Elliptic fibrations and the Hilbert Property

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Abstract

For a number field K, an algebraic variety X/K is said to have the Hilbert Property if $X(K)$ is not thin. We are going to describe some examples of algebraic varieties, for which the Hilbert Property is a new result.

The first class of examples is that of smooth cubic hypersurfaces with a K-rational point in $\mathbb{P}_n/K$, for $n \geq 3$. These fall in the class of unirational varieties, for which the Hilbert Property was conjectured by Colliot-Thélène and Sansuc.

The second class is that of simply connected algebraic surfaces endowed with two elliptic fibrations, subject to a technical condition. This result generalizes earlier work on diagonal quartics in $\mathbb{P}_3$. In this family of examples we find some instances of K3 surfaces for which the Hilbert Property is a new result. Among these we also find some Kummer surfaces, for which the Hilbert Property was suggested to be true by Corvaja and Zannier.

1 Introduction

This paper will be concerned with providing some examples of varieties with the *Hilbert Property*, concerning the set of k-rational points $X(k)$, for an algebraic variety X over a field k.

A geometrically irreducible variety X over a field k is said to have the *Hilbert Property* if, for any finite morphism $\pi : E \to X$, such that $X(k) \setminus \pi(E(k))$ is not Zariski-dense in X, there exists a rational section of π (see [9, Ch. 3] for an introduction of the Hilbert Property).

Motivation for the study of the Hilbert Property comes from the following conjecture, of which a proof would settle the Inverse Galois Problem (as noted in [3]):

Conjecture 1.1 (Colliot-Thélène, Sansuc)**.** *Let X/k be a unirational variety over a number field, then X has the Hilbert Property.*

The results of this paper all concern the proof of the Hilbert Property for some specific classes of varieties, which are characterized by the presence of multiple elliptic fibrations.

The first result, in Section ... is the Hilbert Property for smooth cubic hypersurfaces, of dim ≥ 2 over a number field K with at least one K-rational point. Since, under these hypothesis, cubic hypersurfaces are K-unirational ([8, ...]), this result gives positive examples of Conjecture 1.1. We then turn, in Section ..., onto giving explicit examples of K3 surfaces with the Hilbert Property. This examples are produced starting from a construction presented in [**?**], by ... .

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2 Background

We recall in this section some standard facts and known theorems, that we are going to use in later sections. ...

Notation Throughout this paper, except when otherwise stated, k denotes a perfect field and K a number field. A (k-)*variety* is an algebraic quasi-projective variety (defined over the field k), not necessarily irreducible or reduced. Unless specified otherwise, we will always work with the Zariski topology.

Given a morphism $f : X \to Y$ between k-varieties, and a point $s : \mathrm{Spec}(k(s)) \to Y$, we denote by $f^{-1}(s)$ the scheme-theoretic fibered product $\mathrm{Spec}(k(s)) \times_Y X$, and call it the *fiber* of f in s. Hence, with our notation, this is not necessarily reduced. ...=c'è bisogno?

Cubic hypersurfaces Cubic hypersurfaces are hypersurfaces in $\mathbb{P}_n$ defined by a cubic homogeneous polynomial. We recall the following:

Theorem 2.1. *...*

...

Hilbert Property For a more detailed exposition of the basic theory of the Hilbert Property and thin sets we refer the interested reader to [9, Ch. 3]. We limit ourselves here to recalling some basic definitions and properties.

Definition 2.2. Let X be a geometrically integral variety, defined over a field k. A *thin* subset $S \subset X(k)$ is any set contained in a union $D(k) \cup \bigcup_{i=1,\ldots,n} \pi_i(E_i(k))$, where $D \subsetneq X$ is a subvariety, the E_i's are irreducible varieties and $\pi_i : E_i \to X$ are generically finite morphisms of degree > 1.

Remark 2.3. A k-variety X has the Hilbert Property if and only if $X(k)$ is not thin.

The following proposition summarizes some basic properties of the Hilbert Property.

Proposition 2.4. *Let k be a perfect field, and X be a geometrically irreducible k-variety.*

(i) If X has the Hilbert Property and Y is a k-variety birational to X, then Y has the Hilbert Property.

