BIVARIATE EXPONENTIAL DISTRIBUTIONS

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A bivariate distribution is not determined by the knowledge of the margins. Two bivariate distributions with exponential margins are analyzed and another is briefly mentioned.

In the first distribution (2.1) the conditional expectation of one variable decreases to zero with increasing values of the other one. The coefficient of correlation is never positive and lies in the interval $-.40 \leq \rho \leq 0$, and the correlation ratio varies from $-.48$ to zero.

In the second distribution (3.4) the conditional expectation of one variable increases or decreases with increasing values of the other variable depending on the sign of the correlation. The coefficient of correlation lies in the interval $-.25 \leq \rho \leq .25$, and the correlation ratio is proportional to the coefficient.

THE exponential distribution which is analytically very simple plays a prominent role in physics since it governs radioactive decay [7]. It holds for distances in time, especially between the happening of rare events [1]. In recent years it has served as a first approach to a model for life testing. Finally it can be used as the starting point for the theory of extreme values [5]. It may therefore be of interest to study bivariate distributions where the marginal distributions are exponential.

Most thinking on bivariate distributions is centered about the normal case, which has been studied intensively since the times of Bravais and Karl Pearson. It was believed that its well-known properties may serve as a general model L. The curves of equal probability density are concentric ellipses. The regression curves are straight lines which intersect at the expectations. With increasing values of one variable the conditional expectation of the other one increases (or decreases) without limit. Finally, the coefficient of correlation varies from -1 to $+1$. However, in a previous publication [4] bivariate distributions with marginal normal distributions were constructed where these properties do not hold.

The marginal distributions do not determine the corresponding bivariate distribution. On the contrary, M. Fréchet [2] has proven that for given marginal distributions there exist infinitely many bivariate distributions with these margins. Fréchet's result concerns the existence of these bivariate distributions and does not involve their construction; thus it is of interest to examine specific analytical forms of bivariate distributions.

In the following, two bivariate distributions with exponential margins will be studied. Again, none of the properties of the normal distributions are valid: the curves of equal probability density are not ellipses, the regression curves are not straight lines and do not intersect at the common mean. With increasing values of one variable the conditional expectation of the other remains within finite limits. Finally, the coefficient of correlation varies in a narrower domain.

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1. BIVARIATE DISTRIBUTIONS

Let $F_1(x)$ and $F_2(y)$, $f_1(x)$ and $f_2(y)$ be the probability and density functions of continuous random variables X and Y. Then a bivariate probability function $F(x, y)$ with these marginal distributions is monotonically increasing from zero to unity and is subject to the following conditions:

a) $F(-\infty, y) = F(x, -\infty) = 0$

$$F(x, \infty) = F_1(x); \qquad F(\infty, y) = F_2(y); \qquad F(\infty, \infty) = 1 \tag{1.1}$$

b) The probability content of every rectangle is nonnegative, that is, for every $x_1 < x_2$, $y_1 < y_2$

$$\begin{aligned} &\text{Prob}\{x_1 < X \leq x_2, y_1 < Y \leq y_2\} \\ &\qquad = F(x_2, y_2) - F(x_2, y_1) - F(x_1, y_2) + F(x_1, y_1) \geq 0. \end{aligned} \tag{1.2}$$

If the second cross partial derivative $\partial^2F/\partial x\partial y$ exists everywhere, the bivariate distribution has a density $f(x, y)$ equal to this derivative and the condition (1.2) is then equivalent to

$$\frac{\partial^2 F}{\partial x \partial y} = f(x, y) \geq 0. \tag{1.3}$$

The variables are independent if and only if

$$F(x, y) = F_1(x)F_2(y). \tag{1.4}$$

More generally, the marginal density functions $f_1(x)$ and $f_2(x)$ are related to the bivariate density function $f(x, y)$ by

$$\int_{-\infty}^{\infty} f(x, y)dy = f_1(x); \qquad \int_{-\infty}^{\infty} f(x, y)dx = f_2(y). \tag{1.5}$$

