Field Guide to

Polarization

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SPIE Field Guides
Volume FG05

John E. Greivenkamp, Series Editor

Introduction to the Series

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Each *SPIE Field Guide* addresses a major field of optical science and technology. The concept of these *Field Guides* is a format-intensive presentation based on figures and equations supplemented by concise explanations. In most cases, this modular approach places a single topic on a page, and provides full coverage of that topic on that page. Highlights, insights and rules of thumb are displayed in sidebars to the main text. The appendices at the end of each *Field Guide* provide additional information such as related material outside the main scope of the volume, key mathematical relationships and alternative methods. While complete in their coverage, the concise presentation may not be appropriate for those new to the field.

The *SPIE Field Guides* are intended to be living documents. The modular page-based presentation format allows them to be easily updated and expanded. We are interested in your suggestions for new *Field Guide* topics as well as what material should be added to an individual volume to make these Field Guides more useful to you. Please contact us at **fieldguides@SPIE.org**.

John E. Greivenkamp, *Series Editor*
Optical Sciences Center
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Field Guide to Polarized Light

The polarization of light is one of the most remarkable phenomena in nature and has led to numerous discoveries and applications. Today it continues to play a vital role in optics. Before the 1950s there was very little activity on the foundations of polarized light. For example, answers to questions such as the nature and mathematical formulation of unpolarized light and partially polarized light were not readily forthcoming. Today there is a very good understanding of polarized light. In particular, the mathematical difficulties that had hindered complex polarization calculations were finally overcome with the introduction of the Mueller-Stokes matrix calculus and the Jones matrix calculus. Research in polarized light continues with much vigor as witnessed by the continued appearance of numerous publications and conferences.

The primary objective of this Guide is to provide an introduction to the developments in polarized light that have taken place over the past half-century. In this Guide I have tried to present the most salient topics on the subject. Hopefully, this Field Guide will enable the reader to have a good grasp of the material and most of all to allow him or her to be comfortable and even delighted with the beauty and subject of polarized light.

Finally, this Field Guide is dedicated to my wife, Mary Ann, and my children Ron and Greg. Their encouragement and support greatly simplified the task of writing this Guide.

Edward Collett
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Glossary

Frequently used variables and symbols:

B	birefringence
$\mathbf{B}(\mathbf{r},t)$	magnetic induction vector
c	speed of light in a vacuum
cp	circularly polarized
db	decibels
$\mathbf{D}(\mathbf{r},t)$	electric displacement vector
e-	extraordinary ray
ε	permittivity constant
E_{0x}	maximum amplitude in the *x* direction
E_{0y}	maximum amplitude in the *y* direction
$E_x(\mathbf{r},t)$	*x* component of the optical field
$E_y(\mathbf{r},t)$	*y* component of the optical field
$\mathbf{E}$	Jones vector
$\mathbf{E}(\mathbf{r},t)$	electric field vector
$\mathbf{F}$	force vector
H_0	Transmission of two parallel polarizers
H_{90}	Transmission of two crossed polarizers
$\mathbf{H}(\mathbf{r},t)$	magnetic field vector
HWP	half wave plate
i	angle of incidence
i_B	Brewster angle
$\mathbf{i},\mathbf{j},\mathbf{k}$	Cartesian unit vectors
$\mathbf{J}$	Jones matrix
$\mathbf{j}(\mathbf{r},t)$	electric current density vector,
$\mathbf{J}_{POL}$	Jones matrix for a polarizer
$\mathbf{J}_{WP}$	Jones matrix for a wave plate
$\mathbf{J}_{ROT}$	Jones matrix for a rotator
$\mathbf{J}_{QWP}$	Jones matrix for a quarter-wave plate
$\mathbf{J}_{HWP}$	Jones matrix for a half-wave plate

$\mathbf{J}(\theta)$	Jones matrix for a rotated polarizing element
k	wave number
$\mathbf{k}$	wave vector
k_1	major transmittance of a polarizer
k_2	minor transmittance of a polarizer
KDP	potassium dihydrogen phosphate
L–45P	linear –45 polarization
L+45P	linear +45 polarization
LCP	Left circular polarization
LHP	linear horizontal polarization
LVP	linear vertical polarization
µ	permeability constant
M	Mueller matrix
M_{HWP}	Mueller matrix of a half-wave plate
M_{LP}	Mueller matrix of a linear polarizer
M_{POL}	Mueller matrix of a polarizer
M_{QWP}	Mueller matrix of a quarter-wave plate
M_R	Mueller matrix for reflection
M_{ROT}	Mueller matrix of a rotator
M_T	Mueller matrix for transmission
M_{WP}	Mueller matrix of a wave plate
$M(\theta)$	Mueller matrix of a rotated polarizing element
$\mathbf{n}$	complex refractive index
n_e	refractive index of the extraordinary ray
n_o	refractive index of the ordinary ray
n_p	parallel refractive index
n_s	perpendicular refractive index
n_L	levo-rotary refractive index

n_R	dextro-rotary refractive index
n	refractive index
o-	ordinary ray
OPS	observable polarization sphere
p-	parallel polarization state
p_x	polarizer transmission coefficient (*x*)
p_y	polarizer transmission coefficient (*y*)
$\mathscr{P}$	degree of polarization
PBS	polarizing beam splitter
QWP	quarter wave plate
r	angle of refraction
r	radius vector
RCP	right circular polarization
s-	perpendicular polarization state
S	Stokes vector
S_0	first Stokes parameter
S_1	second Stokes parameter
S_2	third Stokes parameter
S_3	fourth Stokes parameter
S_R	Stokes vector for reflection
S_T	Stokes vector for transmission
TIR	total internal reflection
UNP	unpolarized
v(**r**,t)	velocity vector
$v_x, v_y. v_z$	principal velocities
V	Verdet's constant
V_π	half-wave voltage
V_m	maximum modulation voltage
WP	wave plate
x,y,z	Cartesian coordinate system
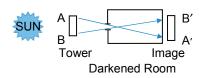
2α	coordinate on the observable polarization sphere
2ψ	coordinate angle on the Poincaré sphere
2χ	coordinate angle on the Poincaré sphere
α	auxiliary angle
$\boldsymbol{\varepsilon}$	complex dielectric constant
ε_x, ε_y, ε_z	principal dielectric constants
δ	coordinate on the observable polarization sphere
δ	phase difference
δ_x	phase of the wave (x)
δ_y	phase of the wave (y)
θ	angle of rotation
κ	absorption coefficient
ψ	orientation angle
χ	ellipticity angle
$\rho_{s,p}$	Fresnel reflection coefficients
$\rho(\mathbf{r},t)$	electric charge density
σ	conductivity
$\tau_{s,p}$	Fresnel transmission coefficients
ϕ	phase shift
ϕ_x	phase shift (x)
ϕ_y	phase shift (y)
ω	angular frequency
ω_c	cyclotron frequency
ω_L	Larmor's frequency
ω_m	modulation frequency
∇	spatial vector operator

The Ray Theory of Light
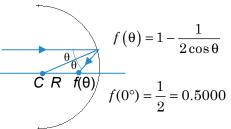
Polarized light has its origins in the **ray theory of light**. In the 11th century **Al-Hazen** examined a tower through a very small hole in a darkened room. On the back wall the tower appeared as an **inverted image**. This demonstrated, along with the **law of reflection** later defined by **Snell**, that light proceeds from points A and B to A′ and B′, respectively, in the form of **rays**.
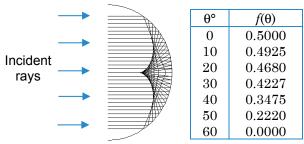
The ray theory appears when light rays are reflected from a concave **spherical surface** as shown below.

Here C is the center of the circle, R is the radius of the spherical surface, and $f(\theta)$ is the **focal point** for a circle of unit radius.

$$f(\theta) = 1 - \frac{1}{2\cos\theta}$$

$$f(0°) = \frac{1}{2} = 0.5000$$

The focal point moves from $f(0°) = 0.5000$ to $f(60°) = 0.0000$ as the incident ray moves away from the axis of symmetry. Plotting the ray paths for $f(\theta)$ from $\theta = 0°$ to $60°$ yields the figure.

$\theta°$	$f(\theta)$
0	0.5000
10	0.4925
20	0.4680
30	0.4227
40	0.3475
50	0.2220
60	0.0000
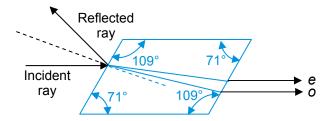
The reflected light rays do not come to a single focus and the rays describe the locus of a **cardioid**, which is also called a **caustic**. The image is readily observed in a drinking cup.

The Polarization of Light

In 1670 **Bartholinus** discovered that when a single ray of natural incident light propagated through a rhombohedral **calcite crystal**, two rays emerged, demonstrating that a single ray of light actually consists of two rays called the **ordinary** or **o-ray** (n_o), and the **extraordinary** or **e-ray** (n_e). Because the two rays refract at different angles, the calcite crystal is said to be doubly refractive or **birefringent**. Both rays obey **Snell's law of refraction** but experience different **refractive indices**.
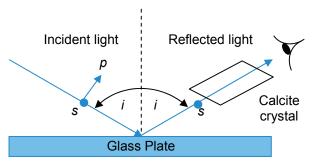
Further investigation by **Huygens**, showed that by rotating a second (analyzing) crystal the intensity of one ray was maximized and the other ray vanished. A further rotation of 90° then showed that the first ray reappeared with a maximum intensity and the second ray vanished. At a rotation angle of 45° the intensities of the two rays were equal. Because of this opposite behavior of intensity, the two rays were said to be **polarized**. Thus, a single ray of **natural light** actually consists of two independent oppositely polarized rays. The two rays are said to represent the **s-** and **p-polarization** states.

Double refraction occurs in calcite because it is an anisotropic crystal and the o-ray wave front propagates as a sphere, whereas the e-ray propagates as an ellipsoid. Furthermore, the refractive indices of the o- and e-rays at λ = 5893 (**sodium D line**) are n_o = 1.6853 and n_e = 1.4864.

Malus's Law
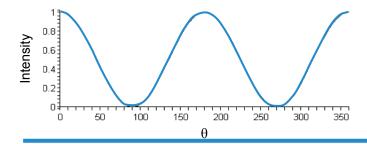
At the beginning of the nineteenth century the only known way to generate polarized light was with a calcite crystal. In 1808, using a calcite crystal, **Malus** discovered that natural incident light became polarized when it was reflected by a glass surface, and that the light reflected close to an angle of incidence of 57° could be extinguished when viewed through the crystal. He then proposed that natural light consisted of the *s*- and *p*-polarizations, which were perpendicular to each other.

Since the intensity of the reflected light varied from a maximum to a minimum as the crystal was rotated, Malus proposed that the amplitude of the reflected beam must be $A = A_0\cos\theta$. However, in order to obtain the intensity, Malus squared the amplitude relation so that the intensity equation $I(\theta)$ of the reflected polarized light was

$$I(\theta) = I_0 \cos^2\theta,$$

where $I_0 = A_0{}^2$; this equation is known as **Malus's Law**. A normalized plot of Malus's Law is shown below.
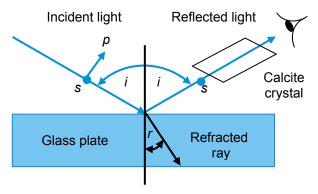
Brewster's Law

Around 1812 **Brewster** discovered that for different glasses the *p*-polarized ray (in the plane of the paper) vanished completely at a particular angle of incidence *i*. Furthermore, by rotating the analyzing calcite crystal through 90°, the *s*-polarized ray (directed out of the plane of the paper) became extinguished. He then discovered that the refracted ray angle *r* was simply related to the incident ray angle *i* by

$$i + r = 90°.$$

Snell's law of refraction between the two media is

$$n_1 \sin i = n_2 \sin r,$$

where n_1 and n_2 are the refractive indices of the media, respectively. For air $n_1 = 1$ and for glass $n_2 = n$. Substituting the first equation into Snell's law leads to

$$\tan i = n_2 / n_1 = n.$$

This equation is known as **Brewster's law**. Its immediate practical use was that it enabled the refractive index of glass to be determined by reflection rather than by refraction; the measurement of the refractive angle *r* by transmission is difficult. The result of this discovery was that it led to the rapid development and measurement of new optical glasses.

Fresnel's Wave Theory
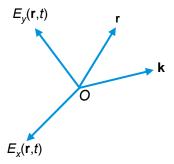
Around 1820 **Fresnel** proposed a theory of light now known as **Fresnel's wave theory** that completely explained the three major phenomena of light: interference, diffraction, and polarization. Furthermore, Fresnel and **Arago** experimentally showed that the optical field consisted of only two **orthogonal components** in the plane transverse to the direction of propagation.

Fresnel's wave theory postulated that the orthogonal components were $u_1(\mathbf{r},t)$ and $u_2(\mathbf{r},t)$, which he called optical disturbances. We now know, however, that the optical disturbances can be represented by the electric field components of the electromagnetic field. Fresnel hypothesized that the field components are described by two equations known as the **wave equations**,

$$\nabla^2 E_x(\mathbf{r},t) = \frac{1}{v^2}\frac{\partial^2 E_x(\mathbf{r},t)}{\partial t^2},$$

$$\nabla^2 E_y(\mathbf{r},t) = \frac{1}{v^2}\frac{\partial^2 E_y(\mathbf{r},t)}{\partial t^2},$$

where $E_x(\mathbf{r},t)$ and $E_y(\mathbf{r},t)$ are the optical-field components, $\mathbf{r}$ is the radius vector to a point in space measured from the origin of a coordinate system, t is the time, v is the velocity of the waves, and ∇^2 is the **Laplacian operator**. The two components of the field and the direction $\mathbf{k}$ form an orthogonal system as shown.

Fresnel's Wave Theory (cont'd)

The solutions of the wave equations are

$$E_x(\mathbf{r},t) = E_{0x}\cos(\omega t - \mathbf{k}\cdot\mathbf{r} + \delta_x)$$

and

$$E_y(\mathbf{r},t) = E_{0y}\cos(\omega t - \mathbf{k}\cdot\mathbf{r} + \delta_y),$$
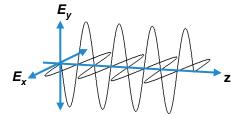
where $\mathbf{k}$ is the (vector) wave number and describes the direction of the propagation, and $\mathbf{r}$ is a point in the field. In practice, the field is taken to be directed along the z-axis. The two components, above, can then be written as

$$E_x(z,t) = E_{0x}\cos(\omega t - kz + \delta_x)$$

and

$$E_y(z,t) = E_{0y}\cos(\omega t - kz + \delta_y),$$

where $\omega = 2\pi f$ is the angular frequency, $k = 2\pi/\lambda$ is the wave number magnitude, E_{0x} and E_{0y} are the maximum amplitudes and δ_x and δ_y are arbitrary phases; the term $\omega t - kz$ is called the **propagator**.

The propagation of these two waves (equations) can be graphically represented as shown in the figure.

The E_y component is in the plane of the paper (also called the plane of incidence) and is called the **p-polarization** component. Similarly, the E_x component is perpendicular to the plane of the paper and is called the **s-polarization** component.

> The *p* and *s* notation come from the German words for parallel (*paralelle*) and perpendicular (*senkrecht*).
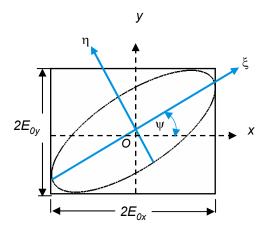
The Polarization Ellipse

According to Fresnel's theory, $E_x(z,t)$ and $E_y(z,t)$ describe sinusoidal oscillations in the *x-z* and *y-z* planes, respectively (see the figure on p. 6) By themselves, these equations are not particularly revealing. However, eliminating the time-space propagator $\omega t - kz$ between the two equations leads to the **equation of an ellipse**, namely,

$$\frac{E_x(z,t)^2}{E_{0x}{}^2} + \frac{E_y(z,t)^2}{E_{0y}{}^2} - \frac{2E_x(z,t)E_y(z,t)}{E_{0x}E_{0y}}\cos\delta = \sin^2\delta\,,$$

where $\delta = \delta_y - \delta_x$. The above equation describes an ellipse in its nonstandard form. Because the equation refers to polarized light, the equation is called the **polarization ellipse**. In the equation, the time-space propagator has been explicitly eliminated. Nevertheless, the field components $E_x(z,t)$ and $E_y(z,t)$ continue to be time-space dependent. A plot of the nonstandard polarization ellipse is shown below.
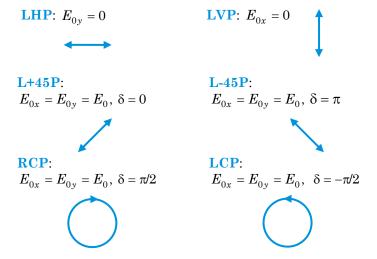
The figure also shows the rotated ξ-η coordinate system. Because of the amplitudes E_{0x} and E_{0y} and the phase δ are constant, the polarization ellipse remains fixed as the polarized beam propagates.

Degenerate Polarization States

In general, the optical field is elliptically polarized, but there are several combinations of amplitude and phase that are especially important. These are called **degenerate polarization states**: (1) linearly horizontal/vertical polarized light (**LHP**/**LVP**), (2) linear ±45° polarized light (L+45P/L–45P), and (3) right/left circularly polarized light (**RCP**/**LCP**). The polarization states along with the mathematical conditions and corresponding figures (polarization ellipses) are as follows.

LHP: $E_{0y} = 0$ **LVP**: $E_{0x} = 0$

L+45P: **L-45P**:
$E_{0x} = E_{0y} = E_0$, $\delta = 0$ $E_{0x} = E_{0y} = E_0$, $\delta = \pi$

RCP: **LCP**:
$E_{0x} = E_{0y} = E_0$, $\delta = \pi/2$ $E_{0x} = E_{0y} = E_0$, $\delta = -\pi/2$

RCP light rotates clockwise and LCP rotates counter-clockwise when propagating toward the observer.

These polarization states are important because (1) they are relatively easy to create in a laboratory using linear and **circular polarizers**, and (2) polarization measurements as well as many polarization calculations are greatly simplified using these specific polarization states. This is especially true when a polarized beam propagates through numerous polarizing elements.

The Parameters of the Polarization Ellipse

The polarization ellipse can be expressed in terms of two angular parameters: the **orientation angle** ψ $(0 \leq \psi \leq \pi)$ and the **ellipticity angle** χ $(-\pi/4 < \chi \leq \pi/4)$.

These angles can be defined in terms of the **parameters of the polarization ellipse**:

$$\tan 2\psi = \frac{2E_{0x}E_{0y}}{E_{0x}^2 - E_{0y}^2}\cos\delta, \qquad 0 \leq \psi \leq \pi,$$

$$\sin 2\chi = \frac{2E_{0x}E_{0y}\sin\delta}{E_{0x}^2 + E_{0y}^2}, \qquad -\pi/4 < \chi \leq \pi/4.$$

The right-hand side of both of these equations consists of algebraic and trigonometric terms. The two equations can be rewritten completely in trigonometric terms by introducing an angle known as the **auxiliary angle** α defined by

$$\tan\alpha = \frac{E_{0y}}{E_{0x}}, \qquad 0 \leq \alpha \leq \pi/2.$$

This leads to purely trigonometric equations

$$\tan 2\psi = (\tan 2\alpha)\cos\delta,$$

$$\sin 2\chi = (\sin 2\alpha)\sin\delta.$$

The conditions on the angles are $0 \leq \alpha \leq \pi/2$ and $0 \leq \delta < 2\pi$.

E ample We determine the orientation and the ellipticity angles ψ and χ for RCP light. We have for RCP light that $E_{0y} = E_{0x} = E_0$ and $\delta = \pi/2$. Then, tan α yields $\alpha = 45°$ and

$$\tan 2\psi = \tan 90° \cos 90° = 0$$
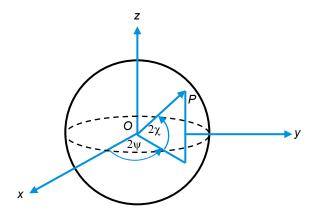
$$\sin 2\chi = \sin 90° \sin 90° = +1$$

Thus, the angles for RCP light are $\psi = 0°$ and $\chi = +45°$.

The Poincaré Sphere

By itself, the polarization ellipse is an excellent way to visualize polarized light. However, except for the degenerate polarization states, it is practically impossible to determine the orientation and ellipticity angles viewing the polarization ellipse. Furthermore, the calculations required to determine the new angles of a polarized beam that propagates through one or more polarizing elements are difficult and tedious.

In order to overcome these difficulties **Poincaré** (1892) suggested using a sphere now known as the **Poincaré sphere** to represent polarized light. The following figure shows the Poincaré sphere and its spherical and Cartesian coordinates.

Here x, y, and z are Cartesian coordinate axes, ψ and χ are the spherical orientation and ellipticity angles (of the polarization ellipse), and P is a point on the surface of the sphere. Note that on the sphere the angles are expressed as 2ψ and 2χ. For a unit sphere the Cartesian coordinates are related to the spherical coordinates by the equation.

$$x = \cos(2\chi)\cos(2\psi),\ 0 \le \psi < \pi,$$

$$y = \cos(2\chi)\sin(2\psi),\ -\pi/4 < \chi \le \pi/4$$

$$z = \sin(2\chi),$$
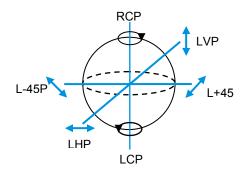
where $x^2+y^2+z^2=1$ for a sphere of unit radius.

Degenerate States on the Poincaré Sphere

From the previous equations any polarization state can be represented by the coordinate pair $(2\psi, 2\chi)$. The degenerate polarization states on the Poincaré sphere are for LHP(0°,0°), for L+45P(+90°,0°), for LVP(180°,0°), for L-45P(270°,0°), for RCP(0°,+90°), and for LCP(0°,–90°). The **degenerate states** on the x, y, and z axes are shown below.
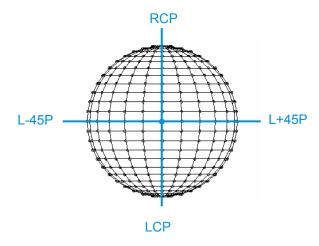
All linear polarization states lie on the equator and right and left circular polarization states are at the north and south poles, respectively. Elliptically polarized states are represented everywhere else on the surface of the sphere.

The following figure shows polarization states plotted at every intersection of the 7.5° latitude and 15° longitude lines.

