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# A note on the Aboav–Weaire law

S.F. Edwards[a] and K.D. Pithia[b]

[a]*Cavendish Laboratory, University of Cambridge, Madingley Road, Cambridge CB3 OHE, UK*

[b]*South Bank University, Wandsworth Road, London SW8 2JZ, UK*

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In this paper we present an alternative derivation of the Aboav–Weaire law. By first making the assumption that the mean of the number of sides surrounding a cell is a function of the time, leads to $M_n$, the mean number of sides surrounding a cell of sides $n$, as a linear function of the second moment $\mu$, and is independent of $n$. This indicates that the local mean increases with time. When written in the form of the general Aboav–Weaire relation we find in their notation that $a = 1$ and $b = 6/7$. The analysis also leads to the relation that the deviation from the ensemble average, 6, is proportional to $\mu$, and $b$ is the coefficient of proportionality.

The initial assumption is removed and the assumption that $M_n$ is now a function of time and the number of sides is made. This assumption leads to $M_n$ as a linear function of the ratio $\mu/(n+1)$. This applies in the limit of small $\mu$ and the assumption that $M_n$ can be expanded as a Maclaurin series. When written in the form of the general Aboav–Weaire law $a = 1$ and $b = 0$. This implies that the deviation from the mean 6 is then proportional to $\mu/(n+1)$. A similar analysis is applied to three dimensions and when the faces of the cells are considered we find that the average number of faces of the cells surrounding a cell of face $f$ is a linear function of $\mu/(f+1)$, where $\mu$ is the second moment of the faces with mean 14.

This analysis of the mean number of sides surrounding a cell is extended to finding the mean area surrounding a cell of sides $n$ and area $A$, $M_n^A$. It is found that the mean area of the cells $M_n^A$ is given by $M_n^A = A_a + (A_a - A)/n$, where $A_a$ is the local cell area average which in the first approximation can be treated as the ensemble average area. Similarly the mean volume of cells surrounding a cell of faces $f$ and volume $v$ is given by $M_f^v = v_a + (v_a - v)/f$, where $v_a$ is the average local volume. In this case cells of small areas (volumes) are surrounded by cells of areas (volumes) greater than the average area (volume). We can reverse the argument to show that the large area (volume) cells are surrounded by small area (volume) cells.

The results of this analysis indicates that the cells are not randomly distributed.

## 1. Introduction

One of the conclusions of the study of the growth of grains in metals by Aboav [1–3] is the derivation of the empirical relation

$$M_n = 5 + \frac{8}{n}, \tag{1}$$

where $M_n$ is the mean of the sides of neighbouring cells surrounding a cell of sides $n$. It must be remembered that such a relation is empirical and applies for a particular metal concerned.

This empirical relation was put on more sure footing by Weaire [4]. In his analysis he considers a general network and derives a rule similar to Aboav,

$$M_n = 5 + \frac{6}{n}. \tag{2}$$

Further work of Weaire [5] has improved the relations (1) and (2). This analysis is based on The T1 process a mechanism which is fundamental in the evolution of the grains in metals.

The results of both Aboav and Weaire can be generalised into a relation, as pointed out by Aboav, of the form

$$M_n = 6 - a + b\mu/6 + [6a + (1-b)\mu]/n\,. \tag{2a}$$

The implications of the empirical and theoretical results is that it suggest that there is some ordering in the arrangement of grains in metals and therefore it is not purely random.

## 2. The Aboav–Weaire law

Given the conclusions of the work of Aboav and Weaire and its implications on the arrangement of cells in a network it is worthwhile to consider an alternative derivation of the results of Aboav and Weaire. This results largely because there is very little literature on the arrangement of cells within a random network and what does exist in the literature needs to be verified by alternative means. In short alternative derivations will provide useful verification of the Aboav–Weaire law.

We now derive the Aboav–Weaire relation. In this derivation we will be focusing on a liquid foam in two dimensions [2,3]. Let $M_n$ = mean of sides of neighbours of a cell of sides $n$. $nM_n$ is then the total number of sides of neighbouring cells. The total number of cells involved is $n + 1$. We then have

$$\langle n \rangle = \frac{nM_n + n}{n+1} \tag{3}$$

for the average of the number of sides in the two dimensional liquid foam. Now the average is six for a large number of cells [2] and therefore

$$M_n = 5 + \frac{6}{n} . \tag{4}$$

This as indicated applies for the whole two dimensional foam. Generally however $\langle n \rangle$ does not equal six locally. Let us write this local average in the foam as

$$\langle n \rangle = f(t) , \tag{5}$$

where $f(t)$ is a function of time only and such that at $t = 0$, $f(0) = 6$. Using (3) we have

$$M_n = f(t) - 1 + f(t)/n . \tag{6}$$

It has been shown that [3]

