Recursive and Fractal Structures in Holography, CFT, and Causal Geometry

Julian Del Bel[1]
[1] *Unaffiliated*
(Dated: February 1, 2025)

I. RECURSIVE HOLOGRAPHIC ENTROPY AND CFT STRUCTURES

A. Recursive Holographic Entropy Scaling

We postulate a recursive relation for the entropy:

$$S_{n+1} = S_n + \phi^{-1} S_{n-1}, \tag{1}$$

where the golden ratio is

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618.$$

The associated characteristic equation is

$$\lambda^2 - \lambda - \phi^{-1} = 0, \tag{2}$$

with solutions

$$\lambda_\pm = \frac{1 \pm \sqrt{1+4\phi^{-1}}}{2}. \tag{3}$$

Remark: For $\phi^{-1} \approx 0.618$, one finds

$$\sqrt{1+4\phi^{-1}} \approx \sqrt{3.472} \approx 1.863, \quad \lambda_+ \approx 1.4315,$$

which does not coincide with ϕ. We note the numerical discrepancy and leave a more detailed analysis for future work.

Assuming dominance of λ_+, the asymptotic behavior is

$$S_n \sim S_0 \, \lambda_+^n. \tag{4}$$

A mapping to horizon area is assumed by writing

$$S_{\text{holo}} \sim A_{\text{horizon}} \, \phi^{D/2}, \tag{5}$$

where D is the spacetime dimension. For $D > 3$, the factor $\phi^{D/2}$ exceeds simple area proportionality, suggesting the emergence of fractal microstates.

B. CFT Entanglement and Central Charge

A modified Cardy formula for the entanglement entropy is proposed:

$$S_A^{(n)} = \frac{c_n}{3} \log\Big(\phi^n \, \ell\Big), \tag{6}$$

with ℓ a characteristic length scale. The effective central charge is assumed to satisfy the recursion

$$c_n = c_0 + \sum_{k=1}^{n} \phi^{-k} \, c_k. \tag{7}$$

Assuming $c_k \sim 24\, \phi^{-k}$ and summing the resulting geometric series, one obtains formally

$$c_\infty = \frac{24\phi}{1-\phi^{-1}}, \tag{8}$$

though the precise numerical factors require further scrutiny.

C. Recursive RG Flow and AdS Geometry

A recursive beta function is assumed:

$$\beta_{n+1} = \phi^{-1}\,\beta_n \quad \Rightarrow \quad \beta_n = \beta_0\,\phi^{-n}. \tag{9}$$

Identifying the AdS radial coordinate discretely as

$$z_n = \phi^{-n}\,z_0, \tag{10}$$

one obtains a self-similar, fractal scaling in the radial flow.

D. Fractal AdS/CFT and Spin Networks

The bulk-boundary correspondence is conjectured to be encoded in a fractal spin network:

$$\Gamma_n = \bigoplus_{k=0}^{n} \mathfrak{su}(2)_k \otimes \phi^{-k}, \tag{11}$$

with an associated geodesic scaling law

$$\ell_n = \phi^n\,\ell_0. \tag{12}$$

The details of these constructions remain to be fully specified.

E. Lean 4 Formalization

An inductive proof is assumed for the entropy scaling:

$$S_n = S_0\,\lambda_+^n. \tag{13}$$

Similarly, the RG flow converges since $\phi^{-1} < 1$, ensuring that

$$\beta_n = \beta_0\,\phi^{-n} \quad \text{converges as } n \to \infty.$$

F. Mirror Symmetry and Fractal Moduli Spaces

The mirror map is defined recursively by

$$F_{n+1}(z) = \phi^{-1}\,F_n(\phi\,z), \tag{14}$$

and the Yukawa couplings by

$$Y_{ijk}^{(n+1)} = \phi^{-1}\,Y_{ijk}^{(n)}. \tag{15}$$

These relations are intended to preserve a self-similar fractal structure in the moduli space.

G. Recursive Picard–Fuchs Equations

Quantum periods are assumed to satisfy

$$\Pi_{n+1}(z) = \phi^{-1}\,\Pi_n(\phi\,z), \tag{16}$$

while monodromy matrices obey

$$M_{n+1} = \phi^{-1}\,M_n. \tag{17}$$

H. Higher–Genus Gromov–Witten Invariants

We postulate the following recursive relations:

$$N_{g,\beta}^{(n+1)} = \phi^{-1}\, N_{g,\beta}^{(n)}, \tag{18}$$

$$F_{g,n+1} = \phi^{-1}\, F_{g,n}. \tag{19}$$

These are conjectured to align with the recursive structure seen in mirror symmetry and topological string theory.

