TECHNOMETRICS, AUGUST 1985, VOL. 27, NO. 3

Estimation of the Generalized Extreme-Value Distribution by the Method of Probability-Weighted Moments

J. R. M. Hosking
Institute of Hydrology
Wallingford, Oxon OX10 8BB
England

J. R. Wallis
IBM Research Center
Yorktown Heights, NY 10598

E. F. Wood
Department of Civil Engineering
Princeton University
Princeton, NJ 08540

We use the method of probability-weighted moments to derive estimators of the parameters and quantiles of the generalized extreme-value distribution. We investigate the properties of these estimators in large samples, via asymptotic theory, and in small and moderate samples, via computer simulation. Probability-weighted moment estimators have low variance and no severe bias, and they compare favorably with estimators obtained by the methods of maximum likelihood or sextiles. The method of probability-weighted moments also yields a convenient and powerful test of whether an extreme-value distribution is of Fisher-Tippett Type I, II, or III.

KEY WORDS: Generalized extreme-value distribution; Hypothesis testing; Order statistics; Probability-weighted moments.

1. INTRODUCTION

The generalized extreme-value distribution of Jenkinson (1955) is widely used for modeling extremes of natural phenomena, and it is of considerable importance in hydrology, since it was recommended by the *Flood Studies Report* [Natural Environment Research Council (NERC) 1975a] for modeling the distribution of annual maxima of daily streamflows of British rivers. Currently favored methods of estimation of the parameters and quantiles of the distribution are Jenkinson's (1969) method of sextiles and the method of maximum likelihood (Jenkinson 1969; Prescott and Walden 1980, 1983). Neither method is completely satisfactory: The justification of the maximum-likelihood approach is based on large-sample theory, and there has been little assessment of the performance of the method when applied to small or moderate samples; whereas the sextile method involves an inherent arbitrariness (why sextiles rather than, say, quartiles or octiles?), requires interpolation in a table of values of a function in order to estimate the shape parameter of the distribution, and has statistical properties that are not known even for large samples.

Probability-weighted moments, a generalization of the usual moments of a probability distribution, were introduced by Greenwood et al. (1979). There are several distributions—for example, the Gumbel, logistic, and Weibull—whose parameters can be conveniently estimated from their probability-weighted moments. The Gumbel distribution, being a special case of the generalized extreme-value distribution, is of particular interest. Landwehr et al. (1979) investigated the small-sample properties of probability-weighted moment (PWM) estimators of parameters and quantiles for the Gumbel distribution and found them superior in many respects to the conventional moment and maximum-likelihood estimators. The estimators used by Landwehr et al. (1979) are identical to Downton's (1966b) linear estimates with linear coefficients, and thus share the asymptotic properties of the latter; in particular, the asymptotic efficiencies of the PWM estimators of the Gumbel scale and location parameters are .756 and .996, respectively.

In this article we summarize some theory for probability-weighted moments and show that they can be used to obtain estimates of parameters and quantiles of the generalized extreme-value distribution. We derive the asymptotic distributions of these estimators and compare, via computer simulation, the small-sample properties of the PWM, sextile, and maximum-likelihood estimators. The method of probability-weighted moments outperforms the other methods in many cases and will usually be preferred to them. We also derive, from the PWM estimator of the shape parameter of the generalized extreme-value distribution, a test of whether this shape parameter is zero, and we assess the performance of this test by computer simulation.

2. PROBABILITY-WEIGHTED MOMENTS

The probability-weighted moments of a random variable X with distribution function $F(x) =$

$P(X \le x)$ are the quantities

$$M_{p,r,s} = E[X^p\{F(X)\}^r\{1 - F(X)\}^s], \quad (1)$$

where p, r, and s are real numbers. Probability-weighted moments are likely to be most useful when the inverse distribution function $x(F)$ can be written in closed form, for then we may write

$$M_{p,r,s} = \int_0^1 \{x(F)\}^p F^r (1 - F)^s \, dF, \quad (2)$$

and this is often the most convenient way of evaluating these moments. The quantities $M_{p,0,0}$ ($p = 1, 2, \ldots$) are the usual noncentral moments of X. The moments $M_{1,r,s}$ may, however, be preferable for estimating the parameters of the distribution of X, since the occurrence of only the first power of X in the expression for $M_{1,r,s}$ means that the relationship between parameters and moments often takes a simpler form in this case than when using the conventional moments. When r and s are integers, $F^r(1 - F)^s$ may be expressed as a linear combination of either powers of F or powers of $(1 - F)$, so it is natural to summarize a distribution either by the moments $M_{1,r,0}$ ($r = 0, 1, 2, \ldots$) or by $M_{1,0,s}$ ($s = 0, 1, 2, \ldots$). Greenwood et al. (1979) generally favored the latter approach, but here we will consider the moments $\beta_r = M_{1,r,0} = E[X\{F(X)\}^r]$ ($r = 0, 1, 2, \ldots$).

Given a random sample of size n from the distribution F, estimation of β_r is most conveniently based on the ordered sample $x_1 \le x_2 \le \cdots \le x_n$. The statistic

$$b_r = n^{-1} \sum_{j=1}^{n} \frac{(j-1)(j-2)\cdots(j-r)}{(n-1)(n-2)\cdots(n-r)} x_j \quad (3)$$

is an unbiased estimator of β_r (Landwehr et al. 1979). Instead one may estimate β_r by

$$\hat{\beta}_r[p_{j,n}] = n^{-1} \sum_{j=1}^{n} p_{j,n}^r x_j, \quad (4)$$

where $p_{j,n}$ is a plotting position—that is, a distribution-free estimate of $F(x_j)$. Reasonable choices of $p_{j,n}$, such as $p_{j,n} = (j - a)/n$, $0 < a < 1$, or $p_{j,n} = (j - a)/(n + 1 - 2a)$, $-\frac{1}{2} < a < \frac{1}{2}$, yield estimators $\hat{\beta}_r[p_{j,n}]$, which are asymptotically equivalent to b_r and, therefore, consistent estimators of β_r.

The estimators b_r are closely related to U-statistics (Hoeffding 1948), which are averages of statistics calculated from all subsamples of size $j < n$ of a given sample of size n. In particular, $b_0 = n^{-1} \sum x_j$ is a trivial example of a U-statistic, and it is a natural estimator of location of a distribution;

$$2b_1 - b_0 = \tfrac{1}{2}U_2 = \tfrac{1}{2}\binom{n}{2}^{-1} \sum_{i>j} (x_i - x_j) \quad (5)$$

is a U-statistic for estimating the scale of a distribution—the statistic U_2, sometimes known as Gini's mean difference, has a history going back at least as far as van Andrae (1872), and $\frac{1}{2}\sqrt{\pi}\, U_2$ is a 98% efficient estimator of the scale parameter of a Normal distribution (Downton 1966a); and

$$6b_2 - 6b_1 + b_0 = \tfrac{1}{3}U_3 = \tfrac{1}{3}\binom{n}{3}^{-1} \sum_{i>j>k} (x_i - 2x_j + x_k) \quad (6)$$

is a U-statistic for estimating skewness, which has been used as the basis of a test for Normality by Locke and Spurrier (1976). U-statistics are widely used in nonparametric statistics (e.g., see Fraser 1957, chap. 4, and Randles and Wolfe 1979, chap. 3), and their desirable properties of robustness to outliers in the sample, high efficiency, and asymptotic normality may be expected to extend to the probability-weighted moment estimators b_r and other quantities calculated from them.

