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Review Article

Extreme Value Distributions: An Overview of Estimation and Simulation

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The generalized extreme value distribution (GEVD) and various extreme value distributions are commonly applied in air pollution, telecommunications, operational risk management, finance, insurance, material sciences, economics, and hydrology, among many other industries that deal with extreme events. Extreme value distributions (EVDs) typically limit the distribution of maximum and minimum values for many random observations drawn from the same arbitrary distribution. Besides that, it is a crucial method for forecasting future events and emerged as critical method for predicting future events. As a result, prior research is required to select the best estimation method to obtain a reliable value for the parameters of extreme value distributions. This study provides an overview of three-parameter estimation methods based on goodness-of-fit statistics and root mean square error (RMSE). This paper reviewed and compared three estimation methods used to approximate values of parameters for simulated observations taken from the EVD and GEVD. The method of moments (MOMs), maximum likelihood estimator (MLE), and maximum product of spacing (MPS) were the methods investigated in this study. Our findings indicated that the MPS performed better based on the mean square errors (MSEs); meanwhile, the MPS had similar goodness-of-fit statistic values compared to the MLE.

1. Introduction

Extreme value distribution (EVD) is used to limit distributions for maximum or minimum [1]. Thus, as the sample size increases with the smallest or largest data in independent identically distributed random variables, the data set density shape will follow one of the three types of EVD [1, 2]. EVD is also used to model tail-related risk measurements such as value at risk, return level, or expected shortfall [3]. Extreme wind speed analysis is used mainly in natural emergency preparedness, mitigation, management, prevention, and various civil engineering, environmental, and ocean applications [4]. An accurate estimate of the parameters for any analyses using the EVD is a must. Hence, there should be a suitable estimation method that provides accurate estimates for the parameters of the EVD. There had been many studies related to various parameter estimation methods on EVD.

Without a doubt, parameter estimation is essential to fit any probability distribution on any data sets. As a result, various estimation methods could provide us with insight into determining the "best-fitting" distribution and estimate the parameters for EVD, such as the scale, shape, and location parameters. The following are some standard parameter estimation methods that are commonly used in probability distribution fitting:

(i) The MOM (Johan Bernoulli, 1667-1748).
(ii) The MLE (Daniel Bernoulli, 1700-1782).
(iii) The MPS (Cheng and Amin, 1979-1983).

Since their introduction, these methods have progressed through several stages and have their drawbacks and benefits

[5–7]. Nonetheless, the MLE method is the most widely used estimation method.

The three methods mentioned above are used in this study to estimate EVD and GEVD parameters. Several studies were comparing the various estimation methods for different distributions. By reviewing other studies, we described the basic idea of each estimation method and their applications on EVD. A simulation study was carried out for reference purposes to assess the performance of the estimators. As it has been widely deployed in many research areas, the EVD is used to represent the distributions of various observations. These include wind speed and energy data [4, 8–12], wave data prediction [13], data on air pollution [14–18], information and communication technology [19], data on flooding [20], financial risk [3, 21], temperature [22], food drying technology [23], and rainfall [24]. It has also been implemented in public health and medical sciences [25, 26].

Therefore, studies comparing MLE, MOM, and MPS estimators for GEVD, two-parameter EVD, and three-parameter EVD were reviewed in this research. The MOM method is the oldest method for estimating parameters, whereas the MLE is the most commonly used. However, MLE can fail in various circumstances, necessitating a less popular alternative (i.e., MPS). This review article aimed to guide selecting the best estimation method for the GEVD and EVD, which will be of great interest to applied statisticians. The novelty of this review stems from the fact that no thorough review of MOM, MLE, and MPS estimators for EVD has been made. The following is how this article was structured. The history of EVD is presented before reviewing the MLE, MOM, and MPS. Next, EVD applications were discussed, followed by a simulation study. Last but not least, conclusions were drawn based on the factors reviewed and discussed above.

2. Extreme Value Theory (EVT) and Extreme Value Distribution (EVD)

An extreme value in a series of observations is either a very large or small value. It can even be described as the outer or outlier points, which are the highest and lowest values. EVT is a theory of modeling and measuring events with the least amount of probability [27]. To be specific, EVT identifies extreme events based on a probability of occurrence and also depicts the extreme events through statistical analysis of the extreme properties. It consists of 3 types of distributions. It only requires three distributions to model the maximum or minimum random observations for the same distribution [2]. Recently, the EVT has emerged as one of the most important statistical disciplines for engineers and applied scientists [14, 28].

If we assume $X_1; X_2; \ldots, X_n$ are independent random variables with a standard distribution function (F). Then, $M_n = \mathrm{Max}\{X_i; \ldots; X_n\}$ for each i with $i = 1, \ldots, n$ denotes the maximum of observational process over n time units of observations. According to Coles [28], the distribution of M_n can be derived as follows:

$$\begin{aligned}\Pr\{M_n \le x\} &= \Pr\{X_1 \le x, X_2 \le x, \ldots, X_n \le x\}\\ &= \Pr(X_1 \le x)\Pr(X_2 \le x)\ldots\Pr(X_n \le x),\end{aligned} \tag{1}$$

$$\Pr\{M_n \le x\} = [F_X(x)]^n. \tag{2}$$

The probability density function (PDF), $f_X(x)$, for EVD distribution derived from the cumulative function $F_X(x)$ can be derived as below:

$$f_X(x) = n[F_X(x)]^{n-1} f_X(x). \tag{3}$$

There is a concern with degeneration of the exact function because the distribution function, F, is unknown, and $n \longrightarrow \infty$. Hence, we pursue approximate families of models for F^n that can be estimated solely on the extreme data. As per the central limit theorem (CLM), the estimation is similar to the usual practice of approximating the sample means for normal distribution. To resolve this situation, we developed a normalized version of M_n to stabilize the function. A normalized M_n could be generated as below with the presence of normalizing constants, a_n and b_n:

$$M_n^* = \frac{M_n - b_n}{a_n}. \tag{4}$$

The relevant and suitable choices of a_n and b_n stabilize the location and scale of M_n as $n \longrightarrow \infty$. M_n^* converges in the form of three EVD distribution types: Type I, Type II, and Type III. If the normalizing constants a_n and b_n exist, thus,

$$\lim_{n\longrightarrow\infty} \Pr\left(\frac{M_n - b_n}{a_n} \le x\right) \approx G(x), \tag{5}$$

$G(x)$ is the nondegenerate cumulative distribution function (CDF), which relates to the three EVD families: Type I, Type II, and Type III.

2.1. Gumbel Distribution (Type I). Emil Gumbel, a German mathematician, invented the Gumbel distribution (1891-1966). The primary focus was on the extensive use of the EVT in various fields for modeling extreme events [2]. The formula includes the following PDF:

$$f(x;\mu,\sigma) = \frac{1}{\sigma} \times \exp\left[-\frac{x-\mu}{\sigma} - \exp\left(-\frac{x-\mu}{\sigma}\right)\right], \tag{6}$$

whereby σ = distribution scale $(\sigma > 0)$ and μ = location parameter. The CDF can then be given as follows:

$$F(x;\mu,\sigma) = \exp\left(-\exp\left(-\frac{x-\mu}{\sigma}\right)\right). \tag{7}$$

2.2. Fréchet Distribution (Type II). A French mathematician, Maurice Fréchet (1878-1973), had derived the Fréchet Distribution. In 1927, he proposed one possible limiting distribution for the maximal order statistics [2]. The Fréchet distribution is also known as the inverse Weibull distribution (IWD). It includes the following CDF and PDF:

2.2.1. Two-Parameter Fréchet Distribution. PDF is as follows:

$$f(x;\alpha,\sigma)=\frac{\alpha}{\sigma}\left(\frac{\sigma}{x}\right)^{(\alpha+1)}\exp\left(-\left(\frac{\sigma}{x}\right)^{\alpha}\right). \tag{8}$$

CDF is as follows:

$$F(x;\alpha,\sigma)=\exp\left(-\left(\frac{\sigma}{x}\right)^{\alpha}\right). \tag{9}$$

2.2.2. Three-Parameter Fréchet Distribution. PDF is as follows:

$$f(x;\alpha,\sigma,\mu)=\frac{\alpha}{\sigma}\left(\frac{\sigma}{x-\mu}\right)^{(\alpha+1)}\exp\left(-\left(\frac{\sigma}{x-\mu}\right)^{\alpha}\right). \tag{10}$$

CDF is as follows:

$$F(x;\alpha,\sigma,\mu)=\exp\left(-\left(\frac{\sigma}{x-\mu}\right)^{\alpha}\right), \tag{11}$$

whereby $\mu=$ location parameter ($\mu=0$ for the two-parameter Fréchet distribution), $\sigma=$ scale parameter ($\sigma>0$), and α $=$ shape parameter ($\alpha>0$).

2.3. Weibull Distribution (Type III). Waloddi Weibull (1887-1979), a Swedish engineer, invented the Weibull distribution. Initially, the distribution was developed to address minima problems in material sciences [2] where

$$\min(X_1,\ldots,X_n)=-\max(-X_1,\ldots,-X_n). \tag{12}$$

The CDF and PDF for this distribution are as below:

2.3.1. Two-Parameter Weibull Distribution. PDF is as follows:

$$f(x;\alpha,\sigma)=\frac{\alpha}{\sigma}\left(\frac{x}{\sigma}\right)^{(\alpha-1)}\exp\left(-\left(\frac{x}{\sigma}\right)^{\alpha}\right). \tag{13}$$

CDF is as follows:

$$F(x;\alpha,\sigma)=1-\exp\left(-\left(\frac{x}{\sigma}\right)^{\alpha}\right). \tag{14}$$

2.3.2. Three-Parameter Weibull Distribution. PDF is as follows:

$$f(x;\alpha,\sigma,\mu)=\frac{\alpha}{\sigma}\left(\frac{x-\mu}{\sigma}\right)^{(\alpha-1)}\exp\left(-\left(\frac{x-\mu}{\sigma}\right)^{\alpha}\right). \tag{15}$$

CDF is as follows:

$$F(x;\alpha,\sigma,\mu)=1-\exp\left(-\left(\frac{x-\mu}{\sigma}\right)^{\alpha}\right), \tag{16}$$

whereby $\mu=$ location parameter ($\mu=0$ for the two-parameter Weibull Distribution), $\sigma=$ scale parameter ($\sigma>0$), and α $=$ shape parameter ($\alpha>0$).

