The Arrangement of Cells in a Net. III

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A topological property is shown to be correlated with a metrical property in several naturally occurring and artificial irregular plane tessellations.

Introduction

In this work we continue to investigate the kind of pattern made by the grains of a metal in plane section, namely that of an irregular planar tessellation whose vertex figures [1] are triangles. It has been established empirically [2, 3] that the shape of a cell in such a tessellation is related statistically to that of its nearest neighbors by the equation

$$m_n = 6 - a + (6a + c)/n, \tag{1}$$

where m_n is the average number of sides of cells in contact with n-sided cells, and a and c are numerical parameters; c is within experimental error equal to μ_2, the variance of n.

In naturally occurring tessellations, for which μ_2, and hence the parameter c, may have different values [2, 3], the parameter a is found to be constant and equal to 1.2; whereas in some artificial irregular tessellations, where Eq. (1) is valid [4], a may have a value as low as 0.5. Our object is to correlate this observed variation of the parameter a with that of another property of the tessellation, for which purpose we introduce the concept of the dual of an irregular tessellation.

The Dual Tessellation

Among the tessellations of the type being investigated there exists one that is *regular*. Its cells, all alike, are regular hexagons. Part of this tessellation is shown in Fig. 1a, where the center of every hexagon is joined

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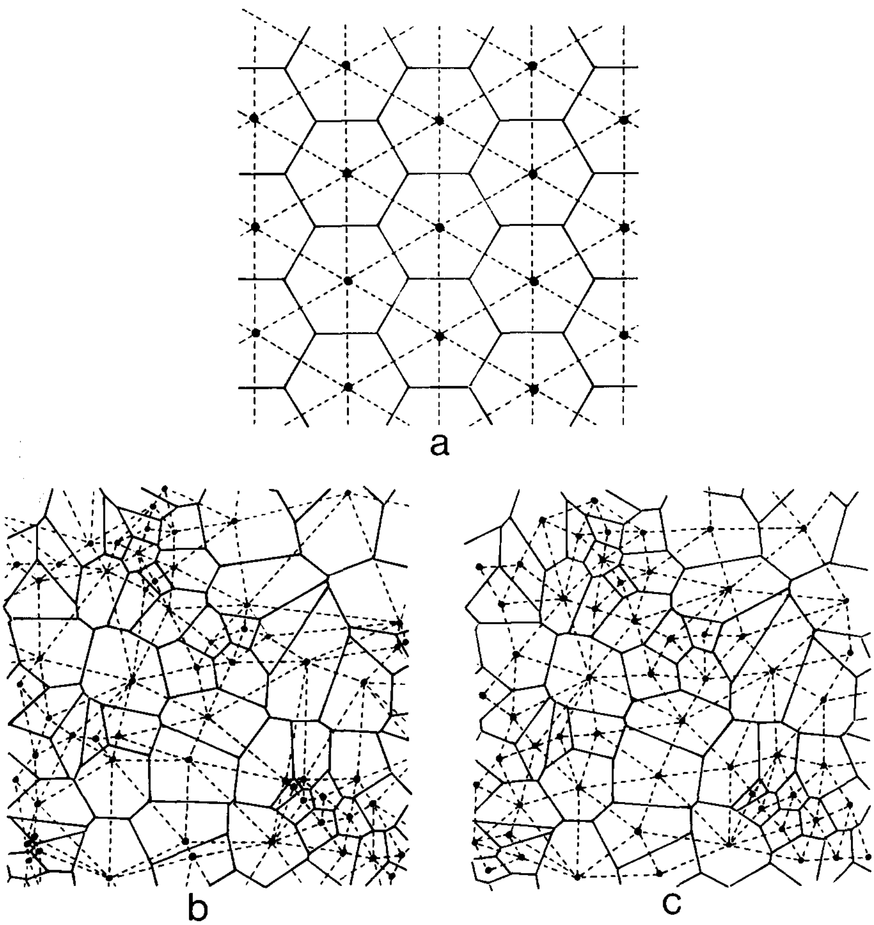

FIG. 1. Tessellations and their duals: (a) regular; (b) irregular, with generating points of cells as vertices of dual tessellations; (c) as (b), but with centroids as vertices.

by a broken line to the center of each of the hexagons in contact with it. The resulting tessellation of equilateral triangles is called the *reciprocal*, or *dual*, tessellation [1].

We may, by analogy, extend the meaning of this term to describe the figure obtained by joining, with a straight line, a point within every cell of an *irregular* tessellation to an equivalent point within each of the cells in contact with it. Defined in this way, however, the dual of an irregular tessellation is not unique, as is evident from Figs. 1b and 1c, where two such duals are seen to belong to the same tessellation of Dirichlet polygons. In Fig. 1b the points chosen as vertices of the dual are the generating points of the polygons, while in Fig. 1c they are their centroids. Since a dual of this definition can in general be drawn in more than one way, its particular construction has in each case to be specified.

