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ORIGINAL PAPER

An extension of the Gumbel–Barnett family of copulas

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Abstract
The Gumbel–Barnett family of bivariate distributions with given marginals, is frequently used in theory and applications. This family has been generalized in several ways. We propose and study a broad generalization by using two differentiable functions. We obtain some properties and describe particular cases.

Keywords Gumbel–Barnett copula · Bivariate copulas · Stochastic dependence · Copula visualization

Mathematics Subject Classification 62H20 (primary) · 60E05 (secondary)

1 Introduction

The construction of copulas and dependence models have interest in statistics, probability, econometrics, informatics, insurance, finance, physics, hydrology, etc. A copula function is a bivariate cdf with uniform (0, 1) marginals that captures the dependence properties of two r.v.'s defined on the same probability space. Many copulas and bivariate families of distributions have been studied in Hutchinson and Lai (1991), Joe (1997), Nelsen (2006), Cuadras (2006) and Durante and Sempi (2016).

Of notable interest is the Gumbel–Barnett family of copulas (see Hutchinson and Lai 1991; Nelsen 2006):

$$C(u, v) = uv \exp(-\theta \ln u \ln v), \quad 0 \leq \theta \leq 1. \tag{1}$$

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This is an example of an Archimedean copula useful in reliability. This family only covers negative dependence and can be extended.

An extension showing a wider range of dependence, was proposed by Celebioglu (1997) and Cuadras (2009):

$$C(u, v) = uv \exp[\theta(1-u)(1-v)], \quad -1 \leq \theta \leq 1. \tag{2}$$

The aim of this article is to study the following extension

$$C_\theta(u, v) = uv \exp[\theta A(u)B(v)], \quad \theta_1 \leq \theta \leq \theta_2, \tag{3}$$

where A, B are differentiable continuous functions such that $A(1) = B(1) = 0$.

This generalized Gumbel–Barnett family of copulas (3), was proposed by Bekrizadeh et al. (2017), as the limit of the family $uv[1 - \frac{\theta}{p}A(u)B(v)]^p$ as $p \to \infty$. However, they illustrate it with examples, but do not study this limit family in some detail, a task that is addressed here.

This family can also be considered as a particular case of the general family studied by Cuadras (2015)

$$C_\theta(u, v) = uvH_\theta[Q(u, v)], \quad \theta \in \mathbf{D},$$

where H_θ is a continuous real function such that $H_\theta(1) = 1$ and Q satisfies $Q(u, 1) = Q(1, v) = 1$. To see this assertion, take $H_\theta(x) = \exp[\theta(1-x)]$ and $Q(u, v) = 1 - A(u)B(v)$.

This article is organized as follows. In Sect. 2 we study the range of the parameter for the family (3). In Sect. 3 we propose and study three sub-families. Section 4 is devoted to the study of the geometric mean of two members of this generalized Gumbel–Barnett family. In Sect. 5 we study Taylor's expansions and approximations to other well-known families. Some concepts and measures of stochastic dependence are studied in Sect. 6. We perform in Sect. 7 approximations to copulas of rank 2 to visualize several sub-families of the family (3).

2 Parameter restrictions

In this section we study the range of the parameter θ for the family (3). Assume that the distribution is absolutely continuous. If C is a copula function, the probability density function is given by (see Cuadras 2009):

$$c(u, v) = C(u, v)\left[\frac{\partial^2 \ln C(u, v)}{\partial u \partial v} + \frac{\partial \ln C(u, v)}{\partial u}\frac{\partial \ln C(u, v)}{\partial v}\right], \tag{4}$$

where $0 \leq u, v \leq 1$.

For the family (3) we have

$$c_\theta(u, v) = C_\theta(u, v)\left\{\theta A'(u)B'(v) + \left[u^{-1} + \theta A'(u)B(v)\right]\left[v^{-1} + \theta A(u)B'(v)\right]\right\}. \tag{5}$$

Multiplying by $\exp[-\theta A(u)B(v)]$, the parameter θ should satisfy the inequality

$$\theta u A'(u) v B'(v) + \left[1 + \theta u A'(u) B(v)\right]\left[1 + \theta A(u) v B'(v)\right] \geq 0. \tag{6}$$

Keeping $u, v \in (0, 1)$ fixed, (6) is a second degree polynomial inequality in the variable θ, difficult to solve in general.

Theorem 1 *The range of the parameter θ for which the generalized Gumbel–Barnett family is a copula, is a finite closed interval $[\theta_1, \theta_2]$, containing $\theta = 0$.*

Proof It is obvious that θ_1, θ_2 must be finite and $\theta = 0$ is an admissible value. Consider the equation $c_\theta(u, v) = 0$ which is written as the second degree polynomial equation $P_2(\theta, u, v) = 0$, where $P_2(\theta, u, v)$ is the left hand side of (6).