The three EVD families can be generalized to form a single distribution called the generalized extreme value distribution (GEVD). The GEVD was an extension of the EVT developed by Fisher–Tippett (1928) and Gnedenko (1943). It is a good choice for representing the distribution of the minimum and maximum sequences of independent identically distributed random variables [1, 2].

2.4. GEVD. The CDF for the three-parameter is as follows:

$$F(x;\mu,\sigma,\alpha)=\exp\left\{-\left[1+\alpha\left(\frac{x-\mu}{\sigma}\right)\right]^{(-1/\alpha)}\right\}. \tag{17}$$

From equation (17), σ and $1+\alpha(x-\mu)/\sigma>0$, where μ and α can take any real value. The three types of EVD can be obtained through GEVD based on the value of alpha where $\alpha=0$ is the Type I EVD (Gumbel distribution), $\alpha>0$ is the Type II EVD (Fréchet distribution), and $\alpha<0$ is the Type II EVD (Weibull distribution).

Meanwhile, for the PDF for GEVD is given in equation (18) with $\sigma>0$, with α and μ, can take any real value.

$$f(x;\mu,\sigma,\alpha)=\begin{cases}\exp-\left(1+\dfrac{\alpha(x-\mu)}{\sigma}\right)^{-(1/\alpha)} & \\ \times\dfrac{1}{\sigma}\left\{1+\alpha\left(\dfrac{x-\mu}{\sigma}\right)\right\}, & \alpha\neq 0,\\ \dfrac{1}{\sigma}\exp\left[-\left(\dfrac{x-\mu}{\sigma}\right)-\exp-\left(\dfrac{x-\mu}{\sigma}\right)\right], & \alpha=0.\end{cases} \tag{18}$$

GEVD is broadly used in hydrology, telecommunications, risk management, economics, finance, material sciences, or insurance that deal with extreme events [29–33].

3. Application Study Review

In this section, we will review and discuss the comparison of MOM, MLE, and MPS estimation methods using actual data or simulation studies as the following:

Hall et al. [34] estimated the generalized Gumbel distribution parameters using the MLE method in 1989.

A comparison study was conducted between the standard MLE and the unbiased MLE estimator, which is derived from MLE linear functions, product spacing method, and quantile estimate method to estimate two exponential distribution parameters. For both the location and scale parameters, the unbiased MLE had the lowest RMSE, followed by MPS and MLE. Overall, both methods performed nearly identically equivalent. However, the unbiased MLE provides better parameter estimates [35].

Hurairah et al. [36] proposed a new Gumbel distribution for handling air pollution data by introducing a new parameter that shapes the parameter α. The MLE method is applied to estimate the parameters of the new Gumbel distribution. The simulated results indicated that the new Gumbel distribution could achieve higher accuracy in fitting carbon monoxide (CO) data and significantly impacting air pollution studies [14].

Other research also used the MLE to estimate the following parameters: Gumbel, generalized Pareto distribution with two and three parameters, Weibull with two and three parameters, and GEVD [17]. The two-year daily maximum data were used to analyze the efficiency of the six distributions using error and accuracy measures as performance indicators. The GEVD was found to be an adequate distribution for maximum daily density of particulate matter (PM10) for all monitoring stations under study. MOM was used in another study to estimate the parameters of the Gumbel and Fréchet distribution instead of lognormal to fit the daily maximum concentration of PM10 in Malaysia. The goodness-of-fit was used to select the distribution that best fits the data for PM10 exceedances based on the Malaysian Ambient Air Quality Guidelines (MAAQG). The work concluded that the EVD fits the actual high value of PM10 better than central fitting distribution [16].

On the other hand, Wong and Li [37] compared the MLE and the MPS in estimating parameters of EVD using samples with small sample sizes. His study found that the MPS functioned satisfactorily. Not only does it performs consistently for data maxima extracted from clusters, but it also accurately estimated more data generated from a known parameter set, whereas the MLE does not. Based on this finding, the MPS is considered one of the best estimation methods for fitting EVD.

Jiang [38] had demonstrated that the location and scale estimator parameters were biased, and MPS underestimated the shape parameter. Hence, he modified an MPS to fit a three-parameter Weibull distribution that could accurately estimate parameters better. Meanwhile, Huang and Lin [39] also altered the MPS method to improve the estimate parameters of the GEVD. The simulations revealed that not only is the suggested method highly efficient and applicable across the entire parameters, but it also outperforms the study's existing parametric and nonparametric methods.

A least square estimation (LSE), MLE, and MPS were used to compare traditional estimation methods to fit the generalized inverted exponential distribution [40]. The study was also intended to analyze the estimates' behavior for small samples. Results showed that MPS outperformed the other two methods with a minor mean square error (MSE). Therefore, the study suggested using MPS since it exceeded both MLE and LSE.

Akram and Hayat [41] compared the performance of fitting a three-parameter Weibull distribution with the following parameter estimation methods in terms of bias and RMSE in a small sample: L-moments, LSE, the modified MLE, MOM, and MPS. Overall, the L-moments method performed well and is the best estimation method. The modified MPS performed well when the shape parameter was less than a specific value. In contrast, the modified MLE method was inefficient and inconsistent because it might not exist.

Next, Soukissian and Tsalis [4] investigated parameter estimation methods for predicting extreme wind speeds in the Atlantic and Pacific ocean basins. A natural wind measurements and simulation study from four buoys were used in the analysis. According to the research, the MPS, elemental percentile (EP), and standard entropy method appeared less accurate than the MLE. Based on the MSE, bias, and variance of the estimated data, the MLE was a much better estimation method.

Meanwhile, Salah et al. [42] used various estimation methods such as weighted least squares, MLE, probability-weighted moments, and LSE for the accelerated life test (ALT) under the family of exponentiated distributions. He chose the best method to estimate the reliability function. The four methods were applied using both simulated and actual-world data. Among other estimation methods, it has been discovered that the MLE produces the best results. Louzada et al. [43] considered the MLE, modified moments, MOM, L-moments, minimum distance estimator percentile estimation, MPS, ordinary, and weighted least squares for estimating unknown parameters of the extended exponential geometric distribution. Compared to its competitors, the MPS estimated the best for the extended exponential geometric distribution parameters.

Singh et al. [44] studied the possibility of estimating the scale and shape parameters for the generalized inverted exponential distribution using progressive type-II censored samples. The MPS was used to estimate the reliability, hazard functions, and parameters of the model. Based on a Monte Carlo simulation study, the MPS was compared to the corresponding MLE. Based on MSE, it is discovered that the MPS method outperforms the MLE. As a result, regardless of sample size, the former method could estimate reliability, hazard function, and distribution parameters well.

Dey et al. [45] investigated various methods and properties for estimating unknown parameters for the following distributions"

(i) Exponentiated Chen distribution.

(ii) Transmuted Rayleigh distribution.

(iii) Exponentiated Gumbel distribution.

The right-tail Anderson–Darling, MLE, percentile estimation, MOM, least squares estimation, Cramér-von-Mises, MPS, and Anderson–Darling methods were used in this study. Extensive simulation studies were used to compare them using Monte Carlo simulations. The results revealed that the MPS is the best estimator for transmuted Rayleigh and exponentiated Chen distributions in terms of biases and RMSE. The MLE method, on the other hand, is the best for estimating the exponentiated Gumbel distribution parameters [7, 46, 47].

The finite sample properties of the Marshall–Olkin extended exponential distribution parameters were obtained by ten estimation methods using Monte Carlo simulations. They were Anderson–Darling, weighted least squares, L-moments, maximum likelihood, right-tail Anderson–Darling, ordinary least squares, modified moments, MPS, percentile estimation, and Cramér-von-Mises. The performance of all the methods was compared using the absolute, bias, and maximum absolute difference between RMSE and the estimated and actual distribution functions. The simulation demonstrated that the MLE and L-moments perform admirably in large sample sizes. Nonetheless, both methods have lower accuracy with small sample sizes than the MPS and Anderson–Darling methods [48].

The MPS was employed in the linear regression model based on Student-t, normal, skewed Student-t, and MLE distributions. A study found that all of the estimates were consistent and, in some cases, outperformed the MLE method. Furthermore, the MPS estimator is likely to exceed MLE when the sample size is small [49].

Vivekanandan [22] conducted Hissar extreme value analysis of rainfall and temperature using a logged Pearson Type-3 probability distribution and two-parameter log-normal fitted to one-day maximum and minimum rainfall and annual temperature series. L-moments, MLE, and MOM estimation methods were used to determine the distribution parameters based on their applicability. The study's tests revealed that the MLE estimated better than other methods for allocating the minimum and maximum rainfall and temperature.

Meanwhile, Nassar et al. [50] proposed a new extension for Weibull distribution. Two shape parameters and one scale parameter were included in the proposed distribution. It also contains submodels such as logarithmic-altered Weibull distribution and exponential distribution and the logarithmic-transformed exponential and logarithmic-transformed Weibull distributions. The research concentrated on the unknown parameters as well as several new mathematical properties. Least squares, MLE, percentile-based, MPS, and weighted-least square estimators have all been used. Monte Carlo simulations were used to compare the proposed estimation methods for large and small samples. Based on the results, percentile-based was the best performing estimator with respect to MSE. The applications on two actual data sets showed that the MPS performed better than the least square estimator for data set I. Meanwhile, the least square method is a better estimator for data set II.

