Including temperature in the extended GPE

Two primary methods:

1. TeGPE — Thermal eGPE with a mean-field term that includes the effects of thermal fluctuations:

   - Juan Sánchez-Baena, Thomas Pohl, and Fabian Maucher, Superfluid-supersolid phase transition of elongated dipolar Bose-Einstein condensates at finite temperatures, Phys. Rev. Res. 6, 023183 (2024).
   - J. Sánchez-Baena, C. Politi, F. Maucher, F. Ferlaino, and T. Pohl, Heating a dipolar quantum fluid into a solid, Nat. Commun. 14, 1868 (2023).
   - Liang-Jun He, Juan Sánchez-Baena, Fabian Maucher, and Yong-Chang Zhang, Accessing elusive two-dimensional phases of dipolar bose-einstein condensates by finite temperature, Phys. Rev. Research 7 023019 (2025).

2. SPeGPE – Stochastic Projected eGPE

TeGPE

Most terms are calculated “as usual”, except for H_th,

$$\mu\psi(\mathbf{r}) = \Bigg(-\frac{\hbar^2\nabla^2}{2m} + U(\mathbf{r}) + \int d\mathbf{r}' V_{dd}(\mathbf{r}-\mathbf{r}')|\psi(\mathbf{r}')|^2 + \frac{4\pi\hbar^2 a}{m}|\psi(\mathbf{r})|^2 + H_{\mathrm{qu}}(\mathbf{r}) + \boxed{H_{\mathrm{th}}(\mathbf{r})}\Bigg)\psi(\mathbf{r}) \quad (1)$$

$$\boxed{H_{\mathrm{th}}(\mathbf{r}) = \int \frac{d\mathbf{k}}{(2\pi)^3}\frac{1}{(e^{\beta\varepsilon_{\mathbf{k}}}-1)}\tilde{V}(\mathbf{k})\frac{\tau_{\mathbf{k}}}{\varepsilon_{\mathbf{k}}(\mathbf{r})},} \quad (3)$$

where $\varepsilon_{\mathbf{k}}(\mathbf{r}) = \sqrt{\tau_{\mathbf{k}}[\tau_{\mathbf{k}} + 2|\psi(\mathbf{r})|^2\tilde{V}(\mathbf{k})]}$ is the Bogoliubov excitation spectrum for a given local density $|\psi(\mathbf{r})|^2$ of the BEC, $\tau_{\mathbf{k}} = \frac{\hbar^2k^2}{2m}$, $\beta = 1/k_BT$, and T denotes temperature. $\tilde{V}(\mathbf{k})$ represents the Fourier transform of the sum of the dipole-dipole interaction and the contact interaction, given by

$$\tilde{V}(\mathbf{k}) = \frac{4\pi\hbar^2 a}{m} + \frac{4\pi\hbar^2 a_d}{m}\left(3\frac{k_z^2}{k^2} - 1\right). \quad (4)$$

The parameter $a_d = mC_{dd}/(12\pi\hbar^2)$ corresponds to the dipolar length, C_{dd} is the strength of the dipolar interaction, and

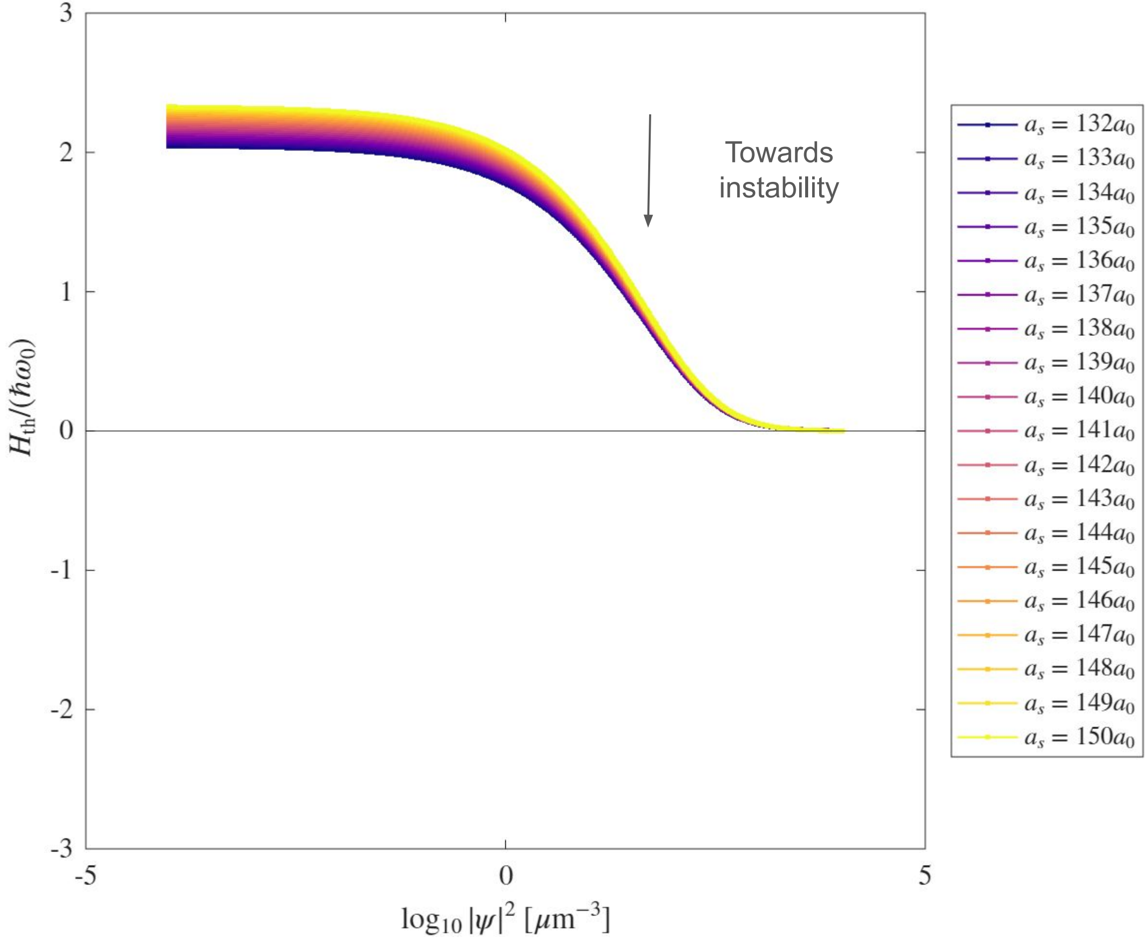
Recipe for calculating H_{th}

In principle, H_{th} is an integral over k-space at all local values of the density $|\psi(r)|^2$