For u, v fixed, the equation $P_2(\theta, u, v) = 0$ has two roots θ_a and θ_b, where a and b depend on u, v. Since $P_2(0, u, v) = 1$, we have $\theta_a \leq 0 \leq \theta_b$. As the intersection of closed intervals is also closed, define

$$[\theta_1, \theta_2] = \bigcap_{u,v} [\theta_a, \theta_b] \qquad (a, b \text{ depend on } u, v).$$

Since $0 \in [\theta_1, \theta_2]$, this intersection is non-empty. Thus $\theta \in [\theta_1, \theta_2]$. □

Some particular cases are next studied.

Theorem 2 *Suppose that C_θ is a copula of the Gumbel–Barnett generalized family. Then θ satisfies:*

$$\frac{-1}{\max\{M_3, M_4\}} \leq \theta \leq \frac{-1}{\min\{M_1, M_2\}}, \tag{7}$$

whenever

$$M_1 = \inf\{A'(1)[B(v) + vB'(v)]\} < 0, \quad M_2 = \inf\{B'(1)[uA'(u) + A(u)]\} < 0,$$
$$M_3 = \sup\{A'(1)[vB'(v) + B(v)]\} > 0, \quad M_4 = \sup\{B'(1)[uA'(u) + A(u)]\} > 0.$$

The parameter θ also satisfies

$$\frac{-1}{\max\{M_3', M_4'\}} \leq \theta \leq \frac{-1}{\min\{M_1', M_2'\}}, \tag{8}$$

whenever

$$M_1' = \inf\{A(0)vB'(v)\} < 0, \qquad M_2' = \inf\{B(0)uA'(u)\} < 0,$$
$$M_3' = \sup\{A(0)vB'(v)\} > 0, \qquad M_4' = \sup\{B(0)uA'(u)\} > 0.$$

Proof Taking $u = 1$ in (6), as $A(1) = 0$, this inequality reduces to

$$\theta A'(1) v B'(v) + 1 + \theta A'(1) B(v) \geq 0,$$

therefore

$$\theta A'(1) \left[v B'(v) + B(v) \right] \geq -1.$$

Suppose $\theta > 0$ and $M_1 = \inf\{A'(1) \left[v B'(v) + B(v) \right]\} < 0$. Then θ should satisfy $\theta \leq -1/M_1$. Similarly, the inequalities $\theta > 0$ and $M_2 = \inf\{B'(1) \left[u A'(u) + A(u) \right]\} < 0$ show that $\theta \leq -1/M_2$. Thus $\theta \leq -1/\min\{M_1, M_2\}$.

Suppose $\theta < 0$. Changing $B(v)$ to $-B(v)$ and θ (assumed positive) to $-\theta$, we have $(-\theta)A'(1) \left[-v B'(v) - B(v) \right] \geq -1$ and the proof that $\theta \geq -1/\max\{M_3, M_4\}$ is analogous. The second part is similarly proved by taking $u = 0$ in (6). □

Corollary 1 *Suppose that* $[\theta_1, \theta_2]$ *is the interval for* θ *such that* C_θ *in (3) is a copula. Assuming that* $M_1, M_2, M_1', M_2' < 0$, *so* $m = \min\{M_1, M_2, M_1', M_2'\} < 0$, *and* $M_3, M_4, M_3', M_4' > 0$, *so* $M = \max\{M_3, M_4, M_3', M_4'\} > 0$, *then* θ *should satisfy:*

$$-\frac{1}{M} \leq \theta \leq -\frac{1}{m}.$$

That is, the interval $[\theta_1, \theta_2]$ *is included in* $[-M^{-1}, -m^{-1}]$.

All these restrictions are necessary conditions for the interval. Under some conditions of monotonicity of $A(u)$, $uA'(u)$, and $B(v)$, $vB'(v)$, it is possible to find necessary and sufficient conditions for θ, allowing us to determine the exact interval range $[\theta_1, \theta_2]$.

In the next Theorem, we suppose monotonicity to guarante that the functions involved in (6), reach the supremum and the infimum at the extremes of the interval [0,1].

Theorem 3 *Suppose that* A *and* B *are monotonic functions, as well as* $uA'(u)$ *and* $vB'(v)$.

