' · '(1,4)' · '(4,6)' · '(5,4)' · '(5,4)' · '(2,4)' · '(4,1)' · '(4,2)' · '(4,4)' · '(4,1)' · '(4,1)' · '(4,2)' · '(4,1)' · '(5,4)' · '(5,4)' · '(4,6)' · '(1,4)' ·
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'(5,4)' · '(1,4)' · '(4,1)' · '(4,3)' · '(4,3)' · '(4,4)' · '(5,4)' · '(2,4)' · '(5,4)' · '(4,6)' · '(4,2)' · '(4,5)' · '(4,3)' · '(4,6)' · '(2,4)' · '(5,4)' · '(6,4)' ·
'(1,4)' · '(4,1)' · '(3,4)' · '(4,2)' · '(4,5)' · '(4,6)' · '(4,4)' · '(1,4)' · '(4,4)' · '(2,4)' · '(1,4)' · '(4,1)' · '(6,4)' · ··· · '(4,1)' · '(4,3)' · '(4,5)' ·
'(3,4)' · '(6,4)' · '(4,6)' · '(6,4)' · '(4,3)' · '(2,4)' · '(2,4)' · '(4,2)' · '(4,4)' · '(4,6)' · '(6,4)' · '(6,4)' · '(4,2)' · '(4,3)' · '(3,4)' · '(3,4)' · '(4,2)' ·
'(3,4)' · '(4,2)' · '(4,1)' · '(4,1)' · '(4,4)' · '(4,5)' · '(4,4)' · '(3,4)' · '(6,4)' · '(4,3)' · '(2,4)' · '(4,6)' · '(4,3)' · '(4,3)' · '(4,2)' · '(3,4)' · '(4,5)' ·
'(4,5)' · '(4,4)' · '(1,4)' · '(4,2)' · '(1,4)' · '(1,4)' · '(1,4)' · '(4,5)' · '(6,4)' · '(5,4)' · '(4,6)' · '(4,4)' · '(6,4)' · '(4,1)' · '(4,2)' · '(2,4)' · '(4,4)' ·
'(2,4)' · '(2,4)' · '(3,4)' · '(6,4)' · '(4,2)' · '(6,4)' · '(2,4)' · '(2,4)' · '(4,1)' · '(3,4)' · '(4,5)' · '(4,5)' · '(4,3)' · '(5,4)' · '(1,4)' · '(1,4)' · '(5,4)' ·
'(4,5)' · '(3,4)' · '(3,4)' · '(4,1)' · '(6,4)' · '(5,4)' · '(3,4)' · '(4,3)' · '(4,4)' · '(4,2)' · '(4,6)' · '(4,3)' · '(3,4)' · '(2,4)' · '(4,2)' · '(4,4)' · '(4,2)' ·
'(2,4)' · '(2,4)' · '(6,4)' · '(2,4)' · '(1,4)' · '(4,6)' · '(3,4)' · '(5,4)' · '(6,4)' · '(2,4)' · '(5,4)' · '(4,4)' · '(4,3)' · '(4,5)' · '(4,5)' · '(4,4)' · '(4,2)' ·
'(5,4)' · '(4,2)' · '(4,2)' · '(4,5)' · '(4,2)' · '(4,4)' · '(6,4)' · '(6,4)' · '(5,4)' · '(4,6)' · '(2,4)' · '(4,1)' · '(4,4)' · '(2,4)' · '(6,4)' · '(4,5)' · '(4,6)' ·
'(3,4)' · '(4,2)' · '(4,3)' · '(4,5)' · '(4,4)' · '(5,4)' · '(5,4)' · '(3,4)' · '(4,2)' · '(4,4)' · '(4,6)' · '(2,4)' · '(4,6)' · '(6,4)' · '(4,3)' · '(5,4)' · '(4,1)' ·
'(4,3)' · '(1,4)' · '(4,4)' · '(1,4)' · '(3,4)' · '(4,1)' · '(6,4)' · '(4,5)' · '(5,4)' · '(6,4)' · '(1,4)' · '(1,4)' · '(4,1)' · '(4,3)' · '(3,4)' · '(3,4)' · '(4,5)' ·
'(4,4)' · '(4,4)' · '(3,4)' · '(4,3)' · '(4,6)' · '(5,4)' · '(2,4)' · '(2,4)' · '(3,4)' · '(4,4)' · '(4,5)' · '(4,1)' · '(3,4)' · '(1,4)' · '(6,4)' · '(4,5)' · '(6,4)' ·
'(4,5)' · '(6,4)' · '(4,6)' · '(6,4)' · '(6,4)' · '(1,4)' · '(4,2)' · '(4,6)' · '(4,4)' · '(4,1)' · '(3,4)' · '(4,5)' · '(2,4)' · '(4,1)' · '(4,3)' · '(4,3)' · '(2,4)' ·
'(6,4)' · '(1,4)' · '(3,4)' · '(3,4)' · '(1,4)' · '(4,3)' · '(4,1)' · '(1,4)' · '(4,4)' · '(3,4)'
```

and the results are:

```
(5,4) -----> 0.0941

(1,4) -----> 0.0884

(3,4) -----> 0.0907

(4,4) -----> 0.0898
```

```
(4,3) -----> 0.0898

(4,2) -----> 0.0923

(4,1) -----> 0.0897

(4,5) -----> 0.0901
```

```
(6,4) -----> 0.0944

(4,6) -----> 0.0898

(2,4) -----> 0.0909
```

```
["(4,1)","(1,4)"] -----> 0.1785
```

Is so easy to verify that this dates

are so close to the theorical values described above.