Eshelby integrals of specific geometries (Mura 1987 page 79) 29.04.19

Note: Eshelby's tensor $\mathbb{S}$ does not depend on absolute values of a_i $i \in [1,2,3]$.

For example: a_i in (11.16) reduce with denominator of I_i with $i \in [11, 22, 33, 12, 23, 31]$ for sphere

$\Rightarrow$ Formulas for $\mathbb{S}$ could be expressed in $w_{ij} = \frac{a_i}{a_j}$ with (i,j) in $[(1,2),(1,3),(2,3)]$

However, the notation of Mura 1987 is adopted to enable explicit comparison of the formulas.

$$\boxed{\begin{aligned} \left(a_1^2 - a_3^2\right)^{\frac{3}{2}} &= \left(\sqrt{a_1^2\left(1 - \frac{1}{w_{13}^2}\right)}\right)^3 = a_1^3\left(1 - \frac{1}{w_{13}^2}\right)^{\frac{3}{2}} \\ a_1^2 a_3 &= a_1^3 \frac{a_3}{a_1} = a_1^3\left(\frac{1}{w_{31}}\right) \\ a_1^2 + a_2^2 &= a_2^2\left(\frac{a_1^2}{a_2^2} + 1\right) = a_2^2\left(w_{12}^2 + 1\right) \end{aligned}}$$

Sphere: $a_1 = a_2 = a_3 = r$

$$I_1 = I_2 = I_3 = \frac{4\pi}{3}$$

$$I_{11} = I_{22} = I_{33}$$

$$= I_{12} = I_{23} = I_{31} = \frac{4\pi}{5r^2}$$

Oblate spheroid : $a_1 = a_2 > a_3$

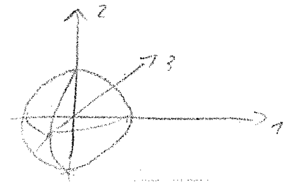
$$I_1 = I_2 = \frac{2\pi a_1^2 a_3}{(a_1^2 - a_3^2)^{\frac{3}{2}}} \left[\cos^{-1}\left(\frac{a_3}{a_1}\right) - \frac{a_3}{a_1}\left(1 - \frac{a_3^2}{a_1^2}\right)^{\frac{1}{2}} \right]$$

$$I_3 = 4\pi - I_1$$

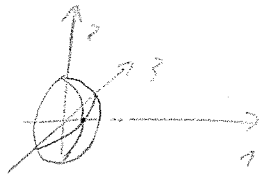
$$I_{11} = I_{22} = I_{12} = \frac{\pi}{a_1^2} - \frac{I_1 - I_3}{4(a_3^2 - a_1^2)}$$

$$I_{13} = I_{23} = \frac{I_1 - I_3}{a_3^2 - a_1^2}$$

$$I_{33} = \frac{4\pi}{3a_3^2} - \frac{2}{3} I_{13}$$



Permute indices to get representation with $a_1 < a_2 = a_3$



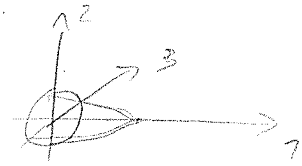
$$I_3 = I_2 = \frac{2\pi a_3^2 a_1}{(a_3^2 - a_1^2)^{\frac{3}{2}}} \left[\cos^{-1}\left(\frac{a_1}{a_3}\right) - \frac{a_1}{a_3}\left(1 - \frac{a_1^2}{a_3^2}\right)^{\frac{1}{2}} \right]$$

$$I_1 = 4\pi - 2I_3$$

$$I_{33} = I_{22} = I_{32} = \frac{\pi}{a_3^2} - \frac{I_3 - I_1}{4(a_1^2 - a_3^2)}$$

$$I_{31} = I_{21} = \frac{I_3 - I_1}{a_1^2 - a_3^2}$$

$$I_{11} = \frac{4\pi}{3a_1^2} - \frac{2}{3} I_{31}$$



Prolate spheroid : $a_1 > a_2 = a_3$

$$I_2 = I_3 = \frac{2\pi\, a_1\, a_3^2}{(a_1^2 - a_3^2)^{\frac{3}{2}}} \left[\frac{a_1}{a_3} \left(\frac{a_1^2}{a_3^2} - 1 \right)^{\frac{1}{2}} - \cosh^{-1}\left(\frac{a_1}{a_3}\right) \right]$$

$$I_1 = 4\pi - 2 I_2$$

$$I_{12} = \frac{I_2 - I_1}{a_1^2 - a_2^2} \qquad \overset{\partial B}{=} I_{13}$$

$$I_{11} = \frac{4\pi}{3 a_1^2} - \frac{2}{3} I_{12}$$

$$I_{22} = I_{33} = I_{23} = \frac{\pi}{a_2^2} - \frac{I_2 - I_1}{4(a_1^2 - a_2^2)}$$

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and

$$3 I_{22} = \frac{4\pi}{a_2^2} - I_{23} - \frac{I_2 - I_1}{a_1^2 - a_2^2}$$

this equation contains no additional information!
Is there a bug?
What is I_{13}?

There is no contradiction as

$$3 I_{22} = \frac{4\pi}{a_2^2} - \left[\frac{\pi}{a_2^2} - \frac{I_2 - I_1}{4(a_1^2 - a_2^2)} \right] - \frac{I_2 - I_1}{a_1^2 - a_2^2}$$

$$= \frac{3\pi}{a_2^2} - \frac{3}{4} \frac{(I_2 - I_1)}{(a_1^2 - a_2^2)}$$

$$\Rightarrow I_{22} = \frac{\pi}{a_2^2} - \frac{1}{4} \frac{(I_2 - I_1)}{(a_1^2 - a_2^2)}$$ see above

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As

$$I_{12} = 2\pi\, a_1\, a_2\, a_3 \int_0^\infty \frac{ds}{(a_1^2 + s)(a_2^2 + s)\Delta(s)} \qquad (11.14)$$

and $a_2 = a_3$ it follows that $I_{13} = I_{12}$