The study of the conditional densities

$$f(x \mid y) = f(x, y)/f_2(y); \qquad f(y \mid x) = f(x, y)/f(x) \tag{1.6}$$

leads to the conditional expectations $E(x \mid y)$ and $E(y \mid x)$ and to the expectation of the cross product

$$E(xy) = \int_{-\infty}^{\infty} yE(x \mid y)f_2(y)dy. \tag{1.7}$$

These values lead to the classical coefficient of correlation

$$\rho = \frac{E(xy) - E(x)E(y)}{\sigma_x \sigma_y}; \tag{1.8}$$

to the squared correlation ratio

$$\eta^2(x \mid y) = \frac{1}{\sigma_y{}^2}\int_{-\infty}^{\infty} [E(x) - E(x \mid y)]^2 f_2(y)dy; \tag{1.9}$$

and the corresponding expression $\eta^2(y \mid x)$. The correlation ratios measure the departure from the regression lines. These well known methods will now be applied to two new distributions and the results will be compared to the corresponding properties of the usual bivariate normal distribution.

2. A BIVARIATE DISTRIBUTION WITH EXPONENTIAL MARGINS

Consider the bivariate function

$$F(x, y) = 1 - e^{-x} - e^{-y} + e^{-x-y-\delta xy}; \qquad x \geq 0, y \geq 0. \tag{2.1}$$

The marginal probabilities are exponential. The case $\delta=0$ leads from (1.4) to independence. The boundary values of (2.1) are

$$F(x, 0) = F(0, y) = F(0, 0) = 0; \qquad F(\infty, \infty) = 1.$$

We are going to prove now that the parameter δ must satisfy the inequalities

$$0 \leq \delta \leq 1. \tag{2.2}$$

From (1.2), the density function $f(x, y)$ is

$$f(x, y) = e^{-x(1+\delta y-y}[(1 + \delta x)(1 + \delta y) - \delta] \tag{2.3}$$

with

$$f(\infty, y) = f(x, \infty) = 0; \qquad f(0, 0) = 1 - \delta.$$

From the last equation and from the nonnegativity of a density function, it follows that $\delta \leq 1$.

From the inequality which is true for all bivariate distributions

$$F(x, y) \leq F_1(x)$$

and from (2.1) it follows after a short simplification that

$$-x(1 + \delta y) \leq 0 \tag{2.4}$$

for all x and y. Since these are always nonnegative it follows that δ cannot be negative. Therefore $\delta \geq 0$.

With the restriction (2.3) the conditions (1.1) and (1.3) are fulfilled. The relation (1.5) is immediately verified. Therefore (2.1) constitutes a bivariate distribution with exponential margins.

Evidently the curves of equal probability density are not ellipses but transcendental functions. Since the probability function (2.1) depends in the same way on x and y it is sufficient to analyze one variable, say x.

The conditional density function $f(x \mid y)$ is from (1.6) and (2.3)

$$f(x \mid y) = e^{-x(1+\delta y)}[(1 + \delta x)(1 + \delta y) - \delta]. \tag{2.5}$$

The boundary density function $f(x \mid 0)$ which is one of the conditional density functions and should not be confused with the marginal density is