In general, the dual of an irregular tessellation is also irregular, and both have unequal cell sides. If x and y represent the side lengths of the cells of a tessellation and its dual, respectively, a measure of their dispersion is given by their standard deviations σ_x and σ_y:

$$\left.\begin{aligned} \sigma_x &= \langle (x - \langle x \rangle)^2 \rangle^{1/2}, \\ \sigma_y &= \langle (y - \langle y \rangle)^2 \rangle^{1/2}. \end{aligned}\right\} \tag{2}$$

Using the normalized deviations s_x and s_y defined by

$$\begin{aligned} s_x &= \sigma_x/\langle x \rangle, \\ s_y &= \sigma_y/\langle y \rangle, \end{aligned} \tag{3}$$

we define the *differential longitudinal dispersion*, α, of a tessellation and its dual as the ratio of the difference, and the sum, of s_x and s_y:

$$\alpha = (s_x - s_y)/(s_x + s_y) \tag{4}$$

Irregular Tessellations in Nature

Figures 2 (a–f) are outline sketches of the following naturally occurring irregular tessellations: (a–d) a single layer of foam cells at four different stages of growth [2]; (e) the domains of a thin film of chalcogenide glass [3]; and (f) a polycrystalline ceramic [5]. The black dots in the sketches represent the approximate centroids of the cells of the tessellations, and the broken lines joining them form, according to our definition, dual tessellations.

In Figs. 3 and 4 the distribution of the side lengths x of the cells of the tessellations of Fig. 2, and the side lengths y of the cells of their duals, are shown as u' and v' against x' and y'. Note that u', v', x' and y' are equal to u/u_m, v/v_m, $x/\langle x \rangle$, and $y/\langle y \rangle$, respectively. The number of sides in the tessellation having length x within a given range is u, and its max imum value is u_m. The number of sides in its dual having length y within a given range is v, and its maximum value is v_m.

Table 1 shows, for each of the above tessellations: z, the approximate number of cells measured; μ_2, the variance of n; the quantities s_x and s_y defined by Eq. (3); and the differential longitudinal dispersion, α, derived from them (4). In the last column of the table we give the values of the parameter a (1) reported in parts I and II of this work [2, 3]. The results

