The Arrangement of Cells in a Net

D. A. ABOAV

29 Clements Road, Chorleywood, Hertfordshire, England

The distribution of cells in a planar section of a soap foam is determined at different stages of growth, and the shape of their sections is related quantitatively to that of the adjoining cell sections by an empirical equation. This observed regularity is compared with that already established for the grain sections of a polycrystal.

Introduction

Our attention was first directed to the subject of the arrangement of cells in a net when it was observed, in the etched surface of a piece of polycrystalline cadmium, that several grain sections with a larger than average number of sides were often arranged in a chain. Such chains were generally separated by grain sections of fewer than average sides similarly strung together, as many as a dozen of these sections being thus juxtaposed in a typical case. The relative average diameter of the grain sections constituting the two kinds of chain appeared, moreover, to be simply related to the different stages into which the grain growth of this metal had been found to divide. The discovery of similar patterns of grains in a polycrystalline ceramic, magnesium oxide, suggested that the observed regularity might be a general one, and an attempt was made to express it quantitatively in terms of the relative shape of adjoining grain sections [1].

As a first step, it was decided to measure the average number of sides, m, of sections adjoining n-sided ones: it seems preferable, however, to write m_n in place of m, to emphasize its dependence on n. The determination of m_n for some 3000 grains in a specimen of magnesium oxide of average grain diameter 12 μm showed that it varied linearly with the reciprocal of n, the variation being expressed with an average error of less than 3% by the equation

$$m_n = 5 + (8/n) \qquad (1)$$

In a specimen of larger grain diameter ($\sim$50 μm), for which the distri-

0026-0800/80/010043 + 16\$01.75

bution of n was found to be the same as for the specimen with the smaller grains, m_n was represented by this equation with the same degree of accuracy, thus indicating that, for the material in question, the relation is independent of grain size.

In a net of three-fold vertices the average number of sides of the polygonal cells defined by the net is six (Euler's theorem), and, if f_n is the fraction of cells with n sides, the moments of distribution about $n = 6$ are defined by

$$\mu_N = \sum_n (n - 6)^N f_n, \tag{2}$$

from which it follows that, unless all the cells have six sides, the second moment $\mu_2 > 0$; and that [2]

$$\sum_n m_n n f_n = \mu_2 + 36. \tag{3}$$

This is a purely topological relation and holds no matter how the cells are arranged.

If, as in the case now being considered, m_n varies linearly with the reciprocal of n, to be consistent with Eq. (3) the variation can be expressed generally as

$$m_n = \left(6 - a + \frac{b\mu_2}{6}\right) + \frac{6a + (1 - b)\mu_2}{n}, \tag{4}$$

where a and b are numerical constants.

The laws of topology, however, do not require m_n to vary linearly with the reciprocal of n, or indeed to depend regularly on n at all; nor if the above dependence exists do they require the constants of Eq. (4) to have a particular value. In general, therefore, the arrangement of cells in a net cannot be determined by considerations of topology alone.

Since Eq. (1) evidently cannot hold generally (that is, for systems with different values of μ_2) Weaire [2] surmised that it might be expressible as

$$m_n = 5 + \frac{6 + \mu_2}{n}. \tag{5}$$

This is equivalent to Eq. (4) with the constants a equal to 1, and b equal to 0; and to Eq. (1) in the particular instance where μ_2 is equal to 2. The object of the investigation now to be described is to determine the dependence of m_n on μ_2, as well as on n, in a naturally occurring net.

To use a polycrystal to establish a general relation of that kind is unsuitable for two reasons: first, because, as was found with an alloy of aluminum and tin [3] and with magnesium oxide [4], a small proportion of the intergranular vertices in a section of a polycrystal do not appear at first sight to be three-fold, so that Eq. (3) may not hold exactly; and second, because in a polycrystal the distribution of n does not usually vary as the grains grow, μ_2 in a typical case [4] acquiring the value 2.4 at an incipient stage of growth and thereafter remaining constant.

What is required for our purpose is a net of three-fold vertices only, defining a system of polygons for which μ_2 may vary over a wide range. A section of a foam, unlike that of a polycrystal, has these properties, the progressive change in the shape and arrangement of the foam cells being particularly easy to observe if they comprise a single layer. Such layers of cells, or "two-dimensional" foams, which we therefore now propose to examine in detail, have been investigated by Smith [5]: the following data are taken from photographs used in those investigations, and I am obliged to Professor Smith for allowing me to use them for the purpose of this enquiry.