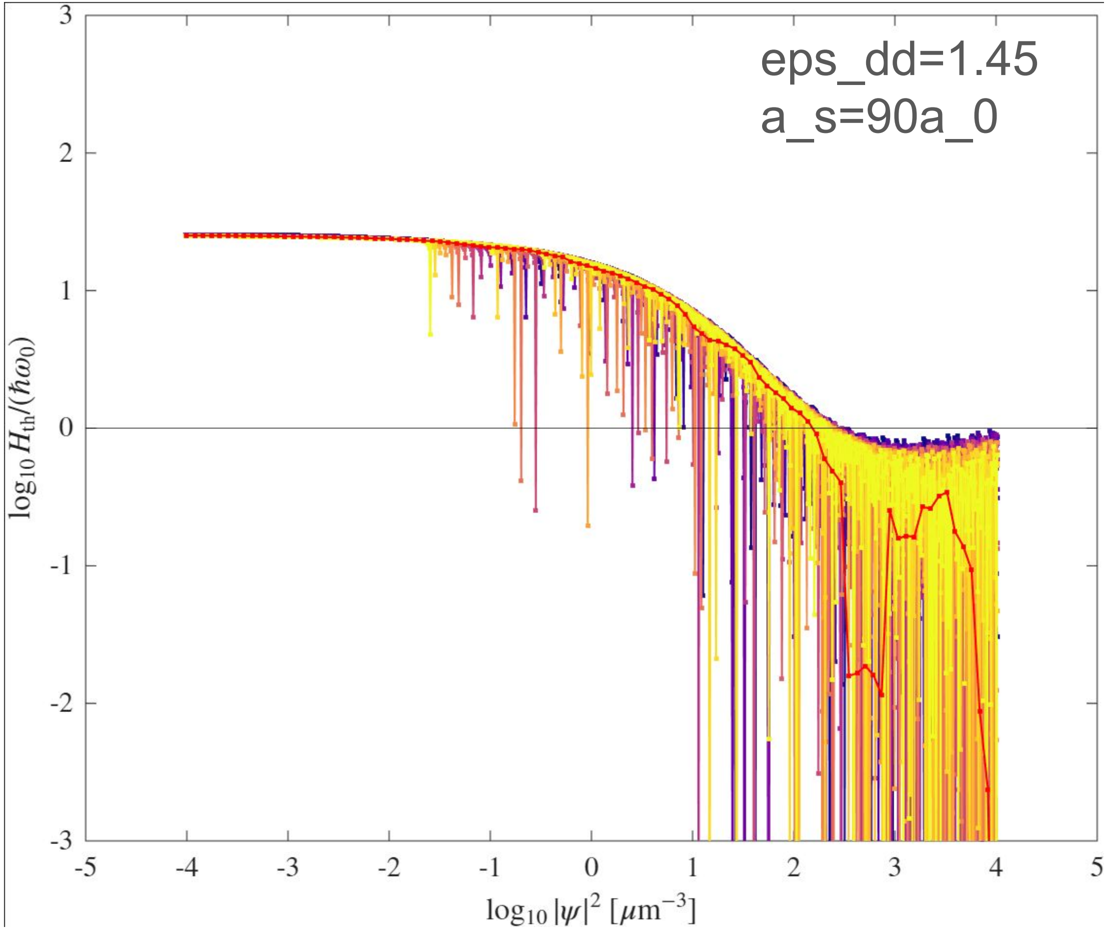
→ This is inefficient so we will precalculate H_{th} at a wide range of densities, and then assuming it’s relatively smooth, interpolate

→ Calculating H_{th} on our numerical (cartesian) grids also isn’t probably the best approach, so we can calculate it either in spherical or cylindrical coordinates (Gaussian quadrature)