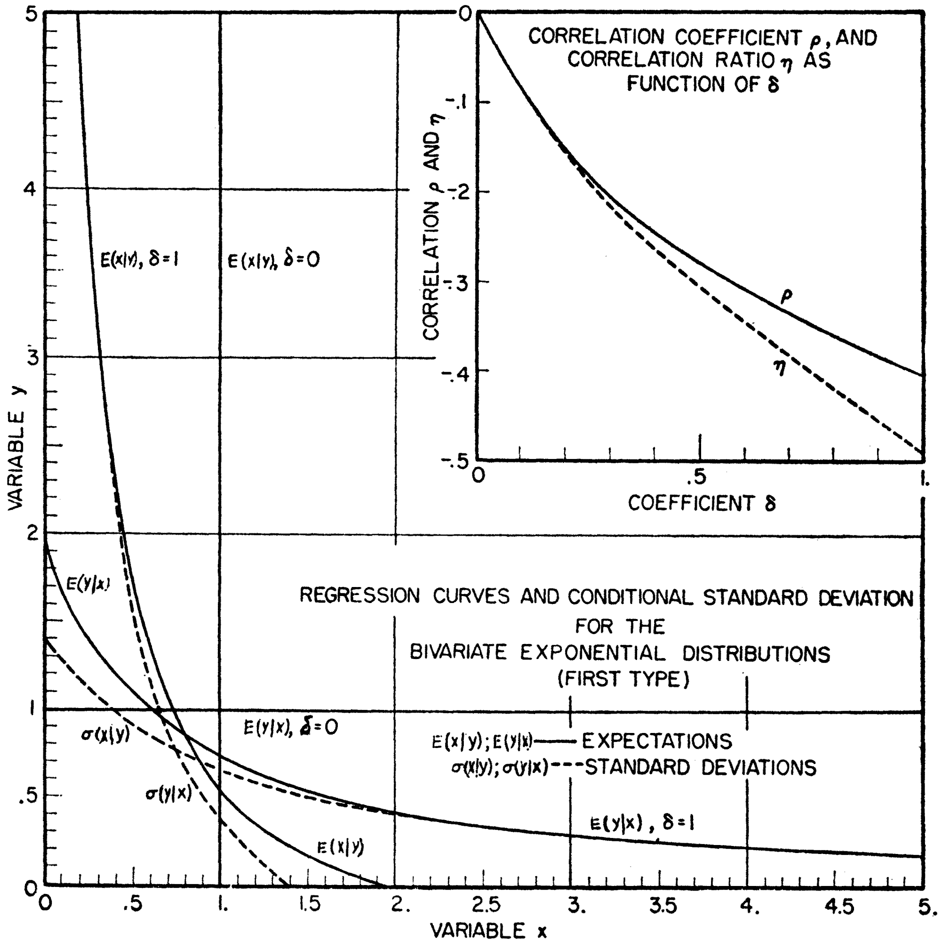
$$f(x \mid 0) = e^{-x}(1 - \delta + \delta x).$$

It has a mode at $\breve{x}=2-1/\delta$, provided that $\delta > \frac{1}{2}$. Otherwise the density decreases with x.

The conditional expectation is

$$E(x \mid y) = \frac{1 + \delta + \delta y}{(1 + \delta y)^2}\,. \tag{2.6}$$

This regression curve is traced in Graph 1 for the two limiting cases $\delta=0$ and $\delta=1$. The conditional expectation (2.6) diminishes for increasing values of y from $1+\delta$ valid for $y=0$ to zero, while the variable y is unlimited to the right. The two regression curves do not intersect at the common mean $E(x)=E(y)=1$, except in the trivial case $\delta=0$.



GRAPH 1

The conditional second moment is, from (2.5)

$$E(x^2 \mid y) = \frac{2}{(1+\delta y)^2} + \frac{4\delta}{(1+\delta y)^3}\,.$$

Subtraction of $E^2(x|y)$ leads from (2.6) to the conditional variance $\sigma^2(x|y)$ of x as a function of y.

$$\sigma^2(x \mid y) = \frac{1}{(1+\delta y)^2} + \frac{2\delta}{(1+\delta y)^3} - \frac{\delta^2}{(1+\delta y)^4} \tag{2.7}$$

which, of course, is equal to unity for $\delta=0$, i.e., the case of independence. In the opposite case $\delta=1$, the expression becomes

$$\sigma^2(x \mid y) = \frac{2 + 4y + y^2}{(1 + y)^4} \cdot \tag{2.8}$$

The conditional standard deviation of x as a function of y,

$$\sigma(x \mid y) = [(1 + y)^2 + 1 + 2y)^{1/2}(1 + y)^{-2} \tag{2.9}$$

decreases for $\delta=1$ with increasing values of y (see Graph 1). The squared conditional coefficient of variation obtained from (2.6) and (2.7),

$$\frac{\sigma^2(x \mid y)}{E^2(x \mid y)} = \frac{(1 + \delta y)^2 + 2\delta(1 + \delta y) - \delta^2}{(1 + \delta y)^2 + 2\delta(1 + \delta y) + \delta^2}$$

converges, for increasing values of y to unity, which is also its value for the marginal distributions. In this sense

$$\sigma(x \mid y) \sim E(x \mid y) \tag{2.10}$$

independent of the value of δ. The cross product is from (2.6) and (1.7)

$$E(xy) = \int_{-\infty}^{\infty} y \frac{1 + \delta + \delta y}{(1 + \delta y)^2} e^{-y} dy$$

$$= \frac{1}{\delta} \int_0^{\infty} \frac{\delta y e^{-y} dy}{1 + \delta y} + \int_0^{\infty} \frac{(1 + \delta y - 1)}{(1 + \delta y)^2} e^{-y} dy.$$