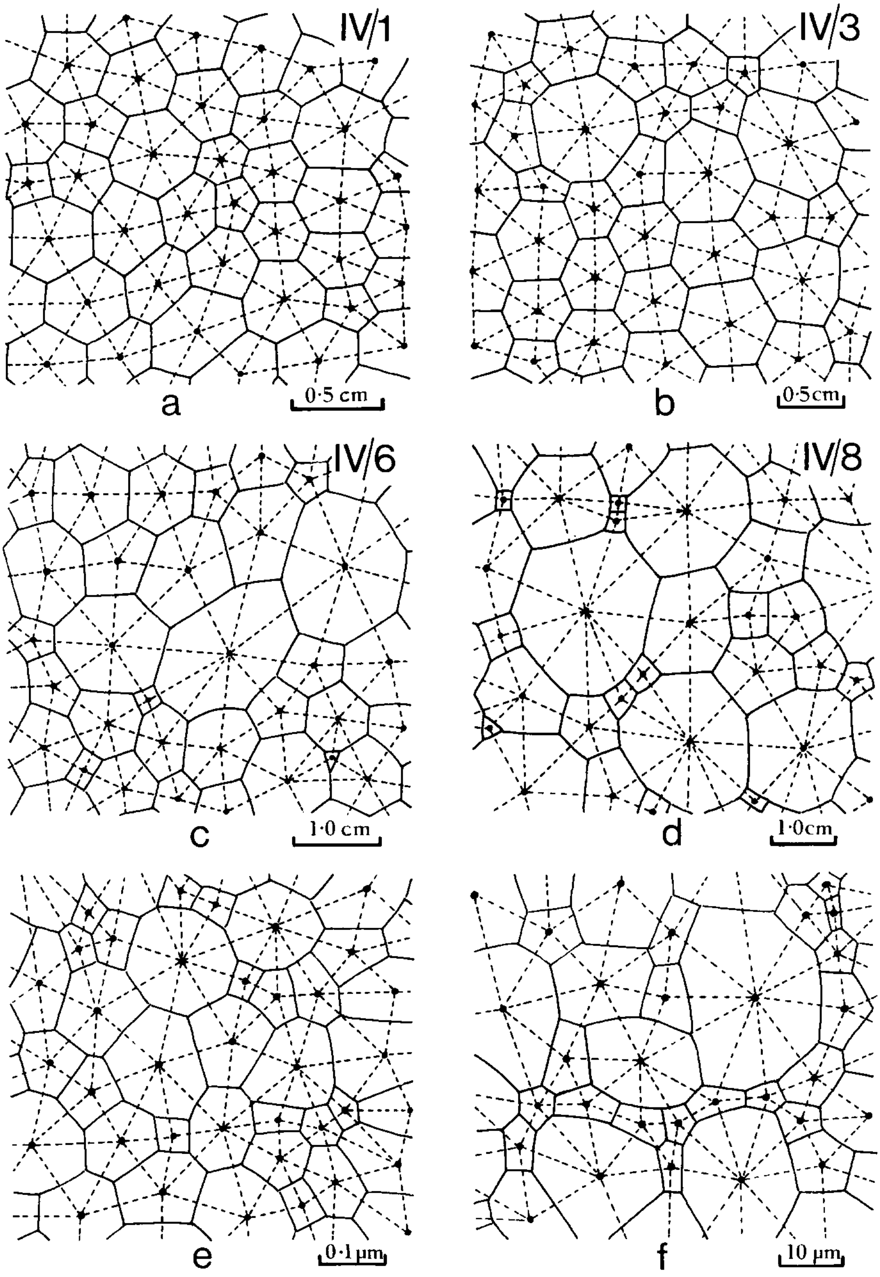

FIG. 2. Naturally occurring tessellations and their duals: (a–d) layer of foam cells at four different stages of growth; (e) domains of a thin film of chalcogenide glass; (f) polycrystalline ceramic.

