Algorithm Analysis

CSE 122 ~ Algorithms & Data Structures

Terminology ~ Sequences and Summation

➔ **Sequence:** A *sequence* is a function from a subset of the integers (usually either the set {0,1,2,...} or {1,2,3,...} to a set *S*.

- Use notation a_n to denote a sequence
- $f(n) = a_n$
- $f(n) = a_m, a_{m+1}, a_{m+2}n, \dots, a_n$
- **term**: each individual element (Read “a sub k”)
- k is called a **subscript** or **index**. ($k \in Z$)

Examples of Sequences

- $a_n = 1/n$
  - $a_1, a_2, a_3, \dots, a_n = 1/1, 1/2, 1/3, \dots, 1/n$
- What are the first five terms of the following:
  - $A_k = k/(k+1)$, $k = 1, 2, 3, \dots, n$
  - $B_i = i-1/i$, $i = 2, 3, 4, \dots, n$

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  - Both generate 1/2, 2/3, 3/4, 4/5, 5/6

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- ➔ What does $c_j = (-1)^j$, $j > 0$ generate?
  - $-1, 1, -1, 1, \ldots, (-1)^j$
- ➔ Find the formula a_k for the sequence:
  - $1, -1/4, 1/9, -1/16, 1/25, -1/36, \ldots$
  - Answer: $a_k = (-1)^{k+1}/k^2$, for all $\forall k$, $k >= 1$, $k \in Z^+$

Examples of Sequences

➔ In CS, for loops are sequences

```
for (int i = 1; i < 101; i++)
    a[i] = 1/i;
```

Examples of Sequences

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Examples of Sequences

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```
for (int i = 1; i < n; i++)
    a[i] = pow((-1), i);
```

- ➔ In CS, these sequences are called *strings*
  - Example: *bit string* ~ a sequence of zero or more bits
  - *Length* ~ number of terms in the string
  - *Empty string* ~ a string that has no terms

Summation

- Notation due Lagrange to describe the sum of terms of a_m, a_{m+1}, … , a_n from the sequence $\{a_n\}$:

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_n$$

- *i* is called the **index of summation** and the index runs through the lower limit *m* to the upper limit *n*. n>=m

Summation

- Let $a_1 = -2$, $a_2 = -1$, $a_3 = 0$, $a_4 = 1$, $a_5 = 2$
- Find the following sums:

$$\sum_{i=1}^{5} a_i \qquad \sum_{i=2}^{2} a_i \qquad \sum_{i=1}^{2} a_{2i}$$

Summation

- Let $a_1 = -2$, $a_2 = -1$, $a_3 = 0$, $a_4 = 1$, $a_5 = 2$
- Find the following sums:

$$\sum_{i=1}^{5} a_i$$

0

$$\sum_{i=2}^{2} a_i$$

-1

$$\sum_{i=1}^{2} a_{2i}$$

0

Summation

- Another example uses the index as part of the formula:

$$\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$$

- How do you write the terms of the sum:

$$\sum_{k=0}^{5} \frac{k+1}{n+k}$$

Summation

➔ How do you write the terms of the sum:

$$\sum_{k=0}^{5} \frac{k+1}{n+k} = \frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n}$$

Properties of Summations

$$\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$$

$$c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k$$

Geometric Progressions

- A **geometric progression** is a sequence of the form
  - $a, ar, ar^2, ar^3, \ldots, ar^k$
  - Examples:

$$2^0 + 2^1 + 2^2 + \cdots + 2^n = \sum_{k=0}^{n} 2^k$$

$$3^0 + 3^1 + 3^2 + \cdots + 3^n = \sum_{k=0}^{n} 3^k$$

Geometric Progressions

- Let's find a formula for the sum of the terms of a geometric progression:

$$S = \sum_{j=0}^{n} ar^j$$

Geometric Progressions

- ➔ Multiply both sides by *r* and manipulate sum

$$rS = r\sum_{j=0}^{n} ar^j$$

$$= \sum_{j=0}^{n} ar^{j+1}$$

$$= \sum_{k=1}^{n+1} ar^k, \text{ where } k = j+1$$

- ➔ Why? Have to replace the upper and lower limits
  - k = j + 1, when j = 0, k = 0 + 1 = 1, and when j = n, k = n + 1
  - Also have to replace the term in the summation j + 1

Geometric Progressions

➔ Complete and Solve for s

$$= \sum_{k=0}^{n} ar^k + (ar^{n+1} - a)$$

$$= S + (ar^{n+1} - a)$$

Now solve for S

$$S = \frac{ar^{n+1} - a}{r-1}$$

Geometric Sequence Example

- What does $2^0 + 2^1 + 2^2 + \ldots + 2^n$ equal?
  - Apply the formula

Geometric Sequence Example

- What does $2^0 + 2^1 + 2^2 + \ldots + 2^n$ equal?
  - Apply the formula
  - a = 1, r = 2, so $S = \frac{2^{n-1}-1}{2-1} = 2^{n+1} - 1$

Problems for Class

➔ Transform the index when i = k + 1

$$\sum_{k=0}^{5} k(k-1) =$$

Problems for Class

➔ Transform the index when i = k + 1

$$\sum_{k=0}^{5} k(k-1) = \sum_{i=1}^{6} (i-1)(i-2)$$

Problems for Class

➔ Transform the index when j = i - 1

$$\sum_{i=1}^{n+1} \frac{(i-1)^2}{i \cdot n} =$$

Problems for Class

➔ Transform the index when j = i - 1

$$\sum_{i=1}^{n+1} \frac{(i-1)^2}{i \cdot n} = \sum_{j=0}^{n} \frac{j^2}{(j+1) \cdot n}$$

Problems for Class

➔ Write as a single summation:

$$3 \cdot \sum_{k=1}^{n} (2k - 3) + \sum_{k=1}^{n} (4 - 5k) =$$

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- Write as a single summation:

$$3 \cdot \sum_{k=1}^{n} (2k - 3) + \sum_{k=1}^{n} (4 - 5k) = \sum_{k=1}^{n} (k - 5)$$

Monday is a holiday

- I won’t be here
- Neither should you