(ii) If X is a rational variety, and k is a number field, then X has the Hilbert Property.

(iii) If X has the Hilbert Property, and L/k is a finite extension, then X_L has the Hilbert Property.

Proof. (i) is an immediate consequence of Remark 2.3. It follows from (i) that, in order to prove (ii), it suffices to prove that $\mathbb{A}_n/k$ has the Hilbert Property. This is a consequence of the Hilbert Irreducibility Theorem. We refer the reader to [9, Ch. 3] for the details, and a proof of (iii). □

The examples in this paper are all derived from the following two general theorems, which are, respectively, Theorem 1.1 of [1] and Theorem ... of [5].

Theorem 2.5 (Bary-Soroker, Fehm, Petersen)**.** *Let $f : X \to S$ be a morphism of K-varieties. Suppose that S/K has the Hilbert Property and that there exists a subset $A \subset S(K)$ not thin such that for each $s \in A$, $f^{-1}(s)$ has the HP. Then X/K has the HP.*

Proof. See [1, Theorem 1.1]. □

Definition 2.6. Let $\mathcal{E}/K$ be a normal algebraic surface, and let $\pi : \mathcal{E} \to \mathbb{P}_1$ be a morphism. We say that π is an elliptic fibration, if its generic fiber is a smooth, geometrically irreducible, genus 1 curve.

Theorem 2.7. *Let K be a number field, and E be a simply connected algebraic surface, endowed with two elliptic fibrations $\pi_i : E \to \mathbb{P}_1/K$, $i = 1, 2$. Suppose that the following hold:*

(a) The K-rational points $E(K)$ are Zariski-dense in E;

(b) Let $\eta_1 \cong \operatorname{Spec} K(\lambda)$ be the generic point of the codomain of π_1. All the diramation points (i.e. the images of the ramification points) of the morphism $\pi_2|_{\pi_1^{-1}(\eta_1)}$ are non-constant in λ, and the same holds upon inverting π_1 and π_2.

Then the surface E/K has the Hilbert Property.

Proof. See [5]. ... □

3 Hilbert Property for cubic hypersurfaces

Theorem 3.1. *Let $X \subset \mathbb{P}_n/K$, $n \geq 3$ be a smooth cubic hypersurface, with a K-rational point. Then X has the Hilbert Property.*

In this section our base field will always be assumed to be a number field K.

We need the following lemma, of which an explicitly computable version can be found in [**?**], and its corollary.

Lemma 3.2. *Let $\pi : \mathcal{E} \to \mathbb{P}_1$ be an elliptic fibration, defined over a number field K. Then, there exists an open Zariski subset $U_\pi \subset \mathcal{E}$ such that, for any $P \in U_\pi(K)$, $\pi^{-1}(\pi(P))$ is smooth and $\#\pi^{-1}(\pi(P))(K) = \infty$.*

Corollary 3.3. *Let $\mathcal{E}/K$ be a smooth projective algebraic surface, with two elliptic fibrations $\pi_1, \pi_2 : \mathcal{E} \to \mathbb{P}_1$, such that the morphism $(\pi_1, \pi_2) : \mathcal{E} \to \mathbb{P}_1 \times \mathbb{P}_1$ is finite. Let U_1, U_2 be as in Lemma 3.2, for $(\mathcal{E}, \pi) = (\mathcal{E}, \pi_1)$, $(\mathcal{E}, \pi_2)$ respectively. Then, if $(U_1 \cap U_2)(K) \neq \emptyset$, the K-rational points of $\mathcal{E}$ are Zariski-dense.*

Proof. See [5, ...]. □

Proof of Theorem 3.1. Before all, let us remark that X is K-unirational by Theorem 2.1, in particular it has Zariski-dense K-rational points.

Keeping the notation of 3.1, we prove the result by induction on n.

Case $n = 3$.

We assume by contradiction that X does not have the HP. Then there exist irreducible covers $\varphi_i : Y_i \to X$, $i = 1, \dots, m$ of degree $\deg \varphi_i > 1$ and a divisor $D \subset X$ such that $X(K) \subset \cup_i \varphi_i(Y_i(K)) \cup D(K)$. We may assume, without loss of generality, that the Y_i's are normal and geometrically integral[1], and that the φ_i's are finite morphisms. Let us denote now by D_i the diramation divisor (see Remark **??**) of φ_i. By Lefschetz' hyperplane Theorem, X is simply connected, hence we know that the D_i's are nonempty for each $i = 1, \dots, m$.