The Stokes Polarization Parameters

The most serious limitation to the Poincaré sphere and the polarization ellipse are (1) the polarization ellipse is an instantaneous representation of polarized light, and (2) neither the rotation angle ψ nor the ellipticity angle χ is directly measurable. In order to overcome these limitations it is necessary to determine the measurables of the polarized field. This can be done by taking a **time average of the polarization ellipse**:

$$\frac{E_x(z,t)^2}{E_{0x}{}^2} + \frac{E_y(z,t)^2}{E_{0y}{}^2} - \frac{2E_x(z,t)E_y(z,t)}{E_{0x}E_{0y}}\cos\delta = \sin^2\delta.$$

The time average $\langle E_i(z,t)E_j(z,t)\rangle$ is defined by

$$\left\langle E_i(z,t)E_j(z,t)\right\rangle = \lim_{T\to\infty}\frac{1}{T}\int_0^T E_i(z,t)E_j(z,t)\,dt\,, \quad i,j = x,y,$$

where T is total averaging time. Applying the time average definition to the polarization ellipse then yields the following equation:

$$S_0{}^2 = S_1{}^2 + S_2{}^2 + S_3{}^2,$$

where

$$S_0 = E_{0x}^2 + E_{0y}^2,$$

$$S_1 = E_{0x}^2 - E_{0y}^2$$

$$S_2 = 2E_{0x}E_{0y}\cos\delta,$$

$$S_3 = 2E_{0x}E_{0y}\sin\delta,\ \delta = \delta_y - \delta_x.$$

The quantities S_0, S_1, S_2, and S_3 are the **observables** of the polarized field. They were introduced by Stokes (1852) and are called the **Stokes polarization parameters**.

The Stokes Polarization Parameters (cont'd)

The first Stokes parameter S_0 describes the total intensity of the optical beam; the second parameter S_1 describes the preponderance of LHP light over LVP light; the third parameter S_2 describes the preponderance of L+45P light over L-45P light and, finally, S_3 describes the preponderance of RCP light over LCP light.

The Stokes parameters can be expressed in **complex notation** (in order to bypass formally the time integration) by suppressing the propagator and writing

$$E_x(t) = E_{0x} \exp(i\delta_x),$$

$$E_y(t) = E_{0y} \exp(i\delta_y).$$

The Stokes parameters are then defined in complex notation by the following equations.

$$S_0 = E_x E_x^{\ *} + E_y E_y^{\ *},$$

$$S_1 = E_x E_x^{\ *} - E_y E_y^{\ *},$$

$$S_2 = E_x E_y^{\ *} + E_y E_x^{\ *},$$

$$S_3 = i\left(E_x E_y^{\ *} - E_y E_x^{\ *}\right),$$

where $i = \sqrt{-1}$ and * represents the complex conjugate.

It is convenient to arrange the Stokes parameters as a column matrix, which is referred to as the **Stokes vector for elliptically polarized light**:

$$S = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} E_{0x}^2 + E_{0y}^2 \\ E_{0x}^2 - E_{0y}^2 \\ 2E_{0x}E_{0y}\cos\delta \\ 2E_{0x}E_{0y}\sin\delta \end{pmatrix}.$$

Stokes Parameter Relations

The **Stokes vectors for the degenerate polarization states** are readily found using the previous definitions and equations:

$$S_{\mathrm{LHP}} = I_0 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad S_{\mathrm{LVP}} = I_0 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \qquad S_{\mathrm{L+45P}} = I_0 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

$$S_{\mathrm{L-45P}} = I_0 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad S_{\mathrm{RCP}} = I_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad S_{\mathrm{LCP}} = I_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix},$$

where I_0 is the intensity and is very often normalized to unity.

The Stokes parameters can be shown to be related to the orientation and ellipticity angles, ψ and χ, associated with the Poincaré sphere as follows:

$$S_1 = S_0 \cos(2\chi)\cos(2\psi),$$

$$S_2 = S_0 \cos(2\chi)\sin(2\psi),$$

$$S_3 = S_0 \sin(2\chi),$$

and

$$\psi = \frac{1}{2}\tan^{-1}\left(\frac{S_2}{S_1}\right), \qquad 0 \le \psi \le \pi,$$

$$\chi = \frac{1}{2}\sin^{-1}\left(\frac{S_3}{S_0}\right), \qquad -\frac{\pi}{4} \le \chi \le \frac{\pi}{4}.$$

Stokes Parameter Relations (cont'd)

The Stokes parameters describe not only completely polarized light but **unpolarized** and **partially polarized light** as well. **The Stokes vector for unpolarized light** is

$$S_{\text{unp}} = S_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where S_0 is the first Stokes parameter (total intensity). Since there are no amplitude or phase relations between the orthogonal components, S_1, S_2, and S_3 are 0. Partially polarized light is a mixture of completely polarized light and unpolarized light, and is represented by

$$S = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = (1 - \mathcal{P}) \begin{pmatrix} S_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mathcal{P} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}, \qquad 0 \le \mathcal{P} \le 1,$$

where $\mathcal{P}$ is called the **degree of polarization (DOP)**. For completely polarized light, $\mathcal{P} = 1$, and the above equation reduces to the Stokes vector for elliptically polarized light. Similarly, for unpolarized light, $\mathcal{P} = 0$, and the above equation reduces to the Stokes vector for unpolarized light.

The DOP $\mathcal{P}$ is defined by the equation

$$\mathcal{P} = \frac{I_{\text{pol}}}{I_{\text{tot}}} = \frac{\sqrt{S_1^{\,2} + S_2^{\,2} + S_3^{\,2}}}{S_0}, \qquad 0 \le \mathcal{P} \le 1,$$

where I_{tot} is the total intensity. These results show that the relation between the Stokes parameters must be broadened to
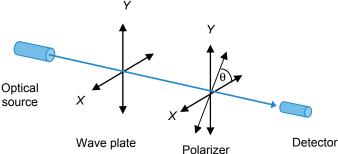
$$S_0^{\,2} \ge S_1^{\,2} + S_2^{\,2} + S_3^{\,2},$$

where the = and > sign indicate completely and unpolarized/partially polarized light, respectively.

Classical Measurement of the Stokes Parameters

The four **Stokes parameters of a polarized beam** can be measured by passing a beam sequentially through two polarizing elements known as a **wave plate** and a **polarizer**. The emerging beam is then incident on an optical detector. The measurement configuration is shown as below.

In the measurement the wave plate introduces a phase shift ϕ between the orthogonal components of the incident optical beam. The polarizer then transmits the resultant field along its transmission axis at an angle θ and the intensity $I(\theta, \phi)$ on the detector is then found to be

$$I(\theta,\phi) = \frac{1}{2}\left[S_0 + S_1 \cos 2\theta + S_2 \sin 2\theta \cos \phi - S_3 \sin 2\theta \sin \phi\right].$$

The Stokes parameters in the above equation are the parameters of the incident beam. By rotating the polarizer to $\theta = 0$, $\pi/4$, and $\pi/2$ without the quarter waveplate and then inserting the waveplate in the final measurement (a total of four measurements)žthe Stokes polarization parameters of the incident beam are found from the above equation to be
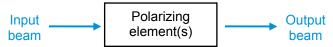
$$S_0 = I(0,0) + I(\pi/2,0),$$

$$S_1 = I(0,0) - I(\pi/2,0),$$

$$S_2 = 2I(\pi/4,0) - S_0,$$

and

$$S_3 = S_0 - 2I(\pi/4,\pi/2).$$

The Mueller Matrices for Polarizing Components

In general, the polarization ellipse is in a nonstandard form. The polarization ellipse (polarization state) can be changed by changing the amplitude(s), the phase, or by rotating the ellipse. **Polarizing materials** are available to do this. A polarized beam with a given polarization state propagates through one or more polarizing elements, where the beam acquires a new polarization state. This process is represented in the following figure.

The input beam is characterized by a Stokes vector S and the output beam by S'. The assumption is made that S and S' are linearly related by a 4×4 transformation matrix known as the **Mueller matrix**, which represents the polarizing element(s):

$$\begin{pmatrix} S_0' \\ S_1' \\ S_2' \\ S_3' \end{pmatrix} = \begin{pmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \\ m_{30} & m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}.$$

All the elements in the 4×4 Mueller matrix are real quantities. The above matrix relation can be written as a matrix equation,

$$\mathbf{S'} = \mathbf{M} \cdot \mathbf{S}$$
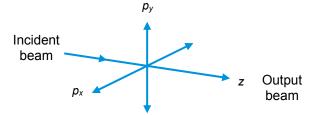
Only two polarizing elements are needed to change the three parameters of the ellipse (the orthogonal amplitudes and phase). The amplitude can be changed by using a polarizing element known as a polarizer. Similarly, the phase of an optical beam can be changed by a wave plate (also called a **retarder** or **phase shifter**). Finally, the polarization ellipse can be changed by rotation using a component called a **rotator**. Using these three polarizing elements, any elliptical polarization state can be obtained.

Polarizers

The polarizing element that changes the amplitude is a **linear polarizer**, which is characterized by two absorption coefficients that differ along the x- and y-axes, respectively. The absorption coefficients in the amplitude domain are defined by p_x and p_y and are

$$0 \le p_x \le 1, \qquad 0 \le p_y \le 1.$$

The value of 0 represents total absorption (no transmission) and the value of 1 represents total transmission (no absorption).

The Mueller matrix for a linear polarizer (the polarization matrix form used in the intensity domain) is found to be
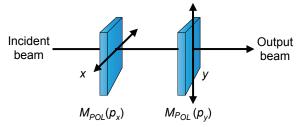
$$M_{\mathrm{POL}}\left(p_x, p_y\right) = \frac{1}{2}\begin{pmatrix} p_x^2 + p_y^2 & p_x^2 - p_y^2 & 0 & 0 \\ p_x^2 - p_y^2 & p_x^2 + p_y^2 & 0 & 0 \\ 0 & 0 & 2p_x p_y & 0 \\ 0 & 0 & 0 & 2p_x p_y \end{pmatrix}.$$

For an **ideal linear polarizer** there is complete transmission along one axis and no transmission along the orthogonal axis. The **Mueller matrix for an ideal linear polarizer** with its transmission axis along the x-axis is $p_x = 1$ and $p_y = 0$. Therefore,

$$M_{\mathrm{POL}} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Polarizers (cont'd)

For an ideal linear polarizer in which the transmission axis is along the y-axis, the Mueller matrix is

$$M_{\text{POL}} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

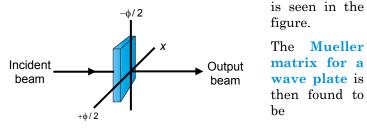The following figure shows a pair of **crossed polarizers** (The transmission axes are orthogonal to each other.)

$M_{POL}(p_x)$ $M_{POL}(p_y)$

The Mueller matrix for the pair is $M = M_{\text{POL}}(p_y) \cdot M_{\text{POL}}(p_x)$

$$M = M_{\text{POL}}\left(p_x\right) \cdot M_{\text{POL}}\left(p_y\right) = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This is a null matrix and shows that no light emerges when the linear polarizers are crossed. Another interesting application occurs for equal absorption along both axes [a neutral density (ND) filter]. In this case, $p_x = p_y = p$, therefore

$$\begin{pmatrix} S_0' \\ S_1' \\ S_2' \\ S_3' \end{pmatrix} = p^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = p^2 \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}.$$

The polarization state of the incident beam remains unchanged, but the intensity is reduced by a factor of p^2.

Wave Plates

Wave plates have the property that along the x-axis (called the **fast axis**) the x component of the field experiences a phase shift of $+\phi/2$ and, similarly, along the y-axis (called the **slow axis**) the y component experiences a phase shift of $-\phi/2$. The configuration for the wave plate

is seen in the figure.

The **Mueller matrix for a wave plate** is then found to be

$$M_{\text{WP}}(\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix}, \qquad \begin{aligned} \phi &= \phi_x - \phi_y, \\ &= +\frac{\phi}{2} - \left(-\frac{\phi}{2}\right), \end{aligned}$$

where ϕ is the total phase shift between the orthogonal components of the beam. Two forms of the **Mueller matrices** that are very important in polarization are the **quarter-wave plate (QWP)** ($\phi = \pi/2$) and the **half-wave plate (HWP)** ($\phi = \pi$):

$$M_{\text{QWP}}(\phi = \pi/2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$M_{\text{HWP}}(\phi = \pi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Wave Plates (cont'd)

The QWP has the interesting property that it transforms L+45P light to RCP light. This is shown in the following Stokes vector calculation:

$$S_{\text{OUT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Similarly, RCP light is transformed to L-45P light:

$$S_{\text{OUT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}.$$

The HWP has the unique property in that it reverses the **ellipticity and orientation angles** of the polarization ellipse:

$$S_{\text{OUT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ \cos 2\chi \cos 2\psi \\ \cos 2\chi \sin 2\psi \\ \sin 2\chi \end{pmatrix} = \begin{pmatrix} 1 \\ \cos 2\chi \cos 2\psi \\ -\cos 2\chi \sin 2\psi \\ -\sin 2\chi \end{pmatrix}.$$

The orientation and ellipticity angles are expressed, respectively, for the output beam by

$$\psi = \frac{1}{2}\sin^{-1}\left(\frac{S_2}{S_1}\right), \qquad \chi = \frac{1}{2}\sin^{-1}\left(\frac{S_3}{S_0}\right).$$

Comparing the elements of the Stokes vector shows that

$$\psi' = \frac{\pi}{2} - \psi, \qquad \chi' = \chi - \frac{\pi}{2}.$$

Rotators

The final method for changing the polarization state of an optical beam is to allow the beam to propagate through a rotator. The Mueller matrix for a rotator is

$$M_{ROT}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta & 0 \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where θ is the angle of rotation. Rotators only rotate the polarization ellipse; they do not affect the ellipticity. This can be seen by the propagation of a beam through a rotator:

$$\begin{pmatrix} S_0' \\ S_1' \\ S_2' \\ S_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta & 0 \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \cos 2\chi \cos 2\psi \\ \cos 2\chi \sin 2\psi \\ \sin 2\chi \end{pmatrix},$$

which yields

$$S' = \begin{pmatrix} S_0' \\ S_1' \\ S_2' \\ S_3' \end{pmatrix} = \begin{pmatrix} 1 \\ \cos 2\chi \cos(2\psi + 2\theta) \\ \cos 2\chi \sin(2\psi + 2\theta) \\ \sin 2\chi \end{pmatrix}$$

The above equation shows that the ellipticity (χ) is not affected by the rotator.

The polarizing elements given above are defined with respect to the x-axis. If either the polarizer or the wave plate M is rotated through an angle θ from the x-axis, the Mueller matrix equation $M(\theta)$ for the rotated component is

$$M(\theta) = M_{ROT}(-\theta) \cdot M \cdot M_{ROT}(\theta)$$

where the angle θ is measured from the x-axis.

Mueller Matrices for Rotated Components

We now consider the Mueller matrices for several polarizing components. The first is the **Mueller matrix for a rotated ideal linear polarizer** which is

$$M_{\text{POL}}(\theta) = \frac{1}{2}\begin{pmatrix} 1 & \cos 2\theta & \sin 2\theta & 0 \\ \cos 2\theta & \cos^2 2\theta & \sin 2\theta \cos 2\theta & 0 \\ \sin 2\theta & \sin 2\theta \cos 2\theta & \sin^2 2\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In particular, the Mueller matrix reduces to the following special forms for $\theta = 0°$, $45°$, $90°$, and $135°$:

$$M_{\text{LHP}} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad M_{\text{L+45P}} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_{\text{LVP}} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_{\text{L-45P}} = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Another important rotated Mueller matrix is the **Mueller matrix of a rotated wave plate**, which is

$$M_{\text{WP}}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2 2\theta + \cos\phi\sin^2 2\theta & (1-\cos\phi)\sin 2\theta\cos 2\theta & \sin\phi\sin 2\theta \\ 0 & (1-\cos\phi)\sin 2\theta\cos 2\theta & \sin^2 2\theta + \cos\phi\cos^2 2\theta & -\sin\phi\cos 2\theta \\ 0 & -\sin\phi\sin 2\theta & \sin\phi\cos 2\theta & \cos\phi \end{pmatrix}.$$

Of special interest is the **Mueller matrix for the rotated HWP,** written as $M_{\textbf{HWP}}(\theta)$, which is

$$M_{\text{HWP}}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 4\theta & \sin 4\theta & 0 \\ 0 & \sin 4\theta & -\cos 4\theta & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Mueller Matrices for Rotated Components (cont'd)

This Mueller matrix looks very similar to the Mueller matrix for rotation. The difference is that the angle θ is doubled and the negative signs show that the orientation and ellipticity angles are reversed. Consequently, this matrix is called a **pseudo-rotator**.

The **Mueller matrix of a rotated QWP** is also of interest and is found to be

$$M_{\text{QWP}}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2 2\theta & \sin 2\theta \cos 2\theta & \sin 2\theta \\ 0 & \sin 2\theta \cos 2\theta & \sin^2 2\theta & -\cos 2\theta \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \end{pmatrix}.$$

Multiplying this matrix by the Stokes vector of the output beam for input L+45P light yields

$$S = \begin{pmatrix} 1 \\ \cos 2\theta \sin 2\theta \\ \sin^2 2\theta \\ \cos 2\theta \end{pmatrix}.$$
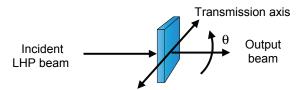
Rotating the QWP from 0° in steps of 45° increments generates the following Stokes vectors:

$$S(0°) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad S(45°) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad S(90°) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad S(135°) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Rotating the QWP generates RCP, L+45P, and LCP light but not LVP, LHP, or L–45P polarized light.

Mueller Matrix Applications: Malus's Law

The first of several applications of the Mueller matrices is the propagation of a LHP beam through a polarizer rotating through an angle.

The Stokes vector of the output beam is

$$S' = \frac{1}{2}\begin{pmatrix} 1 & \cos 2\theta & \sin 2\theta & 0 \\ \cos 2\theta & \cos^2 2\theta & \cos 2\theta \sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta \sin 2\theta & \sin^2 2\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

and

$$S' = \frac{1 + \cos 2\theta}{2}\begin{pmatrix} 1 \\ \cos 2\theta \\ \sin 2\theta \\ 0 \end{pmatrix}.$$
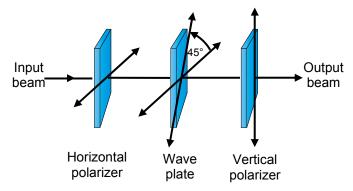
The output beam continues to be linearly polarized. However, its intensity is

$$I(\theta) = \frac{1 + \cos 2\theta}{2} = \cos^2 \theta,$$

which is recognized as Malus's Law.

An important application of polarizers and wave plates is the **optical shutter**. A polarized incident beam is completely blocked by a pair of crossed polarizers. However, by placing a wave plate with its fast axis at +45° between a pair of crossed polarizers and varying the phase of the wave plate the intensity of the light that emerges from the second polarizer can be controlled.

Mueller Matrix Applications – The Optical Shutter

Horizontal polarizer   Wave plate   Vertical polarizer

The shutter configuration is shown in the following figure. The Mueller matrix for the **variable-phase wave plate** with its fast axis at +45° is

$$M_{\text{WP}}(45°) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & 0 & \sin\phi \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\phi & 0 & \cos\phi \end{pmatrix}.$$

For an input beam of arbitrary polarization the Stokes vector of the output beam is then

$$S' = M_{\text{polv}} \cdot M_{\text{WP}}(\phi, 45°) \cdot M_{\text{polh}} \cdot S,$$
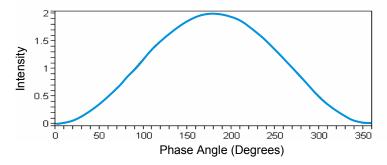
$$S' = \frac{(1 - \cos\phi)(S_0 + S_1)}{4} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

When $\phi = 0°$ (the wave plate is not present) the intensity, as expected, is zero. We also see that the output beam intensity will be zero if the input beam is LVP. The intensity of the beam can be written, in general, as

$$I(\phi) = I_0 \left(1 - \cos\phi\right).$$

Mueller Matrix Applications – Stokes Parameters

A plot of the output intensity of the optical shutter is shown in the following figure.

The maximum intensity occurs at 180°. Thus, by varying the phase from 0° to 180° there is complete control of the output intensity. This behavior is the basis of optical shutters used to prevent "flash blindness."

The final application is of the Mueller calculus to the **classical measurement** of the Stokes parameters. The measurement requires a wave plate with its fast axis fixed at 0° and a linear polarizer rotated through θ.

The input beam is characterized by its four Stokes parameters S_0, S_1, S_2, and S_3. The Mueller matrix $M = M_{\text{pol}}(\theta) \cdot M_{\text{WP}}(\phi)$ and the output intensity are then

$$M = \frac{1}{2}\begin{pmatrix} 1 & \cos 2\theta & \sin 2\theta \cos \phi & -\sin 2\theta \sin \phi \\ \cos 2\theta & \cos^2 2\theta & \sin 2\theta \cos 2\theta \cos \phi & -\sin 2\theta \cos 2\theta \sin \phi \\ \sin 2\theta & \sin 2\theta \cos 2\theta & \sin^2 2\theta \cos \phi & -\sin^2 2\theta \sin \phi \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$I(\theta, \phi) = (1/2)(S_0 + S_1 \cos 2\theta + S_2 \sin 2\theta \cos \phi - S_3 \sin 2\theta \sin \phi).$$

The Observable Polarization Sphere

When an incident beam propagates through several polarizing elements, the mathematical manipulations to determine the polarization state of the output beam are tedious. Poincaré suggested that a graphical solution to this problem could be obtained by plotting the polarization state(s) in terms of the orientation and ellipticity angles, ψ and χ, on the surface of a sphere, where the input and output beams are represented by the points $P(\psi, \chi)$ and $P(\psi', \chi')$. A difficulty arises because measurements of polarized light are made in the intensity (observable) domain, but the Poincaré sphere and its spherical coordinates ψ and χ apply to the amplitude domain. This problem can be overcome by reformulating the Poincaré sphere so that the Stokes parameters S_1, S_2, and S_3, are plotted on the Cartesian axes and the amplitude products $E_{0x}{}^2$, $E_{0y}{}^2$, and $E_{0y}E_{0x}$, and the phase δ are plotted on the surface of the sphere. This requires that the amplitude products be transformed to a spherical angle. The reformulated sphere is called the **observable polarization sphere (OPS)**.

The Stokes parameters can be expressed in terms of spherical coordinate angles by first dividing each of the four equations representing the Stokes parameters by S_0, and then using the auxiliary relation equation $\tan\alpha = E_{0y} / E_{0x}$ transform S_1, S_2, and S_3 to trigonometric terms:

$$S_0 = E_{0x}{}^2 + E_{0y}{}^2 \rightarrow 1,$$

$$S_1 = E_{0x}{}^2 - E_{0y}{}^2 \rightarrow \cos 2\alpha,$$

$$S_2 = 2E_{0x}E_{0y}\cos\delta \rightarrow \sin 2\alpha \cos\delta,$$

$$S_3 = 2E_{0x}E_{0y}\sin\delta \rightarrow \sin 2\alpha \sin\delta, \quad \delta = \delta_y - \delta_x.$$

On the OPS, the Stokes polarization parameters S_1, S_2, and S_3 represent the Cartesian axes, and 2α and δ are the spherical coordinate angles.