Assume $\theta \geq 0$. *Then:*

1. *If the infima of the products* $uA'(u)vB'(v)$, $A(u)vB'(v)$ *and* $uA'(u)B(v)$, *are reached at* $u = 1$, $v = 1$, *the parameter must satisfy:*

$$1 + \theta A'(1) B'(1) \geq 0, \tag{9}$$

where $A'(1)$, $B'(1)$ *are the left-derivatives.*

2. *If the infima of these functions are reached at* $u = 1$, $v = 0$, *the parameter must satisfy*

$$1 + \theta A'(1)[B(0) + 0B'(0)] \geq 0, \tag{10}$$

where $0B'(0) = \lim vB'(v)$ *as* $v \to 0$. *In particular, if* $0B'(0) = 0$,

$$1 + \theta A'(1)B(0) \geq 0. \tag{11}$$

3. *If the infima of these functions are reached at* $u = 0$, $v = 0$, *the parameter must satisfy*

$$1 + \theta 0A'(0)0B'(0)[1 + \theta 0A'(0)B(0)][1 + A(0)0B'(0)] \geq 0$$

Assume $\theta \leq 0$. *Then:*

4. *If the suprema of these functions are reached at* $u = 1$, $v = 1$, *we also obtain the restriction* (9).
5. *If the suprema of these functions are reached at* $u = 1$, $v = 0$, *we also obtain the restriction* (10).

Proof The density (4) must be non-negative. Hence θ should satisfy (6). Assume $\theta \geq 0$ and suppose that the products $uA'(u)vB'(v)$, $A(u)vB'(v)$ and $uA'(u)B(v)$ reach the minimum at $u = 1$, $v = 1$. Then $\theta uA'(u)vB'(v) \geq \theta 1A'(1)1B'(1)$, $1 + \theta uA'(u)B(v) \geq 1 + \theta 1A'(1)1B(1)$, $\left[1 + \theta A(u)vB'(v)\right] \geq \left[1 + \theta A(1)1B'(1)\right]$. Hence, if we consider

$$P_2(\theta, u, v) = \theta uA'(u)vB'(v) + \left[1 + \theta uA'(u)B(v)\right]\left[1 + \theta A(u)vB'(v)\right],$$

then

$$P_2(\theta, u, v) \geq \theta A'(1)B'(1) + \left[1 + \theta A'(1)B(1)\right]\left[1 + \theta A(1)B'(1)\right].$$

As $A(1) = B(1) = 0$, from (6), the first inequality follows.

Now suppose that these product functions reach the minimum at $u = 1$, $v = 0$. Then we similarly have

$$P_2(\theta, u, v) \geq \theta A'(1)B'(0) + \left[1 + \theta A'(1)B(0)\right]\left[1 + \theta A(1)0B'(0)\right].$$

As $A(1) = 0$, from (6), the second inequality follows.

If we assume $\theta \leq 0$ then we should consider the suprema of these functions at $u = 1$, $v = 1$. Since $\theta uA'(u)vB'(v) \geq \theta 1A'(1)1B'(1)$, etc., we have

$$P_2(\theta, u, v) \geq \theta A'(1)B'(1) + \left[1 + \theta A'(1)B(1)\right]\left[1 + \theta A(1)B'(1)\right]$$

and the first inequality also follows. The proof of the second inequality is similar. □

Of course, further constrains can be considered. Suppose that A or B (or both functions) change the sign in the interval $(0, 1)$. If $A(u_0) = 0 = B(v_0)$, from (6) we have

$$\theta A'(u_0)B'(v_0) + 1 \geq 0,$$

which provide more intervals containing θ. However this case is not interesting, since $C(u, v) - uv$ may change the sign, so certain measures of dependence may take small values. See below.

Example 1 Consider the classical Gumbel–Barnett family (1). Then $A(u) = \ln u$, $B(v) = -\ln v$ and $A'(1) = 1$, $B'(1) = -1$. Using Theorem 2, we have

$$M_1 = \inf\{A'(1)(-\ln v - 1)\} = -1, \qquad M_2 = \inf\{B'(1)(\ln u + 1)\} = -1,$$
$$M_3 = \sup\{A'(1)(-\ln v - 1)\} = \infty, \qquad M_4 = \sup\{B'(1)(\ln u + 1)\} = \infty.$$

From (7) we get $0 \leq \theta \leq 1$. The range of the correlation coefficient is $(-0.5238, 0]$. This range is obtained by integrating $C(u, v) - uv$ into $[0, 1]^2$ (which gives the covariance, see below).

Alternatively, as $\ln u$ is monotonic, we can use Theorem 3 with $A(u) = \ln u$, $B(v) = -\ln v$. Suppose $\theta \geq 0$. Then $\inf\{uA'(u)vB'(v)\} = A'(1)B'(1) = -1$, so (9) gives $1 + \theta A'(1)B'(1) = 1 - \theta \geq 0$, i.e., $\theta \leq 1$.

Suppose $\theta \leq 0$. Then $\sup\{uA'(u)B(v)\} = A'(1)B(0) = \infty$. From (10) we have $1 + \theta\infty \geq 0$, so $\theta \geq -1/\infty = 0$. The interval is again $0 \leq \theta \leq 1$.

Example 2 Consider the Celebioglu-Cuadras family (2). We have $A(u) = 1 - u$, $B(v) = 1 - v$. Then $uA'(u) = -u$. Let us use Theorem 3.