Let us denote by $\mathbb{A}_4^*$ the dual affine space of $\mathbb{A}_4$, minus the origin. To each element of $\mathbb{A}_4^*$ corresponds a hyperplane of $\mathbb{P}_3$ in a canonical way[2]. Let $(H_1, H_2) \in \mathbb{A}_4^* \times \mathbb{A}_4^*$ be such that

[1]In fact, possibly by enlarging D, one can substitute Y_i with its normalization. A normal variety that is not geometrically integral over the base field K does not have any K-rational points.

[2]Namely, if $\lambda \in \mathbb{A}_4^*$, the associated hyperplane is $\{\mathbf{x} \in \mathbb{P}_3 \mid \lambda(\mathbf{x}) = 0\}$.

1. $H_1 \cap H_2 \cap X$ is (a scheme consisting of) three distinct points (and hence, as a direct consequence, $H_1 \cap H_2$ is a line), and it is disjoint from the union of the D_i's;

2. $H_1 \cap X$ and $H_2 \cap X$ are smooth curves;

3. The morphism $[H_1 : H_2] : X \setminus H_1 \cap H_2 \to \mathbb{P}_1$ is non-constant on each of the irreducible components of the D_i's.

We note that, since all conditions are open and non-empty, such a couple (H_1, H_2) always exists.

Let $\{P_1, P_2, P_3\}$ be the intersection $H_1 \cap H_2 \cap X$, and let $\pi : X \setminus \{P_1, P_2, P_3\} \to \mathbb{P}_1$ be the following morphism:

$$P \longmapsto [H_1(P) : H_2(P)].$$

The map π extends naturally to a morphism $\hat{\pi} : \hat{X} \to \mathbb{P}_1$, where $\hat{X} = \mathrm{Bl}_{P_1+P_2+P_3} X$ denotes the blowup of X in the (smooth) subscheme $P_1+P_2+P_3 \subset X$. We note that, since X is a cubic surface the morphism $\hat{\pi}$ is an elliptic fibration.

We claim now that the morphisms $\pi \circ \varphi_i : Y_i \setminus \varphi_i^{-1}(\{P_1, P_2, P_3\}) \to \mathbb{P}_1$ have geometrically integral generic fiber for each $i = 1, \dots, m$. In fact, let us assume by contradiction that there exists an $i \in \{1, \dots, m\}$ such that the morphism $\pi \circ \varphi_i$ has a geometrically reducible generic fiber. Then, by the existence of the relative normal factorization (Theorem **??**), there exist a (smooth, geometrically integral) curve C, and morphisms $\pi' : Y_i \setminus \varphi_i^{-1}(\{P_1, P_2, P_3\}) \to C$ and $r : C \to \mathbb{P}_1$ such that $\pi \circ \varphi_i = r \circ \pi'$, and $\deg r > 1$. This, in turn, gives rise to a morphism $\varphi_i' : Y_i \setminus \varphi_i^{-1}(\{P_1, P_2, P_3\}) \to X \setminus \{P_1, P_2, P_3\} \times_{\mathbb{P}_1} C$, and a factorization $\varphi_i = \alpha \circ \varphi_i'$, where $\alpha : X \setminus \{P_1, P_2, P_3\} \times_{\mathbb{P}_1} C \to X \setminus \{P_1, P_2, P_3\}$ denotes the first projection morphism.

Hence, the diramation of φ_i would contain the diramation of α, which is nonempty (since $\mathbb{P}_1$ is simply connected) and contained all in the fibers of π (by Theorem **??**). This contradicts our choice of (H_1, H_2).

