Let $\pi : E \to \mathbb{P}_1$ be the Legendre scheme, i.e. the elliptic surface associated to the following elliptic curve, defined over the function field $\mathbb{C}(\lambda)$ of $\mathbb{P}_1$:

$$y^2 = x(x-1)(x-\lambda).$$

Theorem 0.1. *Let $\sigma : B \to E_B := E \times_{(\pi,r)} B$ be an algebraic section of π, where $r : B \to \mathbb{P}_1$ is a finite morphism, and B is a smooth complete complex curve. Let $S = \{0, 1, \infty\} \subset \mathbb{P}_1(\mathbb{C})$ denote the points of bad reduction for π. Then the following equality holds:*

$$\hat{h}(\sigma) = \int_{B \setminus r^{-1}(S)} \sigma^*(\,\mathrm{d}\beta_1 \wedge \mathrm{d}\beta_2), \tag{0.1}$$

where (β_1, β_2) is a (local branch) of the Betti map (as defined in [CMZ17, Section 1.1], following the notation introduced by D. Bertrand) on E. [1]

Before coming to the proof, we recall the following Lemma, of which a proof can be found in [BW14]:

Lemma 0.2. *Let $X \subset \mathbb{R}^k$ be a bounded definable subset of $\mathbb{R}^k$ in an o-minimal theory. Let, for each $n > 0$:*

$$\alpha_n := \left\{ \left(\frac{a_1}{n}, \dots, \frac{a_k}{n}\right) \in X : \ a_1, \dots, a_k \in \mathbb{Z} \right\}.$$

Then, the limit $\lim_{n\to\infty} n^{-k}\alpha_n$ exists and it is equal to $\mu(X)$, the Lebesgue measure of X.

Proof of Theorem 0.1. Let us first consider the case where σ is torsion. In this case, left and right hand side of (0.1) are both equal to 0, hence there is nothing to prove.

We restrict now to the case where σ is not torsion, and consider, for each $n \geq 1$, the following quantity:

$$A_n := \#\{t \in B(\mathbb{C}) \setminus r^{-1}(S) : \sigma(t) \text{ is } n-\text{torsion in } E_t\}. \tag{0.2}$$

We notice that, since σ is not torsion, this quantity is finite for each $n \geq 1$. We claim the following (which obviously implies the thesis):

1. The limit $\lim_{n\to\infty} \frac{A_n}{n^2}$ exists, is finite, and $\lim_{n\to\infty} \frac{A_n}{n^2} = \hat{h}(\sigma)$;
2. The limit $\lim_{n\to\infty} \frac{A_n}{n^2} = \int_{E \setminus r^{-1}(S)} \sigma^*(\,\mathrm{d}\beta_1 \wedge \mathrm{d}\beta_2)$.

[1] We notice that, although for the Betti map (β_1, β_2) to be well-defined, one needs to restrict oneself to a simply connected fundamental domain, the $(1,1)$-form $\mathrm{d}\beta_1 \wedge \mathrm{d}\beta_2$ is well-defined on $E_B \setminus r^{-1}(\mathcal{S})$.

Let us first prove (1). We know that (see e.g. [BSW13, Sections 2,3]):

$$\hat{h}(\sigma) = \lim_{n\to\infty} \frac{< n\sigma, O >}{n^2},$$

where $n\sigma$ denotes, with a slight abuse of notation, the graph of the section of $n\sigma$, O denotes the zero section of π, and $< n\sigma, O >$ denotes the intersection product in a smooth proper model of E_B. We write:

$$< n\sigma, O >= A_n + \delta_n + s_n,$$

where s_n denotes the intersection of $n\sigma$ and O on the singular fibers of $\pi : E_B \to B$, and δ_n is a correction term that keeps track of the intersection that happens with multiplicity greater than 1. I.e.:

$$\delta_n = \sum_{t\in B\setminus r^{-1}(S)} (\mathrm{ord}_t(n\sigma) - 1),$$

where $\mathrm{ord}_t(n\sigma)$ denotes the multiplicity of intersection of $n\sigma$ and O at t. We will now prove that $\delta_n + s_n = O(1)$, as $n \to \infty$. The fact that $s_n = O(1)$ follows from an analysis of the local intersection numbers on the singular fibers using the formal group law structure[...].

