Machine Learning for Agricultural Applications

Assignment 1

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Task 1 – Calculate Derivatives [10 points]

Calculate the derivatives of the following real-valued functions of a real variable:

a) $f(x) = u(x)\,v(x)$, b) $g(x) = \dfrac{u(x)}{v(x)}$,

c) $h(x) = u(v(x))$, d) $f(x) = x^2 \sin x$,

e) $g(x) = \ln(1 + x^2)$, f) $h(x) = \tanh\left(\dfrac{(x-1)^2}{x^2 + 1 + \cos x}\right)$.

Hints: $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$, $\sinh(x) = \frac{e^x - e^{-x}}{2}$, $\ln(x) = \log_e(x)$ natural logarithm.

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a) $f'(x) = u'(x)v(x) + u(x)v'(x)$

b) $g'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2}$

c) $h'(x) = u'(v(x))v'(x) = \frac{du}{dv}\frac{dv}{dx}$

d) $f'(x) = 2x \sin x + x^2 \cos x$

e) $g'(x) = \frac{2x}{1+x^2}$

$$\text{f)} \quad \tanh' x = \left(\frac{\sinh x}{\cosh x}\right)' = \frac{1}{\cosh^2 x}$$

$$\Rightarrow \tanh' z = \frac{1}{\cosh^2 z} z' = \frac{1}{\cosh^2 z} \frac{2(x-1)(x^2+1+\cos x) - (x-1)^2(2x - \sin x)}{(x^2+1+\cos x)^2}$$

Task 2 – Calculate Gradients [10 points]

Calculate the gradients of the following real-valued functions of three real variables:

a) $f(x, y, z) = x\, z^2 e^y \cos y$,

b) $g(x, y, z) = \ln(\sqrt{x^2 + y^2 + z^2})$.

a) $\nabla f = z\,e^y (z\cos y,\ xz(\cos y - \sin y),\ 2x\cos y)^T = z\,e^y \cos y\ (\,z,\ xz(1-\tan y),\ 2x\,)^T$

b) $\nabla g = \frac{2}{x^2+y^2+z^2}\,(\,x,\ y,\ z\,)^T$

Task 3 – Calculate Minimum [10 points]

a) Calculate the gradient of the following function:

$$f(x,y) = (a-x)^2 + b(y-x^2)^2, \quad \text{with } a,b>0 \text{ and } x,y\in\mathbb{R}$$

b) Calculate the extremum of the function $f(x,y)$ analytically and check if it is a minimum.

The function $f(x,y)$ is the so-called Rosenbrock function, which is often used as a performance test problem for optimization algorithms.

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a) $\nabla f = (\,2(x-a) + 4bx(x^2-y),\ 2b(y-x^2)\,)^T$

b) $\nabla f = (\,0,\ 0\,)^T \implies 2b(y-x^2)=0 \implies y=x^2$
$\implies 2(x-a)+4bx(x^2-y)=0 \implies 2(x-a)=0 \implies x=a \implies y=a^2$

$(x_{\text{min}},\ y_{\text{min}}) = (\,a,\ a^2\,)^T, \quad f(a,a^2)=0$

Check if it is really a minimum: calculate Hesse matrix.

$$H(x=a,\ y=a^2) = \begin{bmatrix} 2+8ba^2 & -4ab \\ -4ab & 2b \end{bmatrix}$$

The determinant $D = f_{xx}+f_{yy}-f_{xy}^2 = 2+8ba^2+2b+16a^2b^2 > 0$, hence it is an extremum. f_{xx} or f_{yy} positive, hence it is a minimum (the matrix is positive definit).

Task 4 – Gradient Descent [20 points]

Write a python program, that calculates the minimum of the Rosenbrock functions $f(x,y)$ given above with the gradient descent method. You need the gradients from task 3.

Compare the analytical result from task 3 with the result from gradient descent. Also think about useful plots.