Dey et al. [45] applied various estimation methods on the Gompertz distribution in a medical application. Fourteen methods were used to estimate the model parameters. A simulation study was conducted to compare these methods, and it was discovered that modified moments and moment estimators outperform others. Nonetheless, MPS estimators can still perform reasonably well and produce good results.

Last but not least, Ramos et al. [51] investigated the estimation of Fréchet distribution parameters. MLE, percentile estimators, MOM, L-moments, MPS, and ordinary and weighted-least squares were compared in this study, focusing on MSE. In terms of RMSE, the results revealed that MPS outperformed the other estimators significantly.

4. Parameter Estimation Methods

The parameters of EVD have been estimated using a variety of methods. Nonetheless, we will only concentrate on MLE, MOM, and MPS for the distributions mentioned above to evaluate the performance of each estimation method.

4.1. Maximum Likelihood Estimator. Maximum likelihood estimator (MLE) is one of the methods used for estimating model parameters [5]. The MLE principle is to use the model with the highest likelihood. It is a necessary tool for many statistical modeling techniques and becomes a favored method of parameter estimation in statistics [52]. There are three advantages of MLE [28]: it has desirable mathematical and optimality properties, it could give a consistent approach to parameter estimation problems, and it is applicable in almost all popular statistical software packages. An example of using MLE to estimate parameters for a probability distribution with 3 parameters μ, σ, and α is as follows:

(i) Step I. The likelihood function of the probability distribution $L(\mu, \sigma, \alpha)$ is obtained and written as follows:

$$\begin{aligned} L(\mu, \sigma, \alpha) &= \prod_{i=1}^{n} f_{\mu,\sigma,\alpha}(x_i), \\ L(\mu, \sigma, \alpha | x) &= L(\mu, \sigma, \alpha | x_1, \wedge, x_n) = \prod_{i=1}^{n} f(x_i | \mu, \sigma, \alpha). \end{aligned} \tag{19}$$

(ii) Step II. Take the natural log of the likelihood and collect terms involving μ, σ, α.

(iii) Step III. The differentiation of $L(\mu, \sigma, \alpha)$ and solve it with respect to μ, σ, and α:

$$\begin{aligned} \frac{\partial}{\partial \mu}\{\log L(\mu)\} &= 0, \\ \frac{\partial}{\partial \sigma}\{\log L(\sigma)\} &= 0, \\ \frac{\partial}{\partial \alpha}\{\log L(\alpha)\} &= 0. \end{aligned} \tag{20}$$

The formulas for estimating μ, σ, and α for various extreme value distributions using MLE are shown in Table 1.

4.2. Method of Moments. Method of moments (MOMs) is one of the conventional estimation methods for fitting statistical distributions [51]. The MOM estimators are usually easy to use and almost always produce some estimate. Unfortunately, MOM frequently generates estimators that could be improved. This method relies on matching the distribution moment to the sample moment. It is built on the presumption that sample moments should provide reasonable estimates of the corresponding population moments [53]. Equations in Table 2 show that $\overline{x}$ and S represent the sample mean and standard deviation, respectively. We define the mean value by $\mu = (1/n) \sum_{i=1}^{n} (X_i)$. The j^{th} sample moment is then computed as follows:

$$\mu_j = \frac{1}{n} \sum_{i=1}^{n} (X_i)^j, \tag{21}$$

Table 1: Estimation parameters by maximum likelihood estimator (MLE).

Extreme value distribution (EVD)	Parameter estimator using MLE
Gumbel (Type I)	$\sigma = \overline{x} - \left(\sum_{i=1}^{n} \exp(-x_i/\sigma) / \sum_{i=1}^{n} \exp(-x_i/\sigma)\right), \mu = -\sigma \ln\left((1/n) \sum_{i=1}^{n} \exp(-x_i/\sigma)\right)$
2-Fréchet (Type II)	$\widehat{\sigma} = \left(n / \sum_{i=1}^{n} x_i^{-\widehat{\alpha}}\right), (n/\alpha) - \sum_{i=1}^{n} \log(x_i) - \left(n / \sum_{i=1}^{n} x_i^{-\widehat{\alpha}}\right) \sum_{i=1}^{n} x_i^{-\widehat{\alpha}} \log(x_i) = 0$
	$\widehat{\sigma} = (1/e^{\alpha}) \sum_{i=1}^{n} (x_i - \mu)^{\alpha}$
3-Fréchet (Type II)	$(1/\alpha) + \ln\ \left(\sigma / \sum_{i=1}^{n} (x_i - \mu)\right)^{*} \left(1 + \left(\sigma / \sum_{i=1}^{n} (x_i - \mu)\right)^{\alpha}\right) = 0$
	$\left(\alpha + 1 / \sum_{i=1}^{n} (x_i - \mu)\right) - \left(\alpha\sigma^{\alpha} / \sum_{i=1}^{n} (x_i - \mu)^{\alpha+1}\right) = 0$
2-Weibull (Type III)	$\widehat{\sigma} = \left((1/n) \sum_{i=1}^{n} x_i^{\alpha}\right)^{1/\alpha}, (1/\alpha) - \left(\sum_{i=1}^{n} x_i^{\alpha} \ln(x_i) / \sum_{i=1}^{n} x_i^{\alpha}\right) + (1/n) \sum_{i=1}^{n} \ln(x_i) = 0$
3-Weibull (Type III)	$\widehat{\sigma} = \left((1/n) \sum_{i=1}^{n} (x_i - \mu)^{\alpha}\right)^{1/\alpha}, (1/\alpha) - \left(\sum_{i=1}^{n} (x_i - \mu)^{\alpha} \ln(x_i - \mu) / \sum_{i=1}^{n} (x_i - \mu)^{\alpha}\right) + \cdots + (1/n) \sum_{i=1}^{n} \ln(x_i - \mu) = 0$
	$\alpha - 1/\alpha \sum_{i=1}^{n} (x_i - \mu)^{-1} - n\left(\sum_{i=1}^{n} (x_i - \mu)^{\alpha-1} / \sum_{i=1}^{n} (x_i - \mu)^{\alpha}\right) = 0$
	$(1/\sigma) \sum_{i=1}^{n} \left(1 - \alpha - (1 - (\alpha/\sigma)(x_i - \mu))^{1/\alpha} / (1 - (\alpha/\sigma)(x_i - \mu))\right) = 0$
GEV-distribution (GEVD)	$-(n/\sigma) + 1/\sigma \sum_{i=1}^{n} \left((1 - \alpha - (1 - (\alpha/\sigma)(x_i - \mu))^{1/\alpha} / (1 - (\alpha/\sigma)(x_i - \mu))) \times (x_i - \mu/\sigma)\right) = 0$
	$-(1/\alpha^2) \sum_{i=1}^{n} \left(\ln(1 - (\alpha/\sigma)(x_i - \mu)) \times [1 - \alpha - (1 - (\alpha/\sigma)(x_i - \mu))^{1/\alpha}] \times (1 - \alpha - (1 - (\alpha/\sigma)(x_i - \mu))^{1/\alpha} / (1 - (\alpha/\sigma)(x_i - \mu))) + \alpha(x_i - \mu/\sigma)\right) = 0$

Table 2: Estimation parameters by method of moments (MOMs).

Extreme value distribution (EVD)	Parameter estimator using MOM
Gumbel (Type I)	$\widehat{\mu} = \overline{x} - \alpha\sigma, \widehat{\sigma} = (\sqrt{6}\, S/\pi)$
2-Fréchet (Type II)	$\widehat{\alpha} = \sqrt{\Gamma(1 - (2/\alpha))/\Gamma^2(1 - \alpha - 1)} - 1 - (8/x), \widehat{\sigma} = (\overline{x}^{\alpha}/\Gamma^{\alpha}(1 - \alpha - 1))$
2-Weibull (Type III)	$\widehat{\sigma} = (\widehat{\alpha}\overline{x}/\Gamma(1/\widehat{\alpha})), \widehat{\alpha} = (\sum_{i=1}^{n} x_i^2 [\Gamma(1/\widehat{\alpha})]^2/2n\overline{x}^2\Gamma(2/\widehat{\alpha}))$
	$\widehat{\sigma} = (\overline{x} - \mu/\Gamma(1 + (1/\alpha)))$
3-Weibull (Type III)	$\widehat{\alpha} = (\Gamma[1 + (1/\alpha)]/\sqrt{[\Gamma[1 + (2/\alpha)] - \Gamma^2[1 + (1/\alpha)]]})$
	$\widehat{\mu} = (x_1 x_n - x_2^2/x_1 + x_n - 2x_2)$
	$\widehat{\mu} = \overline{x} + (\widehat{\sigma}/\alpha[\Gamma(1 + \alpha) - 1])$
GEV-distribution (GEVD)	$\widehat{\sigma} = ((\pm\widehat{\alpha})S(\alpha)/\sqrt{[(\Gamma 1 + 2\alpha) - (\Gamma 1 + 2\alpha)^2]})$
	$\widehat{\alpha} = (\pm\widehat{\alpha}(-\Gamma(1 + 3\widehat{\alpha}) + 3\Gamma(1 + \widehat{\alpha})\Gamma(1 + 2\widehat{\alpha}) - 2[\Gamma(1 + \widehat{\alpha})]^3)/(\Gamma(1 + 2\widehat{\alpha}) - \Gamma(1 + \widehat{\alpha})^2)^{3/2})$

and the population moment by $\mu_j(\theta_1, \ldots, \theta_n) = E(X)^j$, for $j = 1, \ldots, n$, where $\theta_1, \ldots, \theta_n$ are unknown parameters. Next, $m_j = \mu_j(\theta_1, \ldots, \theta_n)$ is set and solved for $\theta_1, \ldots, \theta_n$. The equations are the MOM's estimator for $\widehat{\theta}_1, \ldots, \widehat{\theta}_n$. The formulas to estimate the parameters μ, σ, and α for various extreme value distributions using MOM are shown in Table 2.

