APPLIED PROBABILITY FOR MATHEMATICAL FINANCE PROJECT 2

JULIAN SCHADY

Part 1 Variational PDE and Infinite-Horizon (Perpetual) American Put

1 a) Variational PDE for an American Put.

Under the risk–neutral measure $\mathbb{Q}$ the stock price satisfies

$$dS_t = rS_t\,dt + \sigma S_t\,d\widetilde{W}_t,$$

and the value of an American put option with strike K and maturity T can be formulated as

$$V(t,x) = \sup_{\tau\in[t,T]} \mathbb{E}^{\mathbb{Q}}\left[e^{-r(\tau-t)}(K-S_\tau)^+ \mid S_t = x\right], \tag{1}$$

where $\tau \in [t,T]$ denotes stopping times taking values in $[t,T]$.
The payoff of the American Put Option is defined as

$$\psi(x) := (K-x)^+.$$

Lets define the continuation Region and the exercise region. The continuation region is when it is optimal to continue rather than two exercise the option.

The discounted value of a American put option is a supermatingale. Therefore we have

$$e^{-rt}V(t,x) = \sup_{\tau\in[t,T]} \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}(K-S_T)^+ \mid F_t\right], \tag{2}$$

Plugging this into Ito's Formula we have

$$dV = V_t\,dt + V_x\,dS_t + \tfrac{1}{2}V_{xx}\,d[S,S]_t.$$

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t, \qquad d[S,S]_t = \sigma^2 S_t^2\,dt,$$

$$dV = V_t\,dt + V_x\big(rS_t\,dt + \sigma S_t\,dW_t\big) + \tfrac{1}{2}V_{xx}\sigma^2 S_t^2\,dt.$$

$$dV = \big(V_t + rS_tV_x + \tfrac{1}{2}\sigma^2S_t^2V_{xx}\big)\,dt + \sigma S_tV_x\,dW_t.$$

$$d\left(e^{-rt}V\right) = e^{-rt}dV - re^{-rt}V\,dt.$$

$$d\left(e^{-rt}V\right) = e^{-rt}\left(V_t + rS_tV_x + \tfrac{1}{2}\sigma^2S_t^2V_{xx} - rV\right)dt + e^{-rt}\,\sigma S_tV_x\,dW_t.$$

Supermartingales have a negative drift term, therefore we can write.

$$V_t + rS_tV_x + \tfrac{1}{2}\sigma^2S_t^2V_{xx} - rV \le 0$$

If it is in the continuation region then,

$$V(t,x) = \psi(x) := (K-x)^+$$

Combining the two regions we can write the variational PDE

$$\begin{cases} V_t + rS_tV_x + \frac{1}{2}\sigma^2S_t^2V_{xx} - rV \le 0 \\ (K-x)^+ - V(t,x) \le 0 \end{cases}$$

Which can also be written as

$$\boxed{\max\left\{-rV + V_t + rxV_x + \tfrac{1}{2}\sigma^2 x^2 V_{xx},\ \psi(x) - V\right\} = 0} \qquad (t,x) \in [0,T) \times (0,\infty).$$

The terminal condition is as follow, ar maturity,

$$V(T,x) = \psi(x) = (K-x)^+.$$

1 b) replicating portfolio. Let X_t be the portfolio that can invest in a the stock and a tisk free rate. So $X_t = (X_t - \Delta_t S_t) + \Delta_t S_t$. If this is self financing then

$$\begin{aligned} dX_t &= \Delta_t dS_t + r(X_t - \Delta_t S_t)dt \\ &= \Delta_t(rS_t\,dt + \sigma S_t\,dW_t) + r(X_t - \Delta_t S_t)dt \\ &= rX_t dt + \sigma S_t \Delta_t dW_t \end{aligned}$$

If this is replicating portfolio then the volatility terms of dX and dV should be the same, therefore

$$\sigma S_t \Delta_t dW_t = \sigma S_t V_x dW_t$$
$$\Delta_t = V_x$$

So the delta hedging strategy is the exact same as for a European option. The main difference is that it only matches for the continuation region as if the option is exercised there is no need to hedge any longer.

1 c) Perpetual American Put Option. The perpetual American put is an American put with no maturity date. It has a closed form solution. first lets define

$$\kappa = \frac{r}{\sigma^2}, \qquad \Delta = \sqrt{\left(\kappa - \tfrac{1}{2}\right)^2 + 2\kappa}, \qquad \beta = \frac{\left(\kappa - \frac{1}{2}\right) + \Delta}{2}.$$

It has the closed form solution.

$$V_\infty(x) = \begin{cases} K - x, & x \le \dfrac{K\beta}{\beta+1}, \\ \dfrac{K}{\beta+1}\left(\dfrac{K\beta}{(\beta+1)x}\right)^\beta, & x > \dfrac{K\beta}{\beta+1}, \end{cases} \tag{3}$$

Since there is no maturity the $V_t = 0$. And our variational PDE becomes

$$\begin{cases} rS_t V_x + \frac{1}{2}\sigma^2 S_t^2 V_{xx} - rV \le 0 \\ (K-x)^+ - V(t,x) \le 0 \end{cases}$$

To verify that the closed form solution is correct we can take the derivatives of it and see if it solves are PDE equation.

$$V'_\infty(x) = \begin{cases} -1, & x \le S^*, \\ -\left(\dfrac{K\beta}{(\beta+1)x}\right)^{\beta+1}, & x > S^*, \end{cases} \qquad V''_\infty(x) = \begin{cases} 0, & x \le S^*, \\ \left(\dfrac{K\beta}{\beta+1}\right)^\beta \dfrac{K\beta}{x^{\beta+2}}, & x > S^*, \end{cases}$$

where $S^* = \frac{K\beta}{(\beta+1)}$.

