# LAB REPORT: LAB 5

TNM079, MODELING AND ANIMATION

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**Abstract**

This lab report describes the tasks and results from lab 5 of the course TNM079, Modeling and Animation at Linköping University. The lab covered level sets and how they can be used in time to change a shape of an implicit surface by using vector or scalar fields. It also covered different types of equations which require discretization in different ways to achieve various operations. The operations covered reach from smoothing the surface to morphological changes of the surface. Furthermore, a way to optimize those operations performance-wise by applying a narrow band scheme was discussed.

## 1 Background

A level set function should be as close as possible to a *signed distance function*, which is a function where each point describes the *closest distance* to the surface. A level set implicitly represents an interface

$$S = \{x \in \Re^d : \phi(x) = h\} \tag{1}$$

where $S$ is the levels set, $\phi$ the level set function and $h$ the value which represents the surface.

The sign of the *sigend distance function* determines whether a set of points is *inside* or *outside* of the surface

$$S_{inside} = \{x \in \Re^d : \phi(x) < h\}; S_{outside} = \{x \in \Re^d : \phi(x) > h\}. \tag{2}$$

The normal and curvature of a level set is given by

$$\hat{n} = \frac{\nabla\phi}{||\nabla\phi||}; K = \nabla \times n. \tag{3}$$

To change the level set over time, a time parameter, $t$, is being introduced in the level set function:

$$S(t) = \{x \in \Re^d : \phi(x, t) = h\} \tag{4}$$

In order to move the level set, let the point $\alpha(t)$ on $S$ ($\alpha(t) = h$) for all $t$. By differentiating with respect to the time $t$ the following is obtained:

$$\begin{aligned} \frac{\partial\phi}{\partial t} &= -\nabla\phi \times \frac{d\alpha}{dt} && (5)\\ &= -F|\nabla\phi| && (6) \end{aligned}$$

where F refers to the level set's speed function and describes the speed of $\alpha$ in normal direction

$$F = n * \frac{d\alpha}{dt} = \frac{\nabla\phi}{|\nabla\phi|} \times \frac{d\alpha}{dt}. \tag{7}$$

To be able to use those equations on a computer, the functions need to be discretized both in the *temporal* and *spatial* domain. The temporal discretization (determines how equation 5 and 6 evolve over time) is done with a *forward Euler* scheme. Alternatively, the *total variation diminishing Runge-Kutta* can be used for more accuracy.

The spatial discretization limits the domain of the function $\phi$ to discrete points on a unifrom grid with the spacing $\Delta x$. The notation $\phi_{i,j,k}$ is meaning the value of $\phi$ at the position (i,j,k) on the grid. The *spacial discretization* also strongly depends on the particular PDE which is being used. There are *hyperbolic* and *parabolic* PDEs.

In order to solve *hyperbolic advection* equations 5 and 6 can be rewritten as

$$\frac{\partial\phi}{\partial t} = -V \times \frac{d\alpha}{dt} \tag{8}$$

$$= -F|\nabla\phi| \tag{9}$$

which describes the advection of a surface along a *vector field V* or in the normal directions of the surface. Advection in the normal direction can for example be used to dilate or erode an object by setting $F$ equal to a constant that is smaller (erosion) or larger (dilation) than the isovalue zero.

The calculation of the gradient requires a discretization of the partial spatial derivatives. Due to the fact that the flow direction of the vector field is known, an upwind scheme like following can be used.

$$\frac{\partial\phi}{\partial x} \approx \begin{cases} \phi_x^+ = (\phi_{i+1,j,k} - \phi_{i,j,k})/\Delta x & \text{if } V_x < 0 \\ \phi_x^- = (\phi_{i,j,k} - \phi_{i-1,j,k})/\Delta x & \text{if } V_x > 0 \end{cases} \tag{10}$$

For formulation 9, the direction of the flow is not explicitly known. To calculate this for x the *Godunov's method* is being used:

$$(\frac{\partial\phi}{\partial x})^2 \approx \begin{cases} max[max(\phi_x^-, 0)^2, min(\phi_x^+, 0)^2] & if F > 0 \\ max[min(\phi_x^-, 0)^2, max(\phi_x^+, 0)^2] & if F < 0 \end{cases} \tag{11}$$

The derivatives for y and z can be calculated in the same way.