The Observable Polarization Sphere (cont'd)

The angles 2α and δ on the OPS are related to the Stokes parameters by
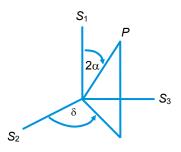
$$2\alpha = \cos^{-1}\left(\frac{S_1}{S_0}\right), \ \ 0 \le 2\alpha < \pi,$$

$$\delta = \tan^{-1}\left(\frac{S_3}{S_2}\right), \ \ 0 \le \delta < 2\pi.$$

The orientation and ellipticity angle s, ψ and χ, can be expressed in terms of 2α and δ by

$$\tan(2\psi) = \tan(2\alpha)\cos\delta,$$

$$\sin(2\chi) = \sin(2\alpha)\sin\delta.$$

Similarly, the above equations can be inverted so 2α and δ can be expressed in terms ψ and χ:

$$\cos(2\alpha) = \cos(2\chi)\cos(2\psi),$$

$$\cot\delta = \cot(2\chi)\sin(2\psi).$$
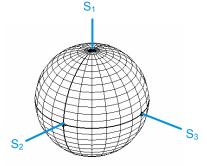
The coordinates axes and angles of the OPS are now arranged as shown. S_2, S_3, and S_1 correspond to the *x*-, *y*- and *z*-axes. The angles 2α and δ are measured from the positive S_1 and S_2, axes, respectively.

The S_1-S_2 is the **prime meridian** plane and S_2-S_3 is the equatorial plane. The latitude lines represent the phase δ and the longitude lines represent 2α.
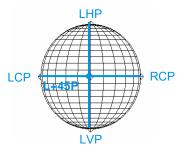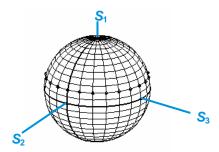

OPS and the Cartesian axes

The Observable Polarization Sphere (cont'd)

The Stokes vectors for beams emerging from a wave plate, a rotator, and an ideal linear polarizer on the OPS are now plotted. For a wave plate we have
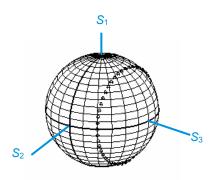
$$S' = M_{\text{WP}}(\phi) \cdot S,$$

$$S' = \begin{pmatrix} S_0' \\ S_1' \\ S_2' \\ S_3' \end{pmatrix} = \begin{pmatrix} 1 \\ \cos 2\alpha' \\ \sin 2\alpha' \cos \delta' \\ \sin 2\alpha' \sin \delta' \end{pmatrix} = \begin{pmatrix} 1 \\ \cos 2\alpha \\ \sin 2\alpha \cos(\delta + \phi) \\ \sin 2\alpha \sin(\delta + \phi) \end{pmatrix}.$$

This equation shows that the phase increases along small circles around the S_1 axis. The equation is plotted in increments of $\phi = 15°$ on the OPS at a fixed latitude angle of $2\alpha = 75°$.

The phase behavior on the sphere is restricted to the latitude lines. Now consider the rotator:

$$S' = M_{\text{ROT}}(\theta) \cdot S,$$

$$S' = \begin{pmatrix} S_0' \\ S_1' \\ S_2' \\ S_3' \end{pmatrix} = \begin{pmatrix} 1 \\ \cos 2\alpha' \\ \sin 2\alpha' \cos \delta' \\ \sin 2\alpha' \sin \delta' \end{pmatrix} = \begin{pmatrix} 1 \\ \cos 2\theta \cos 2\alpha - \sin 2\theta \sin 2\alpha \cos \delta \\ \sin 2\theta \cos 2\alpha + \sin 2\alpha \cos 2\theta \cos \delta \\ \sin 2\alpha \sin \delta \end{pmatrix}.$$

The Observable Polarization Sphere (cont'd)

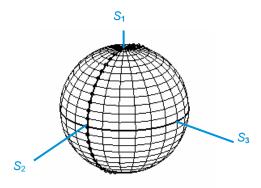The equation for rotation is plotted in increments of $\theta = 10°$ for an input beam with $\alpha = 45°$ and $\delta = 30°$.

On the OPS rotation appears as small circles around the S_3 axis. Finally, for a rotated ideal linear polarizer we have

$$S' = M_{POL}(\theta) \cdot S,$$

$$S' = \frac{1}{2}\left(S_0 + S_1 \cos 2\theta + S_2 \sin 2\theta\right)\begin{pmatrix} 1 \\ \cos 2\theta \\ \sin 2\theta \\ 0 \end{pmatrix}.$$

The output beam is linearly polarized and is always restricted to the prime meridian as shown below on the OPS.

Plotting the Quarter-Wave Plate on the OPS
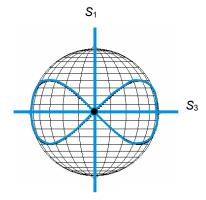
We now consider several plotting applications on the OPS. The first **is rotation of a QWP**.

It is easiest to understand the behavior of the **rotating QWP** using an incident L+45P beam. For this input polarization state the output beam is then $S' = M_{QWP}(\theta) \cdot S_{L+45P}$ and the Stokes vector of the S' output beam is
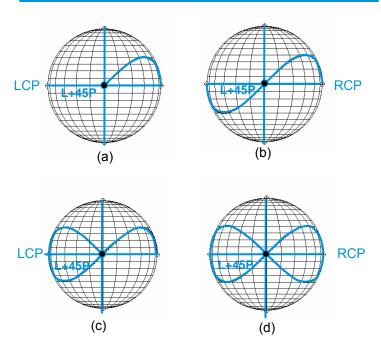
$$S' = \begin{pmatrix} 1 \\ \cos 2\theta \sin 2\theta \\ \sin^2 2\theta \\ \cos 2\theta \end{pmatrix},$$

where S' is now plotted over 180° on the OPS and yields

The boldface dot is the initial L+45P point. A complete rotation through 180° leads to a "**figure 8**." The rotation of the QWP in increments of 45° on the next page.

Plotting the Quarter-Wave Plate on the OPS (cont'd)

(a)

(b)

(c)

(d)

The input L+45P beam is immediately transformed to RCP *prior* to any rotation and appears at the RCP point; this can be confirmed by evaluating the Stokes vector S' for $\theta = 0$. Rotating the QWP through 45° the polarization returns to (a) the L+45P point. Another 45° of rotation moves the beam to (b) the LCP point; an additional 45° rotation brings the polarization back again to (c) the L+45P point, and another rotation through 45° brings the polarization state back to the original starting point (d) RCP light.

It was shown that the only degenerate polarization states generated by rotating the QWP are RCP, L+45P, and LCP. The rotation of the QWP cannot generate LHP, LVP, or L-45P polarization states. This is clearly seen by plotting the rotating QWP on the OPS. We also note that rotation of the QWP appears in the form of phase shifts.
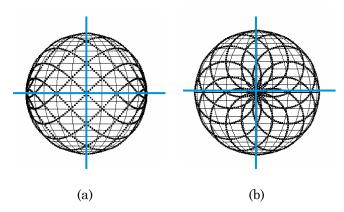
The Rotating Quarter-Wave Plate

The configuration for arbitrary linear polarization input states that are rotated through 180°:

$S \longrightarrow$ | Rotating LP | $\longrightarrow$ | Rotating QWP | $\longrightarrow S'$

The Stokes vector of the output beam S' is

$$S' = \begin{pmatrix} 1 \\ \dfrac{1}{2}\cos(4\theta - 2\alpha) + \dfrac{1}{2}\cos 2\alpha \\ \dfrac{1}{2}\sin(4\theta - 2\alpha) + \dfrac{1}{2}\sin 2\alpha \\ -\sin(2\theta - 2\alpha) \end{pmatrix},$$
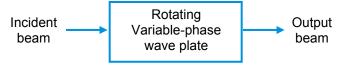
where θ and 2α refer to the rotation of the QWP and orientation of the incident LP beam on the QWP. The equation for S' is now plotted for different LP orientations.

(a)   (b)

The views in (a) and (b) are along the positive S_2 axis and S_3 axes, respectively. The figures show that using a rotating linear polarizer and a rotating QWP any polarization state on the polarization sphere can be obtained.
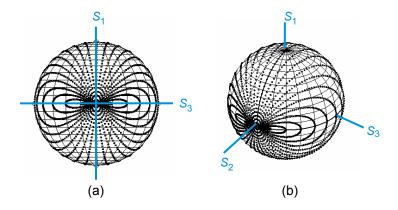
The Babinet-Soleil Compensator

A device that is used to generate and analyze polarized light is the **Babinet-Soleil compensator**. This device is a variable-phase wave plate that can be rotated through 360°. The device is described by the Mueller matrix for a rotated wave plate.

Incident beam → Rotating Variable-phase wave plate → Output beam

For an incident L+45P beam the Stokes vector S′ is

$$S' = \begin{pmatrix} 1 \\ \cos 2\theta \sin 2\theta (1 - \cos \phi) \\ \sin^2 2\theta + \cos^2 2\theta \cos \phi \\ \cos 2\theta \sin \phi \end{pmatrix},$$

where θ is the rotation angle of the compensator and ϕ is the phase shift. The equation for S' is plotted on the OPS.

(a)   (b)

View (a) is along the S_2 axis, and (b) is a nonaxial view. The views show that the Babinet-Soleil compensator can generate any polarization state using incident L+45P light.

Linear and Circular Polarizers

Two polarizing components that are used extensively in an optics laboratory are linear and circular polarizers. It is worthwhile to understand the origins of these names. **The Mueller matrix for a rotated linear polarizer** is

$$M_{\text{LP}}(\theta) = \frac{1}{2} \begin{pmatrix} 1 & \cos 2\theta & \sin 2\theta & 0 \\ \cos 2\theta & \cos^2 2\theta & \cos 2\theta \sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta \sin 2\theta & \sin^2 2\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The Stokes vector of an output beam using the standard form $S = (S_0, S_1, S_2, S_3)$ for the input Stokes vector is then

$$S_{\text{LP}} = \frac{1}{2}\left(S_0 + S_1 \cos 2\theta + S_2 \sin 2\theta\right) \begin{pmatrix} 1 \\ \cos 2\theta \\ \sin 2\theta \\ 0 \end{pmatrix}.$$

Thus, regardless of the polarization state of the input beam the output beam is always linearly polarized.

A circular polarizer is constructed from an L+45P polarizer and a QWP. The **Mueller matrix of a circular polarizer** and the output beam are, respectively,

$$M_{\text{CP}} = M_{\text{QWP}} \cdot M_{\text{L+45P}} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

$$S_{\text{CP}} = \frac{1}{2}\left(S_0 + S_2\right) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The output beam is always circularly polarized regardless of the polarization state of the input beam.

The Generation of Elliptically Polarized Light

As a final application of the OPS we consider the problem of generating elliptically polarized light of any orientation and ellipticity. To do this, consider the following figure:
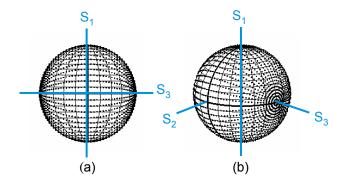
S_{L+45P} ⟶ [Wave plate] ⟶ [Rotator] ⟶ S',

where S_{L+45P} is Stokes vector for L+45P light. The matrix equation for the above diagram and the output S' are

$$S' = M_{ROT}(\theta) \cdot M_{WP}(\phi) \cdot S_{L+45P},$$

$$S' = \begin{pmatrix} 1 \\ \sin 2\theta \cos \phi \\ \cos 2\theta \cos \phi \\ \sin \phi \end{pmatrix}.$$
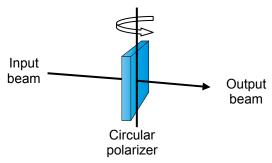
The above equation is plotted on the OPS.

(a)   (b)

The figures show that S' generates every polarization state *very uniformly* on the OPS unlike, for example, the Babinet-Soleil compensator. In addition, the Stokes vector S' shows that the ellipticity and the orientation angles are directly obtained from $\chi = \phi/2$ and $\psi = \pi/4 - \theta$.

Measurement Methods of the Stokes Parameters

In addition to the classical measurement to determine the Stokes parameters, several other measurement methods have been developed. In the classical measurement a QWP, which is also absorbing, must be used in the final measurement. In order to avoid this problem as well as axial alignment problems of the linear polarizer and the

Circular
polarizer

wave plate, a circular polarizer can be used.

A circular polarizer has the property that on one side it behaves as a circular polarizer, whereas on the other side it behaves as a linear polarizer. For an incident beam with an arbitrary polarization, the intensity $I_C(\theta)$ of the output beam on the circular side is

$$I_C(\theta) = \frac{1}{2}(S_0 - S_1 \sin 2\theta + S_2 \cos 2\theta).$$

The circular polarizer is then flipped to the linear side. The intensity $I_L(\theta)$ of the output beam is found to be independent of the rotation of the circular polarizer and is

$$I_L(\theta) = \frac{1}{2}(S_0 + S_3).$$

From the above two equations the intensities are measured and the Stokes parameters are then found to be

$$S_0 = I_C(0°) + I_C(90°), \qquad\qquad S_1 = S_0 - 2I_C(90°),$$

$$S_2 = I_C(0°) - I_C(90°), \qquad\qquad S_3 = 2I_L(0°) - S_0.$$

The Rotating Quarter-Wave Plate Measurement

Another method for measuring the Stokes parameters of a polarized beam is to allow the beam to propagate through a rotating QWP followed by a linear horizontal polarizer; the wave plate rotates at an angular frequency of ω. This arrangement is shown in the following figure.

The **Mueller matrix equation** for the above configuration is

$$S' = M_{\text{LHP}} \cdot M_{\text{QWP}}(\theta) \cdot S,$$

where $\theta = \omega t$. Carrying out the matrix multiplication and using the trigonometric half-angle formulas, the intensity of the output beam is found have the form

$$I(\theta) = \frac{1}{2}\left(A + B\sin 2\theta + C\cos 4\theta + D\sin 4\theta\right)$$

where

$$A = S_0 + \frac{S_1}{2}, \qquad B = S_3, \qquad C = \frac{S_1}{2}, \qquad D = \frac{S_2}{2}.$$

The Stokes parameters are then

$$S_0 = A - C, \qquad S_1 = 2C, \qquad S_2 = 2D, \qquad S_3 = B.$$

The terms A, B, C, and D can be determined from the equation I(θ), which is a truncated **Fourier series**. It shows that there is a dc term (A), a double frequency term (B) and two quadruple frequency terms (C and D). Using Fourier analysis the coefficients A, B, C, and D are then

$$A = \frac{1}{\pi}\int_0^{2\pi} I(\theta)d\theta, \qquad\qquad B = \frac{2}{\pi}\int_0^{2\pi} I(\theta)\sin 2\theta d\theta,$$

$$C = \frac{2}{\pi}\int_0^{2\pi} I(\theta)\cos 4\theta d\theta, \qquad D = \frac{2}{\pi}\int_0^{2\pi} I(\theta)\sin 4\theta d\theta.$$

Birefringent Crystals and Wave Plates

Quartz is a **birefringent crystal** used extensively as a wave plate. A crystal is generally characterized by three orthogonal axes, the *x*-, *y*-, and *z*-axis. Along these axes the propagation velocities of a wave are called the principal velocities v_x, v_y, and v_z, and are expressed by

$$v_x = c/\sqrt{\varepsilon_x}, \qquad v_y = c/\sqrt{\varepsilon_y}, \qquad v_z = c/\sqrt{\varepsilon_z},$$

where ε_x, ε_y, and ε_z are the principal dielectric constants and c is the speed of light. The velocity of propagation v of a wave in a crystal is governed by Fresnel's equation of wave normals s_x, s_y, and s_z:

$$\frac{s_x^{\,2}}{v^2 - v_x^{\,2}} + \frac{s_y^{\,2}}{v^2 - v_y^{\,2}} + \frac{s_z^{\,2}}{v^2 - v_z^{\,2}} = 0.$$

There are two solutions for v and are found to correspond to a spherical wave and an ellipsoidal wave. In the direction of the **optic axis**, however, the ellipsoidal wave becomes spherical.

For a **uniaxial crystal** with its optic axis in the z direction, $v_x = v_y = v_o$ [the ordinary wave (*o*)] and $v_z = v_e$ [the extraordinary wave (*e*)]. Substituting these conditions into Fresnel's equation, above, yields two solutions:

$$v'^{\,2} = v_o^{\,2}, \quad \text{and} \quad v''^{\,2} = v_o^{\,2}\cos^2\theta + v_e^{\,2}\sin^2\theta,$$

where the angle θ is measured from the optics axis. Along the optic axis $v''^{\,2} = v_o^{\,2}$, and perpendicular to the optic axis $v''^{\,2} = v_e^{\,2}$. The maximum phase shift occurs when the propagation is perpendicular to the optic axis. The phase shift ϕ of the wave propagating through a wave plate is then

$$\phi = \frac{2\pi}{\lambda}(n_o - n_e)d,$$

where n_o and n_e are the refractive indices of the ordinary and extraordinary axes, respectively, and d is the path length.

Multiple and Zero-Order Wave Plates

For a QWP and a HWP, the **optical thickness** $(n_o - n_e)d$ can be expressed as

$$(n_o - n_e)d = (4m+1)\frac{\lambda}{4} = m\lambda + \frac{\lambda}{4} = m\lambda + k_1\lambda,$$

$$(n_o - n_e)d = (2m+1)\frac{\lambda}{2} = m\lambda + \frac{\lambda}{2} = m\lambda + k_2\lambda,$$

where $m = 0, 1, 2\cdots$, and $k_1 = 1/4$ and $k_2 = 1/2$, respectively. For $m = 0, 1, 2\cdots$, the wave plates behave like QWPs and HWPs. Because multiple wavelengths can propagate through the wave plate, it is called a **multiple-order wave plate**.

It is difficult to control the optical thickness in the fabrication of wave plates. However, by using two wave plates with their fast axes at +45° and −45° this can be overcome. The Mueller matrix for this combination is

$$M_{\text{WP12}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\phi_1 - \phi_2) & 0 & \sin(\phi_1 - \phi_2) \\ 0 & 0 & 1 & 0 \\ 0 & -\sin(\phi_1 - \phi_2) & 0 & \cos(\phi_1 - \phi_2) \end{pmatrix}.$$
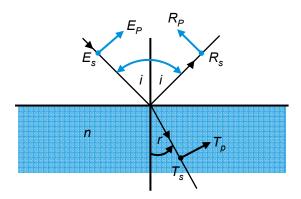
Then, using the above relations the Mueller matrix becomes

$$M_{\text{WP12}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\left[2\pi(k_1 - k)_2\right] & 0 & \sin\left[2\pi(k_1 - k_2)\right] \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\left[2\pi(k_1 - k_2)\right] & 0 & \cos\left[2\pi(k_1 - k_2)\right] \end{pmatrix}.$$

Thus, the multiplicity parameter m cancels out. Because m does not appear, the two-wave-plate combination is commonly called a zero-order wave plate. The parameters k_1 and k_2 appear as $k_1 - k_2 = k$. Since there are now two variables (k_1, k_2), greater control can be exercised when polishing in order to obtain the correct phase shift, ϕ.

Mueller Matrices for Reflection and Transmission

The reflection and transmission of light at an air-dielectric (glass) interface is shown.

Here, $E_{s,p}$, $R_{s,p}$, and $T_{s,p}$ are the incident, reflected, and transmitted field components, s and p refer to the components perpendicular and parallel to the plane of the paper, i and r are the incident and refracted angles, and n is the refractive index of the medium (glass). The reflection and transmission is governed by **Fresnel's equations**:

$$R_p = \frac{\tan(i-r)}{\tan(i+r)} E_p, \qquad R_s = -\frac{\sin(i-r)}{\sin(i+r)} E_s,$$

$$T_p = \frac{2\sin r \cos i}{\sin(i+r)\cos(i-r)} E_p, \qquad T_s = \frac{2\sin r \cos i}{\sin(i+r)} E_s.$$

The Stokes parameters for the reflected field are defined by

$$S_{0R} = \cos i (R_s R_s{}^* + R_p R_p{}^*), \qquad S_{1R} = \cos i (R_s R_s{}^* - R_p R_p{}^*),$$

$$S_{2R} = \cos i (R_s R_p{}^* + R_p R_s{}^*), \qquad S_{3R} = j \cos i (R_s R_p{}^* - R_p R_s{}^*),$$

where the factor $j = \sqrt{-1}$ and $*$ is the complex conjugate.

Reflection and Transmission Stokes Parameters

The **Mueller-Stokes matrix equation for reflection** is given by

$$S_R = M_R \cdot S,$$

where M_R is the Mueller matrix for reflection. The individual Stokes polarization parameters are

$$S_{0R} = f_R[(\cos^2\alpha_- + \cos^2\alpha_+)S_0 + (\cos^2\alpha_- - \cos^2\alpha_+)S_1],$$

$$S_{1R} = f_R[(\cos^2\alpha_- - \cos^2\alpha_+)S_0 + (\cos^2\alpha_- + \cos^2\alpha_+)S_1],$$

$$S_{2R} = -f_R(2\cos\alpha_-\cos\alpha_+)S_2, \qquad S_{3R} = -f_R(2\cos\alpha_-\cos\alpha_+)S_3,$$

$$f_R = \frac{1}{2}\left(\frac{\tan\alpha_-}{\sin\alpha_+}\right)^2,$$

where equations $\alpha_\pm = i \pm r$. The Stokes parameters for reflection correspond to reflection by a polarizing element.

Similarly, the **Mueller-Stokes equation for transmission** is

$$S_T = M_T \cdot S.$$

The Stokes parameters for the transmitted components are

$$S_{0T} = n\cos i(T_s T_s{}^* + T_p T_p{}^*), \quad S_{1T} = n\cos i(T_s T_s{}^* - T_p T_p{}^*),$$

$$S_{2T} = n\cos i(T_s T_p{}^* + T_p T_s{}^*), \quad S_{3T} = jn\cos i(T_s T_p{}^* - T_p T_s{}^*).$$

The Stokes parameters for the transmitted beam are then

$$S_{0T} = f_T[(\cos^2\alpha_- + 1)S_0 + (\cos^2\alpha_- - 1)S_1],$$

$$S_{1T} = f_T[(\cos^2\alpha_- - 1)S_0 + (\cos^2\alpha_- + 1)S_1],$$

$$S_{2T} = -f_T(2\cos\alpha_-)S_2,$$

$$S_{3T} = -f_T(2\cos\alpha_-)S_3, \qquad f_T = \frac{1}{2}\frac{\sin 2i\sin 2r}{(\sin\alpha_+\cos\alpha_-)^2}.$$

The Stokes parameters for transmission correspond to transmission through a polarizer.