Substituting into the pde we get,

$$rxV'_\infty + \frac{1}{2}\sigma^2 x^2 V''_\infty - rV_\infty = rx(-1) + \frac{1}{2}\sigma^2 x^2 (0) - r(K-x) = -rK < 0$$

when $x \leq \frac{K\beta}{\beta+1}$ and

$$
\begin{aligned}
rxV_\infty' + \frac{1}{2}\sigma^2 x^2 V_\infty'' - rV_\infty &= rx[-(\frac{K\beta}{(\beta+1)x})^{\beta+1}] + \frac{1}{2}\sigma^2 x^2[(\frac{K\beta}{\beta+1})^\beta \frac{K\beta(\beta+1)}{x^{\beta+2}}] - r[\frac{K\beta}{\beta+1}(\frac{K\beta}{(\beta+1)x})^\beta] \\
&= -\frac{rK\beta}{\beta+1}(\frac{K\beta}{(\beta+1)x})^\beta + \frac{K\beta\sigma^2}{2}(\frac{K\beta}{(\beta+1)x})^\beta - \frac{rK}{\beta+1}(\frac{K\beta}{(\beta+1)x})^\beta \\
&= (\frac{K\beta}{(\beta+1)x})^\beta[\frac{K\beta\sigma^2}{2} - \frac{rK\beta}{\beta+1} - \frac{rK}{\beta+1}] \\
&= (\frac{K\beta}{(\beta+1)x})^\beta \frac{K(\beta\sigma^2 - 2r)}{2} \\
&= 0
\end{aligned}
$$

when $x > \frac{K\beta}{\beta+1}$ Therefore, it satisfies the PDE, so we have verified the closed form solution to the infinite horizon American put option.

I am now going to compute and plot $V_\infty(S)$ and S* for the parameters, $K = 10$, $\sigma = 20\%$ and $r = 2\%$

$$
\beta = \frac{\left(\kappa - \frac{1}{2}\right) + \Delta}{2} = \frac{2r - \sigma^2}{2\sigma^2} + \frac{1}{2\sigma^2}\sqrt{(\sigma^2 - 2r)^2 + 8r\sigma^2} = 1.
$$

$$
S^* = \frac{K\beta}{\beta+1} = 5
$$

$$
V_\infty(x) = \begin{cases} K - x & if\ x \leq \frac{K\beta}{\beta+1} \\ \frac{K\beta}{\beta+1}(\frac{K\beta}{(\beta+1)x})^\beta & if\ x > \frac{K\beta}{\beta+1} \end{cases} = \begin{cases} 10 - x & if\ x \leq 5 \\ \frac{25}{x} & if\ x > 5 \end{cases}
$$

Figure 1 displays the value of the perpetual american put over different stock prices. And Figure 2 shows how β and $\frac{S*}{K}$. It seems to suggest that they both decrease with higher volatility but increase with higher r.

