

Linear Algebra

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Preface

I wrote these notes principally FOR understanding, they are meant for future reference for a refresher on what I have learnt. As such, certain definitions may not be exactly precise but are rephrased into simpler terms.

I am currently using Schaum's Easy Outlines: Linear Algebra - Crash Course. ISBN 978-0-07-139880-0.

I plan to learn and use Linear Algebra by Hoffman and Kunze in the future. But I am currently clearing the pre-requisites for the book as the time of writing (04/11/2022); I am still learning proofs.

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1 Vectors in \mathbb{R}^n

Although we will restrict ourselves in this chapter to vectors whose elements come from the field of real numbers, denoted by \mathbb{R} , many of our operations also apply to vectors whose entries come from arbitrary field \mathbf{K} . In the context of vectors, the elements of our number fields are called *scalars*.

1.1 List of Numbers

Suppose the height (in centimeters) of eight students are listed as follows:

155 165 175 185 180 170 160 150

One can denote all the values in the list using only one symbol, for example h , but with different subscripts; giving rise to h_i where $1 \leq i \leq 8$.

$h_1 \ h_2 \ h_3 \ h_4 \ h_5 \ h_6 \ h_7 \ h_8$

It is easy to see that each subscript denotes the position of the value in the list; $h_1 = 155$, $h_2 = 165$, etc. Such a list of values, $\mathbf{h} = (h_1, h_2, h_3, \dots, h_8)$ is called a *linear array* or *vector*.

The set of all n -tuples of real numbers, denoted by \mathbb{R}^n , is called *n-space*. A particular n -tuple in \mathbb{R}^n like $\mathbf{u} = (a_1, a_2, \dots, a_n)$ is called a *point* or *elements* of \mathbf{u} . Moreover, when discussing the space/domain of \mathbb{R}^n , we use the term *scalar* for the elements in \mathbb{R} (notice that $n = 1$).

Consider two vectors, \mathbf{u} and \mathbf{v} , are equal, we can express it as $\mathbf{u} = \mathbf{v}$ if they have the same number of elements and if the corresponding elements (position of the elements) are equal. Thi means that while vectors $(1, 2, 3)$ and $(2, 3, 1)$ have the same number of elements, the vectors are not equal because of the corresponding elements do not match each other from one vector to the other.

The vector $(0, 0, \underbrace{\dots}_n, 0)$; where all of the elements are zero, is called the *zero vector*, and it is usually denoted by $\mathbf{0}$.

1.2 Column Vectors

Sometimes a vector in n -space \mathbb{R}^n is written vertically, rather than horizontally as shown previously. Such a vector is called a *column vector* while the ones shown previously are called *row vectors*. For example, the following are column vectors that belong to \mathbb{R}^2 , \mathbb{R}^2 , \mathbb{R}^3 , and \mathbb{R}^3 , respectively:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix} \quad \begin{bmatrix} 1.5 \\ \frac{2}{3} \\ -15 \end{bmatrix}$$

Remark 1.1. We also note that any operation defined for row vectors is defined analogously for column vectors.

1.3 Vector Addition and Scalar Multiplication

Consider two vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^n :

$$\mathbf{u} = (a_1, \dots, a_n) \text{ and } \mathbf{v} = (b_1, \dots, b_n)$$

Their *sum* can be expressed as $\mathbf{u} + \mathbf{v}$. $\mathbf{u} + \mathbf{v}$ is the *vector* that is obtained by adding the corresponding elements from \mathbf{u} and \mathbf{v} . It is as follows:

$$\mathbf{u} + \mathbf{v} = (a_1 + b_1, \dots, a_n + b_n)$$

The *scalar product* of the vector \mathbf{u} by a real number k can be expressed as $k\mathbf{u}$. $k\mathbf{u}$ is the vector obtained by multiplying each element of \mathbf{u} by k . It is as follows:

$$k\mathbf{u} = k(a_1, \dots, a_n) = (ka_1, \dots, ka_n)$$

Remark 1.2. Notice that $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ are also vectors in \mathbb{R}^n . The sum of vectors who have different numbers of elements is not defined.

Negatives and *subtraction* are defined in \mathbb{R}^n as follows:

$$-\mathbf{u} = (-1)\mathbf{u} \quad \text{and} \quad \mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$$

The vector $-\mathbf{u}$ is called the negative of \mathbf{u} , and $\mathbf{u} - \mathbf{v}$ is called the *difference* of \mathbf{u} and \mathbf{v} .