Reflection and Transmission Stokes Parameters (cont'd)

From the previous relations $S_0 = S_{0R} + S_{0T}$; that is, the total intensity (energy) is conserved. The previous relations are now used to consider the behavior of incident unpolarized light, $S = I_0\{1,0,0,0\}$, that is reflected by a dielectric surface. The Stokes vector S_R is

$$S_R = \begin{pmatrix} S_{0R} \\ S_{1R} \\ S_{2R} \\ S_{3R} \end{pmatrix} = \frac{1}{2}\left(\frac{\tan\alpha_-}{\sin\alpha_+}\right)^2 \begin{pmatrix} \cos^2\alpha_- + \cos^2\alpha_+ \\ \cos^2\alpha_- - \cos^2\alpha_+ \\ 0 \\ 0 \end{pmatrix}.$$

The DOP $\mathscr{P}$ of the reflected light is

$$\mathscr{P} = \left|\frac{S_{1R}}{S_{0R}}\right| = \left|\frac{\cos^2\alpha_- - \cos^2\alpha_+}{\cos^2\alpha_- + \cos^2\alpha_+}\right|.$$
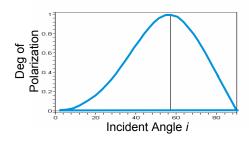
In general, $\mathscr{P}$ is less than 1. If, however, $\cos\alpha_+ = 0$ then $\mathscr{P} = 1 (= 100\%)$ and $i + r = \pi/2$. This is the **Brewster's angle condition** and S_R then reduces to

$$S_R = \begin{pmatrix} S_{0R} \\ S_{1R} \\ S_{2R} \\ S_{3R} \end{pmatrix} = \frac{1}{2}\cos^2 2i_B \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Thus, at the Brewster's angle, i_B, the reflected light is LHP and the LVP component vanishes. When using a LVP polarizer as an analyzer, the intensity of the beam that emerges is 0. The angle of incidence i_B has now been found so that from **Brewster's law** ($n = \tan i_B$) the refractive index of the glass can be determined.

On the following page a figure is shown for $\mathscr{P}$ versus the incident angle i for a glass with a refractive index of $n = 1.50$.

Reflection and Transmission Stokes Parameters (cont'd)
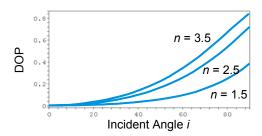
Here, the vertical line represents the Brewster angle. The DOP is 0 at an incident angle of 0°, rises to a maximum of 1 at the Brewster angle (~57.4°) and then returns to 0 at 90°. The above figure also shows that changing the incident angle i allows any DOP to be obtained, a property of reflection that is often overlooked.

The **Stokes vector of a transmitted beam** through a single surface of a dielectric (glass) for incident unpolarized light is

$$\begin{pmatrix} S_{0T} \\ S_{1T} \\ S_{2T} \\ S_{3T} \end{pmatrix} = \frac{1}{2} \frac{\sin 2i \sin 2r}{(\sin\alpha_+ \cos\alpha_-)^2} \begin{pmatrix} \cos^2 2\alpha_- + 1 \\ \cos^2 2\alpha_- - 1 \\ 0 \\ 0 \end{pmatrix},$$

and the **DOP of the transmitted beam** is
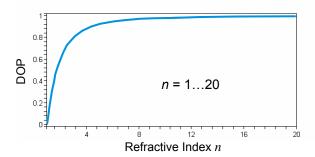
$$\mathscr{P} = \left| \frac{S_1}{S_0} \right| = \left| \frac{\cos^2\alpha_- - 1}{\cos^2\alpha_- + 1} \right|.$$

Reflection and Transmission Stokes Parameters (cont'd)

The transmitted light is always partially polarized, and the incident unpolarized light never becomes completely polarized using a single surface. However, by increasing the refractive index, the DOP increases, and for an incident angle of 90°, the DOP reduces to

$$\mathscr{P} = \left| \frac{n^2 - 1}{n^2 + 1} \right|.$$

The Mueller matrices for reflection and transmission assume simplified forms:

Normal Incidence ($i = 0°$):

$$M_R = \left(\frac{n-1}{n+1} \right)^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad M_T = \frac{4n}{(n+1)^2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where n is the refractive index of the medium. For incident unpolarized light of intensity I_0 the reflected and transmitted intensities are

$$I_R = \left(\frac{n-1}{n+1} \right)^2 I_0, \ I_T = \frac{4n}{(n+1)^2} I_0,$$

and $I_R + I_T = I_0$.

Reflection and Transmission Mueller Matrices

Brewster Angle $(i = i_B)$:

$$M_{R,B} = \frac{1}{2}\cos^2 2i_B \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_{T,B} = \frac{1}{2}\begin{pmatrix} \sin^2 2i_B + 1 & \sin^2 2i_B - 1 & 0 & 0 \\ \sin^2 2i_B - 1 & \sin^2 2i_B + 1 & 0 & 0 \\ 0 & 0 & 2\sin 2i_B & 0 \\ 0 & 0 & 0 & 2\sin 2i_B \end{pmatrix}.$$

$45°$ Incident Angle $(i = 45°)$:

$$M_{R45°} = \frac{1 - \sin 2r}{(1 + \sin 2r)^2}\begin{pmatrix} 1 & \sin 2r & 0 & 0 \\ \sin 2r & 1 & 0 & 0 \\ 0 & 0 & -\cos 2r & 0 \\ 0 & 0 & 0 & -\cos 2r \end{pmatrix}.$$
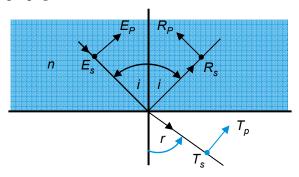
This matrix shows that r can be determined by reflection and the refractive index n can then be found from Snell's law.

Experts say polarized sunglasses can reduce glare that caused by sunlight reflecting off snow. Others say the lenses are not satisfactory for sports such as downhill skiing because they may not provide the contrast the eye needs to distinguish ice patches or moguls.

Polarized lenses may also react adversely with liquid crystal displays (LCDs) found on the dashboards of some cars or in other places such as the digital screens on automatic teller machines (ATMs). When viewed through polarized lenses from a certain angle, LCDs can be invisible.

Total Internal Reflection (TIR)

An interesting phenomenon takes place when an optical beam propagates from the medium into air.

In this arrangement Snell's law becomes $n \sin i = \sin r$. The maximum value of r is $\pi/2$; therefore, the beam *cannot* propagate from the medium to air if $n \sin i > 1$. For this condition the incident beam is reflected back into the medium and there is **total internal reflection (TIR)**. **Fresnel's reflection equations** for the parallel (p) and perpendicular (s) components become

$$R_p = e^{-i\delta_p} E_p, \; R_s = e^{-i\delta_s} E_s,$$

which leads to

$$\tan\frac{\delta}{2} = \frac{\cos i \sqrt{n^2 \sin^2 i - 1}}{n \sin^2 i}, \; \delta = \delta_p - \delta_s.$$
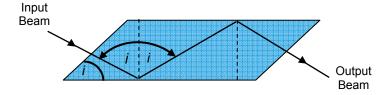
The Mueller matrix for TIR then becomes

$$M_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\delta & -\sin\delta \\ 0 & 0 & \sin\delta & \cos\delta \end{pmatrix}.$$

This matrix is seen to correspond to a wave plate.

The Fresnel Rhomb

The phenomenon of TIR was first used by Fresnel to create circularly polarized light from linearly polarized light. In order to use TIR, Fresnel constructed a glass rhomb.

Using input L+45P light, circularly polarized light can be obtained by creating a phase shift (δ) of 45° at the lower and upper surfaces of the glass rhomb, respectively. The Mueller matrix for two reflections is

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\delta_U + \delta_L) & -\sin(\delta_U + \delta_L) \\ 0 & 0 & \sin(\delta_U + \delta_L) & \cos(\delta_U + \delta_L) \end{pmatrix},$$
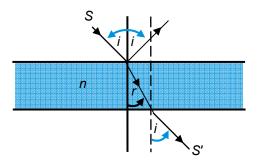
where $\delta_{U,L}$ are the phase shifts at the upper and lower surfaces, respectively. For glass such as BK7 the refractive index is $n = 1.5151$ at 6328 Å (HeNe). From the equation for δ on the previous page the angle of incidence of $I = 55°05'$ yields a phase shift of $\delta = 45.0°$. Then $\delta_U + \delta_L = 90°$, so the **Mueller matrix for the Fresnel rhomb** is

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

For an input L+45P beam $S_{L+45P} = \{1, 0, 1, 0\}$ and the Stokes vector of the output beam is then $S_{RCP} = \{1, 0, 0, 1\}$.

Single and Multiple Dielectric Plates

For a single dielectric plate, there are an infinite number of internal reflections between the two surfaces; however, most of these can usually be ignored.

A single dielectric plate.

For transmission through the upper and lower surface of the dielectric plate the Mueller matrix equation for the plate is

$$S_T = M_T^2 \cdot S.$$
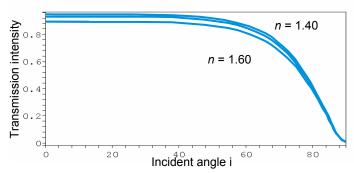
In terms of the Stokes parameters of the incident beam, the stokes parameters of the transmitted beam are

$$S_{0T} = f_T[(\cos^4\alpha_- + 1)S_0 + (\cos^4\alpha_- - 1)S_1],$$

$$S_{1T} = f_T[(\cos^4\alpha_- - 1)S_0 + (\cos^4\alpha_- + 1)S_1],$$

$$S_{2T} = -f_T(2\cos^2\alpha_-)S_2,$$

$$S_{3T} = -f_T(2\cos^2\alpha_-)S_3, \qquad f_T = \frac{1}{2}\left[\frac{\sin 2i \sin 2r}{(\sin\alpha_+ \cos\alpha_-)^2}\right]^2,$$

where equations $\alpha\pm = i\pm r = \sin^{-1}[\sin(i)/n]$. For incident unpolarized light, the equation for the intensity of the output beam is

$$I_T = \frac{1}{2}\left[\frac{\sin 2i \sin 2r}{(\sin\alpha_+ \cos\alpha_-)^2}\right]^2 \left(\cos^4\alpha_- + 1\right).$$

Single and Multiple Dielectric Plates (cont'd)
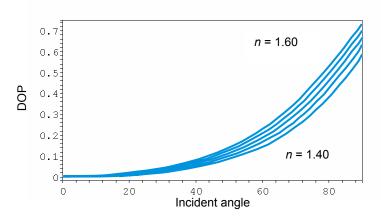
A plot of the output beam intensity equation for a refractive index of n = 1.4, 1.5, and 1.6 from i = 0° to 90°.

The DOP is

$$\mathscr{P} = \left| \frac{S_1}{S_0} \right| = \left| \frac{1 - \cos^4 \alpha_-}{1 + \cos^4 \alpha_-} \right|.$$

This equation is plotted for n = 1.40, 1.45…1.60 from i = 0° to 90°.

Pile of Polarizing Dielectric Plates

The transmitted beam is always partially polarized. An increase in the DOP can be obtained using a configuration known as a **pile of polarizing plates**.

A configuration of 6 dielectric plates.

Input beam → Output beam
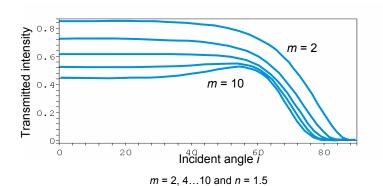
The plates are separated in order to reduce the effects of reflection. The Mueller matrix equation and the Stokes vector for a beam transmitted through **m dielectric plates** are

$$S_T{}^m = M_T{}^{2m} \cdot S,$$

$$S_T{}^m = \frac{1}{2}\left[\frac{\sin 2i \sin 2r}{(\sin \alpha_+ \cos \alpha_-)^2}\right]^{2m} \begin{pmatrix} (\cos^{4m}\alpha_- +1)S_0 + (\cos^{4m}\alpha_- -1)S_1 \\ (\cos^{4m}\alpha_- -1)S_0 + (\cos^{4m}\alpha_- +1)S_1 \\ 2\cos^{2m}\alpha_- S_2 \\ 2\cos^{2m}\alpha_- S_2 \end{pmatrix}.$$

For input unpolarized light the transmitted intensity is

$$I_T = \frac{1}{2}\left[\frac{\sin 2i \sin 2r}{(\sin \alpha_+ \cos \alpha_-)^2}\right]^{2m} (\cos^{4m}\alpha_- +1).$$

m = 2, 4...10 and n = 1.5
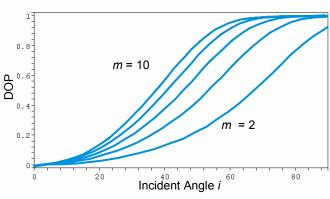
Pile of Polarizing Dielectric Plates (cont'd)

The DOP for *m* dielectric plates   is

$$\mathcal{P} = \left| \frac{S_1}{S_0} \right| = \left| \frac{\cos^{4m}(i-r)-1}{\cos^{4m}(i+r)-1} \right|.$$

m = 2, 4…10 plates and *n* = 1.50

As the number of plates increase the DOP increases. At least 4 plates are required to obtain a DOP of 100%. Also, from the previous page, as the number of plates increase, the intensity decreases and the DOP increases. Thus, there is a trade-off when using a pile of polarizers to obtain linearly polarized light from unpolarized light.

For unpolarized incident light $S_T{}^m$ is

$$S_T{}^m = \frac{1}{2}\left[\frac{\sin 2i \sin 2r}{(\sin\alpha_+ \cos\alpha_-)^2} \right]^{2m} \begin{pmatrix} (\cos^{2m}\alpha_- + 1) \\ (\cos^{2m}\alpha_- - 1) \\ 0 \\ 0 \end{pmatrix}.$$

In the limit as m→∞ the term $\cos^{2m}\alpha_-$ in the Stokes vector vanishes and LVP light $S^T{}_m \to \{1,-1,0,0\}$ is obtained.

Pile of Polarizing Dielectric Plates (cont'd)
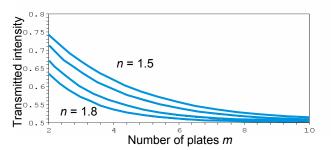
The transmission of unpolarized light through m dielectric plates at the Brewster angle i_B yields the Stokes vector

$$S_{T,B} = \frac{1}{2}\begin{pmatrix} \sin^{4m} 2i_B + 1 \\ \sin^{4m} 2i_B - 1 \\ 0 \\ 0 \end{pmatrix}.$$

The transmitted intensity is

$$I_{T,B} = \frac{1}{2}\left(1 + \sin^{4m} 2i_B\right),$$

where $i_B = \tan^{-1}(n)$. A plot of the above intensity is made for refractive indices of $n = 1.5...1.8$.

Similarly, the **degree of polarization (DOP)** and its plot are

$$\mathcal{P} = \left|\frac{S_1}{S_0}\right| = \left|\frac{1 - \sin^{4m} 2i_B}{1 + \sin^{4m} 2i_B}\right|.$$

Fresnel's Reflection and Transmission Coefficients

A simpler notation for the Mueller matrices for reflection and transmission is to use the **Fresnel reflection and transmission coefficients**. These are defined to be

for reflection:

$$\rho_s = \left(\frac{R_s}{E_s}\right)^2 = \left(\frac{\sin\alpha_-}{\sin\alpha_+}\right)^2, \qquad \rho_p = \left(\frac{R_p}{E_p}\right)^2 = \left(\frac{\tan\alpha_-}{\tan\alpha_+}\right)^2,$$

and for transmission:

$$\tau_s = \frac{n\cos r}{\cos i}\left(\frac{T_s}{E_s}\right)^2 = \frac{\sin 2i \sin 2r}{\sin^2\alpha_+},$$

$$\tau_p = \frac{n\cos r}{\cos i}\left(\frac{T_p}{E_p}\right)^2 = \frac{\sin 2i \sin 2r}{\sin^2\alpha_+ \cos^2\alpha_-},$$

where

$$\rho_s + \rho_p = 1, \qquad \tau_s + \tau_p = 1.$$

The Fresnel coefficients are between 0 and 1 and can be easily expressed as percentages. At the Brewster angle the Fresnel coefficients are
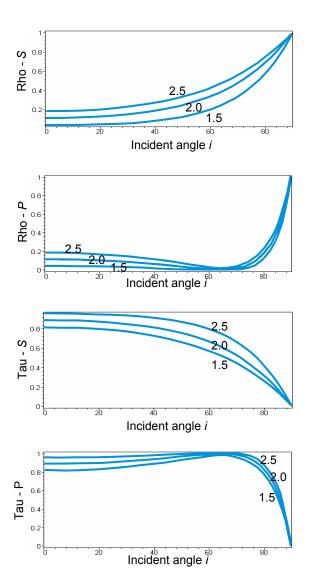
$$\rho_{s,B} = \cos^2 2i_B, \qquad \rho_{p,B} = 0, \qquad \tau_{s,B} = \sin^2 2i_B, \qquad \tau_{p,B} = 1.$$

The **Mueller matrices for reflection and transmission** are then

$$M_\rho = \frac{1}{2}\begin{pmatrix} \rho_s + \rho_p & \rho_s - \rho_p & 0 & 0 \\ \rho_s - \rho_p & \rho_s + \rho_p & 0 & 0 \\ 0 & 0 & 2\sqrt{\rho_s\rho_p} & 0 \\ 0 & 0 & 0 & 2\sqrt{\rho_s\rho_p} \end{pmatrix},$$

$$M_\tau = \frac{1}{2}\begin{pmatrix} \tau_s + \tau_p & \tau_s - \tau_p & 0 & 0 \\ \tau_s - \tau_p & \tau_s + \tau_p & 0 & 0 \\ 0 & 0 & 2\sqrt{\tau_s\tau_p} & 0 \\ 0 & 0 & 0 & 2\sqrt{\tau_s\tau_p} \end{pmatrix}.$$
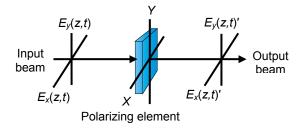
Fresnel's Reflection and Transmission Coefficients (cont'd)

The following four plots are made for the coefficients for refractive indices of $n = 1.5$, 2.0, and 2.5.

The Jones Matrix Calculus

The **Jones matrix calculus** is a matrix formulation of polarized light that consists of 2 × 1 **Jones vectors** to describe the field components and 2 × 2 **Jones matrices** to describe polarizing components. While a 2 × 2 formulation is "simpler" than the Mueller matrix formulation the Jones formulation is limited to treating only completely polarized light; it cannot describe unpolarized or partially polarized light. The Jones formulation is used when treating interference phenomena or in problems where field amplitudes must be superposed. A polarized beam propagating through a polarizing element is shown below.

The 2 × 1 **Jones column matrix** or vector for the field is

$$\mathbf{E} = \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} E_{0x}e^{i\delta_x} \\ E_{0y}e^{i\delta_y} \end{pmatrix},$$

where E_{0x} and E_{0y} are the amplitudes, δ_x and δ_y are the phases, and $i = \sqrt{-1}$. The components E_x and E_y are complex quantities. An important operation in the Jones calculus is to determine the intensity I:

$$I = \begin{pmatrix} E_x{}^* & E_y{}^* \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = E_x E_x{}^* + E_y E_y{}^*.$$

The row matrix is the complex transpose † of the column matrix, so I can be written formally as

$$I = \mathbf{E}^\dagger \cdot \mathbf{E}.$$

It is customary to normalize I to 1.

The Jones Matrix Calculus (cont'd)

The **Jones vectors for the degenerate polarization states** are:

$$\mathbf{E}_{\mathrm{LHP}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{E}_{\mathrm{LVP}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \mathbf{E}_{\mathrm{L+45P}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$\mathbf{E}_{\mathrm{L-45P}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \mathbf{E}_{\mathrm{RCP}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad \mathbf{E}_{\mathrm{LCP}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

The Jones vectors are orthonormal and satisfy the relation $\mathbf{E}_i{}^{\dagger} \cdot \mathbf{E}_j = \delta_{ij}$, where $\delta_{ij}(i = j, 1, \; i \neq j, 0)$ is the Kronecker delta.

The superposition of two orthogonal Jones vectors leads to another Jones vector. For example,

$$\mathbf{E} = \mathbf{E}_{\mathrm{LHP}} + \mathbf{E}_{\mathrm{LVP}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

which, aside from the normalizing factor of $1/\sqrt{2}$, is L+45P light. Similarly, the superposition of RCP and LCP yields

$$\mathbf{E}_{\mathrm{RCP}} + \mathbf{E}_{\mathrm{LCP}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{2}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

which, again, aside from the normalizing factor is seen to be LHP light. Finally, in its most general form, LHP and LVP light are

$$\mathbf{E}_{\mathrm{LHP}} = \begin{pmatrix} E_{0x} e^{i\delta_x} \\ 0 \end{pmatrix}, \qquad \mathbf{E}_{\mathrm{LVP}} = \begin{pmatrix} 0 \\ E_{0y} e^{i\delta_y} \end{pmatrix}.$$

Superposing $\mathbf{E}_{\mathrm{LHP}}$ and $\mathbf{E}_{\mathrm{LVP}}$ yields

$$\mathbf{E} = \mathbf{E}_{\mathrm{LHP}} + \mathbf{E}_{\mathrm{LVP}} = \begin{pmatrix} E_{0x} e^{i\delta_x} \\ E_{0y} e^{i\delta_y} \end{pmatrix}.$$

This shows that two orthogonal oscillations of arbitrary amplitude and phase can yield **elliptically polarized light**.

The Jones Matrix Calculus (cont'd)

A polarizing element is represented by a 2×2 Jones matrix

$$\mathbf{J} = \begin{pmatrix} j_{xx} & j_{xy} \\ j_{yx} & j_{yy} \end{pmatrix}.$$

It is related to the 2×1 output and input Jones vectors by $\mathbf{E'} = \mathbf{J} \cdot \mathbf{E}$. For a linear polarizer the Jones matrix is

$$\mathbf{J}_{\text{POL}} = \begin{pmatrix} p_x & 0 \\ 0 & p_y \end{pmatrix}, \qquad 0 \le p_x, p_y \le 1.$$

For an ideal linear horizontal and linear vertical polarizer the Jones matrices take the form, respectively,

$$\mathbf{J}_{\text{LHP}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{J}_{\text{LVP}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The Jones matrices for a wave plate ($E_{0x} = E_{0y} = 1$) with a phase shift of $\phi/2$ along the x-axis (fast) and $\phi/2$ along the y-axis (slow) are ($i = \sqrt{-1}$)

$$\mathbf{J}_{\text{WP}} = \begin{pmatrix} e^{i(\phi/2)} & 0 \\ 0 & e^{-i(\phi/2)} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\phi} \end{pmatrix}.$$

The Jones matrices for a QWP $\phi = \pi/2$ and HWP $\phi = \pi$ are, respectively,

$$\mathbf{J}_{\text{QWP}} = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}, \qquad \mathbf{J}_{\text{HWP}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For an incident beam that is L-45P the output beam from a QWP aside from a normalizing factor is

$$\mathbf{E'} = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix},$$

which is the Jones vector for RCP light. Finally, the **Jones matrix for a rotator** is

$$\mathbf{J}_{\text{ROT}}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

The Jones Matrix Calculus (cont'd)

For a rotated polarizing element the Jones matrix is given by

$$\mathbf{J}(\theta) = \mathbf{J}_{\text{ROT}}(-\theta) \cdot \mathbf{J} \cdot \mathbf{J}_{\text{ROT}}(\theta).$$

The **Jones matrix for a rotated ideal LHP** is

$$\mathbf{J}_{\text{LHP}}(\theta) = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}.$$

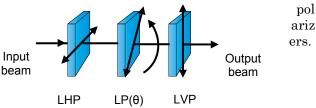Similarly, the **Jones matrix for a rotated wave plate** is

$$\mathbf{J}_{\text{WP}}(\phi,\theta) = \begin{pmatrix} \cos\dfrac{\phi}{2} + i\sin\dfrac{\phi}{2}\cos 2\theta & i\sin\dfrac{\phi}{2}\sin 2\theta \\ i\sin\dfrac{\phi}{2}\sin 2\theta & \cos\dfrac{\phi}{2} - i\sin\dfrac{\phi}{2}\cos 2\theta \end{pmatrix}.$$

For a HWP $\phi = \pi$ the matrix reduces to

$$\mathbf{J}_{\text{WP}}(\theta) = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}.$$

The matrix is *almost* identical to the matrix for a rotator except that the presence of the negative sign with $\cos\theta$ rather than with $\sin\theta$ along with the factor of 2 shows that the matrix is a pseudo-rotator; a rotating HWP reverses the polarization ellipse and doubles the rotation angle.