The integrals may be written, after the appropriate additions and subtractions

$$\frac{1}{\delta} + \left(1 - \frac{1}{\delta}\right) \int_0^{\infty} \frac{e^{-y} dy}{1 + \delta y} + \frac{1}{\delta} \int_0^{\infty} e^{-y} d\left(\frac{1}{1 + \delta y}\right).$$

Partial integration of the last integral leads after the transformation

$$1/\delta + y = z$$

to

$$E(xy) = -\frac{e^{1/\delta}}{\delta} Ei(-\delta^{-1}) \tag{2.11}$$

where Ei stands for the integral logarithm. Since the means and standard deviations of the marginal distributions are equal to unity, the coefficient of correlation ρ is a function of the parameter δ namely

$$\rho = -\frac{e^{1/\delta}}{\delta} Ei(-\delta^{-1}) - 1. \tag{2.12}$$

From (2.11) the coefficient of correlation is zero for $\delta=0$. It decreases for increasing values of δ up to $\rho=-.40365$, valid for $\delta=1$ as shown in the upper part of Graph 1. The correlation is never positive.

Since the regression curves (2.6) are not straight lines, the correlation ratio does not measure the departure from the regression lines. In the present case its value is

$$\eta^2(x \mid y) = \int_0^\infty [1 - E(x \mid y)]^2 e^{-y} dy.$$

From (2.6),

$$\eta^2(x \mid y) = 1 - 2J(1) + (1 - 2\delta)J(2) + 2\delta J(3) + \delta^2 J(4) \tag{2.13}$$

where

$$J(k) = \int_0^\infty \frac{e^{-y} dy}{(1 + \delta y)^k} \cdot \tag{2.14}$$

For decreasing values of k the integrals are linked by the recurrence formula

$$J(k) = \frac{1}{(k-1)\delta} - \frac{J(k-1)}{(k-1)\delta} \cdot$$

The introduction of the values for $k=2$, 3, 4, into (2.13) yields

$$\eta^2(x \mid y) = -\frac{1}{6} - \frac{J(1)}{6\delta} + \frac{1}{6\delta} + \frac{\delta}{3} \cdot$$

On the other hand, the combination of (2.11), (2.12) and (2.13) leads to

$$\rho = J(1) - 1$$

whence

$$\eta^2(x \mid y) = \frac{\delta}{3} - \frac{1}{6} - \frac{\rho}{6\delta} \cdot \tag{2.15}$$

For $\delta = 1$ we obtain

$$\eta^2(x \mid y) = .23394.$$

For $\delta = 0$, the last factor in (2.15) becomes indeterminate. To obtain its value we expand

$$\rho = -1 + \int_0^\infty \frac{e^{-y} dy}{1 + \delta y}$$

in increasing powers of δy. This yields

$$\rho = \sum_{v=1}^{\infty} (-1)^v \delta^v v!$$

whence, for $\delta = 0$

$$\frac{\rho}{\delta} = 1.$$

From (2.15) it follows that $\eta^2(x \mid y) = 0$ for $\delta = 0$, as it should be. For increasing values of δ the correlation ratio $-\sqrt{\eta^2(x \mid y)}$ decreases from zero to $-.4837$, as shown in the upper part of Graph 1.

3. A SECOND BIVARIATE DISTRIBUTION WITH EXPONENTIAL MARGINS

In previous articles [4, 6], it was shown that given two probability functions $F_1(x)$ and $F_2(y)$ a bivariate function $F(x, y)$ can be constructed, by means of the equation

$$F(x, y) = F_1(x)F_2(y)[1 + \alpha\{1 - F_1(x)\}\{1 - F_2(y)\}] \tag{3.1}$$

where

$$-1 \leq \alpha \leq 1. \tag{3.2}$$

The bivariate density function is given by

$$f(x, y) = f_1(x)f_2(y[1 + \alpha(2F_1(x) - 1)(2F_2(y) - 1)]. \tag{3.3}$$

The fact that conditions (1.1) and (1.3) hold, was shown in the publications mentioned.

Let $F_1(x)$ and $F_2(y)$ be exponential functions. Then the bivariate probability and density functions become

$$\begin{aligned} F(x, y) &= (1 - e^{-x})(1 - e^{-y})[1 + \alpha e^{-x-y}]; \qquad x \geq 0; y \geq 0 \\ f(x, y) &= e^{-x-y}[1 + \alpha(2e^{-x} - 1)(2e^{-y} - 1)]. \end{aligned} \tag{3.4}$$

Since these functions depend in the same way on x and y it is sufficient to study one variable, say x. The marginal density function obtained from (1.5) is, of course, the exponential function itself.