If $\theta \geq 0$ then $\inf\{uA'(u)B(v)\} = A'(1)B'(0) = -1$ so $\theta \leq 1$.

If $\theta \leq 0$ then $\sup\{uA'(u)B(v)\} = A'(1)B'(1) = 1$ so $\theta \geq -1$. The interval is $-1 \leq \theta \leq 1$. The range of the correlation coefficient is $(-0.2962, 0.3806)$.

Example 3 Consider the asymmetric family proposed by Bekrizadeh et al. (2017), with $A(u) = 1 - u$, $B(v) = \ln v$. We have $M_1 = M_2 = -1$, so $\theta \leq 1$. Also $1 + \theta A(0)B'(1) = 1 + \theta \geq 0$. The interval for θ is contained in [-1,1]. But $1 + \theta A'(1)B(0) \geq 0$ shows that $\theta \geq 0$. Thus $0 \leq \theta \leq 1$. The range of the correlation coefficient is (-0.4053,0].

3 New sub-families

We propose three sub-families obtained by taking appropriate functions $A(u)$ and $B(v)$ in (3).

3.1 Trigonometric

An interesting sub-family of (3) comes out by considering the functions

$$A(u) = (2/\pi)\cos(\pi u/2), \;\; B(v) = (2/\pi)\cos(\pi v/2)$$

Then $uA'(u) = -u\sin(\pi u/2)$ and similarly $vB'(v)$. We can use Theorem 3.

Suppose $\theta \leq 0$. Then $\sup\{uA'(u)vB(v)\} = 1A'(1)1B'(1) = 1$. From (9), we have $1 + \theta(-1)(-1) \geq 0$, so $\theta \geq -1$.

Suppose $\theta \geq 0$. Now $\inf\{uA'(u)vB(v)\} = 1A'(1)B(0) = -2/\pi$. From (11) we have $1+\theta(-1)(\pi/2) \geq 0$, so $\theta \leq \pi/2$. The interval is $-1 \leq \theta \leq \pi/2$. The range of the correlation coefficient is (-0.2400,0.4694).

3.2 Entropic

Another extension arises by taking $A(u) = u \ln u$ and $B(v) = v \ln v$. The name "entropic" is conventional.

These functions are not monotonic. Using Theorem 2 we have $vB'(v) + B(v) = 2v \ln v + v$. Then

$$M_1 = M_2 = 1, \; M_3 = M_4 = -2e^{-3/2}.$$

We obtain $-1 \leq \theta \leq (1/2)e^{3/2} = 2.2408$. The range of the correlation coefficient is (-0.1425,0.3636).

3.3 Potential

Consider the functions $A(u) = u^a(1-u^b)$, $B(v) = v^c(1-v^d)$. We study some particular cases.

If we take $a = c = 0$ and $b = d = 1$ this family reduces to the Celebioglu-Cuadras family, see (2).

If we take $a = b = c = d = 1$, then $uA'(u) + A(u) = 2u - 3u^2$. From Theorem 2 we have $M_1 = M_2 = -1/3$, ${}_3M_3 = M_4 = 1$. The second part of Theorem 2 does not apply here because $A(0) = 0$, so $M_i' = 0$, $i = 1, \ldots, 4$. Thus the parameter is included in the interval $-1 \leq \theta \leq 3$. Now the functions $A(u)$, $B(v)$ are not monotonic. Then taking $u = v = 1$ in (6) we get $1 + \theta A'(1)B'(1) \geq 0$, which is only a necessary condition. As $A'(1) = B'(1) = -1$, we also obtain $\theta \geq -1$. Thus the interval is $-1 \leq \theta \leq 3$. However, the range of the correlation is quite short.

Take $a = c = 1$, $b = d = 2$. Then $uA'(u) + A(u) = 2u - 4u^3$. We obtain $M_1 = M_2 = -8/(3\sqrt{6})$, $M_3 = M_4 = 4$. The parameter should satisfy $-1/4 \leq \theta \leq (3\sqrt{6})/8$. The range of the correlation is (-0.527,0.2198).

Another polynomial example is $A(u) = u(u-1/2)(u-1)$, which changes the sign at $u = 1/2$. With the same function for B, we obtain a wide interval $-4 \leq \theta \leq 12.77$. However, the copula is not quadrant dependent (see below) and the correlation is quite small.

4 Weighted geometric mean

The weighted arithmetic mean of two copulas is also a copula. In general, the geometric mean does not provide a copula. A property of the generalized family (3) is the invariance when taking the geometric mean (see Cuadras 2009; Zhang et al. 2013).