H_{th} in the stable regime

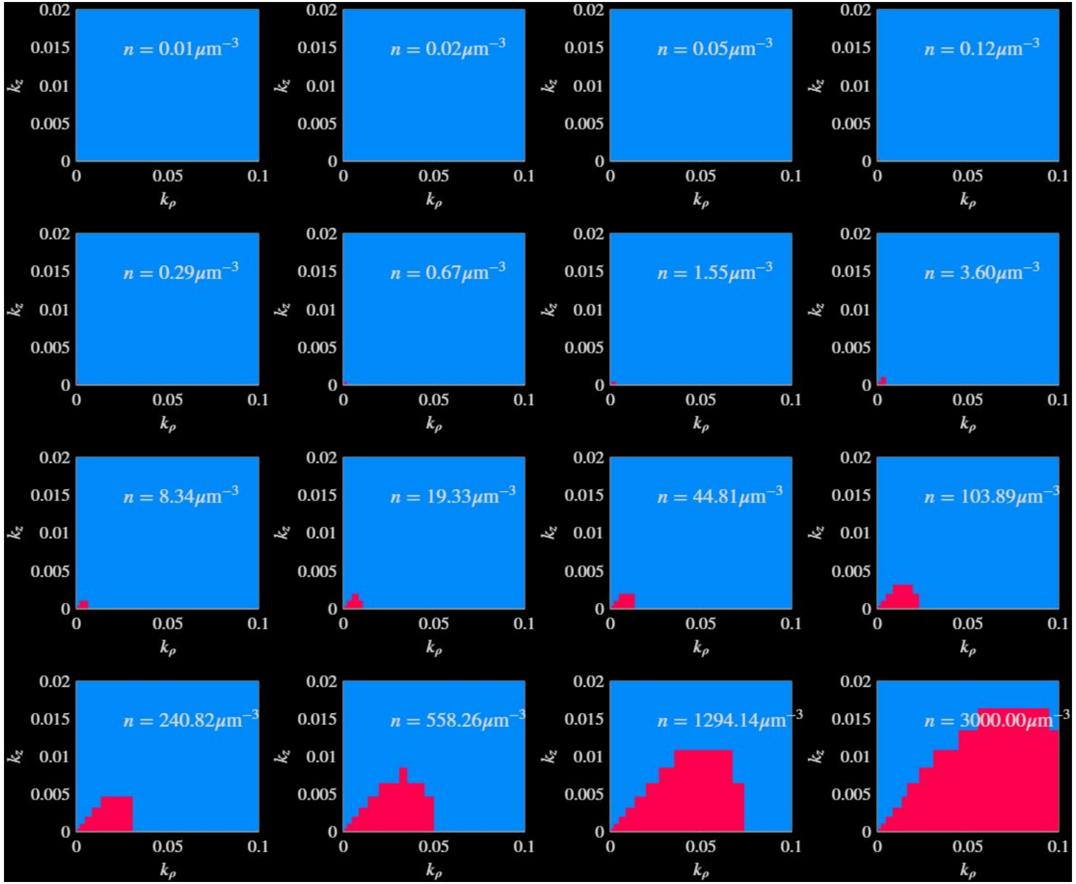


H_{th} in the unstable regime



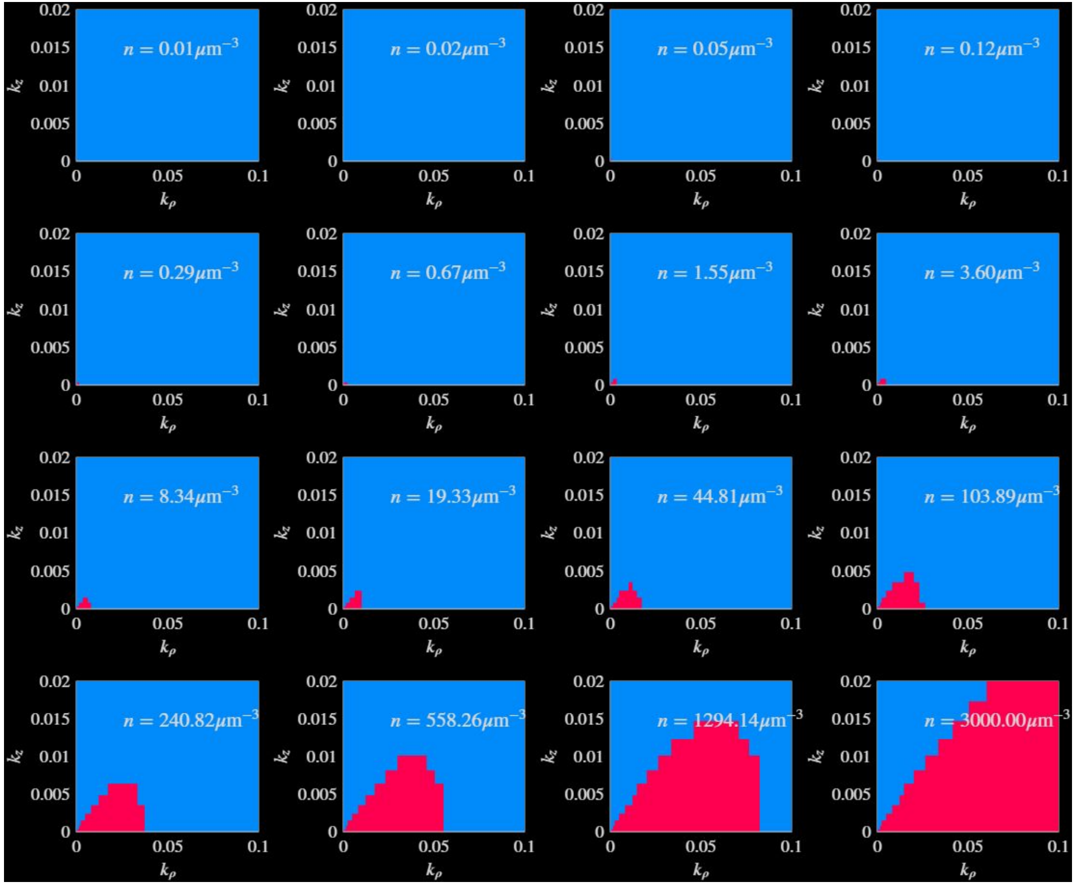
Unstable modes (cylindrical coordinates)

eps_dd=1.31



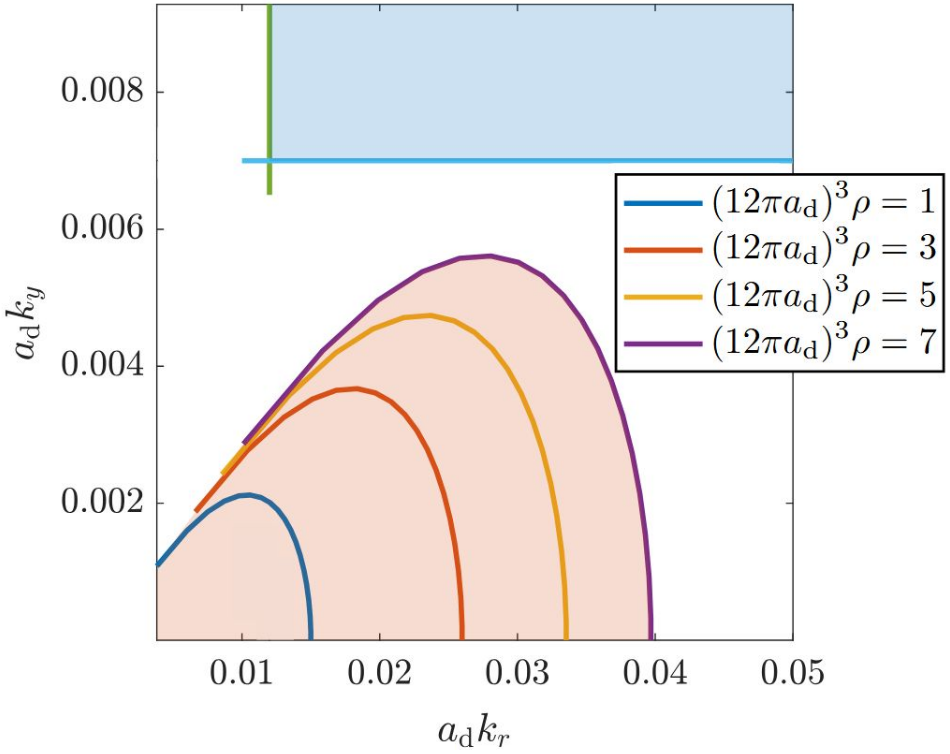
Unstable modes (cylindrical coordinates)

