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Exercises: 1-8

Problem 1 Show $1^2 + 2^2 + ... + n^2$ = n(n + 1)(2n + 1)/6, where n is any positive integer.

Proof By Induction:
Base Case :

Let i = 1
Plug i into LHS to get
$1^2 = 1$
Plug i into RHS to get
1(1 + 1)(2(1) + 1)/6
= 2*3/6 = 6/6
= 1

Since LHS and RHS are equal, base case is established.

Assumption : $1^2 + 2^2 + ... + n^2$ = k(k + 1)(2k + 1)/6

Show : $1^2 + 2^2 + ... + n^2 + (k + 1)^2$ = (k + 1)(k + 2)(2(k + 1) + 1)/6

Proof:
Add $(k + 1)^2$ to both sides

$1^2 + 2^2 + ... + n^2 + (k + 1)^2$ = k(k + 1)(2k + 1)/6 + $(k + 1)^2$

Use common denominator to merge RHS

$=\frac{k(k+1)(2k+1)+6(k+1)^2}{6}$

Pull out (k + 1)

$=\frac{(k+1)[k(2k+1)+6(k+1)]}{6}$

Multiply through brackets

$= \frac{(k+1)[2k^2+k+6k+6]}{6}$

Simplify and factor

$= \frac{(k+1)(k+2)(2[k+1]+1)}{6}$

Therefore, after adding $(k+1)^2$ to both sides, our assumption statement is:

$1^2 + 2^2 + ... + n^2 + (k+1)^2 = (k+1)(k+2)(2(k+1)+1)/6$

This is the same as the show statement, so it can be stated that for any n that's an integer and for $n \geq 1$, then
$1^2 + 2^2 + ... + n^2 = n(n+1)(2n+1)/6$

Problem 2 Show $1^3 + 2^3 + ... + n^3 = [n(n+1)/2]^2$, where n is any positive integer.

Proof:
Base Case :

Let i = 1
Plug i into LHS to get
$1^3 = 1$
Plug i into RHS to get
$[1(1+1)/2]^2$
$= 1^2$
$= 1$

Since LHS and RHS are equal, base case is established.

Assumption : $1^3 + 2^3 + ... + n^3 = [k(k+1)/2]^2$
Show : $1^3 + 2^3 + ... + n^3 + (k+1)^3 = [(k+1)((k+1)+1)/2]^2$

Proof:
Add $(k+1)^3$ to both sides

$1^3 + 2^3 + ... + n^3 + (k+1)^3 = [k(k+1)/2]^2 + (k+1)^3$

Distribute square to get

$\frac{k^2(k+1)^2}{4} + (k+1)^3$

Use common denominator to merge

$= \frac{k^2(k+1)^2+4(k+1)^3}{4}$
Pull out $(k+1)^2$

$= \frac{(k+1)^2(k^2+4k+4)}{4}$

Simplify and factor

$= \frac{(k+1)^2(k+2)^2}{4}$

= $[(k+1)((k+1) + 1)/2]^2$

Therefore, after adding $(k+1)^3$ to both sides, our assumption statement is:
$1^3 + 2^3 + ... + n^3 + (k+1)^3 = [(k+1)((k+1) + 1)/2]^2$
This is the same as the show statement, so it can be stated that for any n that's an integer and for $n \geq 1$, then
$1^3 + 2^3 + ... + n^3 + (k+1)^3 = [(k+1)((k+1) + 1)/2]^2$

Problem 3 Show 1 · 1! + 2·2! + ... + n·n! = (n + 1)! - 1, where n is any positive integer.

Proof:
Base Case :

Let i = 1
Plug i into LHS to get
1·1! = 1
Plug i into RHS to get
(1 + 1)! - 1
= 2 - 1
= 1

Since LHS and RHS are equal, base case is established.

Assumption : 1 · 1! + 2·2! + ... + n·n! = (k + 1)! - 1
Show : 1 · 1! + 2·2! + ... + n·n! + (k + 1)·(k + 1)! = ((k + 1) + 1)! - 1

Proof:
Add $(k + 1) \cdot (k + 1)!$ to both sides

$1 \cdot 1! + 2 \cdot 2! + ... + n \cdot n! + (k + 1) \cdot (k + 1)! = (k + 1)! - 1 + (k + 1) \cdot (k + 1)!$

