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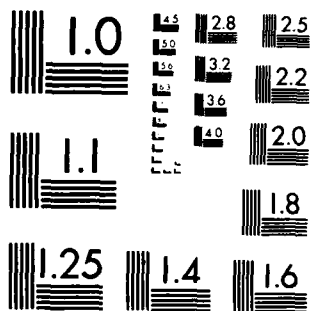
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ESTIMATION OF THE GENERALIZED
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METHOD OF PROBABILITY WEIGHTED MOMENTS

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April 1984

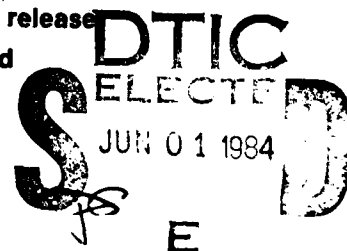
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ESTIMATION OF THE GENERALIZED EXTREME-VALUE DISTRIBUTION
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ABSTRACT

The authors
We use the method of probability weighted moments to derive estimators of the parameters and quantiles of the generalized extreme-value distribution. We investigate the properties of these estimators in large samples, via asymptotic theory, and in small and moderate samples, via computer simulation. Probability weighted moment estimators have low variance and no severe bias, and compare favourably with estimators obtained by the methods of maximum likelihood or sextiles. The method of probability weighted moments also yields a convenient and powerful test of whether an extreme-value distribution is of Fisher-Tippett type I, II or III.

AMS (MOS) Subject Classifications: 62F10, 62G30

Key Words: generalized extreme-value distribution, hypothesis testing, order statistics, probability weighted moments.

Work Unit Number 4 - Statistics and Probability

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SIGNIFICANCE AND EXPLANATION

Many problems in hydrology and civil engineering are related to the properties of extreme events: maximum levels or flow rates of rivers, greatest wave heights or sea levels, or maximum wind loads on buildings. The magnitudes of extreme events are random quantities whose distribution is often well described by the generalized extreme-value distribution.

The method of probability weighted moments is a general procedure for estimating parameters and quantiles of probability distributions. It is here applied to the generalized extreme-value distribution. We derive the large-sample distribution of estimators obtained by the method of probability weighted moments, and compare the performance of these estimators in small samples with the performance of the currently widely-used maximum-likelihood and sextile estimators. The probability weighted moment estimators generally have the best performance.

We also consider the problem of testing hypotheses about the shape parameter of the generalized extreme-value distribution and show that the method of probability weighted moments yields a convenient and powerful testing procedure.



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ESTIMATION OF THE GENERALIZED EXTREME-VALUE DISTRIBUTION
BY THE METHOD OF PROBABILITY WEIGHTED MOMENTS

J. R. M. Hosking¹, J. R. Wallis² and E. F. Wood³

Introduction

The generalized extreme-value distribution of Jenkinson (1955) is widely used for modelling extremes of natural phenomena, and is of considerable hydrological importance, since it is recommended by the Flood Studies Report (NERC 1975) for modelling the distribution of annual maxima of daily streamflows of British rivers. Currently favoured methods of estimation of the parameters and quantiles of the distribution are Jenkinson's (1969) method of sextiles, and the method of maximum likelihood (Jenkinson 1969; Prescott and Walden 1980, 1983). Neither method is completely satisfactory: the justification of the maximum-likelihood approach is based on large-sample theory, and there has been little assessment of the performance of the method when applied to small or moderate samples; while the sextile method involves an inherent arbitrariness (why sextiles rather than, say, quartiles or octiles?), requires interpolation in a table of values of a function in order to estimate the shape parameter of the distribution, and has statistical properties which are not known even for large samples.

Probability weighted moments, a generalization of the usual moments of a probability distribution, were introduced by Greenwood et al (1979). There are several distributions, for example the Gumbel, logistic and Weibull, whose parameters can be conveniently estimated from their probability weighted moments. The Gumbel distribution, being a special case of the generalized extreme-value distribution, is of particular interest. Landwehr et al (1979) investigated the small-sample properties of probability weighted moment

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estimators (PWM estimators) of parameters and quantiles for the Gumbel distribution and found them superior in many respects to the conventional moment and maximum-likelihood estimators. The estimators used by Landwehr et al (1979) are identical to Downton's (1966b) "Linear estimates with linear coefficients", and thus share the asymptotic properties of the latter: in particular, the asymptotic efficiencies of the PWM estimators of the Gumbel scale and location parameters are 0.756 and 0.996 respectively.

In the present work we summarize some theory for probability weighted moments and show that they can be used to obtain estimates of parameters and quantiles of the generalized extreme-value distribution. We derive the asymptotic distributions of these estimators, and compare, via computer simulation, the small-sample properties of the probability weighted moment, sextile, and maximum-likelihood estimators. The method of probability weighted moments outperforms the other methods in many cases and will usually be preferred to them. We also derive from the PWM estimator of the shape parameter of the generalized extreme-value distribution a test of whether this shape parameter is zero, and assess the performance of this test by computer simulation.

Probability weighted moments

The probability weighted moments of a random variable X with distribution function $F(x) = P(X \leq x)$ are the quantities

$$M_{p,r,s} = E[X^p \{F(X)\}^r \{1 - F(X)\}^s] , \quad (1)$$

where p , r and s are real numbers. Probability weighted moments are likely to be most useful when the inverse distribution function $x(F)$ can be written in closed form, for then we may write

$$M_{p,r,s} = \int_0^1 \{x(F)\}^p F^r (1-F)^s dF , \quad (2)$$

and this is often the most convenient way of evaluating these moments. The quantities $M_{p,0,0}$, $p = 1, 2, \dots$, are the usual noncentral moments of X . The moments $M_{1,r,s}$ may, however, be preferable for estimating the parameters of the distribution of X , since the occurrence of only the first power of X in the expression for $M_{1,r,s}$ means that the relationship between parameters and moments often takes a simpler form in this case than

when using the conventional moments. When r and s are integers, $F^r(1-F)^s$ may be expressed as a linear combination either of powers of F or powers of $(1-F)$, so it is natural to summarize a distribution either by the moments $M_{1,r,0}$, $r = 0, 1, 2, \dots$, or by $M_{1,0,s}$, $s = 0, 1, 2, \dots$. Greenwood et al (1979) generally favour the latter approach but here we will consider the moments $\beta_r = M_{1,r,0} = E[X\{F(X)\}^r]$, $r = 0, 1, 2, \dots$.

