Basic meanfield quantities

$\bar{\cdot} \hat{=}$ Effective quantity
$\hat{\cdot} \hat{=}$ Fluctuation
$\langle\cdot\rangle \hat{=}$ Volume average

Effective quantities

$$\bar{\sigma} = \langle \sigma \rangle = \sigma(x) - \hat{\sigma}(x)$$

$$\bar{\varepsilon} = \langle \varepsilon \rangle = \varepsilon(x) - \hat{\varepsilon}(x)$$

$$\exists\ \bar{\mathbb{C}} \text{ with } \bar{\sigma} = \bar{\mathbb{C}}[\bar{\varepsilon}]$$

if:
- no cracks / pores
- local linear elastic material $\sigma(x) = \mathbb{C}(x)\,\varepsilon(x)$

] linear BVP

Decomposition of volume average

$$\langle \psi \rangle = \frac{1}{V} \int_V \psi \, dv$$

$$= \frac{1}{V} \sum_\alpha \int_{V_\alpha} \psi \, dv = \sum_\alpha \frac{V_\alpha}{V} \frac{1}{V_\alpha} \int_{V_\alpha} \psi \, dv$$

$$\boxed{\langle \psi \rangle = \sum_\alpha c_\alpha \langle \psi \rangle_\alpha}$$ exact!

Localization tensors

$$\boxed{\begin{aligned} \varepsilon(x) &= \mathbb{A}(x)[\bar{\varepsilon}] \\ \sigma(x) &= \mathbb{B}(x)[\bar{\sigma}] \end{aligned}}$$

with $\langle \mathbb{A}(x) \rangle = \langle \mathbb{B}(x) \rangle = \mathbb{1}^S$

$\mathbb{A} = \mathbb{A}^{T_R} = \mathbb{A}^{T_L}$

$\mathbb{B} = \mathbb{B}^{T_R} = \mathbb{B}^{T_L}$

and relation: $\sigma = \mathbb{C}[\varepsilon] = \mathbb{C}\mathbb{A}[\bar{\varepsilon}] = \underbrace{\mathbb{C}\mathbb{A}\bar{\mathbb{S}}}_{=\mathbb{B}}[\bar{\sigma}] \Rightarrow \boxed{\mathbb{B} = \mathbb{C}\mathbb{A}(\bar{\mathbb{C}})^{-1}}$

Effective stiffness

$$\bar{\sigma} = \langle \sigma \rangle = \langle \mathbb{C}[\varepsilon] \rangle = \langle \mathbb{C}\mathbb{A}[\bar{\varepsilon}] \rangle = \underbrace{\langle \mathbb{C}\mathbb{A} \rangle}_{=\bar{\mathbb{C}}}[\bar{\varepsilon}] \Rightarrow \boxed{\bar{\mathbb{C}} = \langle \mathbb{C}\mathbb{A} \rangle}$$

$$\bar{\varepsilon} \cdot \bar{\sigma} = \langle \varepsilon \rangle \cdot \langle \sigma \rangle = \langle \varepsilon \cdot \sigma \rangle$$

$$= \langle \mathbb{A}[\bar{\varepsilon}] \cdot \mathbb{C}\mathbb{A}[\bar{\varepsilon}] \rangle = \bar{\varepsilon} \langle \mathbb{A}^{T_H} \mathbb{C}\mathbb{A} \rangle \bar{\varepsilon} \Rightarrow \boxed{\bar{\mathbb{C}} = \mathbb{A}^{T_H} \mathbb{C}\mathbb{A}}$$

$$\langle \mathbb{A} \rangle = \mathbb{1}^s = c_1 \langle \mathbb{A} \rangle_1 + c_2 \langle \mathbb{A} \rangle_2 + c_3 \langle \mathbb{A} \rangle_3 + \dots$$

$$\Rightarrow \langle \mathbb{A} \rangle_1 = \frac{1}{c_1} \left[\mathbb{1}^s - c_2 \langle \mathbb{A} \rangle_2 - c_3 \langle \mathbb{A} \rangle_3 - \dots \right]$$

$$= \frac{1}{c_1} \left[\mathbb{1}^s - \sum_\alpha c_\alpha \langle \mathbb{A} \rangle_\alpha \right]$$

$$\bar{\mathbb{C}} = \langle \mathbb{C} \mathbb{A} \rangle = c_1 \langle \mathbb{C} \mathbb{A} \rangle_1 + c_2 \langle \mathbb{C} \mathbb{A} \rangle_2 + \dots$$ piecewise constant stiffness

$$= c_1 \mathbb{C}_1 \langle \mathbb{A} \rangle_1 + c_2 \mathbb{C}_2 \langle \mathbb{A} \rangle_2 + \dots$$