Theorem 4 *Let $\mathcal{F}_{AB}$ be the generated Gumbel–Barnett family of copulas generated by $A(u)$, $B(u)$, with $\theta_1 \leq \theta \leq \theta_2$. Suppose that C_{θ_a} and C_{θ_b} belong to $\mathcal{F}_{AB}$, where*

$\theta_1 \leq \theta_a, \theta_b \leq \theta_2$. Then the weighted geometric mean

$$C_{\theta_a}^{\alpha} C_{\theta_b}^{1-\alpha}, \quad 0 \leq \alpha \leq 1,$$

also belongs to $\mathcal{F}_{AB}$.

Proof From (3)

$$\{uv \exp[\theta_a A(u)B(v)]\}^{\alpha} \{uv \exp[\theta_b A(u)B(v)]\}^{1-\alpha} = uv \exp[\theta A(u)B(v)],$$

where $\theta = \alpha\theta_a + (1-\alpha)\theta_b$. As the interval $[\theta_1, \theta_2]$ is convex, $uv \exp[\theta A(u)B(v)]$ is a copula belonging to $\mathcal{F}_{AB}$. □

The geometric mean with $\alpha = 1/2$ converts any asymmetric copula to symmetric.

Theorem 5 *Let us consider the family (3) with $A \neq B$. Then*

$$uv \exp\left\{\frac{\theta}{2}\left[A(u)B(v) + B(u)A(v)\right]\right\}, \tag{12}$$

is a symmetric family of copulas for θ belonging to a sub-interval contained in the closed interval

$$\mathbf{I}_1 \cap \mathbf{I}_2 \cap \mathbf{I}_3,$$

where $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_3$ are respectively defined by

$$\begin{aligned} \theta A'(1)B'(1) &\geq -1, \\ \theta\left[A(0)B'(1) + A'(1)B(0)\right] &\geq -2, \\ \theta\left[A'(1)B(0) + A(0)B'(1)\right] &\geq -2. \end{aligned} \tag{13}$$

Proof If $uv \exp[\theta A(u)B(v)]$ belongs to $\mathcal{F}_{AB}$, clearly $uv \exp[\theta B(u)A(v)]$ belongs to $\mathcal{F}_{BA}$. Then the geometric mean ($\alpha = 1/2$) gives rise to the symmetric family (12). In the same way that we obtain (6), some algebra gives the parameter constraints

$$\begin{aligned} &\frac{1}{2}\theta\left[uv\left[A'(u)B'(v) + A'(v)B'(u)\right]\right] + \left\{1 + \frac{1}{2}\theta\left[uA'(u)B(v) + uA(v)B'(u)\right]\right\} \\ &\quad \times \left\{1 + \frac{1}{2}\theta\left[vA(u)B'(v) + vA'(v)B(u)\right]\right\} \geq 0, \end{aligned}$$

which reduces to (13) for $u = v = 1$ and $u = 0$, $v = 1$, defining the intervals $\mathbf{I}_1$, $\mathbf{I}_2$. We get $\mathbf{I}_3$ taking $u = 1$, $v = 0$. □

Example 4 Considering again the asymmetric sub-family with $A(u) = 1-u$, $B(v) = \ln v$, we may construct the symmetric family of copulas

$$uv \exp\left\{\frac{\theta}{2}\left[(1-u)\ln v + (1-v)\ln u\right]\right\}. \tag{14}$$

Then the left and right hand sides of (13) give $\theta \leq 1$ and $\theta \geq 0$. The interval for the parameter is $0 \leq \theta \leq 1$. The range of the correlation coefficient is $(-0.4053, 0)$.

5 Power series expansion

Taylor's expansion of $\exp[\theta A(u)B(v)]$ provides the expansion of (3)

$$C_\theta(u,v) = uv\left[1 + \sum_{k=1}^{\infty} \frac{\theta^k}{k!} A^k(u) B^k(v)\right], \tag{15}$$

which is useful for performing some moment computations.

If we take $A(u) = 1-u$, $B(v) = 1-v$, and consider only the first term in (15), we get the well-known Farlie-Gumbel-Morgenstern (FGM) family of copulas $uv\,[1+\theta(1-u)(1-v)]\,,\quad -1 \leq \theta \leq 1$. Thus the FGM family is the first order approximation to the Celebioglu-Cuadras family.

As for the Gumbel–Barnett family, we have

$$\begin{aligned} uv\exp(-\theta \ln u \ln v) &\simeq uv(1-\theta \ln u \ln v) && \text{(expanding } e^{-x}\text{)} \\ &\simeq uv[1-\theta(1-u)(1-v)] && \text{(expanding } \ln x\text{)}, \end{aligned}$$

so the FGM family is also the first order approximation to this family.

More generally, if we assume the limits $0A(0) = 0B(0) = 0$, the generalized FGM family $uv\,[1+\theta A(u)B(v)]$ is an approximation to the family (3).

6 Stochastic dependence

In this section we find the covariance and three non-parametric measures of association. Suppose that the random vector (U, V) has the distribution given by the cdf C_θ defined in (3).