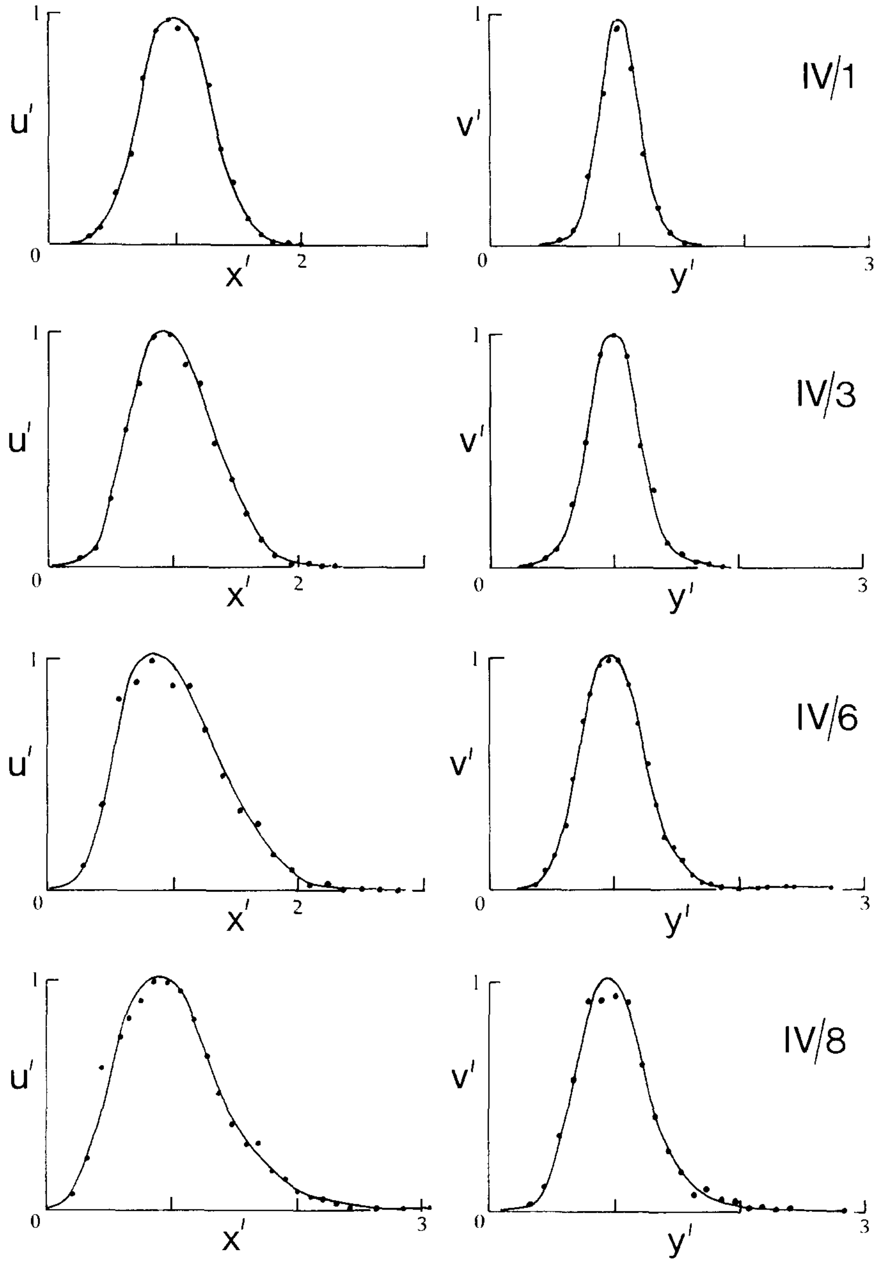

FIG. 3. Distribution of side lengths of cells in foam tessellations (left) and in their duals (right).

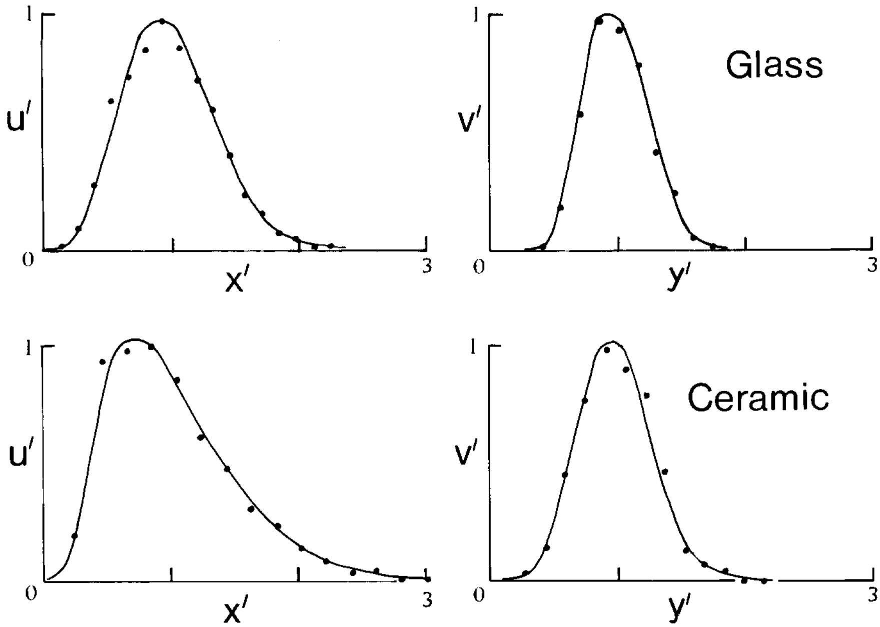

FIG. 4. Distribution of side lengths of cells in glass and ceramic tessellations (left) and in their duals (right).

obtained with the ceramic, being less accurate because of the smaller number of cells available for measurement, are shown in the table in italics.

Artificial Irregular Tessellations

The centers of the cells of a naturally occurring tessellation are generally distributed in its plane at random. But a number of points in a plane can

TABLE 1

Tessellation	z	μ_2	s_x	s_y	α	a
Foam IV/1	1000	0.6	0.258	0.170	0.205	1.14
Foam IV/3	1100	1.3	0.318	0.214	0.195	1.22
Foam IV/6	1000	2.0	0.389	0.261	0.197	1.23
Foam IV/8	500	2.9	0.422	0.284	0.195	1.18
Glass	700	1.8	0.353	0.234	0.202	1.21
Ceramic	*300*	*1.8*	*0.49*	*0.31*	*0.23*	*1.2*

in principle yield more than one kind of random distribution, from each of which a typical, irregular tessellation can be generated artificially. Here we shall consider, as examples of such tessellations, ones whose cells are Dirichlet regions of the generating points—the so-called *Dirichlet tessellations*—because they are easy to draw.

On each of four sheets of white card, of size $A2$ (0.25 m^2 in area), roughly 2000 dots were placed at random, giving a density of about 8000 dots/m^2. The dots, which were used to generate tessellations as described below, were distributed in each of the four cases as follows.

1. From an enlargement of the foam of the previous section, the centroids of a group of about 2000 of its cells were transferred to the card, and there marked as dots. The tessellation generated by them is here referred to as *Tessellation A* (Fig. 5a).
2. For *Tessellation B* the card was divided into a grid of roughly 2000 large squares, each of which was subdivided into 36 smaller square compartments. In each of the large squares a dot was placed at random into one of its compartments, using one throw of a pair of dice for each dot (Fig. 5b).
3. For *Tessellation C* the card was subdivided into roughly 400 squares, each of which was again divided into 36 compartments. In each of the larger squares 5 dots were randomly placed, using one throw of a pair of dice for each dot (Fig. 5c).
4. Finally, a card was divided into 15,552 ($12 \times 36 \times 36$) compartments and 2000 dots distributed at random among them, using three throws of a pair of dice for each dot. The tessellation generated by these dots (Fig. 5d), here referred to as *Tessellation D*,[1] approximates to the RVP tessellation of Boots [4].