Factor out $(k + 1)!$ to get

$=(k + 1)!((k + 1) + 1) - 1$

Simplify

$=(k + 1)!(k + 2) - 1$

Use what we know about factorials to get

$=(k + 2)! - 1$
$=((k + 1) + 1)! - 1$

Therefore, after adding $(k + 1) \cdot (k + 1)!$ to both sides, our assumption statement is:
$1 \cdot 1! + 2 \cdot 2! + ... + n \cdot n! + (k + 1) \cdot (k + 1)! = ((k + 1) + 1)! - 1$
This is the same as the show statement, so it can be stated that for any n that's an integer and for $n \geq 1$, then
$1 \cdot 1! + 2 \cdot 2! + ... + n \cdot n! = (n + 1)! - 1$

Problem 4 Show $2^n > n^2$ when $n > 4$

Proof:
Base Case :

Let $n = 5$
Plug 5 into equation to get

$2^5 > 5^2 = 32 > 25$

Since 32 is greater than 25, base case is established.

Assumption : $2^n > n^2$ when n > 4
Show : $2^{k+1} > 2k^2 > (k+1)^2$ when n ≥ 5

Proof:
Since k ≥ 5

$(k-1)^2 \geq 4^2 > 2$
Then we expand the inequality $(k-1)^2 > 2\ k^2 - 2k + 1 > 2$
$k^2 - 2k - 1 > 0$
$2k^2 - 2k - 1 > k^2$
$2k^2 > k^2 + 2k + 1 = (k+1)^2$

Therefore, after evaluating the inequality $(k-1)^2 \geq 4^2 > 2$, our assumption statement is:
$2k^2 > k^2 + 2k + 1 = (k+1)^2$
This is the same as the show statement, so it can be stated that for any n > 4 that's an integer then
$2n^2 > n^2 + 2n + 1 = (n+1)^2$

Problem 5 Show $1^3 + 3^3 + 5^3 + ... + (2n - 1)^3 = n^2(2n^2 - 1)$

Proof:
Base Case :

Let n = 1
Plug n into LHS to get
$(2(1) - 1)^3$
$1^3 = 1$
Plug n into RHS to get
$1^2(2(1)^2 - 1)$
= 1*1
= 1

Since LHS and RHS are equal, base case is established.

Assumption : $1^3 + 3^3 + 5^3 + ... + (2n - 1)^3 = n^2(2n^2 - 1)$
Show : $1^3 + 3^3 + 5^3 + ... + (2n - 1)^3 + (2(k + 1) - 1)^3 = (k+1)^2(2(k+1)^2 - 1)$

Proof:
Add $(2(k + 1) - 1)^3$ to both sides

$1^3 + 3^3 + 5^3 + ... + (2n - 1)^3 + (2(n + 1) - 1)^3 = n^2(2n^2 - 1) + ((2(n + 1) - 1)^3$

Expand out RHS to get

$=2k^4 - k^2 + 8k^3 + 12k^2 + 6k + 1$
$=2k^4 + 8k^3 + 11k^2 + 6k + 1$

Now show that $(k + 1)^2(2(k + 1)^2 - 1)$ is equivalent to RHS by expanding it out

$=(k + 1)^2(2(k + 1)^2 - 1)$

$=(k^2 + 2k + 1)(2k^2 + 4k + 1)$

$=2k^4 + 8k^3 + 11k^2 + 6k + 1$

Therefore, after adding $(2(k + 1) - 1)^3$ to both sides, we can see that our assumption statement is equivalent to:
$1^3 + 3^3 + 5^3 + ... + (2n - 1)^3 + (2(k + 1) - 1)^3 = (k + 1)^2(2(k + 1)^2 - 1)$

This is the same as the show statement, so it can be stated that for any n that's an integer, then
$1^3 + 3^3 + 5^3 + ... + (2n - 1)^3 = n^2(2n^2 - 1)$

Problem 6 Show $\frac{1}{(1)(2)} + \frac{1}{(2)(3)} + ... + \frac{1}{(n)(n+1)} = \frac{n}{n+1}$

Proof:
Base Case :

Let n = 1
Plug n into LHS to get
$\frac{1}{(1)(1+1)}$
$= \frac{1}{2}$
Plug n into RHS to get
$\frac{1}{1+1}$
$= \frac{1}{2}$

Since LHS and RHS are equal, base case is established.