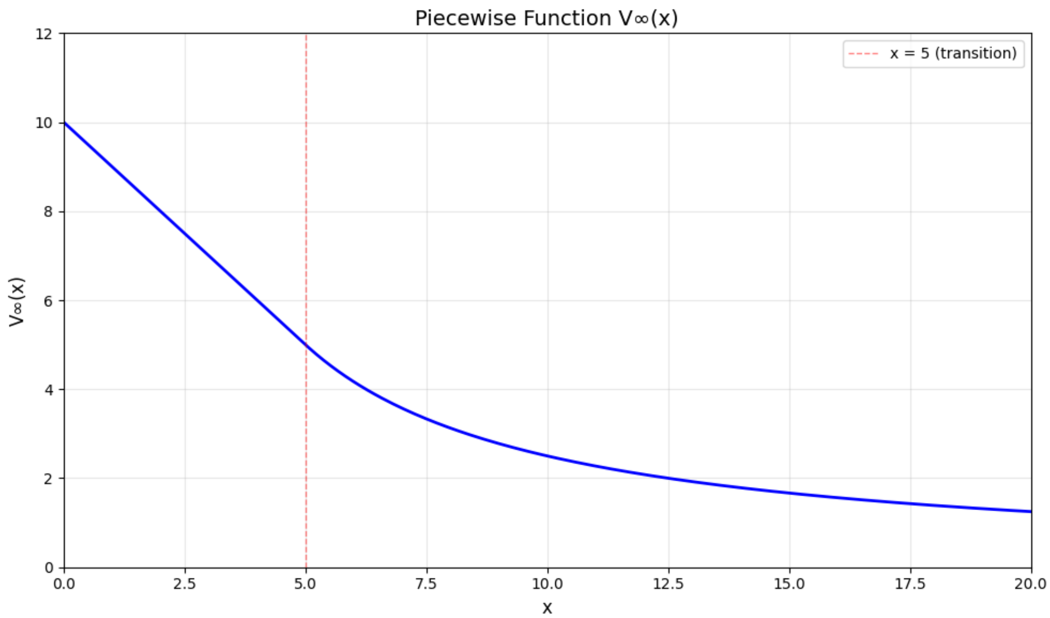

FIGURE 1. $V_\infty(S)$ versus S, and S* $= 5$

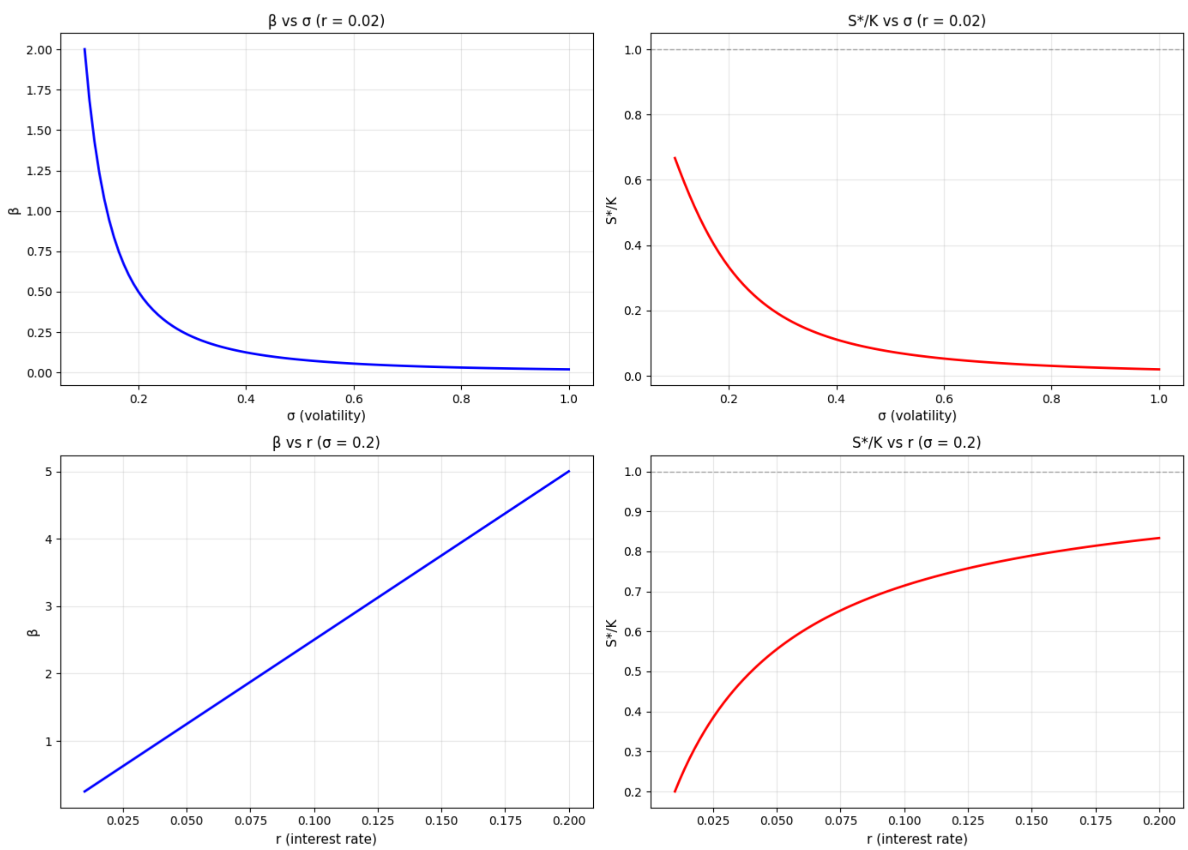

FIGURE 2. β and $\frac{S*}{K}$ with changing r andσ

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Part 2 Binomial (CRR) Model for the American Option

2 a) and b). In this part we constructed a Binomial tree uner the CRR model using the parameters,

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}, \quad p = \frac{e^{r\Delta t} - d}{u - d}.$$

At each node $(t_j, S_{j,i})$: the value of the american put option is

$$V_{j,i} = \max\left\{(K - S_{j,i})^+, e^{-r\Delta t}\left[pV_{j+1,i+1} + (1-p)V_{j+1,i}\right]\right\}.$$

We calculated $V(0, S_0)$ for the parameters $T = 1, S_0 = 10, K = 10, r = 0.02, \sigma = 0.2$. We calculated trees with the number of steps increasing, where $M_{bin} = [10, 50, 100, 200, 400, 800, 1600]$. The result of this graph was the convergence of the initial price of the option as the number of steps increase. This can be seen in Figure 3. The exercised boundary was also tracked, which is the lowest stock price at a given time where $(k - S^+_{i,j} \geq V_{i,j}$. The decision boundary can be seen in Figure 3, where underneath is the exercise region and above is the continuation region.