On each of the four cards, perpendicular bisectors of the lines joining adjacent dots were drawn freehand, in pencil, to produce a tessellation of 2000 Dirichlet polygons. Where it seemed that more than three such lines might meet at the same point, care was taken to slew them slightly, to cause them to meet not more than three together at points separated by at least 1 mm. This precaution, which introduced into the tessellation a non-random error too small to affect the general result of our investigation, had to be taken to ensure that all the vertex figures of the tessellation were triangles.

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[1] A further card was in fact divided into 186,624 ($4 \times 36 \times 36 \times 36$) compartments and 2000 dots distributed among them, using 4 throws of a pair of dice for each dot. This experiment was abandoned, however, when, on drawing the tessellation, it became apparent it was not going to differ appreciably from Tessellation D.

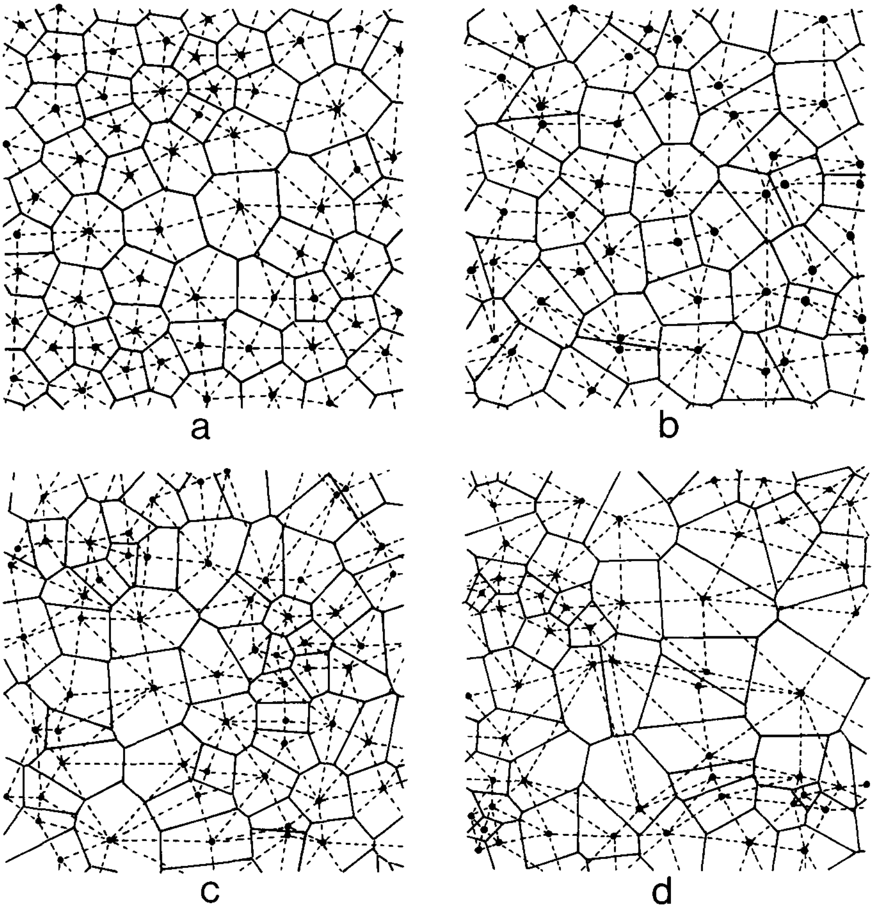

FIG. 5. Dirichlet tessellations generated by random sets of points differently distributed in a plane, and their duals.

On completion of the pencil sketch of the tessellation, the sides of its cells were ruled straight, in black ink, using a Rotring 0.35 mm pen, and at the same time measured with a ruler to the nearest millimeter. The dots within each pair of contiguous cells were then joined by a straight line, drawn in red ink with a Rotring 0.25 mm pen, and also measured to the nearest millimeter. The red lines thus form a dual tessellation, which contrasts well with, and hence is easily distinguished from, the tessellation drawn in black. (This is unfortunately not the case in Fig. 5, where the red lines have had to be shown as broken black lines).