3. PWM ESTIMATORS FOR THE GENERALIZED EXTREME-VALUE DISTRIBUTION

The generalized extreme-value (GEV) distribution, introduced by Jenkinson (1955), combines into a single form the three possible types of limiting distribution for extreme values, as derived by Fisher and Tippett (1928). The distribution function is

$$\begin{aligned} F(x) &= \exp[-\{1 - k(x - \xi)/\alpha\}^{1/k}], \qquad k \ne 0, \\ &= \exp[-\exp\{-(x - \xi)/\alpha\}], \qquad k = 0, \end{aligned} \quad (7)$$

with x bounded by $\xi + \alpha/k$ from above if $k > 0$ and from below if $k < 0$. Here ξ and α are location and scale parameters, respectively, and the shape parameter k determines which extreme-value distribution is represented: Fisher-Tippett Types I, II, and III correspond to $k = 0$, $k < 0$, and $k > 0$, respectively. When $k = 0$ the GEV distribution reduces to the Gumbel distribution. The inverse distribution function is

$$\begin{aligned} x(F) &= \xi + \alpha\{1 - (-\log F)^k\}/k, \qquad k \ne 0 \\ &= \xi - \alpha \log(-\log F), \qquad k = 0. \end{aligned} \quad (8)$$

In practice the shape parameter usually lies in the range $-\frac{1}{2} < k < \frac{1}{2}$. For example, the data base used in NERC (1975a) included 32 annual flood series with sample sizes of 30 or more. GEV distributions were fitted to these 32 samples by the method of maximum likelihood: The estimated shape parameter ranged from $-.32$ to $.48$.

The probability-weighted moments of the GEV distribution for $k \ne 0$ are given by

$$\beta_r = (r + 1)^{-1}[\xi + \alpha\{1 - (r + 1)^{-k}\Gamma(1 + k)\}/k], \qquad k > -1 \quad (9)$$

(for proof, see Appendix A). When $k \le -1$, β_0 (the mean of the distribution) and the rest of the β_r do not exist. From (9) we have

$$\beta_0 = \xi + \alpha\{1 - \Gamma(1 + k)\}/k, \tag{10}$$

$$2\beta_1 - \beta_0 = \alpha\Gamma(1 + k)(1 - 2^{-k})/k, \tag{11}$$

and

$$(3\beta_2 - \beta_0)/(2\beta_1 - \beta_0) = (1 - 3^{-k})/(1 - 2^{-k}). \tag{12}$$

The PWM estimators $\hat{\xi}$, $\hat{\alpha}$, $\hat{k}$ of the parameters are the solutions of (10)–(12) for ξ, α, and k when the β_r are replaced by their estimators b_r or $\hat{\beta}_r[p_{j,n}]$. To obtain $\hat{k}$ we must solve the equation

$$(3b_2 - b_0)/(2b_1 - b_0) = (1 - 3^{-\hat{k}})/(1 - 2^{-\hat{k}}). \tag{13}$$

The exact solution requires iterative methods, but because the function $(1 - 3^{-k})/(1 - 2^{-k})$ is almost linear over the range of values of k $(-\frac{1}{2} < k < \frac{1}{2})$, which is usually encountered in practice, low-order polynomial approximations for $\hat{k}$ are very accurate. We propose the approximate estimator

$$\hat{k} = 7.8590c + 2.9554c^2, \qquad c = \frac{2b_1 - b_0}{3b_2 - b_0} - \frac{\log 2}{\log 3}; \tag{14}$$

the error in $\hat{k}$ due to using (14) rather than (13) is less than .0009 throughout the range $-\frac{1}{2} < \hat{k} < \frac{1}{2}$. Given $\hat{k}$, the scale and location parameters can be estimated successively from Equations (11) and (10) as

$$\hat{\alpha} = \frac{(2b_1 - b_0)\hat{k}}{\Gamma(1 + \hat{k})(1 - 2^{-\hat{k}})}, \quad \hat{\xi} = b_0 + \hat{\alpha}\{\Gamma(1 + \hat{k}) - 1\}/\hat{k}. \tag{15}$$

Equations (13) and (15), or their equivalent forms with b_r replaced by $\hat{\beta}_r[p_{j,n}]$, define the PWM estimators of the parameters of the GEV distribution. Given the estimated parameters, the quantiles of the distribution are estimated using the inverse distribution function (8).

When calculated using b_r as the estimator of β_r, the PWM estimates of the GEV distribution satisfy a feasibility criterion, namely that $\hat{k} > -1$ and $\hat{\alpha} > 0$ almost surely (for proof, see Appendix B). This is clearly a desirable property, since one would like estimates calculated using a set of sample moments to yield an estimated distribution for which the corresponding population moments exist. We have not been able to prove that this feasibility criterion is satisfied when plotting-position estimators $\hat{\beta}_r[p_{j,n}]$ are used, but no examples of the criterion not being satisfied have been discovered in practice.

4. ASYMPTOTIC DISTRIBUTION OF PWM ESTIMATORS

When modeling the properties of extremes of physical processes, it rarely occurs that the available data set is large enough to ensure that asymptotic large-sample theory may be directly applied to the problem. It is nonetheless valuable to investigate the asymptotic properties of a new statistical technique, for two main reasons. First, one may seek to establish the integrity of the technique, in the sense that when a large sample is available, the new method should not be grossly inefficient compared to an established, asymptotically optimal method such as maximum likelihood. Second, asymptotic theory may provide an adequate approximation to some aspect of the distribution of a statistic even in quite small samples. In the present case we shall see that the variance of PWM estimators of parameters and quantiles of the GEV distribution is well approximated by asymptotic theory for sample sizes of 50 or larger.

We consider first the asymptotic distribution of the b_r. From (3), b_r is a linear combination of the order statistics $x_1, \ldots, x_n$, and the results of Chernoff et al. (1967) may be used to prove that the vector $b = (b_0\ b_1 b_2)^T$ has asymptotically a multivariate Normal distribution with mean $\beta = (\beta_0\, \beta_1 \beta_2)^T$ and covariance matrix $n^{-1}V$. The elements of V and details of the proof are given in Appendix C.

The asymptotic distribution of the PWM estimators of the GEV parameters follows from the preceding result. Let $\theta = (\xi \alpha k)^T$, $\hat{\theta} = (\hat{\xi}\hat{\alpha}\hat{k})^T$, and write (13) and (15) as the vector equation $\hat{\theta} = f(b)$. Define the 3×3 matrix $G = (g_{ij})$ by $g_{ij} = \partial f_i/\partial b_j$. Then asymptotically, $\hat{\theta}$ has a multivariate Normal distribution with mean $f(\beta) = \theta$ and covariance matrix $n^{-1}GVG^T$ (Rao 1973, p. 388). The covariance matrix has the form

$$n^{-1}GVG^T = n^{-1}\begin{pmatrix} \alpha^2 w_{11} & \alpha^2 w_{12} & \alpha_{w13} \\ \alpha^2 w_{12} & \alpha^2 w_{22} & \alpha w_{23} \\ \alpha w_{13} & \alpha w_{23} & w_{33} \end{pmatrix}. \tag{16}$$

The w_{ij} are functions of k and have a complicated algebraic form, but they can be evaluated numerically and are given in Table 1 for several values of k. As k approaches $-\frac{1}{2}$, the variance of the GEV distribution becomes infinite and the variances of the b_r and of the PWM parameter estimators are no longer of order n^{-1} asymptotically.