Given a random sample of size n from the distribution F , estimation of β_r is most conveniently based on the ordered sample $x_1 < x_2 < \dots < x_n$. The statistic

$$b_r = n^{-1} \sum_{j=1}^n \frac{(j-1)(j-2)\dots(j-r)}{(n-1)(n-2)\dots(n-r)} x_j \quad (3)$$

is an unbiased estimator of β_r (Landwehr et al 1979). Instead one may estimate β_r by

$$\hat{\beta}_r[p_j] = n^{-1} \sum_{j=1}^n p_j^r x_j \quad (4)$$

where p_j is a plotting position, i.e. a distribution-free estimate of $F(x_j)$. Reasonable choices of p_j , such as $p_j = (j-a)/n$, $0 < a < 1$, or $p_j = (j-a)/(n+1-2a)$, $-\frac{1}{2} < a < \frac{1}{2}$, yield estimators $\hat{\beta}_r[p_j]$ which are asymptotically equivalent to b_r and therefore are consistent estimators of β_r .

The estimators b_r are closely related to U-statistics (Hoeffding 1948), which are averages of statistics calculated from all subsamples of size $j < n$ of a given sample of size n . In particular $b_0 = n^{-1} \sum x_j$ is a trivial example of a U-statistic, and is a natural estimator of location of a distribution;

$$2b_1 - b_0 = \frac{1}{2} U_2 = \frac{1}{2} \binom{n}{2}^{-1} \sum_{i>j} (x_i - x_j) \quad (5)$$

is a U-statistic for estimating the scale of a distribution - the statistic U_2 , sometimes known as Gini's mean difference, has a history going back at least as far as von Andrae (1872), and $\frac{1}{2} \sqrt{n} U_2$ is a 98% efficient estimator of the scale parameter of a Normal distribution (Downton 1966a); and

$$6b_2 - 6b_1 + b_0 = \frac{1}{3} U_3 = \frac{1}{3} \binom{n}{3}^{-1} \sum_{i>j>k} (x_i - 2x_j + x_k) \quad (6)$$

is a U-statistic for estimating skewness which has been used as the basis of a test for Normality by Locke and Spurrier (1976). U-statistics are widely used in nonparametric statistics (see, for example, Fraser, 1957, Chap. 4; Randles and Wolfe, 1979, Chap. 3) and

their desirable properties of robustness to outliers in the sample, high efficiency and asymptotic normality may be expected to extend to the probability weighted moment estimators b_r and other quantities calculated from them.

PWM estimators for the generalized extreme-value distribution

The generalized extreme-value (GEV) distribution, introduced by Jenkinson (1955), combines into a single form the three possible types of limiting distribution for extreme values, as derived by Fisher and Tippet (1928). The distribution function is

$$F(x) = \begin{cases} \exp[-\{1 - k(x-\xi)/\alpha\}^{1/k}], & k \neq 0, \\ \exp[-\exp\{-(x-\xi)/\alpha\}] , & k = 0, \end{cases} \quad (7)$$

with x bounded by $\xi + \alpha/k$ from above if $k > 0$ and from below if $k < 0$. Here ξ and α are location and scale parameters respectively, while the shape parameter k determines which extreme-value distribution is represented: Fisher-Tippet types I, II and III correspond to $k = 0$, $k < 0$ and $k > 0$ respectively. When $k = 0$ the GEV distribution reduces to the Gumbel distribution. The inverse distribution function is

$$x(F) = \begin{cases} \xi + \alpha\{1 - (-\log F)^k\}/k, & k \neq 0, \\ \xi - \alpha \log(-\log F), & k = 0. \end{cases} \quad (8)$$

The probability weighted moments of the GEV distribution for $k \neq 0$ are given by

$$\beta_r = (r+1)^{-1} [\xi + \alpha\{1 - (r+1)^{-k} \Gamma(1+k)\}/k], \quad k > -1 \quad (9)$$

(for proof see Appendix 1). When $k \leq -1$, β_0 (the mean of the distribution) and the rest of the β_r do not exist. From (9) we have

$$\beta_0 = \xi + \alpha\{1 - \Gamma(1+k)\}/k, \quad (10)$$

$$2\beta_1 - \beta_0 = \alpha\Gamma(1+k)(1 - 2^{-k})/k, \quad (11)$$

$$\frac{3\beta_2 - \beta_0}{2\beta_1 - \beta_0} = \frac{1-3^{-k}}{1-2^{-k}}. \quad (12)$$

The PWM estimators $\hat{\xi}$, $\hat{\alpha}$, \hat{k} of the parameters are the solutions of (10) - (12) for ξ , α and k when the β_r are replaced by their estimators b_r or $\hat{\beta}_r[p_j]$. To obtain \hat{k} we

must solve the equation

$$\frac{3b_2 - b_0}{2b_1 - b_0} = \frac{1 - 3^{-\hat{k}}}{1 - 2^{-\hat{k}}} \quad (13)$$

The exact solution requires iterative methods, but because the function $(1 - 3^{-k})/(1 - 2^{-k})$ is almost linear over the range of values of k , $-\frac{1}{2} < k < \frac{1}{2}$, which is usually encountered in practice, low-order polynomial approximations for \hat{k} are very accurate. We propose the approximate estimator

$$\hat{k} = 7.8590c + 2.9554c^2, \quad c = \frac{2b_1 - b_0}{3b_2 - b_0} - \frac{\log 2}{\log 3}, \quad (14)$$

the error in \hat{k} due to using (14) rather than (13) is less than 0.0009 throughout the range $-\frac{1}{2} < \hat{k} < \frac{1}{2}$. Given \hat{k} , the scale and location parameters can be estimated successively as

$$\hat{a} = \frac{(2b_1 - b_0)\hat{k}}{\Gamma(1+\hat{k})(1-2^{-\hat{k}})}, \quad \hat{\xi} = b_0 + \hat{a}\{\Gamma(1+\hat{k}) - 1\}/\hat{k} \quad (15)$$

Equations (13) and (15), or their equivalent forms with b_x replaced by $\hat{b}_x[p_j]$, define the PWM estimators of the parameters of the GEV distribution. Given the estimated parameters, the quantiles of the distribution are estimated using the inverse distribution function (8).

When calculated using b_x as the estimator of β_x , the PWM estimates of the GEV distribution satisfy a feasibility criterion, namely that $\hat{k} > -1$ and $\hat{a} > 0$ almost surely (for proof see Appendix 2). This is clearly a desirable property, since one would like estimates calculated using a set of sample moments to yield an estimated distribution for which the corresponding population moments exist. We have not been able to prove that this feasibility criterion is satisfied when plotting-position estimators $\hat{b}_x[p_j]$ are used, but no examples of the criterion not being satisfied have been discovered in practice.