The conditional density function $f(x|y)$ is, from (1.6) and (3.4)

$$f(x \mid y) = e^{-x}(1 + \alpha - 2\alpha e^{-y}) - 2\alpha e^{-2x}(1 - 2e^{-y}). \tag{3.5}$$

It follows that the boundary density function

$$f(x \mid 0) = e^{-x}(1 - \alpha) + 2\alpha e^{-2x}$$

diminishes with x if $\alpha > 0$. For $\alpha < 0$ it has a mode at

$$\tilde{x} = lg\left[\frac{-4\alpha}{1 - \alpha}\right].$$

The regression curve of x on y is obtained from (3.5) as

$$E(x \mid y) = E_\alpha(x \mid y) = 1 + -\frac{\alpha}{2}\,\alpha e^{-y}. \tag{3.6}$$

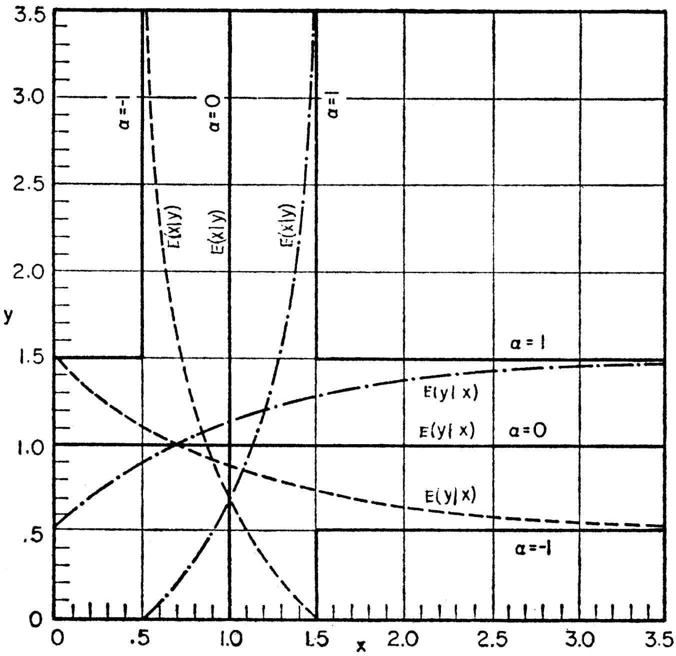
The conditional expectation increases for $\alpha > 0$ (decreases for $\alpha < 0$) with increasing values of y from $1 - \alpha/2$ to $1 + \alpha/2$. Thus it remains finite. The regression curves are not linear but exponential functions traced in Graph 2.

For two values of α which differ only in sign the conditional expectations are related by

$$\tfrac{1}{2}[E_{-\alpha}(x \mid y) + E_\alpha(x \mid y)] = 1. \tag{3.7}$$

This constitutes a symmetry of the conditional expectation about unity.

The conditional expectation $E(x^2|y)$ is easily obtained from (3.5) as



GRAPH 2. Regression curves for bivariate exponential distributions (second type).

$$E(x^2 \mid y) = 2 + \tfrac{3}{2}\alpha - 3\alpha e^{-y}.$$

In analogy to (3.7) we have

$$\tfrac{1}{2}[E_{-\alpha}(x^2 \mid y) + E_{\alpha}(x^2 \mid y)] = 2.$$

The conditional variance $\sigma^2(x|y)$ of x as a function of y is from (3.6)

$$\sigma^2(x \mid y) = 1 + \frac{\alpha}{2}(1 - 2e^{-y}) - \frac{\alpha^2}{4}(1 - 2e^{-y})^2. \tag{3.8}$$

The conditional expectations, the squared expectations, and the variances are equal to the unconditional values 1, 2 and 1 at the medians $\check{y} = \log 2$.