4.3. Maximum Product of Spacing. Cheng and Amin [54] pioneered the maximum product of Spacing (MPS) method for univariate distributions while Ranneby [55] developed this method to approximate the Kullback–Leibler information measure. Both researchers demonstrated that the MPS method could work in situations where the MLE method fails. They also discovered that MPS estimators own nearly all of the MLE properties. The MPS estimator possesses almost all properties, and it gives consistent estimators with asymptotic efficiency equal to MLE estimators. Furthermore, in some cases where MLE fails, it provides consistent, asymptotically efficient estimators [56].

$$D_i(\mu, \sigma, \alpha) = F\left(x_{(i)}; \mu, \sigma, \alpha\right) - F\left(x_{(i-1)}; \mu, \sigma, \alpha\right), \quad i = 1, \ldots, n + 1. \tag{22}$$

The MPS estimators are regarded as values that maximize the logarithm of the sample spacing geometric. The estimated parameters μ, σ, and α.

$$H(\widehat{\mu}, \widehat{\sigma}, \widehat{\alpha}) = \underset{\mu,\sigma,\alpha\in\Theta}{\arg\max}\, S_n(\mu, \sigma, \alpha), \tag{23}$$

where

$$\begin{aligned} S_n(\mu, \sigma, \alpha) &= \ln^{n+1}\sqrt{D_1, D_2, \ldots, D_{n+1}} \\ &= \frac{1}{n+1}\sum_{i=1}^{n+1} \ln D_i(\mu, \sigma, \alpha), \end{aligned} \tag{24}$$

$\widehat{\mu}, \widehat{\sigma}$, and $\widehat{\alpha}$ estimators from the parameter μ, σ, and α could be achieved by solving the nonlinear equations as follows:

$$\begin{aligned} \frac{\partial(\mu, \sigma, \alpha)}{\partial\mu} &= \frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\mu, \sigma, \alpha)}\left[\delta_1(x_i|\mu, \sigma, \alpha) - \delta_1(x_{i-1}|\mu, \sigma, \alpha)\right], \\ \frac{\partial(\mu, \sigma, \alpha)}{\partial\sigma} &= \frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\mu, \sigma, \alpha)}\left[\delta_1(x_i|\mu, \sigma, \alpha) - \delta_1(x_{i-1}|\mu, \sigma, \alpha)\right], \\ \frac{\partial(\mu, \sigma, \alpha)}{\partial\alpha} &= \frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\mu, \sigma, \alpha)}\left[\delta_1(x_i|\mu, \sigma, \alpha) - \delta_1(x_{i-1}|\mu, \sigma, \alpha)\right], \end{aligned} \tag{25}$$

where δ = the derivative of the cumulative function of the extreme distribution with respect to the estimated parameter. Reference [54] demonstrated that maximizing μ, σ, and α in MOM is as efficient as MLE. Compared to the MLE estimator, the MPS is more consistent under general conditions. The equations for estimating μ, σ, and α for various extreme value distributions using MPS are shown in Table 3.

5. Simulation Study

Some experimental results comparing the MOM, MLE, and MPS estimation methods were discussed in this section using a simulated study to investigate the performances of the proposed estimators. We simulated Gumbel distribution (Type I), Fréchet distribution (Type II), Weibull distribution

Table 3: Maximum product of spacings (MPS).

Extreme value distribution (EVD)	Parameter estimator using MPS
Gumbel (Type I)	$[(1/\sigma)\exp((x_{(i:n+1)} - \mu/\sigma) - \exp(x_{(i:n+1)} - \mu/\sigma))] - [(1/\sigma)\exp((x_{(i-1:n+1)} - \mu/\sigma) - \exp(x_{(i-1:n+1)} - \mu/\sigma))] = 0$
	$[(x_{(i:n+1)} - \mu/\sigma^2)\exp((x_{(i:n+1)} - \mu/\sigma) - \exp(x_{(i:n+1)} - \mu/\sigma))] - [(x_{(i-1:n+1)} - \mu/\sigma^2)\exp((x_{(i-1:n+1)} - \mu/\sigma) - \exp(x_{(i-1:n+1)} - \mu/\sigma))] = 0$
2-Fréchet (Type II)	$[(\sigma/x_{(i:n+1)})^{\alpha} \ln(\sigma/x_{(i:n+1)})^{*} - \exp(\sigma/x_{(i:n+1)})^{\alpha}] - [(\sigma/x_{(i-1:n+1)})^{\alpha} \ln(\sigma/x_{(i-1:n+1)})^{*} - \exp(\sigma/x_{(i-1:n+1)})^{\alpha}] = 0$
	$[-(\alpha\sigma^{\alpha-1}/x^{\alpha}_{(i:n+1)})\exp(-(\sigma/x_{(i:n+1)}))^{\alpha}] - [-(\alpha\sigma^{\alpha-1}/x^{\alpha}_{(i-1:n+1)})\exp(-(\sigma/x_{(i-1:n+1)}))^{\alpha}] = 0$
3-Fréchet (Type II)	$[-(\alpha\sigma^{\alpha}/(x_{(i:n+1)} - \mu)^{\alpha+1})\exp(-(\sigma/(x_{(i:n+1)} - \mu)))^{\alpha}] - [-(\alpha\sigma^{\alpha}/(x_{(i-1:n+1)} - \mu)^{\alpha+1})\exp(-(\sigma/(x_{(i-1:n+1)} - \mu)))^{\alpha}] = 0$
	$[-(\sigma/x_{(i:n+1)} - \mu)^{\alpha} \ln(\sigma/x_{(i:n+1)} - \mu)\exp(-(\sigma/x_{(i:n+1)}))^{\alpha}] - [-(\sigma/x_{(i-1:n+1)} - \mu)^{\alpha} \ln(\sigma/x_{(i-1:n+1)} - \mu)\exp(-(\sigma/x_{(i-1:n+1)}))^{\alpha}] = 0$
	$[-(\alpha\sigma^{\alpha-1}/(x_{(i:n+1)} - \mu))\exp(-(\sigma/(x_{(i:n+1)} - \mu)))^{\alpha}] - [-(\alpha\sigma^{\alpha-1}/(x_{(i-1:n+1)} - \mu)^{\alpha})\exp(-(\sigma/(x_{(i-1:n+1)} - \mu)))^{\alpha}] = 0$
2-Weibull (Type III)	$[(-\alpha x^{\alpha}_{(i:n+1)}/\sigma^{\alpha+1})e^{-(x_{(i:n+1)}/\sigma)^{\alpha}} - (-\alpha x^{\alpha}_{(i-1:n+1)}/\sigma^{\alpha+1})e^{-(x_{(i-1:n+1)}/\sigma)^{\alpha}}] = 0$
	$[(x_{i:n+1}/\sigma)^{\alpha} \ln(x_{i:n+1}/\sigma)\exp-(x_{i:n+1}/\sigma)^{\alpha} - (x_{i-1:n+1}/\sigma)^{\alpha} \ln(x_{i-1:n+1}/\sigma)\exp-(x_{i-1:n+1}/\sigma)^{\alpha}] = 0$
3-Weibull (Type III)	$[(x_{i:n+1} - \mu/\sigma)^{\alpha} \ln(x_{i:n+1} - \mu/\sigma)\exp-(x_{i:n+1} - \mu/\sigma)^{\alpha} - (x_{i-1:n+1} - \mu/\sigma)^{\alpha} \ln(x_{i-1:n+1} - \mu/\sigma)\exp-(x_{i-1:n+1} - \mu/\sigma)^{\alpha}] = 0$
	$[(-\alpha(x_{(i:n+1)} - \mu)^{\alpha}/\sigma^{\alpha+1})e^{-(x_{(i:n+1)} - \mu/\sigma)^{\alpha}} - (-\alpha(x_{(i-1:n+1)} - \mu)^{\alpha}/\sigma^{\alpha+1})e^{-(x_{(i-1:n+1)} - \mu/\sigma)^{\alpha}}] = 0$
	$[(-\alpha(x_{(i:n+1)} - \mu)^{\alpha-1}/\sigma^{\alpha})e^{-(x_{(i:n+1)} - \mu/\sigma)^{\alpha}} - (-\alpha(x_{(i-1:n+1)} - \mu)^{\alpha-1}/\sigma^{\alpha})e^{-(x_{(i-1:n+1)} - \mu/\sigma)^{\alpha}}] = 0$
GEV-distribution (GEVD)	$[-\exp-(1 + -\varepsilon(x_{(i:n+1)} - \mu/\sigma))^{-(1/\alpha)}\sigma^{1/\alpha}(\sigma + \alpha(x_{(i:n+1)} - \mu))^{-1-\alpha/\alpha} + \exp-(1 + -\alpha(x_{(i-1:n+1)} - \mu/\sigma))^{-(1/\alpha)}\sigma^{1/\alpha}(\sigma + \alpha(x_{(i-1:n+1)} - \mu))^{-1-\alpha/\alpha}] = 0$
	$[-(x_{(i:n+1)} - \mu/\sigma^2)\exp-(1 + \alpha(x_{(i:n+1)} - \mu/\sigma))^{-1/\sigma}(1 + \alpha(x_{(i:n+1)} - \mu/\sigma))^{-1-\alpha/\alpha} - -(x_{(i-1:n+1)} - \mu/\sigma^2)\exp-(1 + \alpha(x_{(i-1:n+1)} - \mu/\sigma))^{-1/\sigma}(1 + \alpha(x_{(i-1:n+1)} - \mu/\sigma))^{-1-\alpha/\alpha}] = 0$
	$[-\exp-(1 + \alpha(x_{(i:n+1)} - \mu/\sigma))^{-(1/\alpha)}(1 + \alpha(x_{(i:n+1)} - \mu/\sigma))^{-(1/\alpha)}((\ln(1 + \alpha(x_{(i:n+1)} - \mu/\sigma))/\alpha^2) - (x_{(i:n+1)} - \mu/\alpha\sigma + \alpha^2(x_{(i:n+1)} - \mu))) + \exp-(1 + \alpha(x_{(i-1:n+1)} - \mu/\sigma))^{-(1/\alpha)}(1 + \alpha(x_{(i-1:n+1)} - \mu/\sigma))^{-(1/\alpha)}((\ln(1 + \alpha(x_{(i-1:n+1)} - \mu/\sigma))/\alpha^2) - (x_{(i-1:n+1)} - \mu/\alpha\sigma + \alpha^2(x_{(i-1:n+1)} - \mu)))] = 0$

Table 4: Maximum likelihood estimation method (MLE).