Experimental Results

The foams were made by blowing air at atmospheric pressure into a soap solution contained in a rectangular (25 × 33 cm) enclosure, with top and bottom plates about 4 mm apart and with a side drain and reservoir to keep the level of the solution just at the level of the bottom plate. The bubbles, or cells, formed in this way were initially roughly uniform in size. Because of the small differences in the curvature of their faces, the pressure of air within different cells of such an aggregate is not quite equal, being on average a little higher inside the smaller ones, whose faces generally have a greater curvature. This gives rise to a diffusion of air from the smaller cells to their larger neighbors, and hence to a continued shrinking and eventual collapse of the smaller ones. In these experiments the observed growth of foam cells resulted solely from such diffusion, and there were no breakages of the soap film. The foams were kept at 21°C, at which temperature, diffusion was very slow. The changing configuration of the cellular aggregate was photographed at intervals through the glass plate at the top of the enclosure, the lapse of time between successive photographs being typically 15 hours. The observations described below are of sections of cells by this plate, but such cell-sections will for brevity be referred to as cells.

Figure 1 (a–f) shows the general appearance of one of these foams (IV) at five of eight different stages (1–8) in the growth of its cells, and of a similar foam (V) of smaller initial cell size at an early stage of growth where most of the cells are still hexagonal.

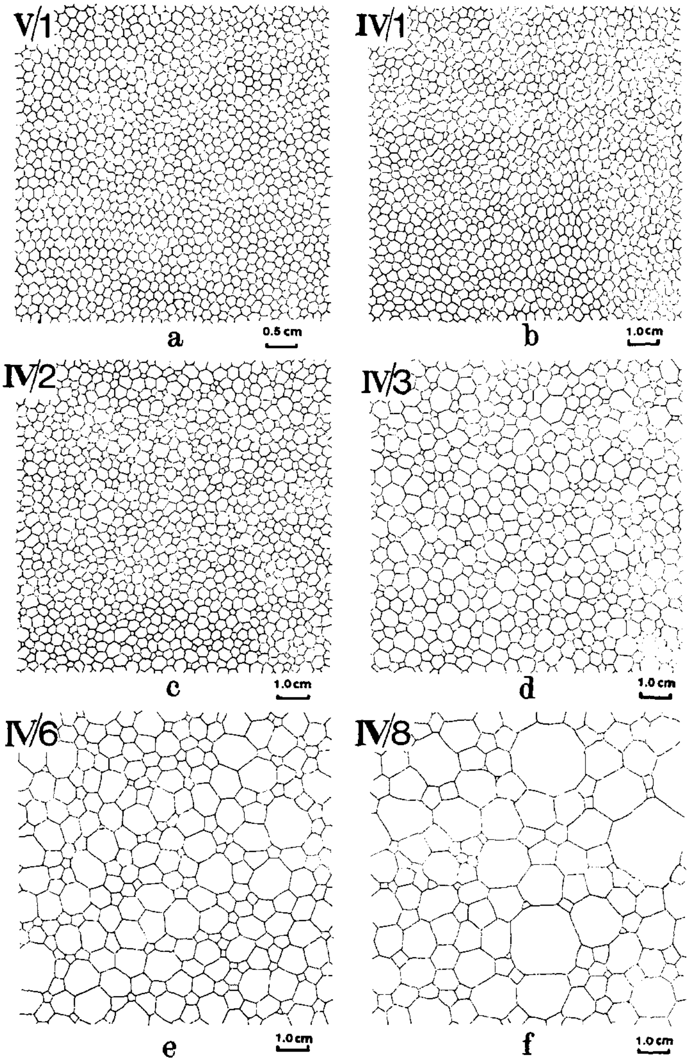

FIG. 1. "Two-dimensional" foam at different stages of growth.

Tables 1-6 show, for a region comprising several thousand cells in each of the above-illustrated cases, the values of z_n, the number of n-sided cells in the region, and of m_n, the average number of sides of the cells adjoining n-sided ones, for each value of n. The value of μ_2 is shown in each case at the head of the table. The total number of cells for which m_n was determined was 19,428.

