Framework for Holographic Entropy and Quantum Gravity

Julian Del Bel

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Abstract

The framework presented in this document explores the intersection of holographic entropy and quantum gravity, incorporating recursive spacetime dynamics and the implications of fractal structures. Through a detailed examination of renormalization, recursive corrections, and scaling behaviors, this work aims to advance the theoretical understanding of entropy bounds and quantum spacetime structure.

Contents

1 Theoretical Framework

1.1 Holographic Entropy and Bekenstein Bound

In the study of holographic systems, the entropy S_{holo} associated with a black hole or other gravitational object plays a central role in understanding its thermodynamic properties. One of the foundational results in this context is the Bekenstein bound, which relates the entropy of a system to its surface area and the gravitational constant G. This bound is given by:

$$S_{\text{Bekenstein}} = \frac{A}{4G}$$

where A is the surface area of the system. However, in the context of fractal spacetimes and recursive geometries, we encounter an issue where the holographic entropy scaling exceeds this bound due to the fractal dimension D of the spacetime.

- **Scaling Issue**: The initial holographic entropy is given by the scaling relation:

  $$S_{\text{holo}} \sim \frac{A}{4G}\phi^{D/2}, \quad D > 3$$

  where D is the fractal dimension, which can exceed 3 in recursive spacetime models. As the fractal dimension increases, the entropy grows beyond the classical Bekenstein bound. This issue arises because the scaling factor ϕ introduces a dimensional expansion that leads to a higher entropy value than the area-based entropy bound allows.

- **Renormalization**: To resolve this issue, we introduce a counterterm series to subtract the divergent contributions and renormalize the entropy. The counterterms are chosen such that the renormalized entropy $S_{\text{holo}}^{\text{ren}}$ remains finite and respects the entropy bounds. The counterterm series is expressed as:

  $$S_{\text{corr}} = \sum_n c_n \phi^{-n}$$

  where c_n are the coefficients of the counterterms. The renormalized entropy is then given by:

  $$S_{\text{holo}}^{\text{ren}} = S_{\text{holo}} - S_{\text{corr}}$$

  By subtracting the counterterm contributions, we ensure that the renormalized entropy satisfies the Bekenstein bound and remains physically consistent with the principles of quantum gravity.

Thus, the issue of scaling entropy beyond the Bekenstein bound in fractal spacetime geometries is addressed through the renormalization procedure, which ensures that the system's entropy remains finite and bounded in accordance with established thermodynamic laws. The renormalization not only makes the entropy compatible with the Bekenstein bound, but also preserves the underlying fractal structure of spacetime that gives rise to the scaling behavior.

The counterterm approach allows us to correct the infinite contributions from the fractal dimension and obtain a well-defined, finite renormalized entropy. This result ensures the physical validity of holographic entropy scaling in recursive geometries and guarantees compliance with the fundamental principles of black hole thermodynamics.

1.2 Renormalization Counterterms

In quantum field theory and gravity, renormalization is an essential process where infinities in physical quantities are systematically controlled and subtracted. In the context of fractal spacetime geometries, we introduce a counterterm series to correct the infinite contributions that may arise from the fractal structure. These counterterms are typically represented as a series of terms that scale with the recursion factor ϕ^{-n}, ensuring that the renormalized quantity remains finite and physically meaningful.

The counterterm series for the corrected entropy is given by:

$$S_{\text{corr}} \sim \sum_n c_n \phi^{-n}$$

where: - S_{corr} represents the series of counterterms, - c_n are coefficients that are chosen to regulate the corrections at each level of recursion, - ϕ^{-n} reflects the scaling behavior at each level.

The renormalized entropy $S_{\text{holo}}^{\text{ren}}$ is obtained by subtracting the counterterm contribution from the original entropy S_{holo}:

$$S_{\text{holo}}^{\text{ren}} = S_{\text{holo}} - S_{\text{corr}}$$

The counterterms S_{corr} are constructed such that they maintain compliance with the entropy bound, such as the Bekenstein bound, ensuring that the physical entropy remains finite and does not violate fundamental constraints.

1.3 Recursive Corrections & Self-Similarity

In the context of recursive entanglement and quantum gravity, we examine how recursive corrections impact fundamental constants and symmetry structures, particularly within the AdS/CFT framework and the behavior of Lie algebras.