The asymptotic biases of the estimators are of order n^{-1} and can be evaluated by methods similar to those of Rao (1973, p. 388). The biases, graphed in Figure 1, are negligible in large samples provided that $k > -.4$.

For comparison, asymptotic biases of the maximum-likelihood estimators of the parameters of the GEV distribution are graphed in Figure 2. These biases were calculated using equation 3.12 of Shenton and Bowman (1977) and are functions of expected values of third derivatives of the log-

Table 1. Elements of the Asymptotic Covariance Matrix of the PWM Estimators of the Parameters of the GEV Distribution

k	w_{11}	w_{12}	w_{13}	w_{22}	w_{23}	w_{33}
−.4	1.6637	1.3355	1.1405	1.8461	1.1628	2.9092
−.3	1.4153	.8912	.5640	1.2574	.4442	1.4090
−.2	1.3322	.6727	.3926	1.0013	.2697	.9139
−.1	1.2915	.5104	.3245	.8440	.2240	.6815
.0	1.2686	.3704	.2992	.7390	.2247	.5633
.1	1.2551	.2411	.2966	.6708	.2447	.5103
.2	1.2474	.1177	.3081	.6330	.2728	.5021
.3	1.2438	−.0023	.3297	.6223	.3033	.5294
.4	1.2433	−.1205	.3592	.6368	.3329	.5880

NOTE: PWM—Probability-weighted moment; GEV—Generalized extreme value.

likelihood function. For $k \geq \frac{1}{3}$ these expectations do not exist and the biases of the maximum-likelihood estimators are not of order n^{-1} asymptotically.

The asymptotic variances of the estimators are graphed in Figure 3, and their asymptotic efficiencies in Figure 4. Asymptotic efficiency is defined as the ratio

$$\text{eff}\,(\hat{\theta}_i) = \lim_{n\to\infty}\,(\text{var}\ \tilde{\theta}_i/\text{var}\ \hat{\theta}_i)$$

for each element θ_i of the parameter vector θ, where $\tilde{\theta}_i$ is the maximum-likelihood estimator of θ_i. Overall efficiency is the ratio of the determinants of the asymptotic covariance matrices of $\tilde{\theta}$ and $\hat{\theta}$. The overall efficiency of the PWM estimators tends to zero at $k = \pm.5$, but for values of k not too far from zero the PWM method is reasonably efficient. Within the range $-.2 < k < .2$, which is valid for many hydrological data sets, each PWM parameter estimator has efficiency of more than .7.

Corresponding results may be obtained for PWM and maximum-likelihood estimators of quantiles of the GEV distribution. These are not presented in full, but Tables 2 and 3 give results for various quantiles when $k = -.2$ and for various values of k at the $F = .98$ quantile. The tables illustrate the main characteristics of PWM quantile estimators, which are (a) high positive bias in extreme upper tail, arising from positive bias in $\hat{k}$; (b) high variance in upper tail when $k < 0$; and (c) fair or high efficiency except when k is close to $\pm.5$. The maximum-likelihood quantile estimators have lower variance than the PWM estimators but have some very large biases, particularly in the extreme upper tail of the distribution.

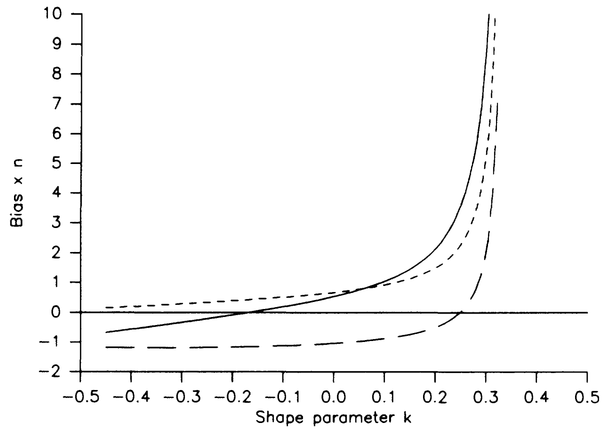

Figure 2. Asymptotic Bias of ML Estimators of Parameters of the GEV Distribution: ——— $\hat{k}$; — — — $\hat{\alpha}$; ----- $\hat{\xi}$.

The results of this section were derived for PWM estimators that use b_r to estimate β_r. If the plotting-position estimates $\hat{\beta}_r[p_{j,n}]$ are used instead, the asymptotic variances and efficiencies remain unchanged, but the asymptotic biases are different and cannot be easily calculated, being affected by the biases in the $\hat{\beta}_r[p_{j,n}]$ themselves.

5. SMALL-SAMPLE PROPERTIES OF ESTIMATES OF THE GEV DISTRIBUTION

A computer simulation experiment was run to compare three methods of estimation of the parame-

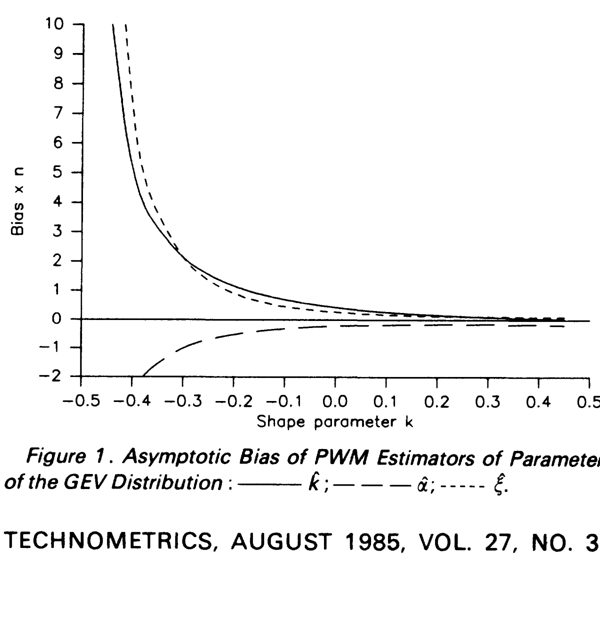

Figure 1. Asymptotic Bias of PWM Estimators of Parameters of the GEV Distribution: ——— $\hat{k}$; — — — $\hat{\alpha}$; ----- $\hat{\xi}$.

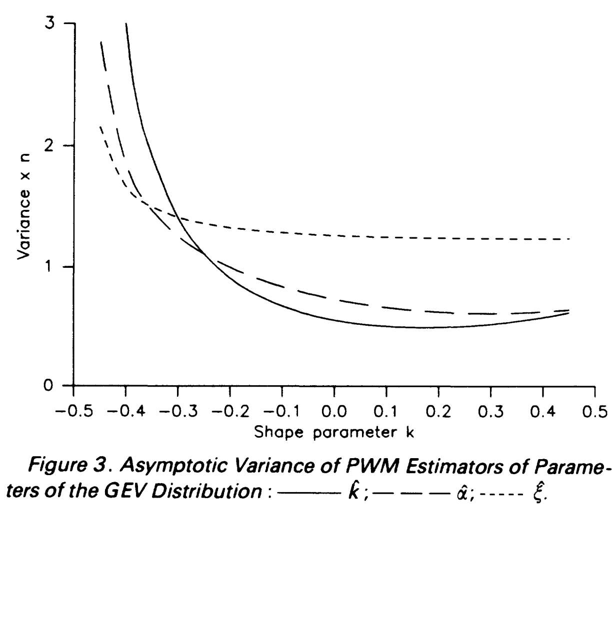

Figure 3. Asymptotic Variance of PWM Estimators of Parameters of the GEV Distribution: ——— $\hat{k}$; — — — $\hat{\alpha}$; ----- $\hat{\xi}$.