eps_dd=1.45



Avoiding instabilities via momentum cutoff

Introduce low momentum cutoff in both k_\rho and k_z



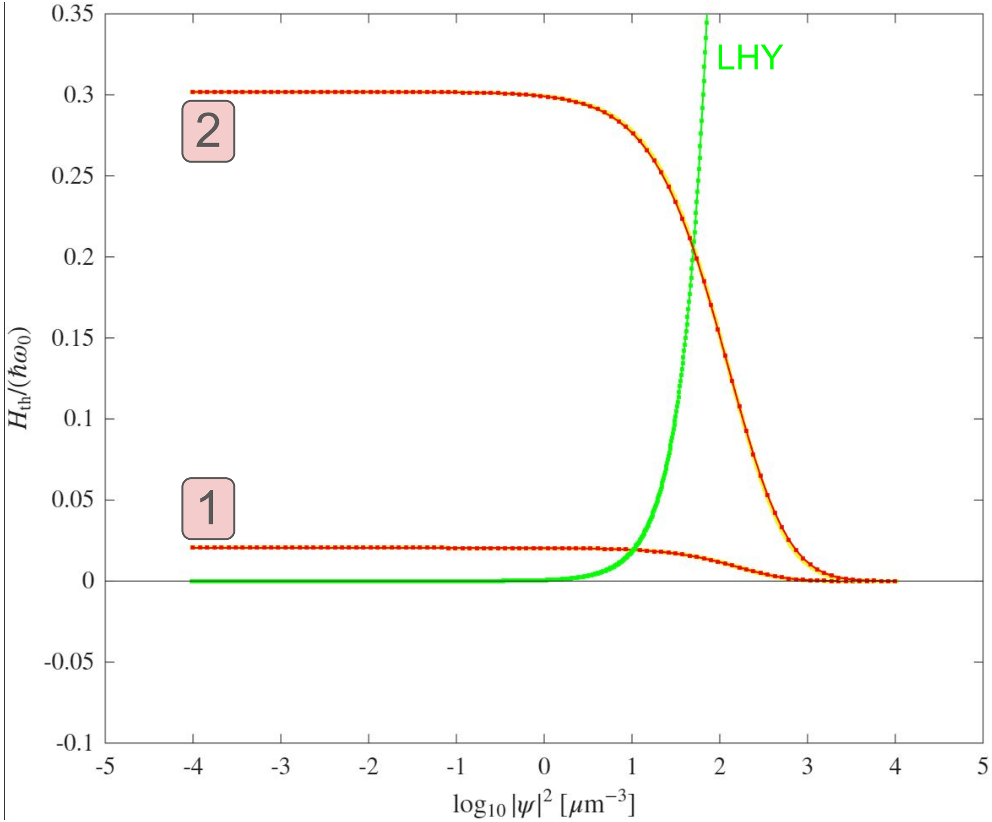
Avoiding instabilities via momentum cutoff

Sánchez-Baena's approach:

1 $$\begin{cases} k_z > \dfrac{2\pi}{l_z} \\ k_\rho > \sqrt{\left(\dfrac{2\pi}{l_x}\right)^2 + \left(\dfrac{2\pi}{l_y}\right)^2} \end{cases}$$

2 $$\begin{cases} k_z > 0.5\dfrac{2\pi}{l_z} \\ k_\rho > 0.5\sqrt{\left(\dfrac{2\pi}{l_x}\right)^2 + \left(\dfrac{2\pi}{l_y}\right)^2} \end{cases}$$

Seems highly cutoff dependent to me…



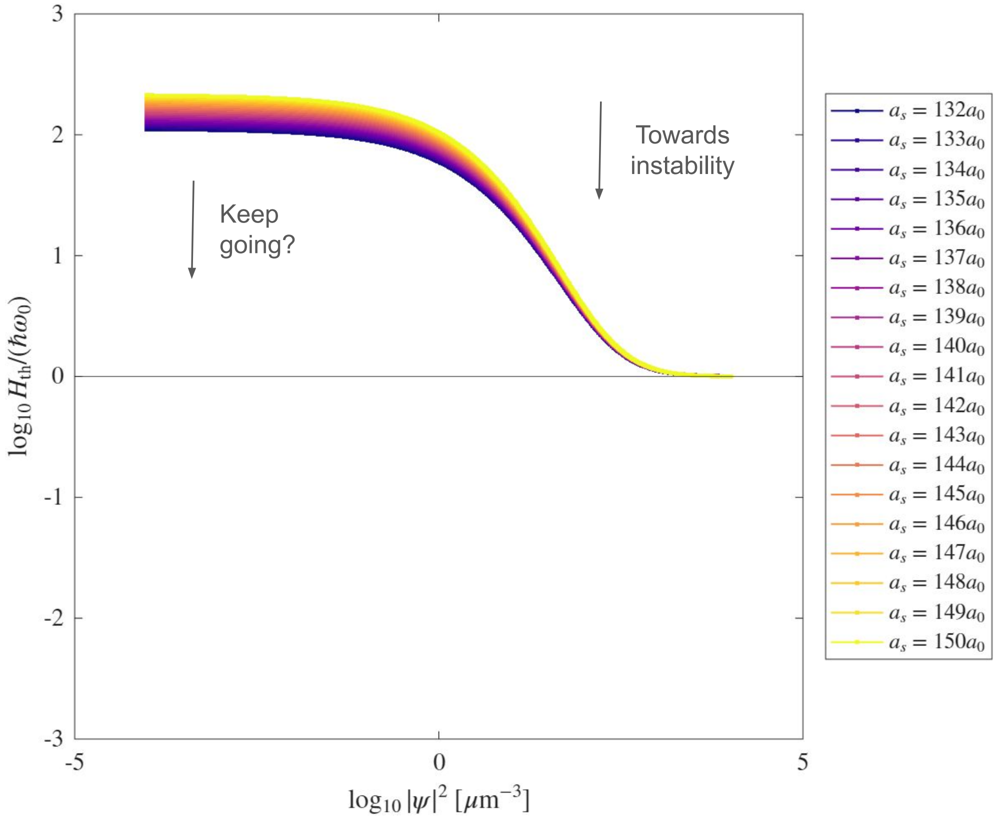
Avoiding instabilities

What about assuming the general trend doesn't stop at the instability?

Introduce a general model that fits at various scattering lengths and temperatures