To ensure stability the CFL condition which is relating to the time step $\Delta t$ is being used. For the vector field formulation of equation 8 the conditions for the lab are given by

$$\Delta t < \frac{\Delta x}{max(|V_x|, |V_y|, |V_z|)} \tag{12}$$

and for equation 9 by

$$\Delta t < \frac{\Delta x}{|F|}. \tag{13}$$

Level set operations also allow smoothing surfaces with *parabolic diffusion* which gives us a new speed function in equation 6 for F:

$$F = -\alpha k \tag{14}$$

where $\alpha$ is a scaling parameter which defines the intensity of the smoothing, and k represents the curvature.

Since the information flow has not a specified direction, a *second-order accurate central difference scheme*

$$\frac{\partial \phi}{\partial x} \approx \pi_x^{\pm} = \frac{\phi_{i+1,j,k} - \phi_{i-1,j,k}}{2\Delta x} \tag{15}$$

is needed. By using the notation $\phi_x = \partial\phi/\partial x$ and $\phi_{xy} = \partial^2\phi/\partial x \partial y$, the mean curvature can be calculated by

$$K = \frac{1}{2}\nabla \times \frac{\nabla\phi}{|\nabla\phi|} = \frac{\phi_x^2(\phi_{yy} + \phi_{zz}) - 2\phi_y\phi_z\phi_{yz}}{2(\phi_x^2 + \phi_y^2 + \phi_z^2)^{\frac{3}{2}}} + \frac{\phi_y^2(\phi_{xx} + \phi_{zz}) - 2\phi_x\phi_z\phi_{xz}}{2(\phi_x^2 + \phi_y^2 + \phi_z^2)^{\frac{3}{2}}} + \frac{\phi_z^2(\phi_{xx} + \phi_{yy}) - 2\phi_x\phi_y\phi_{xy}}{2(\phi_x^2 + \phi_y^2 + \phi_z^2)^{\frac{3}{2}}} \tag{16}$$

To discretize this expression a second order central difference scheme is used:

$$\frac{\partial^2 \phi}{\partial^2 x} \approx \frac{\phi_{i+1,j,k} - 2\phi_{i,j,k} + \phi_{i-1,j,k}}{\Delta_{x^2}} \tag{17}$$

$$\frac{\partial^2 \phi}{\partial x} \approx \frac{\phi_{i+1,j+1,k} - \phi_{i+1,j-1,k} + \phi_{i-1,j-1,k} + \phi_{i-1,j+1,k}}{4\Delta_{xy}} \tag{18}$$

Derivatives for the other dimensions are calculated analogously. The time step restriction for this scheme is

$$\Delta t < \frac{\Delta x^2}{6\alpha} \tag{19}$$

In order to assure stability of a level set function, the function should be a signed distance function. For that, it has to approximately satisfy the *Eikonal equation*

$$|\nabla\phi| = 1. \tag{20}$$

This is needed, because the size of the gradient can change over time when modifying the level set. Hence, there is a need for an action called *reinitialization* to restore the signed distance property. Reinitialization goes along with the equation

$$\frac{\partial \phi}{\partial t} = S(\phi)(1 - |\nabla\phi|) \tag{21}$$

where $S(\phi)$ is the sign of $\phi$ with a smooth approximation given by

$$S(\phi) = \frac{\phi}{\sqrt{\phi^2 + |\nabla\phi|^2 \Delta x^2}} \tag{22}$$

Equation 21 i a hyperbolic PDE which needs to be discretized by an up-wind scheme.

