Problem 1

Modify A by adding two constraints which force the solution to a vertex, or to something non-feasible. Specifically, add two rows, $-a_i$ and $-a_j$ to A and add the negatives of the constraining values to the bottom of b, too. There are m choose 2 ways to do this. Take the maximum value of the constraint equation you find by this method, it will be at a vertex.

Problem 2

We have a source vertex s and an end vertex e. We want to find the value of a shortest path. The linear program is written as:

$$\max d_t \begin{cases} d_v \leq d_u + w(u,v), \forall (u,v) \in E \\ d_s = 0 \end{cases}$$

Let d denote the length of an optimal path from s to e. If for all v, d_v is replaced by the Dv, the length of a shortest path to v, then the constraints are satisfied. (The distance from s to itself is zero, and certainly the shortest distance from s to v is shorter than or equal to a distance from s to u plus the edge length from u to v, by the triangle inequality. So, $d_e \geq d$. Now, let's see if we can flip the inequality to derive an equality.

Consider the path $(s, v_1, v_2, \ldots, v_n = e)$ of minimal length d. Due to the constraints, $d_s = 0$, $d_1 \leq w(s, v_1,$ $d_2 \leq d_1 + w(v_1, v_2)$, and $d_i \leq \sum_{i=1,n} w(v_{i-1}, v_i)$ by the same reasoning. Hence, $d_n = d_e \leq d$.

Now we need to formulate this in a form which we can feed to an LP solver. We can rewrite constraint (1) as $d_v - d_u \leq w(u,v)$, and we can rewrite constraint (2) as $d_S \geq 0$ and $d_s \leq 0$.

So, we have $O(|E|)$ constraints. We can set this up as a matrix equation. Each row in our matrix A has two non zero entries, a 1 and a -1. We are solving for $\vec{d}$. Our right hand side is the vector of weights.

Some bad things could happen in the case where the weights were negative, and there were a cycle. But, we have the weights in $R+$, so there is no need to worry.