TABLE 1
Foam V/1. $\mu_2 = 0.24$.

n	z_n	m_n	m_n'	$m_n - m_n'$
5	580	6.29	6.29	0
6	3511	6.03	6.04	−0.01
7	516	5.84	5.86	−0.02
8	5			

TABLE 2
Foam IV/1. $\mu_2 = 0.62$.

n	z_n	m_n	m_n'	$m_n - m_n'$
3	12			
4	124	6.68	6.75	−0.07
5	1205	6.33	6.36	−0.03
6	3055	6.09	6.10	−0.01
7	1159	5.91	5.92	−0.01
8	125	5.76	5.78	0.02
9	7			

TABLE 3
Foam IV/2. $\mu_2 = 0.98$

n	z_n	m_n	m_n'	$m_n - m_n'$
3	42	7.49	7.53	−0.04
4	177	6.82	6.84	−0.02
5	771	6.44	6.44	0
6	1636	6.11	6.16	−0.05
7	731	5.93	5.97	−0.04
8	176	5.80	5.82	−0.02
9	15			
10	1			
11	1			

TABLE 4
Foam IV/3. μ_2 = 1.30.

n	z_n	m_n	m_n'	$m_n - m_n'$
3	39	7.86	7.63	0.23
4	215	6.96	6.92	0.04
5	960	6.49	6.50	−0.01
6	1352	6.19	6.22	−0.03
7	716	5.99	6.01	−0.02
8	268	5.87	5.86	0.01
9	66	5.76	5.74	0.02
10	4			
11	2			
13	1			

The variation of f_n, the fraction of n-sided cells, with n is shown in Fig. 2, with the mean value of n denoted by a vertical full line and the mode, or most probable value, by a broken line. In Fig. 3, the value of μ_2, calculated from these and similar measurements covering all eight of the above-mentioned stages (1–8) in the growth of foam IV, is plotted against the average linear intercept, d, of the foam cells at each stage.

In Fig. 4, the distribution of the length of the cell sides in the foams of Fig. 2 is shown as u_w/u_{wm} against w/w_m, where u_w is the number of cell sides having length w within a given range and w_m is the value of w for

TABLE 5
Foam IV/6. μ_2 = 1.98.

n	z_n	m_n	m_n'	$m_n - m_n'$
3	35	7.97	7.86	0.11
4	110	7.09	7.09	0
5	390	6.59	6.64	−0.05
6	423	6.28	6.33	−0.05
7	239	6.10	6.11	−0.01
8	105	5.95	5.95	0
9	48	5.78	5.82	−0.04
10	15			
11	5			
12	1			
14	1			

TABLE 6
Foam IV/8. $\mu_2 = 2.86$.

n	z_n	m_n	m_n'	$m_n - m_n'$
3	20	8.4	8.2	0.2
4	65	7.4	7.3	0.1
5	170	6.7	6.8	−0.1
6	153	6.4	6.5	−0.1
7	87	6.2	6.2	0
8	45	6.1	6.1	0
9	19	6.0	5.9	0.1
10	10			
11	6			
12	6			
13	2			
18	1			

which u_w has the maximum value u_{wm}; and Fig. 5 shows the distribution of cell diameters for the same foam with u_x/u_{xm} plotted against x/x_m, where u_x is the number of cells having diameter x within a given range and x_m is the value of x for which u_x has the maximum value u_{xm}. Here again the full and broken vertical lines in the figures represent the mean and the mode, respectively.

For two of the foams, V/1 and IV/6, the product $m_n n$ is shown against n in Fig. 6, the data for the others being omitted to avoid an excessive overlap of points in the figure.

Discussion

The "two-dimensional" foam used in these experiments may be regarded as part of a three-dimensional one, with all but two of its cells forming a single layer sandwiched between a pair of polyhedral cells of infinite diameter and infinitely many faces. The fact that these faces being of glass are planar, instead of curved as they would be if they were of the same material as the walls of the foam cells in the layer, enables the layer to be easily observed without affecting the geometrical relations that are here the object of our attention, though it influences it in other respects, causing it, for example, to appear unlike the section of a foam consisting of several layers of cells [5,6]. A foam of this kind is, however, appropriate for our present purpose because of the extensive change of shape its cells may undergo if allowed to stand.