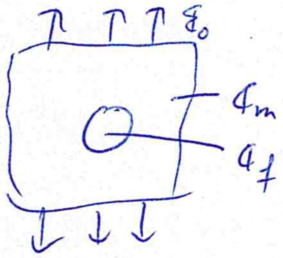
$$= \mathbb{C}_1 \left[\mathbb{1}^s - \sum_\alpha c_\alpha \langle \mathbb{A} \rangle_\alpha \right] + c_2 \mathbb{C}_2 \langle \mathbb{A} \rangle_2 + \dots$$

$$= \mathbb{C}_1 + c_2 (\mathbb{C}_2 - \mathbb{C}_1) \langle \mathbb{A} \rangle_2 + \dots$$

$$\boxed{\bar{\mathbb{C}} = \mathbb{C}_m + \sum_\alpha^{\text{inclusion}} c_\alpha (\mathbb{C}_\alpha - \mathbb{C}_m) \langle \mathbb{A} \rangle_\alpha}$$

useful if $\langle \mathbb{A} \rangle_m$ is unknown
but $\langle \mathbb{A} \rangle_\alpha \ \forall \alpha \in$ inclusions are known

<u>Single ellipsoidal inhomogeniety</u>



$$\langle \varepsilon \rangle_f = \langle \mathbb{A} \rangle_f [\bar{\varepsilon}_0] = \text{const.}$$

$$\langle \mathbb{A} \rangle_f = \boxed{\mathbb{A}^{SI}} = \text{const.}$$

$$\langle \sigma \rangle_f = \langle \mathbb{B} \rangle_f [\bar{\sigma}_0] = \text{const.}$$

$$\langle \mathbb{B} \rangle_f = \boxed{\mathbb{B}^{SI} = \mathbb{C}_f \mathbb{A}^{SI} \mathbb{C}_m^{-1}}$$

$$\boxed{\mathbb{A}^{SI} = (\mathbb{P} (\mathbb{C}_f - \mathbb{C}_m) + \mathbb{1}^s)^{-1}}$$

$$\boxed{\mathbb{P} = \mathbb{E} \mathbb{C}_m^{-1}}$$

$\mathbb{P} \hat{=}$ Hill polarization tensor

$\mathbb{E} \hat{=}$ Eshelby tensor

$= \mathbb{E}(\text{geometry}, \nu_{matrix})$

$\nu \hat{=}$ Poissons ratio

Mori-Tanaka

Sought: $\langle \varepsilon \rangle_m = \mathbb{A}_m^{MT} \langle \varepsilon \rangle$

$\langle \varepsilon \rangle_f = \mathbb{A}_f^{MT} \langle \varepsilon \rangle$

Ansatz: $\boxed{\langle \varepsilon \rangle_f = \mathbb{A}^{SI} [\langle \varepsilon \rangle_m]}$

$$\Rightarrow \langle \varepsilon \rangle = c_m \langle \varepsilon \rangle_m + c_f \langle \varepsilon \rangle_f$$

$$= c_m \langle \varepsilon \rangle_m + c_f \mathbb{A}^{SI} \langle \varepsilon \rangle_m$$

$$= \left(c_m \mathbb{1} + c_f \mathbb{A}^{SI} \right) \langle \varepsilon \rangle_m$$

$$\Rightarrow \boxed{\mathbb{A}_m^{MT} = \left(c_m \mathbb{1} + c_f \mathbb{A}^{SI} \right)^{-1}}$$

$$\langle \varepsilon \rangle = c_m \langle \varepsilon \rangle_m + c_f \langle \varepsilon \rangle_f$$

$$= c_m \left(\mathbb{A}^{SI} \right)^{-1} \langle \varepsilon \rangle_f + c_f \langle \varepsilon \rangle_f$$

$$= \left(c_m \left(\mathbb{A}^{SI} \right)^{-1} + c_f \mathbb{1} \right) \langle \varepsilon \rangle_f$$

$$\Rightarrow \boxed{\mathbb{A}_f^{MT} = \left(c_m \left(\mathbb{A}^{SI} \right)^{-1} + c_f \mathbb{1} \right)^{-1}}$$

$$= \left(\left(\mathbb{A}^{SI} \right)^{-1} \left(c_m \mathbb{1} + c_f \mathbb{A}^{SI} \right) \right)^{-1}$$

$$= \mathbb{A}^{SI} \underbrace{\left(c_m \mathbb{1} + c_f \mathbb{A}^{SI} \right)^{-1}}_{= \mathbb{A}_m^{MT}}$$