Firstly, let us consider two real functions $\nu(x), \xi(y)$ of bounded variation on the interval [0,1]. Cuadras (2002) proved that

$$\operatorname{cov}(\nu(U), \xi(V)) = \int_0^1\int_0^1 [C_\theta(u,v) - uv]d\nu(u)d\xi(v), \tag{16}$$

where cov means covariance. This formula has been generalized by Diaz and Cuadras (2017).

If $\nu(u) = u$, $\xi(v) = v$, then (16) reduces to Hoeffding's lemma:

$$\operatorname{cov}(U, V) = \int_0^1\int_0^1 [C_\theta(u,v) - uv]dudv.$$

1. **Covariance**. The covariance of (U, V) with joint cdf given by (3) is

$$\operatorname{cov}(U, V) = \sum_{k=1}^{\infty} \frac{\theta^k}{k!} I_k J_k,$$

where $I_k = \int_0^1 u A^k(u) du$, $J_k = \int_0^1 v B^k(v) dv$. The proof follows by using (16) and step-wise integration of (15).

2. **Spearman's rho**. If (X, Y) is a random vector with joint cdf $H(x, y)$ and marginal cdf's $F(x)$, $G(y)$, Spearman's rho coefficient of correlation is defined by Pearson's correlation between $U = F(X)$ and $V = G(Y)$. Note that U and V have (0,1) uniform distribution.
Thus $\rho_S = \operatorname{cor}(F(X), G(Y)) = 12\operatorname{cov}(F(X), G(Y)) = 12\operatorname{cov}(U, V)$. For the copula (3) ρ_S is given by

$$\begin{aligned} \rho_S &= 12 \int_0^1 \int_0^1 [C_\theta(u, v) - uv] du dv \\ &= 12 \sum_{k=1}^{\infty} \frac{\theta^k}{k!} I_k J_k. \end{aligned}$$

3. **Kendall's tau**. The rank correlation or Kendall's tau is another measure of association given by $\tau = 4 \int_a^b \int_c^d H(x, y) dH(x, y) - 1$. For a copula function this coefficient is

$$\tau = 4 \int_0^1 \int_0^1 C_\theta(u, v) dC_\theta(u, v) - 1.$$

However, it is not possible to find a closed expression giving τ for the copula (3). A relatively treatable expression may be possible when C_θ is Archimedean, i.e., when

$$\phi[C_\theta(u, v)] = \phi(u) + \phi(v), \tag{17}$$

where $\phi : [0, 1] \to [0, \infty)$ is a decreasing differentiable function such that $\phi(0) = 1$. Then Kendall's tau is given by (see Nelsen 2006):

$$\tau = 1 + 4 \int_0^1 \frac{\phi(t)}{\phi'(t)} dt.$$

For example, if $A(u) = \ln u$, $B(v) = -\ln v$, copula (3) reduces to the Archimedean copula (1), which satisfies (17) for $\phi(t) = \ln(1 - \theta \ln t)$. Then

$$\tau_\theta = 1 - \frac{4}{\theta} \int_0^1 \frac{1}{t[\ln(1 - \theta \ln t)]^2} dt.$$

However, this is probably the only Archimedean copula of the generalized family (3).

4. **Fréchet-Hoeffding bounds**. Any copula $C(u, v)$ satisfies

$$\max\{u + v - 1, 0\} \leq C(u, v) \leq \min\{u, v\},$$

where the copulas $\max\{u+v-1, 0\}$ and $\min\{u, v\}$ are the lower and upper Fréchet-Hoeffding bounds. When $C(u, v)$ reaches the lower bound, then $\rho_S = \tau = -1$. When $C(u, v)$ reaches the upper bound then $\rho_S = \tau = 1$.
The generalized Gumbel–Barnett family (3) neither reaches the lower bound nor the upper bound, showing a weak dependence.

Theorem 6 *Suppose that the joint cdf C_θ of (U, V) belongs to the generalized Gumbel–Barnett family of copulas (3). Then Kendall's tau satisfies the strict inequality*

$$|\tau_\theta| < 1,$$

uniformly in θ. Similarly, Spearman's rho satisfies $|\rho_S| < 1$.

Proof Suppose that A, B are continuous functions. If both functions are derivable in (0,1), the distribution is absolutely continuous and the proof is trivial. Thus we assume only continuity.

Let us prove that C_θ can not reach the Fréchet-Hoeffding lower bound. Suppose that $C_\theta(u, v) = \max\{u + v - 1, 0\}$ as a limit distribution for some θ. Set $u = t$, $v = 1 - t$ with $0 < t < 1$. Then

$$t(1 - t) \exp[\theta A(t)B(1 - t)] = \max\{t + (1 - t) - 1, 0\} = 0,$$

which is impossible. Hence $C_\theta(u, v) > \max\{u+v-1, 0\}$ for $0 < u, v < 1$. Therefore $-1 < \tau_\theta$. We also have $-1 < \rho_S$.