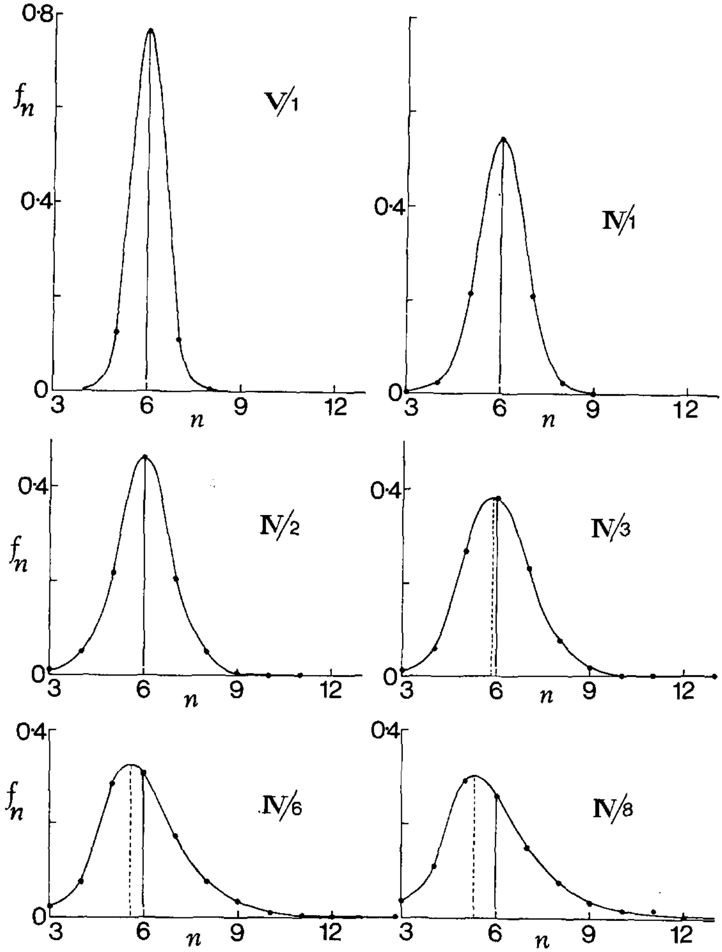

FIG. 2. Distribution of f_n, the fraction of n-sided cells, at different stages of growth.

When the foam is first made most of its cells are hexagonal, so that initially μ_2 is nearly equal to zero. Fig. 1(a) shows that at this stage the relatively few nonhexagonal cells are nearly all 5- or 7-sided occurring in 5–7 pairs, and this suggests that the early evolution of the cellular aggregate may be determined by properties peculiar to such pairs. The values of μ_2 at which 4- (and 8-) and 3- (and 9-) sided cells first appear, during the adjustment that takes place before individual cells collapse, are found to be roughly $\frac{1}{4}$ and $\frac{1}{2}$; and the first 10-sided cells appear when μ_2 is roughly equal to 1 [Fig. 2 (a–c)]. The observed symmetry of the distri-

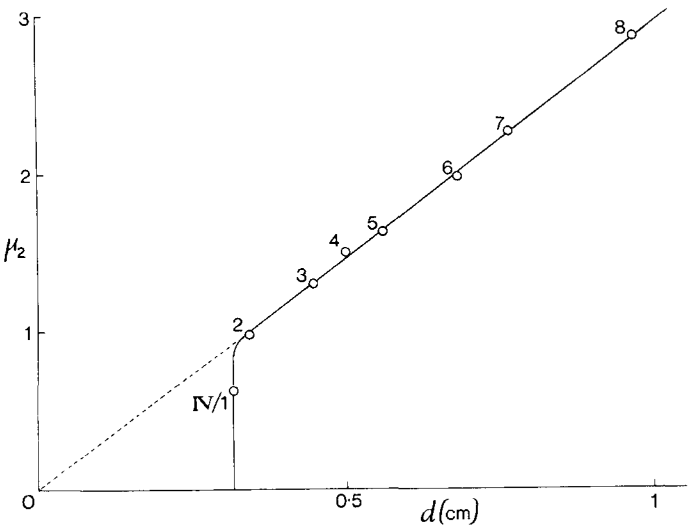

FIG. 3. Variation of μ_2, the second moment of the distribution of n about $n = 6$, with d, the average linear intercept of the cells at different stages of growth.

bution of n during this stage implies that the formation of each cell of $(6 + p)$ sides is generally accompanied by that of another of $(6 - p)$ sides, where $p = 1, 2$, or 3. Further, if there is no breaking of the soap film, the total number of cells during this stage, and hence their average linear intercept, should remain unchanged; and this is found to be roughly the case, since, as Fig. 3 shows, there is little difference in the average linear intercepts of foams IV/1 ($\mu_2 = 0.62$) and IV/2 ($\mu_2 = 0.98$), the variation of μ_2 in this region being therefore represented in the figure by a vertical line.