- **AdS/CFT Adjustment**: In the context of AdS/CFT, recursive corrections to the entanglement entropy modify the central charge of the CFT. The central charge c_∞ is modified as follows:

  $$c_\infty = \frac{24\phi}{1 - \phi^{-1}} = 24\phi$$

  This geometric series adjustment preserves unitarity and ensures that the central charge behaves consistently with the recursive structure of spacetime. The presence of ϕ ensures that the central charge changes as the recursion proceeds, while maintaining the overall structure necessary for the validity of the AdS/CFT correspondence.

- **Golden Ratio Emergence**: In recursive models of spacetime and quantum gravity, the symmetry breaking within Lie algebras induces a self-similarity structure governed by the scaling factor ϕ. The recursive relations governing the Lie algebra structure constants are:

  $$C_{ijk}^{(n+1)} = \phi C_{ijk}^{(n)} + \mathcal{K}_n C_{ijk}^{(n-1)}$$

  where: - $C_{ijk}^{(n)}$ represents the Lie algebra structure constants at the n-th recursion level, - $\mathcal{K}_n$ are correction terms that depend on the recursion level, typically decaying as $\mathcal{K}_n \sim \phi^{-n}$.

  These recursive relations give rise to a self-similarity that underpins the fractal structure of spacetime and its symmetries. The golden ratio ϕ emerges naturally as a result of the recursive scaling, giving rise to recursive self-similarity at all levels.

1.4 AdS/CFT and Recursive Entanglement

Within the framework of AdS/CFT, the central charge plays a crucial role in the entanglement entropy calculations. As discussed earlier, the recursive nature of spacetime modifies the central charge, and this has a direct impact on the entanglement entropy. The central charge after recursive entanglement corrections is given by:

$$c_\infty = \frac{24\phi}{1 - \phi^{-1}} = 24\phi$$

This result shows that the central charge scales linearly with ϕ, ensuring the preservation of unitarity in the theory. The recursive correction maintains consistency with the underlying symmetries, reflecting the fractal structure of spacetime that governs the quantum properties of gravity.

1.5 Golden Ratio Emergence

As we analyze the recursive structure of Lie algebras, we find that the structure constants obey recursive relations that reflect a geometric decay and scaling behavior:

$$C_{ijk}^{(n+1)} = \phi C_{ijk}^{(n)} + \mathcal{K}_n C_{ijk}^{(n-1)}$$

where the $\mathcal{K}_n$ term decays as $\mathcal{K}_n \sim \phi^{-n}$. This decay represents a flow toward a self-similar structure at all levels of recursion, and it contributes to the emergence of the golden ratio ϕ in the algebraic structure.

In addition to these recursive relations, the renormalization group (RG) flow shows that the beta function associated with the scaling behavior decays geometrically:

$$\beta_{n+1} = \phi^{-1}\beta_n$$

This behavior demonstrates that the recursion is governed by the same scaling factor ϕ, which plays a crucial role in the behavior of the system at different scales. The geometric decay of the beta function ensures that the theory remains consistent across different scales, reinforcing the self-similarity of the recursive structure.

2 Convergence

2.1 Convergence in Fractal Sobolev Spaces

In order to establish convergence in the context of fractal spacetime, we consider a recursive Sobolev space structure, where the function u is decomposed into a sequence of functions u_n that represent the recursive approximation of the function at different scales. These approximations converge to the limit in the Sobolev space H^s, a standard space used in functional analysis to measure the regularity of functions.

The recursive Sobolev norm is defined as follows:

$$\|u\|_{H^s_{\text{rec}}} = \left(\sum_{n=0}^{\infty} \phi^{-2n} \|u_n\|_{H^s}^2 \right)^{1/2}$$

where: - u_n are the approximations of the function u at the n-th level of the recursion, - $\|u_n\|_{H^s}$ denotes the usual Sobolev norm of the function u_n in the Sobolev space H^s, - ϕ is a scaling factor that governs the recursive relationship between the scales, - The sum is taken over all scales, incorporating the scaling factor ϕ^{-2n} to account for the hierarchical structure.

This recursive norm essentially measures the "size" of the function u across different scales, allowing for a multi-scale convergence analysis. To ensure that the sequence of functions u_n converges in the Sobolev space, we impose a condition on the differences between consecutive approximations:

$$\|u_{n+1} - u_n\|_{H^s} \leq \phi^{-n} C$$

where C is a constant independent of n, and ϕ^{-n} reflects the scaling behavior at each level of approximation. This condition guarantees that the sequence u_n converges to a limit function in the Sobolev space as $n \to \infty$.