Asymptotic distribution of PWM estimators

When modelling the properties of extremes of hydrological variables it rarely occurs that the available data set is large enough to ensure that asymptotic large-sample theory may be directly applied to the problem. It is nonetheless valuable to investigate the asymptotic properties of a new statistical technique, for two main reasons. First, one may seek to establish the integrity of the technique, in the sense that when a large sample is available, the new method should not be grossly inefficient compared to an established, asymptotically optimal method such as maximum likelihood. Second, asymptotic theory may provide an adequate approximation to some aspect of the distribution of a statistic even in quite small samples. In the present case we shall see that the variance of PWM estimators of parameters and quantiles of the GEV distribution is well approximated by asymptotic theory for sample sizes of 50 or larger.

We consider first the asymptotic distribution of the b_r . From (3), b_r is a linear combination of the order statistics x_1, \dots, x_n , and the results of Chernoff et al (1967) may be used to prove that the vector $b = (b_0, b_1, b_2)^T$ has asymptotically a multivariate Normal distribution with mean $\beta = (\beta_0, \beta_1, \beta_2)^T$ and covariance matrix $n^{-1}V$. The elements of V and details of the proof are given in Appendix 3.

The asymptotic distribution of the PWM estimators of the GEV parameters follows from the preceding result. Let $\theta = (\xi, \alpha, k)^T$, $\hat{\theta} = (\hat{\xi}, \hat{\alpha}, \hat{k})^T$, and write (13) and (15) as the vector equation $\hat{\theta} = f(b)$. Define the 3×3 matrix $G = (g_{ij})$ by $g_{ij} = \partial f_i / \partial b_j$. Then asymptotically $\hat{\theta}$ has a multivariate Normal distribution with mean $f(\beta) = \theta$ and covariance matrix $n^{-1}GVG^T$ (Rao, 1973, p. 388). The covariance matrix has the form

$$n^{-1}GVG^T = n^{-1} \begin{bmatrix} \alpha^2 w_{11} & \alpha^2 w_{12} & \alpha w_{13} \\ \alpha^2 w_{12} & \alpha^2 w_{22} & \alpha w_{23} \\ \alpha w_{13} & \alpha w_{23} & w_{33} \end{bmatrix} \quad (16)$$

The w_{ij} are functions of k and have a complicated algebraic form, but they can be evaluated numerically and are given in Table 1 for several values of k . As k approaches

$-\frac{1}{2}$, the variance of the GEV distribution becomes infinite and the variances of the b_i and of the PWM parameter estimators are no longer of order n^{-1} asymptotically.

Table 1. Elements of the asymptotic covariance matrix of the PWM estimators of the parameters of the GEV distribution.

k	w ₁₁	w ₁₂	w ₁₃	w ₂₂	w ₂₃	w ₃₃
-0.4	1.6637	1.3355	1.1405	1.8461	1.1628	2.9092
-0.3	1.4153	0.8912	0.5640	1.2574	0.4442	1.4090
-0.2	1.3322	0.6727	0.3926	1.0013	0.2697	0.9139
-0.1	1.2915	0.5104	0.3245	0.8440	0.2240	0.6815
0.0	1.2687	0.3705	0.2995	0.7395	0.2249	0.5635
0.1	1.2551	0.2411	0.2966	0.6708	0.2447	0.5103
0.2	1.2474	0.1177	0.3081	0.6330	0.2728	0.5021
0.3	1.2438	-0.0023	0.3297	0.6223	0.3033	0.5294
0.4	1.2433	-0.1205	0.3592	0.6368	0.3329	0.5880

The asymptotic biases of the estimators are of order n^{-1} and can be evaluated by methods similar to those of Rao (1973, p. 388). The biases, graphed in Figure 1, are negligible in large samples provided that $k > -0.4$.

The asymptotic variances of the estimators are graphed in Figure 2, and their asymptotic efficiencies in Figure 3. Asymptotic efficiency is defined as the ratio

$$\text{eff}(\hat{\theta}_i) = \lim_{n \rightarrow \infty} (\text{var } \tilde{\theta}_i / \text{var } \hat{\theta}_i)$$

for each element θ_i of the parameter vector θ , where $\tilde{\theta}_i$ is the maximum-likelihood estimator of θ_i . Overall efficiency is the ratio of the determinants of the asymptotic covariance matrices of $\tilde{\theta}$ and $\hat{\theta}$. The overall efficiency of the PWM estimators tends to zero at $k = \pm 0.5$ but for values of k not too far from zero the PWM method is reasonably efficient. Within the range $-0.2 < k < 0.2$, which is valid for many hydrological data sets, each PWM parameter estimator has efficiency of over 0.7.

Figure 1. Asymptotic bias of PMM estimators of parameters of the
 GEV distribution: — \hat{k} , ---- \hat{a} , ---- \hat{c} .