Now suppose we are given vectors $\mathbf{u}_1, \dots, \mathbf{u}_m$ in \mathbb{R}^n and scalars k_1, \dots, k_m in \mathbb{R} . We can multiply the vectors by corresponding scalars and then add the resultant scalar products to form the vector:

$$\mathbf{v} = k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \dots + k_m\mathbf{u}_m$$

The vector \mathbf{v} is called a *linear combination* of the vectors $\mathbf{u}_1, \dots, \mathbf{u}_m$.

Theorem 1.3 (Basic properties of vectors under the operations of vector addition and scalar multiplication)

For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and any scalars $k, k' \in \mathbb{R}$,

$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	$k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
$\mathbf{u} + \mathbf{0} = \mathbf{u}$	$(k + k')\mathbf{u} = k\mathbf{u} + k'\mathbf{u}$
$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$	$(kk')\mathbf{u} = k(k'\mathbf{u})$
$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	$1(\mathbf{u}) = \mathbf{u}$

Suppose \mathbf{u} and \mathbf{v} are vectors in \mathbb{R}^n for which $\mathbf{u} = k\mathbf{v}$ for some non-zero scalar k in \mathbb{R} . It follows that \mathbf{u} is a *multiple* of \mathbf{v} . Also, \mathbf{u} is said to be in the *same* or *opposite direction* as \mathbf{v} for $k > 0$ or $k < 0$.

1.4 Dot Product

Consider the following arbitrary vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^n :

$$\mathbf{u} = (a_1, \dots, a_n) \text{ and } \mathbf{v} = (b_1, \dots, b_n)$$

The *dot product* or *inner product* or *scalar product* of \mathbf{u} and \mathbf{v} is denoted and defined by $\mathbf{u} \cdot \mathbf{v} = a_1 b_1 + \dots + a_n b_n$. $\mathbf{u} \cdot \mathbf{v}$ is obtained by multiplying corresponding components and taking the sum of the resulting products. The vectors \mathbf{u} and \mathbf{v} are said to be *orthogonal* (or *perpendicular*) if their *dot product* is zero; $\mathbf{u} \cdot \mathbf{v} = 0$.

Theorem 1.4 (Basic properties of the dot product in \mathbb{R}^n)

For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and any scalar $k \in \mathbb{R}$,

$(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$	$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
$(k\mathbf{u}) \cdot \mathbf{v} = k(\mathbf{u} \cdot \mathbf{v})$	$\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if $\mathbf{u} = \mathbf{0}$

To be somewhat complete (and rigorous) with the properties, consider $\mathbf{u} \cdot (k\mathbf{v})$. It is similar to the cell on row 2, column 1. It follows that $\mathbf{u} \cdot (k\mathbf{v}) = (k\mathbf{v}) \cdot \mathbf{u} = k(\mathbf{v} \cdot \mathbf{u}) = k(\mathbf{u} \cdot \mathbf{v})$.

The space \mathbb{R}^n with the above operations of vector addition, scalar multiplication, and dot product is usually called **Euclidean n -space**.

1.5 Norm (Length) of a Vector

The *norm* or *length* of a vector \mathbf{u} in \mathbb{R}^n can be expressed as $\|\mathbf{u}\|$. It is defined to be the non-negative square root of $\mathbf{u} \cdot \mathbf{u}$ (the inner product). For example, if $\mathbf{u} = (a_1, \dots, a_n)$, then $\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{a_1^2 + \dots + a_n^2}$. Therefore, $\|\mathbf{u}\|$ is the square root of the sum of the squares of the elements of \mathbf{u} . Thus, $\|\mathbf{u}\| > 0$, and $\|\mathbf{u}\| = 0$ iff $\mathbf{u} = \mathbf{0}$.

A vector \mathbf{u} is called a *unit vector* if $\|\mathbf{u}\| = 1$ or if $\mathbf{u} \cdot \mathbf{u} = 1$ (both are equivalent).

For any non-zero vector \mathbf{v} in \mathbb{R}^n , the vector $\hat{\mathbf{v}} = \frac{1}{\|\mathbf{v}\|} \mathbf{v} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$ is the unique unit vector in the same direction as \mathbf{v} . The process of finding $\hat{\mathbf{v}}$ from \mathbf{v} is called *normalizing* \mathbf{v} .

Theorem 1.5 (Cauchy-Schwarz Inequality)

For any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\|$$

Theorem 1.6 (Triangle Inequality or Minkowski's Inequality)

For any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

2 Algebra of Matrices

2.1 Matrices

A matrix \mathbf{A} over a field \mathbf{K} is a rectangular array of scalars:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

The rows of \mathbf{A} are the m horizontal lists of scalars:

$$(a_{11}, \dots, a_{1n}), (a_{21}, \dots, a_{2n}), \dots, (a_{m1}, \dots, a_{mn})$$

A matrix with m rows and n columns is called an $m \times n$ matrix which is read as m by n matrix. The pair of numbers m and n is called the size of the matrix. Suppose two matrices \mathbf{A} and \mathbf{B} are *equal*, $\mathbf{A} = \mathbf{B}$. This is true when they have the same size and if the corresponding elements are equal. Thus, the equality of two $m \times n$ matrices is equivalent to a system of mn equalities; one for each corresponding pair of elements.