Analogie

vgl. Durchbauer, Schemmann

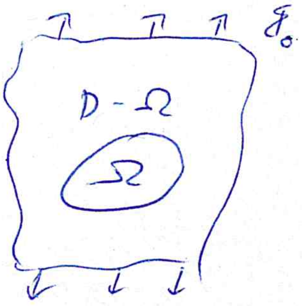
$$\langle \sigma \rangle_f = \mathbb{B}^{SI} [\langle \sigma \rangle_m]$$

$$\mathbb{B}_m^{MT} = \left(c_m \mathbb{1} + c_f \mathbb{B}^{S} \right)^{-1}$$

$$\mathbb{B}_f^{MT} = \mathbb{B}^{SI} \mathbb{B}_m^{MT}$$

$\Rightarrow$ vergleiche $\bar{\mathbb{C}}$ mit $\bar{\mathbb{S}}^{-1}$

<u>Task</u> : Identify stress in single fiber in infinite matrix



<u>Ansatz</u> : Eshelby

<u>Assumptions</u> : - Fiber $\hat{=}$ Ω

- Compatible inhomogeniety
- Ellipsoidal shape (approximation e.g. Müller 2015 p.44)
- Perfectly bonded
- Isotropic linear elastic — Jöchen 2013

$$\mathbb{C}_f = 3k_f \mathbb{P}_1 + 2G_f \mathbb{P}_2$$

- Matrix $\hat{=}$ D − Ω
  - Infinite
  - Homogeneous far field strain ε_0
  - Isotropic linear elastic

$$\mathbb{C}_m = 3k_m \mathbb{P}_1 + 2G_m \mathbb{P}_2$$

<u>Solution</u> : Identification of the stress in the ellipsoidal fiber reduces to the <u>Equivalent inclusion problem</u> :

$$\forall (\mathbb{C}_f, \mathbb{C}_m) \; \exists \; [\text{eigenstrain } \varepsilon^*(x) = \text{const} = \varepsilon^* \quad \forall x \in \Omega]$$

yielding a strain field inside D which is equivalent to the ~~...~~ strain field of the inhomogeniety problem.

Inside the inhomogeniety, the stress reads as

$$\sigma_{\text{inhom.}} = \mathbb{C}_f \left[\varepsilon_0 + \tilde{\varepsilon}(x) \right]$$

↳ $\mathbb{C}_f$

└ fluctuation due to the inhomogeniety

The stress inside an equivalent inclusion of matrix stiffness reads as

$$\sigma_{\text{eq. inclusion}} = \mathbb{C}_m \left[\varepsilon_0 + \tilde{\varepsilon}(x) - \varepsilon^*(x) \right]$$

For an ellipsoidal inclusion the Eshelby tensor is constant and maps the eigenstrain ε^* to the strain fluctuation $\tilde{\varepsilon}$

$$\tilde{\varepsilon} = \mathbb{E}[\varepsilon^*]$$

The inclusion problem is equivalent to the inhomogeniety problem if

$$\sigma_{\text{Inhom.}} = \sigma_{\text{Eq. inclusion}}$$

$$\mathbb{C}_f[\varepsilon_0 + \tilde{\varepsilon}] = \mathbb{C}_m[\varepsilon_0 + \tilde{\varepsilon} - \varepsilon^*]$$

$$\mathbb{C}_f[\varepsilon_0 + \mathbb{E}[\varepsilon^*]] = \mathbb{C}_m[\varepsilon_0 + \mathbb{E}[\varepsilon^*] - \varepsilon^*]$$

$$\mathbb{C}_f[\varepsilon_0] + \mathbb{C}_f\mathbb{E}[\varepsilon^*] = \mathbb{C}_m\mathbb{E}[\varepsilon^*] - \mathbb{C}_m[\varepsilon^*] + \mathbb{C}_m[\varepsilon_0]$$

$$(\mathbb{C}_f - \mathbb{C}_m)[\varepsilon_0] = -[(\mathbb{C}_f - \mathbb{C}_m)\mathbb{E} + \mathbb{C}_m][\varepsilon^*] \quad |(\mathbb{C}_f-\mathbb{C}_m)^{-1}$$

$$\varepsilon_0 = -[\mathbb{E} + (\mathbb{C}_f - \mathbb{C}_m)^{-1}\mathbb{C}_m][\varepsilon^*]$$

$$\Rightarrow \quad \boxed{\varepsilon^* = -\underbrace{[\mathbb{E} + (\mathbb{C}_f - \mathbb{C}_m)^{-1}\mathbb{C}_m]^{-1}}_{=:\ \mathbb{G}\ \hat{=}\ \text{temporarily useful}}[\varepsilon_0]}$$