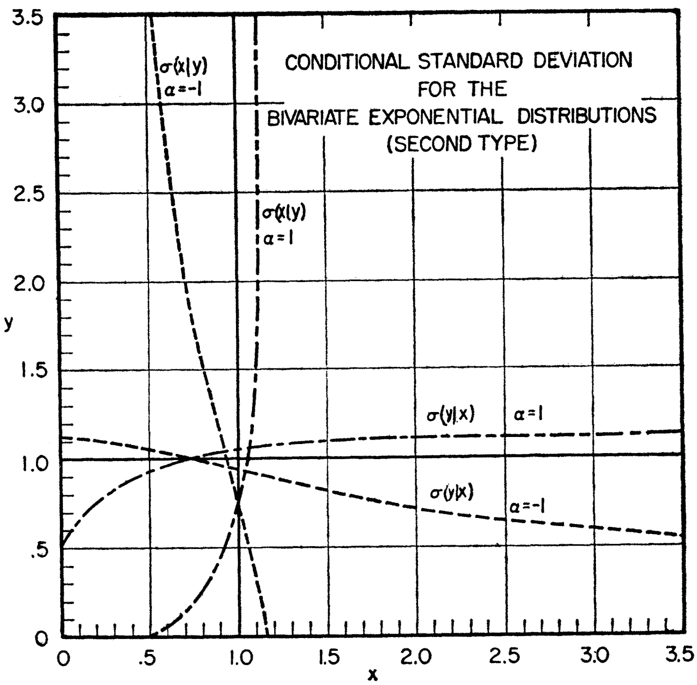
For positive (negative) values of α the variance increases (decreases) with increasing values of y from

$$1 - \frac{\alpha^2}{4} - \frac{\alpha}{2}$$

up to

$$1 - \frac{\alpha^2}{4} + \frac{\alpha}{2}\,.$$

In particular, for $\alpha = 1$, it increases from 1/4 to 5/4 and decreases for $\alpha = -1$ from 5/4 to 1/4. Consequently the standard deviations vary between .5 and 1.118 as shown in Graph 3. The relation (2.10) holds for this system only in the case $\alpha = -1$. Then the expression



GRAPH 3

$$\frac{\sigma^2(x \mid y)}{E^2(x \mid y)} = \frac{\frac{1}{4} + 2e^{-y} - e^{-2y}}{\frac{1}{4}(1 - 2e^{-y} + e^{-2y})} \tag{3.9}$$

converges, with increasing y, towards unity, while for $\alpha = 1$, the expression converges to 5/9.

The expectation of the cross product becomes from (1.7) and (3.6)

$$E(xy) = 1 + \frac{\alpha}{4}.$$

Therefore the coefficient of correlation is

$$\rho = \frac{\alpha}{4}. \tag{3.10}$$

From (3.2) the correlation varies only within the narrow domain

$$-.25 \leq \rho \leq .25. \tag{3.11}$$

In contrast to the previous case it can also be negative. The correlation ratio $\eta(x \mid y)$ becomes, from (3.6)

$$\eta(x \mid y) = \frac{\alpha}{2\sqrt{3}}. \tag{3.12}$$

Thus the corresponding ratio

$$\eta(x \mid y) = 2\rho/\sqrt{3} \tag{3.13}$$

is a multiple of the coefficient of correlation, and varies in the interval $\pm.28867$.

A third bivariate distribution with exponential margins is given by

$$F(x, y) = 1 - e^{-x} - e^{-y} + P(x, y) \tag{3.14}$$

where

$$P(x, y) = \exp\left[-(x^m + y^m)^{1/m}\right]. \tag{3.15}$$

The case $m=1$ leads to independence. The density function

$$f(x, y) = P(x, y)(x^m + y^m)^{1/m-2}x^{m-1}y^{m-1}\left[(x^m + y^m)^{1/m} + m - 1\right] \tag{3.16}$$

is nonnegative only if $m \geq 1$.

CONCLUSION

The fact that many of the properties of bivariate normal distribution do not hold here may serve as a warning against the indiscriminate use of normal correlation and regression analysis; prior investigation of the nature of the bivariate distributions is necessary.

If we know that the marginal distributions are exponential, then the different behavior of the conditional distributions (2.5) or (3.5) and in particular, of the boundary distributions, of the regression curves (2.6) or (3.6), of the conditional standard deviations (2.7) or (3.8) and the different domains of the coefficient of correlation and the correlation ratio may be used as criteria for the acceptance of one of the two systems for a given set of observations. However, it has to be realized that other bivariate distributions with exponential margins exist which are outside of the systems considered here.

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