Let us now denote by $\hat{Y}_i$ the desingularization of $Y_i' = Y_i \times_X \hat{X}$, and by $\psi_i : \hat{Y}_i \to \mathbb{P}_1$ the composition of the desingularization morphism $\hat{Y}_i \to Y_i'$, the projection $Y_i' \to \hat{X}$ and the map $\hat{\pi} : \hat{X} \to \mathbb{P}_1$. By the Theorem of generic smoothness [7, Corollary 10.7] we know that there exists an open subset $V_i \subset \mathbb{P}_1$ such that, for each $t \in V_i(\overline{K})$, $\psi_i^{-1}(t)$ is smooth (and we may assume, by further restricting V_i, irreducible as well, beacuse ψ_i has geometrically irreducible generic fiber). Let us denote now by D' the proper subscheme $D' \subset \hat{X}$, which is the union of all the following:

1. the fibers $\hat{\pi}^{-1}(x)$, for each $x \notin V_i$, for some $i = 1, \dots, m$;

2. the proper transform of $D \subset X$, and the exceptional locus of $\hat{X} \to X$;

3. the proper transform of $X \setminus U$, where U is defined as in Lemma 3.2 for $(\mathcal{E}, \pi) = (\hat{X}, \hat{\pi})$.

Let us now choose a K-rational point $P \in (\hat{X} \setminus D')(K)$, and let us denote by E_P the fiber $\hat{\pi}^{-1}(\hat{\pi}(P))$. We know, by Lemma 3.2, that E_P has infinitely many K-rational points. We have assumed, however, that $X(K) \subset \cup_i \varphi_i(Y_i(K)) \cup D(K)$, from which follows that $\hat{X}(K) \subset \cup_i \varphi'_i(Y_i(K)) \cup D'(K)$, and hence

$$E_P(K) \subset \cup_i \psi_i^{-1}(\hat{\pi}(P))(K) \cup (E_P \cap D')(K). \tag{3.1}$$

We claim that the right hand side of 3.1 is finite. In fact, for each $i = 1, \dots, m$, the morphism $\psi_i^{-1}(\hat{\pi}(P)) \to E_P$ is ramified by Proposition **??**, and, since the curve $\psi_i^{-1}(\hat{\pi}(P))$ is a smooth curve, it is of genus > 1. As a consequence, $\psi_i^{-1}(\hat{\pi}(P))(K)$ is finite for each $i = 1, \dots, m$ by Falting's theorem. Moreover, $(E_P \cap D')$ is obviously finite, hence we have proved that the right hand side of 3.1 is finite. As we noted before, however, $E_P(K)$ is infinite, hence we have reached an absurd, proving the theorem in the case $n = 3$.

Case $n \geq 4$.

By Bertini's theorem [7, Remark 10.9.2], we know that there exists a Zariski-open subset $U \subset X$ such that, for each $P \in U(\overline{K})$, the generic hyperplane of $\mathbb{P}_n$ passing through P cuts X in a smooth irreducible cubic of dimension $n-2$.

We choose now a K-rational point $P \in U(K)$, and two K-rational (distinct) hyperplanes H_0, H_∞ passing through P such that $H_0 \cap X$ is smooth.

Let us consider now the following morphism:

$$\varphi : X \setminus L \cap X \longrightarrow \mathbb{P}_1, \quad P \longmapsto [H_0(P) : H_\infty(P)],$$

which extends naturally to a morphism $\hat{\varphi} : \mathrm{Bl}_{L \cap X} X \to \mathbb{P}_1$. For $t = [t_1 : t_2] \in P_1$, the scheme-theoretic fiber $\hat{\varphi}^{-1}(t)$ is isomorphic to the intersection $H_t \cap X$, where H_t denotes the hyperplane $t_1 H_0 + t_2 H_\infty = 0$ in $\mathbb{P}_n$. Since $H_0 \cap X$ is smooth, for generic $t \in \mathbb{P}_1$, the intersection $H_t \cap X$ will be smooth. Hence, there exists a Zariski open subset $U_C \subset \mathbb{P}_1(\overline{K})$, such that, for $x \in U_C$, the fiber $\hat{\varphi}^{-1}(x)$ is a smooth cubic in an $(n-1)$-dimensional projective space, with a K-rational point in it (namely, P). Hence, by induction hypothesis, this fiber has the HP, and hence, since $\mathbb{P}_1/K$ has the HP, X has the HP as well by Theorem 2.5. We have hence proved the theorem. $\square$

4 A family of K3 surfaces with the Hilbert Property

In this section we look at applications of Theorem **??**, and we are particularly interested in K3 surfaces, as this represent a limiting case for the study of rational points in dimension 2 [3].