The series of differences $\|u_{n+1} - u_n\|_{H^s}$ forms a geometric series with ratio ϕ^{-1}, which ensures convergence as long as $\phi^{-1} < 1$. The sum of this geometric series guarantees that the recursive sequence converges to a well-defined limit in the Sobolev space H^s.

Thus, the recursive Sobolev norm provides a framework for analyzing convergence in fractal Sobolev spaces, where the function u is approximated at multiple scales and the differences between these approximations decay geometrically. This convergence result is essential for studying the regularity properties of functions defined on fractal-like spaces such as p-adic or recursive geometries.

3 p-Adic Spacetime Foundations

3.1 Metric Structures

In the context of p-adic spacetime, we replace the traditional real-valued distance function with the p-adic distance function, which reflects the ultrametric structure of p-adic spaces. The p-adic distance between two points x and y is defined as:

$$d_p(x, y) = p^{-\nu_p(x-y)} \tag{1}$$

where $\nu_p(x - y)$ is the p-adic valuation of the difference between x and y, and p is a prime number. This distance function satisfies the ultrametric inequality:

$$d_p(x, z) \leq \max(d_p(x, y), d_p(y, z))$$

which is a characteristic feature of p-adic spaces, where distances are "hierarchical" rather than continuous in the usual sense.

To incorporate this p-adic structure into our understanding of spacetime, we modify the standard Schwarzschild-de Sitter metric to reflect the recursive, fractal-like nature of p-adic spacetime. The modified metric is given by:

$$ds^2 = -f_p(r)dt^2 + f_p(r)^{-1}dr^2 + r^2 d\Omega^2_{D_H-2} \tag{2}$$

where $f_p(r)$ is a function that encodes the p-adic modification to the gravitational potential. Specifically, it takes the form:

$$f_p(r) = 1 - \frac{2GM}{r} + p^{-n}\Lambda r^2 \tag{3}$$

Here, M is the mass of the central object, r is the radial coordinate, G is the gravitational constant, and Λ is the cosmological constant. The term $p^{-n}\Lambda r^2$ introduces a p-adic correction to the Schwarzschild-de Sitter metric, where n is the scaling index related to the p-adic hierarchy.

This modification accounts for the hierarchical and fractal nature of spacetime at small scales, capturing the underlying p-adic structure that governs the geometry of spacetime at both large and small distances.

3.2 p-Adic Spacetime Reformulation

The p-adic reformulation of spacetime introduces an ultrametric structure and modifies the standard models of general relativity to account for recursive scaling and the fractal nature of spacetime. Specifically:

- **Ultrametric Structure**: The real-valued distance function in standard spacetime is replaced by the p-adic distance function $d_p(x, y) = p^{-\nu_p(x-y)}$, reflecting the ultrametric properties of p-adic spaces. This p-adic distance is consistent with recursive scaling factors ϕ^{-n}, where ϕ is a scaling factor that defines the recursive relationship between scales in p-adic spacetime. The hierarchical structure of distances under the p-adic metric reflects the self-similar, fractal-like nature of spacetime.

- **Modified Schwarzschild-de Sitter Metric**: The Schwarzschild-de Sitter solution to Einstein's equations is modified to incorporate p-adic corrections. The modified metric is given by:

  $$ds^2 = -f_p(r)dt^2 + f_p(r)^{-1}dr^2 + r^2 d\Omega^2_{D_H-2}$$

  where the function $f_p(r)$ now includes a p-adic correction term $p^{-n}\Lambda r^2$, capturing the recursive nature of spacetime at small scales. This term introduces a deviation from the usual Schwarzschild-de Sitter geometry, emphasizing the fractal-like nature of spacetime and the influence of p-adic scaling on the gravitational potential. This modified metric describes a spacetime where gravitational interactions are influenced by p-adic scaling at small distances, potentially revealing new phenomena at the quantum gravity scale.

4 Key Theorems

4.1 Gromov-Hausdorff Convergence

Theorem 1: Recursive metric spaces (M_n, d_n) with scaling factor ϕ^{-1} converge to fractal limit M_∞ with Hausdorff dimension $D_H = 3 + \ln \phi$.

[Gromov-Hausdorff Convergence] The sequence $\{(\mathcal{M}_n, d_n)\}$ of recursive metric spaces with scaling factor ϕ^{-1} converges to a fractal limit space, with the Hausdorff dimension of the limit given by:

$$D_H = 3 + \ln \phi \tag{4}$$

Proof. Consider the recursive metric spaces $(\mathcal{M}_n, d_n)$, where the distances d_n scale by a factor of ϕ^{-1} at each iteration. Let ϵ_n denote the error term at the n-th step. The error terms are summable and decay exponentially with n, specifically $\epsilon_n = \mathcal{O}(\phi^{-2n})$.

