Algorithm Analysis ~ 06

CSE/IT 122 ~ Algorithms & Data Structures

Combinations

- Given a set S with n elements, how many subsets of size r can be chosen from S?
  - Ex: How many different starting lineups are there on a basketball team consisting of 12 players?
- Order does not matter as it does in permutations
- Thm: Let n and r be nonnegative integers with r <= n. An r-combination of a set of n elements is a subset of r of the n elements. We say “n choose r” and use $\binom{n}{r}$ to denote r-combinations.
  - The formula to determine n choose r is: $$\binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$$

Ex: BBall Starting Lineup

➔ Ex: How many different starting lineups?

$$\binom{12}{5} = \frac{12!}{5!(12-5)!} = \frac{12!}{5!7!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!}{5!7!}$$

$$= \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 6 \cdot 11 \cdot 2 \cdot 3 \cdot 2 = 792$$

Ex: 8 Bit Strings

- Ex: How many eight bit strings have exactly three 1's?
  - Fix 8 positions, set 1's in any of the three places and then the rest must be 0's (one way to do the 0's)
  - How many ways to choose three things from eight possible choices?

$$\binom{8}{3} = \frac{8!}{3!5!} = 56$$

The Case of Repetition

- What happens when you allow repeats?
  - Example: there are 4 ways to choose 3 elements of the set {1,2,3,4}
    - {1,2,3}, {1,2,4}, {1,3,4}, {2, 3, 4}
- But how many multisets (lists or bags) can you create?
  - That is, allow repetition
  - Now how many ways can you choose 3 things with repetition
  - [1,1,1], [1,2,2], etc
  - Note: still unordered ~ [1,2,2] is the same as [2,1,2]

The Case Of Repetition

- Definition: An **r-combination with repetition allowed** of multiset of size r, chosen from a set X of n elements is an unordered selection of elements taken from X with repetition allowed.
  - If $X = \{x_1, x_2, x_3, ..., x_n\}$, we write an r-combination with repetition allowed as $\{x_{i1}, x_{i2}, ..., x_{ir}\}$ where each x_{ij} is in X and some of the x_{ij} may equal each other.

Ex: 4 Choose 3

- For the 4 choose 3 with repetition create the list:
  - [1,1,1],[1,1,2],[1,1,3],[1,1,4] all combinations of 1,1
  - [1,2,2],[1,2,3],[1,2,4] all combinations of 1,2
  - [1,3,3],[1,3,4],[1,4,4] all combinations of 1,3 or 1,4
  - [2,2,2],[2,2,3],[2,2,4] all combinations of 2,2
  - [2,3,3],[2,3,4],[2,4,4] all combinations of 2,3 or 3,4
  - [3,3,3],[3,3,4],[3,4,4] all combinations of 3,3 or 3,4
  - [4,4,4] all combinations of 4,4
- A total of 20
- Any easier way to do this then enumeration?

Ex: 4 Choose 3

➔ Consider the numbers as categories and imagine choosing a total of 3 numbers from the categories with multiple selections allowed from any category allowed.

Cat 1	Cat 2	Cat 3	Cat 4	Result
|	x |	|	x x	1 from cat. 2, 2 from Cat 4
x |	|	x |	x	1 from cat. 1, 3, and 4
xxx |	|	|		3 from cat. 1

Ex: 3 Choose 4

- So you can represent the selection of three numbers from four categories as a string of vertical bars and crosses
- The vertical bars are used to separate the categories
- The crosses are used to indicate how many items from each category are chosen
- Each distinct string represents a distinct selection
- xx||x| represents the selection two from category 1 and 1 from category 3

Ex: 3 Choose 4

- So the number of distinct selections of three elements that can be formed from the set {1,2,3,4} with repetition equals the number of distinct strings of six symbols consisting of 3 x and 3 |.
  - This equals the number of ways to select 3 positions out of 6 because once 3 positions have been chosen for the x's the | are placed in the remaining three positions
  - Thus the answer is:

$$\binom{6}{3} = \frac{6!}{3!(6-3)!} = 20$$

r-Combinations With Repetition

- In general, to count the number of r-combinations with repetition allowed, or multisets of size r, that can be selected from a set of n elements, think of the elements of the sets as categories
  - Each r-combination with repetition allowed can be represented as a string of:
    - n-1 bars, to separate the n categories
    - r crosses to represent the r elements to be chosen
- Theorem: The number of r-combinations with repetition allowed that can be selected from a set of n elements is

$$\binom{r+n-1}{r}$$

Different Way of Counting Loops

```
for i = 1 to 3
    for j = 1 to i
        x = i * j
```

- How many times does the body of the j loop execute?
- Draw a table:

i	1	2		3		
j	1	1	2	1	2	3

Different Way of Counting Loops

- ➔ Can think of the i,j as ordered pairs (j,i)
  - (1,1),(1,2,),(2,2),(1,3),(2,3),(3,3)
  - Or as a string of vertical bars and crosses: (1,1) = xx|||
- ➔ So how many ordered pairs are there in this case?
  - r = 2
  - n = 3

$$\binom{2+3-1}{2} = \frac{4!}{2!(4-2)!} = \frac{4!}{2!2!} = \frac{4 \cdot 3 \cdot 2!}{2!2!} = 6$$

Different Way of Counting Loops

➔ So in general for two for loops and n elements:

```
for i = 1 to n
    for j = 1 to j
        x = i * j
```

➔ The number of times the body of the for loop executes is:

$$\binom{r+n-1}{r} = \binom{2+n-1}{2} = \binom{n+1}{2} = \frac{(n+1)!}{2!(n+1-2)!} = \frac{(n+1)n(n-1)!}{2!(n-1)!} = \frac{n(n+1)}{2}$$

Different Way of Counting Loops

- This is the same answer if you work it explicitly with a table approach
  - The i-loop happened n times
  - The j-loop happened 2+3+...+n+1=(n+1)(n+2)/2 times
  - The body of the j-loop executes n(n+1)/2 times

i	j	body
1	2	1
2	3	2
3	4	3
...	...	...
n	$n+1$	n