We are going to describe here a family of examples to which Theorem **??** applies. All these examples are birational a variety $X'_\lambda(f_1, f_2) \subset \mathbb{P}_1 \times \mathbb{P}_2$, where:

$$X'_\lambda(f_1, f_2) \coloneqq \{([w_0 : w_1], [x : y : z]) \in \mathbb{P}_1 \times \mathbb{P}_2 \mid w_0^2 f_1(x, y, z) = \lambda w_1^2 f_2(x, y, z)\}, \tag{4.1}$$

for some $\lambda \in K^*$, and $f_1, f_2 \in K[x, y, z]$ cubic homogeneous polynomials.

Remark 4.1. When $f_1(x, y, z) = f_1(x, z)$ does not depend on y, $f_2(x, y, z) = f_2(y, z)$ does not depend on x and both f_1 and f_2 do not have multiple roots, equation (4.1) describes a Kummer surface (i.e. a quotient of an abelian surface by the group of isomorphisms $\{\pm 1\}$).

In fact, equation (4.1) describes exactly the quotient of $E_1 \times E_2$ by the group $\{\pm 1\}$, where E_1 and E_2 are the elliptic curves defined by the following Weierstrass equations:

$$E_1 : \ w^2 = f_1(x, z), E_2 : \ w^2 = f_2(y, z). \tag{4.2}$$

The K3 surfaces that we are going to describe are introduced in [6], and they are constructed in such a way that they are naturally endowed with multiple elliptic fibrations.

Let $P_1, \ldots, P_9$ be nine (distinct) points in $\mathbb{P}_2(\overline{K})$ such that:

1. $P_1, \ldots, P_4$ are the four points of intersection of two smooth conics in $\mathbb{P}_2$, defined over K;

2. $P_5, \ldots, P_8$ are the four points of intersection of two smooth conics in $\mathbb{P}_2$, defined over K;

3. The eight points $P_1, \ldots, P_8$ are in generic position[4];

[3]K3 surfaces (and, in general, Calabi-Yau varieties) represent a "limiting case" for the study of rational points, at least conjecturally. In fact, the conjectures of Vojta suggest that on algebraic varieties there should be "less" rational points as the canonical bundle gets "bigger". Hence, since for K3 surfaces the canonical bundle is trivial by definition, we expect the rational points here not to be "too much", yet their existence (and Zariski-density) is not precluded. In fact, proving the HP, we are providing some examples of abudance of rational points in such surfaces.

[4]By this, we mean that no three of these points lie on a line, and no six of these points lie on a quartic.

4. $P_1, \dots, P_9$ are the nine points of intersection of two smooth cubics (say C_1, C_2) in $\mathbb{P}_2$, defined over K.

Definition 4.2. We say that a ninetuple $(P_1, \dots, P_9) \in P_2(\bar{K})^9$ is *good* if it satisfies the four conditions above.

Remark 4.3. Nine such points may always be constructed in the following way. Let Q_1, Q_2 be two (distinct) smooth conics (defined over K) in $\mathbb{P}_2/K$, and let $P_1, \dots, P_4$ be their four points of intersection.

Let then Q_3, Q_4 be other two different smooth conics, defined over K, such that the base locus of the pencil generated by them is made by four points, say $P'_1, \dots, P'_4$, such that the points $P_1, \dots, P_4, P'_1, \dots, P'_4$ are in generic position[4]. We note that this condition is satisfied if Q_3 and Q_4 are two generic conics.

Let now $\{P_5, \dots, P_8\} = \{P'_1, \dots, P'_4\}$, and let C_1, C_2 be two (distinct) smooth cubics of the pencil of cubics[5] passing through the points $P_1, \dots, P_8$. We let now P_9 be the ninth point of intersection (different from $P_1, \dots, P_8$) of C_1 and C_2, which is a base point for the pencil of cubics passing through $P_1, \dots, P_8$.