Fit this model to many points in the stable region

Assume the model holds past the instability

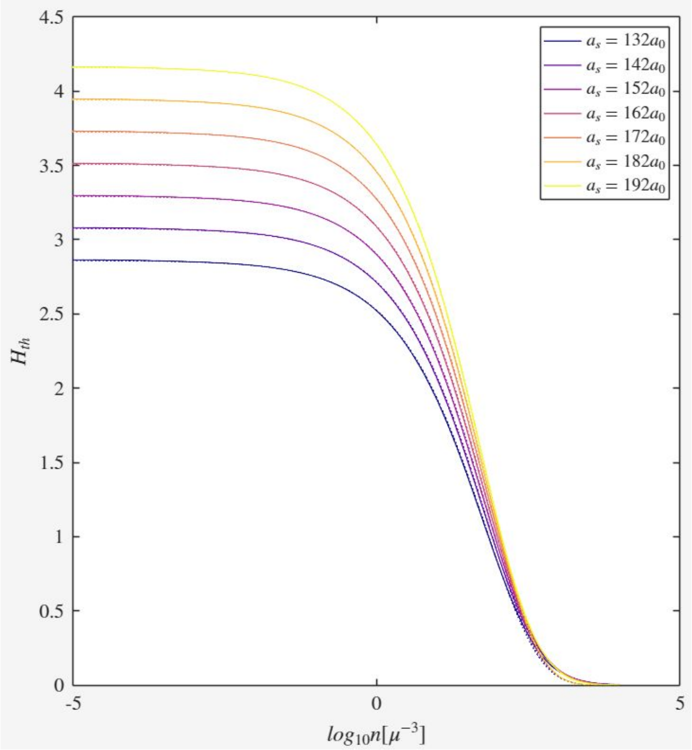


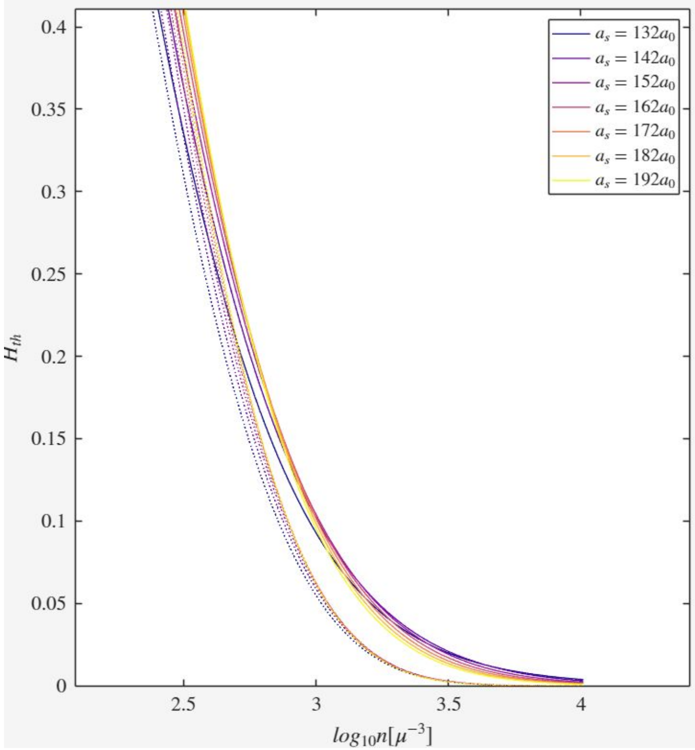
Avoiding instabilities

Try the following model: $$H_2 = A\mathrm{e}^{-B\sqrt{n}}$$

Where *A* and *B* are fitted functions of the scattering length and temperature

T=100nK