An application of the **Jones matrix calculus** is to determine the intensity of an output beam when a **rotating polarizer** is placed between two crossed polarizers.

Input beam   Output beam

LHP   LP(θ)   LVP
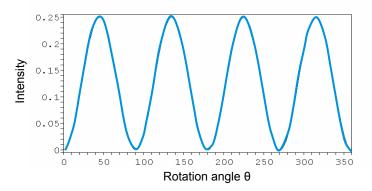
The Jones Matrix Calculus (cont'd)
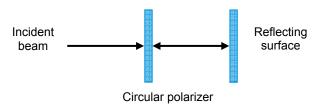
The Jones vector for the output beam is $\mathbf{E'} = \mathbf{J} \cdot \mathbf{E}$ and the Jones matrix for the three polarizer configuration is

$$\mathbf{J} = \mathbf{J}_{\mathrm{LVP}} \cdot \mathbf{J}_{\mathrm{LP}}(\theta) \cdot \mathbf{J}_{\mathrm{LHP}} = \begin{pmatrix} 0 & 0 \\ \sin\theta\cos\theta & 0 \end{pmatrix}.$$

For input LHP light the intensity of the output beam is $I = \mathbf{E'}^{\dagger} \cdot \mathbf{E'} = [1 - \cos(4\theta)]/8$.

An important optical device is an **optical isolator**.

Circular polarizer

The Jones matrix equation and its expansion is

$$\mathbf{J} = \left(\mathbf{J}_{\mathrm{L\text{-}45P}} \cdot \mathbf{J}_{\mathrm{QWP}}\right) \cdot \mathbf{J}_{\mathrm{REFL}} \cdot \left(\mathbf{J}_{\mathrm{QWP}} \cdot \mathbf{J}_{\mathrm{L+45P}}\right),$$

$$\mathbf{J} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus, no light is returned to the optical source and the circular polarizer acts as an ideal optical isolator.

Wolf's Coherency Matrix Calculus

Another 2×2 polarization matrix calculus is **Wolf's coherency matrix calculus**. This matrix calculus serves as a useful bridge between the Mueller and Jones matrix calculi. The coherency matrix $\mathbf{C}$ is defined in terms of the complex products of the optical field:

$$\mathbf{C} = \begin{pmatrix} \langle E_x E_x{}^* \rangle & \langle E_x E_y{}^* \rangle \\ \langle E_y E_x{}^* \rangle & \langle E_y E_y{}^* \rangle \end{pmatrix} = \begin{pmatrix} C_{xx} & C_{xy} \\ C_{yx} & C_{yy} \end{pmatrix}.$$

The matrix elements are related to the Stokes parameters by

$$\mathbf{C} = \begin{pmatrix} \langle E_x E_x{}^* \rangle & \langle E_x E_y{}^* \rangle \\ \langle E_y E_x{}^* \rangle & \langle E_y E_y{}^* \rangle \end{pmatrix} = \frac{1}{2}\begin{pmatrix} S_0 + S_1 & S_2 + iS_3 \\ S_2 - iS_3 & S_0 - S_1 \end{pmatrix}.$$

The coherency matrix can be expanded as a linear superposition of the Stokes parameters,

$$\mathbf{C} = \frac{1}{2}\sum_{i=0}^{3} \sigma_i S_i,$$

where

$$\sigma_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; \sigma_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

are the **Pauli spin matrices**. The common states of polarized light expressed in terms of the coherency matrix are

$$C_{\text{LHP}} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C_{\text{LVP}} = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_{\text{L+45P}} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$C_{\text{L-45P}} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad C_{\text{RCP}} = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \quad C_{\text{LCP}} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix},$$

$$C_{\text{UNP}} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_{\text{ELP}} = \frac{1}{2}\begin{pmatrix} 1 + \cos 2\alpha & \sin 2\alpha e^{i\delta} \\ \sin 2\alpha e^{-i\delta} & 1 - \cos 2\alpha \end{pmatrix}.$$

where $\alpha = \arctan(E_{0y}/E_{0x})$ and δ is the phase for ELP light.

Optical Activity and Optical Rotation
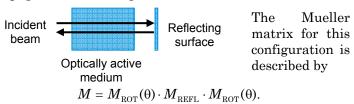
When polarized light propagates through a quartz crystal the polarization ellipse rotates. This behavior also takes place in many liquids such as sugars and fruit acids. This is called **natural optical activity**. For liquids, the angle of rotation θ is given by

$$\theta = \frac{\gamma \rho\, l}{10},$$

where γ is the rotary power of the liquid, ρ is the density (gms/cc), and l is the path length (cm). Natural optical activity can be described by the **Mueller matrix for rotation**

$$M_{\mathrm{ROT}}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta & 0 \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

A beam propagates through an optically active medium, which is then reflected by an ideal reflecting surface and propagates back through the medium.

The Mueller matrix for this configuration is described by

$$M = M_{\mathrm{ROT}}(\theta) \cdot M_{\mathrm{REFL}} \cdot M_{\mathrm{ROT}}(\theta).$$

The rotation angle θ in the reflected path remains unchanged since the optically active medium is independent of direction (The rotation angle is $\theta = kz$ and on the return path $k \to -k$, $z \to -z$ so $(-k)(-z) \to kz$; k is the propagator and z is the distance.) Carrying out the matrix multiplication yields

$$M = M_{\mathrm{ROT}}(\theta) \cdot M_{\mathrm{REFL}} \cdot M_{\mathrm{ROT}}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$
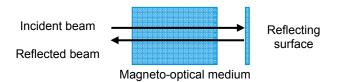
Thus, the effect of natural rotation is cancelled.

Faraday Rotation

When a polarized beam propagates through a block of glass that is subjected to a very strong magnetic field, the direction of the beam's propagation is parallel to the direction of the magnetic field and the polarization ellipse rotates. Materials that exhibit this behavior are called Faraday, **magneto-optical media**, or, more commonly, **Faraday rotators**. The rotation angle is given by

$$\theta = VHl,$$

where V is Verdet's constant, H is magnetic field intensity, and l is the propagation distance. **Faraday rotation** is described by a Mueller rotation matrix that is identical to that of optical activity except the magnetic field intensity is directional: H is $+H$ when propagation is left to right, and $-H$ when the propagation is right to left. From the relation the rotation angle becomes $-\theta$ on the return path. The configuration for propagation of a beam that is reflected back through a magneto-optical medium is shown.

Incident beam — Reflecting surface

Reflected beam

Magneto-optical medium

The Mueller matrix for this configuration is

$$M = M_{\mathrm{ROT}}(-\theta) \cdot M_{\mathrm{REFL}} \cdot M_{\mathrm{ROT}}(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(4\theta) & \sin(4\theta) & 0 \\ 0 & \sin(4\theta) & -\cos(4\theta) & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

The above matrix is the Mueller matrix of a pseudo-rotation matrix; that is, in a single trip the rotation angle doubles and the ellipticity is reversed.

Faraday Rotation (cont'd)

If an additional reflector is placed before the Faraday medium (the medium is now in an optical cavity,) then for N trips the Mueller rotation matrix becomes

$$M^N = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 4N\theta & \sin 4N\theta & 0 \\ 0 & -\sin 4N\theta & \cos 4N\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By slightly tilting the reflecting surface on the right side of the optical cavity, the effect of allowing the beam to make N passes can be used to measure the small rotation angle θ.

For propagation in a Faraday medium the field can be expressed as a **superposition of two circularly polarized waves** propagating with different wave numbers:

$$k' = n_L k_0, \qquad k'' = n_R k_0,$$

where k_0 is the propagation constant in free space and n_L and n_R are the refractive indices associated with each of the circular field components, respectively. The **circular birefringence** is then defined to be

$$n_L - n_R = \frac{\beta}{n},$$
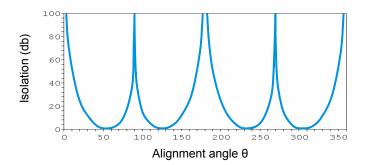
where β is a parameter and n is the mean refractive index. The refractive indices are also called **dextro-rotary** and **levo-rotatory** or, simply, R- and L-rotatory, and indicates that LCP and RCP waves propagate with different phase velocities.

Optical Isolators

The advent of coherent lasers required the development of optical isolators in order to prevent reflections from returning to the laser source. The **classical optical isolator** is the circular polarizer constructed from a QWP and a L+45P polarizer. If the axial alignment between the wave plate and the polarizer is not exact, then the intensity of the beam returned to the laser is
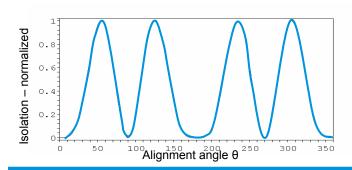
$$I(\theta) = 2\sin^2 2\theta(1 - \cos 2\theta),$$

where θ is the angle of alignment. This equation shows that at θ = 0, the reflected intensity is zero so there is total (ideal) isolation.
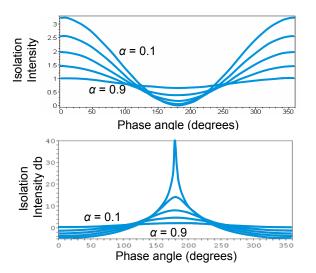
The isolation intensity can range over a very wide range so it is useful to express the isolation in terms of db (decibels).

$$\mathrm{db} = -10\log_{10}\big[I(\theta)\big].$$

Optical Isolators (cont'd)
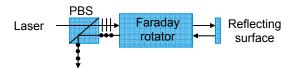
Reflectors are imperfect due to phase shifts other than 180° and attenuation (α); ideally the reflected intensity should be 0. For nonideal conditions the phase shift for the isolation are plotted directly and in db.

There is maximum isolation for a phase shift of 180°; the plots are for attenuations (α) from 10% to 90%. The actual performances of optical isolators attenuate the reflected beam by ~ 30 to 40 db.

The classical optical isolator is limited to operate at a single wavelength. This can be overcome by using a **Faraday isolator** configuration.

A polarizing beam splitter (PBS) transmits LHP light directly through the prism and reflects LVP light out through the sides of the prism. The dots and the vertical lines represent LHP and LVP light, respectively.

Optical Isolators (cont'd)

The **Mueller matrix for the Faraday rotator**-reflecting surface combination is

$$M_{\text{FARADAY}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 4\theta & \sin 4\theta & 0 \\ 0 & \sin 4\theta & -\cos 4\theta & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

In the forward direction the PBS behaves as a LHP polarizer, so if the return beam propagates along this path back to the laser:

$$M_{\text{LASER}} = M_{\text{LHP}} \cdot M_{\text{FARADAY}} \cdot M_{\text{LHP}}$$

$$= \frac{1 + \cos 4\theta}{4} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$
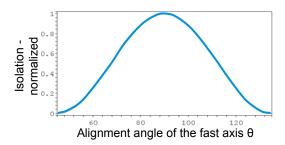
Similarly, for a beam that is reflected by the prism so that it propagates out of the side of the prism:

$$M_{\text{SIDE}} = M_{\text{LVP}} \cdot M_{\text{FARADAY}} \cdot M_{\text{LHP}}$$

$$= \frac{1 - \cos 4\theta}{4} \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If the fast axis of the Faraday rotator is rotated to $\theta = 45°$, the above two matrices are reduced to

$$M_{\text{LASER}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ and } M_{\text{SIDE}} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

No energy is returned to the optical source (laser), and all of the reflected energy emerges out of the lower side of the PBS.

Optical Isolators (cont'd)

The optical isolation can be expressed as a function of the rotation angle of the Faraday rotator. The intensity of the beam returned to a laser that emits LHP light of unit intensity is
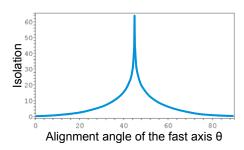
$$I(\theta) = \frac{1 + \cos 4\theta}{2}.$$

When the fast axis of the Faraday rotator is at an angle of $\theta = 45°$ the isolation (intensity) is 0. However, if the fast axis is at 90°, the intensity returned to the laser is unity; that is, there is no isolation.

The isolation can also be expressed in terms of db:
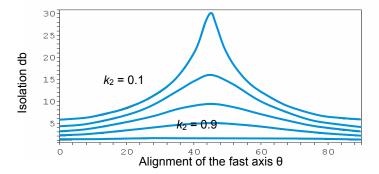
$$\text{db} = -10 \log_{10}[(1 + \cos 4\theta)/2].$$
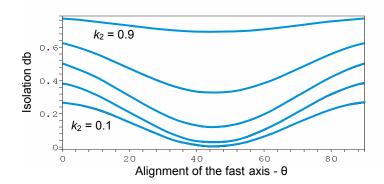
These two plots represent an ideal isolator. In practice, linear polarizers are not ideal and Mueller matrices for imperfect polarizers must be used.

Optical Isolators (cont'd)

Linear polarizers can also be expressed as major and minor **transmittances** k_1 and k_2, respectively, where they are related to the elements of the polarizer matrix by $k_1 = p_x^2$ and $k_2 = p_x^2$.
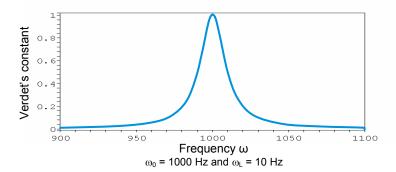
An analysis of Faraday isolators is readily made using imperfect polarizers. The following two plots show the behavior of the isolation in terms of intensity and expressed also in db for $k_1 = 0.95$ and $k_2 = 0.9, 0.7 \ldots 0.1$; the smaller the value of k_2, the better the polarizer ($k_2 = k_1 = k$ corresponds to a **neutral density filter**). The best isolation is for $k_2 = 0.1$.

Optical Isolators (cont'd)

The maximum isolation is approximately 30 db. Unlike classical optical isolators that are fixed frequency QWP, Faraday rotators are tunable over frequency ω. This is due to the fact that Verdet's constant V actually varies over ω. The rotation angle of the Faraday rotator is $\theta = VHz$. Verdet's constant actually is described by the Lorentz spectrum and is given by

$$V = f \frac{2\omega}{(\omega^2 - \omega_0^2)^2 + (2\omega_L\omega)^2},$$
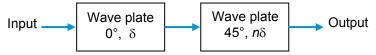
where ω is the frequency, ω_0 is the resonance frequency, ω_L is the **Larmor frequency**, and f is a constant.

Verdet's constant vs. Frequency ω

$\omega_0 = 1000$ Hz and $\omega_L = 10$ Hz

Wave Plate Depolarizers

An optical **depolarizer** is an element that transforms polarized light to depolarized light. In order to obtain depolarized light the Mueller matrix and the Stokes vector relation must be

$$
S = \begin{pmatrix} m_{00} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = I_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \left(I_0 = m_{00} S_0 \right).
$$

Thus, all the elements except for m_{00} must be 0 in the matrix of a depolarizer. Depolarizers can be created by spatial or temporal averaging of the matrix elements. One method is to use two variable-phase wave plates whose fast axes are oriented at +45° from one another as shown below, where n is an integer.

Input $\longrightarrow$ | Wave plate 0°, δ | $\longrightarrow$ | Wave plate 45°, $n\delta$ | $\longrightarrow$ Output

If the phase of the first wave plate is taken to be δ and the second is $n\delta$ the matrix equation is
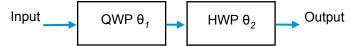
$$
M = M_{\mathrm{WP}}(n\,\delta) \cdot M_{\mathrm{WP}}(\delta),
$$

$$
M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos n\delta & \sin n\delta \sin\delta & \sin n\delta \cos\delta \\ 0 & 0 & \cos\delta & -\sin\delta\sin\delta \\ 0 & -\sin n\delta & \cos n\delta \sin\delta & \cos n\delta \cos\delta \end{pmatrix}.
$$

Averaging over the phase δ that varies linearly in time the **Mueller matrix for a depolarizer** is obtained for $n = 2$:

$$
M_{\mathrm{Depolarizer}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
$$

Wave Plate Depolarizers (cont'd)

Another method to depolarize polarized light is to rotate a quarter- and half-wave plate at rotation angles of θ_1 and θ_2.

In this method the previous variable phase shifting (an **electro-optical effect**) is replaced by a mechanical rotation of each of the wave plates; the rotation angles are expressed in terms of angular frequencies $\theta_1 = \omega t$ and $\theta_2 = n\omega t$, where n is an integer. For an input beam that is LHP, the Stokes vector of the output beam is then found to be

$$S_{\text{out}} = \begin{pmatrix} 1 \\ \frac{1}{2}\cos\left[4(n-1)\omega t\right] + \frac{1}{2}\cos(4n\,\omega t) \\ \frac{1}{2}\sin\left[4(n-1)\omega t\right] + \frac{1}{2}\sin(4n\,\omega t) \\ \sin(2\omega t) \end{pmatrix},$$
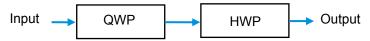
where $\omega = 2\pi/T$ and T is the period of rotation. Taking the time average of each Stokes parameter over a period T, the Stokes vector (S_{out}) is found to be for $n = 2$,

$$S_{\text{out}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

which is the Stokes vector for depolarized (unpolarized) light. Thus, a QWP and a HWP in which the rotation rate of the HWP is *twice* that of the QWP can be used as a depolarizer. For the mechanical method to work requires, however, that the rotation rate significantly exceed the response time of the optical detector.

The Lyot Crystal Depolarizer

The **Lyot crystal depolarizer** operates in the wavelength domain. This depolarizer consists of two plates of quartz whose thicknesses have a ratio of 2:1 (corresponding to a QWP and a HWP); their *x*-axes differ by an angle of 45°.

The **Mueller matrix for the Lyot crystal depolarizer** is

$$M_{\text{LYOT}}(\phi, \theta = \pi/4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos n\phi & \sin n\phi \sin \phi & \sin n\phi \cos \phi \\ 0 & 0 & \cos \phi & -\sin \phi \\ 0 & -\sin n\phi & \cos n\phi \sin \phi & \cos n\phi \sin \phi \end{pmatrix}.$$

The phase shift is assumed to vary linearly with wavelength:

$$\phi(\lambda) = a\lambda + b = \frac{2\pi(\lambda - \lambda_1)}{\lambda_2 - \lambda_1},$$

where a and b are evaluated at λ_1 where $\phi(\lambda_1) = 0$ and at λ_2 where $\phi(\lambda_2) = 2\pi$. The average value of each matrix element m_{ij} is determined by integrating from λ_1 to λ_2 using
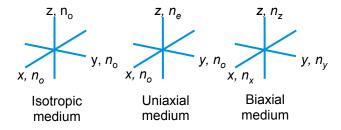
$$\overline{m_{ij}} = \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} m_{ij}(\lambda)\, d\lambda.$$

Evaluating all the matrix elements yields 0 only for $n = 2$, and so the Mueller matrix reduces to a depolarizer:

$$M_{\text{LYOT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Polarizers

For polarizers the most commonly used materials are naturally occurring calcite and synthetically produced **polaroid** (sheets) and **PolaCor**. For wave plates the most common materials are quartz and synthetic materials. Finally, rotators are usually made using quartz.

The best polarizing material for use in the visible spectrum is calcite. Materials can be characterized as **isotropic**, e.g., glass, **uniaxial**, e.g., calcite, and **biaxial**, e.g., **mica**. For each of these structures an **orthogonal triad axis** system can be constructed.

| Isotropic medium | Uniaxial medium | Biaxial medium |

Both calcite and quartz are uniaxial crystals. In order to understand the behavior of the propagation of *e*- and *o*-rays we restrict the discussion to isotropic and uniaxial media. In an **isotropic medium** all three axes have the same refractive index whereas in a uniaxial medium the refractive index along the *z*-axis is different from the other two.

In an isotropic medium for an unpolarized beam propagating along the *z*-axis, the transverse components are directed along the *x*- and *y*-axes and the refractive indices are both n_o. Thus, as the beam propagates along the *z*-axis the transverse waves *both* experience the same refractive index, so they propagate at the same velocity. Similarly, the same behavior is observed if the propagation is along the *y*- and *z*-axes. Thus, the polarization state of an optical beam is unaffected when propagating in an isotropic medium.

Polarizing Prisms

For a uniaxial crystal the transverse components of a propagating beam along the *z*-axis *both* experience the same refractive index. The *z*-axis is the optic axis where the *e*- and *o*-rays do not divide and the beam propagates through the crystal unaffected. However, when the beam propagates along the *x*- or *y*-axis, one component experiences a refractive index of n_o and the other experiences n_e. Thus, each component travels at different velocities and divides according to Snell's law.
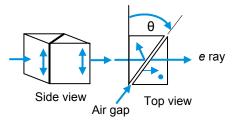
The velocity of the wave is $v = c/n$, where c is the speed of light and n is the refractive index.
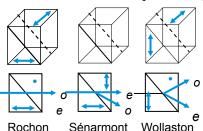
In the **Glan-Foucault polarizer,** the refractive indices of calcite are $n_o = 1.658$ and $n_e = 1.486$ (NaD line). The vertical double arrow and the dot show the direction of the optic axis. The incident beam is perpendicular to the optic axis so the unpolarized beam divides into *o*- and *e*-rays. By choosing an apex angle of $\theta = 38.5°$ the *o*-ray is internally reflected and *e*-ray is refracted into the air gap. The second prism bends the *e*-ray back to the original path and emerges perpendicular to the exit face; the ray is 100% polarized.

Side view   Air gap   Top view   *e* ray

In the Glan-Foucault prism only one ray appears at the exit of the polarizer. Other **polarizing prisms** have been designed in which both rays can be used, and include the **Rochon,** the **Sénarmont**, and the **Wollaston**.

Rochon   Sénarmont   Wollaston
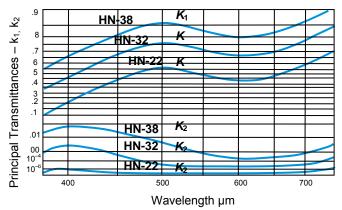
Polarizing Prisms (cont'd)

In most polarizing prisms, the optic axes are parallel in both halves, but in the Rochon, Sénarmont, and Wollaston prisms they are at right angles. Polarizing beam splitters can also be made using quartz for the ultraviolet regime. In most applications, however, the preferred material is calcite because it yields a greater angular separation of the beams, typically 10° as compared to 0.5° for quartz. The Wollaston prism is especially useful because it generates two orthogonal linear polarized beams of equal intensity with a large angle of separation.