$$\langle nM_n \rangle = \mu + 36 , \tag{7}$$

where $\mu$ is the second moment of the sides of the cells. Using (6) we have

$$\langle nM_n \rangle = 6f(t) - 6 + f(t) , \tag{8}$$

which by (7) gives

$$\mu + 36 = 7f(t) - 6 \tag{9}$$

and this gives

$$f(t) = (\mu + 42)/7 \tag{10}$$

and

$$M_n = 5 + \mu/7 + (6 + \mu/7)/n , \tag{11}$$

which we write as eq. (2a),

$$M_n = 6 - a + b\mu/6 + [6a + (1 - b)\mu]/n ,$$

with $a = 1$ and $b = 6/7$. We can also rewrite (9) as

$$M_n = A + B/n , \tag{12}$$

with $A = B - 1 = 5 + \mu/7 = f(t) - 1$. Thus the coefficients $A$ and $B$ should be related. Returning back to eq. (10) we have

$$f(t) = 6 + \mu/7 , \tag{13}$$

which we now re-write in the form

$$f(t) - 6 = (1 - b)\mu , \tag{14}$$

with $b = 6/7$. An interpretation of the coefficient $b$ in (13) can now be made. Remembering that $f(t)$ is a measure of the local mean about a cell and six is the global mean, then $(1 - b)\mu$ is a measure of the deviation of the local mean from the global mean six.

For a more general description consider

$$\frac{nM_n + n}{n + 1} = f(n, t) , \tag{15}$$

where $f(n, t)$ is now a function of the number of sides $n$ and of the time $t$. We have

$$nM_n = (n + 1)f(n, t) - n . \tag{16}$$

Using eq. (7) we arrive at

$$\mu + 36 = \int (n + 1)f(n, t)\, P(n, t)\, \mathrm{d}n - \langle n \rangle , \tag{17}$$

where $P(n, t)$ is the probability distribution for the number of the sides, $n$ at time $t$. From the condition that $\langle n \rangle$ is 6 globally it follows that

$$\mu + 42 = \int (n + 1)f(n, t)\, P(n, t)\, \mathrm{d}n . \tag{18}$$

The first approximation of $f(n, t)$ is 6. This is at $\mu = 0$. Thus if we rewrite $f(n, t)$ as $f(n, \mu)$ with $\mu = 0$ at $t = 0$ and expand for small $\mu$,

$$f(n, \mu) = f(n, 0) + \mu f'(n, \mu)\big|_{\mu=0} + \cdots , \tag{19}$$

we consider the first approximation

$$f(n, \mu) = 6 + \mu f'(n, \mu)\big|_{\mu=0} \tag{20}$$

as $\mu$ is assumed small.

Using eq. (18) and eq. (20) we find

$$1 = \int (n+1) f'(n, \mu)\, P(n, \mu)\, \mathrm{d}n \,. \tag{21}$$

Now $f'(n, \mu)$ is only a function of $n$ since $\mu$ is defined (i.e. zero). Let this function be denoted by $g(n)$. Then we have

$$\int g(n)\,(n+1) P(n, \mu)\, \mathrm{d}n = 1 \,. \tag{22}$$

A solution for $g(n)$ is such that

$$g(n) = \frac{1}{(n+1)} \,, \tag{23}$$

with

$$f(n, \mu) = 6 + \mu/(n+1) \,. \tag{24}$$

This solution for $f(n, \mu)$ applies only for small $\mu$ and

$$M_n = 5 + \frac{6+\mu}{n} \,. \tag{25}$$

Using eq. (2a) we find that $a = 1$ and $b = 0$. In this case the deviation from six is proportional to $\mu/(n+1)$.

## 3. Three dimensions

Proceeding in a similar manner as above and using the average number of faces as 14 [3, 5], if $M_f$ is the mean of the faces surrounding a cell of faces $f$,

$$\frac{fM_f + f}{f+1} = 14 \,, \tag{26}$$

$$M_f = 13 + 14/f \,, \tag{27}$$

provided the local average is a function of time only. We can write as

$$M_f = A + B/f \,, \tag{28}$$

with $A = B - 1$ and $B = 14$ and for the more general case we have

$$h(f, \mu) = 14 + \mu/f + 1 \,, \tag{29}$$

where $h(f, \mu)$ is the local mean of the number of sides of the three dimensional foam. $\mu$ is the second moment of the number of faces with mean fourteen. This is assumed to be small. Thus the deviation from the assumed ensemble average is proportional to $\mu/(f+1)$,

$$M_n = 13 + \frac{14+\mu}{f} . \tag{30}$$

## 4. Area correlation in two dimensions

Consider a cell of sides $n$ and area $A$. Then let $M_n^A$ be the mean area of cells neighbouring $A$. Then

$$nM_n^A = \text{total area of neighbouring cells} . \tag{31}$$

The total area $= nM_n^A + A$, the number of cells $= n + 1$. Thus the average is

$$\frac{nM_n^A + A}{n+1} = A_a , \tag{32}$$

so that

$$M_n^A = A_a + (A_a - A)/n . \tag{33}$$

Thus to a first approximation, if $A_a$ is the ensemble average area then if $A > A_a$, then $M_n^A$ is less than $A_a$. This implies that cells less than the ensemble average surround cell of area $A$. That is small cells should to first approximation surround large cells and reversing the argument small cells are surrounded by large cells.