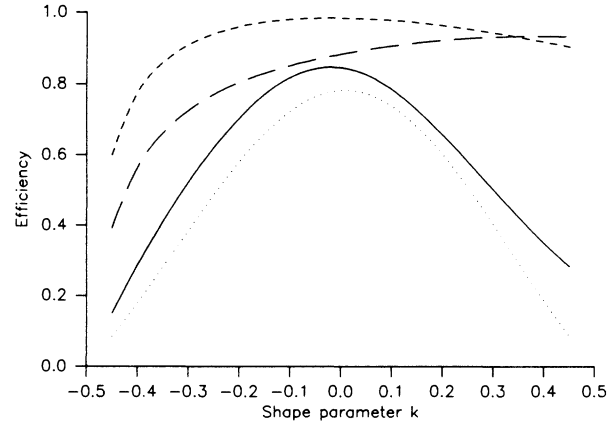

Figure 4. Asymptotic Efficiency of PWM Estimators of Parameters of the GEV Distribution: ——— $\hat{k}$; — — — $\hat{\alpha}$; ----- $\hat{\xi}$; ······ *overall efficiency (i.e., ratio of determinants of asymptotic covariance matrices of ML and PWM estimators).*

ters and quantiles of the GEV distribution. Simulations were performed for sample sizes $n = 15, 25, 50, 100$ with the shape parameter of the distribution taking values $k = -.4, -.2, .0, .2, .4$. All the methods of estimation are invariant under linear transformations of the data, so without loss of generality the location and scale parameters $\xi = 0$ and $\alpha = 1$ were used throughout. For each combination of values of n and k, 10,000 random samples were generated from the GEV distribution, and for each sample the parameters ξ, α, and k, and the quantiles $x(F)$, where $F = .001, .01, .1, .2, .5, .8, .9, .98, .99, .998, .999$, were estimated by each of three methods: (a) the method of probability-weighted moments (PWM), described before; (b) the method of maximum likelihood (ML), using Newton–Raphson iteration to maximize the likelihood function, as recommended by Prescott and Walden (1983); and (c) Jenkinson's (1969) method of

Table 2. Asymptotic Bias and Variance of PWM and ML Estimators of GEV Quantiles, and Efficiency of the PWM Estimators Compared With ML (parameters $\xi = 0$, $\alpha = 1$, $k = -.2$)

		$n \times$ Bias		$n \times$ Variance		
F	*x(F)*	*PWM*	*ML*	*PWM*	*ML*	*Efficiency*
.001	−1.60	−1.2	1.7	3.78	2.29	.61
.01	−1.32	−.2	1.6	2.06	1.35	.66
.1	−.77	.8	1.2	.86	.79	.92
.2	−.45	1.0	.9	.88	.88	1.00
.5	.38	.6	−.1	1.92	1.79	.93
.8	1.75	−1.3	−1.2	6.10	6.00	.98
.9	2.84	−3.1	−1.1	16.1	15.9	.99
.98	5.91	−4.4	5.5	147	131	.89
.99	7.55	−1.6	13.4	336	289	.86
.998	12.33	23.9	54.8	1,760	1,430	.81
.999	14.90	49.1	88.2	3,310	2,630	.80

NOTE: PWM—Probability-weighted moment; ML—Maximum likelihood; GEV—Generalized extreme value.

Table 3. Asymptotic Bias, Variance, and Efficiency of the PWM and ML Estimators of the F = .98 Quantile of the GEV Distribution, and the Efficiency of the PWM Estimators (parameters $\xi = 0$, $\alpha = 1$)

		$n \times$ Bias		$n \times$ Variance		
k	*x(F)*	*PWM*	*ML*	*PWM*	*ML*	*Efficiency*
−.4	9.41	−64.8	31.5	1,170	574	.49
−.3	7.41	−15.3	15.3	369	275	.75
−.2	5.91	−4.4	5.5	147	131	.89
−.1	4.77	−1.1	−.3	64.8	62.0	.96
.0	3.90	−.1	−3.8	30.2	28.8	.95
.1	3.23	.3	−6.2	14.7	13.0	.88
.2	2.71	.5	−8.9	7.53	5.62	.75
.3	2.30	.5	−23.2	4.04	2.28	.56
.4	1.98	.6	—	2.28	.83	.36

NOTE: PWM—Probability-weighted moment; ML—Maximum likelihood; GEV—Generalized extreme value.

sextiles (JS). The PWM method requires a choice of a suitable estimator of β_r. Several possibilities were investigated, including the unbiased estimator b_r and a number of plotting-position estimators $\hat{\beta}_r[p_{j,n}]$. The best overall results were given by the estimator $\hat{\beta}_r[p_{j,n}]$ with $p_{j,n} = (j - .35)/n$, and the simulation results presented for the PWM method refer to this version of the PWM estimators.

Maximum-likelihood estimation of the GEV distribution is not always satisfactory. The log-likelihood function for a sample $\{x_1, \ldots, x_n\}$ is

$$\log L = -n \log \alpha - (1 - k) \sum y_i - \sum e^{-y_i},$$
$$y_i = -k^{-1} \log \{1 - k(x_i - \xi)/\alpha\},$$

and $\log L$ can be made arbitrarily large by setting $k > 1$ and choosing ξ and α so that the upper bound $\xi + \alpha/k$ of the distribution is sufficiently close to the largest data value. In practice maximum-likelihood estimates of the parameters are obtained by finding a local maximum of $\log L$. For some samples, however, it appears that $\log L$ does not have a local maximum. In our simulations of the GEV distribution this non-regularity of the likelihood function caused occasional nonconvergence of the Newton–Raphson iteration that was used to maximize the log-likelihood; such samples were omitted from the simulations. Such occurrences were rare except when the sample size was very small and when $k = .4$ (see Table 4), and in our opinion they have no significant effect on the conclusions that may be drawn regarding the relative merit of the different estimation procedures. Similar difficulties with maximum-likelihood estimation are encountered with other distributions whose range depends on their parameters, such as the three-parameter lognormal, Weibull, and gamma distributions (e.g., see Griffiths 1980 or Cheng and Amin 1983).

The simulation results for estimation of the param-

Table 4. Failure Rate of Maximum-Likelihood Estimation for the GEV Distribution

	k				
n	−.4	−.2	.0	.2	.4
15	.1	.6	1.7	3.8	12.4
25	.0	.0	.0	.3	1.9
50	.0	.0	.0	.0	.0
100	.0	.0	.0	.0	.0

NOTE: GEV—Generalized extreme value. Tabulated values are numbers of failures to converge of Newton-Raphson iterations per 100 simulated samples.

eters of the GEV distribution are summarized in Tables 5 and 6. Results for the estimator of k are of the greatest importance, since this parameter determines the overall shape of the GEV distribution and the rate of increase of the upper quantiles $x(F)$ as F approaches 1. Apart from the case $n = 100$, when all the methods have comparable performance, the PWM estimator has, consistently, the lowest standard deviation of the three estimators of k, its advantage being particularly marked in small samples, $n = 15$ and $n = 25$. The PWM estimator has in general a larger bias than the other estimators, but its bias is small near the important value $k = 0$ and is in any case relatively insignificant compared to the standard deviation in its contribution to the mean squared error of $\hat{k}$. The sextile estimator of k has a large positive bias in small samples when $k < 0$, and its standard deviation is generally larger than that of the PWM estimator.

