NEW MEXICO TECH, COMPUTER SCIENCE AND ENGINEERING

Induction Examples

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1 EXERCISE A

Prove the following: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Proof by Induction

Base Case: Let

$$i = 1$$

Plug i into the left hand side (LHS) to get

$$\sum_{i=1}^{1} i^2 = 1^2 = 1$$

Now plug i into the right hand side (RHS) to get

$$\begin{aligned} \frac{n(n+1)(2n+1)}{6} &= \frac{1(1+1)(2*1+1)}{6} \\ &= \frac{1(2)(3)}{6} \\ &= \frac{6}{6} \\ &= 1 \end{aligned} \tag{1.1}$$

Since the LHS and the RHS are equal, we have established out base case.

Assumption

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

Show

$$\sum_{i=1}^{k} i^2 + (k+1)^2 = \frac{(k+1)(k+2)(2[k+1]+1)}{6}$$

Proof
Begin with the assumption, and add $(k+1)^2$ to both sides.

$$\sum_{i=1}^{k} i^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

Combine the two pieces into a single fraction by creating a common denominator of 6.

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

Extract the common factor $(k+1)$.

$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}$$

Multiply through the portion in brackets.

$$= \frac{(k+1)[2k^2 + k + 6k + 6]}{6}$$

Now let's start by simplifying the above equation.

$$= \frac{(k+1)[2k^2 + 7k + 6]}{6}$$

If we use a little factorization for the portion in brakets we get

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

And from there, we can see that

$$= \frac{(k+1)(k+2)(2[k+1]+1)}{6}$$

Therefore, our assumption statement after adding $(k+1)^2$ is

$$\sum_{i=1}^{k} i^2 + (k+1)^2 = \frac{(k+1)(k+2)(2[k+1]+1)}{6}$$

which is the same as our *Show* statement. Therefore, we can state that for all n where n is an integer and $n \geq 1$, then

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

□

2 Exercise B

Prove the following: $\sum_{j=1}^{n} 2^{j-1} = 2^n - 1$.

Proof by Induction

Base Case: Let

$$j = 1$$

Plug j into the left hand side (LHS) to get

$$\sum_{j=1}^{1} 2^{j-1} = \sum_{j=1}^{1} 2^{1-1} = 2^0 = 1$$

Plug j into the right hand side (RHS) to get

$$\begin{aligned} 2^n - 1 &= 2^1 - 1 \\ &= 2 - 1 \\ &= 1 \end{aligned} \tag{2.1}$$

Since the LHS and the RHS are equal, we have established out base case.

Assumption

$$\sum_{j=1}^{k} 2^{j-1} = 2^k - 1$$

Show

$$\sum_{j=1}^{k} 2^{j-1} + 2^{k+1-1} = 2^{k+1} - 1$$

Proof

Begin with the assumption, and add 2^{k+1-1} to both sides.

$$\sum_{j=1}^{k} 2^{j-1} + 2^{k+1-1} = 2^k - 1 + 2^{k+1-1}$$

We can simplify the exponent because $k+1-1 = k$, therefore

$$= 2^k - 1 + 2^k$$

Now pull out the common factor of 2^k.

$$= 2^1 * 2^k - 1$$

Now we can use the Product Rule of exponents to get

$$= 2^{k+1} - 1$$

Therefore, our assumption statement after adding 2^{k+1-1} is

$$\sum_{j=1}^{k} 2^{j-1} + 2^{k+1-1} = 2^{k+1} - 1$$

which is the same as our *Show* statement. Therefore, we can state that for all n where n is an integer and $n \geq 1$, then

$$\sum_{j=1}^{k} 2^{j-1} = 2^k - 1$$

□

3 Exercise C

Prove the following: $\sum_{j=0}^{n} x^j = \frac{1-x^{n+1}}{1-x}, for x \neq 1$

Proof by Induction

Base Case: Let

$$j = 0$$

Plug j into the left hand side (LHS) to get

$$\sum_{j=0}^{0} x^j = x^0 = 1$$

Plug j into the right hand side (RHS) to get

$$\begin{aligned}\frac{1-x^{n+1}}{1-x} &= \frac{1-x^{0+1}}{1-x} \\ &= \frac{1-x}{1-x} \\ &= 1\end{aligned} \tag{3.1}$$

Since the LHS and the RHS are equal, we have established out base case.