A matrix with only one row or with only one column is called a *row matrix/vector* and *column matrix/vector* respectively. A matrix whose entries are all zero is called a *zero matrix* and is written as $\mathbf{0}$.

2.2 Matrix Addition and Scalar Multiplication

Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be two matrices with the same size; $m \times n$ matrices. The *sum* of \mathbf{A} and \mathbf{B} can be expressed as $\mathbf{A} + \mathbf{B}$. It is the matrix obtained by adding the corresponding elements from \mathbf{A} and \mathbf{B} :

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{bmatrix}$$

The *product* of a matrix \mathbf{A} by a scalar k can be expressed as $k \cdot \mathbf{A} = k\mathbf{A}$. It is the matrix obtained by multiplying each element of \mathbf{A} by k :

$$k\mathbf{A} = \begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{bmatrix}$$

Negatives and *subtractions* are defined in a similar fashion as vectors as seen in the first section; $-\mathbf{A} = (-1)\mathbf{A}$ and $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$. The matrix $-\mathbf{A}$ is called the *negative* of the matrix \mathbf{A} . The matrix $\mathbf{A} - \mathbf{B}$ is called the *difference* of \mathbf{A} and \mathbf{B} . The sum of matrices with different sizes is not defined.

Consider λ and μ where $\lambda, \mu \in \mathbb{R}$, and two $m \times n$ matrices \mathbf{A} and \mathbf{B} . $\lambda\mathbf{A} + \mu\mathbf{B}$ is called a *linear combination* of \mathbf{A} and \mathbf{B} .

Theorem 2.1 (Basic properties of matrices under the operations of matrix addition and scalar multiplication)

For any matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ (with the same size) and any scalars k and k' ,

$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$	$k(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B}$
$\mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A} = \mathbf{A}$	$(k + k')\mathbf{A} = k\mathbf{A} + k'\mathbf{A}$
$\mathbf{A} + (-\mathbf{A}) = (-\mathbf{A}) + \mathbf{A} = \mathbf{0}$	$(kk')\mathbf{A} = k(k'\mathbf{A})$
$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$	$1 \cdot \mathbf{A} = \mathbf{A}$

2.3 Matrix Multiplication

Suppose $f(k)$ is an algebraic expression involving the letter k . It follows that the expression $\sum_{k=1}^n f(k)$ has the following meaning; \sum is the *summation symbol*, and k is called the index where 1 and n are the *lower* and *upper bounds* respectively. As such, this means we are taking the sum of $f(k)$ from 1 to n . First, we set $k = 1$ in $f(k)$ which yields $f(1)$. Next, we set $k = 2$ in $f(k)$ which yields $f(2)$, and take sum with $f(1)$ which yields $f(1) + f(2)$. We increase the value of k by 1 until we reach n ; taking the sum of the result at each step. We should obtain the sum $f(1) + f(2) + \cdots + f(n)$ at the end.

We also generalize our definition by allowing the sum to range from any integer n to N .

$$\sum_{k=n}^N f(k) = f(n) + f(n+1) + f(n+2) + \cdots + f(N)$$

The *product* of matrices \mathbf{A} and \mathbf{B} can be expressed as \mathbf{AB} . Matrix multiplication is quite convoluted as many things are happening but we will try to break it down slowly.

For a start, we begin with a special case. The product \mathbf{AB} of a row matrix $\mathbf{A} = [a_i]$ and column matrix $\mathbf{B} = [b_i]$ with the same number of elements is defined to be the scalar (or 1×1 matrix) obtained by multiplying corresponding entries and adding them together:

$$\mathbf{AB} = [a_1, a_2, \dots, a_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1b_1 + a_2b_2 + \cdots + a_nb_n = \sum_{k=1}^n a_kb_k$$

It is important to note that \mathbf{AB} is a scalar (or a 1×1 matrix). The product \mathbf{AB} is not defined when \mathbf{A} and \mathbf{B} have different numbers of elements.