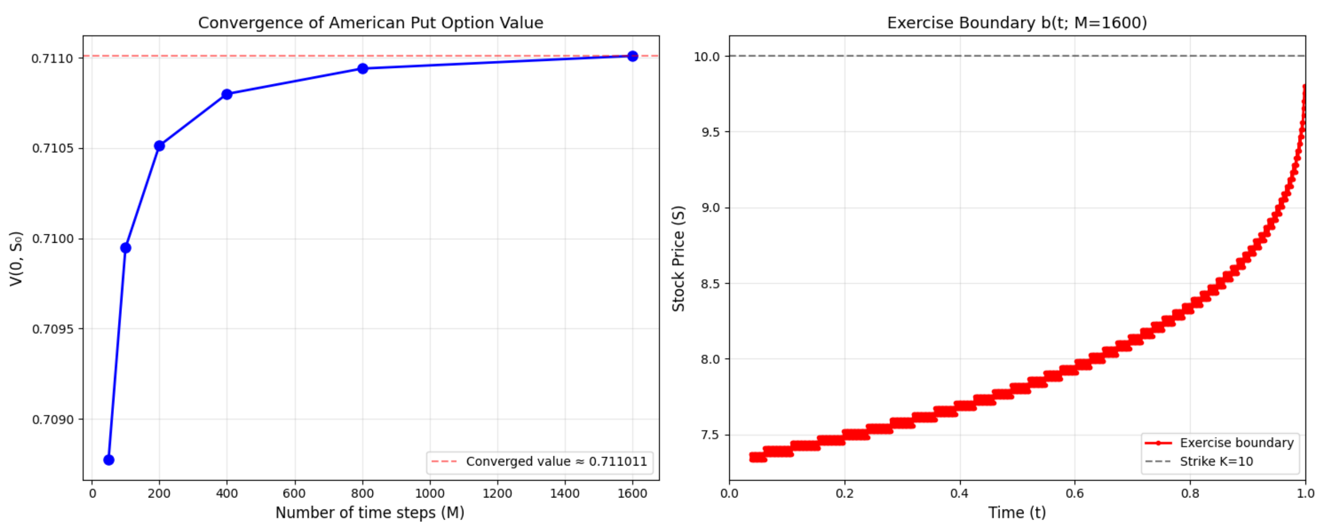

Figure 3. Price Convergence and Exersize Boundary

2 c) Generated Paths. Figure 4 shows the two generated stock paths where path 1 doesnt ever cross the exercize boundary so it would have the chance to be exercised at maturity. Path 2 crosses the exercise boundary at around tie step 900, so the option would be exercised there. I tried to stop the path at that point but was unable to visualize this.

2 d) Delta Hedging Value Over Time. Figure 5 shows the delta hedging amount for both stocks over time. When Stock two is exercized it is no longer needed to be hedge so the path stops.