Let $R := \mathrm{Bl}_{P_1 + \dots + P_9} \mathbb{P}_2$ be the blowup of $\mathbb{P}_2$ in the nine points $P_1, \dots, P_9$. The two cubics C_1, C_2 define an elliptic fibration on R, which, on $\mathbb{P}_2 \setminus \{P_1, \dots, P_9\}$ is defined as:

$$\pi : \mathbb{P}_2 \setminus \{P_1, \dots, P_9\} \to \mathbb{P}_1, \quad \pi([x : y : z]) = [f_1(x, y, z) : f_2(x, y, z)] \tag{4.3}$$

The fibres of π are by construction the proper transform of the elements of the pencil generated by C_1, C_2.

Let now $\lambda \in K^*$ be a constant (we are going to fix its value later), and $f_\lambda : \mathbb{P}_1 \to \mathbb{P}_1$ be the morphism defined by $f_\lambda([w_0 : w_1]) = [w_0^2 : \lambda w_1^2]$. Let also X_λ be the smooth surface defined as the fibered product $R \times_{\pi, f_\lambda} \mathbb{P}_1$, $\alpha_\lambda : X_\lambda \to R$ be the projection on the first factor, and $\varphi_\lambda : X_\lambda \to \mathbb{P}_1$ the projection on the second factor. The surface X_λ is a K3 surface (see [6]), and it is endowed with at least three elliptic fibrations. These are, namely, the proper transforms of the two pencil of conics generated by $\{Q_1, Q_2\}$ and by $\{Q_3, Q_4\}$, and the fibration defined by φ_λ. We denote by π_1, π_2 the first two. I.e., $\pi_i = \alpha_\lambda \circ \pi'_i$, where the maps $\pi'_i : R \to \mathbb{P}_1$ are the unique morphisms whose restrictions on $\mathbb{P}_2 \setminus \{P_1, \dots, P_9\}$ are:

$$\pi'_1|_U : \mathbb{P}_2 \setminus \{P_1, \dots, P_9\} \longrightarrow \mathbb{P}_1, \quad [x : y : z] \longmapsto [Q_1(x, y, z) : Q_2(x, y, z)], \tag{4.4}$$

$$\pi'_2|_U : \mathbb{P}_2 \setminus \{P_1, \dots, P_9\} \longrightarrow \mathbb{P}_1, \quad [x : y : z] \longmapsto [Q_3(x, y, z) : Q_4(x, y, z)]. \tag{4.5}$$

[5]It is well known that, under our assumption of genericity on the points $P_1, \dots, P_8$, there is a dimension 1 pencil of cubics (generically non-singular) passing through these.

Proposition 4.4. *The fibrations $\pi_1, \pi_2 : X_\lambda \to \mathbb{P}_1$, as defined above, satisfy condition* **??**.

Proof. To lighten the notation, let $X \coloneqq X_\lambda$.

Let $E_1/K(t)$ be the generic fiber of $\pi_1 : X \to \mathbb{P}_1$, where $\operatorname{Spec} K(t)$ denotes the generic point of the target $\mathbb{P}_1$ of π_1, and $\pi_2^{gen} = \pi_2|_{E_1}$. We then have that $\pi_2^{gen} = \pi_2'^{gen} \circ r_1^{gen}$, where $r_1^{gen} = r|_{E_1}$, and $\pi_2'^{gen}$ is the restriction of π_2' to the generic fiber $R_1/K(t)$ of π_1'.

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First of all, we note that the diramation points of π_2^{gen} are the union of the diramation points of $\pi_2'^{gen}$ and the image under $\pi_2'^{gen}$ of the diramation points of r_1^{gen}. Secondly, we note that $r_1^{gen} \coloneqq r|_{E_1}$ has diramation in the intersection of R_1 and the two cubics C_1 and C_2 (by Theorem **??**...). These intersection points are non-constant in t (if they were constant, they would be base points for the pencil of quadrics generated by Q_1, Q_2 in R, which is base point free). Morover, since neither C_1 or C_2 are contained in the fibers of π_2', the image under $\pi_2'^{gen}$ of these diramation points are again non-constant in t.