The form of the distribution of n changes, however, as the smallest cells collapse and cells with more than 9 sides are formed, since, for the average number of sides to remain equal to 6 in accordance with Euler's theorem, the formation of a cell with more than 9 sides necessitates that of at least two others with fewer than 6 sides. The distribution curve consequently loses its symmetry, becoming, as Fig. 2 (d–f) shows, gradually more skew as cells with 10 or more sides are formed; and this is accompanied by a similar change in the distribution of the length of cell sides (Fig. 4) and of cell diameters (Fig. 5). Moreover, as a result of the continual collapse of some cells, the average size of the remaining ones gradually increases. Figure 3 shows how in this region of change μ_2 increases proportionally to the average linear intercept of the cells; and that within the limits of measurement, this increase, like that of the

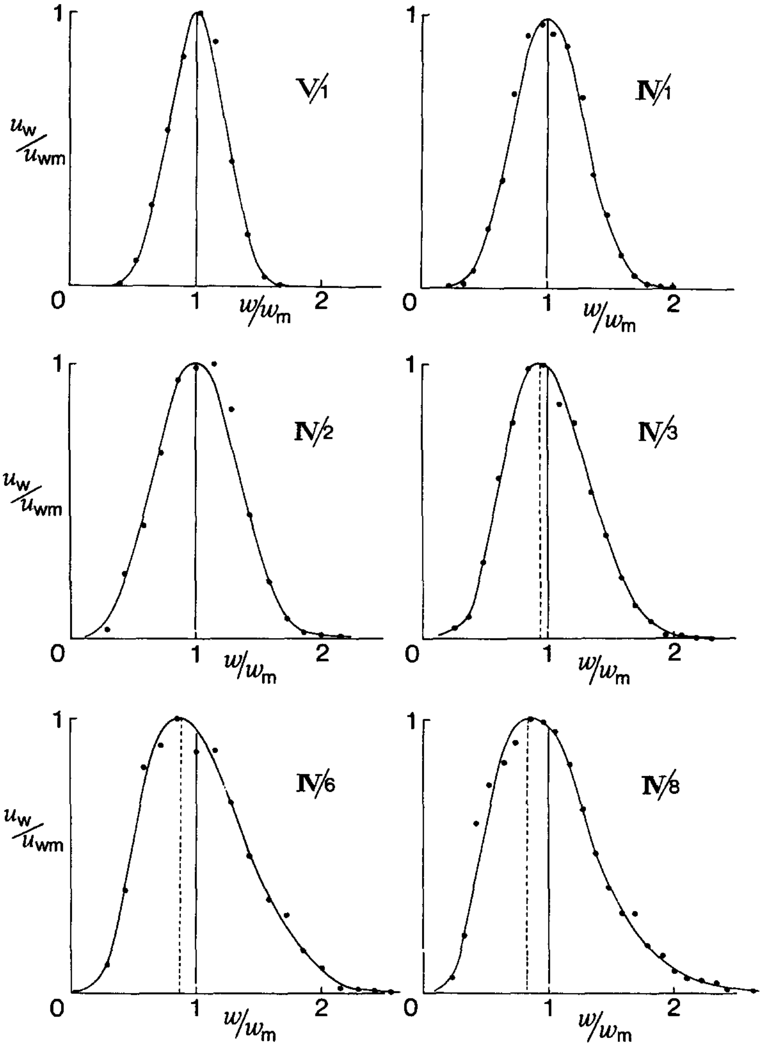

FIG. 4. Distribution of the length of cell sides at different stages of growth.

average size of the cells with time [5], shows no sign of coming to an end. This behavior contrasts with that of a typical polycrystal, for which as we have seen the form of the distribution of n during grain growth does not usually vary with the average grain size [4] and for which, at a given temperature, a stage is generally reached at which the grains cease to grow [7].