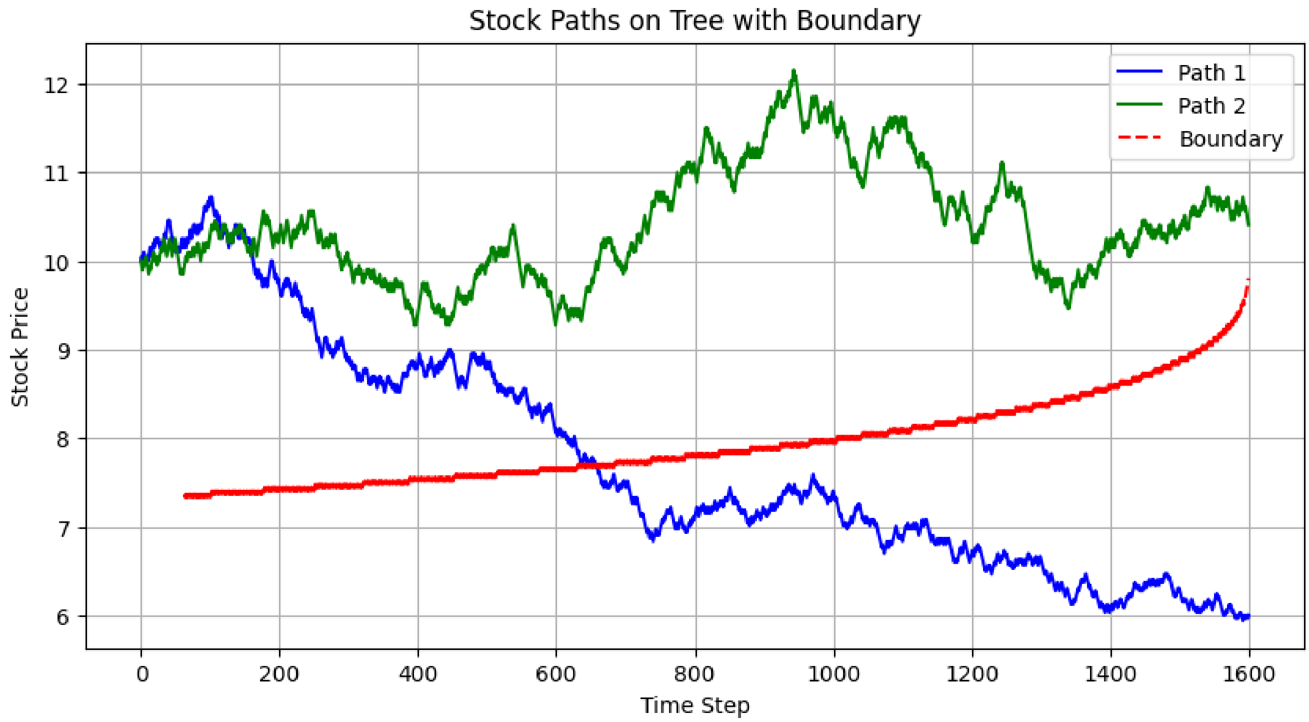

FIGURE 4. Generated Paths

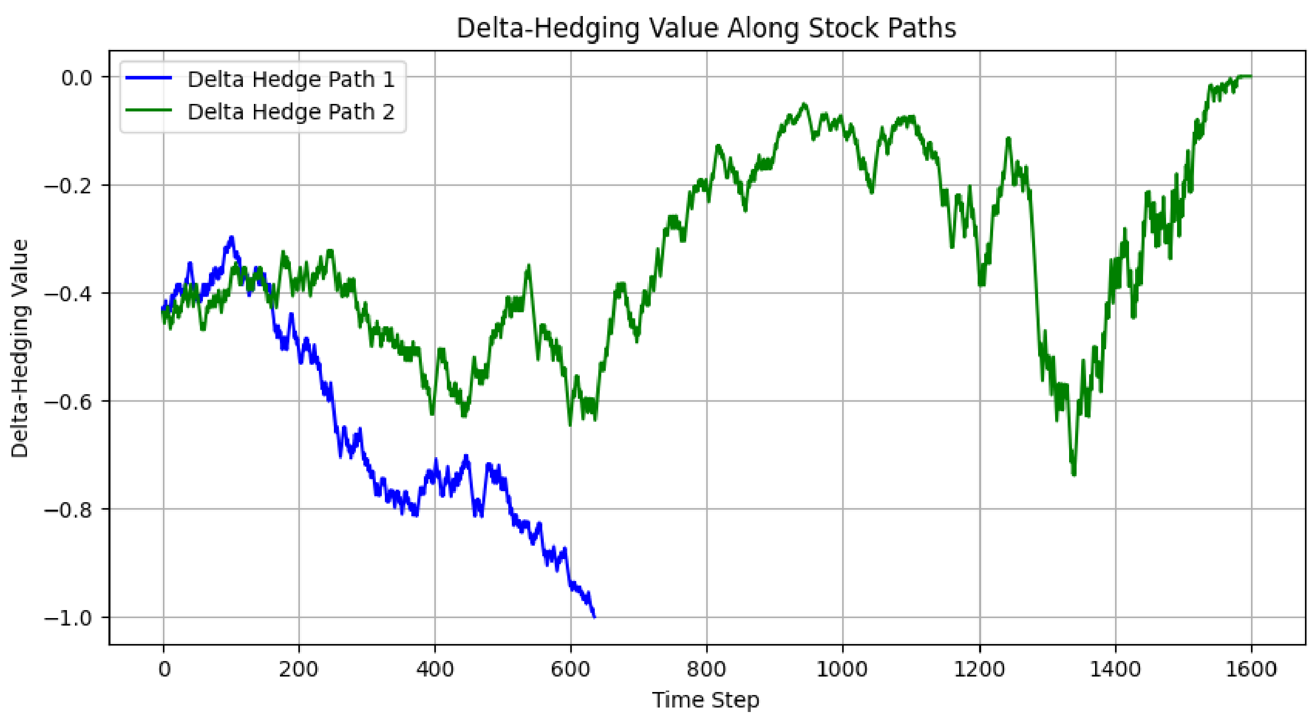

FIGURE 5. Delta Hedging Path for Two Stocks

2 e). Th decision boundary was recalculated for different volatility and returns. Each combination over volatilites of 10, 20, 30 percent and returns from 0 2 and 4 percent, were tried. the exercise