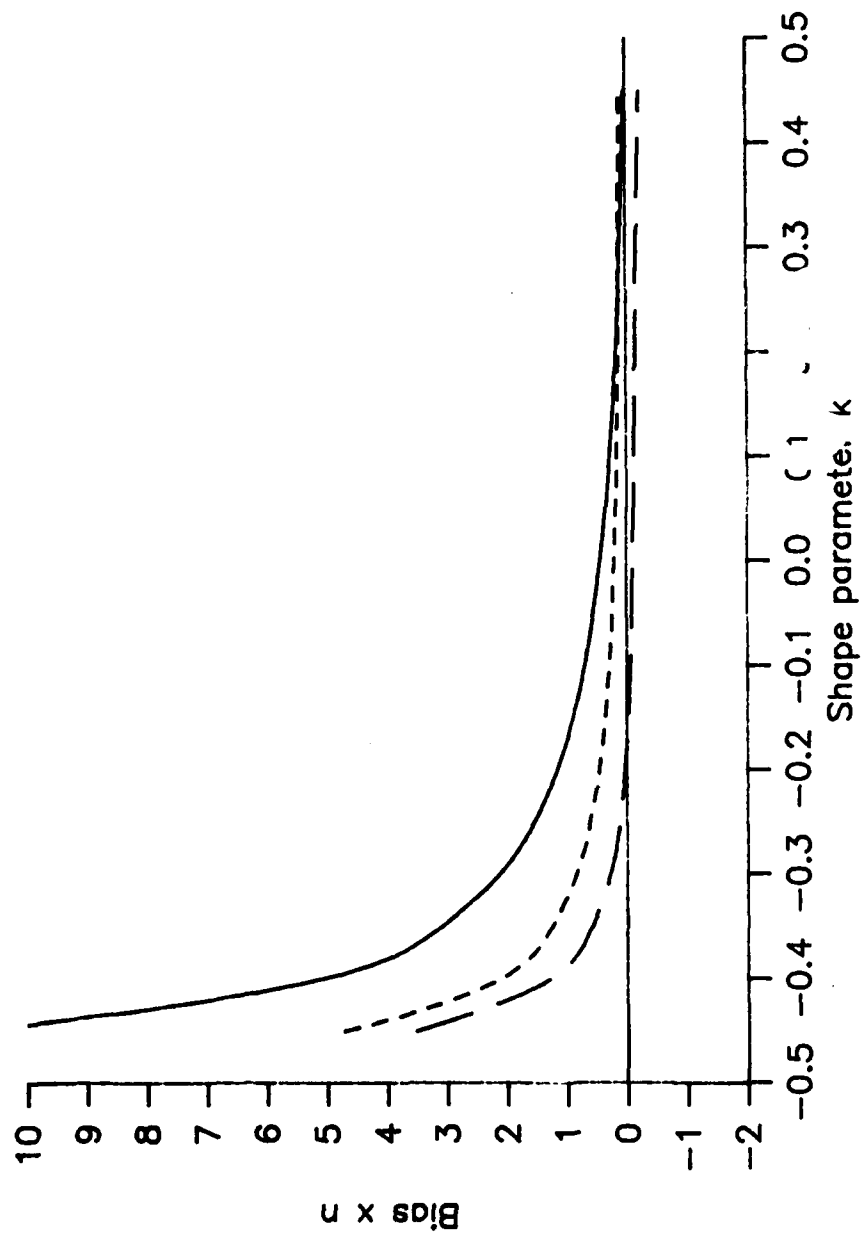
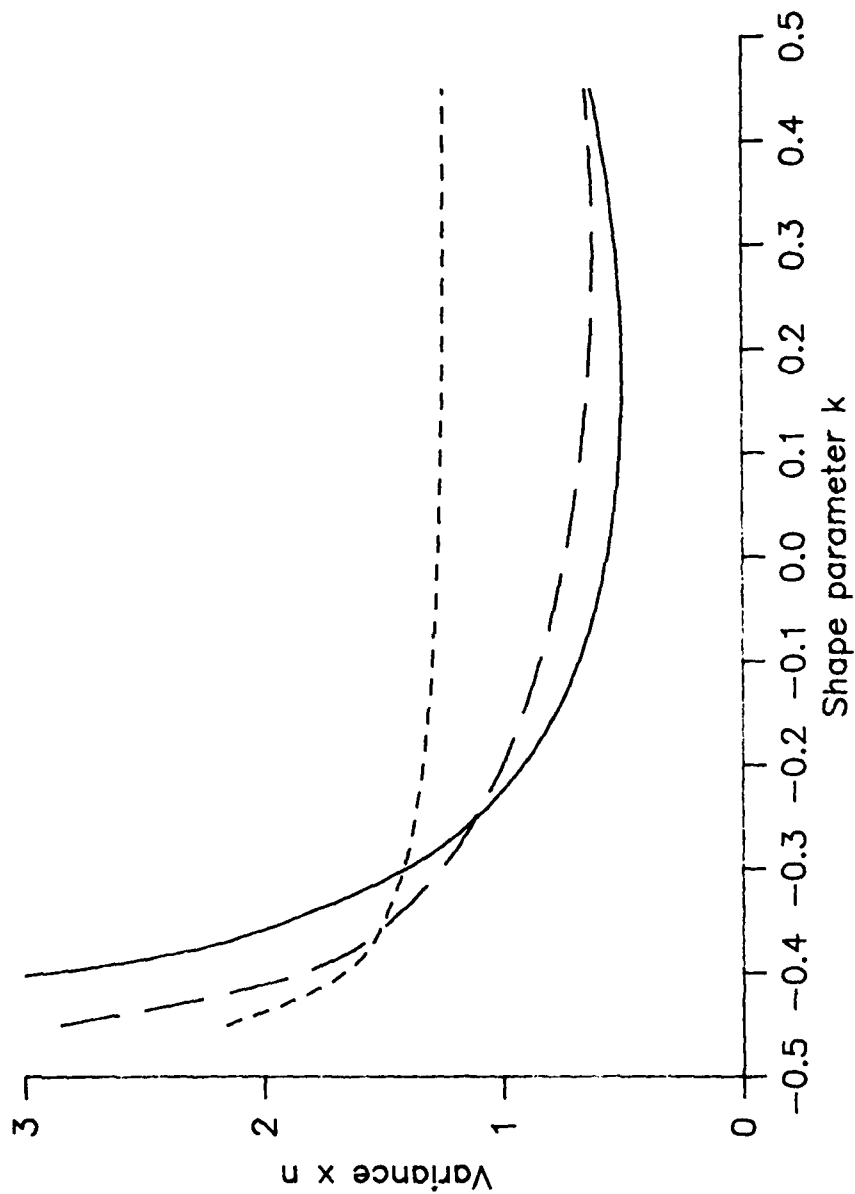
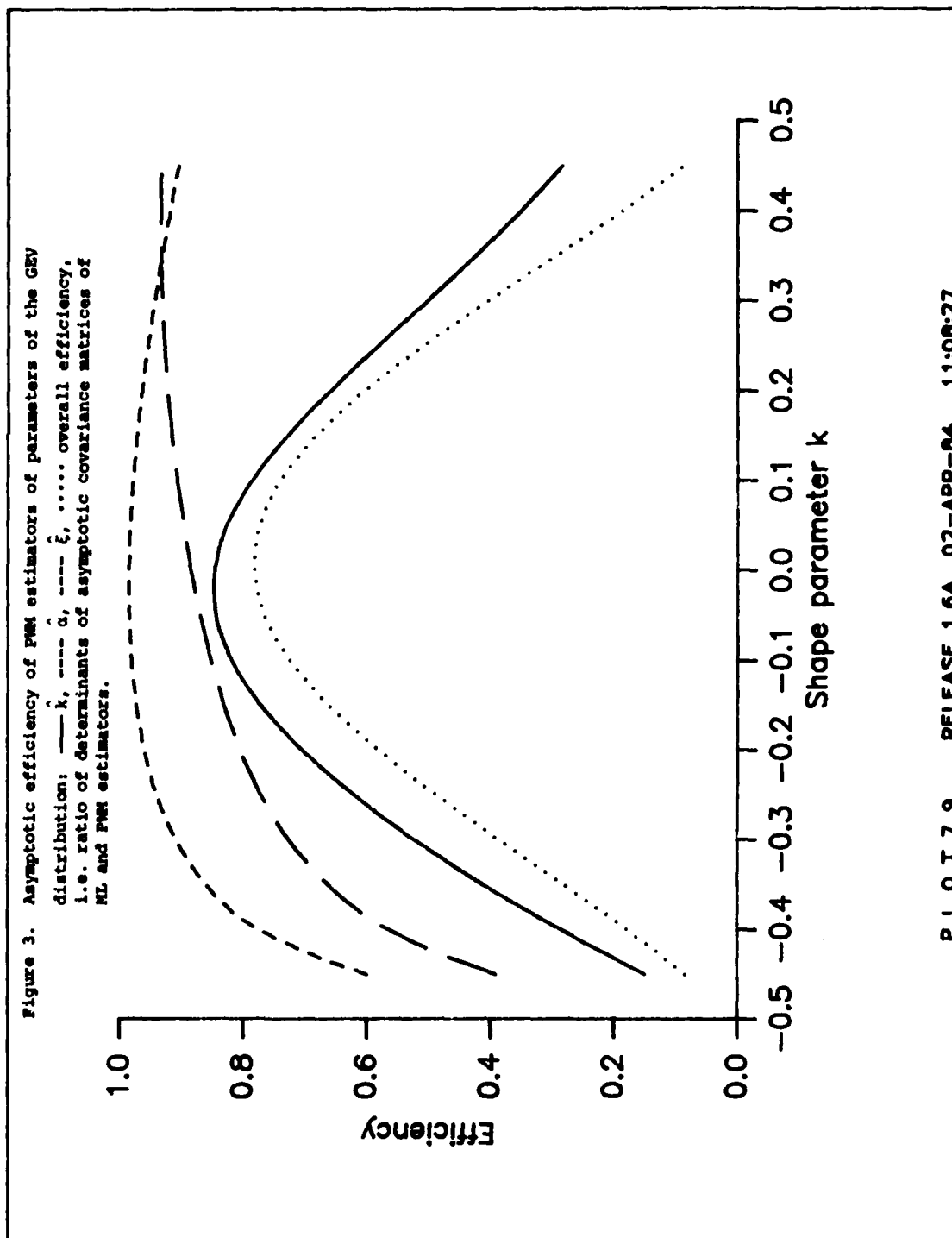