We prove now that δ_n is uniformly bounded (for any $n > 0$). We notice that, in order for the intersection of $n\sigma$ and O to be of order greater than 1 at a certain point $t \in B$, the differential of (a local branch of) the Betti map $(b_1, b_2) \circ \sigma$ would have to be 0 at the point t. As proven by P. Corvaja and U. Zannier in [...], this happens only for a finite amount of base points t in the base. We denote them by $t_1, \dots, t_k$. For each $i = 1, \dots, k$, let λ_i denote a uniformizer for $t_i \in \mathbb{P}_1(\mathbb{C})$, and let ρ_1^i, ρ_2^i denote a local choice (in a neighbourhood of t_i) of periods for the elliptic logarithm (see e.g. [CMZ17, Section 1.1]). We have then that (by standard intersection theory on complex surfaces, see e.g. [Bea96, Chapter I]):

$$\mathrm{ord}_{t_i}(n\sigma) = \dim_{\mathbb{C}} \mathbb{C}\{\lambda_i\} \big/ (n\sigma(\lambda_i) - nb_1\rho_1^i(\lambda_i) - nb_2\rho_2^i(\lambda_i)), \tag{0.3}$$

where $\mathbb{C}\{\lambda_i\}$ denotes the ring of locally analytic functions in the variable λ_i, and $b_1, b_2 \in \mathbb{C}$ are defined through the following condition:

$$\sigma(t_i) = b_1\rho_1^i(t_i) + b_2\rho_2^i(t_i).$$

Since the ideal $(n\sigma(\lambda_i) - nb_1\rho_1^i(\lambda_i) - nb_2\rho_2^i(\lambda_i)) \subset \mathbb{C}\{\lambda\}$ does not depend on $n > 0$, we have that the right hand side of (0.3) does not depend on $n > 0$, and we denote this quantity by R_i. As an immediate consequence of the above argument, we have that:

$$\delta_n \leq R_1 + \dots + R_k - k,$$

from which we conclude that $\delta_n = O(1) = o(n^2)$, thus proving point (1).

We prove now point (2). We use the fact that the Betti map is (locally) definable and bounded (by definable, we will always mean definable in $(\mathbb{R}_{an,exp})$). Namely, it is proven in [JS17, Section 10] that there exists a (definable) partition $B \setminus r^{-1}(S) = Y_1 \cup \dots Y_l$, and, on each Y_j there exists a well defined branch of the Betti map, which we will denote by $B^j = (\beta_1^j, \beta_2^j) \circ \sigma : Y_j \to \mathbb{R}^2$, such that B_j is definable, and in [JS17, Proposition 4] that it is bounded. Let us consider now, for each $j = 1, \dots, l$, the following definable set in $\mathbb{R}^2 \times \mathbb{R}^2$:

$$X_j \coloneqq \{(y, t) \in \mathbb{R}^2 \times \mathbb{R}^2 : t \in Y_j \text{ and } y = B_j(t)\},$$

i.e. X_j is the transpose of the graph of B_j. Applying Hardt's Theorem ([VdDCH98, Theorem 9.1.2]) to X_j, we know that there exists a finite partition of $\mathbb{R}^2$, say $A_j^1 \cup \dots \cup A_j^{d(j)} = \mathbb{R}^2$, such that X_j is definably trivial over each A_j^m. We recall that this means that, for each $m \leq d(j)$, there exists a definable set F_j^m, and a (definable) isomorphism $h_{A_j^m} : X_j \cap (A_j^m \times \mathbb{R}^2) \to A_j^m \times F_j^m$, that commutes with the projection to A_j^m. Morover, since the Betti map is bounded on each Y_j, we may assume that, for each $m \leq d(j)$, either A_j^m is bounded, or $X_j \cap (A_j^m \times \mathbb{R}^2) = \emptyset$. We define now, for each $j = 1, \dots, l$, $m \leq d(j)$:

$$f_j^m \coloneqq \# F_j^m \tag{0.4}$$

We notice that f_i^j is always finite. In fact, this is a direct consequence of the fact that the fibers of the Betti map B_j are isolated points (see [CMZ17, Proposition 1.1]). Then, the following equality holds:

$$\sum_{m \leq d(j)} f_j^m \mu(A_j^m) = \int_{B \setminus r^{-1}(S)} \sigma^*(\mathrm{d}\beta_1 \wedge \mathrm{d}\beta_2). \tag{0.5}$$

In fact, notice that both left and right hand side of (0.5) are equal to the measure of the graph of the section $\sigma : \mathbb{P}_1(\mathbb{C}) \setminus S \to E \setminus \mathcal{S}_B$, given by the integration of the restriction of the $(1,1)$-form $\mathrm{d}\beta_1 \wedge \mathrm{d}\beta_2 \in \Omega^2 E \setminus \mathcal{S}_B$.