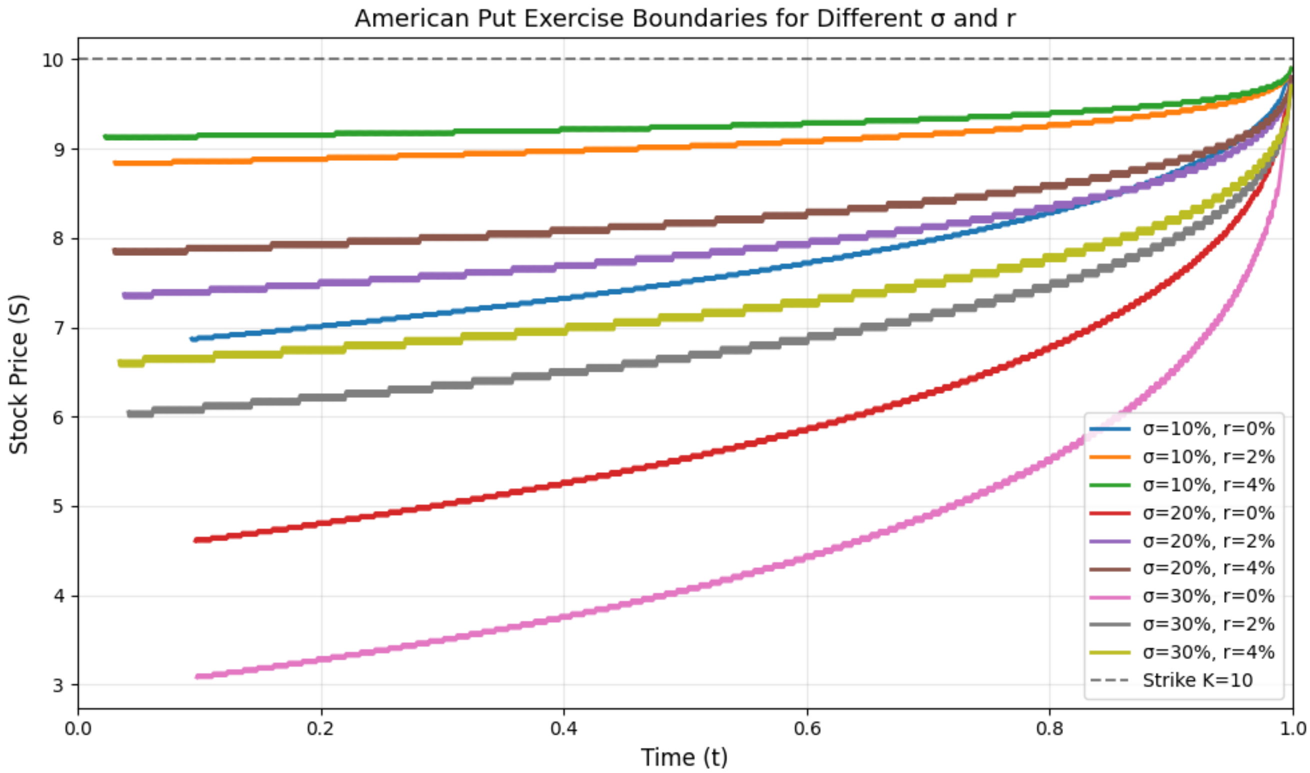

FIGURE 6. Exercise Boundaries for Different Volatilitys and Returns

boundaries can be seen in Figure 6. From the graph when σ is held constant, as the return increases the exercise boundary moves higher. When the Return is held constant we see that as volatility increases the exercise boundary drops lower.

Volatility and the risk-free rate have a direct impact on the delta of an option and, consequently, on the behavior of a delta-hedging strategy. Delta measures the sensitivity of the option's value to changes in the underlying stock price, and a higher volatility increases the likelihood of large stock price movements. This means that the option's delta fluctuates more rapidly, requiring more frequent adjustments to the hedge to maintain a neutral position. Similarly, the risk-free rate affects the present value of future payoffs and the probability weighting of upward versus downward movements in a binomial tree. A higher r increases the expected growth of the underlying, which generally shifts the delta of put options lower at a given stock price and time. They also affect the exercise boundary, with a higher return and lower volatility the boundary is much higher, so it is more likely to be exercised. Meaning that the option will needed to be hedge for a shorter amount of time.

2 f). The Price and Decision boundary were found for increasing time horizons where with T : $[10, 20, 50, 100]$. I was going to try T = 1000, but it made the graph of the other decision boundaries less visible. The number of steps had to be set to 5000 to accurately depict the decision boundaries. This can be seen in Figure 7, where it seems the price does approach a value of approximately 2.47 and the decision boundaries seem to be getting close to a constant decision boundary of 5, which is the optimal decision boundary for the perpetual put option.