Calcite is expensive, so **synthetic polarizers** have been developed. The most common is **H-sheet Polaroid**, where polarization occurs because of unequal absorption along the orthogonal transmission axes. **Polaroid** consists of polyvinyl-alcohol that is stretched, cemented to a sheet of plastic, and dipped into a liquid solution of iodine, fixing the long-chain molecules in the plastic. Three of the most common polarizers and their transmissions are HN-38 (38%), HN-32 (32%), and HN-22 (22%).

> It is the iodine that gives Polaroid its color.

Characterizing Polarizers

Commercial linear polarizers use a different form of the Mueller matrix:

$$M_{POL} = \frac{1}{2}\begin{pmatrix} k_1 + k_2 & k_1 + k_2 & 0 & 0 \\ k_1 + k_2 & k_1 + k_2 & 0 & 0 \\ 0 & 0 & 2\sqrt{k_1 k_2} & 0 \\ 0 & 0 & 0 & 2\sqrt{k_1 k_2} \end{pmatrix} \quad 0 \le k_{1,2} \le 1$$

where $k_1 = p_x^2$ is called the **major principal transmittance** and $k_2 = p_y^2$ is called the **minor principal transmittance**. For an ideal linear horizontal polarizer $k_1 = 1$ and $k_2 = 0$. The Stokes vector for the output beams are

$$S_{UNP}' = \frac{k_1 + k_2}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad S_{LHP}' = \frac{k_1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad S_{LVP}' = \frac{k_2}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

The **transmittance** T of the linear polarizer is

$$T_{UNP} = \frac{k_1 + k_2}{2} \qquad T_{LHP} = \frac{k_1}{2} \qquad T_{LVP} = \frac{k_2}{2}$$
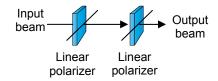
The major and minor principal transmittances k_1 and k_2 are obtained using LHP and LVP light, respectively. The transmittance of the linear polarizer shown on the previous page using input unpolarized light is

$$T_{UNP} = \frac{k_1 + k_2}{2}$$

The unpolarized light condition can be produced by sequentially using incident LHP and LVP light and then adding the respective intensities; superposing the transmittances T_{LHP} and T_{LVP} forms the transmittance T_{UNP}.

Polarizer characteristics are usually evaluated using pairs of polarizers. The transmittances when the horizontal transmission axes of *both* polarizers are in the *x*-direction (parallel).

Characterizing Polarizers (cont'd)

The light the transmittances are measured when the transmission axes of both polarizers are in the x-direction:

$$T_{\text{UNP}} = H_0 = \frac{1}{2}\left(k_1^{\,2} + k_2^{\,2}\right), \qquad T_{\text{LHP}} = \frac{k_1^{\,2}}{2}, \qquad T_{\text{LVP}} = \frac{k_2^{\,2}}{2},$$

where H_0 refers to the **transmittances for parallel polarizers**. Similarly, for polarizers that are crossed:

$$T_{\text{UNP}} = H_{90} = k_1 k_2.$$

For an ideal polarizer $k_2 = 0$, so $H_{90} = 0$.

The **transmittance for a single rotated polarizer** with its transmission axis at an angle θ for input UNP light is

$$T(\theta) = k_1 \sin^2\theta + k_2 \cos^2\theta.$$

For a polarizer pair consisting of a single fixed polarizer followed by a rotated polarizer:

$$T(\theta) = k_1 k_2 \sin^2\theta + \frac{1}{2}(k_1^{\,2} + k_2^{\,2})\cos^2\theta$$

$$= H_{90} + (H_0 - H_{90})\cos^2\theta.$$

Because transmittance values vary so widely it is common to express them as **logarithms:**

$$D_0 = -\log_{10} H_0, \qquad D_{90} = -\log_{10} H_{90}.$$

Finally, an important means of characterizing linear polarizers is by means of the **contrast ratio**.

The transmission for a single linear polarizer for the values of $k_1 = 0.9$ and $k_2 = 0.1$ is

$$T(\theta) = 0.9\sin^2\theta + 0.1\cos^2\theta.$$

The contrast ratio is a measure of the maximum to the minimum transmittance and is defined by
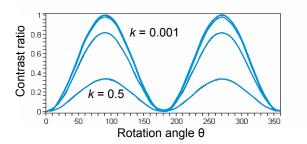
$$C = \frac{T_{\text{max}} - T_{\text{min}}}{T_{\text{max}} + T_{\text{min}}} = \frac{k_1 - k_2}{k_1 + k_2}.$$

Characterizing Polarizers (cont'd)

For a neutral density filter $k_1 = k_2$, $C = 0$. Similarly, for $k_1 \gg k_2$ the contrast ratio approaches unity (an ideal linear polarizer). Thus, the greater the contrast ratio is the better the polarizer is. The ratio of the minor principal transmittance to the major principal transmittance is defined to be $k = k_2 / k_1$, so $C = (1-k)/(1+k)$ The contrast ratio for a rotated linear polarizer is:
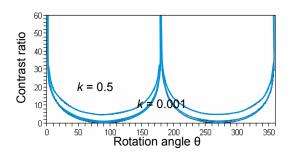
$$C(\theta) = \frac{T(\theta) - T_{min}}{T_{max} + T_{min}} = \left(\frac{1-k}{1+k}\right)\sin^2\theta = C(0)\sin^2\theta.$$

The contrast ratio can be changed by rotating the polarizer.

The graph clusters around a contrast ratio of 1, where most polarizers perform. Because the contrast ratio varies so widely, it is also defined in terms of db:
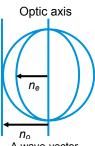
$$C(\text{db}) = -10\log_{10}C(\theta).$$

Wave Plate Materials
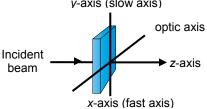
The wave plate (retarder) changes the polarization state of an optical beam by introducing a phase shift between the orthogonal components of a polarized beam. Wave plates materials include calcite, quartz, or synthetic retarders.

Optic axis

A wave-vector surface for a negative uniaxial crystal such as calcite.

For a calcite crystal the ordinary wave expands as a circle whereas the extraordinary wave expands as an ellipse. For propagation *parallel* to the optic axis the refractive indices are identical and the phases of the beam components are unaffected. However, for beam propagation perpendicular to the optic axis the phase difference ϕ for a wavelength λ and a path length of d is

$$\phi = \frac{2\pi}{\lambda}(n_e - n_o)d.$$

Calcite wave plates are rarely used even though their **birefringence** is large; the wave plate would be extremely thin and fragile. Quartz is a positive uniaxial crystal and has refractive indices, $n_o = 1.544$ and $n_e = 1.553$; its birefringence is about $1/20^{\text{th}}$ of that of calcite, making it a much more suitable wave plate material. Also, inexpensive QWPs and HWPs are made from birefringent **polyvinyl** are available.

When an unpolarized beam propagates through a wave plate, the emerging beam remains unpolarized. This shows that the *o*- and *e*-rays of the unpolarized beam are *independent* of each other. Wave plates, therefore, can onl*y* affect completely or partially polarized light.

Incoherent Superposition and Decomposition

The **superposition and decomposition of optical beams** in the intensity and amplitude domains are very different. In the intensity domain a beam is described by the Stokes vector and in the amplitude domain by the Jones vector. The **Stokes vector for unpolarized light** is

$$S_{\text{UNP}} = I_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where I_0 is the intensity of the optical beam. The first Stokes parameter is the intensity and the remaining three parameters describe the polarization. Because the polarization parameters are 0 there is no amplitude or phase relation between the orthogonal components of the unpolarized light. Thus, the Stokes vector for unpolarized light consists of two **independent beams** of equal intensity; furthermore, the beams are orthogonal to each other. This allows the Stokes vector for unpolarized light to be decomposed into orthogonal polarization states in the following ways:
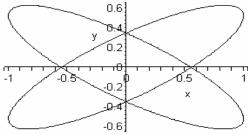
$$S_{\text{UNP}} = I_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{I_0}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \frac{I_0}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix},$$

$$S_{\text{UNP}} = I_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{I_0}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \frac{I_0}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix},$$

$$S_{\text{UNP}} = I_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{I_0}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \frac{I_0}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

Incoherent Decomposition—Ellipses

Unpolarized light can be decomposed into the Stokes vectors for LHP and LVP light, L+45P and L-45P light and RCP and LCP light, respectively. Conversely, two orthogonal independent beams can be superposed to form unpolarized light. It is this property that is used to determine the transmission coefficients of linear polarizers, using two independent beams of LHP and LVP light to simulate unpolarized light.

Unpolarized light can, in general, be decomposed (**incoherent decomposition**) into two **orthogonal ellipses**,

$$S_{\mathrm{UNP}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ \cos 2\psi \cos 2\chi \\ \sin 2\psi \cos 2\chi \\ \sin 2\chi \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ \cos 2\psi \cos(2\chi - \pi) \\ \sin 2\psi \cos(2\chi - \pi) \\ \sin(2\chi - \pi) \end{pmatrix}.$$

The orthogonal polarization ellipses for $\psi = 30°$ and $\chi = 15°$.

Coherent Superposition and Decomposition

Coherent superposition and decomposition applies only when there is an amplitude and phase relation between the orthogonal components. Consider the vector representation $\mathbf{E}(z,t)$ of the optical field in the *x-y* plane:

$$\mathbf{E}(z,t) = E_x(z,t)\,\mathbf{i} + E_y(z,t)\,\mathbf{j}$$
$$= E_{0x}\cos(kz - \omega t)\,\mathbf{i} + E_{0y}\cos(kz - \omega t + \delta)\,\mathbf{j},$$

where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors along the *x*- and *y*-axes, respectively, and $\delta = \delta_y - \delta_x$ is the phase shift between the orthogonal components. The above equation is the **vectorial representation for elliptically polarized light**. Where amplitudes $E_{0x} = E_{0y} = E_0$ and the phase shift is $\delta = \pm 90°$:

$$\mathbf{E}_1(z,t) = E_0\left[\cos(kz - \omega t)\,\mathbf{i} - \sin(kz - \omega t)\,\mathbf{j}\right],$$
$$\mathbf{E}_2(z,t) = E_0\left[\cos(kz - \omega t)\,\mathbf{i} + \sin(kz - \omega t)\,\mathbf{j}\right],$$

describing RCP and LCP light, respectively. Together they form the vectorial equation for LHP light:

$$\mathbf{E}(z,t) = 2E_0\cos(kz - \omega t)\,\mathbf{i}.$$

Thus, the coherent superposition of RCP and LCP light yields LHP light, and the coherent decomposition of LHP yields RCP and LCP light. In terms of the Jones vectors, LHP light can also be decomposed for RCP and LCP light:

$$\mathbf{E} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix}.$$
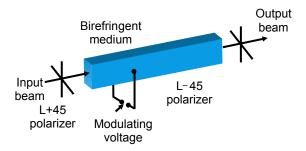
Another example of coherent superposition is of L+45P and L-45P light to obtain LHP light:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{2}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

In the intensity domain only unpolarized light can be decomposed into two independent elliptically polarized beams. In the amplitude domain, elliptically or degenerate polarized light can be decomposed or superposed to form other polarization states.

The Electro-Optical Effect—Modulators

Certain liquids and crystals exhibit a phenomenon known as the electro-optical effect. The **Kerr effect** is proportional to the *square* of the applied electric field and and the **Pockels effect** is *linearly* proportional to the electric field. In both phenomena the electro-optical effect causes a *phase shift* between the orthogonal field components. This indicates that the Mueller matrix for the effect can be represented by a wave plate. In the Kerr effect the field is applied perpendicular to the incident light.
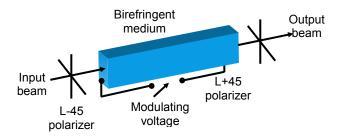
For the Kerr quadratic effect, the field is applied *transversely* to the direction of the incident field. When an isotropic liquid is placed in the electric field it behaves like a uniaxial crystal with the optic axis in the direction of the propagation. The birefringence is given by

$$n_p - n_s = \lambda B \mathcal{E}^2,$$

where n_p and n_s are the refractive indices parallel and perpendicular to the plane of the paper, λ is the wavelength, B is Kerr's constant and $\mathcal{E}$ is the applied field. Kerr cells have serious drawbacks: they require 5 to 10 times the voltage needed by a Pockels cell and many liquids are toxic. As a result crystals that exhibit the Pockels effect are used much more frequently.

The Electro-Optical Effect—Modulators (cont'd)

For the Pockels effect the phase shift varies with the applied *linear* field. The most used crystal is **potassium dihydrogen phosphate (KDP)** in its deuterated form.

In the Pockels cell the applied field is longitudinal and in the direction of the crystal optics axis; the incident light propagates along the same axis. The phase shift between the **transverse field components** is

$$\phi = \phi_y - \phi_x = \frac{\omega n_0{}^3 r_{63}}{c} V,$$

where r_{63} is a constant of the crystal and V is the applied modulating voltage; the phase ϕ is often called the **retardation**. The constants can be eliminated and the equation rewritten as $\phi = \pi(V/V_\pi)$, where V_π is the "**half-wave voltage**", which occurs at $\phi = \pi$.

The **Mueller matrix for Pockels electro-optical modulator** is

$$M_{\mathrm{PEOM}} = M_{\mathrm{L+45P}} \cdot M(\phi) \cdot M_{\mathrm{L\text{-}45P}}$$
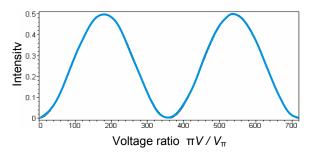
$$M_{\mathrm{PEOM}} = \frac{1-\cos\phi}{4} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The Pockels Cell

The intensity of the Pockels cell output beam is

$$I(\phi) = \frac{I_0}{4}\left(1 - \cos\phi\right) = \frac{I_0}{4}\left[1 - \cos\left(\pi V / V_\pi\right)\right],$$

where V is the applied voltage. V_π is the half-wave voltage and the maximum intensity also occurs at this voltage.
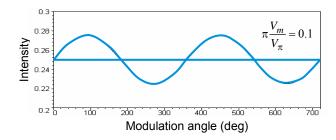
The applied **amplitude modulation voltage** is

$$V = \frac{V_\pi}{2} + V_m \sin\left(\omega_m t\right),$$

where V_m is the maximum modulation voltage and ω_m is the modulation frequency. Assuming $V_m \ll V_\pi$ and the small angle approximation ($\sin\theta \approx \theta$), the intensity and its plot is
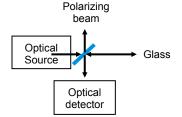
$$I(V) = \left(\frac{I_0}{2}\right)\left[1 + \pi\left(\frac{V_m}{V_\pi}\right)\sin(\omega_m t)\right],$$

Refractive Index Measurement

The refractive index n is the most important optical characteristic of glass. To measure n, the most accurate transmission method is **minimum deviation** where a transparent, extremely homogeneous prism of glass must be made, and accurately cut and polished. An alternate measurement method is to use reflection from an optical surface, though it not as accurate as transmission methods. The most common reflection methods are:

(1) The **normal incidence reflection**
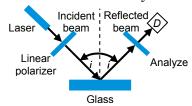
The Mueller matrix for the reflected beam is

$$M_{\mathrm{REFL}} = \left(\frac{n-1}{n+1}\right)^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The beam intensity of the reflecting surface is

$$I_{\mathrm{REFL}} = \left(\frac{n-1}{n+1}\right)^2 I_0,$$

where I_0 is the intensity of the beam incident on the glass. The beam splitter prevents direct **measurement of the refractive index** of the glass and this reduces the measurement accuracy.

(2) The **Brewster angle measurement**

The **Mueller matrix for reflection at the Brewster angle** i_B is

$$M_B = \frac{1}{2}\cos^2 2i_B \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which is the matrix for a linear horizontal polarizer LHP.

Refractive Index Measurement (cont'd)

The Mueller matrix shows that the Stokes vector S′ of the reflected beam is
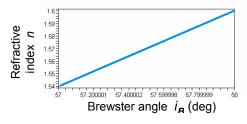
$$S' = \frac{1}{2}(S_0 + S_1)\cos^2 2i_B \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

where S_0 and S_1 are the Stokes parameters associated with the incident beam. If the incident beam is LVP light ($S_1 = -1$), then the Stokes vector S′ becomes 0; that is, the LVP component vanishes and only the LHP component of the incident beam remains. At the Brewster angle the analyzing linear polarizer is rotated until a null condition is reached and the refractive index n is found using Brewster's law $n = \tan i_B$.

The slow rise in this curve indicates that a very good mechanical mount is required to find the location of the Brewster angle. Then, even when this angle or surrounding region is found, the refractive index varies very slowly, and is practically linear over a range of small angles, making the refractive index measurement accurate to about three decimal places.

(3) The **45° incidence angle measurement**

This method is to irradiate the glass surface with an incident beam at 45° form the surface normal.

Refractive Index Measurement (cont'd)

At this angle the Mueller matrix for reflection can be expressed solely in terms of the angle of refraction:

$$M(i = 45°) = \frac{1 - \sin 2r}{(1 + \sin 2r)^2} \begin{pmatrix} 1 & \sin 2r & 0 & 0 \\ \sin 2r & 1 & 0 & 0 \\ 0 & 0 & -\cos 2r & 0 \\ 0 & 0 & 0 & -\cos 2r \end{pmatrix}.$$

If the surface of the glass is irradiated with LHP and then with LVP light the intensities are found to be maximum and minimum intensities, respectively, and are

$$\mathrm{I}_{max} = I_0 \frac{1 - \sin 2r}{1 + \sin 2r},$$

$$\mathrm{I}_{min} = I_0 \frac{(1 - \sin 2r)^2}{(1 + \sin 2r)^2},$$
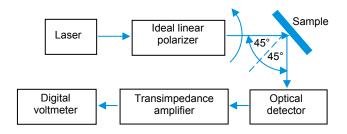
which yields

$$\sin 2r = \frac{\mathrm{I}_{max} - \mathrm{I}_{min}}{\mathrm{I}_{max} + \mathrm{I}_{min}}.$$

Calling the intensity ratio $(I_{max} - I_{min}) / (I_{max} + I_{min}) = f$ the refractive index n can then be determined:

$$n = \frac{\left(1 + \sqrt{1 - f^2}\right)^{1/2}}{f}.$$

Using this measurement method the expensive mechanical mount is eliminated and there is no mechanical motion other than flipping the polarizer from LHP to LVP.

Maxwell's Equations

A complete description of polarized light requires that the nature of the optical source and its relation to the propagating wave and polarization be included. This requires the use of Maxwell's electrodynamic equations (expressed here in Gaussian units):

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{1}{c}\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t},$$

$$\nabla \times \mathbf{H}(\mathbf{r},t) = \frac{4\pi}{c}\mathbf{j}(\mathbf{r},t) + \frac{1}{c}\frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t},$$

$$\nabla \cdot \mathbf{D}(\mathbf{r},t) = 4\pi\rho(\mathbf{r},t),$$

$$\nabla \cdot \mathbf{B}(\mathbf{r},t) = 0.$$

$\mathbf{E}(\mathbf{r},t)$ and $\mathbf{H}(\mathbf{r},t)$ are the electric and magnetic field vectors, $\mathbf{D}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ are the displacement and magnetic induction vectors, $\mathbf{j}(\mathbf{r},t)$ is the electric **current density** vector, and $\rho(\mathbf{r},t)$ is the electric charge density. ∇ is the (spatial) "del" operator, and c is the speed of light. The field vectors are related by the **constitutive equations** for the medium, $\mathbf{D}(\mathbf{r},t) = \varepsilon\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t) = \mu\mathbf{H}(\mathbf{r},t,)$, where ε and μ are permittivity and permeability constants of the medium.

Maxwell's equations can be recast only in terms of $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$,

$$\nabla^2 \mathbf{E}(\mathbf{r},t) - \frac{\mu\varepsilon}{c^2}\frac{\partial^2 \mathbf{E}(\mathbf{r},t)}{\partial t^2} = \frac{4\pi\mu}{c^2}\frac{\partial \mathbf{j}(\mathbf{r},t)}{\partial t}.$$

The current density is $\mathbf{j}(\mathbf{r},t) = e\mathbf{v}(\mathbf{r},t)$, where e is the electric charge and $\mathbf{v}(\mathbf{r},t)$ is the velocity. The above equation is then

$$\nabla^2 \mathbf{E}(\mathbf{r},t) - \frac{\mu\varepsilon}{c^2}\frac{\partial^2 \mathbf{E}(\mathbf{r},t)}{\partial t^2} = \frac{4\pi\mu e}{c^2}\frac{\partial \mathbf{v}(\mathbf{r},t)}{\partial t}.$$

The above equation shows that the radiated electric field $\mathbf{E}(\mathbf{r},t)$ arises from accelerating charges, $e\partial\mathbf{v}(\mathbf{r},t)/\partial t$.
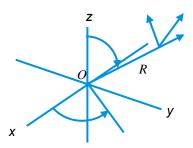
The Radiation Equation and the Stokes Parameters

The above equation is known as the **radiation equation**. We note that in free space there are no current sources so the radiation equation reduces to the **classical wave equation**. The radiation equation can be solved for its transverse field components. In spherical coordinates these are:

$$E_\theta = \frac{e}{c^2 R}[\ddot{x}\cos\theta - \ddot{z}\sin\theta],$$

$$E_\phi = \frac{e}{c^2 R}[\ddot{y}],$$

where E_θ and E_ϕ are the transverse field components, R is the distance of the observation point P from the origin of the coordinate system, and $\ddot{x}, \ddot{y},$ and $\ddot{z}$ are the accelerations of the charged particle in a Cartesian coordinate system.

The Stokes parameters for the radiation field expressed in spherical coordinates are defined by

$$S_0 = E_\theta E_\theta{}^* + E_\phi E_\phi{}^*,$$

$$S_1 = E_\theta E_\theta{}^* - E_\phi E_\phi{}^*,$$

$$S_2 = E_\theta E_\phi{}^* + E_\phi E_\theta{}^*,$$

$$S_3 = i(E_\theta E_\phi{}^* - E_\phi E_\theta{}^*).$$
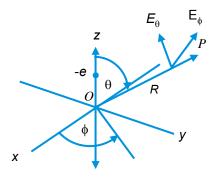
The Linear Oscillating Bound Charge

We now determine the Stokes vectors for several radiation systems. The first is the **Stokes vector for a bound charge** oscillating along the z-axis.

The motion of the bound charge along z is described by

$$\frac{d^2z(t)}{dt^2} + \omega_0^2 z(t) = 0,$$

where ω_0 is the frequency of oscillation of the charge. In complex notation the solution of this differential equation is

$$z(t) = z_0 \, e^{i(\omega_0 t + \alpha)},$$

where z_0 is the amplitude. Differentiating the above equation and substituting into the radiation field components yields

$$E_\theta = \frac{e z_0 \omega_0^2}{c^2 R} \sin\theta \, e^{i(\omega_0 t + \alpha)}, \quad E_\phi = 0.$$

The Stokes vector is
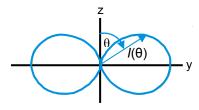
$$S = \left(\frac{e z_0}{c^2 R}\right)^2 \sin^2\theta \, \omega_0^4 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Thus, the radiation corresponds to LHP light and the frequency appears as a fourth power. The intensity of the radiation is

$$I(\theta) = I_0 \sin^2\theta.$$

The Randomly Oscillating Bound Charge
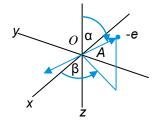
The z-axis angle is $\theta = 0$ and the intensity is zero. Perpendicular to the z-axis the angle is $\theta = 90°$ and the intensity is maximum. The oscillating bound charge emits a radiation pattern that corresponds to a **dipole radiator**.

