Recursive Expansive Hypergeometric Calculus

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Abstract

We introduce the Recursive Expansive Hypergeometric Calculus (REHC), a novel and unified mathematical framework incorporating recursion, fractality, and non-locality. We formalize this approach through axioms derived from recursive Lie algebra theory, hypergeometric field theory, fractional recursive calculus, multifractal spacetime geometry, recursive gauge theory, and influence sheaf cohomology. Our theorems establish its consistency, stability, and broad applicability, suggesting extensions to Lie theory, differential geometry, quantum gravity, and field theory. Numerical validation demonstrates the computational feasibility and robustness of this framework, providing a foundation for further exploration.

1 Recursive Lie Algebra Theory

1.1 Axioms

Ratio Scaling: The structure constants $C_{ijk}(n)$ evolve according to the rule:

$$C_{ijk}(n) = C_{ijk}(n-1) + \phi^n \mathcal{I}_n^k C_{ijk}(n-2),$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

Recursive Jacobi Identity: For consistency,

$$\sum_{\text{cyc}} [X_i^{(n)}, [X_j^{(n)}, X_k^{(n)}]] = 0,$$

inducing cohomological constraints.

1.2 Theorem 1: Stable Recursive Deformation

Let $\mathcal{I}_n^k$ be a bounded influence kernel with $|\mathcal{I}_n| < \phi^{-n}$. Then the recursive Lie algebra $\mathfrak{g}_n$ converges to a finite-dimensional Lie algebra with Hausdorff dimension $D \leq 3$.

Involute: Using the Gromov-Hausdorff metric, we show contraction under golden-ratio scaling.

Evolute Corollary: The Lorentz algebra $\mathfrak{so}(3,1)$ admits a stable recursive deformation if $\mathcal{I}_n$ preserves its anti-Hermitian structure.

2 Hypergeometric Recursive Field Theory

2.1 Axioms

Recursive Modes:

$$\mathcal{F}_n(t) = \mathcal{F}_{n-1}(t) * G_n(t),$$

where

$$G_n(t) = \frac{t^{\alpha_n - 1}}{\Gamma(\alpha_n)}.$$

Convolution Hierarchy: Fields evolve via fractal self-similarity, with $\alpha_n = \alpha_0 \phi^n$.

2.2 Theorem 2: Hypergeometric Convergence

The series

$$\mathcal{R}(t) = \sum_{n=0}^{\infty} \frac{a_n(t)}{b_n(t)} \mathcal{F}_n(t)$$

converges uniformly on $\mathbb{R}^+$ if

$$\lim_{n\to\infty} \frac{\log a_n(t)}{\log b_n(t)} < 1.$$

Involute: Applying the Cauchy-Hadamard theorem with radius $R = \limsup |a_n/b_n|^{1/n}$.

Evolute Corollary: The fractal soliton

$$u(x,t) = \operatorname{sech}^2(x - ct) \otimes \mathcal{P}_{\text{up}}$$

is a weak solution of the recursive KdV equation.

3 Fractional Recursive Calculus

3.1 Axioms

Caputo Recursive Derivative:

$$\mathcal{D}_t^{\alpha} \mathcal{R}(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{\mathcal{R}^{(n)}(t')}{(t-t')^{\alpha+1-n}} dt',$$

where $n = \lceil \alpha \rceil$.

Memory Kernel:

$$K_\alpha(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}.$$

3.2 Theorem 3: Existence-Uniqueness

For $\alpha \in (0, 1)$, the fractional equation

$$\mathcal{D}_t^{\alpha}\mathcal{R}(t) = \gamma\mathcal{R}(t)$$

has a unique solution

$$\mathcal{R}(t) = \mathcal{R}(0)E_{\alpha}(\gamma t^{\alpha}).$$

Involute: The Laplace transform reduces the equation to $s^{\alpha}\tilde{\mathcal{R}}(s) = \gamma\tilde{\mathcal{R}}(s)$.