Similar results can be seen for estimators of ξ and α, although the differences between the variances of the estimators are less marked than is the case with estimators of k. In general, PWM estimators have smallest standard deviation, particularly for $n = 15$ and $n = 25$, and their bias, though often larger than for the ML or sextile estimators, is not severe. The standard deviations of the PWM estimators for $n \geq 50$ are well approximated by their large-sample values given by (16) and Table 1. Maximum-likelihood estimators are the least biased but are more variable than PWM estimators in small samples. Even at sample size 100, the asymptotic inefficiency of the PWM method compared to maximum likelihood is not apparent in the simulation results. Sextile estimators in general have larger standard deviations than PWM estimators and have some significant biases in small samples when $k < 0$.

The statistical properties of estimators of quantiles of the GEV distribution were evaluated for many combinations of quantiles and values of the shape parameter k, and only a few representative simulation results are presented in Table 7. The most important aspect of quantile estimation in hydrological applications is estimation of the extreme upper quantiles, particularly for heavy-tailed GEV distributions with $k < 0$. Table 7 gives the bias and standard deviation of the estimated upper quantiles for two GEV distributions, one with $k < 0$ and one with $k > 0$. Results are presented for the ratios $\hat{x}(F)/x(F)$ rather than for the $\hat{x}(F)$ themselves, since the former quantities are more easily compared at different F values. For sample size 100 the three methods have comparable performance. In small samples the upper quantiles obtained by PWM estimation are rather biased, but they are still preferable to the maximum-likelihood estimators, since these have very large biases and standard deviations. The errors in the maximum-likelihood quantile estimators arise chiefly from a small number of simulated series that yield large negative estimates of k and consequently give very large estimates of extreme upper quantiles.

Table 5. Bias of Estimators of GEV Parameters

		k														
		Bias ($\hat{\xi}$)					Bias ($\hat{\alpha}$)					Bias ($\hat{k}$)				
n	Method	−.4	−.2	.0	.2	.4	−.4	−.2	.0	.2	.4	−.4	−.2	.0	.2	.4
15	PWM	.10	.05	−.02	.00	−.03	.00	−.06	−.10	−.11	−.12	.11	.03	−.03	−.08	−.12
	ML	.03	.03	.05	.05	.04	−.07	−.07	−.07	−.06	−.07	−.04	−.02	.02	.04	.03
	JS	.11	.08	.06	.04	.02	−.05	−.06	−.07	−.08	−.08	.10	.06	.03	.01	−.01
25	PWM	.06	.03	.01	−.01	−.02	.00	−.04	−.06	−.07	−.07	.08	.02	−.02	−.05	−.07
	ML	.01	.02	.03	.03	.04	−.04	−.04	−.04	−.03	−.03	−.02	−.01	.02	.04	.05
	JS	.06	.04	.03	.02	.01	−.03	−.03	−.04	−.04	−.05	.07	.04	.02	.01	−.00
50	PWM	.04	.02	.01	.00	−.01	.01	−.02	−.03	−.04	−.04	.05	.02	−.01	−.02	−.04
	ML	.01	.01	.02	.02	.02	−.02	−.02	−.02	−.02	−.01	−.01	.00	.01	.02	.03
	JS	.04	.02	.02	.01	.01	−.02	−.02	−.02	−.02	−.02	.04	.02	.01	.00	.00
100	PWM	.02	.01	.00	.00	−.01	.00	−.01	−.02	−.02	−.02	.03	.01	.00	−.01	−.02
	ML	.00	.00	.01	.01	.01	−.01	−.01	−.01	−.01	.00	−.01	.00	.00	.01	.02
	JS	.02	.01	.01	.00	.00	−.01	−.01	−.01	−.01	−.01	.02	.01	.00	.00	.00

NOTE: GEV—Generalized extreme value; PWM—Probability-weighted moment; ML—Maximum likelihood; JS—Jenkinson's (1969) sextiles.

Table 6. Standard Deviation of Estimators of GEV Parameters

		Standard Deviation ($\hat{\xi}$)					Standard Deviation ($\hat{\alpha}$)					Standard Deviation ($\hat{k}$)				
n	Method	−.4	−.2	.0	.2	.4	−.4	−.2	.0	.2	.4	−.4	−.2	.0	.2	.4
15	PWM	.32	.30	.29	.28	.28	.33	.25	.21	.19	.19	.20	.19	.18	.18	.19
	ML	.32	.32	.31	.30	.28	.28	.25	.23	.22	.21	.36	.32	.29	.27	.23
	JS	.33	.31	.30	.29	.28	.32	.28	.24	.21	.21	.25	.24	.23	.23	.23
25	PWM	.24	.23	.22	.22	.22	.24	.19	.17	.15	.16	.18	.16	.14	.14	.15
	ML	.24	.24	.23	.23	.22	.21	.19	.17	.16	.17	.24	.21	.20	.18	.17
	JS	.24	.23	.23	.22	.22	.24	.21	.18	.17	.16	.19	.18	.17	.17	.17
50	PWM	.17	.16	.16	.16	.16	.17	.14	.12	.11	.11	.14	.12	.11	.10	.11
	ML	.17	.16	.16	.16	.16	.15	.13	.12	.11	.11	.15	.13	.12	.11	.11
	JS	.17	.16	.16	.16	.16	.17	.14	.13	.12	.11	.14	.13	.12	.11	.12
100	PWM	.12	.12	.11	.11	.11	.12	.10	.09	.08	.08	.11	.09	.07	.07	.08
	ML	.12	.11	.11	.11	.11	.10	.09	.08	.08	.08	.10	.09	.08	.07	.07
	JS	.12	.12	.11	.11	.11	.12	.10	.09	.08	.08	.10	.09	.08	.08	.08

NOTE: GEV—Generalized extreme value; PWM—Probability-weighted moment; ML—Maximum likelihood; JS—Jenkinson's (1969) sextiles.

Estimation of extreme lower quantiles tends to be less important in practice than estimation of upper quantiles, so simulation results for this case are not given in detail. All three methods give comparable results when $n \geq 50$, but for small samples the PWM estimators have smallest standard deviation and small or moderate bias, and are generally to be preferred.

All the methods of quantile estimation are very inaccurate when estimating extreme quantiles in small samples with $k < 0$. It is of course to be expected that a quantile $x(F)$ cannot be estimated reliably from a sample of size n if $F > 1 - 1/n$. Nonetheless it is sometimes possible to obtain useful estimates of extreme quantiles from short data sets, by combining information from a number of independent data sets. Such a "regionalization" procedure, based on the PWM estimation method for the GEV distribution, is described in Hosking et al. (1985).