Suppose $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{kj}]$ are matrices such that the number of columns of \mathbf{A} is equal to the number of rows of \mathbf{B} ; Suppose \mathbf{A} is an $m \times p$ matrix and \mathbf{B} is a $p \times n$ matrix. Then the product \mathbf{AB} is an $m \times n$ matrix whose ij -entry is obtained by multiplying the i th row of \mathbf{A} by the j th column of \mathbf{B} .

$$\begin{bmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & \vdots & \vdots \\ a_{i1} & \cdots & a_{ip} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mp} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{p1} & \cdots & b_{pj} & \cdots & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \vdots & \vdots \\ \cdots & c_{ij} & \cdots \\ \vdots & \vdots & \vdots \\ c_{m1} & \cdots & c_{mn} \end{bmatrix}$$

where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^p a_{ik}b_{kj}$

The product \mathbf{AB} is not defined if \mathbf{A} is an $m \times p$ matrix and \mathbf{B} is a $q \times n$ matrix, where $p \neq q$.

Example 2.2 (Schaum's Easy Outlines: Linear Algebra Crash Course - Example 2.5)

(a) Find \mathbf{AB} where $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 0 & -4 \\ 5 & -2 & 6 \end{bmatrix}$.

Since \mathbf{A} is 2×2 and \mathbf{B} is 2×3 , the product \mathbf{AB} is defined and \mathbf{AB} is a 2×3 matrix. To obtain the first row of \mathbf{AB} , multiply the first row of \mathbf{A} by each column of \mathbf{B} . Then multiply the second row of \mathbf{A} by each column of \mathbf{B} .

$$\mathbf{AB} = \begin{bmatrix} 2+15 & 0-6 & -4+18 \\ 4-5 & 0+2 & -8-6 \end{bmatrix} = \begin{bmatrix} 17 & -6 & 14 \\ 4-5 & 2 & -14 \end{bmatrix}$$

(b) Suppose $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 6 \\ 0 & -2 \end{bmatrix}$.

$$\mathbf{AB} = \begin{bmatrix} 5+0 & 6-4 \\ 15+0 & 18-8 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 15 & 10 \end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix} 5+18 & 10+24 \\ 0-6 & 0-8 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ -6 & -8 \end{bmatrix}$$

In part (b) of the example above, it shows that matrix multiplication is not commutative; the products \mathbf{AB} and \mathbf{BA} of matrices need not be equal.

Theorem 2.3 (Basic properties that matrix multiplication does satisfy) • $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ (associative law),

- $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ (left distributive law),
- $(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$ (right distributive law),
- $k(\mathbf{AB}) = (k\mathbf{A})\mathbf{B} = \mathbf{A}k\mathbf{B}$, where k is a scalar.

We note that $\mathbf{0A} = \mathbf{0}$ and $\mathbf{B0} = \mathbf{0}$, where $\mathbf{0}$ is the zero matrix.

2.4 Transpose of a Matrix

The *transpose* of matrix \mathbf{A} can be expressed as \mathbf{A}^T . It is the matrix obtained by writing the columns of \mathbf{A} , in order, as rows. Consider the following,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & -3 & -5 \end{bmatrix}^T = \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix}$$

In other words, if $\mathbf{A} = [a_{ij}]$ is an $m \times n$ matrix, then $\mathbf{A}^T = [b_{ij}]$ is the $n \times m$ matrix where $b_{ij} = a_{ij}$.

Theorem 2.4 (Basic properties of the transpose operation)

Let \mathbf{A} and \mathbf{B} be matrices and let k be a scalar. Then, whenever the sum and product are defined:

$(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$	$(k\mathbf{A})^T = k\mathbf{A}^T$
$(\mathbf{A}^T)^T = \mathbf{A}$	$(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$

In the last cell – row 2, column 2 – we see that the transpose of a product is the product of the transposes, but in the reverse order.

2.5 Square Matrices

A *square matrix* is a matrix with the same number of rows as columns. An $n \times n$ square matrix is said to be of *order* n and is sometimes called an *n-square matrix*.

Recall that not every two matrices can be added or multiplied. However, if we only consider square matrices of some given order n , then this inconvenience disappears. Specifically, the operations of addition, multiplication, scalar multiplication, and transpose can be performed on any $n \times n$ matrices, and the result is again an $n \times n$ matrix.

2.6 Diagonal and Trace

Let $\mathbf{A} = [a_{ij}]$ be an n -square matrix. The *diagonal* or *main diagonal* of \mathbf{A} consists of the elements with the same subscripts:

$$a_{11}, a_{22}, a_{33}, \dots, a_{nn}$$

The *trace* of \mathbf{A} can be expressed as $\text{tr}(\mathbf{A})$. It is the sum of the diagonal elements; $\text{tr}(\mathbf{A}) = a_{11} + a_{22} + a_{33} + \dots + a_{nn}$.