Figure 6 shows that $m_n n$ varies linearly with n for widely differing

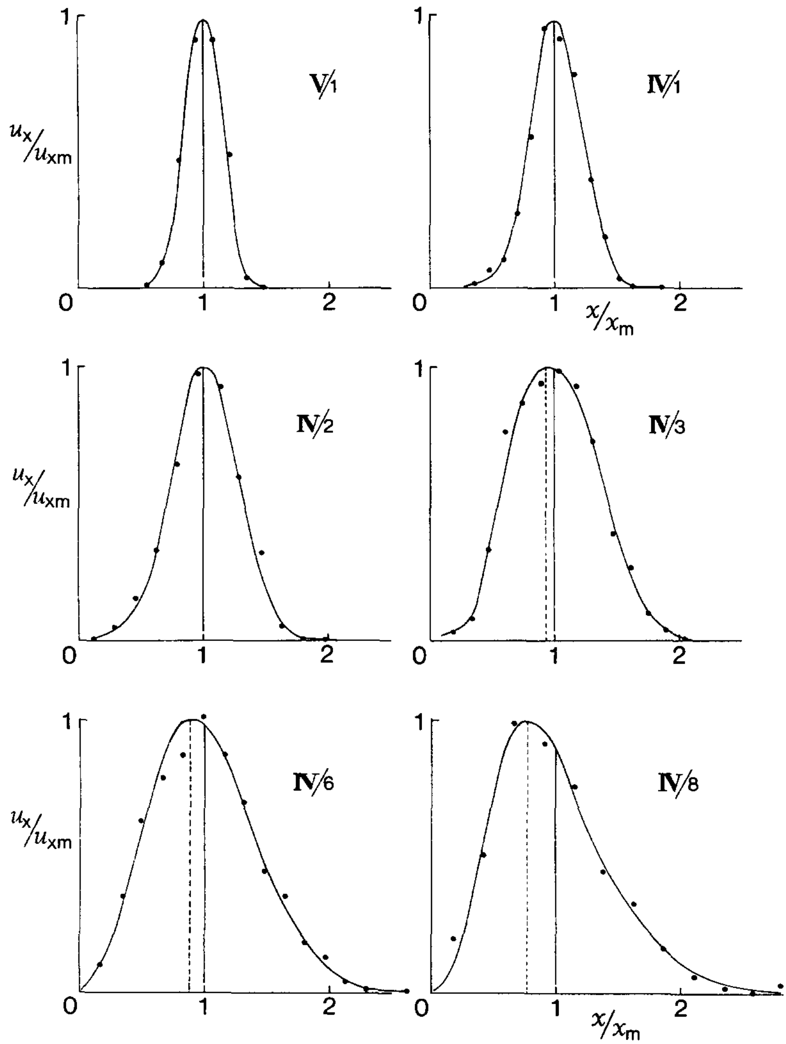

FIG. 5. Distribution of cell diameters at different stages of growth.

values of μ_2: the foam thus resembles a polycrystal in that the dependence of m_n on n can be expressed as

$$m_n = A + (B/n), \tag{6}$$

where A and B are numerical coefficients. The values of these coefficients, calculated by the method of least squares from the data of Tables 1-6, are shown together with the values of μ_2 for each of the six foams

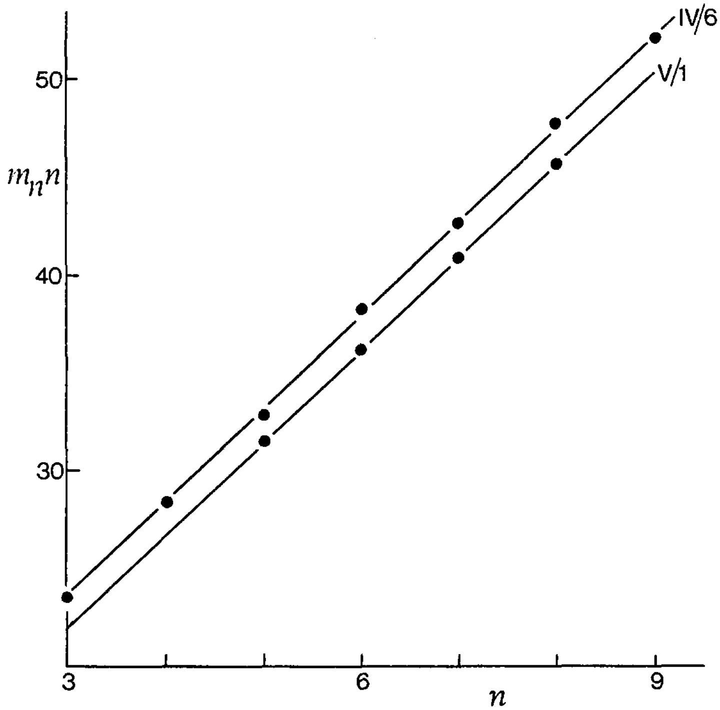

FIG. 6. Variation of $m_n n$ with n for foams V/1 (μ_2 = 0.24) and IV/6 (μ_2 = 1.98).

in Table 7, from which it is seen that A is, within experimental error, independent of μ_2. Its constant b being therefore equal to 0, Eq. (4) may be expressed more simply as

$$m_n = (6 - \mathrm{a}) + \frac{6a + \mu_2}{n} \tag{7}$$

and, since the average value of A from Table 7 is 4.8, the constant a is equal to 1.2.