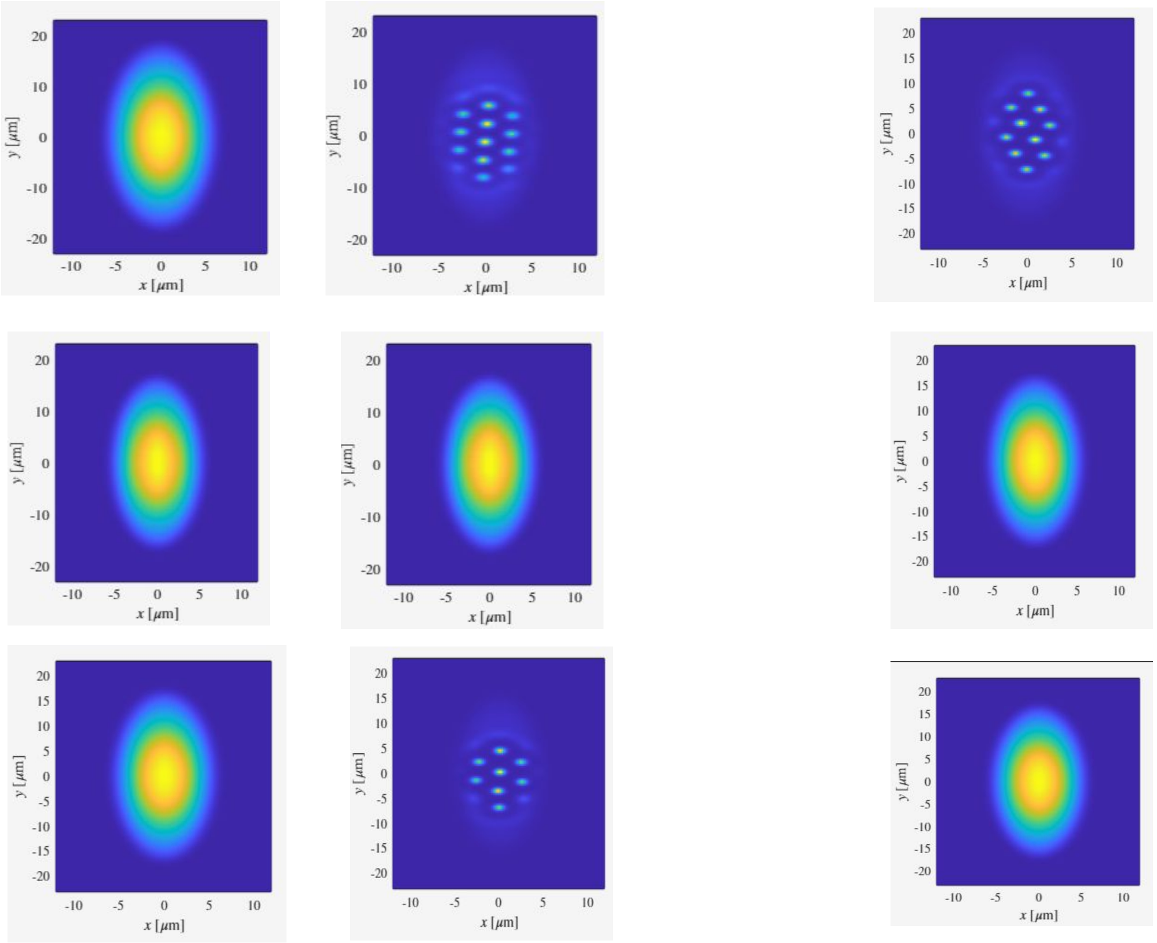


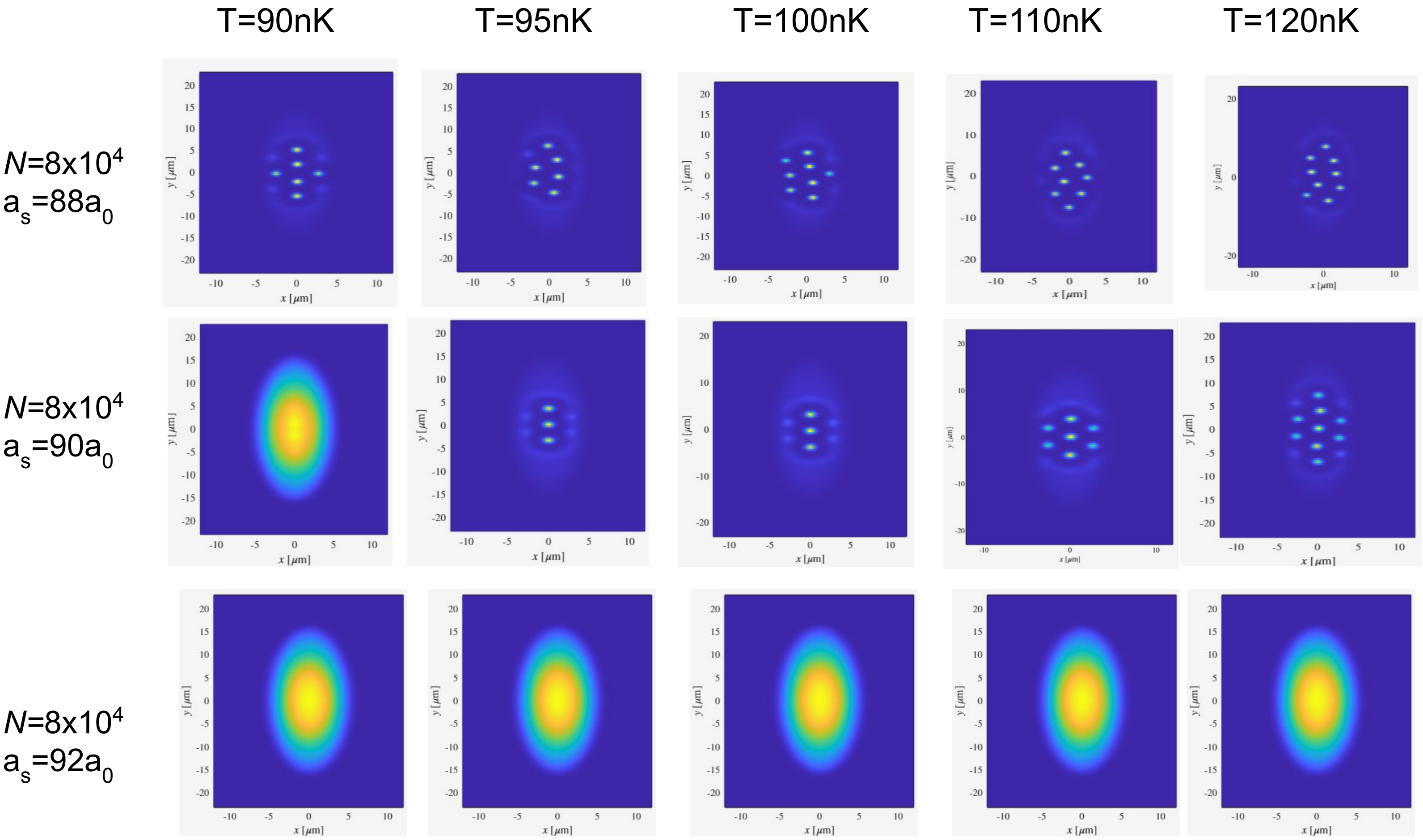
T=0nK | T=100nK | T=100nK (cutoff at 0.5 HO length)

N=1.5x10^5 a_s=92a_0

N=1.0x10^5 a_s=92a_0

N=1.0x10^5 a_s=90a_0

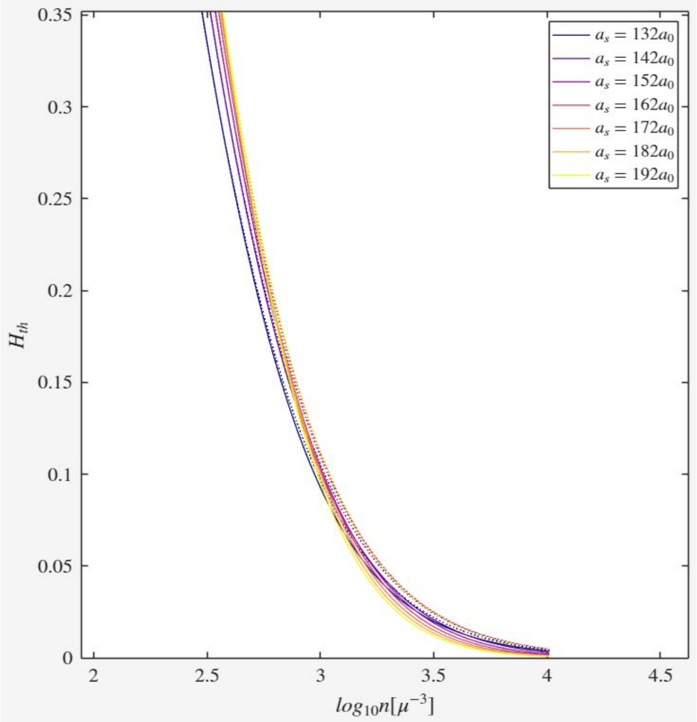


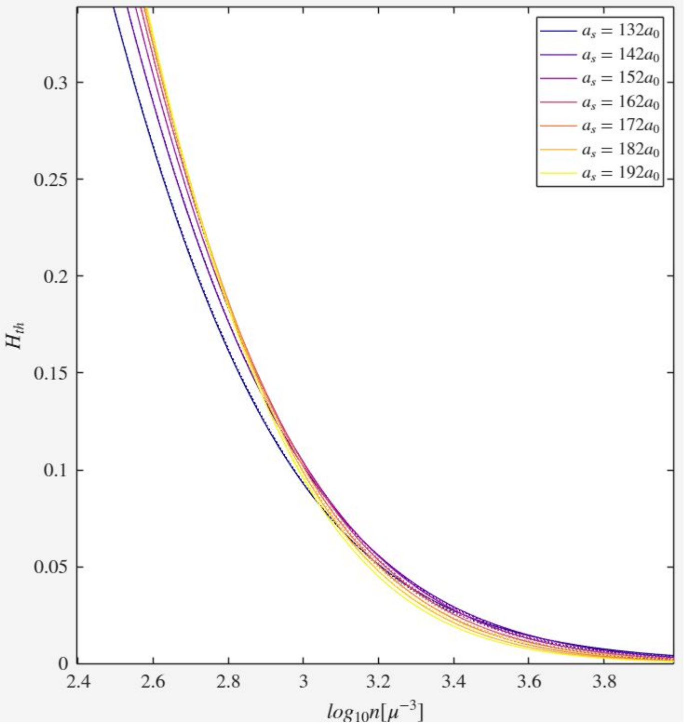

T=90nK
T=95nK
T=100nK
T=110nK
T=120nK
N=8x10^4 a_s=88a_0
N=8x10^4 a_s=90a_0
N=8x10^4 a_s=92a_0
x [μm]
y [μm]