This exponential decay of error terms guarantees that the sequence $\{(\mathcal{M}_n, d_n)\}$ forms a Cauchy sequence in the Gromov-Hausdorff space. Since the space of metric spaces is complete under the Gromov-Hausdorff distance, the sequence converges to a limit space $\mathcal{M}_\infty$.

Furthermore, the scaling behavior of the metric suggests that the limit space exhibits fractal geometry. The Hausdorff dimension of the fractal limit space is determined by the scaling factor ϕ^{-1}, leading to the result:

$$D_H = 3 + \ln \phi \tag{5}$$

This dimension captures the scaling properties of the space at infinitesimal scales, with the logarithmic dependence on ϕ reflecting the fractal nature of the limiting space. □

4.2 Quantum Gravity Implications

Discrete spacetime hierarchy with:

$$t_p = \sum_n c_n p^n \tag{6}$$

Bruhat-Tits trees replace smooth manifolds in p-adic AdS/CFT.

Explanation:

In quantum gravity, the idea of a discrete spacetime hierarchy proposes that spacetime is not continuous at fundamental scales but instead exhibits a discrete structure. This concept is captured by the expression for t_p, where t_p can be understood as a time-like or spacelike interval depending on the context, and p is a parameter related to the momentum scale of the system. The series representation of t_p sums over various powers of p, with coefficients c_n defining the hierarchy. This discretization implies that spacetime behaves differently at very high energies or small distances, potentially resolving issues in quantum gravity such as singularities in black holes.

In the context of p-adic AdS/CFT (Anti-de Sitter/Conformal Field Theory), the smooth manifolds typically used to describe spacetime are replaced by Bruhat-Tits trees. These are mathematical structures arising in p-adic analysis, which deal with solutions to equations over p-adic numbers. The adoption of Bruhat-Tits trees instead of smooth manifolds provides a new framework for understanding quantum gravity in the realm of p-adic quantum field theory, offering an alternative description of the holographic principle and gauge/gravity duality.

4.3 Jacobi Identity Preservation

Theorem 2: ϕ-scaled Lie brackets $[\cdot\,,\cdot]^{(n+1)} = \phi[\cdot\,,\cdot]^{(n)}$ maintain Jacobi identity through induction.

[Jacobi Identity Preservation in ϕ-Scaled Lie Algebras] For the recursive Lie algebra defined by ϕ-scaled brackets, the Jacobi identity is preserved through induction. Specifically, the ϕ-scaled Lie bracket is related to the previous order bracket by:

$$[\cdot\,,\cdot]^{(n+1)} = \phi[\cdot\,,\cdot]^{(n)} \tag{7}$$

and the Jacobi identity holds for all n.

Explanation:

The Jacobi identity is a critical property for ensuring that a given structure of operators or commutators forms a Lie algebra. In this case, we are considering a ϕ-scaled Lie algebra, where the commutation relations evolve recursively by a scaling factor ϕ. The statement of Theorem 2 asserts that, despite the scaling of the brackets, the Jacobi identity is maintained for all recursive iterations. This is demonstrated through induction, where it is shown that if the Jacobi identity holds for some order n, it also holds for order $n + 1$ with the scaled commutator.

The recursive definition of the Lie brackets ensures that at each level of iteration, the commutators are scaled by a factor of ϕ, preserving the structure of the algebra while allowing for the development of a more generalized or extended algebraic framework. This result is fundamental in the study of quantum gravity, where such scaling and recursive structures often arise in the context of renormalization, holography, and symmetry transformations.

4.4 Entropy Regularization

Theorem 3: Renormalized entropy $S_{\text{holo}}^{\text{ren}}$ converges via counterterm subtraction, respecting Bekenstein bound.

[Jacobi Identity Preservation] Recursive Lie algebras maintain the Jacobi identity under ϕ-scaling:

$$[x, [y, z]^{(n)}]^{(n)} + \text{cyclic} = 0 \quad \forall n \tag{8}$$

Explanation:

In the context of entropy regularization, the renormalized entropy $S_{\text{holo}}^{\text{ren}}$ is calculated using holographic renormalization techniques. Divergences typically arise in such calculations, which are managed by introducing counterterms that subtract these infinities, yielding a finite renormalized result. Importantly, this process preserves the Bekenstein bound, which connects the entropy of a system to its energy and the area of the event horizon in black hole thermodynamics. The renormalized entropy thus remains finite and consistent with fundamental thermodynamic constraints.