6. TESTING WHETHER THE SHAPE PARAMETER IS ZERO

The Type I extreme-value distribution, or Gumbel distribution, is a particularly simple special case of the GEV distribution, and it is often useful to test whether a given set of data is generated by a Gumbel rather than a GEV distribution. This is equivalent to testing whether the shape parameter k is zero in the GEV distribution. A test of this hypothesis may be based on the PWM estimator of k. On the null hy-

Table 7. Bias and Standard Deviation of Estimators of GEV Quantiles

		k = −.2						k = .2					
		F = .9, x(F) = 2.84		F = .99, x(F) = 7.55		F = .999, x(F) = 14.90		F = .9, x(F) = 1.81		F = .99, x(F) = 3.01		F = .999, x(F) = 3.74	
n	Method	Bias	SD	Bias	SD	Bias	SD	Bias	SD	Bias	SD	Bias	SD
15	PWM	−.06	.34	−.02	.55	.15	1.12	−.04	.23	.08	.32	.25	.56
	ML	.01	.50	*	*	*	*	−.06	.23	.02	.79	.44	8.10
	JS	−.08	.34	−.05	.56	.14	1.20	−.06	.22	−.01	.33	.11	.64
25	PWM	−.04	.27	−.01	.45	.11	.88	−.02	.18	.05	.24	.14	.39
	ML	−.01	.32	.16	.97	.74	9.59	−.04	.17	−.04	.29	.02	.86
	JS	−.05	.27	−.02	.46	.09	.89	−.04	.17	−.01	.25	.05	.42
50	PWM	−.02	.19	−.01	.33	.06	.61	−.01	.12	.02	.16	.07	.25
	ML	−.01	.22	.05	.40	.18	.86	−.03	.12	−.03	.16	−.02	.28
	JS	−.02	.19	−.01	.33	.04	.59	−.02	.12	−.01	.17	.02	.27
100	PWM	−.01	.14	.00	.24	.04	.42	−.01	.09	.01	.12	.03	.17
	ML	.00	.15	.02	.25	.08	.43	−.01	.09	−.02	.11	−.01	.15
	JS	−.01	.14	.00	.24	.03	.40	−.01	.09	.00	.12	.01	.18

* Values that varied widely between different sets of 1,000 simulations and consequently could not be estimated accurately.
NOTE: GEV—Generalized extreme value; SD—Standard deviation; PWM—Probability-weighted moment; ML—Maximum likelihood; JS—Jenkinson's (1969) sextiles. Tabulated values are bias and standard deviation of the ratio $\hat{x}(F)/x(F)$ rather than of the quantile estimator $\hat{x}(F)$ itself.

pothesis H_0: $k = 0$, the PWM estimator $\hat{k}$ is asymptotically distributed as $N(0, .5633/n)$, so the test may be performed by comparing the statistic $Z = \hat{k}(n/.5633)^{1/2}$ with the critical values of a standard Normal distribution. Significant positive values of Z imply rejection of H_0 in favor of the alternative $k > 0$, and significant negative values of Z imply rejection in favor of $k < 0$.

The size of the test based on Z for various sample sizes is given in Table 8. In Table 9 the power of the Z test is compared with that of the modified likelihood-ratio test, which was found by Hosking (1984) to be the best test of the hypothesis $k = 0$. Tables 8 and 9 are based on computer simulations of 50,000 samples for each value of n and k. The Z test has power almost as high as the likelihood-ratio test, and for samples of size 25 or more its distribution on H_0 is adequately approximated by the standard Normal significance levels. Since the statistic Z is very simple to compute, the Z test can be strongly recommended as a convenient and powerful indicator of the sign of the shape parameter of the GEV distribution.

7. EXAMPLE

As an example we fit extreme-value distributions to the 35 annual maximum floods of the river Nidd at Hunsingore, Yorkshire, England. The data are taken from NERC (1975b, p. 235). In Figure 5 the ordered data values $x_1 \leq \cdots \leq x_n$ are plotted against the corresponding Gumbel reduced variates $-\log(-\log F_i)$, $i = 1, \ldots, n$, where $F_i = (i - .44)/(n + .12)$ is the Gringorten (1963) plotting position for the ith smallest of n observations from a Gumbel distribution. The return period of x_i is $1 - 1/F_i$. Gumbel and GEV distributions were fitted to the data by the method of PWM, and the estimated parameters were as follows (figures in parentheses are standard errors of estimated parameters): Gumbel—$\xi = 108.6$ (8.6), $\alpha = 48.5$ (7.4); GEV—$\xi = 105.8$ (8.2), $\alpha = 42.5$ (6.7), $k = -.13(.14)$. The fitted distributions

Table 8. Empirical Significance Levels of the Statistic Z for Testing the Hypothesis H: k = 0 Against One-Sided and Two-Sided Alternatives

	Alternatives					
	$k < 0$		$k > 0$		$k \neq 0$	
Sample Size	10%*	5%*	10%*	5%*	10%*	5%*
15	10.3	4.3	7.3	3.7	8.0	3.5
25	10.4	4.6	8.4	4.3	8.9	4.1
50	10.5	4.9	8.9	4.6	9.6	4.7
100	10.4	5.1	9.4	4.9	10.0	5.1
200	10.4	5.0	9.7	5.1	10.2	5.2
500	10.5	5.3	9.6	4.9	10.2	5.1

*Nominal level.

Table 9. Powers of Two Tests of the Hypothesis k = 0 Against One-Sided and Two-Sided Alternatives (sample size 50, nominal significance level 5%)

	Applications					
	For Z Test			For Modified Likelihood-Ratio Test		
k	$k < 0$	$k > 0$	$k \neq 0$	$k < 0$	$k > 0$	$k \neq 0$
−.5	.96	—	.94	.97	—	.96
−.4	.90	—	.85	.93	—	.88
−.3	.77	—	.68	.80	—	.72
−.2	.54	—	.43	.57	—	.46
−.1	.25	—	.17	.25	—	.17
.0	.05	.05	.05	.04	.06	.05
.1	—	.18	.11	—	.25	.17
.2	—	.50	.37	—	.57	.43
.3	—	.83	.73	—	.89	.80
.4	—	.96	.93	—	.98	.96
.5	—	1.00	.99	—	1.00	1.00

are also plotted in Figure 5. The Z test of the hypothesis $k = 0$ in the GEV distribution yields a test statistic $Z = 1.00$. This value is not significant and suggests that the Nidd data may reasonably be assumed to come from a Gumbel distribution.

8. CONCLUSIONS

Estimators of parameters and quantiles of the GEV distribution have been derived using the method of probability-weighted moments. These estimators have several advantages over existing methods of estimation. They are fast and straightforward to compute and always yield feasible values for the estimated parameters. The biases of the estimators are small, except when estimating quantiles in the extreme tails of the GEV distribution, and they decrease rapidly as the sample size increases. The standard deviations of the PWM estimators are comparable with those of the maximum-likelihood estimators for moderate samples sizes ($n = 50$, 100) and are often substantially less than those of the maximum-likelihood estimators for small samples

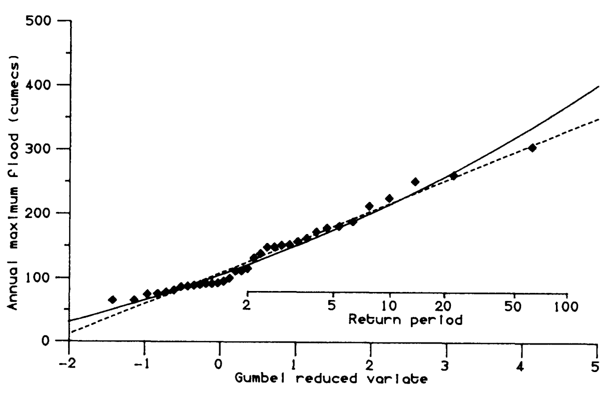

Figure 5. Distributions Fitted by the Method of PWM to 35 Annual Maximum Floods of the River Nidd: ——— GEV distribution; ----- Gumbel distribution.