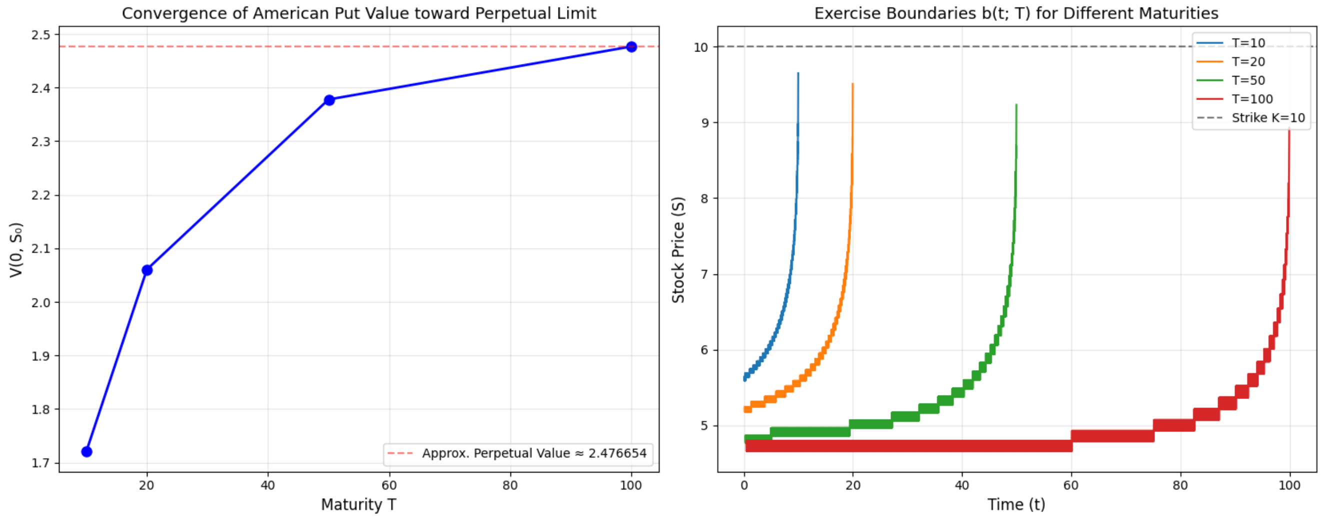

FIGURE 7. Decision Boundary and Prices for Increasing Time Horizons.

PART 3 PROFIT AND LOSS DISTRIBUTION AND STOPPING TIME DISTRIBUTION

3 a) Simulation of 10000 Paths. In this section it is assumed that an American put option was bought at time zero for its fair price being approximately 0.711. Then I Simulated 10000 stock paths. I used the same initial assumptions from part 2. The Profit was calculated as the discounted profit from the put option minus the original put option price. The PnL distribution can be and stopping time distribution can be seen in Figure 8. The most common value is -0.711, which occurs when the option reaches maturity above the strike price. This means the value from the option is 0 leaving you with just the negative price you paid for the option. The rest of the distribution is pretty evenly distributed over a range of values, with a lot being profitable. The most common value for stopping time is maturity or time 1. The other values are distributed increasingly over time. with early stopping times being the least common. Overall the probability of stopping was 47.26%.

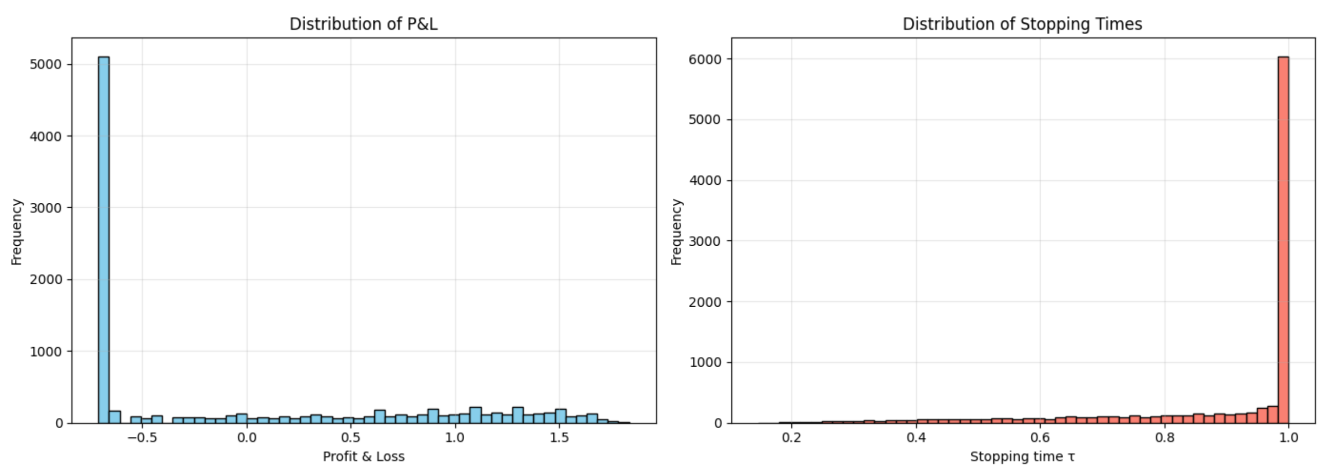

FIGURE 8. Profit and Loss Distribution and Stopping Time Distribution

3 b) Differing σ and r. The PnL distributions, stopping tim distributions and percentage of early stops was calculated for different volatilitys and returns. The results can be seen in Table 1. It seems the Mean of the profit and loss distribution is always around zero. This makes sense as if the fair price is calculated correctly the expected profit and loss should be zero. The Mean of the stopping time distributions does change with differing volatility and return. It seems as though raising the volatility raises the mean of the stopping time. Conversely it seems raising the return lowers the mean of the stopping time. This is exactly what is expected as we seen from part two how the exercise boundary changes with differing volatility and return. Finally the probability of exercising early all seem to be around 50%. However the seems to be a little trend of the probability increasing with increasing return and it decreases with increasing volatility.

σ	r	Mean P&L	Mean τ	p_{early}
10%	0%	-0.0249	0.975	0.460
10%	2%	0.0491	0.845	0.495
10%	4%	0.0953	0.764	0.538
20%	0%	-0.0768	0.976	0.475
20%	2%	0.0043	0.888	0.486
20%	4%	0.0646	0.842	0.5
30%	0%	-0.177	0.976	0.473
30%	2%	-0.0923	0.904	0.473
30%	4%	0.0024	0.869	0.489

TABLE 1. Simulation results for American put option (fill in your values).