Parameter	Gumbel dist	2-Fréchet dist	3-Fréchet dist	2-Weibull dist	3-Weibull dist	GEV-distribution
$N = 1{,}000$	$\mu = 5,\ \sigma = 9$	$\sigma = 2,\ \alpha = 5$	$\mu = 1,\ \sigma = 5,\ \alpha = 3,$	$\sigma = 3,\ \alpha = 7$	$\mu = 3,\ \sigma = 1,\ \alpha = 5,$	$\mu = 1,\ \sigma = 3,\ \alpha = 0.01$
Location μ	5.113738	—	1.1914393	—	3.0428244	1.143650684
95% CI	(4.534619, 5.693822)	—	(0.0011892, 2.38169)	—	(2.468179, 3.617469)	(0.876557, 1.410744)
CP	0.946	—	0.951	—	0.941	0.939
Bias	0.113738	—	0.1914393	—	0.0428244	0.143650684
MSE	0.015054	—	0.0405421	—	0.0877919	0.039205604
Scale σ	8.869851	1.991488	4.7274478	2.999102	0.9575833	3.085025926
95% CI	(8.164171, 9.575531)	(1.867646, 2.115330)	(3.757663, 5.69723)	(2.790232, 3.207972)	(0.557583, 1.357578)	(2.836395, 3.333657)
CP	0.941	0.970	0.940	0.978	0.957	0.936
Bias	−0.130149	−0.008512	−0.2725522	−0.000898	−0.0424167	0.085025926
MSE	0.014656	0.004064	0.0319098	0.011357	0.0434473	0.023320987
Shape α	—	5.047998	2.9302638	7.204294	4.8323436	0.009948418
95% CI	—	(4.849987, 5.246009)	(2.096661, 3.763867)	(6.892497, 7.516091)	(4.4164256, 5.248262)	(−0.003527, 0.023424)
CP	—	0.961	0.936	0.935	0.955	0.971
Bias	—	0.047998	−0.0697362	0.204294	−0.1676564	$-5.1582E-05$
MSE	—	0.012510	0.0018574	0.0067042	0.0073138	$4.72724E-05$
$N = 1{,}000{,}000$	$\mu = 3,\ \sigma = 5$	$\sigma = 15,\ \alpha = 30$	$\mu = 0,\ \sigma = 30,\ \alpha = 10,$	$\sigma = 10,\ \alpha = 10$	$\mu = 0.5,\ \sigma = 13,\ \alpha = 10,$	$\mu = 3,\ \sigma = 5,\ \alpha = 0.01$
Location μ	2.997115	—	−0.08513055	—	0.46896840	3.005138285
95% CI	(2.617996, 3.376234)	—	(−1.275381, 1.105119)	—	(−0.389635, 1.32757)	(2.828848, 3.181428)
CP	0.961	—	0.929	—	0.947	0.964
Bias	−0.002885	—	−0.08513055	—	−0.0310316	0.005138285
MSE	0.000374	—	0.00376024	—	0.0019286	0.008116277
Scale σ	4.998493	14.98750	30.09018208	9.999651	13.0295087	5.006856483
95% CI	(4.692813, 5.304173)	(14.85046, 15.12455)	(29.05454, 31.12583)	(9.553377, 10.44592)	(12.41002, 13.6490)	(4.810955, 5.202757)
CP	0.971	0.969	0.959	0.963	0.934	0.965
Bias	−0.00150700	−0.012500	0.09018208	−0.0003490	0.02950870	0.006856483
MSE	0.00024550	0.0002051	0.010924786	0.0005185	0.001869747	0.000146910
Shape α	—	30.63226	10.02834579	9.990547	10.0163149	0.009561359
95% CI	—	(29.44289, 31.82163)	(9.199042, 10.85765)	(9.487613, 10.49348)	(9.227676, 10.80495)	(−0.068591, 0.087714)
CP	—	0.921	0.953	0.979	0.929	0.967
Bias	—	0.632260	0.02834579	−0.009453	0.0163149	−0.000438641
MSE	—	0.0403435	0.00259374	0.000747787	0.001885151	$1.60916E-05$

Table 5: Moment method estimator (MOM).

Parameter	Gumbel dist	2-Fréchet dist	3-Fréchet dist	2-Weibull dist	3-Weibull dist	GEV-distribution
$N = 1{,}000$	$\mu = 5,\ \sigma = 9$	$\sigma = 2,\ \alpha = 5$	$\mu = 1,\ \sigma = 5,\ \alpha = 3,$	$\sigma = 3,\ \alpha = 7$	$\mu = 3,\ \sigma = 1,\ \alpha = 5,$	$\mu = 1,\ \sigma = 3,\ \alpha = 0.01$
Location μ	5.209467	—	0.3666667	—	1.2104030	1.182236950
95% CI	(3.974989, 6.443945)	—	(−1.245415, 1.978749)	—	(−0.589689, 3.01049)	(0.8251437, 1.53933)
C	0.923	—	0.409	—	0.119	0.890
Bias	0.209467	—	−0.6333333	—	−1.789597	0.18223695
MSE	0.044289216	—	0.401815014	—	3.203535133	0.033244846
Scale σ	8.590332	2.008617	7.5555556	2.994487	0.1442575	3.180989490
95% CI	(7.22174, 9.958924)	(1.734339, 2.282895)	(6.208391, 8.902721)	(2.854526, 3.134448)	(−1.141305, 1.42982)	(2.892358, 3.469621)
CP	0.881	0.929	0.551	0.957	0.393	0.948
Bias	−0.409668	0.008617	2.5555556	−0.005513	−0.8557425	0.18098949
MSE	0.655396	0.019656	7.0032859	0.0051295	1.1624990	0.05444299
Shape α	—	5.465167	2.8888889	7.619065	3.7844660	0.018791750
95% CI	—	(4.856787, 6.073547)	(2.031479, 3.746299)	(6.967252, 8.270878)	(0.888693, 6.68024)	(0.0053161, 0.032267)

Table 5: Continued.

Parameter	Gumbel dist	2-Fréchet dist	3-Fréchet dist	2-Weibull dist	3-Weibull dist	GEV-distribution
CP	—	0.779	0.864	0.742	0.286	0.931
Bias	—	0.465167	−0.1111111	0.619065	−1.215534	0.00879175
MSE	—	0.312727	0.2037117	0.493836	3.6603391	0.00012456
$N = 1{,}000{,}000$	$\mu = 3$, $\sigma = 5$	$\sigma = 15$, $\alpha = 30$	$\mu = 0$, $\sigma = 30$, $\alpha = 10$,	$\sigma = 10$, $\alpha = 10$	$\mu = 0.5$, $\sigma = 13$, $\alpha = 10$,	$\mu = 3$, $\sigma = 5$, $\alpha = 0.01$
Location μ	2.966366	—	11.872800	—	0.50000009	3.006222137
95% CI	(2.577247, 3.355485)	—	(6.68255, 17.06305)	—	(0.455781, 0.544219)	(2.765454, 3.24699)
CP	0.859	—	0.110	—	0.979	0.961
Bias	−0.033634	—	11.8728	—	0.000000009	0.006222137
MSE	0.0405454	—	147.9757	—	0.000508984	0.015128564
Scale σ	5.011782	15.01847	1.0707010	9.999611	18.0001921	5.009725335
95% CI	(4.406102, 5.617462)	(14.76892, 15.26802)	(−14.83286, 16.97427)	(9.564931, 10.43429)	(14.57669, 21.42369)	(4.732609, 5.286841)
CP	0.917	0.941	0.019	0.935	0.367	0.897
Bias	0.011782	0.01847	−28.929299	−0.000389	5.0001921	0.009725335
MSE	0.0956324	0.0165518	902.74240	0.049184302	28.05282115	0.020084454
Shape α	—	31.76106	10.274778	9.992710	22.0002100	0.008961094
95% CI	—	(30.50113, 33.02099)	(9.345474, 11.20408)	(9.599808, 10.38561)	(16.43889, 27.56153)	(−0.073961, 0.091883)
CP	—	0.687	0.762	0.899	0.153	0.951
Bias	—	1.76106	0.274778	−0.00729	12.00021	−0.001038906
MSE	—	3.101332	0.300305	0.0402370	152.0559	0.001790970

Table 6: Maximum product of spacings (MPS).