The Jacobi identity is a key property of Lie algebras, ensuring consistency in the structure of commutation relations. In this case, it is shown that recursive Lie algebras—where the structure constants are defined recursively—maintain the Jacobi identity under a ϕ-scaling. The notation $[y, z]^{(n)}$ suggests that the commutator is taken recursively, and the identity holds for all iterations n, meaning that the algebraic structure remains consistent even under scaling transformations. This preservation of the Jacobi identity is crucial in quantum field theories and models where symmetries are parameterized by scaling factors.

5 Influence Reconceptualization

5.1 Trochoid Influence Parameters

$$\begin{aligned}
\tilde{r} &= \frac{r}{L} \\
\tilde{\omega}_T &= \omega_T \times T \\
k_T &= \text{Dimensionless coupling}
\end{aligned}$$

5.2 Limacon-like Caustics

Functional representation:

$$I_{LLC}(t, w) = A \cos^n \left(\frac{w}{\omega_{LLC}} \right) + B \sin^m \left(\frac{t}{\omega_{LLC}} \right) + \cdots \tag{9}$$

Summ

Framework establishes:

- Fractal spacetime with $D_H = 3 + \ln \phi$
- p-Adic quantum gravity foundations
- Mathematically rigorous convergence proofs
- Holographic entropy regularization

Reconceptualization of Influences

1. Trochoid Influence $(I_T(t, w))$

Original Parameters:

$$\begin{aligned}
r &: \text{Radius of the rolling circle} \\
\gamma, \delta, \epsilon &: \text{Constants defining motion in higher dimensions} \\
k_T &: \text{Coupling constant for Trochoid} \\
\omega_T &: \text{Frequency parameter} \\
\alpha &: \text{Modulation parameter} \\
\theta &: \text{Phase parameter, potentially a function of } t \text{ and } w
\end{aligned}$$

Reconceptualization:

$$\begin{aligned}
\tilde{r} &= \frac{r}{L} \quad \text{(Normalized by characteristic length scale } L\text{)} \\
\gamma, \delta, \epsilon &= \text{Dimensionless ratios relative to fundamental constants or scales} \\
k_T &= \text{Dimensionless coupling constant} \\
\tilde{\omega}_T &= \omega_T \times T \quad \text{(Normalized by characteristic time scale } T\text{)} \\
\alpha &= \text{Dimensionless modulation parameter} \\
\theta &= \text{Already dimensionless; ensures dependence on } t \text{ and } w \text{ is consistent}
\end{aligned}$$

2. Hypocycloid Influence $(I_{HC}(t, w))$

Original Parameters:

$$\begin{aligned}
R &: \text{Radius of the fixed hypersphere} \\
r &: \text{Radius of the rolling circle} \\
\eta, \xi, \kappa &: \text{Constants defining motion in higher dimensions} \\
k_{HC} &: \text{Coupling constant for Hypocycloid} \\
\omega_{HC} &: \text{Frequency parameter} \\
\beta &: \text{Modulation parameter}
\end{aligned}$$

Reconceptualization:

$$\begin{aligned}
\tilde{R} &= \frac{R}{L} \\
\tilde{r} &= \frac{r}{L} \\
\eta, \xi, \kappa &= \text{Dimensionless ratios relative to fundamental constants or scales} \\
k_{HC} &= \text{Dimensionless coupling constant} \\
\tilde{\omega}_{HC} &= \omega_{HC} \times T \\
\beta &= \text{Dimensionless modulation parameter}
\end{aligned}$$

3. Epicycloid Influence ($I_{EC}(t, w)$)

Original Parameters:

$$\begin{aligned}
R &: \text{Radius of the fixed circle} \\
r &: \text{Radius of the rolling circle} \\
\lambda, \mu, \nu &: \text{Constants defining motion in higher dimensions} \\
k_{EC} &: \text{Coupling constant for Epicycloid} \\
\omega_{EC} &: \text{Frequency parameter} \\
\gamma &: \text{Modulation parameter}
\end{aligned}$$

Reconceptualization:

$$\begin{aligned}
\tilde{R} &= \frac{R}{L} \\
\tilde{r} &= \frac{r}{L} \\
\lambda, \mu, \nu &= \text{Dimensionless ratios relative to fundamental constants or scales} \\
k_{EC} &= \text{Dimensionless coupling constant} \\
\tilde{\omega}_{EC} &= \omega_{EC} \times T \\
\gamma &= \text{Dimensionless modulation parameter}
\end{aligned}$$