Theorem 2.5 (Basic properties of the trace of a matrix)

Suppose $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ are n -square matrices and k is a scalar.

$\text{tr}(\mathbf{A} + \mathbf{B}) = \text{tr}(\mathbf{A}) + \text{tr}(\mathbf{B})$	$\text{tr}(\mathbf{A}^T) = \text{tr}(\mathbf{A})$
$\text{tr}(k\mathbf{A}) = k \text{tr}(\mathbf{A})$	$\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA})$

Although $\mathbf{AB} \neq \mathbf{BA}$, the traces are equal.

2.7 Identity Matrix, Scalar Matrices

The n -square *identity* or *unit matrix*, denoted by \mathbf{I}_n , or simply \mathbf{I} , is the n -square matrix with 1s on the diagonal and 0s everywhere. The identity matrix \mathbf{I} is similar to the scalar 1 in that, for any n -square matrix \mathbf{A} , $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$. More generally, if \mathbf{B} is an $m \times n$ matrix:

$$\mathbf{BI}_n = \mathbf{I}_m \mathbf{B} = \mathbf{B}$$

For any scalar k , the matrix $k\mathbf{I}$ that contains ks on the diagonal and 0s everywhere else is called the scalar matrix corresponding to the scalar k . Multiplying a matrix \mathbf{A} by the scalar matrix $k\mathbf{I}$ is equivalent to multiplying \mathbf{A} by the scalar k ; $(k\mathbf{I})\mathbf{A} = k(\mathbf{IA}) = k\mathbf{A}$.

The *Kronecker delta* function δ_{ij} can also be used to define the identity matrix;

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Therefore, the identity matrix, $\mathbf{I} = [\delta_{ij}]$.

2.8 Polynomials in Matrices

Let \mathbf{A} be an n -square matrix over a field K . *Powers of \mathbf{A} :*

$$\mathbf{A}^2 = \mathbf{AA}, \quad \mathbf{A}^3 = \mathbf{A}^2\mathbf{A}, \quad \dots, \quad \mathbf{A}^{n+1} = \mathbf{A}^n\mathbf{A}, \quad \dots, \quad \text{and} \quad \mathbf{A}^0 = \mathbf{I}$$

Polynomials in the matrix \mathbf{A} are also defined. For any polynomial $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ where a_i are scalars in K , $f(\mathbf{A})$ is defined to be the following matrix:

$$f(\mathbf{A}) = a_0\mathbf{I} + a_1\mathbf{A} + a_2\mathbf{A}^2 + \dots + a_n\mathbf{A}^n$$

If $f(\mathbf{A})$ is the zero matrix, then \mathbf{A} is called a *zero* or *root* of $f(x)$.

2.9 Invertible (Non-singular) Matrices

A square matrix \mathbf{A} is said to be *invertible* or *non-singular* if there exists a matrix \mathbf{B} such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$ where \mathbf{I} is the identity matrix. If $\mathbf{AB}_1 = \mathbf{B}_1\mathbf{A} = \mathbf{I}$ and $\mathbf{AB}_2 = \mathbf{B}_2\mathbf{A} = \mathbf{I}$, then $\mathbf{B}_1 = \mathbf{B}_1\mathbf{I} = \mathbf{B}_1(\mathbf{AB}_2) = (\mathbf{B}_1\mathbf{A})\mathbf{B}_2 = \mathbf{IB}_2 = \mathbf{B}_2$.

We call such a matrix \mathbf{B} the *inverse* of \mathbf{A} . It can be expressed as \mathbf{A}^{-1} . Do note that the above relation is symmetric; if \mathbf{B} is the inverse of \mathbf{A} , then \mathbf{A} is the inverse of \mathbf{B} .

It is known that $\mathbf{AB} = \mathbf{I}$ iff $\mathbf{BA} = \mathbf{I}$. Therefore it is sufficient to test only one product to determine whether or not two given matrices are inverses.

Now suppose \mathbf{A} and \mathbf{B} are invertible. It follows that \mathbf{AB} is invertible and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$. More generally, if $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_k$ are invertible, then their product is invertible, and $(\mathbf{A}_1 \mathbf{A}_2 \dots \mathbf{A}_k)^{-1} = \mathbf{A}_k^{-1} \dots \mathbf{A}_2^{-1} \mathbf{A}_1^{-1}$, the product of the inverses in the reverse order.