Other models

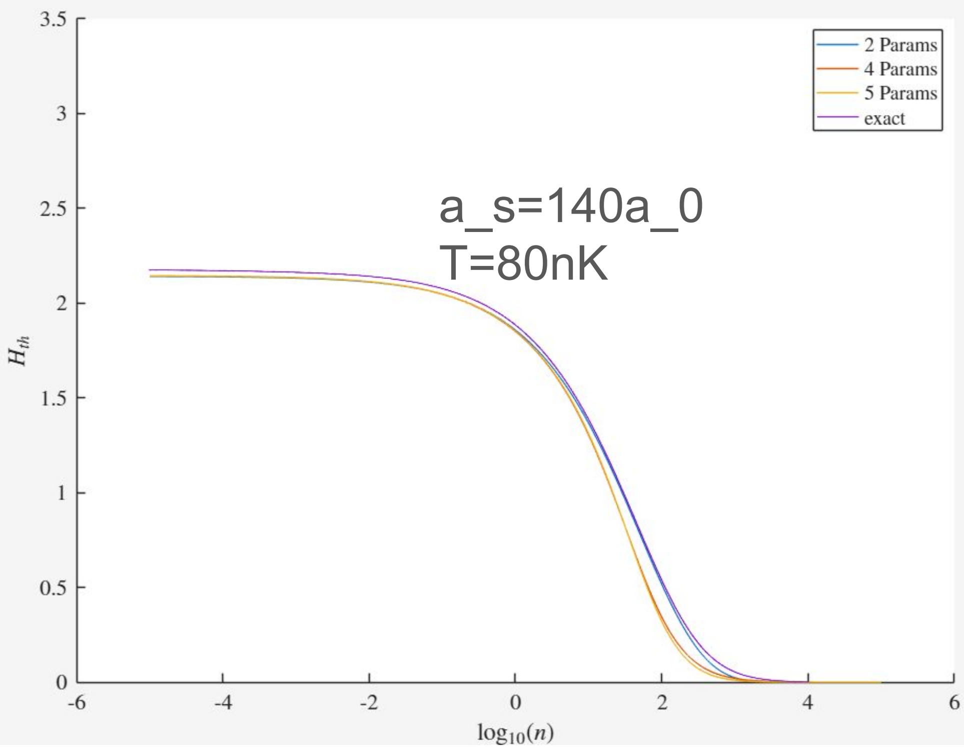
$$H_4 = \frac{A}{1 + Bn^{1/2} + Cn + Dn^{3/2}}$$

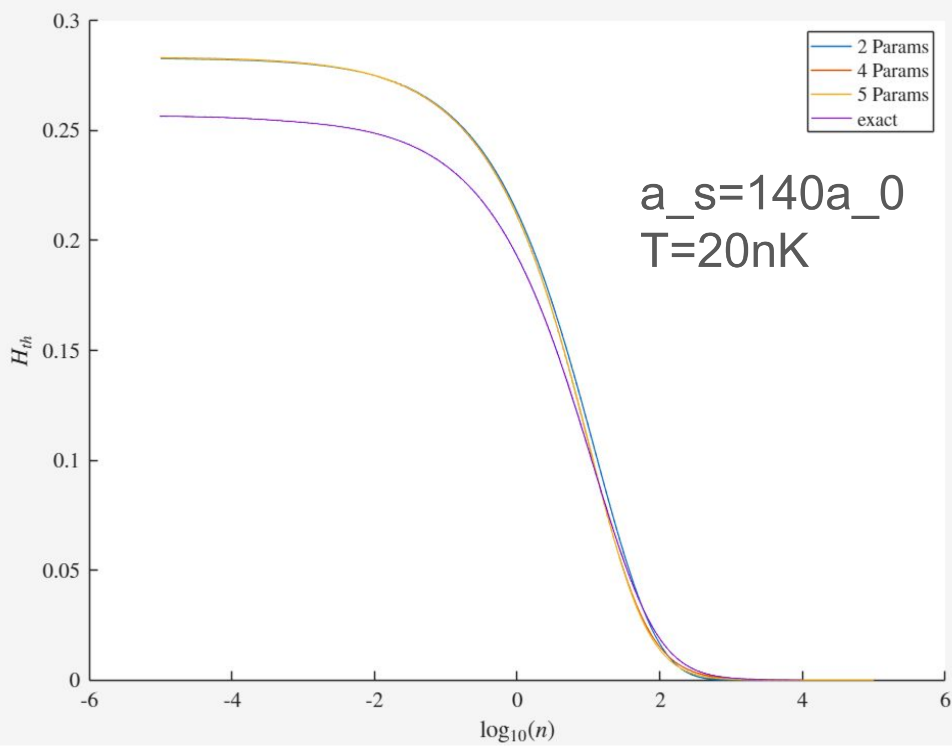
$$H_5 = \frac{A}{1 + Bn^{1/2} + Cn + Dn^{3/2} + En^2}$$





