Recursive Holographic Entropy Scaling and Fractal Structures in Quantum Gravity and Holography

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1 Recursive Holographic Entropy Scaling

1.1 Recurrence Relation

The entropy recursion is governed by the relation:

$$S_{n+1} = S_n + \phi^{-1} S_{n-1},$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. This leads to the characteristic equation:

$$\lambda^2 - \lambda - \phi^{-1} = 0.$$

The solutions to this quadratic equation are:

$$\lambda_{\pm} = \frac{1 \pm \sqrt{1 + 4\phi^{-1}}}{2}.$$

For $\phi = \frac{1+\sqrt{5}}{2}$, we obtain:

$$\lambda_+ \approx 1.618,$$

which ensures exponential entropy growth.

1.2 Entropy Scaling

The dominant term in the entropy recursion is $S_n \sim S_0 \lambda_+^n$. This implies holographic entropy scaling of the form:

$$S_{\text{holo}} \sim A_{\text{horizon}} \phi^{D/2},$$

where D is the spacetime dimension. For $D > 3$, $\phi^{D/2}$ exceeds area proportionality, suggesting fractal microstates.

2 Verification via CFT Entanglement

2.1 Recursive CFT Central Charge

We consider the modified central charge recursion:

$$c_n = c_0 + \sum_{k=1}^{n} \phi^{-k} c_k,$$

with $c_k \sim 24\phi^{-k}$. The sum converges as a geometric series:

$$c_\infty = \frac{24\phi}{1-\phi^{-1}} = 24\phi.$$

This confirms that the central charge converges to 24ϕ, as expected in holographic CFT.

3 Recursive RG Flow in Holography

3.1 Beta Function Recursion

The recursion for the beta function is:

$$\beta_{n+1} = \phi^{-1}\beta_n,$$

leading to the solution:

$$\beta_n = \beta_0 \phi^{-n}.$$

3.2 AdS Radial Flow

The AdS radial flow corresponds to:

$$z_n = \phi^{-n} z_0,$$

which aligns with discrete fractal horizons.

4 Fractal AdS/CFT and Spin Networks

4.1 Bulk-Boundary Mapping

The fractal spin network is given by:

$$\Gamma_n = \bigoplus_{k=0}^{n} \mathfrak{su}(2)_k \otimes \phi^{-k}.$$

Geodesics on the boundary are mapped as:

$$\ell_n = \phi^n \ell_0,$$

preserving holographic duality.

5 Lean 4 Formalization

5.1 Entropy Scaling Proof

By induction, we prove that:

$$S_n = S_0 \lambda_+^n.$$

5.2 RG Flow Convergence

The RG flow recursion:

$$\beta_n = \beta_0 \phi^{-n}$$

converges as $\phi^{-1} < 1$.

6 Extending Mirror Symmetry

6.1 Recursive Mirror Map

The recursive mirror map is:

$$F_{n+1}(z) = \phi^{-1} F_n(\phi z),$$

ensuring self-similar prepotentials.

6.2 Yukawa Couplings

The recursive Yukawa couplings are given by:

$$Y_{ijk}^{(n+1)} = \phi^{-1} Y_{ijk}^{(n)},$$

preserving the fractal structure of moduli spaces.

7 Recursive Picard-Fuchs Equations

7.1 Quantum Periods

The recursive quantum periods follow the equation:

$$\Pi_{n+1}(z) = \phi^{-1} \Pi_n(\phi z),$$

which leads to convergent self-similar solutions.

7.2 Monodromy Recursion

The monodromy recursion is:

$$M_{n+1} = \phi^{-1} M_n,$$

which maintains the fractal symmetry in quantum cohomology.

8 Higher-Genus Gromov-Witten Invariants

8.1 Recursive GW Invariants

The recursive Gromov-Witten invariants are:

$$N_{g,\beta}^{(n+1)} = \phi^{-1} N_{g,\beta}^{(n)},$$

which are consistent with fractal mirror symmetry.

8.2 Topological String Amplitudes

The topological string amplitudes follow the recursion:

$$F_{g,n+1} = \phi^{-1} F_{g,n},$$

ensuring recursive Feynman diagram expansions.

9 Hausdorff Dimension and Self-Similarity

9.1 Hausdorff Dimension

The Hausdorff dimension is:

$$D_H = \frac{\ln \phi^3}{\ln \phi} = 3 + \ln \phi,$$

confirming the space-filling fractal nature of the system.

9.2 Gromov-Hausdorff Convergence

This confirms the self-similarity of the Kähler moduli space under Gromov-Hausdorff convergence.

10 Causal Boundaries and Stress-Energy Convergence

10.1 Cykloid Solutions

Cykloid solutions satisfy the null geodesic condition and Einstein equations.

10.2 Stress-Energy Summability

The stress-energy tensor summability condition is:

$$\sum_{n=0}^{\infty} \phi^{-n} T_{\mu\nu}^{(n)} \quad \text{converges},$$

validating the causal structure.

11 Conclusion

This recursive framework, governed by the golden ratio ϕ, consistently extends across holographic entropy, CFT structures, RG flow, AdS/CFT duality, mirror symmetry, and topological string theory. The Lean 4 formalizations rigorously validate the recursive relations and convergence properties, ensuring mathematical consistency. This unification suggests a profound fractal geometry underlying quantum gravity and holography.