To determine the Stokes vector for an ensemble of **randomly oriented bound oscillating charges** of amplitude A, the vector equation is

$$\ddot{\mathbf{r}}(t) + \omega_0^{\,2}\,\mathbf{r}(t) = 0,$$
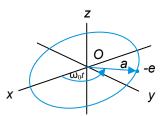
where the charge oscillates along A through the origin O.

The equations of motion along the x-, y-, and z-axes are

$$x(t) = A \sin\alpha \cos\beta \; e^{i\omega_0 t}, \quad y(t) = A \sin\alpha \sin\beta \; e^{i\omega_0 t}, \quad \text{and}$$

$$z(t) = A \cos\alpha \; e^{i\omega_0 t}.$$

The **accelerating charges** for $x(t)$, $y(t)$, and $z(t)$ are determined, substituted into the field equations, and the Stokes parameters are then found. The random behavior of the oscillating charge is described by taking an ensemble average over the angles α and β of the Stokes parameters, leading to a Stokes vector for unpolarized light:

$$S = \frac{8\pi}{3}\left(\frac{eA}{c^2 R}\right)^2 \omega_0^{\,4} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The **Stokes vector for a charge moving in a circular path** of radius a in the x-y plane with an angular frequency of ω_0 is shown.

A Charge Moving in a Circle

The equations of motion of the charge are

$$x(t) = a \cos(\omega_0 t),$$

$$y(t) = a \sin(\omega_0 t), \text{ and}$$

$$z(t) = 0.$$

The accelerations are determined along with the field equations, which then lead to the following Stokes vector for elliptically polarized light:

$$S = \left(\frac{eA}{c^2 R} \right)^2 \omega_0^{\,4} \begin{pmatrix} 1 + \cos^2 \theta \\ \sin^2 \theta \\ 0 \\ 2 \cos \theta \end{pmatrix},$$

which corresponds to the polarization ellipse is in its standard form. The angle θ refers to the observer's viewing angle measured from the z-axis. This result shows that the observed polarization states depend on the position of the observer. Along the z-axis at the angles $\theta = 0°$ or $180°$ the Stokes vector reduce, respectively, to

$$S_{\text{RCP}} = 2 \left(\frac{eA}{c^2 R} \right)^2 \omega_0^{\,4} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \text{ and } \quad S_{\text{LCP}} = 2 \left(\frac{eA}{c^2 R} \right)^2 \omega_0^{\,4} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$
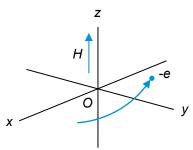
Similarly, viewing the radiation at $\theta = 90°$ or $270°$ the Stokes vector for LHP light is

$$S_{\text{LHP}} = \left(\frac{eA}{c^2 R} \right)^2 \omega_0^{\,4} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

A Charge Moving in a Magnetic Field

The **motion of a charged particle moving in a constant magnetic field** is directed along the *z*-axis so $\mathbf{H} = H\mathbf{k}$, where $\mathbf{k}$ is the unit vector.

In the magnetic field the motion of the charge is governed by the **Lorentz force equation**,

$$\mathbf{F} = m\ddot{\mathbf{r}} = \frac{e}{c}(\mathbf{v} \times \mathbf{H}).$$

The components for the motion along the three axes are

$$\ddot{x}(t) = \frac{eH}{mc}\dot{y}(t),$$

$$\ddot{y}(t) = -\frac{eH}{mc}\dot{x}(t)$$

$$\ddot{z}(t) = 0,$$

where the "over dots" represent differentiation with respect to time. The equations show that the charge moves only in the *x-y* plane. The differential equation pair must be solved simultaneously. The equations are solved with the initial conditions taken for the charge at the origin (0, 0) moving with a velocity v_0 along the *x*-axis and 0 along the *y*-axis (v_0, 0):

$$x(0) = 0, \quad y(0) = 0$$

and

$$\dot{x}(0) = v_0, \quad \dot{y}(0) = 0.$$

A Charge Moving in Magnetic Field (cont'd)

The solutions are

$$x(t) = \frac{v_0}{\omega_c} \sin(\omega_c t)$$

and

$$y(t) = \frac{v_0}{\omega_c}\left[1 - \cos(\omega_c t)\right],$$
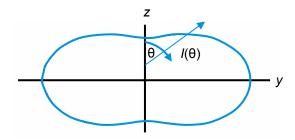
where $\omega_c = eH/mc$ is the **cyclotron frequency**. The equations are added and squared:

$$x(t)^2 + \left(y(t) - \frac{v_0}{\omega_c}\right)^2 = \left(\frac{v_0}{\omega_c}\right)^2,$$

which is the equation of a circle with a radius v_0 / ω_c and center at $x = 0$ and $y = v_0 / \omega_c$. From the above equations for $x(t)$ and $y(t)$ the Stokes vector is
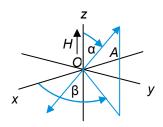
$$S = \left(\frac{ea}{c^2 R}\right)^2 \omega_c^4 \begin{pmatrix} 1 + \cos^2\theta \\ -\sin^2\theta \\ 0 \\ -2\cos\theta \end{pmatrix}.$$

Thus, in a constant magnetic field the emitted radiation is elliptically polarized and the intensity is proportional to ω_c^4. The radiation pattern for the intensity, S_0 is referred to as "**peanut" radiation**.

The Classical Zeeman Effect

When a sodium flame is examined with a spectroscope a very brilliant pair of yellow lines known as the D lines appear. If this same source is then placed between the poles of a very powerful magnet with its magnetic field directed parallel to the z-axis a single line splits into two lines if the radiation is observed parallel to the z-axis, or three lines if the radiation is observed perpendicular to the z-axis. This phenomenon is known as the **classical Zeeman effect**. To analyze this effect the Stokes vectors of these lines must be determined. The model of the sodium atom is assumed to behave as a bound oscillator.

The double-headed arrow describes the motion of the charged particle, where A is the maximum oscillation distance and **H** is the magnetic field. The **equation of motion of the bound charge**d particle is described by the Lorentz force equation

$$m\,\ddot{\mathbf{r}}(t) + k\,\mathbf{r}(t) = \frac{e}{c}[\mathbf{v} \times \mathbf{H}].$$

The components of this equation are

$$\ddot{x}(t) + \omega_0{}^2 x(t) = \left(\frac{eH}{mc}\right)\dot{y}(t),$$

$$\ddot{y}(t) + \omega_0{}^2 y(t) = -\left(\frac{eH}{mc}\right)\dot{x}(t),$$

$$\ddot{z}(t) + \omega_0{}^2 z(t) = 0,$$

where $\omega_0 = \sqrt{k/m}$ is the **natural frequency of oscillation of the electron in the sodium atom**.

The solutions to the differential equations are

$$x(t) = A\sin\alpha\cos(\omega_L\,t)\cos(\omega_0\,t),$$

$$y(t) = A\sin\alpha\sin(\omega_L\,t)\cos(\omega_0\,t),$$

$$z(t) = A\cos\alpha\cos(\omega_0 t),$$

The Classical Zeeman Effect (cont'd)
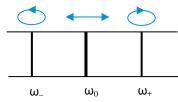
where $\omega_L = eH / 2mc$ is known as the **Larmor precession frequency**, because in the magnetic field the oscillating electron is found to precess around the z-axis. The corresponding Stokes vector is

$$
S = \frac{2}{3}\left(\frac{eA}{2c^2R}\right)^2
\begin{bmatrix}
\frac{2}{3}(\omega_+{}^4 + \omega_-{}^4)(1+\cos^2\theta) + \frac{4}{3}\omega_0{}^4\sin^2\theta \\[4pt]
-\frac{2}{3}(\omega_+{}^4 + \omega_-{}^4)\sin^2\theta + \frac{4}{3}\omega_0{}^4\sin^2\theta \\[4pt]
0 \\[4pt]
\frac{4}{3}(\omega_+{}^4 - \omega_-{}^4)\cos\theta
\end{bmatrix},
$$

where $\omega_\pm = \omega_0 \pm \omega_L$ and θ is the spherical polar angle measured from the z-axis. The equation can be decomposed in terms of its frequencies ω_-, ω_0, and ω_+:

$$
S = \frac{2}{3}\left(\frac{eA}{2c^2R}\right)^2
\left[
\omega_-{}^4
\begin{pmatrix}
1+\cos^2\theta \\
-\sin^2\theta \\
0 \\
-2\cos\theta
\end{pmatrix}
+ 2\omega_0{}^4
\begin{pmatrix}
\sin^2\theta \\
\sin^2\theta \\
0 \\
0
\end{pmatrix}
+ \omega_+{}^4
\begin{pmatrix}
1+\cos^2\theta \\
-\sin^2\theta \\
0 \\
2\cos\theta
\end{pmatrix}
\right].
$$

The **Lorentz-Zeeman model** predicts that three **spectral lines** will be observed at frequencies of $\omega_- = \omega_0 - \omega_L$, ω_0, and $\omega_+ = \omega_0 - \omega_L$. Furthermore, the spectral lines are each polarized.

$$\omega_- \qquad \omega_0 \qquad \omega_+$$
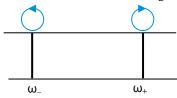
The line associated with ω_- is left elliptically polarized, for ω_0 the line is linear horizontally polarized and for ω_+ the line is right elliptically polarized. In a spectroscope the spectral pattern is observed. The central spectral line ω_0 is *twice* as bright as the left and right spectral lines ω_- and ω_+.

When the radiation is viewed parallel and perpendicular to the magnetic field the former the observation angle is $\theta = 0°$ and the Stokes vector reduces to

The Classical Zeeman Effect (cont'd)

$$S = \frac{4}{3}\left(\frac{eA}{2c^2R}\right)^2 \left[\omega_-{}^4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + \omega_+{}^4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right].$$
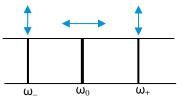
Thus, in a spectroscope viewing the radiation parallel to the magnetic field only two spectral lines will be observed at ω_- and ω_+ and the lines are left- and right-circularly polarized.

Viewing the radiation perpendicular to the magnetic field the observation angle is $\theta = 90°$ and the Stokes vector reduces to

$$S = \frac{2}{3}\left(\frac{eA}{2c^2R}\right)^2 \left[\omega_-{}^4 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + 2\omega_0{}^4 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \omega_+{}^4 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \right].$$

The spectral lines are vertically, horizontally, and vertically polarized. The central spectral line is twice as bright as the adjacent linear vertically polarized lines. When the magnetic field is removed ($\omega_\pm \to \omega_0$) the three Stokes vectors reduce to a single Stokes vector:
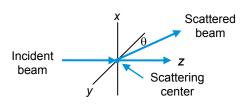
$$S = \frac{8}{3}\left(\frac{eA}{2c^2R}\right)^2 \omega_0{}^4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This is the original Stokes vector (unpolarized) that is observed in a spectroscope at the natural frequency of ω_0.

Optical Scattering

The basic idea behind the phenomenon of scattering is that radiation incident on an electron is **reradiated** or **scattered**. Scattering can occur with either free electrons in **Thomson scattering,** or bound electrons in **Rayleigh scattering**.

In Thomson scattering, the incident beam propagates along the z-axis and is then scattered by the electron. The motion of a free electron is

$$m\ddot{\mathbf{r}}(t) = -e\,\mathbf{E}(t),$$

where $\mathbf{E}(t)$ describes the transverse field of the incident beam. The vector components are

$$\ddot{x}(t) = \frac{-e}{m}E_x(t),$$

$$\ddot{y}(t) = \frac{-e}{m}E_y(t),$$

and the spherical components of the scattered field are

$$E_\theta = \frac{-e^2}{mc^2R}\cos\theta\,E_x,$$

$$E_\phi = \frac{-e^2}{mc^2R}E_y.$$

The Stokes vector of the scattered beam S' is related to the Stokes vector S of the incident beam by the matrix relation

$$S' = M \cdot S,$$

where M is the Mueller matrix of the scatter and is

$$M = \frac{1}{2}\left(\frac{e^2}{mc^2R}\right)^2 \begin{pmatrix} 1+\cos^2\theta & -\sin^2\theta & 0 & 0 \\ -\sin^2\theta & 1+\cos^2\theta & 0 & 0 \\ 0 & 0 & 2\cos\theta & 0 \\ 0 & 0 & 0 & 2\cos\theta \end{pmatrix}.$$

Optical Scattering (cont'd)

The term e^2/mc^2 (= 2.82×10^{-13} cm) is the **classical radius of an electron**. Because this term appears as a squared quantity the scattering is directly related to the area of the electron, that is, scattering is proportional to the area of the scatterer.

The scattering matrix M reduces to two special forms for the viewing angles of $\theta = 0°$ and $\theta = 90°$, respectively. These are

$$M(0°) = \left(\frac{e^2}{mc^2R}\right)^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M(90°) = \frac{1}{2}\left(\frac{e^2}{mc^2R}\right)^2 \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Viewing the scattered radiation along the z-axis ($\theta = 0°$) the polarization state of the incident beam remains unchanged. Viewing the scattered radiation perpendicular to the z-axis ($\theta = 90°$) the Mueller matrix takes the form of a linear vertical polarizer.
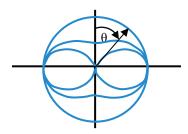
At any viewing angle the observed intensity of the scattered radiation is

$$I(\theta) = \frac{1}{2}[S_0(1 + \cos^2\theta) - S_1 \sin^2\theta].$$

Here, the constant factor has been dropped. For incident linearly vertically polarized light, LVP, unpolarized light, UNP, and linearly horizontally polarized light, LHP, the intensities of the scattered beam are

$$I_{LVP}(\theta) = 1,$$

$$I(\theta)_{UNP} = \frac{1}{2}(1 + \cos^2\theta),$$

$$I(\theta)_{LHP} = \cos^2\theta.$$
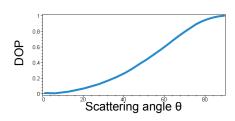
Optical Scattering (cont'd)

In the figure, intensity polar plots of the above equations are shown. The innermost plot corresponds to LHP light and corresponds to dipole radiation. The next plot is for unpolarized light and corresponds to "peanut" radiation. Finally, the outermost plot corresponds to LVP radiation and is a circle.

For incident unpolarized light the DOP for the scattered light is
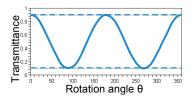
$$\mathscr{P} = \left| \frac{S_1{}'}{S_0{}'} \right| = \left| \frac{\sin^2\theta}{1+\cos^2\theta} \right|.$$

Thus, unpolarized light becomes partially polarized for $0° < \theta < 90°$ and completely polarized for $\theta = 90°$.

The Rayleigh scattering phenomenon is scattering by a bound charge. The vector equation of motion of the charge and it components are

$$m\,\ddot{\mathbf{r}}(t) + k\mathbf{r}(t) = -e\mathbf{E}(t),$$

$$\ddot{x}(t) + \omega_0{}^2 x(t) = -\frac{e}{m} E_x(t),$$

$$\ddot{y}(t) + \omega_0{}^2 y(t) = -\frac{e}{m} E_y(t)$$

$$\ddot{z}(t) + \omega_0{}^2 z(t) = 0,$$

where $\omega_0 = \sqrt{k/m}$. The last equation is ignored since it does not contribute to the reradiated field.

Optical Scattering (cont'd)

The solutions of the above equations are

$$\ddot{x}(t) = \frac{-e\,\omega^2}{m(\omega^2 - \omega_0{}^2)} E_x(t)$$
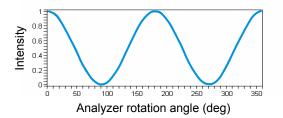
$$\ddot{y}(t) = \frac{-e\,\omega^2}{m(\omega^2 - \omega_0{}^2)} E_y(t).$$

The Stokes vector S' of the scattered beam for an incident beam with an arbitrary **Stokes vector for Rayleigh scattering** is

$$S' = \frac{1}{2}\left[\frac{e^2}{mc^2(\omega^2 - \omega_0{}^2)} \right]^2 \omega^4 \begin{pmatrix} S_0(1 + \cos^2\theta) - S_1\sin^2\theta \\ -S_0\sin^2\theta + S_1(1 + \cos^2\theta) \\ 2S_2\cos\theta \\ 2S_3\cos\theta \end{pmatrix}.$$

For $\omega_0 = 0$ the Stokes vector reduces to scattering by a free electron. However, for the bound electron model the scattering intensity is inversely proportional to the fourth power of the wavelength since $\omega^4 = (2\pi c/\lambda)^4$. Thus, as the wavelength of light decreases from the red region to the blue region of the optical spectrum, the intensity of the scattered light increases; this explains why the sky is blue.

Sunlight is unpolarized and perpendicular to its direction. The scattered light is linearly polarized. The observed intensity varies by using a rotating.

The Optics of Metals and Semiconductors

Metals and semiconductors are absorbing media and are very good conductors of electricity and heat (conducting media), due to the relatively large number of **free electrons** in metals whereas in dielectrics there are relatively few free electrons. Consequently, most metals and semiconductors are almost always highly reflective and are practically opaque. Dielectrics almost always have smaller reflectivities than metals. For metals the reflection of light from the surface is one of the few ways to investigate the optical constants ε and κ.

The interaction of polarized light (optical fields) with metals and semiconductors is described by the relation $\mathbf{j}(r,t) = \sigma \mathbf{E}(r,t)$, where $\mathbf{j}(r,t)$ is the electric current density, $\mathbf{E}(r,t)$ is the electric field, and σ is the conductivity of the medium. Maxwell's equations for the electric field in the metallic-semiconductor medium are found to be

$$\nabla^2 \mathbf{E}(r,t) = \frac{\mu\,\varepsilon}{c^2} \frac{\partial^2 \mathbf{E}(r,t)}{\partial t^2} + \frac{4\pi\mu\,\sigma}{c^2} \frac{\partial \mathbf{E}(r,t)}{\partial t}.$$

The term $\partial \mathbf{E}(r,t)/\partial t$ corresponds to an attenuated (or damped) wave. The terms can be regrouped and lead to a dielectric constant for metals:

$$\boldsymbol{\varepsilon} = \varepsilon - i\left(\frac{4\pi\sigma}{\omega}\right),$$

where $\boldsymbol{\varepsilon}$ is a **complex dielectric constant**, ε is a real dielectric constant, and ω is the angular frequency of the incident wave. The refractive index can be as $\boldsymbol{\varepsilon} = \mathbf{n}^2$. This relation can be written in terms of a real refractive index n and attenuation or absorption coefficient κ by a complex relation $\mathbf{n} = (1 - i\kappa)$.

Refractive Index and Absorption Coefficient

Using $\boldsymbol{\varepsilon} = \mathbf{n}^2$ the real and imaginary parts of this last equation yield

$$\boldsymbol{\varepsilon} = \mathbf{n}^2 = n^2(1 - i\kappa)^2 = \varepsilon - i\left(\frac{4\pi\sigma}{\omega}\right),$$

where

$$n^2(1 - \kappa^2) = \varepsilon,$$

$$n^2\kappa = \frac{2\pi\sigma}{\omega} = \frac{\sigma}{\nu},$$

and $nu \rightarrow \nu$ is the (linear) frequency. These last two equations can be solve simultaneously and yield

$$n^2 = \frac{1}{2}\left[\sqrt{\varepsilon^2 + \left(\frac{2\sigma}{\nu}\right)^2} + \varepsilon\right], \text{ and } n^2\kappa^2 = \frac{1}{2}\left[\sqrt{\varepsilon^2 + \left(\frac{2\sigma}{\nu}\right)^2} - \varepsilon\right].$$

From these two parameters ε and σ can then be found from the equations on the previous page, which also serve as a check on the measurements.

The equations for the *s*- and *p*-components of the reflected intensity are

$$\mathcal{R}_s = \left|\frac{R_s}{E_s}\right|^2, \text{ and } \mathcal{R}_p = \left|\frac{R_p}{E_p}\right|^2.$$
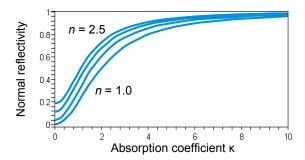
At normal incidence the *s*- and p-components of the reflected intensity are equal and are

$$\mathcal{R}_s = \mathcal{R}_p = \left[\frac{(n-1)^2 + (n\kappa)^2}{(n+1)^2 + (n\kappa)^2}\right].$$

For $\kappa = 0$, this equation reduces to the familiar equation for the reflectivity from a dielectric at normal incidence.

Incidence Angle Reflectivity

The previous equation for n = 1.0, 1.5, 2.0, and 2.5 plotted over a range of κ from 0 to 10 is shown in the graph.

For absorbing media with increasing κ (metals) the reflectivity approaches unity. Thus, highly reflecting metals are characterized by high values of κ.

For non-normal incidence the s- and p-intensity components are

$$\mathcal{R}_s = \left|\frac{R_s}{E_s}\right|^2 = \left[\frac{(n - \cos i)^2 + (n\kappa)^2}{(n + \cos i)^2 + (n\kappa)^2}\right],$$

$$\mathcal{R}_p = \left|\frac{R_p}{E_p}\right|^2 = \left[\frac{(n - 1/\cos i)^2 + (n\kappa)^2}{(n + 1/\cos i)^2 + (n\kappa)^2}\right].$$

For i = 0° the above equations reduce to the normal incidence equation shown above.
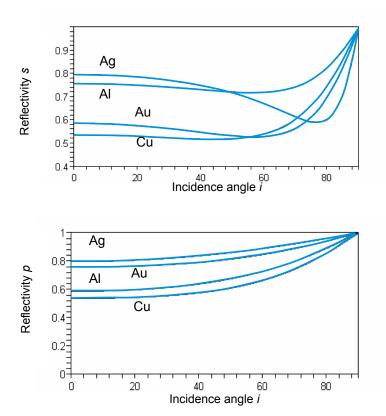
The values of n and κ are listed for λ = 0.60 µm.

Metal	n	κ
Gold (Au)	0.47	6.02
Silver (Ag)	0.20	17.2
Copper (Cu)	0.62	4.15
Aluminum (Al)	1.44	3.63

Incidence Angle Reflectivity (cont'd)
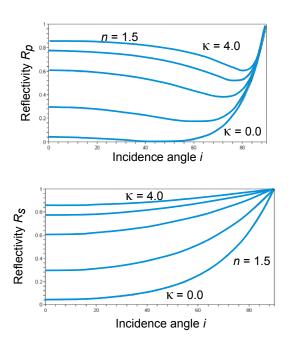
The reflection equations for the *s*- and *p*-components are plotted below using the values for n and κ in the table.

