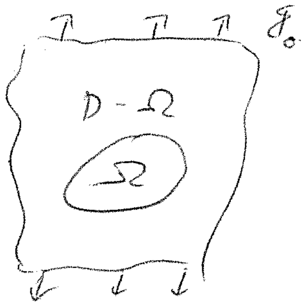
Task : Identify stress in single fiber in infinite matrix



Ansatz : Eshelby

Assumptions : - Fiber $\hat{=}$ Ω

- Compatible inhomogeniety
- Ellipsoidal shape (approximation e.g. Müller 2015 p.44)
- Perfectly bonded
- Isotropic linear elastic — Jöchen 2013

$$\mathbb{C}_f = 3K_f \mathbb{P}_1 + 2G_f \mathbb{P}_2$$

- Matrix $\hat{=}$ $D - \Omega$
  - Infinite
  - Homogeneous far field strain ε_0
  - Isotropic linear elastic

$$\mathbb{C}_m = 3K_m \mathbb{P}_1 + 2G_m \mathbb{P}_2$$

Solution : Identification of the stress in the ellipsoidal fiber reduces to the Equivalent inclusion problem :

$\forall \; (\mathbb{C}_f, \mathbb{C}_m) \; \exists \; [\text{eigenstrain } \varepsilon^*(x) = \text{const} = \varepsilon^* \quad \forall x \in \Omega]$
yielding a strain field inside D which is equivalent to the strain field of the inhomogeniety problem.

Inside the inhomogeniety, the stress reads as

$$\sigma_{\text{Inhom.}} = \underbrace{\mathbb{C}_f}_{\mathbb{C}_f} [\, \varepsilon_0 + \underbrace{\tilde{\varepsilon}(x)}_{\text{fluctuation due to the inhomogeniety}} \,]$$

The stress inside an equivalent inclusion of matrix stiffness reads as

$$\sigma_{\text{eq. inclusion}} = \mathbb{C}_m [\, \varepsilon_0 + \tilde{\varepsilon}(x) - \varepsilon^*(x) \,]$$

For an ellipsoidal inclusion the Eshelby tensor is constant and maps the eigenstrain ε^* to the strain fluctuation $\tilde{\varepsilon}$

$$\tilde{\varepsilon} = \mathbb{E}[\varepsilon^*]$$

The inclusion problem is equivalent to the inhomogeniety problem if

$$\sigma_{\text{inhom.}} = \sigma_{\text{Eq. inclusion}}$$

$$\mathbb{C}_f [\varepsilon_0 + \tilde{\varepsilon}] = \mathbb{C}_m [\varepsilon_0 + \tilde{\varepsilon} - \varepsilon^*]$$

$$\mathbb{C}_f [\varepsilon_0 + \mathbb{E}[\varepsilon^*]] = \mathbb{C}_m [\varepsilon_0 + \mathbb{E}[\varepsilon^*] - \varepsilon^*]$$

$$\mathbb{C}_f [\varepsilon_0] + \mathbb{C}_f \mathbb{E}[\varepsilon^*] = \mathbb{C}_m \mathbb{E}[\varepsilon^*] - \mathbb{C}_m [\varepsilon^*] + \mathbb{C}_m [\varepsilon_0]$$

$$(\mathbb{C}_f - \mathbb{C}_m) [\varepsilon_0] = -[(\mathbb{C}_f - \mathbb{C}_m)\mathbb{E} + \mathbb{C}_m][\varepsilon^*] \quad \Big| (\mathbb{C}_f - \mathbb{C}_m)^{-1}$$

$$\varepsilon_0 = -[\mathbb{E} + (\mathbb{C}_f - \mathbb{C}_m)^{-1} \mathbb{C}_m][\varepsilon^*]$$

$$\Rightarrow \quad \boxed{\varepsilon^* = -\underbrace{[\mathbb{E} + (\mathbb{C}_f - \mathbb{C}_m)^{-1} \mathbb{C}_m]^{-1}}_{=: \mathbb{G} \hat{=} \text{ temporarily useful}} [\varepsilon_0]}$$

The strain field of the inhomogeniety problem inside Ω is linear in ε_0 with

$$\begin{aligned}
\varepsilon &= \varepsilon_0 + \tilde{\varepsilon} \\
&= \varepsilon_0 + \mathbb{E}[\varepsilon^*] \\
&= \varepsilon_0 - \mathbb{E}\,\mathbb{G}[\varepsilon_0]
\end{aligned}$$

$$\varepsilon - \varepsilon_0 = -\mathbb{E}\,\mathbb{G}[\varepsilon_0] \quad | (-1) \quad | \mathbb{E}^{-1}$$

$$\mathbb{E}^{-1}(\varepsilon_0 - \varepsilon) = \mathbb{G}[\varepsilon_0] \quad | \mathbb{G}^{-1}$$

$$\mathbb{G}^{-1}\mathbb{E}^{-1}(\varepsilon_0 - \varepsilon) = \varepsilon_0$$

$$[\mathbb{E} + (\mathbb{C}_f - \mathbb{C}_m)^{-1}\mathbb{C}_m][\mathbb{E}^{-1}(\varepsilon_0 - \varepsilon)] = \varepsilon_0$$

$$(\varepsilon_0 - \varepsilon) + (\mathbb{C}_f - \mathbb{C}_m)^{-1}\mathbb{C}_m\mathbb{E}^{-1}(\varepsilon_0 - \varepsilon) = \varepsilon_0$$

$$(\mathbb{C}_f - \mathbb{C}_m)^{-1}\mathbb{C}_m\mathbb{E}^{-1}(\varepsilon_0 - \varepsilon) = \varepsilon$$

$$(\varepsilon_0 - \varepsilon) = \mathbb{E}\,\mathbb{C}_m^{-1}(\mathbb{C}_f - \mathbb{C}_m)\,\varepsilon$$

$$\varepsilon_0 = [\mathbb{1} + \mathbb{E}\,\mathbb{C}_m^{-1}(\mathbb{C}_f - \mathbb{C}_m)]\,\varepsilon$$

$$\Rightarrow \quad \boxed{\varepsilon = \Big(\underbrace{\underbrace{\mathbb{E}\, \mathbb{C}_m^{-1}}_{=:\, \mathbb{P}_m} (\mathbb{C}_f - \mathbb{C}_m) + \mathbb{1}}_{=:\, \mathbb{A}_m} \Big)^{-1} [\varepsilon_0]}$$

with

$\mathbb{E} \mathrel{\hat{=}}$ Eshelbys tensor mapping eigenstrain to strain due to eigenstrain $\qquad \tilde{\varepsilon} = \mathbb{E}[\varepsilon^*]$

$\mathbb{P}_m \mathrel{\hat{=}}$ Hills polarization tensor mapping stress due to eigenstrain to strain due to eigenstrain $\qquad \mathbb{P}_m = \mathbb{E}\, \mathbb{C}_m^{-1}$

$\tilde{\varepsilon} = \mathbb{P}_m \mathbb{C}_m [\varepsilon^*]$

$\mathbb{A}_m \mathrel{\hat{=}}$ Strain localization tensor $\qquad \varepsilon = \mathbb{A}[\varepsilon_0]$