In general, solving $\phi$ for the entire domain is expensive and unnecessary since only the surface and its immediate surroundings are of interest. For this, a *narrow band scheme* can be formed. The idea is to define two narrow band tubes around the surface

$$T_{ß=\{x\in\Re^d:|\phi(x)|<ß\};\, T_\gamma=\{x\in\Re^d:|\phi(x)|<\gamma\}} \tag{23}$$

where $0 < ß < \gamma$. The level set equations are then only being solve within these tubes. To avoid *numerical oscillations* at the tube's boundary, a cut-off function is used for smoother transition between the tubes:

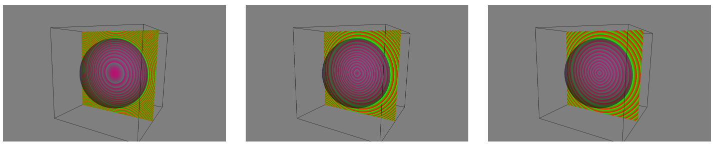
$$c(\phi) = \begin{cases} 1 & if|\phi| \leq ß \\ \frac{(2|\phi|+\gamma-3ß)(|\phi|-\gamma)^2}{(\gamma-ß)^3} & ifß < |\phi| \leq \gamma 0 \\ if|\phi| > \gamma \end{cases} \tag{24}$$

# 2 Results

The results of the lab implementations are presented in following sub chapters.

## 2.1 The signed distance property

In figure 1 the reinitialization operator is visualized. At the beginning the functions deviates from the signed distance property. After 40 iterations the reinitialization corrects the deviation as expected. Reinitialization should be run frequently to ensure this property.

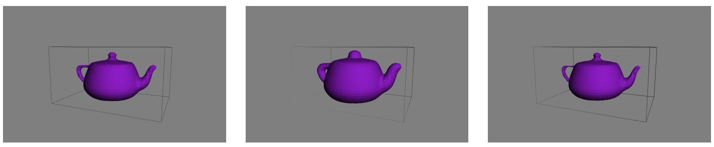


(a) Starting point (b) After 40 iterations (c) After 80 iterations

*Figure 1:* Demonstration of the reinitializaion operator on a grid with spacing 0.1 and repeating color scale with period of 0.1.

## 2.2 Erosion and dilation

A demonstration of the erosion and dilation operators can be seen in figure 2.



(a) Object befor dialation and erode (b) After dialation (c) After erode

*Figure 2:* Demonstration of the dilation and erosion operator.

## 2.3 Surface Advection

The result of the surface advection along a vector field is demonstrated in figure 3. Depending on the vector field different outcomes can be generated.

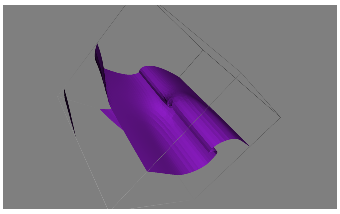


*Figure 3:* Surface advection on a level set plane

### 2.4 Narrow band optimization

In table 1 the computation times of the dilation operation on different grid sizes/resolutions with and without narrow band optimization can be seen. The narrow band width was set to the standard value of 16. The performance without optimization grows very quickly along the increased spatial resolution due to the application of the operation onto the whole grid domain (cubic complexity). The overall performance with the narrow band optimization has a more steady run-time increase because the operation is only applied to the level set surface and some space around it. Therefor, the run-time increases with the size of the surface and not with the growth of the gird domain.

| Narrworw band performance | | |
|---|---|---|
| $\Delta x$ | Time without narrow band opt. | Time with narrow band opt. |
| 0.05 | 0.013 | 0.005 |
| 0.04 | 0.025 | 0.007 |
| 0.03 | 0.059 | 0.014 |
| 0.02 | 0.195 | 0.033 |
| 0.01 | 1.583 | 0.167 |

*Table 1:* Results of the narrow band optimization for the dilation operator on a sphere with different grid sizes/resolution.

## 3 Conclusion

Level set representations of surfaces allow different ways to change a surface by applying scalar fields as well as vector fields. Such fields can be used to perform morphological operations and surface smoothing.

However, the overall performance of these operations can be incredible poor in higher spatial resolutions. Applying a narrow band optimization opposes this deficit by shrinking the domain the operations are applied on.

## 4 Lab partner and grade

The lab and all tasks were solved together with William Toft(wilto938). This report aims for grade 5.