Figure 2. Asymptotic variance of PM estimators of parameters of the GEV distribution: — \hat{k} , ---- $\hat{\alpha}$, - - - $\hat{\xi}$.





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Corresponding results may be obtained for PWM estimators of quantiles of the GEV distribution. These are not presented in full, but Tables 2 and 3 give results for various quantiles when $k = -0.2$, and for various values of k at the $F = 0.98$ quantile. The tables illustrate the main characteristics of PWM quantile estimators, which are: high positive bias in extreme upper tail, arising from positive bias in \hat{k} ; high variance in upper tail when $k < 0$; fair or high efficiency except when k is close to ± 0.5 .

Table 2. Asymptotic bias, variance and efficiency of PWM estimators of GEV quantiles. Parameters $\xi = 0$, $\alpha = 1$, $k = -0.2$.

F	x(F)	n x	n x	efficiency
		bias	variance	
0.001	-1.60	-1.2	3.78	0.60
0.01	-1.32	-0.2	2.06	0.66
0.1	-0.77	0.8	0.86	0.92
0.2	-0.45	1.0	0.88	1.00
0.5	0.38	0.6	1.92	0.93
0.8	1.75	-1.3	6.10	0.98
0.9	2.84	-3.1	16.1	0.99
0.98	5.91	-4.4	147	0.89
0.99	7.55	-1.6	336	0.86
0.998	12.33	23.9	1760	0.81
0.999	14.90	49.2	3310	0.80

Table 3. Asymptotic bias, variance and efficiency of the PWM estimator of the $F = 0.98$ quantile of the GEV distribution. Parameters $\xi = 0, \alpha = 1$.

k	x(F)	n x	n x	efficiency
		bias	variance	
-0.4	9.41	-64.8	1170	0.49
-0.3	7.41	-15.3	369	0.75
-0.2	5.91	-4.4	147	0.89
-0.1	4.77	-1.1	65.8	0.96
0.0	3.90	-0.1	29.6	0.95
0.1	3.23	0.3	14.7	0.88
0.2	2.71	0.5	7.53	0.75
0.3	2.30	0.5	4.04	0.56
0.4	1.98	0.6	2.28	0.36

The results of this section were derived for PWM estimators which use b_x to estimate β_x . If the plotting-position estimates $\hat{\beta}_x[p_j]$ are used instead, the asymptotic variances and efficiencies remain unchanged, but the asymptotic biases are different and cannot be easily calculated, being affected by the biases in the $\hat{\beta}_x[p_j]$ themselves.

Small-sample properties of estimates of the GEV distribution

A computer simulation experiment was run to compare three methods of estimation of the parameters and quantiles of the GEV distribution. Simulations were performed for sample sizes $n = 15, 25, 50, 100$ with the shape parameter of the distribution taking values $k = -0.4, -0.2, 0.0, 0.2, 0.4$. All the methods of estimation are invariant under linear transformations of the data, so without loss of generality the location and scale parameters $\xi = 0$ and $\alpha = 1$ were used throughout. For each combination of values n and k , 1000 random samples were generated from the GEV distribution, and for each sample the parameters ξ, α and k , and the quantiles $x(F)$, where $F = 0.001, 0.01, 0.1, 0.2, 0.5, 0.8, 0.9, 0.98, 0.99, 0.998, 0.999$, were estimated by each of three methods: (1) the method of probability weighted moments (PWM), described above; (2) the method of maximum likelihood

(ML), using Newton-Raphson iteration to maximize the likelihood function, as recommended by Prescott and Walden (1983) and implemented by Hosking (1984b); and (3) Jenkinson's (1969) method of sextiles (JS). The PWM method requires a choice of a suitable estimator of β_r . Several possibilities were investigated, including the unbiased estimator b_r and a number of plotting-position estimators $\hat{\beta}_r[p_j]$. The best overall results were given by the estimator $\hat{\beta}_r[p_j]$ with $p_j = (j-0.35)/n$, and the simulation results presented below for the PWM method refer to this version of the PWM estimators.

For some simulated samples, the maximum-likelihood and sextile estimates could not be found. The cause of this problem for the maximum-likelihood method was nonconvergence of the Newton-Raphson iteration, usually due to an extreme outlier in the sample; for the sextile method, a ratio of sextile means used to estimate the shape parameter of the GEV distribution sometimes lay outside the range of a table of values in which it was to be interpolated. Such cases were omitted from the simulations. No such problems were encountered with the PWM estimators, which could always be calculated.

The simulation results for estimation of the parameters of the GEV distribution are summarized in Table 4 and 5. Results for the estimator of k are of the greatest importance, since this parameter determines the overall shape of the GEV distribution and the rate of increase of the upper quantiles $x(F)$ as F approaches 1. The PWM estimator has the lowest standard deviation of the three methods, except in the case $k = 0.4$, and its advantage is particularly marked in small samples, $n = 15$ and $n = 25$. The PWM estimator is more biased than the maximum-likelihood estimator but its bias is small near the important value $k = 0$. The sextile estimator of k has a large positive bias in small samples when $k < 0$ and its standard deviation is generally larger than that of the PWM estimator.