The strain field of the inhomogeniety problem inside Ω is linear in ε_0 with

$$\varepsilon = \varepsilon_0 + \tilde{\varepsilon}$$

$$= \varepsilon_0 + \mathbb{E}[\varepsilon^*]$$

$$= \varepsilon_0 - \mathbb{E}\,\mathbb{G}[\varepsilon_0]$$

$$\varepsilon - \varepsilon_0 = -\mathbb{E}\,\mathbb{G}[\varepsilon_0] \quad |(-1)\ |\mathbb{E}^{-1}$$

$$\mathbb{E}^{-1}(\varepsilon_0 - \varepsilon) = \mathbb{G}[\varepsilon_0] \quad |\mathbb{G}^{-1}$$

$$\mathbb{G}^{-1}\mathbb{E}^{-1}(\varepsilon_0 - \varepsilon) = \varepsilon_0$$

$$[\mathbb{E} + (\mathbb{C}_f - \mathbb{C}_m)^{-1}\mathbb{C}_m][\mathbb{E}^{-1}(\varepsilon_0 - \varepsilon)] = \varepsilon_0$$

$$(\varepsilon_0 - \varepsilon) + (\mathbb{C}_f - \mathbb{C}_m)^{-1}\mathbb{C}_m\mathbb{E}^{-1}(\varepsilon_0 - \varepsilon) = \varepsilon_0$$

$$(\mathbb{C}_f - \mathbb{C}_m)^{-1}\mathbb{C}_m\mathbb{E}^{-1}(\varepsilon_0 - \varepsilon) = \varepsilon$$

$$(\varepsilon_0 - \varepsilon) = \mathbb{E}\,\mathbb{C}_m^{-1}(\mathbb{C}_f - \mathbb{C}_m)\,\varepsilon$$

$$\varepsilon_0 = [\mathbb{1} + \mathbb{E}\,\mathbb{C}_m^{-1}(\mathbb{C}_f - \mathbb{C}_m)]\,\varepsilon$$

$$\Rightarrow \quad \boxed{\varepsilon = \big(\underbrace{\underbrace{\mathbb{E}\, \mathbb{C}_m^{-1}}_{=:\mathbb{P}} (\mathbb{C}_f - \mathbb{C}_m) + \mathbb{1}\big)^{-1}}_{=:\mathbb{A}} [\varepsilon_0]}$$

with

$\mathbb{E} \,\hat{=}$ Eshelbys tensor mapping eigenstrain to strain du to eigenstrain $\qquad \tilde{\varepsilon} = \mathbb{E}[\varepsilon^*]$

$\mathbb{P} \,\hat{=}$ Hills polarization tensor mapping stress due to eigenstrain to strain du to eigenstrain $\qquad \mathbb{P} = \mathbb{E}\, \mathbb{C}_m^{-1}$

$\tilde{\varepsilon} = \mathbb{P}\, \mathbb{C}_m [\varepsilon^*]$

$\mathbb{A} \,\hat{=}$ Strain localization tensor $\qquad \varepsilon = \mathbb{A}[\varepsilon_0]$

Stress localization in single inhomogeniety

Given: $$\varepsilon = \mathbb{A}[\varepsilon_0]$$

Sought: $\mathbb{B}$ with $$\sigma = \mathbb{B}[\sigma_0]$$

Solution: with $$\sigma = \mathbb{C}_f[\varepsilon]$$

$$\sigma_0 = \mathbb{C}_m[\varepsilon_0]$$

$$\Rightarrow \quad \sigma = \mathbb{B}[\sigma_0]$$

$$\mathbb{C}_f[\varepsilon] = \mathbb{B}\,\mathbb{C}_m[\varepsilon_0]$$

$$\varepsilon = \underbrace{\mathbb{C}_f^{-1}\,\mathbb{B}\,\mathbb{C}_m}_{=:\mathbb{A}^{SI}}[\varepsilon_0] \qquad \Rightarrow \boxed{\mathbb{B}^{SI} = \mathbb{C}_f\,\mathbb{A}\,\mathbb{C}_m^{-1}}$$