Now suppose $C_\theta(u, v) = \min\{u, v\}$ as a limit distribution for some θ. Since $\min\{u, v\} \max\{u, v\} = uv$, it is clear that

$$uv \exp [- \ln \max\{u, v\}] = \min\{u, v\},$$

so $\theta A(u)B(v) = - \ln \max\{u, v\}$, in contrast with the independence of the functions A, B. Thus, if $u \leq v = e^{-\varepsilon}$, with $\varepsilon > 0$ arbitrarily small, we would have $\theta A(u) = \varepsilon / B(e^{-\varepsilon})$, so $A(u)$ is constant in $(0, e^{-\varepsilon})$, where $e^{-\varepsilon} \to 1$ as $\varepsilon \to 0$. Since A is continuous and $A(1) = 0$, the function A is the constant 0, hence the model (3) reduces to uv. Consequently C_θ, see (3), must satisfy $C_\theta(u, v) < \min\{u, v\}$ for $0 < u, v < 1$. Hence $\tau_\theta < 1$ and $\rho_S < 1$. □

5. **Medial correlation coefficient**. This measure of association is also called Blomqvist's beta. For (X, Y) with joint cdf H, this beta is defined by

$$\beta = 4H(x_m, y_m) - 1,$$

where x_m, y_m are the medians. Comparing beta with Kendall's tau, Blomqvist's beta may be interpreted as an approximation of τ.
For the copula (3) we obtain

$$\beta_\theta = \exp\left[\theta A\left(1/2\right) B\left(1/2\right)\right] - 1.$$

6. **Quadrant dependence**. As a consequence of Hoeffding's lemma, the covariance is positive if $C_\theta(u, v) \geq uv$ for all $u, v \in [0, 1]$. Then the distribution is called positive quadrant dependent (PQD).
For the copula (3), the inequality $uv \exp[\theta A(u)B(v)] \geq uv$ is equivalent to $\theta A\left(u\right) B\left(v\right) \geq 0$. In particular, if $A\left(u\right) B\left(v\right) \geq 0$, the distribution is PQD if $\theta \geq 0$.
The negative quadrant dependent distribution (NQD) is similarly defined by the inequality $C_\theta(u, v) \leq uv$. Then (3) is NQD if $\theta A\left(u\right) B\left(v\right) \leq 0$. In particular, if $A\left(u\right) B\left(v\right) \geq 0$, the distribution is NQD if $\theta \leq 0$.
Conversely, if $A\left(u\right) B\left(v\right) \leq 0$ then the distribution is PQD (NQD) if $\theta \leq 0$ $(\theta \geq 0)$.
Notice that Spearman's rho, Kendall's tau and Blomqvist's beta association coefficients, are positive if the distribution is PQD and negative if the distribution is NQD.
For example, (1) satisfies $\ln u(-\ln v) \leq 0$, hence this distribution is NQD for any $\theta \in (0, 1]$. Instead of $\ln u$, let us take $1 - u$. As $(1 - u)(1 - v) \geq 0$, copula (2) is PQD if $\theta \geq 0$ and NQD if $\theta \leq 0$.
7. **Tail dependence**. The measures λ_L and λ_U, called lower and upper tail dependence parameters, are studied in Nelsen (2006). They describe the degree of dependence in the corners of the distribution near to $(0, 0)$ and $(1, 1)$, respectively. It can be proved that

$$\lambda_L = \lim_{t \to 0^+} C(t, t)/t, \quad \lambda_U = 2 - \lim_{t \to 1^-} [1 - C(t, t)]/(1 - t).$$

For the family (3), since $t^2 C(t, t)/t = tC(t, t) \to 0$ as $t \to 0^+$ and also

$$2 - \frac{1 - C(t, t)}{1 - t} = 2 - \frac{1 - t^2 \exp[\theta A(t)B(t)]}{1 - t} \to 0,$$

as $t \to 1^-$, we have $\lambda_L = \lambda_U = 0$.

7 Visualization

In this section we propose a procedure for visualizing the family of copulas (3) for particular cases of $A(u)$ and $B(v)$. To do this, firstly we approximate $C_\theta(u, v)$ to the copula

$$C(u, v) = uv + r_1 3u(1-u)v(1-v) + r_2 5(2u^3 - 3u^2 + u)(2v^3 - 3v^2 + v). \tag{18}$$

That is, the probability density is approximated by

$$c(u, v) = 1 + r_1 g_1(u) g_1(v) + r_2 g_2(u) g_2(v),$$

where $g_1(u) = \sqrt{3}(1-2u)$, $g_2(u) = \sqrt{5}(6u^2 - 6u + 1)$ are orthonormal functions on the interval [0, 1]. This approximation is studied in Cuadras and Diaz (2012), Cuadras et al. (2020).