We notice now that, if $t \in \mathbb{P}_1(\mathbb{C}) \setminus S$, $\sigma(t) \in E_t$ is n-torsion if and only if the Betti coordinates $(\beta_1(\sigma(t)), \beta_2(\sigma(t)))$ are rational with denominator dividing n. Hence, the following equality holds, for each $n > 0$:

$$A_n = \sum_{m \leq d(j)} f_j^m \alpha_{n,j}^m, \tag{0.6}$$

where:

$$\alpha_{n,j}^m \coloneqq \left\{ \left(\frac{a}{n}, \frac{b}{n}\right) \in A_j^m : \ a, b \in \mathbb{Z} \right\}.$$

Hence:

$$\frac{A_n}{n^2} = \sum_{m \le d(j)} \frac{\alpha^m_{n,j}}{n^2},$$

Letting $n \to \infty$, the thesis follows from (0.5), (0.6) and Lemma 0.2. □

Corollary 0.3. *Let $\sigma : B \to E_B$ denote a section of the elliptic surface $E_B \to B$. Then the integral $\int_{B \setminus r^{-1}(S)} \sigma^*(\mathrm{d}\beta_1 \wedge \mathrm{d}\beta_2)$ has a rational value.*

Proof. We know by [SS09, Section 11.8] that $\hat{h}(\sigma) \in \mathbb{Q}$. Hence, the thesis is an immediate consequence of Proposition 0.1. □

We give here another expression of the $(1,1)$-form $\mathrm{d}\beta_1 \wedge \mathrm{d}\beta_2 \in \Omega^2(E \setminus \mathcal{S})$ that might be more suitable for calculations. To fix some notation, we assume that a local choice of periods $\rho_1(\lambda), \rho_2(\lambda)$, such that $< \rho_1(\lambda), \rho_2(\lambda) >= \Lambda_\lambda$, where the latter denotes the unique lattice in $\mathbb{C}$ corresponding to the elliptic curve E_λ, defined through the Weierstrass form $y^2 = x(x-1)(x-\lambda)$. Let $d(\lambda) := \rho_1(\lambda)\overline{\rho_2(\lambda)} - \rho_2(\lambda)\overline{\rho_1(\lambda)} = 2iV(\lambda)$, where $V(\lambda)$ denotes the (oriented) area of the fundamental domain of Λ_λ. Let $\eta_\lambda(\zeta)$ denote the extension to $\mathbb{C}$ of the function η (as defined in [Sil94, VI.3.1]), defined on the lattice Λ_λ. Then, the following equality holds:

$$\mathrm{d}\beta_1(\lambda) \wedge \mathrm{d}\beta_2(\lambda) = \frac{1}{d(\lambda)} \left(\mathrm{d}z \wedge \mathrm{d}\bar{z} + \frac{1}{2\lambda}(\eta \,\mathrm{d}\lambda \wedge \mathrm{d}\bar{z} + \bar{\eta}\,\mathrm{d}z \wedge \mathrm{d}\bar{\lambda}) + \frac{1}{4\lambda^2}\,\mathrm{d}\lambda \wedge \mathrm{d}\bar{\lambda} \right). \tag{0.7}$$

Example 0.4. Let us do an explicit computation of the terms of 0.1, in the case of a specific section, for instance:

$$\sigma(\lambda) = (2, \sqrt{2(2-\lambda)}).$$

This section is not defined over the base curve $\mathbb{P}_1$. It is, however, well defined as a section of the elliptic surface $E_B := E \times_{(\pi,\varphi)} B \to B$, where $B \cong \mathbb{P}_1$, and $\varphi(t) = t^2 + 2$.

One can compute $\hat{h}(\sigma)$ explicitly, by using the intersection product on a proper regular model of E_B (see [SS09, Section 11.8])[2]. A simple application of Tate's algorithm reveals that the fibration $\pi : E_B \to B$ has 5 singular fibers, four of which are of type I_2 (the ones over $\lambda = 0, 1$) and one of type I_4 (the one over $\lambda = \infty$). Looking at the intersection of the section σ with the singular fibers reveals that the canonical height $\hat{h}(\sigma) = \frac{11}{8}$.

[2]Our normalization for the height function differs by a multiplicative factor of $\frac{1}{2}$ from that of [SS09].

Using 0.7, the $(1,1)$-form $\sigma^*(\mathrm{d}\beta_1 \wedge \mathrm{d}\beta_2)$ is equal to the following:

$$\mathrm{d}\beta_1(\lambda) \wedge \mathrm{d}\beta_2(\lambda) = \tag{0.8}$$

$$\frac{1}{d(\lambda)}\left(\mathrm{d}z(\lambda) \wedge \mathrm{d}\overline{z(\lambda)} + \frac{1}{2\lambda}(\eta\,\mathrm{d}\lambda \wedge \mathrm{d}\overline{z(\lambda)} + \overline{\eta}\,\mathrm{d}z(\lambda) \wedge \mathrm{d}\overline{\lambda}) + \frac{1}{4\lambda^2}\,\mathrm{d}\lambda \wedge \mathrm{d}\overline{\lambda}\right). \tag{0.9}$$