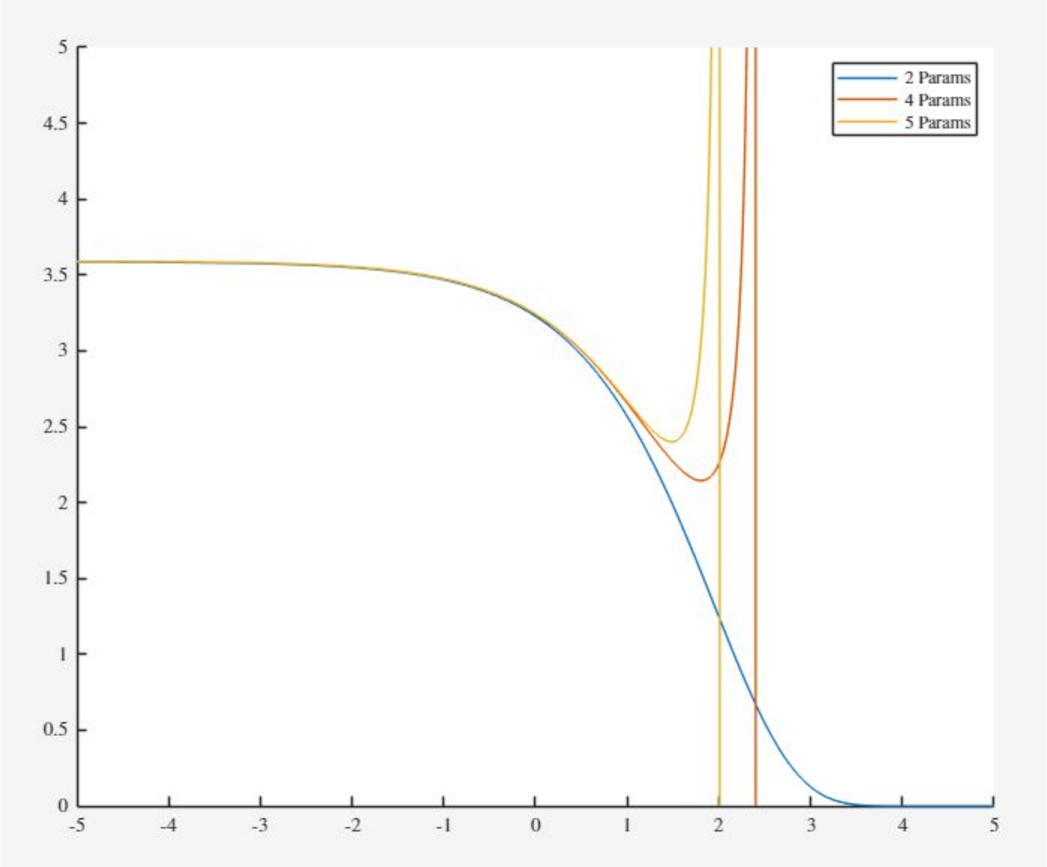
$$H_2 = A\mathrm{e}^{-B\sqrt{n}} \qquad H_4 = \frac{A}{1 + Bn^{1/2} + Cn + Dn^{3/2}} \qquad H_5 = \frac{A}{1 + Bn^{1/2} + Cn + Dn^{3/2} + En^2}$$





H_2 actually sometimes performs better than the other models (seems to be at high temperature)

Other models

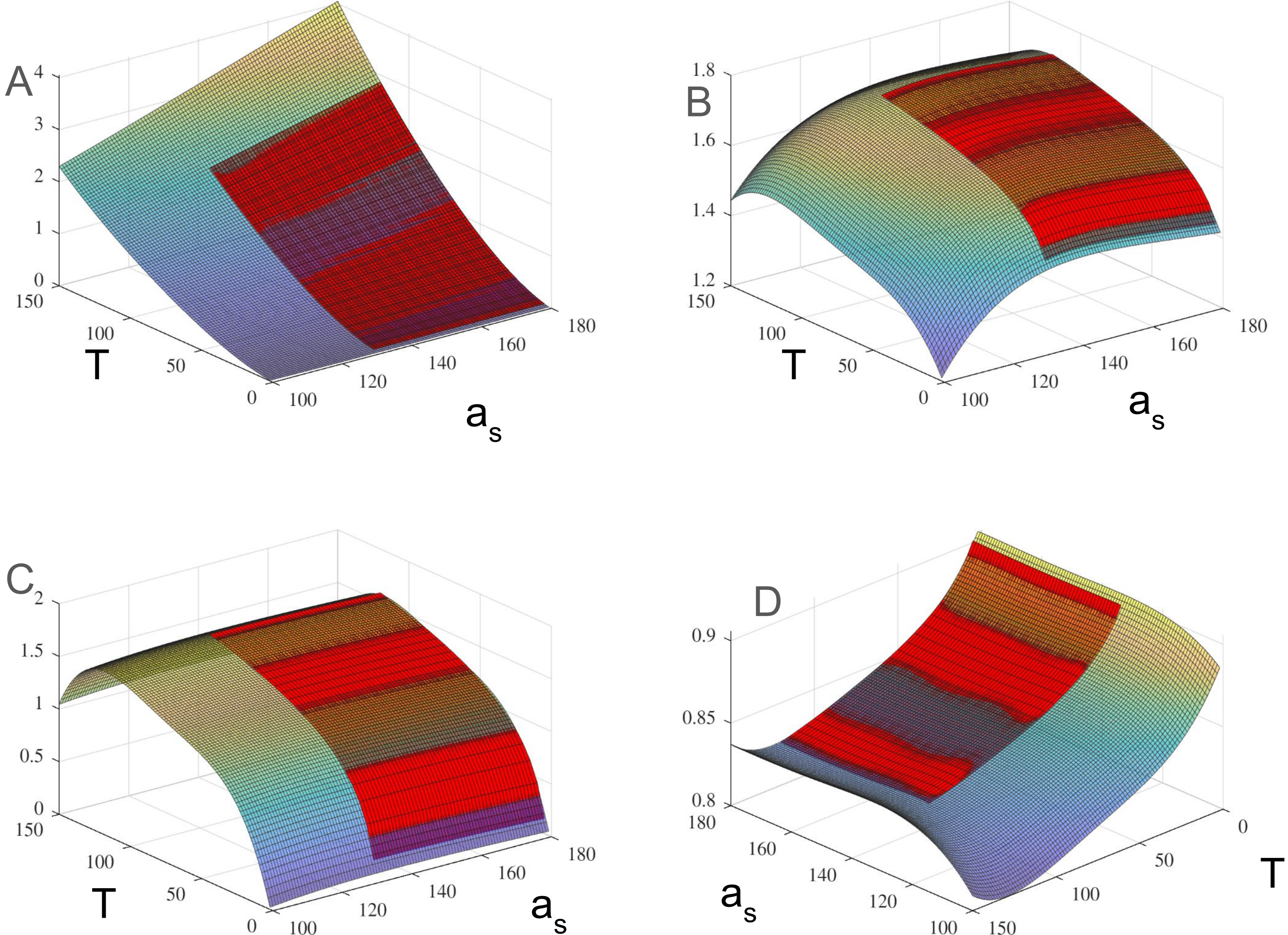


$$H_2 = A\mathrm{e}^{-B\sqrt{n}}$$

$$H_4 = \frac{A}{1 + Bn^{1/2} + Cn + Dn^{3/2}}$$

$$H_5 = \frac{A}{1 + Bn^{1/2} + Cn + Dn^{3/2} + En^2}$$

Unfortunately, the rational functions aren't so easy to control and keep stable for all scattering lengths and temperatures…



A
T
a_s
B
T
a_s
C
T
a_s
D
a_s
T

SPeGPE