In literature:

$$\mathbb{B}^{SI} = \left[\mathbb{1}^S + \mathbb{C}_m\left(\mathbb{1}^S - \mathbb{P}_0\mathbb{C}_m\right)\left(\mathbb{C}_f^{-1} - \mathbb{C}_m^{-1}\right)\right]^{-1}$$

e.g. Schemmann 2018

Duschlbauer 2002

Proof: $$\mathbb{A}^{SI} = \left(\mathbb{P}_0\left(\mathbb{C}_f - \mathbb{C}_m\right) + \mathbb{1}\right)^{-1} = \mathbb{C}_f^{-1}\,\mathbb{B}\,\mathbb{C}_m$$

$$\mathbb{1} = \left(\mathbb{P}_0\left(\mathbb{C}_f - \mathbb{C}_m\right) + \mathbb{1}\right)\mathbb{C}_f^{-1}\,\mathbb{B}\,\mathbb{C}_m$$

$$= \left(\mathbb{P}_0\left(\mathbb{C}_f - \mathbb{C}_m\right)\mathbb{C}_f^{-1} + \mathbb{C}_f^{-1}\right)\mathbb{B}\,\mathbb{C}_m$$

$$= \left(\mathbb{P}_0\left(\mathbb{1} - \mathbb{C}_m\mathbb{C}_f^{-1}\right) + \mathbb{C}_f^{-1}\right)\mathbb{B}\,\mathbb{C}_m$$

$$\mathbb{B} = \left(\mathbb{P}_0\left(\mathbb{1} - \mathbb{C}_m\mathbb{C}_f^{-1}\right) + \mathbb{C}_f^{-1}\right)^{-1}\mathbb{C}_m^{-1}$$

$$= \left[\mathbb{C}_m\left[\mathbb{P}_0\left(\mathbb{1} - \mathbb{C}_m\mathbb{C}_f^{-1}\right) + \mathbb{C}_f^{-1}\right]\right]^{-1}$$

$$\mathbb{B}^{-1} = \mathbb{C}_m\left(\mathbb{C}_f^{-1} - \mathbb{P}_0\left(\mathbb{C}_m\mathbb{C}_f^{-1} - \mathbb{1}\right)\right)$$

$$= \mathbb{C}_m\mathbb{C}_f^{-1} - \mathbb{C}_m\mathbb{P}_0\mathbb{C}_m\mathbb{C}_f^{-1} + \mathbb{C}_m\mathbb{P}_0$$

$$= \mathbb{C}_m\mathbb{C}_f^{-1} - \mathbb{C}_m\mathbb{P}\,\mathbb{C}_m\mathbb{C}_f^{-1} + \mathbb{C}_m\mathbb{P}_0\left(\mathbb{C}_m\mathbb{C}_m^{-1}\right) + \left[\mathbb{1}^S - \mathbb{C}_m\mathbb{C}_m^{-1}\right]$$

$$\mathbb{B}^{-1} = \mathbb{1}^S + \mathbb{C}_m \mathbb{C}_f^{-1} - \mathbb{C}_m \mathbb{C}_m^{-1} - \mathbb{C}_m \mathbb{P}_0 \mathbb{C}_m \mathbb{C}_f^{-1} + \mathbb{C}_m \mathbb{P}_0 (\mathbb{C}_m \mathbb{C}_m^{-1})$$

$$= \mathbb{1}^S + (\mathbb{C}_m - \mathbb{C}_m \mathbb{P}_0 \mathbb{C}_m)(\mathbb{C}_f^{-1} - \mathbb{C}_m^{-1})$$

$$\mathbb{B}^{SI} = \Big[\mathbb{1}^S + \underbrace{\mathbb{C}_m(\mathbb{1}^S - \mathbb{P}_0 \mathbb{C}_m)}_{\text{similar to } \mathbb{P}_0 \text{ in expression for } \mathbb{A}}(\mathbb{C}_f^{-1} - \mathbb{C}_m^{-1})\Big]^{-1}$$

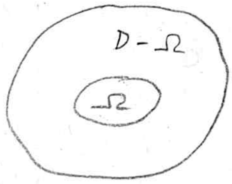
Toshio Mura / Micromechanics of Defects in Solids / 2. Edition

General theory of eigenstrains

1. Definition

Eigenstrain $\hat{=}$ Nonelastic strain

Eigenstress $\hat{=}$ self-equilibrated internal stress caused by eigenstrains in bodies free from external loads or surface constraints



Inclusion $\hat{=}$ subdomain Ω with eigenstrain in eigenstrain-free homogeneous part D

