Algorithm Analysis ~ 03

CSE/IT 122 ~ Algorithms and Data Structures

Induction

- Principle of Mathematical Induction:
  - Let $P(n)$ be a property that is defined for integers n and let a be a fixed integer.
  - Suppose the following two statements are true:
    - P(a) is true
    - For all integers $k >= a$ if $P(k)$ is true the $P(k + 1)$ is true.
  - Then the statement, ∀ integers $n >= a$, $P(n)$ is true.

Method of Proof

1. Basis step.
   - The proposition $P(a)$ is shown to be true.
2. Inductive step.
   - Show the implication $P(k) \Rightarrow P(k + 1)$ is true.

➔ $P(k)$ is called the inductive hypothesis.

Dominoes Example

P(n) = For all integers *n >= 8*, *n* cents can be obtained using 3 cent and 5 cent coins.

3 + 5 = 8

3 + (3 + 3) = 9

5 + 5 = 10

5 + (3 + 3) = 11

(3 + 3) + (3 + 3) = 12

5 + 5 + 3 = 13

5 + (3 + 3) + 3 = 14

5 + 5 + 5 = 15

Main idea: replace 5 with (3 + 3) in the next term.

Dominoes: Proof

Show $P(8)$ is true. $5 + 3 = 8$

Show if $P(k)$ is true then $P(k + 1)$ is true.
Suppose $P(k)$ is true, that is you can get any k from a combination of 5 cent and 3 cent coins.

Have to show $P(k + 1)$ can be made from 5 and 3 cent coins.

DomiNoes: Proof

Two Cases:

- ➔ Case 1: Assume *P(k)* involves a 5 cent coin. If that is the case replace the 5 cent piece with two 3 cent pieces and you have the *(k + 1)*
- ➔ Case 2: Assume *P(k)* does not include a 5 cent coin. Since *n >= 8* then there must be at least three 3 cent coins. Replace the coins with 2 five cent coins. The result is *(k + 1)* cents.

Sum The First n Odd Positive Integers

- ➔ Use mathematical induction to prove the sum of the first n odd integers is n^2.
- ➔ **Basis Step:**
  - *P(1)* is true as the sum of the first odd positive integer is 1^2, and $1 = 1^2$
- ➔ **Inductive Step:**
  - Assume *P(n)* is true, that is $1 + 3 + 5 + \ldots + (2n - 1) = n^2$
  - $(2n - 1)$ is the *n*th odd positive integer

Sum The First n Odd Positive Integers

- Have to show:
  - $1 + 3 + 5 + \ldots + (2n - 1) + (2n + 1) = (n + 1)^2$
- How we show that:
  - $1 + 3 + 5 + \ldots + (2n - 1) + (2n + 1)$
  - $= [1 + 3 + 5 + \ldots + (2n - 1)] + (2n + 1)$
  - $= n^2 + (2n + 1) = n^2 + 2n + 1 = (n + 1)^2$

Ex: $1 + 2 + 3 + \ldots + N = N(N+1)/2$

Show $1 + 2 + 3 + \ldots + n = n(n + 1)/2$ using induction.

$$\sum_{j=1}^{n} i = \frac{n(n+1)}{2}$$

➔ Let’s walk through the steps Gauss used at age 9 to solve this problem (:

Ex: $1 + 2 + 3 + \ldots + N = N(N+1)/2$

Proof:

Basis Step. $P(1) = 1 = \frac{1(1+1)}{2} = \frac{1(2)}{2} = \frac{2}{2} = 1$

Inductive Step:

Ex: 1 + 2 + 3 + … + N = N(N+1)/2

Inductive Step:

Assume it holds for some *n*.

1 + 2 + 3 + … + *n* = *n(n+1)/2*

Have to show its true for (*n + 1*)

$$1+2+3+\cdots+n+(n+1)=\frac{(n+1)((n+1)+1)}{2}=\frac{(n+1)((n+2)}{2}$$

Know $1+2+3+\cdots+n+(n+1)=\frac{n(n+1)}{2}+(n+1).$ This is the assumption P(n).

By manipulating the RHS:

$$\frac{n(n+1)}{2}+(n+1)=\frac{n^2+n}{2}+\frac{2n+2}{2}$$

$$=\frac{n^2+n+2n+2}{2}=\frac{n^2+3n+2}{2}=\frac{n+1)(n+2)}{2}$$

For You To Do:

Show $1 + 2^1 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1$

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Show $1 + 2^1 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1$

Basis Step: $2^{(0+1)} - 1 = 2^1 - 1 = 1$

Show its true for (n + 1): $1 + 2^1 + 2^2 + \ldots + 2^n + 2^{n+1} = 2^{n+2} - 1$

$$1 + 2^1 + 2^2 + 2^3 + \ldots + 2^n + 2^{n+1} = [2^{n+1} - 1] + 2^{n+1}$$

$$= 2^{n+2} - 1$$

More For You To Do:

Show $1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2(2n^2-1)$

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Base Case: Let n = 1; LHS: $(2(1)-1)^3 = (2-1)^3 = 1$

RHS: $1^2(2(1^2)-1) = 1(2-1) = 1$

Assumption: $1^3 + 3^3 + 5^3 + ... + (2k-1)^3 = k^2(2k^2-1)$

Show: $1^3 + 3^3 + 5^3 + ... + (2k-1)^3 + (2(k+1)-1)^3 = (k+1)^2(2(k+1)^2-1)$

More For You To Do: Show $1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2(2n^2-1)$

Proof:

Begin with the assumption, and add $(2(k+1)-1)^3$ to both sides.

$$1^3 + 3^3 + 5^3 + \ldots + (2k-1)^3 + (2(k+1)-1)^3 = k^2(2k^2-1) + (2(k+1)-1)^3$$

Which we can rewrite as:

$$1^3 + 3^3 + 5^3 + \ldots + (2k-1)^3 + (2(k+1)-1)^3 = k^2(2k^2-1) + (2k+1)^3$$

Since the LHS side of this equation matches the LHS in the *Show* statement, now all we need to do is modify the RHS of the *Show* statement. To do this, algebraically expand the RHS.

More For You To Do:

Show $1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2(2n^2 - 1)$

$(2k^4 - k^2) + (2k)^3 + 3*(2k)^2 + 3*(2k) + 1$

Now we can combine like terms. $2k^4 + 8k^3 + 11k^2 + 6k + 1$

I'm just going to work with the RHS of the *Show* statement, and demonstrate that it's equal to the equation above.

$$(k+1)^2(2(k+1)^2 - 1)$$

Expand the squares: $(k^2 + 2k + 1)(2(k^2 + 2k + 1) - 1)$

Clean up the second expression: $(k^2 + 2k + 1)(2k^2 + 4k + 1)$

Now distribute: $2k^4 + 4k^3 + k^2 + 4k^3 + 8k^2 + 2k + 2k^2 + 4k + 1$

And lastly combine: $2k^4 + 8k^3 + 11k^2 + 6k + 1$