In Tables 1-6, the observed value of m_n for the different foams are compared with m_n', the value of m_n calculated from the formula

$$m_n = 4.8 + [(7.2 + \mu_2)/n] \tag{8}$$

obtained by substituting the above value of a in Eq. (7). The comparison shows that for each foam the average error ($m_n - m_n'$) lies within 2 or 3% of the observed range of m_n.

The dependence of m_n on μ_2 and n is thus found to be roughly as

TABLE 7

Foam	μ_2	A	B
V/1	0.24	4.72	7.87
IV/1	0.62	4.86	7.31
IV/2	0.98	4.77	8.18
IV/3	1.30	4.78	8.64
IV/6	1.98	4.77	9.20
IV/8	2.86	4.82	9.90

Weaire suggested it might be for a polycrystal [2], his Eq. (5) (above) differing from Eq. (7) only in the value of the constant a. The available data suggest, but do not suffice to establish, that Eq. (7) with the value of a determined here holds for a polycrystal as well as for a foam.

Figure 7(a) is an example of a net for which the general Eq. (6) does not hold, as can be seen from Fig. 7(b), which shows for this net the product $m_n n$ against n, and a straight line representing Eq. (7) with its constant a equal to 1.2 and μ_2 equal to 4. The net is not a random one since it has a repeating pattern of cells, though this is not obvious from a conformal mapping of it [Fig. 8(a)] made to look like a section of a typical polycrystal [Fig. 8(b)]. As has already been noted, and as this

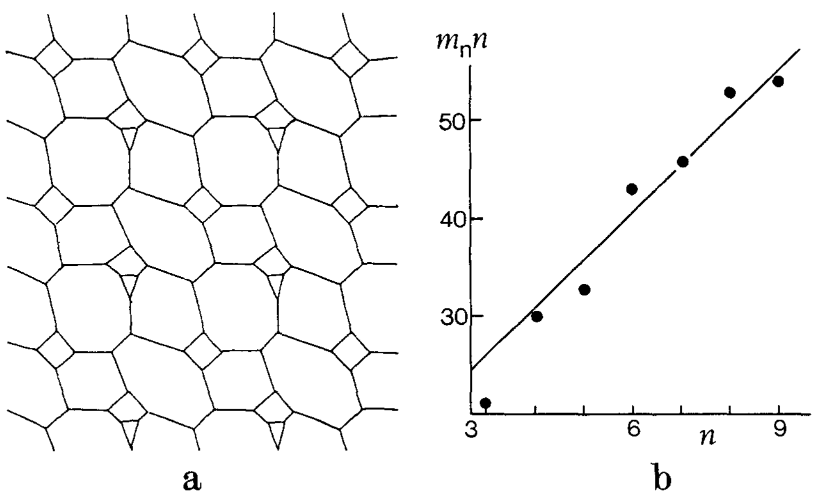

FIG. 7. (a). A net with a repeating pattern of cells. (b). The variation of $m_n n$ with n appertaining to it.

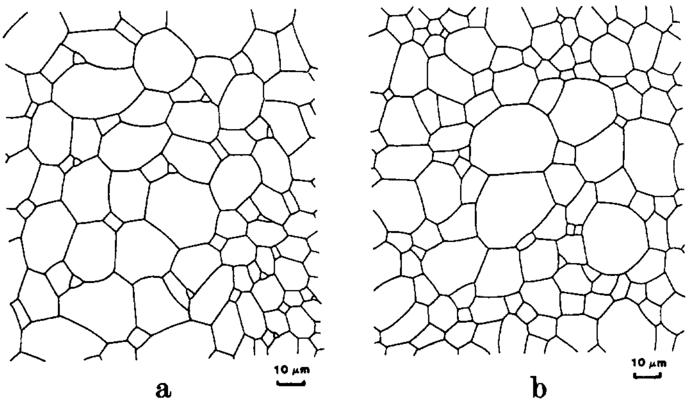

FIG. 8. (a). A conformal mapping of Fig. 7(a). (b). A planar section of polycrystalline magnesium oxide.

example illustrates, the laws of topology are not violated by assuming a dependence of m_n on n different from that given by Eq. (7), and this raises the question of the significance of that equation and of the value of its constant a. In particular, it is important to ascertain whether the empirical value $a = 1.2$ is a purely geometrical property of a random net, or whether it depends on a physical property of the system engendering it.