Parameter	Gumbel dist	2-Fréchet dist	3-Fréchet dist	2-Weibull dist	3-Weibull dist	GEV-distribution
$N = 1{,}000$	$\mu = 5$, $\sigma = 9$	$\sigma = 2$, $\alpha = 5$	$\mu = 1$, $\sigma = 5$, $\alpha = 3$,	$\sigma = 3$, $\alpha = 7$	$\mu = 3$, $\sigma = 1$, $\alpha = 5$,	$\mu = 1$, $\sigma = 3$, $\alpha = 0.01$
Location μ	5.107336	—	1.181352	—	3.0144140	1.136788370
95% CI	(4.03858, 6.176092)	—	(−0.023294, 2.385998)	—	(2.439769, 3.589059)	(0.882695, 1.39088)
CP	0.979	—	0.965	—	0.961	0.959
Bias	0.1073360	—	0.1813520	—	0.0144140	0.13678837
MSE	0.0118304	—	0.0332816	—	0.0002972	0.01872854
Scale σ	8.923836	1.990858	4.737531	2.999377	0.9868423	3.085039300
95% CI	(8.006222, 9.84145)	(1.905312, 2.076404)	(3.488991, 5.986071)	(2.781011, 3.217743)	(0.679259, 1.294426)	(2.845176, 3.324902)
CP	0.981	0.978	0.968	0.984	0.974	0.979
Bias	−0.0761640	−0.00914200	−0.26246900	−0.00062300	−0.01315770	0.08503930
MSE	0.00602903	$8.55584E-5$	0.069312226	$1.33043E-05$	0.000198752	0.00724726
Shape α	—	5.015951	2.918228	7.162783	4.9621254	0.009968900
95% CI	—	(4.93862, 5.093282)	(1.991924, 3.844532)	(6.938631, 7.386935)	(4.539782, 5.384468)	(−0.003507, 0.023445)
CP	—	0.959	0.964	0.958	0.972	0.967
Bias	—	0.01595100	−0.08177200	0.16278300	−0.037874600	$-3.110E-05$
MSE	—	0.00025605	0.00691907	0.02651191	0.001482802	$5.0159E-08$
$N = 1{,}000{,}000$	$\mu = 3$, $\sigma = 5$	$\sigma = 15$, $\alpha = 30$	$\mu = 0$, $\sigma = 30$, $\alpha = 10$,	$\sigma = 10$, $\alpha = 10$	$\mu = 0.5$, $\sigma = 13$, $\alpha = 10$,	$\mu = 3$, $\sigma = 5$, $\alpha = 0.01$
Location μ	2.998510	—	0.01248723	—	0.4794425	3.003037123
95% CI	(2.640391, 3.356629)	—	(−0.506537, 0.531512)	—	(−0.281934, 1.240819)	(2.863204, 3.14287)
CP	0.984	—	0.978	—	0.968	0.988
Bias	−0.00149000	—	0.012487230	—	−0.020557500	0.003037123
MSE	$2.25348E-06$	—	0.000156001	—	0.000422762	$9.22921E-06$
Scale σ	4.997087	14.99970	29.99258016	9.999803	13.018940	5.004417502
95% CI	(4.611407, 5.382767)	(14.85415, 15.14525)	(29.25693, 30.72823)	(9.650049, 10.34956)	(12.46146, 13.57642)	(4.828128, 5.180707)
CP	0.989	0.997	0.991	0.998	0.987	0.986
Bias	−0.00291300	−0.00030000	−0.00741984	−0.00019700	0.018940000	0.004417502

Table 6: Continued.

Parameter	Gumbel dist	2-Fréchet dist	3-Fréchet dist	2-Weibull dist	3-Weibull dist	GEV-distribution
MSE	8.52429$E-06$	9.55146$E-08$	5.51949$E-05$	7.06525$E-08$	0.000358804	1.95224$E-05$
Shape α	—	30.02264	9.99751933	9.992998	10.010710	0.009833649
95% CI	—	(29.84531, 30.19997)	(9.271215, 10.72382)	(9.550806, 10.43519)	(9.568518, 10.4529)	(−0.057776, 0.077444)
CP	—	0.953	0.994	0.992	0.943	0.998
Bias	—	0.02264000	−0.00248067	−0.00700200	0.0107100	−0.000166351
MSE	—	0.00051257	6.29104$E-06$	4.90789$E-05$	0.0001147	2.88626$E-08$

Table 7: Test of goodness of fit and root mean square error with N = 1,000.

Model	Log likelihood	AIC	BIC	A	W	RMSE
Gumbel-MLE	−3752.731	7509.463	7519.278	0.46348167	0.08595688	0.0002449354
Gumbel-MOM	−3753.938	7511.875	7521.691	0.82281230	0.12025590	0.0002451353
Gumbel-MPS	−3752.765	7509.531	7519.346	0.45361000	0.08321900	0.0002449302
Fréchet-MLE	−750.0248	1504.050	1513.865	0.37583648	0.05649818	0.0484279100
Fréchet-MOM	−759.2681	1522.536	1532.352	2.62954654	0.24126612	0.0488245900
Fréchet-MPS	−748.1436	1500.287	1510.103	0.14331630	0.01800061	0.0483957900
Fréchet-3-MLE	−2251.918	4509.837	4524.560	0.32955505	0.03942906	0.0003658070
Fréchet-3-MOM	−2715.859	5437.718	5452.441	416.150933	63.9804000	0.0706939100
Fréchet-3-MPS	−2251.556	4509.112	4523.835	0.31835000	0.04096900	0.0003511651
Weibull- MLE	−620.1875	1244.375	1254.191	0.33785956	0.05217847	0.0060328220
Weibull-MOM	−623.5558	1251.112	1260.927	0.90885173	0.07155323	0.0429038775
Weibull-MPS	−620.2193	1244.439	1254.254	0.37032000	0.06027300	0.0054950800
Weibull-3- MLE	157.6339	−309.2678	−294.5446	0.32842518	0.04926565	0.0162757710
Weibull-3-MOM	Inf	Inf	Inf	Inf	18.3143900	1.1018382408
Weibull-3-MPS	157.5496	−309.0991	−294.3759	0.2807572	0.04370577	0.0162663061
GEVD-MLE	−2711.417	5428.834	5443.557	0.24270623	0.03737667	0.0007444118
GEVD-MOM	−2800.544	5607.088	5621.811	0.42912000	0.06136600	0.0009445907
GEVD-MPS	−2711.411	5428.822	5443.545	0.21240000	0.03254400	0.0007406615

Table 8: Test of goodness of fit and root mean square error with N = 1,000,000.

Model	Log likelihood	AIC	BIC	A	W	RMSE
Gumbel-MLE	−3186828	6373660	6373683	0.5766605925	0.0723973654	0.0002133718
Gumbel-MOM	−3185937	6371879	6371902	16.477515229	2.6486539940	0.0002133731
Gumbel-MPS	−3185902	6371809	6371833	0.2699600000	0.0393380000	0.0002133710
Fréchet-MLE	−902633.6	1805271	1805295	388.21043208	74.546302650	0.0021526790
Fréchet-MOM	−906994.7	1813993	1814017	1342.0150000	148.35070000	0.0021630910
Fréchet-MPS	−902160.6	1804325	1804349	0.1119400000	0.5669100000	0.0021503496
Fréchet-3-MLE	−2733422	5466850	5466885	0.2180080150	0.0315607600	0.0001907248
Fréchet-3-MOM	−2807887	5615779	5615815	18333.010000	1797.1600000	0.0091505427
Fréchet-3-MPS	−2733422	5466850	5466885	0.2612600000	0.0366090000	0.0001906232
Weibull-MLE	−1520320	3040645	3040668	0.3530187557	0.0480795187	0.0020330810
Weibull-MOM	−1520320	3040645	3040668	0.4048516230	0.0580089449	0.0020330760
Weibull-MPS	−1520320	3040645	3040668	0.4294500000	0.0662080000	0.0020330750
Weibull-3-MLE	−1782463	3564932	3564967	0.3671212475	0.0553202892	0.0015759554
Weibull-3-MOM	Inf	Inf	Inf	Inf	18149.390000	1.1149331907
Weibull-3-MPS	−1782463	3564932	3564967	0.3011628000	0.0419733100	0.0015754390
GEVD-MLE	−3193448	6386901	6386937	0.2156288283	0.0336231660	0.0002303789
GEVD-MOM	−3291039	6582086	6582121	0.2515600000	0.0255580000	0.0002313981
GEVD-MPS	−3193448	6386901	6386937	0.1927100000	0.0260980000	0.0002303411

(Type III), and the GEVD with different values for the parameters.

The number of sample sizes (N) used in previous studies varies greatly. Dey et al. [57] generated $N = 1,000$ samples of transformed generalized exponential distribution, whereas Ramos et al. [51] chose $N = 500,000$ for the fitted Fréchet distribution to compare the performance of the various estimation methods. Meanwhile, Dey et al. [58] simulated $N = 100,000$ samples of Kumaraswamy distribution. Rodrigues [49] chose $N = 10,000$ to simulate the Poisson-exponential distribution with various estimation methods. Soukissian and Tsalis

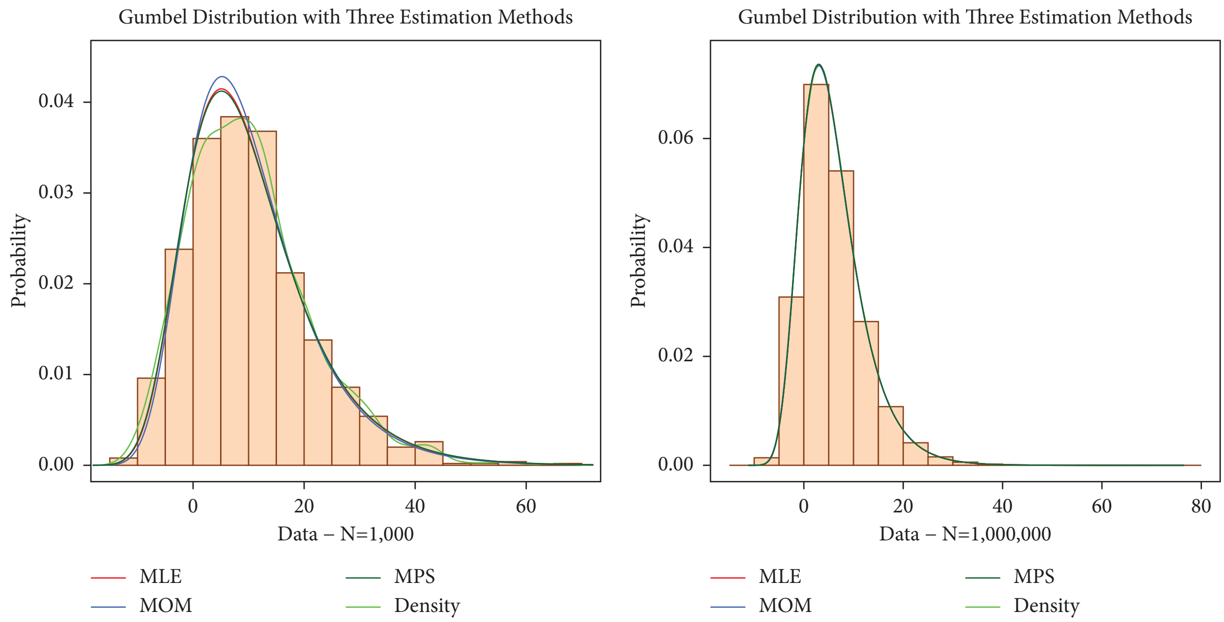

Figure 1: Density histograms and Gumbel distribution for three methods of parameter estimates.