Assumption

$$\sum_{j=0}^{k} x^j = \frac{1-x^{k+1}}{1-x}, for x \neq 1$$

Show

$$\sum_{j=0}^{k} x^j + x^{k+1} = \frac{1-x^{k+2}}{1-x}, for x \neq 1$$

Proof
Begin with the assumption, and add x^{k+1} to both sides.

$$\sum_{j=0}^{k} x^j + x^{k+1} = \frac{1-x^{k+1}}{1-x} + x^{k+1}$$

We want to simplify this equation by using a common denominator of $(1-x)$.

$$= \frac{1-x^{k+1}+(1-x)(x^{k+1})}{1-x}$$

Now we can distribute out to see

$$= \frac{1-x^{k+1}+x^{k+1}-[x^1 * x^{k+1}]}{1-x}$$

We can use the Product rule of exponentials to simplify this just a little bit.

$$= \frac{1-x^{k+1}+x^{k+1}-x^{k+2}}{1-x}$$

We lose the two terms in the middle because they cancel each other out, so we are left with

$$= \frac{1-x^{k+2}}{1-x}$$

Therefore, our assumption statement after adding x^{k+1} is

$$\sum_{j=0}^{k} x^j + x^{k+1} = \frac{1-x^{k+2}}{1-x}, for x \neq 1$$

which is the same as our *Show* statement. Therefore, we can state that for all n where n is an integer and $n \neq 1$ and $n \geq 0$, then

$$\sum_{j=0}^{k} x^j = \frac{1-x^{k+1}}{1-x}, for x \neq 1$$

□

4 Exercise D

Prove the following: $1^3 + 3^3 + 5^3 + ... + (2n-1)^3 = n^2(2n^2 - 1)$

Proof by Induction

Base Case: Let

$$n = 1$$

Plug n into the left hand side (LHS) to get

$$(2(1)-1)^3 = (2-1)^3 = 1$$

Plug n into right hand side (RHS) to get

$$1^2 * (2 * (1^2) - 1) = 1 * (2-1) = 1$$

Since the LHS and the RHS are equal, we have established our base case.

Assumption

$$1^3 + 3^3 + 5^3 + ... + (2k-1)^3 = k^2(2k^2 - 1)$$

Show

$$1^3 + 3^3 + 5^3 + ... + (2k-1)^3 + (2(k+1)-1)^3 = (k+1)^2(2(k+1)^2 - 1)$$

Proof

Begin with assumption, and add $(2(k+1)-1)^3$ to both sides

$$1^3+3^3+5^3+...+(2k-1)^3+(2(k+1)-1)^3 = k^2(2k^2-1)+(2(k+1)-1)^3$$

which we can rewrite as

$$1^3+3^3+5^3+...+(2k-1)^3+(2(k+1)-1)^3 = k^2(2k^2-1)+(2k+1)^3$$

Since the LHS side of this equation matches the LHS in the *Show* statement, now all we need to do is modify the RHS of the *Show* statement. To do this, algebraically expand the RHS

$$(2k^4-k^2)+(2k)^3+3*(2k)^2+3*(2k)+1$$

Now we can combine all the like terms

$$2k^4+8k^3+11k^2+6k+1$$

I'm just going to work with the RHS of the *Show* statement, and demonstrate that it's equal to the equation above.

$$(k+1)^2(2(k+1)^2-1)$$

Expand the squares

$$(k^2+2k+1)(2(k^2+2k+1)-1)$$

Clean up the second expression

$$(k^2+2k+1)(2k^2+4k+1)$$

Now distribute

$$2k^4+4k^3+k^2+4k^3+8k^2+2k+2k^2+4k+1$$

Now combine

$$2k^4+8k^3+11k^2+6k+1$$

Since this expression is the same as the one above, we know we can manipulate the one above to the form

$$(k+1)^2(2(k+1)^2-1)$$

Therefore, our assumption statement after adding $(2k+1)^3$ is

$$1^3+3^3+5^3+\ldots+(2k-1)^3+(2(k+1)-1)^3=(k+1)^2(2(k+1)^2-1)$$

This is our *Show* expression, and now we can state that for all n where n is an integer and $n \geq 1$

$$1^3+3^3+5^3+\ldots+(2n-1)^3=n^2(2n^2-1)$$

$\square$

5 Notes

In the base case, evaluate the LHS and the RHS separately. You cannot say $1^3=1^2(2(1^2)-1)$, even if that is trivial. Instead, show both sides are equal by evaluating the sides separately.

This isn't so much an error, but a comment. Please use words describing what you are doing. Proofs are powerful, but they can't speak for themselves. If I took out every comment in the proofs above, no one would want to read it. Use comments to let people know what you're doing, and why you're able to do it.