Hence, it remains to prove that the diramation points of $\pi_2'^{gen}$ are non-constant in t. We prove this by reducing to an absurd. Suppose by contradiction that there were a constant diramation point $[t_1^0 : t_2^0] \in \mathbb{P}_1$. This means that the conic $Q^0 \coloneqq t_1^0 Q_3 + t_2^0 Q_4$ is tangent (in $\mathbb{P}_2$) to each conic in the pencil generated by Q_1, Q_2. This is equivalent to saying that the intersection of Q^0 with the generic element of the pencil generated by Q_3, Q_4 is not smooth. By the Theorem of Bertini [7, Remark 10.9.2], the singular points of this intersection must generically lie in the base locus of the pencil generated by Q_3, Q_4, i.e. the four points $P_5, \dots, P_9$.

Let us denote by P such a singular point. We know that, since $P = P_i$ for some $i = 5, \dots, 9$, each conic of the pencil generated by Q_3, Q_4 passes through P and is smooth at this point, and its tangent direction is non-constant. Whence, the intersection of Q^0 (which is smooth at P, being an irreducible conic in $\mathbb{P}_2$[6]) and the generic conic in the pencil generated by Q_3, Q_4 is smooth at P, hence we have reached an absurd. □

To apply Theorem **??** to X_λ we still need to prove that X_λ has Zariski-dense K-rational points. This is not guaranteed to be true for a generic λ, but we will choose λ appropriately, so that this holds:

Proposition 4.5. *Let $P_1, \dots, P_9$ be a good ninetuple of points in $P_2(\bar{K})$. Let C_1, C_2 be two smooth cubics passing through these points, and, for $\lambda \in K^*$, let*

[6] In fact, Q^0 is a conic passing through P_1, P_2, P_3, P_4 and P; whence, if it was reducible, it would be a union of two lines passing through these five points, but we know that, by construction, no three of these points are collinear, hence this is absurd.

$X_\lambda := R \times_{\pi, f_\lambda} \mathbb{P}_1$, *using the notation above. Then there exist infinitely many* $\lambda \in K^*$ *such that* X_λ *has Zariski-dense* K*-rational points.*

Proof. Let $X = X_1$, i.e. X_λ for $\lambda = 1$. Let $\iota : X \to X$ denote the involution given by:

$$\iota : R \times_{\pi, f_1} \mathbb{P}_1 \to R \times_{\pi, f_1} \mathbb{P}_1, \quad \iota((r,t)) = (r, -t). \tag{4.6}$$

We note that, for $\lambda \in K^*$, X_λ is a twist of X by the isomorphism $\mathrm{Gal}(\mathbb{Q}(\sqrt{\lambda})/\mathbb{Q}) \cong G$, where $G \cong \mathbb{Z}/2\mathbb{Z}$ denotes the automorphism group of X generated by ι.

Let H be a G-invariant very ample line bundle of positive degree on X[7]. Let now $U_X \subset X(\overline{K})$ be a Zariski-open subscheme, as in Corollary 3.3, applied to the two elliptic fibrations $\pi_1^\lambda, \pi_2^\lambda$, where π_i^λ is the fibration on X_λ induced by the fibration π_i' on R (i.e. $\pi_i^\lambda = \alpha_\lambda \circ \pi_i'$, where $\alpha_\lambda : X_\lambda \to R$ is the projection on R). Because of Remark **??**, $U_X \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, \overline{K}$ can be chosen, without loss of generality, to be independent from λ.

Let analogously $U_R \subset R$ be a Zariski-open subscheme, as in Lemma 3.2 applied to (R, π). Let now $V^0 \subset \mathbb{P}_1(\overline{K})$ be the Zariski-open subset such that, for each $t \in V^0$, $\pi^{-1}(t) \cap U_X \neq \emptyset$ and $\pi^{-1}(t) \cap U_R \neq \emptyset$.

We claim now that, if $[w_0 : w_1] \in \mathbb{P}_1(K)$ and $\lambda \in K^*$ are such that:

$$[w_0^2 : \lambda w_1^2] \in \pi(R(K)) \cap V_0, \tag{4.7}$$

then X_λ has Zariski-dense K-rational points. In fact, if $[w_0^2 : \lambda w_1^2] \in \pi(R(K)) \cap V_0$, then $\pi^{-1}([w_0^2 : \lambda w_1^2])$ has infinitely many K-rational points by Lemma 3.2, hence $\pi^{-1}([w_0^2 : \lambda w_1^2])(K) \cap U_X \neq \emptyset$, and hence, by Corollary 3.3, X_λ has Zariski-dense K-rational points.