Table 4. Bias of estimators of GEV parameters

n	Method	Bias($\hat{\xi}$)					Bias($\hat{\alpha}$)					Bias(\hat{k})				
		k: -0.4	-0.2	0.0	0.2	0.4	-0.4	-0.2	0.0	0.2	0.4	-0.4	-0.2	0.0	0.2	0.4
15	PMM	.06	.02	-.01	-.04	-.06	.00	-.06	-.09	-.10	-.10	.11	.04	.02	-.07	-.11
	ML	.00	.01	.01	.01	.00	-.08	-.07	-.05	-.04	-.03	-.05	-.02	.02	.03	.04
	JS	.29	.25	.22	.17	.10	.23	.12	.06	.00	-.05	.18	.13	.11	.07	-.02
25	PMM	.06	.03	.01	.00	-.02	.01	-.04	-.06	-.07	-.07	.07	.02	-.02	-.05	-.08
	ML	.02	.02	.03	.04	.04	-.04	-.04	-.03	-.03	-.02	-.02	-.01	.01	.03	.05
	JS	.12	.12	.10	.09	.07	.13	.03	-.01	-.04	-.08	.10	.06	.04	.02	-.01
50	PMM	.04	.02	.01	.00	-.01	.01	-.02	-.03	-.03	-.03	.05	.02	.00	-.02	-.04
	ML	.01	.01	.01	.02	.02	-.02	-.02	-.02	-.01	-.01	-.01	.00	.01	.02	.03
	JS	.05	.03	.02	.01	.01	.04	.01	-.01	-.02	-.02	.07	.04	.02	.01	.00
100	PMM	.01	.00	.00	.00	.00	.00	.00	.00	.00	.00	.03	.01	.00	.00	.00
	ML	.00	.00	.00	.00	.01	.01	.01	.01	.01	.00	.00	.00	.00	.01	.02
	JS	.04	.03	.03	.03	.03	.10	.04	.03	.01	.00	.05	.02	.02	.02	.02

Table 5. Standard deviation of estimators of GEV parameters

n	Method	St.dev.($\hat{\xi}$)					St.dev($\hat{\alpha}$)					St.dev(\hat{k})				
		k: -0.4	-0.2	0.0	0.2	0.4	-0.4	-0.2	0.0	0.2	0.4	-0.4	-0.2	0.0	0.2	0.4
15	PMM	.30	.29	.28	.28	.28	.32	.25	.20	.18	.18	.21	.20	.20	.20	.21
	ML	.33	.32	.31	.30	.29	.27	.25	.23	.22	.22	.38	.36	.34	.31	.29
	JS	.37	.33	.30	.28	.25	.43	.31	.26	.21	.18	.25	.25	.24	.21	.17
25	PMM	.25	.24	.23	.23	.23	.25	.20	.17	.15	.15	.17	.16	.14	.14	.15
	ML	.24	.24	.24	.24	.23	.21	.19	.17	.16	.17	.24	.21	.19	.17	.17
	JS	.28	.25	.24	.23	.22	.32	.21	.18	.15	.14	.17	.18	.17	.16	.14
50	PMM	.17	.17	.16	.16	.16	.17	.14	.12	.11	.11	.14	.12	.11	.10	.11
	ML	.17	.17	.17	.16	.16	.15	.13	.12	.11	.11	.15	.13	.12	.11	.11
	JS	.18	.17	.16	.16	.16	.16	.14	.12	.11	.11	.15	.13	.11	.11	.11
100	PMM	.12	.12	.12	.11	.11	.12	.10	.09	.08	.08	.11	.09	.07	.07	.07
	ML	.12	.12	.12	.12	.11	.10	.09	.08	.08	.08	.10	.09	.08	.07	.06
	JS	.12	.12	.12	.12	.11	.16	.11	.09	.08	.08	.09	.09	.08	.08	.08

Similar results can be seen for estimators of ξ and α . In general, PWM estimators have smallest standard deviation, particularly for $n = 15$ and $n = 25$, and their bias is not large. The standard deviations of the PWM estimators for $n > 50$ are well approximated by their large-sample values given by (16) and Table 1. Maximum-likelihood estimators are the least biased but are more variable than PWM estimators in small samples. Even at sample size 100, the asymptotic inefficiency of the PWM method compared to maximum likelihood is not apparent in the simulation results. Sextile estimators in general have larger standard deviations than PWM estimators and have some large biases in small samples when $k < 0$.

The statistical properties of estimators of quantiles of the GEV distribution were evaluated for many combinations of quantiles and values of the shape parameter k , and only a few representative simulation results are presented in Table 6. The most important aspect of quantile estimation in hydrological applications is estimation of the extreme upper quantiles, particularly for heavy-tailed GEV distributions with $k < 0$. Table 5 gives the bias and standard deviation of the estimated upper quantiles for two GEV distributions, one with $k < 0$ and one with $k > 0$. Results are presented for the ratios $\hat{x}(F)/x(F)$ rather than for the $\hat{x}(F)$ themselves, since the former quantities are more easily compared at different F values. For sample size $n > 50$ the three methods have comparable performance. In small samples the upper quantiles obtained by PWM estimation are rather biased, but are still preferable to the maximum-likelihood estimators since these have very large biases and standard deviations. The errors in the maximum-likelihood quantile estimators arise chiefly from a small number of simulated series which yield large negative estimates of k , and consequently give very large estimates of extreme upper quantiles.

Estimation of extreme lower quantiles tends to be less important in practice than estimation of upper quantiles, so simulation results for this case are not given in detail. All three methods give comparable results when $n > 50$, but for small samples the PWM estimators have smallest standard deviation and small or moderate bias, and are generally to be preferred.