## 5. Volume correlations in three dimensions

From arguments similar to that for the area correlations, we define a function $M_f^v$ as the mean volume of cells surrounding a cell of faces $f$ and volume $v$. Then

$$M_f^v = v_a + (v_a - v)/f , \tag{34}$$

where $v_a$ is the ensemble volume average and thus to first approximation small volumes are surrounded by large volumes and large volume cells are surrounded by small volumed cells.

## 6. Conclusion

The work of Aboav and Weaire has revealed that the arrangement of cells in terms the number of sides may not be as random as first thought. Indeed the work suggests that the mean number of sides surrounding a cell is well defined by eq. (2a). This has been verified by the empirical relation of Aboav and by the theoretical work of Weaire. In this analysis we have rederived the Aboav–Weaire relation with the assumption that the local mean of the network is not equal to that of the global or ensemble value of six as results from a large number of cells.

This assumption is first implemented by assuming that the local deviation from the mean six is purely a function of time. Following the eq. (5)–(14) we find that eq. (14) defines the behaviour of the deviation from the mean six. This implies when written in the form eq. (2a) that the coefficient $b$ can be established to have the value 6/7. Eq. (14) allows an interpretation of the coefficient of $b$. We see from eq. (14) that the deviation from the mean six is proportional to $\mu$, the second moment of the number of the sides. The constant of proportionality is $1 - b$. This term $1 - b$ is always positive as $b$ is defined as 6/7 and thus the deviation from the mean six is positive and increases with increasing $\mu$. That is the local mean increases with increasing $\mu$ and if the experimental results are used which indicates that $\mu$ increases with time [5] then this implies that the local mean will always exceed the mean six provided the second moment increases with time. Indicating therefore that the mean of the whole ensemble will increase above six.

This situation warrants a new assumption incorporated into eq. (3). This new assumption is that the local mean can be written as a function of the number of sides, $n$ and the time, $t$. That is the local average is no longer a function of time only but also a function of the number of sides. This is expressed by eq. (15). The Taylor series expansion and the subsequent calculation of the function $f(\mu, t)$ is dependent on the assumption that $\mu$ is small. The coefficients $a$ and $b$ are determined in this analysis $a = 1$ and $b = 0$ and $1 - b = 1$. This therefore indicates that the average of the number of sides of cells surrounding a cell of side $n$ is dependent inversely on $n$ and proportional to the second moment $\mu$. This implies that the cells are not arranged randomly. Eq. (24) defines the local average function. The deviation from the global mean six is now proportional to the second moment $\mu$ and inversely proportional to $n + 1$. This therefore indicates that the local average will increase with time and will be dependent on the size of the cell considered. For small cells the deviation from the mean six will be large initially and will increase with time. For large cells the deviation will be small initially but increase with time.

This analysis is extended to three dimensions where we have considered the

faces of the cells and taken the average of the faces on a cell as fourteen. Proceeding with the same analysis we find that the mean number of faces on the neighbouring cells around a cell of faces $f$ is proportional to the second moment of the faces and inversely on the number of faces the cell possesses $f$. The local average of the number of faces on a cell is proportional to the second moment of the faces and inversely on $f + 1$. This again implies that cells do not arrange themselves randomly in three dimensions.

These local averages are useful also in determining the arrangement of cells of different areas. In this analysis we proceed in an analogous manner to finding the average number of sides or faces surrounding a cell of sides, $n$ or faces, $f$. We find from eq. (33) that if the area of the cell in question is greater than the average area and by average area we mean the local average which in the first approximation is the ensemble average; then the mean of the area of the cells surrounding the cell is less than the local average or in the first approximation is less than the ensemble average. This implies that on average cells of area less than the cell will surround the cell. This again indicates that the cells are not totally arranged randomly.

A similar analysis of the mean of the volumes of the cells surrounding a cell, again shows, eq. (34) that the cell will be surrounded in general by cells of small volumes provided that the cell has a volume greater than the local average or in the first approximation greater than the ensemble average, indicating again that the distribution of the cells is not totally random.

This lack of randomness can be argued to be a result of the need to maintain the averages of either the number of sides or faces, six sides in two dimensions and fourteen faces in case of three dimensions or the ensemble average area in two dimensions or ensemble average volume in three. That is, if we have a large cell in terms of sides (faces) or area (volume) then in order to maintain the respective averages we have to naturally surround the cell with cells of small number of sides (faces) or small area (volume) cells. The relations defined above are such that they allow this type of arrangement to occur with an indication of the relations that are necessary.

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