Inhomogeneity $\hat{=}$ subdomain Ω with other elastic moduli

2 Fundamental equations of elasticity

Free body D $\iff$ D is free from any external surface or body force

Lookeslow

$\epsilon \mathrel{\hat{=}}$ total strain
$e \mathrel{\hat{=}}$ elastic strain
$\epsilon^* \mathrel{\hat{=}}$ eigenstrain

(2.1) $$\epsilon_{ij} = e_{ij} + \epsilon^*_{ij}$$

2.2) $$\epsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$ $\iff$ total strain must be compatible

(2.3) $$\sigma_{ij} = C_{ijkl}\, e_{kl} \overset{(2.1)}{=} C_{ijkl}\left(\epsilon_{kl} - \epsilon^*_{kl}\right)$$
dB: because $C \in$ Sym
$$= C_{ijkl}\left(u_{k,l} - \epsilon^*_{kl}\right)$$

(Inverse of 2.3)

+ expressions for (2.3) & (2.6) for isotropic case

2. $$\epsilon_{ij} - \epsilon^*_{ij} = \underbrace{C^{-1}_{ijkl}}_{=:\ \text{elastic compliance}} \sigma_{kl}$$

⋮ representations of (2.3) and (2.6) for 3D
2D
with notes on plane stress
plane strain

Equilibrium conditions

If D is not free $\Rightarrow$ Use superposition of eigenstress and stress of boundary value problem]

.10) $$\sigma_{ij,j} = 0$$ and

.11) $$\sigma_{ij}\, n_j = 0$$ for free external surfaces

[external unit normal vector

2.3) in (2.10)

$$(2.12) \qquad (C_{ijkl}\, u_{k,l} - C_{ijkl}\, \epsilon^*_{kl})_{,j} = 0 = \sigma_{ij,j}$$

$\iff \quad C_{ijkl}\, u_{k,lj} = C_{ijkl}\, \epsilon^*_{kl,j}$ similar to $\operatorname{div}(\sigma) + b = 0$

with $C_{ijkl}\,\epsilon^*_{kl,j} \hat{=} -b_i$

acting as <u>body force</u>

2.3) in (2.11)

$$(2.13) \qquad \sigma_{ij}\, n_j = (C_{ijkl}\, u_{k,l} - C_{ijkl}\, \epsilon^*_{kl})\, n_j = 0$$

$\iff \quad C_{ijkl}\, u_{k,l}\, n_j = C_{ijkl}\, \epsilon^*_{kl}\, n_j$ with $C_{ijkl}\, \epsilon^*_{kl}\, n_j$

acting as <u>surface force</u>

$\Rightarrow$ The elastic displacement field caused by ϵ^*_{ij} in a free body is <u>equivalent</u> to that caused by a body force $(-C_{ijkl}\, \epsilon^*_{kl,j})$ and surface force $(C_{ijkl}\, \epsilon^*_{kl}\, n_j)$

If D is an infinetily extended body,

use $\sigma_{ij}(x) \to 0$ for $x \to \infty$ instead of (2.11)

<u>Compatibility conditions</u>

$$(2.14) \qquad \underbrace{\epsilon_{pki}\, \epsilon_{qlj}}_{\text{Permutation symbol}}\, \underbrace{\epsilon_{ij,kl}}_{\text{2. derivative of total strain}} = 0 \quad \overset{\partial B}{\iff} \quad \operatorname{rot}(\operatorname{rot}(\epsilon)) = 0$$

Fundamental equation: (2.12)

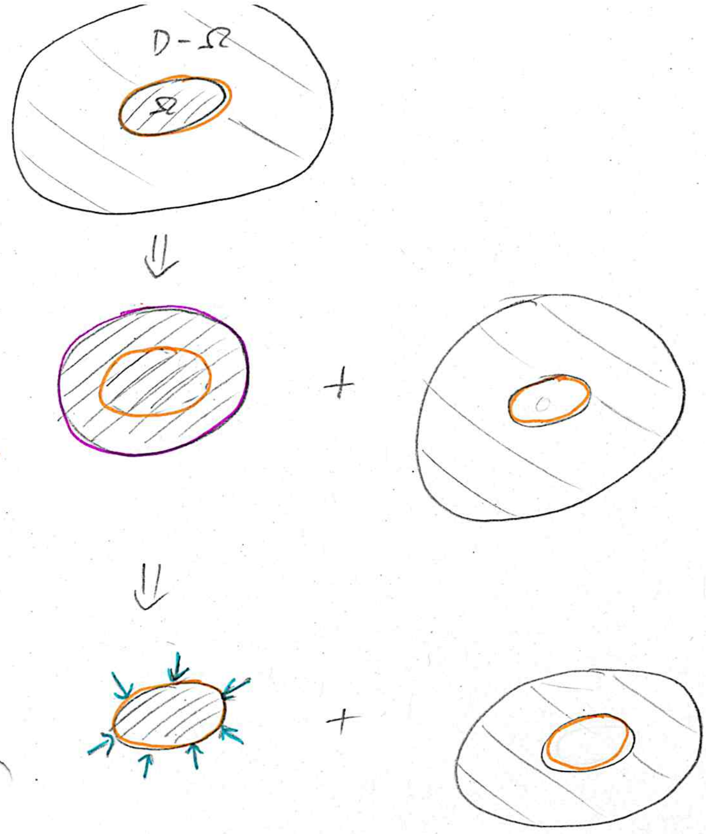
Sometimes useful alternative: (2.10) + (2.3) + (2.14)

