# Linear Algebra

Julian Dominic

04 November 2022

# Preface

I wrote these notes principally FOR understanding, they are meant for future reference for a refresher on what I have learnt. As such, certain definitions may not be exactly precise but are rephrased into simpler terms.
I am currently using Schaum's Easy Outlines: Linear Algebra - Crash Course. ISBN 978-0-07-139880-0.
I plan to learn and use Linear Algebra by Hoffman and Kunze in the future. But I am currently clearing the pre-requisities for the book as the time of writing (04/11/2022); I am still learning proofs.

# Contents

# 1 Vectors in $\mathbb{R}^n$

Although we will restrict ourselves in this chapter to vectors whose elements come from the field of real numbers, denoted by $\mathbb{R}$, many of our operations also apply to vectors whose entries come from arbitrary field $\boldsymbol{K}$. In the context of vectors, the elements of our number fields are called *scalars*.

## 1.1 List of Numbers

Suppose the height (in centimeters) of eight students are listed as follows:

$$155\ 165\ 175\ 185\ 180\ 170\ 160\ 150$$

One can denote all the values in the list using only one symbol, for example $h$, but with different subscripts; giving rise to $h_i$ where $1 \leq i \leq 8$.

$$h_1\ h_2\ h_3\ h_4\ h_5\ h_6\ h_7\ h_8$$

It is easy to see that each subscript denotes the position of the value in the list; $h_1 = 155$, $h_2 = 165$, etc. Such a list of values, $\boldsymbol{h} = (h_1, h_2, h_3, \ldots, h_8)$ is called a *linear array* or *vector*.

The set of all $n$-tuples of real numbers, denoted by $\mathbb{R}^n$, is called *$n$-space*. A particular $n$-tuple in $\mathbb{R}^n$ like $\boldsymbol{u} = (a_1, a_2, \ldots, a_n)$ is called a *point* or *elements* of $\boldsymbol{u}$. Moreover, when discussing the space/domain of $\mathbb{R}^n$, we use the term *scalar* for the elements in $\mathbb{R}$ (notice that $n = 1$).

Consider two vectors, $u$ and $v$, are equal, we can express it as $\boldsymbol{u} = \boldsymbol{v}$ if they have the same number of elements and if the corresponding elements (position of the elements) are equal. Thi means that while vectors $(1, 2, 3)$ and $(2, 3, 1)$ have the same number of elements, the vectors are not equal because of the corresponding elements do not match each other from one vector to the other.

The vector $(0, 0, \underbrace{\ldots}_{n \text{ dots}}, 0)$; where all of the elements are zero, is called the *zero vector*, and it is usually denoted by $\mathbf{0}$.

## 1.2 Column Vectors

Sometimes a vector in $n$-space $\mathbb{R}^n$ is written vertically, rather than horizontally as shown previously. Such a vector is called a *column vector* while the ones shown previously are called *row vectors*. For example, the following are column vectors that belong to $\mathbb{R}^2$, $\mathbb{R}^2$, $\mathbb{R}^3$, and $\mathbb{R}^3$, respectively:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix} \quad \begin{bmatrix} 1.5 \\ \frac{2}{3} \\ -15 \end{bmatrix}$$

**Remark 1.1.** We also note that any operation defined for row vectors is defined analogously for column vectors.

## 1.3 Vector Addition and Scalar Multiplication

Consider two vectors $u$ and $v$ in $\mathbb{R}^n$:

$$\boldsymbol{u} = (a_1, \ldots, a_n) \text{ and } \boldsymbol{v} = (b_1, \ldots, b_n)$$

Their *sum* can be expressed as $\boldsymbol{u}+\boldsymbol{v}$. $\boldsymbol{u}+\boldsymbol{v}$ is the *vector* that is obtained by adding the corresponding elements from $\boldsymbol{u}$ and $\boldsymbol{v}$. It is as follows:

$$\boldsymbol{u}+\boldsymbol{v} = (a_1 + b_2, \ldots, a_n + b_n)$$

The *scalar product* of the vector $\boldsymbol{u}$ by a real number $k$ can be expressed as $k\boldsymbol{u}$. $k\boldsymbol{u}$ is the vector obtained by multiplying each element of $\boldsymbol{u}$ by $k$. It is as follows:

$$k\boldsymbol{u} = k(a_1, \ldots, a_n) = (ka_1, \ldots, ka_n)$$

**Remark 1.2.** Notice that $\boldsymbol{u}+\boldsymbol{v}$ and $k\boldsymbol{u}$ are also vectors in $\mathbb{R}^n$. The sum of vectors who have different numbers of elements is not defined.

*Negatives* and *subtraction* are defined in $\mathbb{R}^n$ as follows:

$$-\boldsymbol{u} = (-1)\boldsymbol{u} \qquad \text{and} \qquad \boldsymbol{u}-\boldsymbol{v} = \boldsymbol{u}+(-\boldsymbol{v})$$

The vector $-u$ is called the negative of $\boldsymbol{u}$, and $\boldsymbol{u}-\boldsymbol{v}$ is called the *difference* of $\boldsymbol{u}$ and $\boldsymbol{v}$.

Now suppose we are given vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ in $\mathbb{R}^n$ and scalars $k_1, \ldots, k_m$ in $\mathbb{R}$. We can multiply the vectors by corresponding scalars and then add the resultant scalar products to form the vector:

$$\boldsymbol{v} = k_1\boldsymbol{u}_1 + k_2\boldsymbol{u}_2 + \cdots + k_m\boldsymbol{u}_m$$

The vector $\boldsymbol{v}$ is called a *linear combination* of the vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$.