($n = 15$, 25). PWM estimators of GEV parameters and quantiles have asymptotic Normal distributions, and the large-sample approximation to the variance of the estimators is adequate for sample sizes of 50 or more. Although PWM estimators are asymptotically inefficient compared to maximum-likelihood estimators, no inefficiency is detectable in samples of size 100 or less. The PWM estimator of the shape parameter k of the GEV distribution may be used as the basis of a test of the hypothesis H_0: $k = 0$, and this test is simple to perform, powerful, and accurate.

ACKNOWLEDGMENTS

This cooperative work was supported by the Institute of Hydrology, Wallingford, England. We are greatly indebted to J. S. G. McCulloch (Director of the Institute of Hydrology), P. E. O'Connell, and R. T. Clarke for initiating and sustaining this research effort. Part of J. R. M. Hosking's research was carried out at the Mathematics Research Center, University of Wisconsin–Madison, with the support of U.S. Army Contract DAAG29-80-C-0041.

APPENDIX A: PROBABILITY-WEIGHTED MOMENTS FOR THE GEV DISTRIBUTION

For the GEV distribution we have from (2) and (3)

$$\beta_r = M_{1,r,0}$$
$$= \int_0^1 [\xi + \alpha\{1 - (-\log F)^k\}/k] F^r \, dF$$
$$= \int_0^\infty \{\xi + \alpha(1 - u^k)/k\} e^{-(r+1)u} \, du,$$

substituting $u = -\log F$,

$$= (\xi + \alpha/k) \int_0^\infty e^{-(r+1)u} \, du - (\alpha/k) \int_0^\infty u^k e^{-(r+1)u} \, du$$
$$= (\xi + \alpha/k)(r + 1)^{-1} - (\alpha/k)(r + 1)^{-1-k}\Gamma(1 + k)$$

provided that $k > -1$,

$$= (r + 1)^{-1}[\xi + \alpha\{1 - (r + 1)^{-k}\Gamma(1 + k)\}/k]. \quad \text{(A.1)}$$

APPENDIX B: FEASIBILITY OF PWM ESTIMATES OF THE GEV PARAMETERS

The PWM estimator $\hat{k}$ satisfies (13), and therefore $\hat{k} > -1$ provided that

$$(2b_1 - b_0)/(3b_2 - b_0) > \tfrac{1}{2}. \quad \text{(B.1)}$$

Now

$$2b_1 - b_0 = \frac{1}{n(n-1)} \sum_{i>j} (x_i - x_j) \quad \text{(B.2)}$$

and

$$3b_2 - b_0 = \frac{2}{n(n-1)(n-2)} \sum_{i>j>k} (2x_i - x_j - x_k) \quad \text{(B.3)}$$

are both positive, so (B.1) reduces to $b_0 - 4b_1 + 3b_2 < 0$. But we can write

$$b_0 - 4b_1 + 3b_2 = \frac{2}{n(n-1)(n-2)} \sum_{i>j>k} (-x_j + x_k); \quad \text{(B.4)}$$

thus $b_0 - 4b_1 + 3b_2 < 0$ almost surely and, therefore, $\hat{k} > -1$ almost surely. Results (B.2)–(B.4) are easily proved by induction on the sample size n.

Furthermore, since

$$\hat{\alpha} = (2b_1 - b_0)\hat{k}/\{\Gamma(1 + \hat{k})(1 - 2^{-\hat{k}})\} \quad \text{(B.5)}$$

and $2b_1 - b_0 > 0$ as noted before, $k/(1 - 2^{-k}) > 0$ for all k, and $\Gamma(1 + \hat{k}) > 0$ because $\hat{k} > -1$, it follows that we must have $\hat{\alpha} > 0$.

APPENDIX C: ASYMPTOTIC DISTRIBUTION OF THE b_r

The statistic b_r may be written as a linear combination of the order statistics of a random sample: We have

$$b_r = n^{-1} \sum_{j=1}^{n} c_{nj}^{(r)} x_j, \quad \text{(C.1)}$$

where $c_{nj}^{(r)} = (j - 1) \cdots (j - r)/\{(n - 1) \cdots (n - r)\}$ and $x_1 \le x_2 \le \cdots \le x_n$ is the ordered sample. As $n \to \infty$ and $j \to \infty$ with $j/n \to \theta$ $(0 < \theta < 1)$, $c_{nj}^{(r)}$ is asymptotically a function of the plotting position $j/(n + 1)$: in fact, $c_{nj}^{(r)} \sim \{j/(n + 1)\}^r$. It is straightforward to verify that b_r satisfies the conditions of Theorem 1 of Chernoff et al. (1967), and from that theorem it follows that b_r is asymptotically Normally distributed with mean β_r and variance

$$n^{-1}v_{rr} = 2n^{-1} \iint_{x<y} \{F(x)\}^r\{F(y)\}^r \cdot F(x) \times \{1 - F(y)\} \, dx \, dy. \quad \text{(C.2)}$$

A similar argument applies to any linear combination of the b_r ($r = 0, 1, 2, \ldots$), and it follows that the b_r are asymptotically jointly Normally distributed with covariance given by

$$v_{rs} = \lim_{n\to\infty} n \operatorname{cov}(b_r, b_s) = \tfrac{1}{2}(g_{rs} + g_{sr}), \quad \text{(C.3)}$$

where

$$g_{rs} = 2 \iint_{x<y} \{F(x)\}^r\{F(y)\}^s \cdot F(x)\{1 - F(y)\} \, dx \, dy. \quad \text{(C.4)}$$

To evaluate the g_{rs} for the GEV distribution we consider first the case $k > 0$ and let

$$I_{rs} = 2 \iint_{x<y} \{F(x)\}^r\{F(y)\}^s \, dx \, dy \quad \text{(C.5)}$$

so that

$$g_{rs} = I_{r+1,s} - I_{r+1,s+1}. \tag{C.6}$$

Substituting (7) in (C.5) and making the further substitution $u = \{1 - k(x - \xi)/\alpha\}^{1/k}$, $v = \{1 - k(y - \xi)/\alpha\}^{1/k}$, we have

$$\begin{aligned} I_{rs} &= 2\alpha^2 \int_0^\infty \int_v^\infty u^{k-1}e^{-ru}\,du \cdot v^{k-1}e^{-sv}\,dv \\ &= 2\alpha^2 r^{-k} \int_0^\infty v^{k-1}e^{-sv}\Gamma(k, rv)\,dv \\ &= \alpha^2 k^{-2}(r+s)^{-2k}\Gamma(1+2k) \\ &\quad \times {}_2F_1[1, 2k; 1+k; s/(r+s)] \end{aligned} \tag{C.7}$$