2.10 Inverse of a 2×2 Matrix

Let \mathbf{A} be an arbitrary 2×2 matrix; $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We want to derive a formula for \mathbf{A}^{-1} . In this case, we seek $2^2 = 4$ scalars – x_1, y_1, x_2, y_2 – such that:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} ax_1 + by_1 & ax_2 + by_2 \\ cx_1 + dy_1 & cx_2 + dy_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Setting the four entries equal to the corresponding entries in the identity matrix yields four equations, which can be partitioned into 2×2 systems as follows:

$$\begin{aligned} ax_1 + by_1 &= 1, & ax_2 + by_2 &= 0 \\ cx_1 + dy_1 &= 0, & cx_2 + dy_2 &= 1 \end{aligned}$$

Suppose we let $|\mathbf{A}| = ad - bc$ ($|\mathbf{A}|$ is the *determinant* of \mathbf{A}). Assuming $|\mathbf{A}| \neq 0$ (if $|\mathbf{A}| = 0$, \mathbf{A} is not invertible), we can solve uniquely for the above unknowns x_1, y_1, x_2, y_2 which obtains:

$$x_1 = \frac{d}{|\mathbf{A}|}, \quad y_1 = \frac{-c}{|\mathbf{A}|}, \quad x_2 = \frac{-b}{|\mathbf{A}|}, \quad y_2 = \frac{a}{|\mathbf{A}|}$$

Accordingly,

$$\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} d/|\mathbf{A}| & -b/|\mathbf{A}| \\ -c/|\mathbf{A}| & a/|\mathbf{A}| \end{bmatrix} = \frac{1}{|\mathbf{A}|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In other words, when $|\mathbf{A}| \neq 0$, the inverse of a 2×2 matrix \mathbf{A} ; \mathbf{A}^{-1} is obtained by:

1. Interchange the two elements on the diagonal,
2. Take the negatives of the other two elements,
3. Multiply the resulting matrix by $1/|\mathbf{A}|$ or divide each element by $|\mathbf{A}|$.

To find the inverse of a matrix in general, it is finding the solution of a collection of $n \times n$ systems of linear equations for an arbitrary n -square matrix.

2.11 Special Types of Square Matrices

2.11.1 Diagonal Matrices

A square matrix $\mathbf{D} = [d_{ij}]$ is *diagonal* if its non-diagonal entries are all zero. It can be expressed as:

$$\mathbf{D} = \text{diag}(d_{11}, d_{22}, \dots, d_{nn})$$

where some or all the d_{ii} may be zero.

2.11.2 Triangular Matrices

A square matrix is *upper triangular* or simply *triangular* if all entries below the main diagonal are equal to 0; If $a_{ij} = 0$ for all $i > j$.

A *lower triangular* matrix is a square matrix whose entries above the diagonal are all zero.

Theorem 2.6

Suppose $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ are $n \times n$ upper/lower triangular matrices.

1. $\mathbf{A} + \mathbf{B}, k\mathbf{A}, \mathbf{AB}$ are triangular with respective diagonals.
 $(a_{11} + b_{11}, \dots, a_{nn} + b_{nn}), (ka_{11}, \dots, ka_{nn}), (a_{11}b_{11}, \dots, a_{nn}b_{nn})$
2. For any polynomial $f(x)$, the matrix $f(\mathbf{A})$ is triangular with diagonal $(f(a_{11}), f(a_{22}), \dots, f(a_{nn}))$
3. \mathbf{A} is invertible *iff* each diagonal element $a_{ii} \neq 0$, and when \mathbf{A}^{-1} exists it is also triangular.

2.11.3 Symmetric Matrices

A matrix \mathbf{A} is *symmetric* if $\mathbf{A}^T = \mathbf{A}$. Equivalently, $\mathbf{A} = [a_{ij}]$ is symmetric if *symmetric elements* (mirror elements wrt. the diagonal) are equal; if each $a_{ij} = a_{ji}$.

A matrix \mathbf{A} is *skew-symmetric* if $\mathbf{A}^T = -\mathbf{A}$. Equivalently, $a_{ij} = -a_{ji}$. Clearly the diagonal elements of such a matrix must be zero, since $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$.

2.11.4 Orthogonal Matrices

A real matrix \mathbf{A} is *orthogonal* if $\mathbf{A}^T = \mathbf{A}^{-1}$; $\mathbf{AA}^T = \mathbf{A}^T\mathbf{A} = \mathbf{I}$. Therefore, \mathbf{A} must necessarily be square and invertible.