**Theorem 1.3** (Basic properties of vectors under the operations of vector addition and scalar multiplication)
For any vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w} \in \mathbb{R}^n$ and any scalars $k$, $k' \in \mathbb{R}$,

| | |
|---|---|
| $(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w} = \boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$ | $k(\boldsymbol{u}+\boldsymbol{v}) = k\boldsymbol{u}+k\boldsymbol{v}$ |
| $\boldsymbol{u}+\boldsymbol{0} = \boldsymbol{u}$ | $(k+k')\boldsymbol{u} = k\boldsymbol{u}+k'u$ |
| $\boldsymbol{u}+(-\boldsymbol{u}) = \boldsymbol{0}$ | $(kk')\boldsymbol{u} = k(k'\boldsymbol{u})$ |
| $\boldsymbol{u}+\boldsymbol{v} = \boldsymbol{v}+\boldsymbol{u}$ | $1(\boldsymbol{u}) = \boldsymbol{u}$ |

Suppose $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in $\mathbb{R}^n$ for which $\boldsymbol{u} = kv$ for some non-zero scalar $k$ in $\mathbb{R}$. It follows that $\boldsymbol{u}$ is a *multiple* of $\boldsymbol{v}$. Also, $\boldsymbol{u}$ is said to be in the *same* or *opposite direction* as $\boldsymbol{v}$ for $k > 0$ or $k < 0$.

## 1.4 Dot Product

Consider the following arbitrary vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^n$:

$$\boldsymbol{u} = (a_1, \ldots, a_n) \text{ and } \boldsymbol{v} = (b_1, \ldots, b_n)$$

The *dot product* or *inner product* or *scalar product* of $\boldsymbol{u}$ and $\boldsymbol{v}$ is denoted and defined by $\boldsymbol{u} \cdot \boldsymbol{v} = a_1 b_1 + \ldots a_n b_n$. $\boldsymbol{u} \cdot \boldsymbol{v}$ is obtained by multiplying corresponding components and taking the sum of the resulting products. The vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are said to be *orthogonal* (or *perpendicular*) if their *dot product* is zero; $\boldsymbol{u} \cdot \boldsymbol{v} = 0$.

**Theorem 1.4** (Basic properties of the dot product in $\mathbb{R}^n$)

For any vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w} \in \mathbb{R}^n$ and any scalar $k \in \mathbb{R}$,

| | |
|---|---|
| $(\boldsymbol{u} + \boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \boldsymbol{w}$ | $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$ |
| $(k\boldsymbol{u}) \cdot \boldsymbol{v} = k(\boldsymbol{u} \cdot \boldsymbol{v})$ | $u \cdot \boldsymbol{u} \geq 0$, and $\boldsymbol{u} \cdot \boldsymbol{u} = 0$ if $\boldsymbol{u} = 0$ |

To be somewhat complete (and rigorous) with the properties, consider $\boldsymbol{u} \cdot (k\boldsymbol{v})$. It is similar to the cell on row 2, column 1. It follows that $\boldsymbol{u} \cdot (k\boldsymbol{v}) = (k\boldsymbol{v}) \cdot \boldsymbol{u} = k(\boldsymbol{v} \cdot \boldsymbol{u}) = k(\boldsymbol{u} \cdot \boldsymbol{v})$.

The space $\mathbb{R}^n$ with the above operations of vector addition, scalar multiplication, and dot product is usually called **Euclidean $\boldsymbol{n}$-space**.

## 1.5 Norm (Length) of a Vector

The *norm* or *length* of a vector $u$ in $\mathbb{R}^n$ can be expressed as $||\boldsymbol{u}||$. It is defined to be the non-negative square root of $\boldsymbol{u} \cdot \boldsymbol{u}$ (the inner product). For example, if $\boldsymbol{u} = (a_1, \ldots, a_n)$, then $||\boldsymbol{u}|| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}} = \sqrt{a_1^2 + \ldots, a_n^2}$. Therefore, $||\boldsymbol{u}||$ is the square root of the sum of the squares of the elements of $\boldsymbol{u}$. Thus, $||\boldsymbol{u}|| > 0$, and $||\boldsymbol{u}|| = 0$ *iff* $\boldsymbol{u} = 0$.

A vector $\boldsymbol{u}$ is called a *unit vector* if $||\boldsymbol{u}||$ or if $\boldsymbol{u} \cdot \boldsymbol{u} = 1$ (both are equivalent).
For any non-zero vector $v$ in $\mathbb{R}^n$, the vector $\hat{\boldsymbol{v}} = \frac{1}{||\boldsymbol{v}||}\boldsymbol{v} = \frac{\boldsymbol{v}}{||\boldsymbol{v}||}$ is the unique unit vector in the same direction as $\boldsymbol{v}$. The process of finding $\hat{\boldsymbol{v}}$ from $\boldsymbol{v}$ is called *normalizing* $\boldsymbol{v}$.

**Theorem 1.5** (Cauchy-Schwarz Inequality)

For any vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$,

$$|\boldsymbol{u} \cdot \boldsymbol{v}| \leq ||\boldsymbol{u}|| \; ||\boldsymbol{v}||$$

**Theorem 1.6** (Triangle Inequality or Minkowski's Inequality)

For any vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$,

$$||\boldsymbol{u} + \boldsymbol{v}|| \leq ||\boldsymbol{u}|| + ||\boldsymbol{v}||$$

# 2 Algebra of Matrices

# 3 Systems of Linear Equations

# 4 Vector Spaces

# 5 Inner Product Spaces; Orthogonality

# 6 Determinants

# 7 Diagonalization; Eigenvalues and Eigenvectors

# 8 Linear Mappings

# 9 Canonical Forms