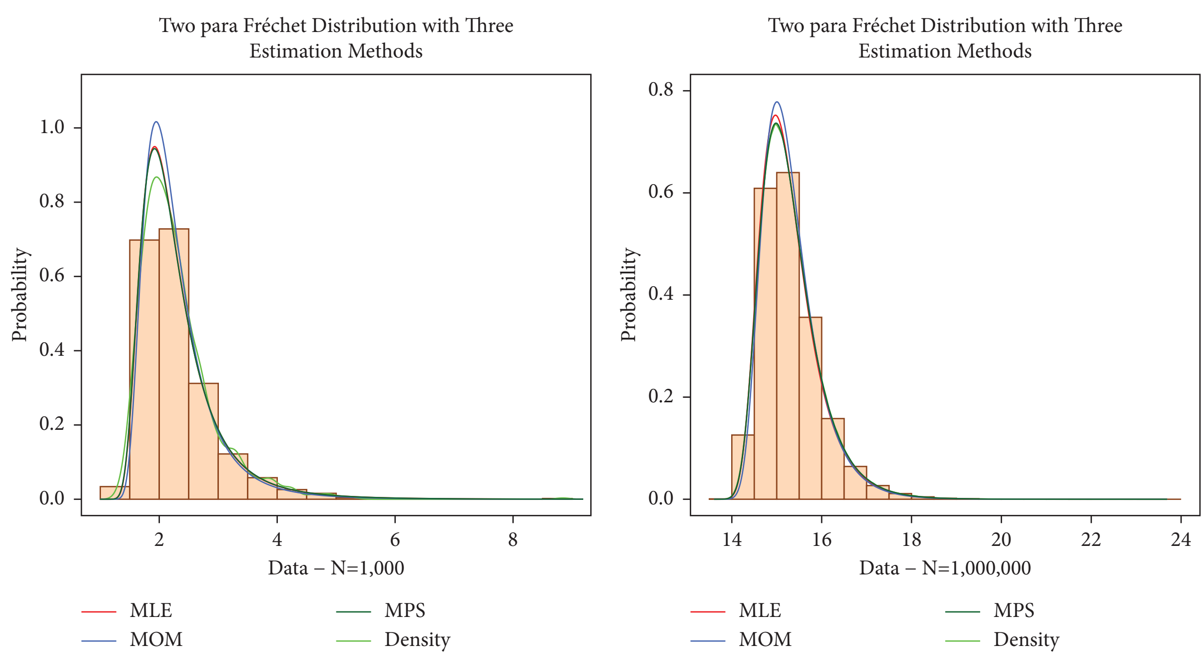

Figure 2: Density histograms and two para Fréchet distribution for three methods of parameter estimates.

[59] studied the effects of the sample size for the GEVD on the design values of wind speed. The assessment was based on a simulation study which includes each simulation is being run for 1000 random samples of each size of maxima as well as an analysis of real wind speed data. It is also reported that over 28 years in the Czech Republic, frequency analysis for two-component GEVD was applied to analyze 6-hour precipitation data from 11 stations [60].

It is critical to differentiate between two types of required sample sizes based on MOM, MPS, or MLE. As a result, we considered two different sample size values: small (N = 1,000) and large (N = 1,000,000). Each sample size has a different set of parameter values chosen at random, with the idea that the randomly chosen parameter values determine the shape of the extreme distributions. The goodness-of-fit statistics to compare the fitted distributions were also calculated: L (maximized log likelihood), AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion), Anderson–Darling (A), the Cramér-von Mises (W), and RMSE. The model with the lowest values for these

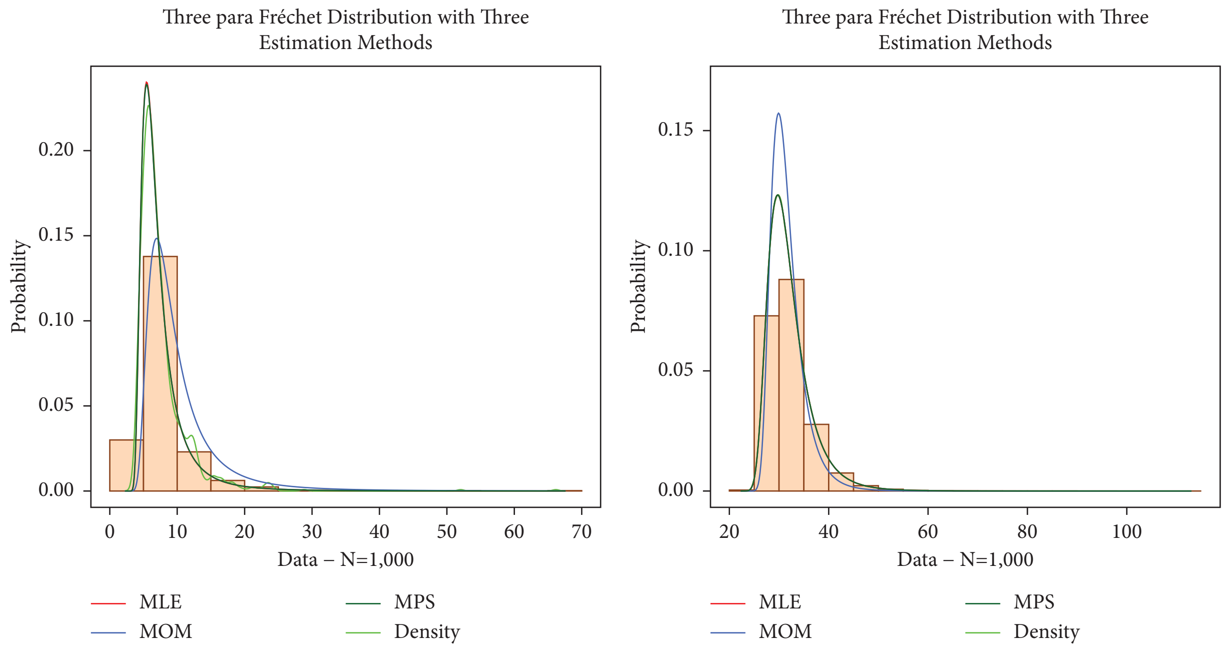

Figure 3: Density histograms and three para Fréchet distribution for three methods of parameter estimates.

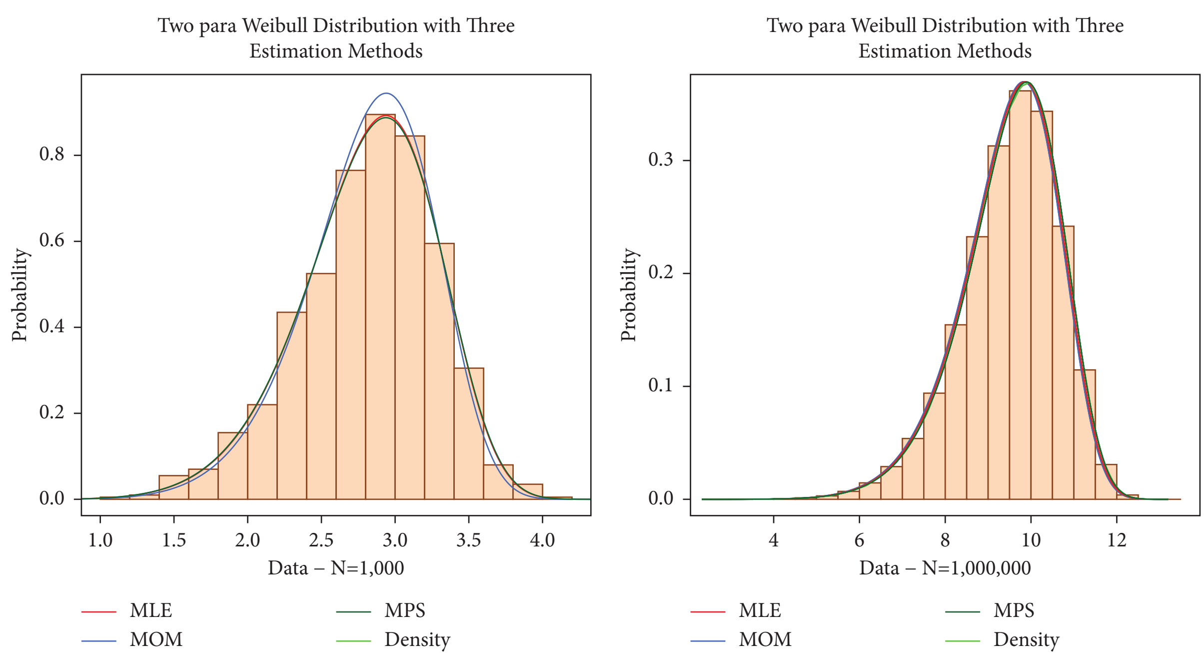

Figure 4: Density histograms and two para Weibull distribution for three methods of parameter estimates.

statistics was chosen as the best fit for the data. A distribution with the smallest AIC and BIC values was found to fit the data better.