3 c) Realized vs Actual Volatility. Suppose the actual volatility differs from that used to calculate the fair price of a American put option, as well as the decision boundary. We Found the Profit and loss distributions for this scenario over 10,000 simulations. Using realized volatilitys of $\sigma 10\%, 15\%, 20\%, 25\%, 30\%$. Figure 9 sows the mean profit and loss, mean stopping time and the probability of exercise against the different realized volatilitys. It seems the mean PnL increases with the volatility. So if the realized volatility is less then that used to price the option then one should expect a loss. But if the volatility is higher then that used to price then you should expect a profit. Figure 10 shows the distribution of the lowest and highest volatilizes. What we can see is that for the low volatility, many more of the stock paths ended above the strike price whereas the high volatility is spread much more out with lots more profit and less paths ending above the strike price. Moving on to mean stopping time, it seems as tho the mean stopping time decreases with higher volatility. This makes sense as a higher volatility would make it more likely to cross the exercise boundary. This is also confirmed by the graph of th exercise probability increasing with the volatility.

In conclusion,

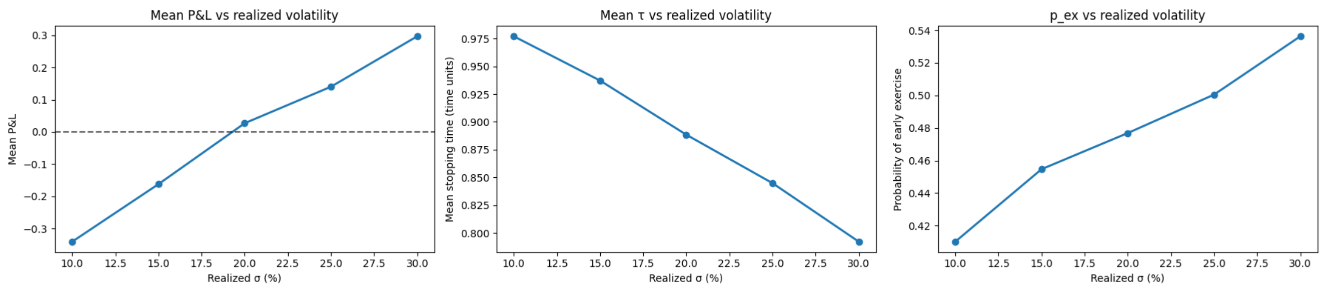

Figure 9

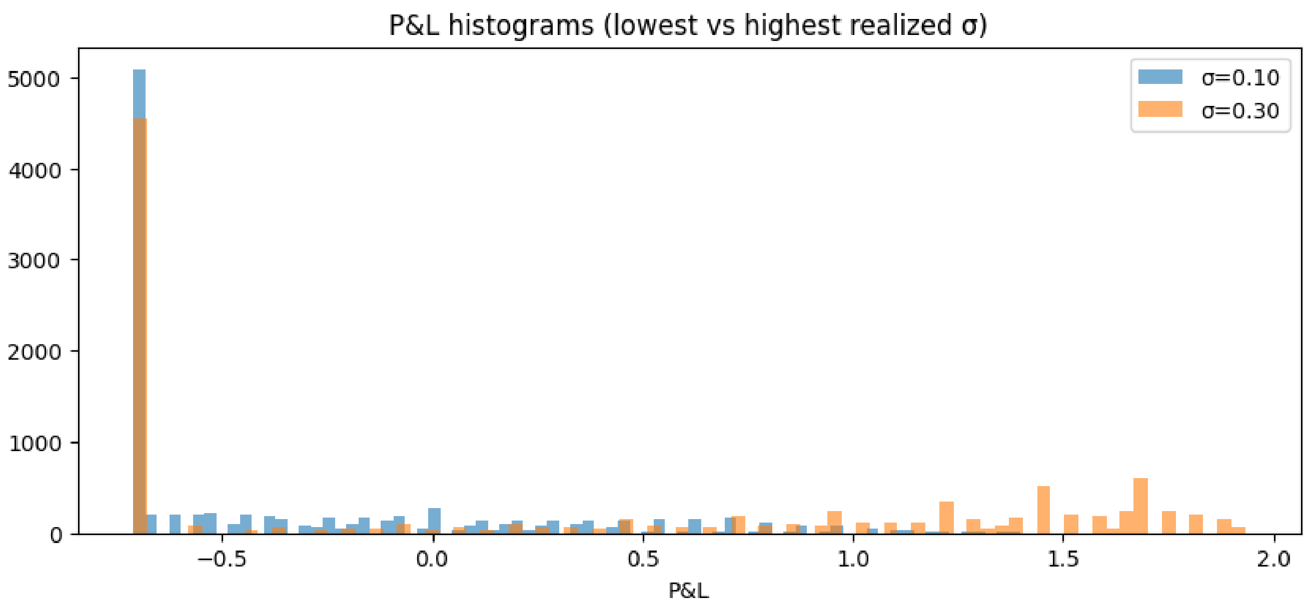

Figure 10