The number of sides n of each cell, and the average number m_n of sides

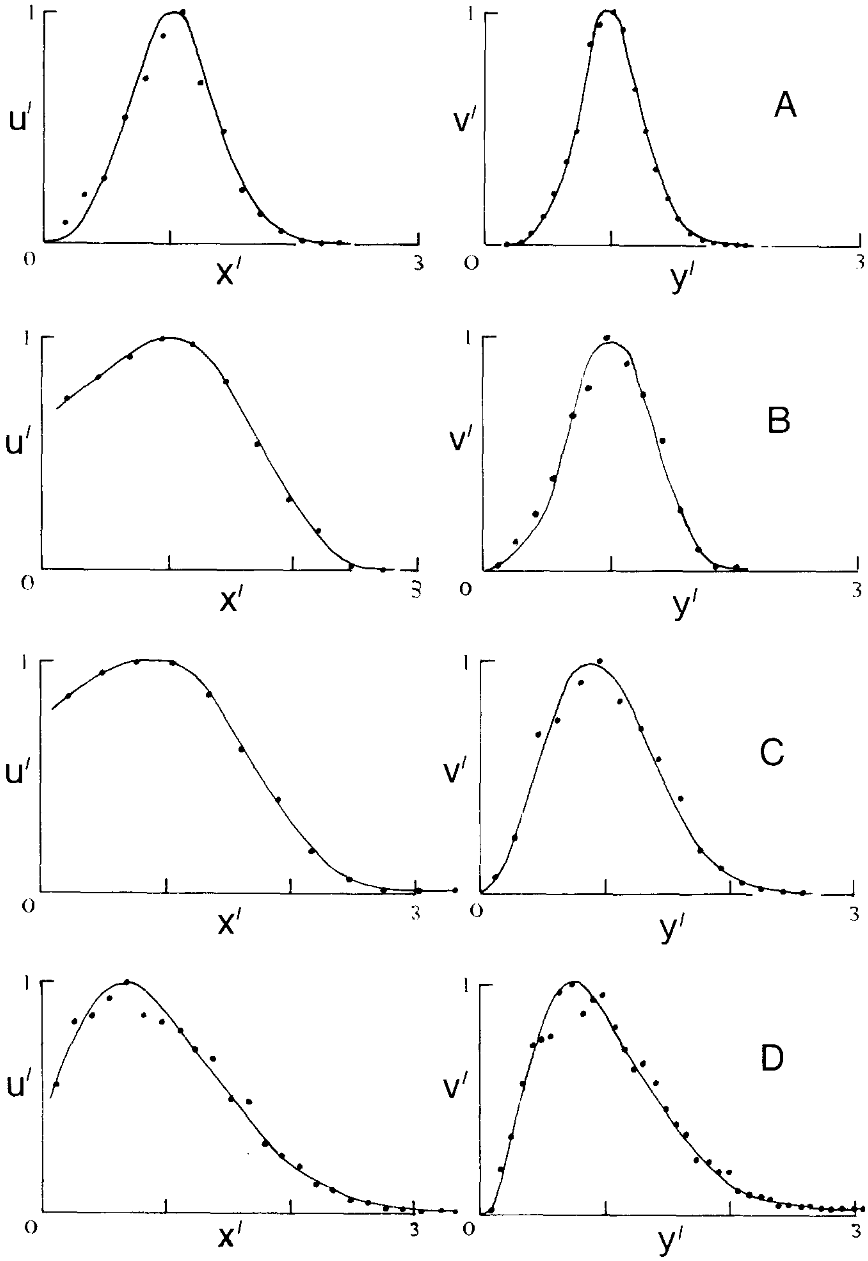

FIG. 6. Distribution of side lengths of cells in Dirichlet tessellations A–D (*left*) and in their duals (*right*).

of cells adjoining n-sided cells, were determined for the resulting tessellations. In each case a value of the parameter a, calculated from Eq. (1), was found to give values of m_n differing on average from the observed values by about 2 percent of the observed range of m_n. To save space, the calculated and observed values of m_n are not, as on previous occasions [2, 3], tabulated here.

In Fig. 6 the distribution of the side lengths x of the cells of the tessellations of Fig. 5, and the side lengths y of the cells of their duals, are shown as u' and v' against x' and y', where these symbols have the same meaning as in the previous section.