For the *s*-component the reflectivity has a pseudo-Brewster angle minimum value, because, unlike the Brewster angle behavior for dielectrics, the intensity does not go to zero for metals.

Complex Reflection Coefficients

The effect of κ on the reflected p- and s-components are shown for $n = 1.5$ and $\kappa = 0, 1 \ldots 4$.

For the p-component, plot $\kappa = 0$ corresponds to the Brewster angle null condition for a dielectric.

To deal with light of any polarization state it is necessary to determine the Mueller matrix for metals. Since a complex refractive index is required to describe metals and semiconductors, the reflection coefficients (complex) are defined to be

$$r_p = R_p / E_p = \rho_p \exp(i\,\phi_p),$$

$$r_s = R_s / E_s = \rho_s \exp(i\,\phi_s),$$

where $\rho_{s,p}$ and $\phi_{s,p}$ are the absolute values of the reflection coefficients and the phases, respectively.
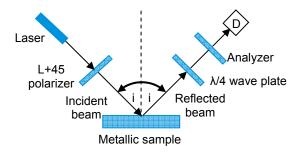
The Principal Angle of Incidence Measurement

The reflection coefficients describing metals and semiconductors can be immediately used to determine the form of the **Mueller matrix for reflection from metals:**

$$M = \frac{1}{2}\begin{pmatrix} \rho_s^{\,2} + \rho_p^{\,2} & \rho_s^{\,2} - \rho_p^{\,2} & 0 & 0 \\ \rho_s^{\,2} - \rho_p^{\,2} & \rho_s^{\,2} + \rho_p^{\,2} & 0 & 0 \\ 0 & 0 & 2\rho_s\rho_p\cos\Delta & -2\rho_s\rho_p\sin\Delta \\ 0 & 0 & 2\rho_s\rho_p\sin\Delta & 2\rho_s\rho_p\cos\Delta \end{pmatrix},$$

where $\Delta = \phi_s - \phi_p$.

A measurement method for measuring n and κ known as the **principal angle of incidence measurement**.

The measurement configuration is identical to that for the Brewster angle measurement except that a $\lambda/4$ wave plate is now in the reflection arm. The measurement method is very similar to that used in ellipsometry, a method that determines not only the optical constants but the thickness of thin films as well.

Incident L+45P polarized light is reflected from the metal sample. In general, the reflected light is elliptically polarized so the corresponding polarization ellipse is in a nonstandard form. The angle of the incident beam is changed until the phase becomes 90° and is observed in the reflected beam. The incident angle where this takes places is the principal angle of incidence $\bar{i}$.

The Principal Angle of Incidence Measurement (cont'd)

The polarization ellipse is in its standard form, and the orthogonal field components are parallel and perpendicular to the plane of incidence. The reflected beam then proceeds through the QWP and the beam of light that emerges is linearly polarized with its azimuthal angle at an unknown angle. The beam is incident on an analyzing polarizer which is rotated until a null intensity is found. This null takes place at the principal azimuthal angle $\bar{\psi}$. From the measurement of the principal angle of incidence and the principal azimuth angle the optical constants n and κ are determined.

For the incident L+45P light that is reflected from the metal surface and using the above Mueller matrix the Stokes vector of the reflected beam is

$$S' = I_0 \begin{bmatrix} 1 + P^2 \\ -(1 - P^2) \\ 2P\cos\Delta \\ 2P\sin\Delta \end{bmatrix},$$

where $P = \rho_s / \rho_p = \tan\bar{\psi}$, and $\bar{\psi}$ is the principal azimuthal angle. The reflected beam passes through the QWP where the elliptically polarized beam is transformed to its standard form, by changing the incident angle until Δ becomes 90°.

The Stokes vector becomes

$$S' = I_0 \begin{bmatrix} 1 + P^2 \\ -(1 - P^2) \\ 0 \\ 2P \end{bmatrix},$$

which describes the polarization ellipse in its standard form.

The Principal Angle of Incidence Measurement (cont'd)

χ and ψ of the polarization ellipse are

$$\chi = \frac{1}{2}\sin^{-1}\left(\frac{S_3{}'}{S_0{}'}\right) = \frac{1}{2}\sin^{-1}\left(\frac{2P}{1+p2}\right), \text{ and}$$

$$\psi = \frac{1}{2}\tan^{-1}\left(\frac{S_2{}'}{S_1{}'}\right) = 0,$$

so the orientation angle of the polarization ellipse is 0, as required. The QWP now transforms this Stokes vector to linear polarized light:

$$S' = I_0 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1+P^2 \\ -(1-P^2) \\ 0 \\ 2P \end{pmatrix} = I_0 \begin{pmatrix} 1+P^2 \\ -(1-P^2) \\ 2P \\ 0 \end{pmatrix}.$$

The final step is to relate the principle angle of incidence $\bar{i}$ ($\Delta = \pi/2$) and the principal azimuth angle $\bar{\psi}$ to the optical constants n and κ. By rotating the analyzer until a null intensity is obtained the value of P is determined and is expressed as

$$\frac{1+jP}{1-jP} = \frac{-\sin\bar{i}\,\tan\bar{i}}{n(1-j\kappa)},$$

where $j = \sqrt{-1}$. Multiplying this equation by its complex conjugate then yields

$$n\sqrt{1+\kappa^2} = \sin\bar{i}\,\tan\bar{i}.$$

From this equation and using the relation $P = \tan\bar{\psi}$ it is straightforward to determine the refractive index n and the absorption coefficient κ in terms of $\bar{i}$ and $\bar{\psi}$. The optical constants of the metal are then found to be

$$n = -\sin\bar{i}\,\tan\bar{i}\,\cos 2\bar{\psi},$$

and

The Principal Angle of Incidence Measurement (cont'd)

$$\kappa = \tan 2\bar{\psi},$$

where $\bar{i}$ and $\bar{\psi}$ are the principal angles of incidence and azimuth, respectively. The above two equations can be written as a single equation in terms of the complex refractive index:

Example The principal angles of incidence and azimuth for Au are found to be $\bar{i} = 108.3°$ and $\bar{\psi} = 40.3°$ at a wavelength of 6000 Å (1 Å = 10^{-8} cm).

$$n = -\sin(108.3°)\tan(108.3°)\cos(80.6°) = 0.477,$$

and

$$\kappa = \tan(80.6°) = 6.17.$$

The dielectric constant ε and conductivity o for Au can be determined at a frequency of $\nu = 0.5 \times 10^{14}$ Hz (which corresponds to 6000 Å):

$$\varepsilon = n^2(1 - \kappa^2) = -8.46,$$

and

$$\sigma = n^2\kappa\nu = 7.05 \times 10^{14} \text{ sec}^{-1}.$$

For metals, negative values of ε can occur.

$$\mathbf{n} = n(1 - j\kappa) = -\sin\bar{i}\,\tan\bar{i}\,\exp(-j2\bar{\psi}).$$

Equation Summary

General equations

Malus' Law and Brewster's Law:

$$I(\theta) = I_0 \cos^2\theta \qquad \tan i = n_2 / n_1 = n$$

Field components propagating in the z direction:

$$E_x(z,t) = E_{0x} \cos(\omega t - kz + \delta_x)$$

$$E_y(z,t) = E_{0y} \cos(\omega t - kz + \delta_y)$$

Equation of the polarization ellipse:

$$\frac{E_x(z,t)^2}{E_{0x}^2} + \frac{E_y(z,t)^2}{E_{0y}^2} - \frac{2E_x(z,t)E_y(z,t)}{E_{0x}E_{0y}}\cos\delta = \sin^2\delta$$

Parameters of the polarization ellipse:

$$\tan 2\psi = \frac{2E_{0x}E_{0y}}{E_{0x}^2 - E_{0y}^2}\cos\delta \qquad 0 \le \psi \le \pi$$

$$\sin 2\chi = \frac{2E_{0x}E_{0y}\sin\delta}{E_{0x}^2 + E_{0y}^2} \qquad -\pi/4 < \chi \le \pi/4$$

Auxilliary angle definition and polarization parameters:

$$\tan\alpha = \frac{E_{0y}}{E_{0x}} \ , \quad 0 \le \alpha \le \pi/2$$

$$\tan 2\psi = (\tan 2\alpha)\cos\delta$$

$$\sin 2\chi = (\sin 2\alpha)\sin\delta \quad 0 \le \delta < 2\pi$$

Stokes polarization parameters:

$$S_0 = E_{0x}^2 + E_{0y}^2$$

$$S_1 = E_{0x}^2 - E_{0y}^2$$

$$S_2 = 2E_{0x}E_{0y}\cos\delta$$

Equation Summary

Stokes polarization parameters (cont'd):

$$S_3 = 2E_{0x}E_{0y}\sin\delta \ , \ \ \delta = \delta_y - \delta_x$$

$$S_0{}^2 = S_1{}^2 + S_2{}^2 + S_3{}^2$$

Stokes vector:

$$S = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} E_{0x}^2 + E_{0y}^2 \\ E_{0x}^2 - E_{0y}^2 \\ 2E_{0x}E_{0y}\cos\delta \\ 2E_{0x}E_{0y}\sin\delta \end{pmatrix}$$

Stokes parameters in complex notation:

$$E_x(t) = E_{0x}\exp(i\delta_x) \ , \ \ E_y(t) = E_{0y}\exp(i\delta_y)$$

$$S_0 = E_x E_x{}^* + E_y E_y{}^*$$

$$S_1 = E_x E_x{}^* - E_y E_y{}^*$$

$$S_2 = E_x E_y{}^* + E_y E_x{}^*$$

$$S_3 = i\left(E_x E_y{}^* - E_y E_x{}^*\right)$$

Stokes vector on the Poincaré sphere:

$$S = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{bmatrix} 1 \\ \cos(2\chi)\cos(2\psi) \\ \cos(2\chi)\sin(2\psi) \\ \sin(2\chi) \end{bmatrix}$$

$$\psi = \frac{1}{2}\tan^{-1}\left(\frac{S_2}{S_1}\right) \ 0 \le \psi \le \pi, \ \ \chi = \frac{1}{2}\sin^{-1}\left(\frac{S_3}{S_0}\right) -\frac{\pi}{4} \le \chi \le \frac{\pi}{4}$$

Equation Summary

Stokes vector on the observable polarization sphere:

$$S = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} 1 \\ \cos 2\alpha \\ \sin 2\alpha \cos \delta \\ \sin 2\alpha \sin \delta \end{pmatrix}$$

Angle relation of $2a$ and δ to the Stokes parameters:

$$2\alpha = \cos^{-1}\left(\frac{S_1}{S_0}\right) \quad 0 \le 2\alpha < \pi$$

$$\delta = \tan^{-1}\left(\frac{S_3}{S_2}\right) \quad 0 \le \delta < 2\pi$$

Ellipticity and orientation angles relations:

$$\tan(2\psi) = \tan(2\alpha)\cos\delta \qquad \sin(2\chi) = \sin(2\alpha)\sin\delta$$

$$\cos(2\alpha) = \cos(2\chi)\cos(2\psi) \qquad \cot\delta = \cot(2\chi)\sin(2\psi)$$

Stokes vectors for unpolarized and partially polarized light:

$$S_{\text{UNP}} = S_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$S = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \left(1 - \mathscr{P}\right)\begin{pmatrix} S_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mathscr{P}\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} \quad 0 \le \mathscr{P} \le 1$$

Definition of the degree of polarization $\mathscr{P}$:

$$\mathscr{P} = \frac{I_{\text{pol}}}{I_{\text{tot}}} = \frac{\sqrt{S_1^{\,2} + S_2^{\,2} + S_3^{\,2}}}{S_0} \quad 0 \le \mathscr{P} \le 1$$

$$S_0^{\,2} \ge S_1^{\,2} + S_2^{\,2} + S_3^{\,2}$$

Equation Summary

Mueller matrices

Mathematical form of the Mueller matrix:

$$M = \begin{pmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \\ m_{30} & m_{31} & m_{32} & m_{33} \end{pmatrix}$$

Mueller matrix for a polarizer:

$$M_{POL}\left(p_x, p_y\right) = \frac{1}{2}\begin{pmatrix} p_x^2 + p_y^2 & p_x^2 - p_y^2 & 0 & 0 \\ p_x^2 - p_y^2 & p_x^2 + p_y^2 & 0 & 0 \\ 0 & 0 & 2p_x p_y & 0 \\ 0 & 0 & 0 & 2p_x p_y \end{pmatrix}$$

$$0 \le p_x \le 1 \qquad 0 \le p_y \le 1$$

Mueller matrices for a linear horizontal and vertical polarizer:

$$M_{POL} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad M_{POL} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Mueller matrix for a linear polarizer with its transmission axis at +45°:

$$M_{POL} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Mueller matrix for a wave plate:

$$M_{WP}(\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix} \qquad \phi = \frac{2\pi}{\lambda}(n_o - n_e)d$$

Equation Summary

Mueller matrices for a quarter- and half-wave plate:

$$M_{\text{QWP}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad M_{\text{HWP}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Mueller matrix for a rotator:

$$M_{\text{ROT}}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta & 0 \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Mueller matrix for a rotated polarizing element:

$$M(\theta) = M_{\text{ROT}}(-2\theta) \cdot M \cdot M_{\text{ROT}}(2\theta)$$

Mueller matrix for a rotated linear polarizer:

$$M_{POL}(\theta) = \frac{1}{2} \begin{pmatrix} 1 & \cos 2\theta & \sin 2\theta & 0 \\ \cos 2\theta & \cos^2 2\theta & \sin 2\theta \cos 2\theta & 0 \\ \sin 2\theta & \sin 2\theta \cos 2\theta & \sin^2 2\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Mueller matrix of a circular polarizer:

$$M_{\text{CP}} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Reflection and transmission

Fresnel's Equations for reflection and refraction:

$$R_p = \frac{\tan(i-r)}{\tan(i+r)} E_p \qquad R_s = -\frac{\sin(i-r)}{\sin(i+r)} E_s$$

$$T_p = \frac{2\sin r \cos i}{\sin(i+r)\cos(i-r)} E_p \quad T_s = \frac{2\sin r \cos i}{\sin(i+r)} E_s$$

Equation Summary

Stokes parameters for reflection:

$$S_{0R} = f_R[(\cos^2\alpha_- + \cos^2\alpha_+)S_0 + (\cos^2\alpha_- - \cos^2\alpha_+)S_1]$$

$$S_{1R} = f_R[(\cos^2\alpha_- - \cos^2\alpha_+)S_0 + (\cos^2\alpha_- + \cos^2\alpha_+)S_1]$$

$$S_{2R} = -f_R(2\cos\alpha_-\cos\alpha_+)S_2$$

$$S_{3R} = -f_R(2\cos\alpha_-\cos\alpha_+)S_3 \quad f_R = \frac{1}{2}\left(\frac{\tan\alpha_-}{\sin\alpha_+}\right)^2$$

Stokes parameters for transmission (refraction):

$$S_{0T} = f_T[(\cos^2\alpha_- + 1)S_0 + (\cos^2\alpha_- - 1)S_1]$$

$$S_{1T} = f_T[(\cos^2\alpha_- - 1)S_0 + (\cos^2\alpha_- + 1)S_1]$$

$$S_{2T} = -f_T(2\cos\alpha_-)S_2$$

$$S_{3T} = -f_T(2\cos\alpha_-)S_3 \quad f_T = \frac{1}{2}\frac{\sin 2i \sin 2r}{(\sin\alpha_+\cos\alpha_-)^2}$$

Mueller matrices for reflection and transmission:

$$M_R = \left(\frac{n-1}{n+1}\right)^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad M_T = \frac{4n}{(n+1)^2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Mueller matrices for Brewster angle reflection and transmission:

$$M_{R,B} = \frac{1}{2}\cos^2 2i_B \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$M_{T,B} = \frac{1}{2}\begin{pmatrix} \sin^2 2i_B + 1 & \sin^2 2i_B - 1 & 0 & 0 \\ \sin^2 2i_B - 1 & \sin^2 2i_B + 1 & 0 & 0 \\ 0 & 0 & 2\sin 2i_B & 0 \\ 0 & 0 & 0 & 2\sin 2i_B \end{pmatrix}$$

Equation Summary

Mueller matrix for total internal reflection:

$$M_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\delta & -\sin\delta \\ 0 & 0 & \sin\delta & \cos\delta \end{pmatrix} \qquad \tan\frac{\delta}{2} = \frac{\cos i \sqrt{n^2 \sin^2 i - 1}}{n \sin^2 i}$$

Mueller matrix of the Fresnel rhomb:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\delta_U + \delta_L) & -\sin(\delta_U + \delta_L) \\ 0 & 0 & \sin(\delta_U + \delta_L) & \cos(\delta_U + \delta_L) \end{pmatrix}$$

Jones matrix calculus

Jones vector:

$$\mathbf{E} = \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} E_{0x} e^{i\delta_x} \\ E_{0y} e^{i\delta_y} \end{pmatrix}$$

Jones matrix formulation of optical intensity:

$$I = \mathbf{E}^\dagger \cdot \mathbf{E} \qquad I = \begin{pmatrix} E_x{}^* & E_y{}^* \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = E_x E_x{}^* + E_y E_y{}^*$$

Jones matrix definition:

$$\mathbf{J} = \begin{pmatrix} j_{xx} & j_{xy} \\ j_{yx} & j_{yy} \end{pmatrix}$$

Jones matrices for linear polarizers:

$$\mathbf{J}_{\text{POL}} = \begin{pmatrix} p_x & 0 \\ 0 & p_y \end{pmatrix} \quad 0 \le p_x, p_y \le 1$$

$$\mathbf{J}_{\text{LHP}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{J}_{\text{LVP}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Equation Summary

Jones matrices for a wave plate, quarter-wave plate, and a half-wave plate:

$$\mathbf{J}_{WP} = \begin{pmatrix} e^{i(\phi/2)} & 0 \\ 0 & e^{-i(\phi/2)} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$$

$$\mathbf{J}_{QWP} = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \qquad \mathbf{J}_{HWP} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Jones matrix for a rotator:

$$\mathbf{J}_{ROT}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Jones matrix for a rotated polarizing element:

$$\mathbf{J}(\theta) = \mathbf{J}_{ROT}(-\theta) \cdot \mathbf{J} \cdot \mathbf{J}_{ROT}(\theta)$$

Polarizer characterization

Transmittance of a single linear polarizer:

$$T_{UNP} = \frac{k_1 + k_2}{2} \qquad T_{LHP} = \frac{k_1}{2} \qquad T_{LVP} = \frac{k_2}{2}$$

Transmittance of a pair of linear polarizers:

$$T_{UNP} = H_0 = \frac{1}{2}(k_1^2 + k_2^2) \qquad T_{LHP} = \frac{k_1^2}{2} \qquad T_{LVP} = \frac{k_2^2}{2}$$

Transmittance of a pair of crossed polarizers:
$$T_{UNP} = H_{90} = k_1 k_2$$

Transmittance for a single rotated polarizer at an angle θ:

$$T(\theta) = k_1 \sin^2\theta + k_2 \cos^2\theta$$

Transmittance for a fixed and rotated polarizer at an angle θ:

$$T(\theta) = k_1 k_2 \sin^2\theta + \frac{1}{2}(k_1^2 + k_2^2)\cos^2\theta$$
$$= H_{90} + (H_0 - H_{90})\cos^2\theta$$

Equation Summary

Logarithmic definition of transmittance (density):

$$D_0 = -\log_{10} H_0 \qquad D_{90} = -\log_{10} H_{90}$$

Contrast ratio of a polarizer:

$$C = \frac{T_{\max} - T_{\min}}{T_{\max} + T_{\min}} = \frac{k_1 - k_2}{k_1 + k_2}$$

Kerr and Pockel cell characterization

Birefringence of the Kerr effect:

$$n_p - n_s = \lambda B \mathcal{E}^2$$

Pockel cell phase shift:

$$\phi = \phi_y - \phi_x = \frac{\omega n_0{}^3 r_{63}}{c} V$$

Stokes vectors for radiating charges

Stokes vector for a linearly oscillating charge:

$$S = \left(\frac{e z_0}{c^2 R}\right)^2 \sin^2\theta \, \omega_0{}^4 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Stokes vector for a randomly oriented oscillating charge:

$$S = \frac{8\pi}{3} \left(\frac{eA}{c^2 R}\right)^2 \omega_0{}^4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Stokes vector for a charge moving in a circle:

$$S = \left(\frac{eA}{c^2 R}\right)^2 \omega_0{}^4 \begin{pmatrix} 1 + \cos^2\theta \\ \sin^2\theta \\ 0 \\ 2\cos\theta \end{pmatrix}$$

Equation Summary

Stokes vector for the Zeeman effect:

$$S = \frac{2}{3}\left(\frac{eA}{2c^2R}\right)^2 \left[\omega_-^{\ 4}\begin{pmatrix} 1+\cos^2\theta \\ -\sin^2\theta \\ 0 \\ -2\cos\theta \end{pmatrix} + 2\omega_0^{\ 4}\begin{pmatrix} \sin^2\theta \\ \sin^2\theta \\ 0 \\ 0 \end{pmatrix} + \omega_+^{\ 4}\begin{pmatrix} 1+\cos^2\theta \\ -\sin^2\theta \\ 0 \\ 2\cos\theta \end{pmatrix}\right]$$

Mueller matrix for Thomson scattering:

$$M = \frac{1}{2}\left(\frac{e^2}{mc^2R}\right)^2 \begin{pmatrix} 1+\cos^2\theta & -\sin^2\theta & 0 & 0 \\ -\sin^2\theta & 1+\cos^2\theta & 0 & 0 \\ 0 & 0 & 2\cos\theta & 0 \\ 0 & 0 & 0 & 2\cos\theta \end{pmatrix}$$

Stokes vector for Rayleigh scattering:

$$S' = \frac{1}{2}\left[\frac{e^2}{mc^2(\omega^2-\omega_0^{\ 2})}\right]^2 \omega^4 \begin{pmatrix} S_0(1+\cos^2\theta) - S_1\sin^2\theta \\ -S_0\sin^2\theta + S_1(1+\cos^2\theta) \\ 2S_2\cos\theta \\ 2S_3\cos\theta \end{pmatrix}$$

Absorbing media—semiconductors and metals

Complex dielectric constant and refractive index for metals:

$$\boldsymbol{\varepsilon} = \varepsilon - i\left(\frac{4\pi\sigma}{\omega}\right) \qquad \mathbf{n} = n(1-i\kappa)$$

Reflectivity at normal incidence for absorbing media:

$$\mathcal{R}_s = \mathcal{R}_p = \left[\frac{(n-1)^2 + (n\kappa)^2}{(n+1)^2 + (n\kappa)^2}\right]$$

s- and p- reflectivity at non-normal incidence:

$$\mathcal{R}_s = \left|\frac{R_s}{E_s}\right|^2 = \left[\frac{(n-\cos i)^2 + (n\kappa)^2}{(n+\cos i)^2 + (n\kappa)^2}\right]$$

$$\mathcal{R}_p = \left|\frac{R_p}{E_p}\right|^2 = \left[\frac{(n-1/\cos i)^2 + (n\kappa)^2}{(n+1/\cos i)^2 + (n\kappa)^2}\right]$$

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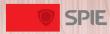
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