Table 6. Bias and standard deviation of estimators of GEV quantiles.
Tabulated values are bias and standard deviation of the ratio
 $\hat{x}(F)/x(F)$ rather than of the quantile estimator $x(F)$ itself.

n	Method	k=-0.2						k=0.2					
		F=0.9		F=0.99		F=0.999		F=0.9		F=0.99		F=0.999	
		$x(F)=2.84$		$x(F)=7.55$		$x(F)=14.90$		$x(F)=1.81$		$x(F)=3.01$		$x(F)=3.74$	
		bias	s.d.	bias	s.d.	bias	s.d.	bias	s.d.	bias	s.d.	bias	s.d.
15	PMM	-.08	.32	-.03	.49	.12	.93	-.04	.21	.09	.31	.28	.57
	ML	.01	.39	.64	2.78	*	*	-.05	.21	.17	1.49	*	*
	JS	.08	.39	.00	.59	.06	1.00	.04	.22	.00	.31	.05	.53
25	PMM	-.03	.27	.01	.45	.13	.78	-.02	.18	.06	.24	.16	.38
	ML	.01	.30	.17	.72	.59	1.93	-.04	.17	.04	.24	.00	.38
	JS	.01	.29	-.01	.45	.05	.76	-.01	.18	-.01	.24	.02	.37
50	PMM	-.02	.19	-.05	.34	-.03	.66	-.02	.12	-.02	.15	-.01	.22
	ML	-.02	.19	-.02	.34	.02	.67	-.01	.12	.00	.16	.01	.23
	JS	-.02	.19	-.04	.34	.00	.69	-.02	.12	-.01	.16	.00	.22
100	PMM	-.01	.14	.00	.23	.03	.38	-.01	.09	.01	.12	.03	.16
	ML	.00	.14	.02	.25	.08	.44	-.01	.09	-.02	.11	-.02	.15
	JS	.03	.15	.01	.23	.01	.36	.01	.09	.00	.12	-.01	.17

* indicates values which varied widely between different sets of 1000 simulations and consequently could not be estimated accurately.

All the methods of quantile estimation are very inaccurate when estimating extreme quantiles in small samples with $k < 0$. It is of course to be expected that a quantile $x(F)$ cannot be estimated reliably from a sample of size n if $F > 1-1/n$. The implication of this fact for hydrological practice is that when estimating the upper quantiles of a flood frequency distribution for a site with scanty data one should seek to incorporate information from other nearby sites. Such a regionalization procedure can be based on the PMM estimation method for the GEV distribution: see Hosking et al (1984).

Testing whether the shape parameter is zero

The type I extreme-value distribution, or Gumbel distribution, is a particularly simple special case of the GEV distribution, and it is often useful to test whether a given set of data is generated by a Gumbel rather than a GEV distribution. This is equivalent to testing whether the shape parameter k is zero in the GEV distribution. A test of this

hypothesis may be based on the PWM estimator of k . On the null hypothesis $H_0 : k = 0$ the PWM estimator \hat{k} is asymptotically distributed as $N(0, 0.5635/n)$ so the test may be performed by comparing the statistic $Z = \hat{k}(n/0.5635)^{1/2}$ with the critical values of a standard Normal distribution. Significant positive values of Z imply rejection of H_0 in favour of the alternative $k > 0$, and significant negative values of Z imply rejection in favour of $k < 0$.

The size of the test based on Z for various sample sizes and its power for sample size 50 are given in Tables 7 and 8. These results are based on computer simulations of 1000 samples for each value of n and k . The results may be compared with Hosking's (1984a) survey of tests of this hypothesis: the Z-test has power almost as high as the likelihood-ratio test and for samples of size 25 or more its distribution on H_0 is accurately approximated by the standard Normal significance levels. Since the statistic Z is very simple to compute, the Z-test can be strongly recommended as a convenient and powerful indicator of the sign of the shape parameter of the GEV distribution.

Table 7. Empirical significance levels of the statistic Z for testing the hypothesis $H : k=0$ against one-sided and two-sided alternatives.

Alternative: Nominal level: Sample size	$k < 0$		$k > 0$		$k \neq 0$	
	10%	5%	10%	5%	10%	5%
15	10.5	4.0	7.3	3.4	7.4	2.5
25	9.8	4.8	9.4	4.9	9.7	4.4
50	11.5	5.4	8.4	4.7	10.1	4.5
100	10.5	4.9	10.3	5.4	10.3	4.5
200	10.0	5.7	9.0	4.7	10.4	5.2
500	10.3	4.9	9.8	5.0	9.9	5.6

Table 8. Power of the Z-test of the hypothesis $k=0$ against one-sided and two-sided alternatives. Sample size 50, nominal significance level 5%

Alternative: k	k<0	k>0	k≠0
-0.5	.96	-	.94
-0.4	.91	-	.85
-0.3	.78	-	.71
-0.2	.55	-	.45
-0.1	.27	-	.18
0.0	.05	.05	.05
0.1	-	.17	.10
0.2	-	.49	.36
0.3	-	.83	.72
0.4	-	.96	.93
0.5	-	1.00	.99

Conclusions

Estimators of parameters and quantiles of the GEV distribution have been derived using the method of probability weighted moments. These estimators have several advantages over existing methods of estimation. They are fast and straightforward to compute and always yield feasible values for the estimated parameters. The biases of the estimators are small, except when estimating quantiles in the extreme tails of the GEV distribution, and decrease rapidly as the sample size increases. The standard deviations of the PMM estimators are comparable with those of the maximum-likelihood estimators for moderate sample sizes ($n = 50, 100$) and often substantially less than those of the maximum likelihood estimators for small samples ($n = 15, 25$). PMM estimators of GEV parameters and quantiles have asymptotic Normal distributions and the large-sample approximation to the variance of the estimators is adequate for sample sizes of 50 or more. Although PMM estimators are asymptotically inefficient compared to maximum-likelihood estimators, no inefficiency is detectable in samples of size 100 or less. The PMM estimator of the shape parameter k of the GEV distribution may be used as the basis of a test of the hypothesis $H_0 : k = 0$, and this test is simple to perform, powerful and accurate.

Appendix 1. Probability weighted moments for the GEV distribution

For the GEV distribution we have from (2) and (3)

$$\begin{aligned}
 \beta_r &= M_{0,r,0} = \int_0^1 [\xi + \alpha(1 - (-\log F)^k)/k] F^r dF \\
 &= \int_0^\infty [\xi + \alpha(1-u^k)/k] e^{-(r+1)u} du \quad \text{substituting } u = -\log F, \\
 &= (\xi + \alpha/k) \int_0^\infty e^{-(r+1)u} du - (\alpha/k) \int_0^\infty u^k e^{-(r+1)u} du \\
 &= (\xi + \alpha/k)(r+1)^{-1} - (\alpha/k)(r+1)^{-1+k} \Gamma(1+k) \quad \text{provided that } k > -1, \\
 &= (r+1)^{-1} [\xi + \alpha(1 - (r+1)^{-k} \Gamma(1+k))/k]. \quad (A1)
 \end{aligned}$$

Appendix 2. Feasibility of PWM estimates of the GEV parameters

The PWM estimator \hat{k} satisfies (13), and therefore $\hat{k} > -1$ provided that

$$(2b_1 - b_0)/(3b_2 - b_0) > \frac{1}{2}. \quad (A2)$$

Now

$$2b_1 - b_0 = \frac{1}{n(n-1)} \sum_{i>j} (x_i - x_j) \quad (A3)$$

and

$$3b_2 - b_0 = \frac{2}{n(n-1)(n-2)} \sum_{i>j>k} (2x_i - x_j - x_k) \quad (A4)$$

are both positive, so (A2) reduces to $b_0 - 4b_1 + 3b_2 < 0$. But we can write

$$b_0 - 4b_1 + 3b_2 = \frac{2}{n(n-1)(n-2)} \sum_{i>j>k} (-x_j + x_k), \quad (A5)$$

thus $b_0 - 4b_1 + 3b_2 < 0$ almost surely and therefore $\hat{k} > -1$ almost surely. Results

(A3) - (A5) above are easily proved by induction on the sample size n .