(Gradshteyn and Ryzhik 1980, pp. 317 and 663); here $\Gamma(\cdot, \cdot)$ is the incomplete gamma function and ${}_2F_1$ is the hypergeometric function. It is convenient to transform the hypergeometric function in (C.7), using results from Gradshteyn and Ryzhik (1980, p. 1043):

$$\begin{aligned} &{}_2F_1[1, 2k; 1+k; s/(r+s)] \\ &\quad = 2^{2k}\{\Gamma(1+k)\}^2/\Gamma(1+2k) \quad \text{if } r = s, \\ &\quad = \{r/(r+s)\}^{-2k}G(s/r) \quad \text{if } r > s, \\ &\quad = -\{s/(r+s)\}^{-2k}G(r/s) + 2r^{-k}s^{-k}(r+s)^{2k} \\ &\qquad \times \{\Gamma(1+k)\}^2/\Gamma(1+2k) \quad \text{if } r < s, \end{aligned} \tag{C.8}$$

where G denotes the hypergeometric function $G(x) = {}_2F_1(k, 2k; 1+k; -x)$; note that $G(0) = 1$. Substituting back into (C.6) and (C.3) we obtain the following expressions for the v_{rs}:

$$\begin{aligned} v_{rr} = \alpha^2 k^{-2}(r+1)^{-2k}[\Gamma(1+2k)G\{r/(r+1)\} \\ - \{\Gamma(1+k)\}^2], \end{aligned} \tag{C.9}$$

$$\begin{aligned} v_{r,r+1} &= \tfrac{1}{2}\alpha^2 k^{-2}[(r+2)^{-2k}\Gamma(1+2k)G\{r/(r+2)\} \\ &\quad + (r+1)^{-k}\{(r+1)^{-k} - 2\cdot(r+2)^{-k}\}\{\Gamma(1+k)\}^2], \end{aligned} \tag{C.10}$$

$$\begin{aligned} v_{r,r+s} &= \tfrac{1}{2}\alpha^2 k^{-2}[(r+s+1)^{-2k}\Gamma(1+2k)G\{r/(r+s+1)\} \\ &\quad - (r+s)^{-k}\Gamma(1+2k)G\{(r+1)/(r+s)\} + 2(r+1)^{-k} \\ &\quad \times \{(r+s)^{-k} - (r+s+1)^{-k}\}\{\Gamma(1+k)\}^2], \qquad s \ge 2. \end{aligned} \tag{C.11}$$

When $k \le 0$ the foregoing argument is not valid because the integral in (C.5) does not converge. However, the expressions (C.9)–(C.11) are analytic functions of k for all $k > -\frac{1}{2}$; hence by analytic continuation expressions (C.9)–(C.11) are valid solutions of the integral representation (C.3)–(C.4) throughout the domain $-\frac{1}{2} < k < \infty$. At the value $k = 0$, the v_{rs} are given by the limits of (C.9)–(C.11) as $k \to 0$; these limits are well defined.

The results stated in this Appendix are valid for arbitrary positive integers r and s, though only the cases r, $s = 0, 1, 2$ are required for deriving the asymptotic distributions of PWM estimators.

[*Received June 1984. Revised December 1984.*]

REFERENCES

Cheng, R. C. H., and Amin, N. A. K. (1983), "Estimating Parameters in Continuous Univariate Distributions With a Shifted Origin," *Journal of the Royal Statistical Society*, Ser. B, 45, 394–403.

Chernoff, H., Gastwirth, J. L., and Johns, M. V. (1967), "Asymptotic Distribution of Linear Combinations of Functions of Order Statistics With Applications to Estimation," *Annals of Mathematical Statistics*, 38, 52–72.

Downton, F. (1966a), "Linear Estimates With Polynomial Coefficients," *Biometrika*, 53, 129–141.

——— (1966b), "Linear Estimates of Parameters in the Extreme Value Distribution," *Technometrics*, 8, 3–17.

Fisher, R. A., and Tippett, L. H. C. (1928), "Limiting Forms of the Frequency Distribution of the Largest or Smallest Member of a Sample," *Proceedings of the Cambridge Philosophical Society*, 24, 180–190.

Fraser, D. A. S. (1957), *Nonparameteric Methods of Statistics*, New York: John Wiley.

Gradshteyn, I. S., and Ryzhik, I. W. (1980), *Table of Integrals, Series and Products* (4th ed.), New York: Academic Press.

Greenwood, J. A., Landwehr, J. M., Matalas, N. C., and Wallis, J. R. (1979), "Probability Weighted Moments: Definition and Relation to Parameters of Several Distributions Expressable in Inverse Form," *Water Resources Research*, 15, 1049–1054.

Griffiths, D. A. (1980), "Interval Estimation for the Three-Parameter Lognormal Distribution via the Likelihood Function," *Applied Statistics*, 29, 58–68.

Gringorten, I. I. (1963), "A Plotting Rule for Extreme Probability Paper," *Journal of Geophysical Research*, 68, 813–814.

Hoeffding, W. (1948), "A Class of Statistics With Asymptotically Normal Distribution," *Annals of Mathematical Statistics*, 19, 293–325.

Hosking, J. R. M. (1984), "Testing Whether the Shape Parameter Is Zero in the Generalized Extreme-Value Distribution," *Biometrika*, 71, 367–374.

Hosking, J. R. M., Wallis, J. R., and Wood, E. F. (1985), "An Appraisal of the Regional Flood Frequency Procedure in the UK Flood Studies Report," *Hydrological Sciences Journal*, 30, 85–109.

Jenkinson, A. F. (1955), "The Frequency Distribution of the Annual Maximum (or Minimum) of Meteorological Elements," *Quarterly Journal of the Royal Meteorological Society*, 81, 158–171.

——— (1969), "Statistics of Extremes," Technical Note 98, World Meteorological Office, Geneva.

Landwehr, J. M., Matalas, N. C., and Wallis, J. R. (1979), "Probability Weighted Moments Compared With Some Traditional Techniques in Estimating Gumbel Parameters and Quantiles," *Water Resources Research*, 15, 1055–1064.

Locke, C., and Spurrier, J. D. (1976), "The Use of *U*-Statistics for Testing Normality Against Non-Symmetric Alternatives," *Biometrika*, 63, 143–147.

Natural Environment Research Council (1975a), *Flood Studies Report* (Vol. 1), London: Author.

——— (1975b), *Flood Studies Report* (Vol. 4), London: Author.

Prescott, P., and Walden, A. T. (1980), "Maximum-Likelihood Estimation of the Parameters of the Generalized Extreme-Value Distribution," *Biometrika*, 67, 723–724.

——— (1983), "Maximum-Likelihood Estimation of the Parameters of the Three-Parameter Generalized Extreme-Value Distribution From Censored Samples," *Journal of Statistical Computing and Simulation*, 16, 241–250.

Randles, R. H., and Wolfe, D. A. (1979), *Introduction to the Theory of Nonparametric Statistics*, New York: John Wiley.

Rao, C. R. (1973), *Linear Statistical Inference and Its Applications* (2nd ed.), New York: John Wiley.

Shenton, L. R., and Bowman, K. O. (1977), *Maximum-Likelihood Estimation in Small Samples*, London: Charles W. Griffin.

Von Andrae (1872), "Uber die Bestimmung das Wahrscheinlichen Fehlers Durch die Gegebenen Differenzen von ein Gleich Genauen Beobachtungen Einer Unbekannten," *Astron. Nach.*, 79, 257–272.