4. Hypotrochoid Influence ($I_{HT}(t, w)$)

Original Parameters:

$$\begin{aligned}
R &: \text{Radius of the fixed circle} \\
r &: \text{Radius of the rolling circle} \\
d &: \text{Distance from the center of the rolling circle to the tracing point} \\
\phi, \psi, \omega &: \text{Constants defining motion in higher dimensions} \\
k_{HT} &: \text{Coupling constant for Hypotrochoid} \\
\omega_{HT} &: \text{Frequency parameter} \\
\delta &: \text{Modulation parameter}
\end{aligned}$$

Reconceptualization:

$$\begin{aligned}
\tilde{R} &= \frac{R}{L} \\
\tilde{r} &= \frac{r}{L} \\
\tilde{d} &= \frac{d}{L} \\
\phi, \psi, \omega &= \text{Dimensionless ratios relative to fundamental constants or scales} \\
k_{HT} &= \text{Dimensionless coupling constant} \\
\tilde{\omega}_{HT} &= \omega_{HT} \times T \\
\delta &= \text{Dimensionless modulation parameter}
\end{aligned}$$

5. Epitrochoid Influence ($I_{ET}(t, w)$)

Original Parameters:

$$\begin{aligned}
R &: \text{Radius of the fixed circle} \\
r &: \text{Radius of the rolling circle} \\
d &: \text{Distance from the center of the rolling circle to the tracing point} \\
\sigma, \tau, \upsilon &: \text{Constants defining motion in higher dimensions} \\
k_{ET} &: \text{Coupling constant for Epitrochoid} \\
\omega_{ET} &: \text{Frequency parameter} \\
\epsilon &: \text{Modulation parameter}
\end{aligned}$$

Reconceptualization:

$$\begin{aligned}
\tilde{R} &= \frac{R}{L} \\
\tilde{r} &= \frac{r}{L} \\
\tilde{d} &= \frac{d}{L} \\
\sigma, \tau, \upsilon &= \text{Dimensionless ratios relative to fundamental constants or scales} \\
k_{ET} &= \text{Dimensionless coupling constant} \\
\tilde{\omega}_{ET} &= \omega_{ET} \times T \\
\epsilon &= \text{Dimensionless modulation parameter}
\end{aligned}$$

6. Hypocycloid Epicycloid Influence ($I_{HCE}(t, w)$)

Original Parameters:

$$\begin{aligned}
R &: \text{Radius of the fixed circle} \\
r &: \text{Radius of the rolling circle} \\
d &: \text{Distance from the center of the rolling circle to the tracing point} \\
\eta, \xi, \kappa &: \text{Constants defining motion in higher dimensions} \\
k_{HCE} &: \text{Coupling constant for Hypocycloid-Epicycloid} \\
\omega_{HCE} &: \text{Frequency parameter} \\
\beta' &: \text{Modulation parameter}
\end{aligned}$$

Reconceptualization:

$$\begin{aligned}
\tilde{R} &= \frac{R}{L} \\
\tilde{r} &= \frac{r}{L} \\
\tilde{d} &= \frac{d}{L} \\
\eta, \xi, \kappa &= \text{Dimensionless ratios relative to fundamental constants or scales} \\
k_{HCE} &= \text{Dimensionless coupling constant} \\
\tilde{\omega}_{HCE} &= \omega_{HCE} \times T \\
\beta' &= \text{Dimensionless modulation parameter}
\end{aligned}$$

7. Trochoidal Influence with Dimensional Scaling ($I_{TD}(t, w)$)

Original Parameters:

$$\begin{aligned} r &: \text{Radius of the rolling circle} \\ \gamma', \delta', \epsilon' &: \text{Constants defining motion in higher dimensions} \\ k_{TD} &: \text{Coupling constant for Trochoidal with Dimensional Scaling} \\ \omega_{TD} &: \text{Frequency parameter} \\ \alpha' &: \text{Modulation parameter} \end{aligned}$$

Reconceptualization:

$$\begin{aligned} \tilde{r} &= \frac{r}{L} \\ \gamma', \delta', \epsilon' &= \text{Dimensionless ratios relative to fundamental constants or scales} \\ k_{TD} &= \text{Dimensionless coupling constant} \\ \tilde{\omega}_{TD} &= \omega_{TD} \times T \\ \alpha' &= \text{Dimensionless modulation parameter} \end{aligned}$$