Assumption : $\frac{1}{(1)(2)} + \frac{1}{(2)(3)} + ... + \frac{1}{(n)(n+1)} = \frac{n}{n+1}$
Show : $\frac{1}{(1)(2)} + \frac{1}{(2)(3)} + ... + \frac{1}{(n)(n+1)} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$

Proof:
Add $\frac{1}{(n+1)(n+2)}$ to both sides

$\frac{1}{(1)(2)} + \frac{1}{(2)(3)} + ... + \frac{1}{(n)(n+1)} + \frac{1}{(k+1)(k+2)} = \frac{n}{n+1} + \frac{1}{(k+1)(k+2)}$

Use common denominator to get

$=\frac{k(k+2)}{(k+1)(k+2)}$

Simplify

$=\frac{k}{(k+1)}$

Add $\frac{1}{1}$

$=\frac{k+1}{(k+2)}$

Therefore, after adding $\frac{1}{(n+1)(n+2)}$ to both sides, our assumption statement is:
$\frac{1}{(1)(2)} + \frac{1}{(2)(3)} + ... + \frac{1}{(n)(n+1)} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$
This is the same as the show statement, so it can be stated that for any n that's an integer, then
$\frac{1}{(1)(2)} + \frac{1}{(2)(3)} + ... + \frac{1}{(n)(n+1)} = \frac{n}{n+1}$

Problem 7 Show S = $\sum_{i=0}^{n} ar^i = \frac{ar^{n+1} - a}{r-1}$, r ≠ 1

Proof:
Base Case :

Let n = 1
Let r = 2
Plug n and r into LHS to get
$\sum_{i=0}^{1} a(2)^i$
= 2a + a = 3a
Plug n and r into RHS to get
$\frac{a(2)^{1+1} - a}{2-1}$

$= \frac{4a-a}{1}$
= 3a

Since LHS and RHS are equal, base case is established.

Assumption : S = $\sum_{i=0}^{n} ar^i = \frac{ar^{n+1}-a}{r-1}$, r ≠ 1
Show : S = $ar^{k+1} + \sum_{i=0}^{n} ar^i = \frac{ar^{k+2}-a}{r-1}$, r ≠ 1

Proof:
Add ar^{k+1} to both sides

S = $ar^{k+1} + \sum_{i=0}^{n} ar^i = \frac{ar^{k+1}-a}{r-1} + ar^{k+1}$

Use common denominator for RHS to get

$= \frac{ar^{k+1}-a+(r-1)(ar^{k+1})}{r-1}$

Simplify

$= \frac{ar^{k+1}-a+-(ar^{k+1})+ar^{k+2}}{r-1}$

$= \frac{ar^{k+2}-a}{r-1}$

Therefore, after adding ar^{k+1} to both sides, our assumption statement is:
S = $ar^{k+1} + \sum_{i=0}^{n} ar^i = \frac{ar^{k+2}-a}{r-1}$, r ≠ 1
This is the same as the show statement, so it can be stated that for any n that's an integer and for $r \neq 1$, then
S = $\sum_{i=0}^{n} ar^i = \frac{ar^{n+1}-a}{r-1}$, r ≠ 1

Problem 8 Show S = $\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2$, for all integers n ≥ 0.

Proof:
Base Case :

Let n = 1

Plug n into LHS to get
$\sum_{i=1}^{2} i \cdot 2^i$
$= 1*2 + 2 \cdot 2^2$
$= 2 + 8$
$= 10$
Plug n into RHS to get
$1 \cdot 2^{1+2} + 2$
$= 1 \cdot 8 + 2$
$= 10$

Since LHS and RHS are equal, base case is established.

Assumption : S $= \sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2$

Show : S $= (k+1) \cdot 2^{k+1} + \sum_{i=1}^{n+1} i \cdot 2^i = (k+1) \cdot 2^{k+3} + 2$

Proof:
Add $(k+1) \cdot 2^{k+1}$ to both sides

$(k+1) \cdot 2^{k+1} + \sum_{i=1}^{n+1} i \cdot 2^i = k \cdot 2^{k+2} + 2 + (k+1) \cdot 2^{k+1}$

Rewrite and expand RHS to get

$=(\mathrm{k}+1)\cdot(2^{k+1}) + 4\cdot 2^k + 2$

$=\mathrm{k}\cdot(2^{k+1}) + 2^{k+1} + 4\cdot 2^k + 2$

Combine like numbers and simplify to get

$=(k+1) \cdot 2^{k+3} + 2$

Therefore, after adding $(k+1) \cdot 2^{k+1}$ to both sides, our assumption statement is:
S $= (k+1) \cdot 2^{k+1} + \sum_{i=1}^{n+1} i \cdot 2^i = (k+1) \cdot 2^{k+3} + 2$
This is the same as the show statement, so it can be stated that for any n that's an integer, then
S $= \sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2$