I. Hausdorff Dimension and Self–Similarity

The Hausdorff dimension is given by

$$D_H = \frac{\ln \phi^3}{\ln \phi} = 3. \tag{20}$$

(An earlier claim of $D_H = 3 + \ln \phi$ is revised by the elementary calculation above.) Gromov–Hausdorff convergence is invoked to support the self-similarity of the underlying Kähler moduli space.

J. Causal Boundaries and Stress–Energy Convergence

Finally, we assume the existence of "cykloid" solutions (to be precisely defined) that satisfy the null geodesic condition and the Einstein equations. The weighted stress–energy sum

$$\sum_{n=0}^{\infty} \phi^{-n}\, T_{\mu\nu}^{(n)} \tag{21}$$

is assumed to converge, ensuring a well–defined causal structure.

II. MATHEMATICAL FOUNDATIONS OF HYPERFOLD GEOMETRY

A. Hyperfold Geometry

The recursive hyperfold equation is decomposed as:

$$\begin{aligned} \mathcal{F}_k(\Psi) = &\int_0^\infty e^{-\mathscr{S}_k\, t}\, \Psi_{k-1}(t)\, dt \\ &+ \phi^{-k} \Lambda\, \nabla^2 \Psi_k, \end{aligned} \tag{22}$$

where $\mathscr{S}_k$ represents a damping operator and Λ is the cosmological constant.

B. Recursive Stress–Energy Tensor

The stress–energy tensor is assumed to decompose recursively as

$$\begin{aligned} T_{\mu\nu}^{(k)} = &\, \phi^{-k} T_{\mu\nu}^{(0)} \\ &+ \sum_{i=1}^{k} \mathcal{O}_i \Big(\nabla^2 \Psi_{k-i} \Big), \end{aligned} \tag{23}$$

where $\mathcal{O}_i$ are nonlocal operators acting on the fractal structure of the fields.

III. CAUSAL STRUCTURE IN FRACTAL SPACETIMES

A. Causal Hypersphere (Mass)

The gravitational potential is modified by fractal scaling:

$$\Phi(r,t) = \frac{GM}{r}\, e^{-r^2/\sigma^2} \times \begin{cases} \phi^{D_H/2}, & r < \sigma, \\ 1, & r \geq \sigma, \end{cases} \tag{24}$$

where the fractal correlation length is defined by

$$\sigma = \phi^{-k}\Lambda^{-1/2}.$$

B. Causal Hypercone (Light)

The lightcone structure is modified and given by

$$\begin{aligned} ds^2 &= -dt^2 + \phi^{-k}\, dr^2 \\ &\quad + r^2 \Bigg(d\theta^2 + \sin^2\theta\, d\phi^2 \\ &\quad + \sum_{n=4}^{D_H} \prod_{i=1}^{n-3} \sin^2\theta_i\, d\theta_{n-2}^2 \Bigg), \end{aligned} \tag{25}$$

extending the spacetime effectively to $D_H = 3 + \ln\phi$ dimensions in the original claim, though the Hausdorff calculation above suggests $D_H = 3$. (This discrepancy indicates an area for further investigation.)

IV. PHOGARITHMIC DYNAMICS

A. Temporal Scaling

A new time coordinate system, termed the PHOGarithmic time, is introduced:

$$t_{\text{PHOG}} = t_0 \ln\left(1 + \phi^{-k} t\right) \times \left[1 - \frac{\phi^{-2k}}{\left(1 + \phi^{-k} t\right)^2}\right], \tag{26}$$

which incorporates self–regulating terms to prevent temporal divergences.

B. Fractal Entropy

The generalized entropy is given by a product of geometric and temporal contributions:

$$\begin{aligned} S_{\text{rec}} = &\underbrace{\frac{A}{4G}\, \phi^{D_H/2}}_{\text{Geometric term}} \\ &\times \underbrace{\left[1 - \mathcal{N}(t)\right]}_{\text{Temporal correction}}, \end{aligned} \tag{27}$$

where $\mathcal{N}(t)$ encodes causal asymmetry.

V. CONCLUSIONS

We have presented a framework that combines recursive structures, fractal scaling, and self–similar dynamics across various domains:

- A recursive approach to holographic entropy and RG flows,
- Modifications of CFT entanglement and central charge recursions,
- Recursive constructions in mirror symmetry, Picard–Fuchs equations, and Gromov–Witten invariants,
- A hyperfold formulation for nonlocal dynamics,
- A causal structure modified by fractal scaling, and
- A novel PHOGarithmic time coordinate that self–regulates temporal divergences.

While many of the ideas are speculative and the derivations are schematic, this document lays out a formalism.