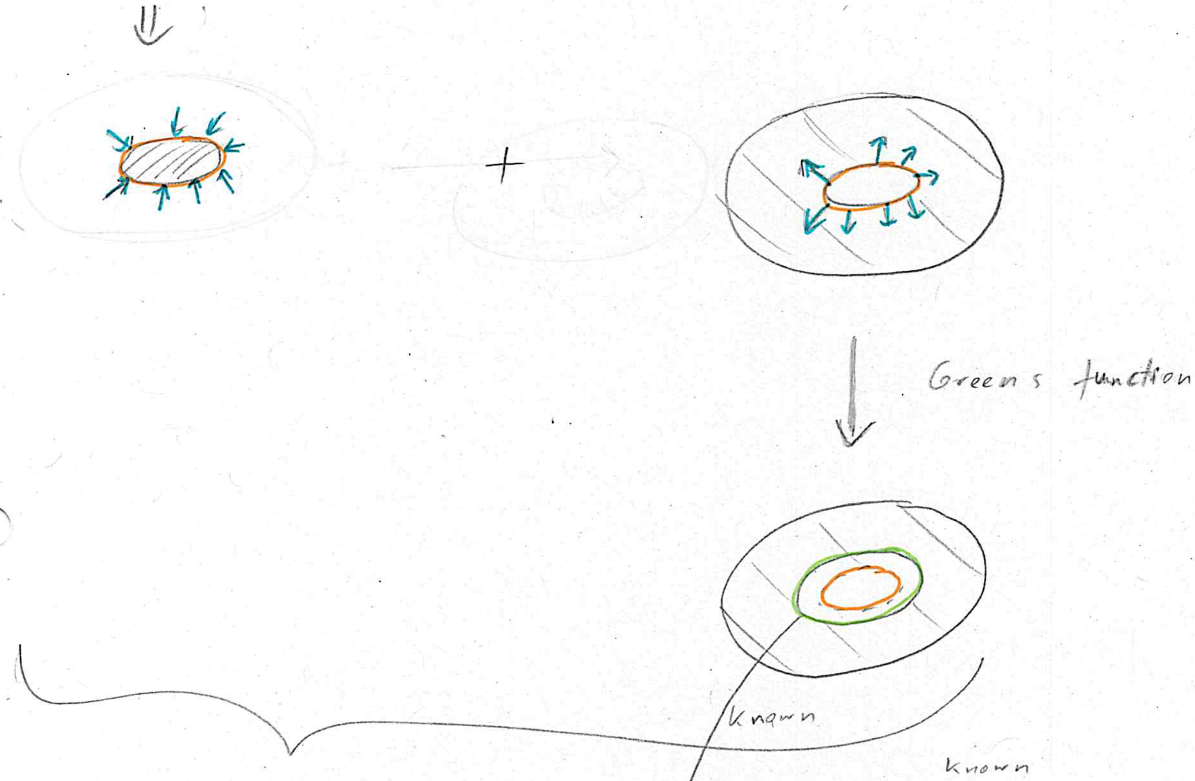
Eigenstresses are <u>caused by constraint</u> from the surrounding elastic medium, which prohibits the geometrically incompatible deformation of ϵ^*_{ij}.