Table 2 shows, for each of these tessellations: z, the approximate number of cells measured; μ_2, the variance of n; the quantities s_x and s_y

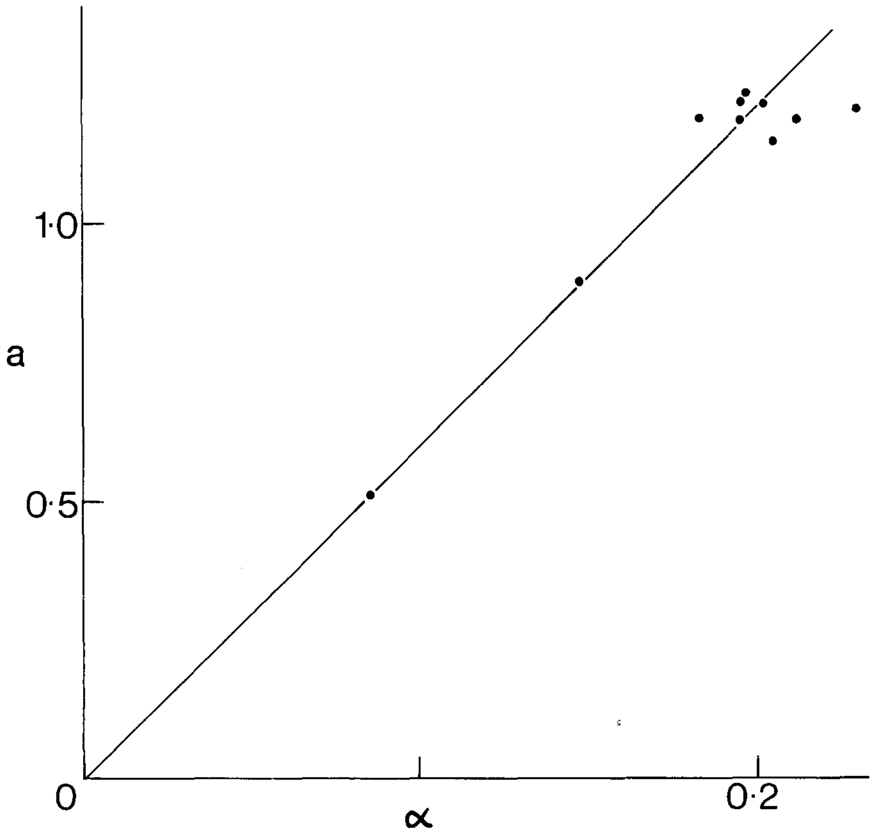

FIG. 7. Variation of parameter a of Eq. (1) with differential longitudinal dispersion, α, defined by Eq. (4).

TABLE 2

Tessellation	z	μ_2	s_x	s_y	α	a
A	2100	1.3	0.378	0.261	0.183	1.18
B	1600	1.0	0.534	0.347	0.212	1.18
C	2000	1.5	0.570	0.424	0.147	0.89
D	2000	1.6	0.590	0.498	0.085	0.51

defined by Eq. (3); the differential longitudinal dispersion, α, derived from them (4); and the parameter a (1).

In Fig. 7, a is shown against α for each of the six naturally occurring and four artificial tessellations considered here. The line in the figure is given by the equation

$$a = 6\alpha. \tag{5}$$

The total number of cell-sides measured in these experiments was 77,078.

Discussion

We have already seen [2] that, in an irregular tessellation for which Eq. (1) holds, the parameter c is within experimental error equal to μ_2, the variance of n. It has now been shown that the parameter a of the same equation is proportional to the differential longitudinal dispersion, α, defined by Eq. (4). Since $a = 6\alpha$ (5), Eq. (1) can be written as

$$m_n = 6(1 - \alpha) + (36\alpha + \mu_2)/n. \tag{6}$$

Equation (6), which expresses a geometrical property of a tessellation, is found to be valid with the same accuracy in all the cases considered here, and appears to be independent of the physical properties of the tessellation, and of its mode of generation. As these results show, it holds for a range of α from 0.08 to 0.2, and of μ_2 from 0.6 to 2.9.

It has already been established empirically [2] that μ_2 and the parameter a are independent. The following example indicates that s_x and s_y, in terms of which α is defined (3), and μ_2, are in principle also independent. In Fig. 8a the tessellation of Fig. 1a is transformed to make the triangles of its dual isosceles, leaving the hexagons equilateral; so that $s_x = 0$, and $s_y > 0$. In Fig. 8b the hexagons of the same tessellation are transformed to make their sides unequal, leaving the triangles equilateral; hence $s_x > 0$, and $s_y = 0$. Further, μ_2 is not changed by the transformations, since they are conformal.