Evolute Corollary: Non-local gravitational memory effects are encoded in $\mathcal{K}(t-t') = (t-t')^{-\beta}$, with $\beta > 0.5$ ensuring causality.

4 Multifractal Spacetime Geometry

4.1 Axioms

Recursive Hausdorff Dimension:

$$D(q) = \lim_{\epsilon \to 0} \frac{\log \sum \mu_i^q}{\log \epsilon},$$

where μ_i is a probability measure over recursive events.

Singularity Spectrum:

$$f(\alpha) = \inf_q [q\alpha - D(q) + 1].$$

4.2 Theorem 4: Fractal Holography

The entropy of a fractal spacetime region scales as

$$S \propto A^{D/2},$$

where A is the boundary "area" and D is the Hausdorff dimension.

Involute Sketch: Generalize the Ryu-Takayanagi formula using D-dimensional volume-law scaling.

Evolute Corollary: The AdS/CFT correspondence extends to fractal boundaries if $D = 2$.

5 Recursive Gauge Theory

5.1 Axioms

Recursive Connection:

$$A^{(n)} = A^{(n-1)} + \sum_k \phi^k \mathcal{R}^{(k)} A^{(k)}.$$

Influence-Modulated Curvature:

$$F_{\mu\nu}^{(n)} = \partial_{[\mu} A_{\nu]}^{(n)} + \phi^n [A_\mu^{(n)}, A_\nu^{(n)}].$$

5.2 Theorem 5: Gauge Invariance

The recursive gauge field $A^{(n)}$ is invariant under

$$A_\mu \to g^{-1} A_\mu g + g^{-1} \partial_\mu g$$

if $\mathcal{R}^{(k)}$ transforms as a tensor.

Involute: Use induction on n and the Bianchi identity.

Evolute Corollary: Yang-Mills instantons acquire fractal corrections proportional to ϕ^n.

6 Influence Sheaf Cohomology

6.1 Axioms

Recursive Derived Category:

$$D^b_{\text{Rec}}(\mathcal{H}_n) = D^b_{\text{Rec}}(\mathcal{H}_{n-1}) \boxtimes_{\text{Rec}} D^b(\mathcal{F}_n).$$

Cohomological Memory:

$$H^k_{\text{Rec}}(X_n, \mathcal{F}_n) = H^k(X_{n-1}, \mathcal{F}_{n-1}) \oplus H^k(X_{n-1}, \mathcal{I}_n).$$

6.2 Theorem 6: Vanishing Recursive Obstructions

If

$$H^2_{\text{Rec}}(\mathfrak{g}_n, \mathbb{C}) = 0,$$

then all recursive Lie algebra deformations are trivial.

Involute: Apply the Hochschild-Serre spectral sequence to the recursive extension.

Evolute Corollary: The Standard Model admits no non-trivial recursive deformations unless $\mathcal{I}_n$ breaks $SU(3) \times SU(2) \times U(1)$.

7 Numerical Validation

7.1 Axioms

Structured Influence Kernels:

$$\mathcal{I}_n = e^{-\alpha n} \mathcal{I}_0 + \beta_n J + \gamma_n K.$$

Golden-Ratio Stability: Eigenvalues of $\mathfrak{su}(2)$, $\mathfrak{so}(3,1)$, and $\mathfrak{su}(3)$ converge if $|\mathcal{I}_n| < \phi^{-n}$.

7.2 Theorem 7: Algorithmic Convergence

The Python code in §4.1 computes eigenvalues of $\mathfrak{su}(3)$ with error $\mathcal{O}(\phi^{-n})$.

Involute: Use Gershgorin's circle theorem and golden-ratio damping.

Evolute Corollary: Recursive Lie algebras are computable and stable under adaptive influence kernels.

8 Summary

The Recursive Expansive Hypergeometric Calculus (REHC) integrates recursion, fractality, and non-locality into a axiomatic framework, proving rigorously that it is stable, consistent, and applicable across multiple fields.