If (U, V) has joint cdf C_θ we should find the correlations between $g_1(U)$, $g_1(V)$ and between $g_2(U)$, $g_2(V)$ with respect to C_θ. Using (16) we compute

$$r_1(\theta) = \int_0^1 \int_0^1 [C_\theta(u, v) - uv] d[\sqrt{3}(1-2u)] d[\sqrt{3}(1-2v)],$$

and

$$r_2(\theta) = \int_0^1 \int_0^1 [C_\theta(u, v) - uv] d[\sqrt{5}(6u^2 - 6u + 1)] d[\sqrt{5}(6v^2 - 6v + 1)].$$

Then we represent the parametric curve $(r_1(\theta), r_2(\theta))$, with θ varying along the interval such that C_θ is a proper copula. Figure 1 shows the curves for some subfamilies belonging to the generalized family (3). All lines pass through point $(0, 0)$ corresponding to the independence copula uv.

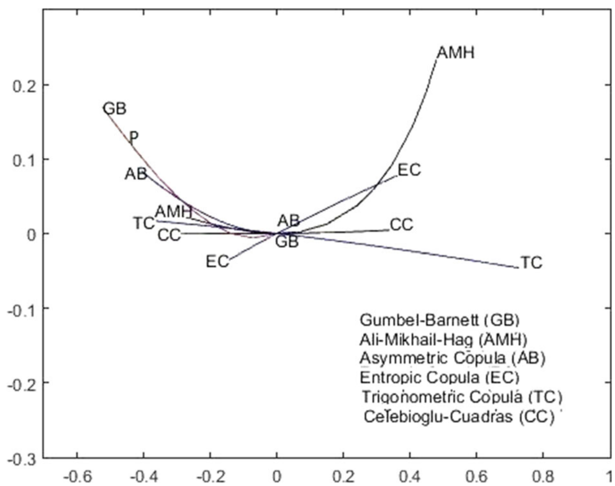

Fig. 1 Six curves visualizing the Gumbel–Barnett, the Celebioglu-Cuadras and two new families of copulas, called trigonometric and entropic. We also include the Ali-Mikhail-Haq family. Each curve visualizes the full family. The point P represents the Gumbel–Barnett copula with $\theta = 0.8$

Since (18) is symmetric, this procedure may be inappropriate if $A \neq B$. To overcome this case, we change the asymmetric copula (3) to a symmetric one considering (12). In accordance with this symmetrization, we replace the sub-family (with $A(u) = 1 - u$, $B(v) = \ln v$) to the symmetric family (14). Actually the difference (in absolute value) between both copulas is quite small, so we represent the symmetrized copula in the Fig. 1.

For comparison purposes, we also represent in the same figure the curve corresponding to the Ali-Mikhail-Haq family of copula (see Nelsen 2006):

$$C_\theta(u, v) = \frac{uv}{[1 - \theta(1 - u)(1 - v)]}, \quad -1 \leq \theta \leq 1.$$

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References

Bekrizadeh H, Parham G, Jamshidi B (2017) A new asymmetric class of bivariate copulas for modeling dependence. Commun Stat Simul Comput 46(7):5594–5609

Celebioglu S (1997) A way of generating comprehensive copulas. J Inst Sci Technol 10(1):57–61

Cuadras CM (2002) On the covariance between functions. J Multivar Anal 81(1):19–27

Cuadras CM (2006) The importance of being the upper bound in the bivariate family. SORT 30(1):55–84

Cuadras CM (2009) Constructing copula functions with weighted geometric means. J Stat Plan Infer 139(11):3766–3772

Cuadras CM (2015) Contributions to the diagonal expansion of a bivariate copula with continuous extensions. J Multivar Anal 139:28–44

Cuadras CM, Diaz W (2012) Another generalization of the bivariate FGM distribution with two-dimensional extensions. Acta Comment Univ Tartu Math 16(1):3–12

Cuadras CM, Diaz W, Salvo-Garrido S (2020) Two generalized bivariate FGM distributions and rank reduction. Commun Stat Theory Methods 49(23):5639–5665

Diaz W, Cuadras CM (2017) On a multivariate generalization of the covariance. Commun Stat Theory Methods 46(9):4660–4669

Durante F, Sempi C (2016) Principles of copula theory, vol 474. CRC Press, Boca Raton, FL

Hutchinson T, Lai C (1991) The engineering statistician's guide to continuous bivariate distributions. Rumsby Scientific Pub

Joe H (1997) Multivariate models and dependence concepts. Chapman & Hall/CRC Monographs on Statistics & Applied Probability, Taylor & Francis

Nelsen R (2006) An introduction to copulas. Springer Series in Statistics, Springer New York

Zhang K, Lin J, Huang C (2013) Some new results on weighted geometric mean for copulas. Int J Uncertain Fuzziness Knowl Based Syst 21(02):277–288