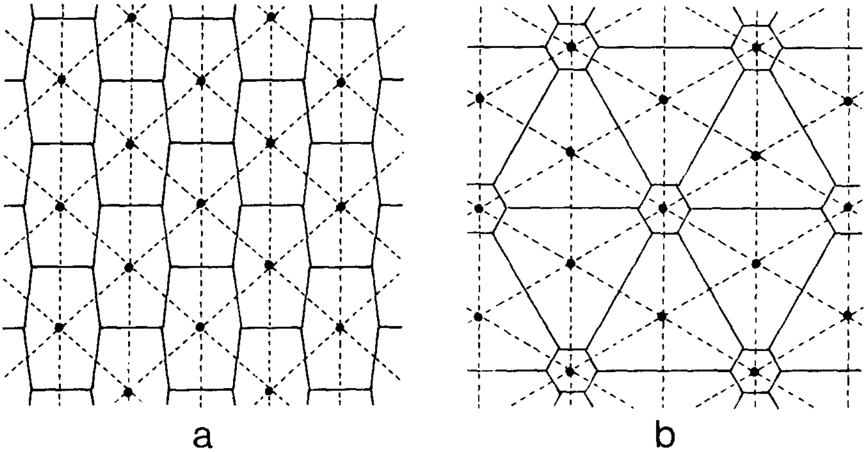

FIG. 8. Conformal transformations of Fig. 1a.

But it does not follow that, because s_x and s_y can vary independently of each other, and of μ_2, they may not in particular cases be correlated. Table 1, for example, shows that for naturally occurring tessellations s_x and s_y are correlated, s_y being in all cases equal to $2s_x/3$. From Table 3, which shows the values of μ_2, s_x, and the ratio $(s_x/\mu_2)^{1/3}$ for five of the natural tessellations (the foam and the film of glass), and the three artificial ones B, C, and D, generated by throws of dice, it will be seen that s_x and μ_2 are also correlated: s_x is proportional to the cube root of μ_2, the factor of proportionality being 0.30 for the foam and the glass, and 0.51 for the artificial tessellations. If, instead of μ_2, the standard deviation σ_n is used to express the dispersion of n, it follows that, for the foam and the film

TABLE 3

Tessellation	μ_2	s_x	$(s_x/\mu_2)^{1/3}$
Foam IV/1	0.6	0.26	0.31
Foam IV/3	1.3	0.32	0.29
Foam IV/6	2.0	0.39	0.31
Foam IV/8	2.9	0.42	0.29
Glass	1.8	0.35	0.29
B	1.0	0.53	0.53
C	1.5	0.57	0.50
D	1.6	0.59	0.50

of glass,

$$\left.\begin{aligned} s_x &= s_n^{2/3}, \\ s_y &= (2s_n^{2/3})/3, \end{aligned}\right\} \tag{7}$$

where $s_n = \sigma_n/\langle n \rangle$ ($= \sigma_n/6$), and $\sigma_n^2 = \mu_2$.

Rather than dwell on relations that hold in particular cases, however, we seek by further observation and experiment to widen the scope of those, like Eq. (6), that have been found to apply generally. Our immediate object is to extend the relatively small range (0.5 through 1.2) of the parameter a realized in these experiments (Tables 1 and 2). We may find that, over a wider range, this parameter is no longer simply expressible in terms of the differential longitudinal dispersion, α, defined by Eq. (4). If so, a function more suitable for this purpose than α will be sought when further data are to hand. This and related matters will be taken up in the next instalment of this work.

Conclusions

The concept of the *dual* of an irregular tessellation, corresponding to the dual, or reciprocal, of a regular tessellation [1], is introduced.

For a wide range of naturally occurring and artificial irregular planar tessellations, whose vertex figures are triangles, the dependence of the shape of a cell on that of its nearest neighbors is found to be well represented by the equation

$$m_n = 6 - a + (6a + c)/n$$

where m_n is the average number of sides of cells adjoining n-sided ones; and a and c are numerical parameters.

It has already been established [2, 3] that c is within experimental error equal to μ_2, the variance of n. The present investigation shows that a is within experimental error equal to 6α, where $\alpha = (s_x - s_y)/(s_x + s_y)$. Note that s_x and s_y represent $\sigma_x/\langle x \rangle$ and $\sigma_y/\langle y \rangle$, and that σ_x and σ_y are the standard deviations of x and y, which are the lengths of cell-sides in the tessellation and its dual.

The above dependence may therefore be expressed as

$$m_n = 6(1 - \alpha) + (36\alpha + \mu_2)/n.$$

This equation holds generally, over a range of α from 0.08 to 0.2, and of μ_2 from 0.6 to 2.9.

Two relations that do not hold generally, but that apply to certain naturally occurring tessellations (foam and glass film) only are

$$s_x = s_n^{2/3},$$

and

$$s_y = (2s_n^{2/3})/3$$

where $s_n = \sigma_n/\langle n\rangle$, and σ_n is the standard deviation of n ($\sigma_n^2 = \mu_2$).

Part IV of The Arrangement of Cells in a Net will be appearing in a later issue of Metallography.

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Received January 1984; accepted March 1984.