A Formal Recursive Approach to Holographic Entropy, CFT Entanglement, and Fractal Structures

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1 Recursive Holographic Entropy Scaling

Recurrence Relation

We postulate a recursive relation for the entropy:

$$S_{n+1} = S_n + \phi^{-1} S_{n-1}, \tag{1}$$

where ϕ is taken as the golden ratio,

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618.$$

Characteristic Equation

The characteristic equation associated with (1) is

$$\lambda^2 - \lambda - \phi^{-1} = 0. \tag{2}$$

Its solutions are

$$\lambda_\pm = \frac{1 \pm \sqrt{1+4\phi^{-1}}}{2}. \tag{3}$$

Note: For $\phi \approx 1.618$, one computes $\phi^{-1} \approx 0.618$ so that

$$\sqrt{1+4\phi^{-1}} \approx \sqrt{3.472} \approx 1.863,$$

and hence

$$\lambda_+ \approx \frac{1+1.863}{2} \approx 1.4315,$$

which differs from the stated value of 1.618.

Entropy Scaling

Assuming the dominant behavior is given by λ_+, we have

$$S_n \sim S_0 \, \lambda_+^n. \tag{4}$$

By further assuming a mapping between the index n and the horizon area, we write

$$S_{\text{holo}} \sim A_{\text{horizon}} \, \phi^{D/2}, \tag{5}$$

with D the spacetime dimension. For $D > 3$, the factor $\phi^{D/2}$ exceeds simple area proportionality, suggesting the emergence of fractal microstate structure. (A more detailed derivation is needed.)

2 CFT Entanglement and Central Charge

Modified Cardy Formula

We introduce a modified Cardy formula for the entanglement entropy:

$$S_A^{(n)} = \frac{c_n}{3} \, \log\Big(\phi^n \, \ell\Big), \tag{6}$$

where ℓ is a characteristic length and c_n is the effective central charge at level n.

Central Charge Recursion

Assume the recursion

$$c_n = c_0 + \sum_{k=1}^{n} \phi^{-k} \, c_k. \tag{7}$$

If we assume $c_k \sim 24 \, \phi^{-k}$, then summing the geometric series yields

$$c_\infty = \frac{24\phi}{1 - \phi^{-1}}, \tag{8}$$

which under further assumptions (e.g., $c_0 = 0$) is claimed to converge to a fixed value. (Note: check consistency of factors.)

3 Recursive RG Flow and AdS Geometry

Beta Function Recursion

The beta function is assumed to obey

$$\beta_{n+1} = \phi^{-1}\,\beta_n, \tag{9}$$

so that

$$\beta_n = \beta_0\,\phi^{-n}. \tag{10}$$

AdS Radial Flow

Identifying the radial coordinate as

$$z_n = \phi^{-n}\,z_0, \tag{11}$$

one obtains a discrete sequence of scales that aligns with the notion of fractal horizons in AdS geometry.

4 Fractal AdS/CFT and Spin Networks

We propose a bulk-boundary mapping based on a fractal spin network:

$$\Gamma_n = \bigoplus_{k=0}^{n} \mathfrak{su}(2)_k \otimes \phi^{-k}. \tag{12}$$

A corresponding geodesic scaling is assumed:

$$\ell_n = \phi^n\,\ell_0. \tag{13}$$

The details of these constructions require further elaboration.

5 Lean 4 Formalization

Entropy Scaling Proof

An inductive proof is assumed to yield

$$S_n = S_0\,\lambda_+^n. \tag{14}$$

RG Flow Convergence

Since $\phi^{-1} < 1$, the RG flow

$$\beta_n = \beta_0 \, \phi^{-n}$$

converges as $n \to \infty$.

6 Mirror Symmetry and Fractal Moduli Spaces

The mirror map is defined recursively by

$$F_{n+1}(z) = \phi^{-1} \, F_n(\phi \, z), \tag{15}$$

with Yukawa couplings scaling as

$$Y^{(n+1)}_{ijk} = \phi^{-1} \, Y^{(n)}_{ijk}. \tag{16}$$

This recursion is meant to preserve a fractal structure in the moduli space.

7 Recursive Picard–Fuchs Equations

The quantum periods obey

$$\Pi_{n+1}(z) = \phi^{-1} \, \Pi_n(\phi \, z), \tag{17}$$

and the monodromy matrices satisfy

$$M_{n+1} = \phi^{-1} \, M_n. \tag{18}$$

8 Higher–Genus Gromov–Witten Invariants

We assume the following recursive relations:

$$N^{(n+1)}_{g,\beta} = \phi^{-1} \, N^{(n)}_{g,\beta}, \tag{19}$$
$$F_{g,n+1} = \phi^{-1} \, F_{g,n}. \tag{20}$$

These are posited to be consistent with mirror symmetry and recursive topological string expansions.

9 Hausdorff Dimension and Self–Similarity

The Hausdorff dimension is claimed to be

$$D_H = \frac{\ln \phi^3}{\ln \phi} = 3. \tag{21}$$

Remark: The algebra immediately shows $D_H = 3$, contrary to the stated $3 + \ln \phi$. A re–examination of the assumptions leading to a fractal (non–integer) dimension is needed.

Gromov–Hausdorff convergence is invoked to argue for self–similarity of the Kähler moduli space.

10 Causal Boundaries and Stress–Energy Convergence

Cykloid Solutions

We assume the existence of cykloid solutions (to be defined precisely) that satisfy the null geodesic condition along with the Einstein equations.

Stress–Energy Summability

The weighted sum

$$\sum_{n=0}^{\infty} \phi^{-n} \, T_{\mu\nu}^{(n)} \tag{22}$$

is assumed to converge, thereby validating the causal structure of the spacetime.

11 Concluding Remarks

While the above formalism presents an intriguing recursive structure linking holographic entropy, RG flows, mirror symmetry, and fractal properties, several points require further clarification or correction:

- The numerical value of λ_+ in (3) does not match the stated value.
- The connection between the discrete recursion and continuum geometric quantities (such as area and moduli) needs a rigorous derivation.

- Several recursions (for the central charge, mirror map, Picard–Fuchs equations, and Gromov–Witten invariants) are postulated without derivation.

- The Hausdorff dimension calculation in (21) is internally inconsistent with the claim of a fractal (non–integer) dimension.