Suppose \mathbf{A} is a real orthogonal 3×3 matrix with rows:

$$u_1 = (a_1, a_2, a_3), \quad u_2 = (b_1, b_2, b_3), \quad u_3 = (c_1, c_2, c_3).$$

Since \mathbf{A} is orthogonal, we must have $\mathbf{A}\mathbf{A}^T = \mathbf{I}$.

$$\mathbf{A}\mathbf{A}^T = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

Multiplying \mathbf{A} by \mathbf{A}^T and setting each entry equal to the corresponding entry in \mathbf{I} yields a 3×3 system of equations.

$$\begin{aligned} a_1^2 + a_2^2 + a_3^2 &= 1, & a_1b_1 + a_2b_2 + a_3b_3 &= 0, & a_1c_1 + a_2c_2 + a_3c_3 &= 0 \\ b_1a_1 + b_2a_2 + b_3a_3 &= 0, & b_1^2 + b_2^2 + b_3^2 &= 1, & b_1c_1 + b_2c_2 + b_3c_3 &= 0 \\ c_1a_1 + c_2a_2 + c_3a_3 &= 0, & c_1b_1 + c_2b_2 + c_3b_3 &= 0, & c_1^2 + c_2^2 + c_3^2 &= 1 \end{aligned}$$

This shows that $u_1 \cdot u_1 = 1, u_2 \cdot u_2 = 1, u_3 \cdot u_3 = 1$, and $u_i \cdot u_j = 0$ for all $i \neq j$. It follows that the rows u_1, u_2, u_3 are unit vectors and are orthogonal to each other.

In general, vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ in \mathbb{R}^n are said to form an *orthonormal* set of vectors if the vectors are unit vectors and are orthogonal to each other:

$$u_i \cdot u_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

You can see that $u_i \cdot u_j = \delta_{ij}$ where δ_{ij} is the Kronecker delta function.

We have shown that the condition $\mathbf{A}\mathbf{A}^T = \mathbf{I}$ implies that the rows of \mathbf{A} form an orthonormal set of vectors. The condition $\mathbf{A}^T\mathbf{A} = \mathbf{I}$ similarly implies that the columns of \mathbf{A} also form an orthonormal set of vectors. Moreover, since each step is reversible, the converse is true. The above results for 3×3 matrices is true in general.

Theorem 2.7

Let \mathbf{A} be a real matrix. Then the following are equivalent:

1. \mathbf{A} is orthogonal,
2. The rows of \mathbf{A} form an orthonormal set,
3. The columns of \mathbf{A} form an orthonormal set.

2.11.5 Normal Vectors

A real matrix \mathbf{A} is *normal* if it *commutes* with its transpose \mathbf{A}^T ; $\mathbf{A}\mathbf{A}^T = \mathbf{A}^T\mathbf{A}$. If \mathbf{A} is symmetric, orthogonal, or skew-symmetric, then \mathbf{A} is normal.

2.12 Block Matrices

Using a system of horizontal and vertical lines, we can partition a matrix \mathbf{A} into sub-matrices called *blocks/cells* of \mathbf{A} . Any given matrix may be divided into blocks in different ways.

$$\left[\begin{array}{cc|cc|c} 1 & -2 & 0 & 1 & 3 \\ 2 & 3 & 5 & 7 & -2 \\ \hline 3 & 1 & 4 & 5 & 9 \\ 4 & 6 & -3 & 1 & 8 \end{array} \right] \quad \left[\begin{array}{cc|cc|c} 1 & -2 & 0 & 1 & 3 \\ 2 & 3 & 5 & 7 & -2 \\ \hline 3 & 1 & 4 & 5 & 9 \\ 4 & 6 & -3 & 1 & 8 \end{array} \right] \quad \left[\begin{array}{ccc|cc} 1 & -2 & 0 & 1 & 3 \\ 2 & 3 & 5 & 7 & -2 \\ \hline 3 & 1 & 4 & 5 & 9 \\ 4 & 6 & -3 & 1 & 8 \end{array} \right]$$

The convenience of block matrices; \mathbf{A} and \mathbf{B} into blocks is that the result of operations on \mathbf{A} and \mathbf{B} can be obtained by carrying out the computation with the blocks, just as if they were the actual elements of the matrices.

Suppose that $\mathbf{A} = [\mathbf{A}_{ij}]$ and $\mathbf{B} = [\mathbf{B}_{ij}]$ are block matrices with the same number of row and column blocks. Suppose that the corresponding blocks have the same size. It follows that adding the corresponding blocks of \mathbf{A} and \mathbf{B} also adds the corresponding elements of \mathbf{A} and \mathbf{B} and multiplying each block of \mathbf{A} by a scalar k multiplies each element of \mathbf{A} by k .