We notice now that condition 4.7 is equivalent to the following:

$$\lambda \frac{w_1^2}{w_0^2} = \frac{f_1(x,y,z)}{f_2(x,y,z)},$$

when $f_2(x,y,z) \neq 0$. By choosing now, any $[x : y : z] \in \mathbb{P}_2(\mathbb{Q})$ such that $f_2(x,y,z) \neq 0$, $\frac{f_1(x,y,z)}{f_2(x,y,z)} \in V^0$, and by choosing any $[w_0 : w_1]$ and $\lambda \in K^*$ such that $\lambda \frac{\lambda w_1^2}{w_0^2} = \frac{f_1(x,y,z)}{f_2(x,y,z)}$, condition 4.7 is satisfied, and X_λ has infinitely many K-rational points. Of course, the possible choices of λ are infinite. $\square$

Corollary 4.6. *There exist infinitely many* $\lambda \in K^*$ *such that* X_λ *has the HP.*

Proof. This follows immediately from Theorem **??** applied to X_λ, endowed with the fibrations π_1, π_2, when $\lambda \in K^*$ is such that X_λ has infinitely many K-rational points (as Proposition in 4.5). We note that the hypothesis **??** of Theorem **??** is satisfied because of Proposition 4.4, while the other hypothesis are satisfied by construction. $\square$

[7] Such a line bundle may always be constructed. For instance, let H' be a very ample line bundle on X, then $H = H' + \iota^* H'$ is such a line bundle.

Remark 4.7. One may use a similar argument[8] to prove the HP for $X'_\lambda(f_1, f_2)$, when $\lambda = 1$, and f_1, f_2 are as in 4.1, if the roots of f_1 and f_2 are all in the base field K, and the elliptic curves E_1 and E_2 (defined as in Remark 4.1) have infinitely many K-rational points. Corvaja and Zannier have suggested in [4] that the HP holds for these surfaces.

References

[1] Bary-Soroker, L., Fehm, A., Petersen, S.: On varieties of Hilbert type. Ann. Inst. Fourier (Grenoble) **64**(5), 1893–1901 (2014)

[2] Clemens, C.H., Griffiths, P.A.: The intermediate Jacobian of the cubic threefold. Ann. of Math. (2) **95**, 281–356 (1972)

[3] Colliot-Thélène, J.L.: Points rationnels sur les fibrations. In: Higher dimensional varieties and rational points (Budapest, 2001), *Bolyai Soc. Math. Stud.*, vol. 12, pp. 171–221. Springer, Berlin (2003)

[4] Corvaja, P., Zannier, U.: On the hilbert property and the fundamental group of algebraic varieties. Mathematische Zeitschrift pp. 1–24 (2016)

[5] Demeio, J.L.: Non-rational varieties with the Hilbert Property. ArXiv e-prints (2018)

[6] Garbagnati, A., Salgado, C.: Linear systems on rational elliptic surfaces and elliptic fibrations on K3 surfaces. ArXiv e-prints (2017)

[7] Hartshorne, R.: Algebraic geometry. Springer-Verlag, New York (1977). Graduate Texts in Mathematics, No. 52

[8] Kollár, J.: Unirationality of cubic hypersurfaces. Journal of the Institute of Mathematics of Jussieu **1**(3), 467–476 (2002)

[9] Serre, J.P.: Topics in Galois theory, *Research Notes in Mathematics*, vol. 1, second edn. A K Peters, Ltd., MA (2008)

[8] In this case, Zariski density of rational points on $X = X_\lambda(f_1, f_2)$ follows immediately from the fact that both E_1 and E_2 have infinitely many K-rational points. The elliptic fibrations that play the role of π_1 and π_2 in Proposition 4.4 are still constructed from conic bundles on the projective plane, in a very similar manner. In this case, however, instead of there being "no constant diramation points" there is "exactly one constant diramation point" for both fibrations.