Furthermore, since

$$\hat{\alpha} = \frac{(2b_1 - b_0)\hat{k}}{\Gamma(1+\hat{k})(1-2^{-\hat{k}})} \quad (A6)$$

and $2b_1 - b_0 > 0$ as noted above, $k/(1-2^{-k}) > 0$ for all k and $\Gamma(1+k) > 0$ because $\hat{k} > -1$, it follows that we must have $\hat{\alpha} > 0$.

Appendix 3. Asymptotic distribution of the b_r

The statistic b_r may be written as a linear combination of the order statistics of a random sample: we have

$$b_r = n^{-1} \sum_{j=1}^n c_{nj}^{(r)} x_j \quad (A7)$$

where $c_{nj}^{(r)} = (j-1)\dots(j-r)/((n-1)\dots(n-r))$ and $x_1 < x_2 < \dots < x_n$ is the ordered sample. As $n \rightarrow \infty$ and $j \rightarrow \infty$ with $j/n \rightarrow \theta$, $0 < \theta < 1$, $c_{nj}^{(r)}$ is asymptotically a function of the plotting position $j/(n+1)$: in fact $c_{nj}^{(r)} \sim \{j/(n+1)\}^r$. It is straightforward to verify that b_r satisfies the conditions of Theorem 1 of Chernoff et al (1967), and from that theorem it follows that b_r is asymptotically Normally distributed with mean β_r and variance

$$n^{-1} v_{rr} = 2n^{-1} \int \int_{x < y} \{F(x)\}^r \{F(y)\}^r F(x) \{1 - F(y)\} dx dy \quad (A8)$$

A similar argument applies to any linear combination of the b_r , $r = 0, 1, 2, \dots$, and it follows that the b_r are asymptotically jointly Normally distributed with covariance given by

$$v_{rs} = \lim_{n \rightarrow \infty} n \text{cov}(b_r, b_s) = \frac{1}{2} (g_{rs} + g_{sr}) \quad (A9)$$

where

$$g_{rs} = 2 \int \int_{x < y} \{F(x)\}^r \{F(y)\}^s F(x) \{1 - F(y)\} dx dy \quad (A10)$$

To evaluate the g_{rs} for the GEV distribution we consider first the case $k > 0$ and let

$$I_{rs} = 2 \int \int_{x < y} \{F(x)\}^r \{F(y)\}^s dx dy \quad (A11)$$

so that

$$g_{rs} = I_{r+1,s} - I_{r+1,s+1} \quad (A12)$$

Substituting (7) in (A11) and making the further substitution $u = \{1 - k(x-\xi)/\alpha\}^{1/k}$, $v = \{1 - k(y-\xi)/\alpha\}^{1/k}$, we have

$$\begin{aligned}
I_{rs} &= 2\alpha^2 \int_0^\infty \int_v^\infty u^{k-1} e^{-ru} du \cdot v^{k-1} e^{-sv} dv \\
&= 2\alpha^2 r^{-k} \int_0^\infty v^{k-1} e^{-sv} \Gamma(k, rv) dv \\
&= \alpha^2 k^{-2} (r+s)^{-2k} \Gamma(1+2k) {}_2F_1[1, 2k; 1+k; s/(r+s)]
\end{aligned} \tag{A13}$$

(Gradshteyn and Ryzhik, 1980, pp. 317, 663); here $\Gamma(\cdot, \cdot)$ is the incomplete gamma function and ${}_2F_1$ is the hypergeometric function. It is convenient to transform the hypergeometric function in (A13), using results from Gradshteyn and Ryzhik (1980, p. 1043):

$$\begin{aligned}
{}_2F_1[1, 2k; 1+k; s/(r+s)] &= 2^{2k} \{\Gamma(1+k)\}^2 / \Gamma(1+2k) \quad \text{if } r = s, \\
&= \{r/(r+s)\}^{-2k} G(s/r) \quad \text{if } r > s, \\
&= -\{s/(r+s)\}^{-2k} G(r/s) + 2r^{-k} s^{-k} (r+s)^{2k} \{\Gamma(1+k)\}^2 / \Gamma(1+2k) \quad \text{if } r < s,
\end{aligned} \tag{A14}$$

where G denotes the hypergeometric function $G(x) = {}_2F_1(k, 2k; 1+k; -x)$ - note that $G(0) = 1$. Substituting back into (A12) and (A9) we obtain the following expressions for the v_{rs} :

$$v_{rr} = \alpha^2 k^{-2} (r+1)^{-2k} [\Gamma(1+2k) G(r/(r+1)) - \{\Gamma(1+k)\}^2], \tag{A15}$$

$$\begin{aligned}
v_{r, r+1} &= \frac{1}{2} \alpha^2 k^{-2} [(r+2)^{-2k} \Gamma(1+2k) G(r/(r+2)) + \\
&\quad (r+1)^{-k} \{(r+1)^{-k} - 2 \cdot (r+2)^{-k}\} \{\Gamma(1+k)\}^2],
\end{aligned} \tag{A16}$$

$$\begin{aligned}
v_{r, r+s} &= \frac{1}{2} \alpha^2 k^{-2} [(r+s+1)^{-2k} \Gamma(1+2k) G(r/(r+s+1)) - (r+s)^{-k} \Gamma(1+2k) G\{(r+1)/(r+s)\} + \\
&\quad 2(r+1)^{-k} \{(r+s)^{-k} - (r+s+1)^{-k}\} \{\Gamma(1+k)\}^2], \quad s > 2.
\end{aligned} \tag{A17}$$

When $k < 0$ the foregoing argument is not valid because the integral in (A11) does not converge. However, the expressions (A15) - (A17) are analytic functions of k for all $k > -\frac{1}{2}$ and hence by analytic continuation expressions (A15) - (A17) are valid solutions of the integral representation (A9) - (A10) throughout the domain $-\frac{1}{2} < k < \infty$. At the value $k = 0$, the v_{rs} are given by the limits of (A15) - (A17) as $k \rightarrow 0$; these limits are well-defined.

The results stated in this Appendix are valid for arbitrary positive integers r and s , though only the cases $r, s = 0, 1, 2$ are required for deriving the asymptotic distributions of PWM estimators.

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