Notes from How to Prove It

Julian Dominic

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Preface

I wrote these notes principally FOR understanding, they are meant for future reference for a refresher on what I have learnt.
Honestly, I don't have much to say, I just thought that a preface would be cool. Let's start our journey now.

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1 Sentential Logic

1.1 Deductive reasoning and logical connectives

In an *argument*, we arrive at **valid** *conclusions* assuming that the *premises* are **true**.
Premises and conclusiosn are often referred to as conditions and outcomes respectively.
If all the premises are **true**, then the conclusion should be **true**. However, for the case where the conclusion is **false** while the premises are **true**, the argument is **invalid**.

1.1.1 Logical Operators

Symbol	Meaning	Description
$\vee$	OR	Disjunction
$\wedge$	AND	Conjunction
$\neg$	NOT	Negation

1.2 Truth tables

A truth table must be able to represent all possible combinations of the variables, premises and conclusions.

P	Q	$P \wedge Q$
F	F	F
F	T	F
T	F	F
T	T	T

In this case, we see that our variables (or statements), P and Q have their individual column to assign a value – **True** or **False** – to them.
We use our logical operators to make a new statement from P and Q which is $P \wedge Q$, and assign a value to the new statement.

It is important note that the number of variables will dictate the number of rows that the truth table will have. Construct a truth table for the following set of variables, $\{P\}$, $\{P, Q\}$, $\{P, Q, R\}$.
The pattern that we find is that as the number of variables increases, the number of rows increases two-fold.

$$\text{Number of Rows} = 2^{\text{Number of Variables}}$$

There are some special truth tables where the column for the conclusion always has the same value (either all *true* or all *false*) for every combination of the variables' values.

When the conclusion is always *true*, we say that the conclusion's statement is a **tautologies**. Construct a truth table for $P \vee \neg P$.
Similarly, when the conclusion is always *false*, we say that the conclusion's statement is a **contradiction**. Construct a truth table for $P \wedge \neg P$.

> **Remark 1.1.** *Tautologies* and *Contradictions* are not the only laws that govern logic. Do see the logic document for more.

1.3 Variables and sets

1.3.1 Sets

A set is a collection of elements. The order of the elements in the set does not matter. If an element appears more than once, it is still the same set.

$$\{3, 7, 14\} \equiv \{7, 3, 14\} \equiv \{14, 3, 7, 7\}$$

When the set is infinite or has too many elemnts to list, we will define it explicitly. Suppose we have the following set P.

$$P = \{x \mid x \text{ is a prime number}\}$$

How we read $P = \{x|x\text{is a prime number}\}$ is "P is equal to the set of all x such that x is a prime number." Which also means that P contains all values of x that make the statement "x is a prime number" true.
Some direct translations of the symbols into words would be (i) "{}" means "the set of", and (ii) "|" means "such that".

Sets like P have an **elementhood test** for the set; in this case, the *elementhood test* is being a prime number. Any value of x that makes the statement come out true, passes the test and is an element of the set.
The **Truth set** of a statement $P(x)$ is the set of all values of x that make the statement $P(x)$ true. In other words, it is the set defined by using the statement $P(x)$ as the elementhood test:

$$\text{Truth set of } P(x) = \{x \mid P(x)\}$$

1.3.2 Understanding variables used in sets

There are two types of variables that can appear in a set; **Free variables** and **Bound variables**. *Free variables* are variables that will make the statement either *True* or *False* while *Bound variables* are variables whose values we do not need to know (they can be considered *dummy variables*). Lets consider the following example,

$$y \in \{x \mid x^2 < 9\}$$

For any number y, to verify $y \in \{x \mid x^2 < 9\}$, we have to check $y^2 < 9$. Since $y \in \{x \mid x^2 < 9\}$ is just a roundabout way of saying $y^2 < 9$, it follows that we do not need to care about the value of x. Rather, only the value of y is required. Thus, we can say that y is a *free variable* while x is a *bound variable.* As such, we can go further and replace x with any other variable except y because we do not need to care what x is. As such, it can even be $y \in \{w \mid w^2 < 9\}$ where y is *free* and w is *bound.* Notice that $x^2 < 9$ makes x a free variable. We can say that that statement $P(x)$ in $\{x \mid P(x)\}$ **binds** the variable x.

> **Remark 1.2.** In general, the statement $y \in \{x \mid P(x)\} \Rightarrow P(y)$ and $y \notin \{x \mid P(x)\} \Rightarrow \neg P(y)$.
> It is also important to note that $x \mid P(x)$ is not a statement. It is a set because of the curly brackets "{}"

The **Universe of Discourse**, U, is the set of all possible values for the variables. We can say things such as $\{x \in U \mid P(x)\}$: The set of all x in U such that $P(x)$. For a set that has the *universe of discourse* defined, an element of the set has to pass two tests, $x \in U$ and $P(x)$. Therefore, in general, $y \in \{x \in A \mid P(x)\} \Rightarrow y \in A \wedge P(y)$.
If $P(x)$ is false for every possible value of x, it yields a truth set with no elements. As such, we get the **empty set/null set**; $\emptyset$ or $\{\}$ where the contents inside the curly brackets are blank. For example,

$$\{x \in \mathbb{Z} \mid x \neq x\} = \emptyset = \{\}$$

Remark 1.3. $\emptyset \neq \{\emptyset\}$. $\emptyset$ is a set while $\{\emptyset\}$ is a set of a set.

1.4 Operations on sets

1.5 The conditional and biconditional connectives