Here $z(\lambda) = \int_\infty^2 dx/\sqrt{x(x-1)(x-\lambda)}$ (the determination chosen for the $\sqrt{}$ is irrelevant). Hence, we obtain the following integral identity from Theorem 0.1:

$$\int_{\lambda \in \mathbb{C}} \mathrm{d}\beta_1(\lambda) \wedge \mathrm{d}\beta_2(\lambda) = \frac{1}{2}^{3} \cdot \frac{11}{8} = \frac{11}{16}.$$

Comparison with a measure coming from dynamics Let us cite the following result of Laura de Marco and Myrto Mavraki from [DM17, Section 3]:

Theorem 0.5. *Let $\pi : E \to B$ be an elliptic surface and $P : B \to E$ a non-torsion section, both defined over $\mathbb{Q}$. Let $S \subset E$ be the union of the finitely many singular fibers in E. There is a positive, closed $(1,1)$-current T on $E(\mathbb{C}) \setminus S$ with locally continuous potentials so that $T|_{E_t}$ is the Haar measure on each smooth fiber, and P^*T is equal to a measure μ_P, that satisfy the following property. For any infinite non-repeating sequence of $t_n \in B(\bar{\mathbb{Q}})$, such that $\hat{h}_{E_{t_n}}(P_{t_n}) \to 0$ as $n \to \infty$, the discrete measures*

$$\frac{1}{\#\,\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})t_n} \sum_{t \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})t_n} \delta_{t_n}$$

converge weakly on $B(\mathbb{C})$ to μ_P.

The restriction of the current T to the open set $E \setminus (\mathcal{S} \cup O)$ (where we are denoting by O, with a slight abuse of notation, the image of the zero section of $\pi : E \to B$), can be written down explicitly as follows:

$$T = \mathrm{d}\,\mathrm{d}^c\lambda, \tag{0.10}$$

where λ denotes the local (archimedean) height function. We recall that the following formula holds [Sil94, p. 466]:

$$\lambda = -\log|e^{-\frac{1}{2}z\eta_l(z)}\sigma_l(z)\Delta(\Lambda_l)^{\frac{1}{12}}|. \tag{0.11}$$

Here Λ_l denotes a lattice in $\mathbb{C}$ such that $\mathbb{C}/\Lambda_l \cong E_l := \pi^{-1}(l)$, z denotes the complex variable of $\mathbb{C}/\Lambda_l$ and η_l and σ_l indicate the elliptic functions η and, resp., σ (as

[3] The coefficient $\frac{1}{2}$ is needed here because we should actually integrate not on the parameter λ, but on the parameter $\mu := \sqrt{2-\lambda}$.

defined in [Sil94, VI.3.1,I.5.4]) associated to the lattice Λ_l. Since $\sigma_l(z)\Delta(\Lambda_l)^{\frac{1}{12}}$ is a holomorphic function in both variables l and z, this gives the following expression for T:

$$T = \frac{1}{2}\,\mathrm{d}\,\mathrm{d}^c(\Re(z\eta_\lambda(z))). \tag{0.12}$$

We expect that the current T is equal to $(1,1)$-current defined by the $(1,1)$-form $\mathrm{d}\beta_1 \wedge \mathrm{d}\beta_2$ on $E \setminus \mathcal{S}$. In fact, we know that both currents restrict to the Haar measure on the fibers.

... If one chooses local coordinates z, λ on the surface E, such that λ is just a (local) coordinate on the base, and z is the complex coordinate on the fiber (i.e. the value, up to periods, of the elliptic integral on the fibers), then one can (locally) decompose every $(1,1)$-form in the following way:

$$\omega = \alpha_{z,\bar{z}}\,\mathrm{d}z \wedge \mathrm{d}\bar{z} + \alpha_{\bar{z},\lambda}\,\mathrm{d}\bar{z} \wedge \mathrm{d}\lambda + \alpha_{z,\bar{\lambda}}\,\mathrm{d}z \wedge \mathrm{d}\bar{\lambda} + \alpha_{\lambda,\bar{\lambda}}\,\mathrm{d}\lambda \wedge \mathrm{d}\bar{\lambda}.$$

If we let ω be $(1,1)$-form $T - \mathrm{d}\beta_1 \wedge \mathrm{d}\beta_2$, we know that the coefficient $\alpha_{z,\bar{z}}$ is 0 (since it is so when restricted to the fibers). One might expect that the remaining pieces are, for instance, not closed for the operator d or d^c (while, $T - \mathrm{d}\beta_1 \wedge \mathrm{d}\beta_2$ is so). ...

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