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} \mathbf{A}_{11} + \mathbf{B}_{11} & \mathbf{A}_{12} + \mathbf{B}_{12} & \dots & \mathbf{A}_{1n} + \mathbf{B}_{1n} \\ \mathbf{A}_{21} + \mathbf{B}_{21} & \mathbf{A}_{22} + \mathbf{B}_{22} & \dots & \mathbf{A}_{2n} + \mathbf{B}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{m1} + \mathbf{B}_{m1} & \mathbf{A}_{m2} + \mathbf{B}_{m2} & \dots & \mathbf{A}_{mn} + \mathbf{B}_{mn} \end{bmatrix}$$

$$k\mathbf{A} = \begin{bmatrix} k\mathbf{A}_{11} & k\mathbf{A}_{12} & \dots & k\mathbf{A}_{1n} \\ k\mathbf{A}_{21} & k\mathbf{A}_{22} & \dots & k\mathbf{A}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ k\mathbf{A}_{m1} & k\mathbf{A}_{m2} & \dots & k\mathbf{A}_{mn} \end{bmatrix}$$

The case of matrix multiplication is less obvious. Suppose $\mathbf{U} = [\mathbf{U}_{ik}]$ and $\mathbf{V} = [\mathbf{V}_{kj}]$ are block matrices such that the number of columns of each block \mathbf{U}_{ik} is equal to the number of rows of each block \mathbf{V}_{kj} ; the condition for matrix multiplication has been satisfied.

$$\mathbf{UV} = \begin{bmatrix} \mathbf{W}_{11} & \mathbf{W}_{12} & \dots & \mathbf{W}_{1n} \\ \mathbf{W}_{21} & \mathbf{W}_{22} & \dots & \mathbf{W}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{W}_{m1} & \mathbf{W}_{m2} & \dots & \mathbf{W}_{mn} \end{bmatrix}$$

where $\mathbf{W}_{ij} = \mathbf{U}_{i1}\mathbf{V}_{1j} + \mathbf{U}_{i2}\mathbf{V}_{2j} + \dots + \mathbf{U}_{ip}\mathbf{V}_{pj}$.

2.13 Square Block Matrices

Theorem 2.8

Let \mathbf{M} be a block matrix. For \mathbf{M} to be a *square block matrix*:

1. \mathbf{M} is square matrix,
2. The blocks form a square matrix,
3. The diagonal blocks are also square matrices.

The last two conditions will occur *iff* there are the same number of horizontal and vertical lines and they are placed symmetrically.

$$\mathbf{A} = \left[\begin{array}{cc|cc|c} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \\ \hline 3 & 1 & 4 & 5 & 9 \\ 4 & 4 & 4 & 4 & 4 \\ \hline 3 & 5 & 3 & 5 & 3 \end{array} \right] \quad \text{and} \quad \mathbf{B} = \left[\begin{array}{cc|cc|c} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \\ \hline 3 & 1 & 4 & 5 & 9 \\ 4 & 4 & 4 & 4 & 4 \\ \hline 3 & 5 & 3 & 5 & 3 \end{array} \right]$$

The block matrix \mathbf{A} is not a square block matrix, since the second and third diagonal blocks are not square. On the other hand, the block matrix \mathbf{B} is a square block matrix.

2.14 Block Diagonal Matrices

Let $\mathbf{M} = [\mathbf{A}_{ij}]$ be a square block matrix such that the non-diagonal blocks are all zero matrices; $\mathbf{A}_{ij} = 0$ when $i \neq j$. Then \mathbf{M} is called a *block diagonal matrix*. It can be expressed as:

$$\mathbf{M} = \text{diag}(\mathbf{A}_{11}, \mathbf{A}_{22}, \dots, \mathbf{A}_{rr}) \quad \text{or} \quad \mathbf{M} = \mathbf{A}_{11} \oplus \mathbf{A}_{22} \oplus \dots \oplus \mathbf{A}_{rr}$$

The importance of block diagonal matrices is that the algebra of the block matrix is frequently reduced to the algebra of the individual blocks. Specifically, suppose $f(x)$ is a polynomial and \mathbf{M} is the above block diagonal matrix. It follows that $f(\mathbf{M})$ is a block diagonal matrix and

$$f(\mathbf{M}) = \text{diag}(f(\mathbf{A}_{11}), f(\mathbf{A}_{22}), \dots, f(\mathbf{A}_{rr}))$$

Moreover, \mathbf{M} is invertible *iff* each \mathbf{A}_{ii} is invertible, and in this case, \mathbf{M}^{-1} is a block diagonal matrix:

$$\mathbf{M}^{-1} = \text{diag}(\mathbf{A}_{11}^{-1}, \mathbf{A}_{22}^{-1}, \dots, \mathbf{A}_{rr}^{-1})$$

3 Systems of Linear Equations

4 Vector Spaces

5 Inner Product Spaces; Orthogonality

6 Determinants

7 Diagonalization; Eigenvalues and Eigenvectors

8 Linear Mappings

9 Canonical Forms