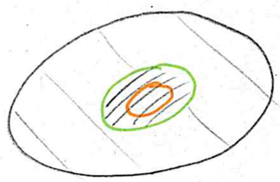
Initial shape inclusion

Shape inclusion with eigenstrain ε^*

Equilibrium shape inclusion







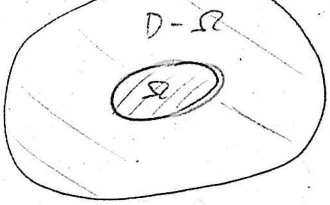
$$\varepsilon(x) = \varepsilon_e(x) + \overbrace{\varepsilon^*(x)}^{\text{known}} \quad \forall x \in \Omega$$

$$\varepsilon(x) = \varepsilon_e(x) \quad \forall x \in D-\Omega$$

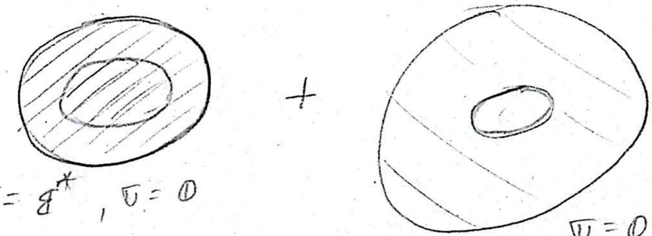
Initial shape inclusion

Shape inclusion with eigenstrain ε^*

Equilibrium shape inclusion



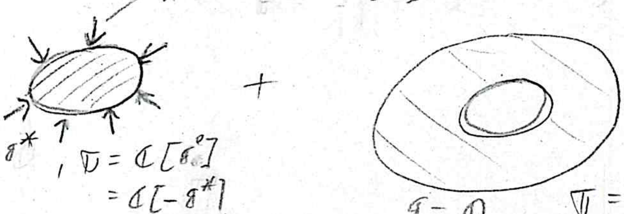
$\Downarrow$



$\varepsilon = \varepsilon^*, \ \sigma = 0$ + $\varepsilon = 0, \ \sigma = 0$

$\Downarrow$

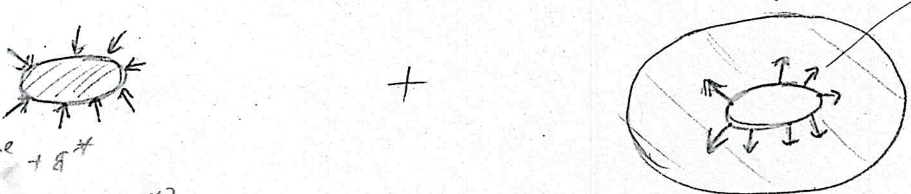
$t = -\sigma^* [n]$



$\varepsilon = \varepsilon^e + \varepsilon^*, \ \sigma = \mathbb{C}[\varepsilon^e] = \mathbb{C}[-\varepsilon^*] =: -\sigma^*$ + $\varepsilon = 0, \ \sigma = 0$

$\Downarrow$

$t = \mathbb{C}\,\sigma^* [n]$

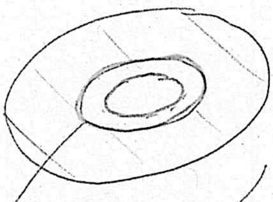


$\varepsilon = 0 = \varepsilon^e + \varepsilon^*$

$\sigma = \mathbb{C}[\varepsilon^e] = \mathbb{C}[-\varepsilon^*]$

$\Downarrow$ Greens function

$$u^c(x) = \int_S U(x - x', \sigma^*[n]\,dS)\,dS'$$

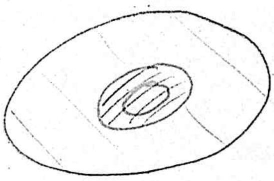


$$\varepsilon^c(x) = \text{sym grad}\,(u^c(x)) \quad \forall x \in D - \Omega$$

$$\sigma^c(x) = \mathbb{C}[\varepsilon^c(x)]$$

known

known



$$\varepsilon^c(x) = \varepsilon(x) = \varepsilon^e(x) + \overline{\varepsilon^*(x)} \quad \forall x \in \Omega$$

$$\varepsilon^c(x) = \varepsilon(x) = \varepsilon^e(x) \quad \forall x \in D - \Omega$$

$$\left.\begin{aligned} \varepsilon(x) &= \varepsilon^c(x) = \varepsilon^e(x) + \varepsilon^*(x) \\ \sigma(x) &= \mathbb{C}[\varepsilon^e] = \mathbb{C}[\varepsilon^c - \varepsilon^*] \end{aligned}\right\} \forall x \in \Omega$$

$$\sigma(x) = \mathbb{C}[\varepsilon^c(x)] \quad \forall x \in D - \Omega$$