$$\begin{aligned} \mathrm{AIC} &= 2K - 2LL, \\ \mathrm{BIC} &= K * \log(N) - 2LL, \end{aligned} \tag{26}$$

where N = sample size, K = number of parameters in the statistical model, and LL = the maximized value of the logarithmic likelihood function for the estimated model. Meanwhile, the calculation for RMSE is as follows:

$$\mathrm{RMSE} = \sqrt{\frac{\left(\widehat{f}(x_i) - f(x_i)\right)^2}{N}}, \tag{27}$$

where $f(x_i)$ = fitted distribution, $i = 1$ until N, $\widehat{f}(x_i)$ interval is the observed frequency distribution, and x_i is the mid-value for the i^{th}. RMSE was calculated for all

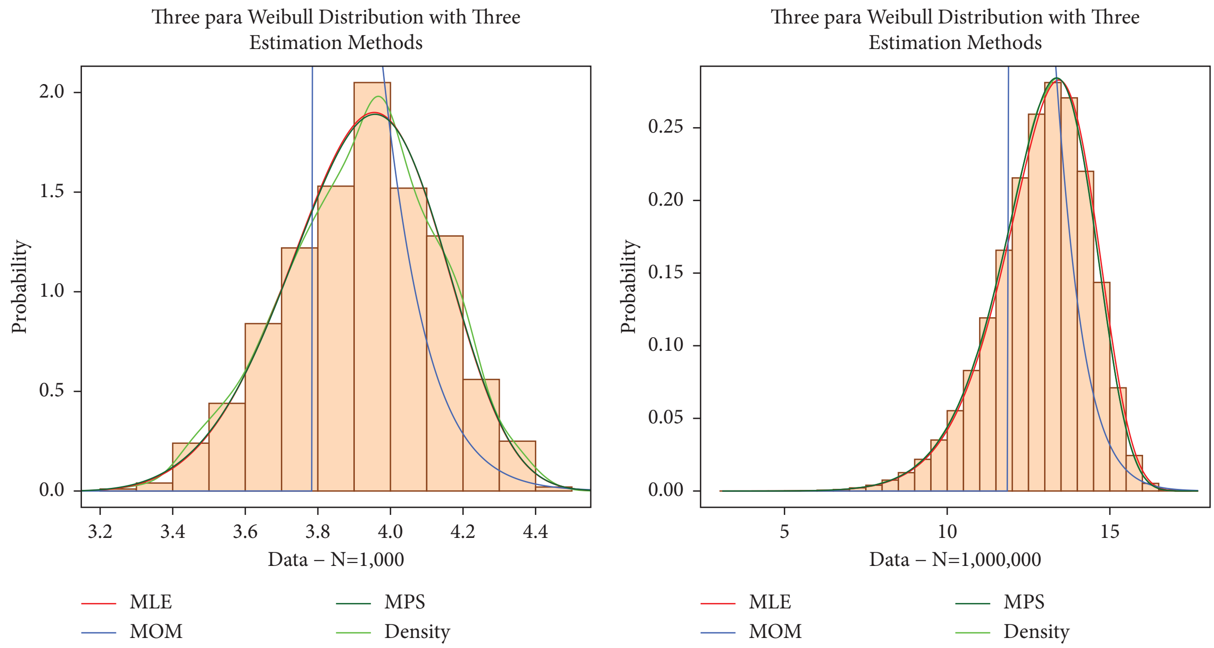

Figure 5: Density histograms and three para Weibull distribution for three methods of parameter estimates.

EVDs using the three-parameter estimation methods considered. The estimation method which provides the smallest value of RMSE is considered as the best estimation method. Finally, a histogram with a density plot was used to compare the MPS with MLE and MOM graphically.

5.1. Estimation of Parameters. The simulation results for all EVDs of MLE, MOM, and MPS estimation methods as well as the 95% confidence intervals are presented in Tables 4-6.

The goodness-of-fit statistics for all EVDs of MLE, MOM, and MPS are presented in Tables 7 and 8 with small and large sample sizes, respectively.

Based on the tables, there are virtually no significant differences in the estimates obtained using the MLE and MPS methods. In other words, MLE and MPS variations for all EVDs were approximately 0.06% difference (for μ), 0.04% difference (for σ), and 0.02% difference (for α). The narrowest 95% CI widths are provided by MPS and MLE, respectively. Moreover, the MPS estimator provides the lowest values for MSE and bias of estimated parameters. Similarly, the values of the goodness-of-fit tests performed, Akaike's Information Criterion (AIC), the Bayesian Information Criterion (BIC), Anderson–Darling (A) test, and the Cramér-von Mises (W) test are shown in Tables 7 and 8. The estimates obtained by the MPS consistently showed lower values of goodness-of-fit statistics than those obtained by other methods for both sample sizes with different parameter values. The MOM had lower accuracy in estimating almost all of the parameters for the EVD. Nonetheless, it provided a better estimate for GEVD than the MLE method.

The root mean square error (RMSE) of each parameter estimation method for all EVDs of both sample sizes is also shown in Tables 7 and 8. For both sample sizes, the MPS method has the lowest RMSE estimates for all EVDs. However, in some distributions, the difference in RMSE values for MPS and MLE estimation methods is considered almost nonexistent. RMSE values for estimates using the MOM method, on the other hand, are significantly higher for almost all distributions. This indicates that the MPS is a better fit for the EVD simulated data. As a result, various estimation methods provide a comprehensive view of the validity and performance of the estimation methods in multiple situations of extreme value analysis. Again, it is shown that MPS could be the best estimation method for fitting EVD.

5.2. Graphical Results. A histogram is considered one of the best tools for observed data to represent the goodness-of-fit of theoretical models. It virtually provides a visual interpretation of the proposed estimation methods. Consequently, the asymptotic behavior of the proposed estimation methods is established, and their performances are investigated in the simulation study using the extreme distribution density plot.

Figures 1-6 show the fitted models for all EVDs with N = 1,000 and N = 1,000,000, indicating that the MPS estimation method fitted the data well for almost all EVDs. Meanwhile, for some distributions, the MOM fitted the data to EVD with less accuracy. As illustrated in Figure 5, the MOM consistently provides a poor fit for the three-parameter Weibull distribution. This outcome is consistent with the goodness-of-fit test results for all EVDs, shown in Tables 7 and 8. As a result, the histograms show that the MPS method remains prominent for all extreme distributions.

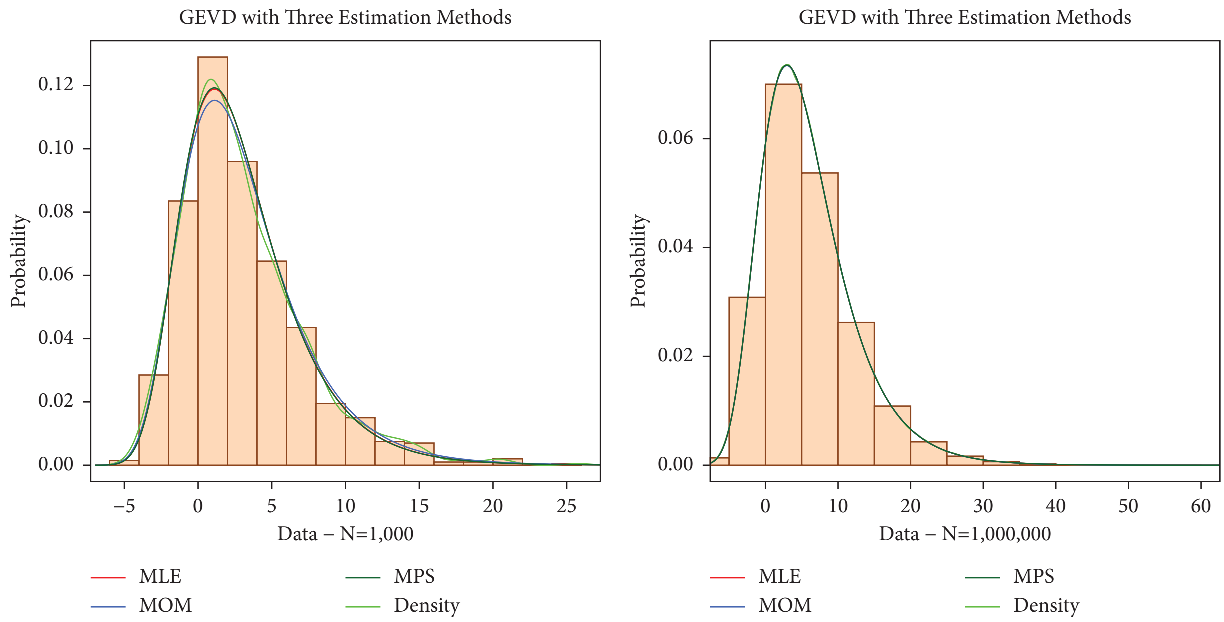

Figure 6: Density histograms and GEVD for three methods of parameter estimates.

6. Conclusions

This review article provides an overview of fitting EVD using MOM, MLE, and MPS. The methods' efficiency is evaluated by comparing the RMSE and several goodness-of-fit indices for two sample sizes. Three types of distributions, namely, the Gumbel, the Fréchet, and Weibull, were used to represent the distributions of extreme events. Nonetheless, determining which distribution is best suited for all extreme statistical events remains difficult. All of the examined methods can give point estimates of the GEVD parameters. However, proposing a unique parameter estimation method for all data sets and types of cases is difficult.

Based on this study, the MPS method is highly recommended regardless of the sample size because it provides better estimates for the unknown parameters and the reliability function. This review article also revealed that the majority of the related publications used MPS and other estimation methods to simulate real-life data, which offers more accurate parameter estimates. To conclude, the MPS performed better than MOM and MLE estimation methods in the majority of cases with the smallest values of RMSE and the narrowest 95% CI widths. However, the MPS provided very similar values with regard to goodness-of-fit statistics to the MLE method. Therefore, the improvement of the performance of the MPS method could be taken into consideration for future studies.

Data Availability

The simulated data sets used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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