As to the observation of chains of similarly shaped grain sections that prompted the present enquiry it must be remembered that in a polycrystal not every grain section with few sides is a section of a grain with few faces, since it may result from cutting a many-faced grain near one of its corners. So it does not follow that in the body of the polycrystal, grains with few faces, which are in effect small grains, are also strung together in chains. Nevertheless, similar concatenations of small cell sections are found in a "mature" two-dimensional foam (Fig. 9). where they certainly represent small cells; and there is some indication from the observed distribution of grains and progress of their growth in a metal [1], that the smallest grains in the body of a polycrystal may also be arranged in that way. This suggests that the chains may be a necessary, topological feature of random nets for which μ_2 exceeds a certain value.

Since it may be questioned whether such a directional property of cell sections is best characterized by the variable m_n, whose magnitude is computed by averaging over all directions in the plane of section, alternative ways of describing this phenomenon are being considered. It is proposed eventually to examine other systems, for example, the cells of living tissues and polycrystals of more than one phase, with the object

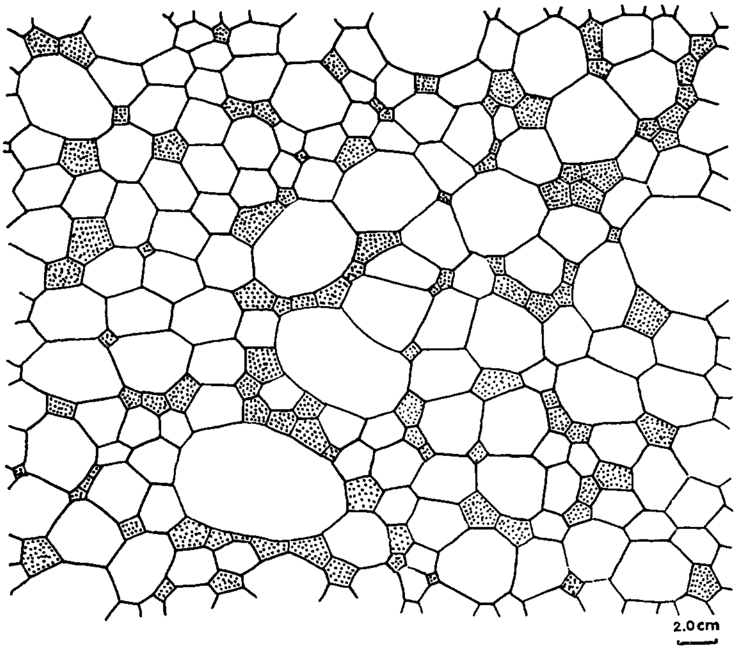

FIG. 9. Section of a "mature" two-dimensional foam (μ_2 = 2.4) with cell-sections of fewer than 6 sides shaded.

of settling the above points definitely and, ultimately, of discovering a practical application of the above findings to problems of metallurgy.

Conclusions

In a section of a foam consisting of a single layer of cells the distribution of n, the number of sides of a cell section, varies continuously as the cells grow, the average linear intercept of the cell sections being simply related to the second moment, μ_2, of the distribution about $n = 6$.

The dependence on n of m_n, the average number of sides of cell-sections in contact with n-sided ones, is accurately represented over a wide range of n and μ_2 by the equation

$$m_n = (6 - a) + \frac{6a + \mu_2}{n}$$

where a is a constant equal to 1.2.

In a "mature" foam smaller cell sections may occur linked together in chains.

The available data do not suffice to determine whether the last two of these regularities, which unlike the first are also observed in polycrystals, are a geometrical feature of random nets, or whether they depend on a physical property of the system engendering them.

I am indebted to C. S. Smith for placing at my disposal the photographs of a soap foam referred to in this work.

References

1. D. A. Aboav, *Metallography* **3**:383 (1970).
2. D. Weaire, *Metallography* **7**:157 (1974).
3. W. M. Williams and C. S. Smith, *J. Metals* **4**:755 (1952).
4. D. A. Aboav and T. G. Langdon, *Metallography* **2**:171 (1969).
5. C. S. Smith, *Metal Interfaces,* Cleveland ASM (1952), pp. 65–108.
6. P. L. White and L. H. Van Vlack, *Metallography* **3**:241 (1970).
7. E. N. da C. Andrade and D. A. Aboav, *Proc. Roy. Soc. (London)* **291A**:18 (1966).

Received May 1979; revised September 1979.