LAB REPORT: LAB 4

TNM079, MODELING AND ANIMATION

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Abstract

This lab report describes the tasks and results from lab 4 of the course TNM079, Modeling and Animation at Linköping University. The lab covered implicit surfaces and their differential properties including approximations of their gradients/normals and curvature. In more detail, different types of implicit quadric surfaces and how to easily implement them were shown. Based on the properties of those surfaces, constructive solid geometry techniques such as the union, intersection, and difference operators can be implemented to create more complex objects.

1 Background

Implicit surfaces are surface which are not defined as a explicit triangle mesh but represented in the form of a mathematical function which has to be solved in order to find the surface geometry. Given a real valued function

$$f(x) \rightarrow \eta, x \in \Re^n, \eta \in \Re \tag{1}$$

a implicit surface can be defined by defining the surface in a way that all points result into the same value for f:

$$S(C) \equiv \{\{x\} : f(x) = C\} \tag{2}$$

where C is the iso-value. Usually this value is set to zero, so that we can use the result of f to determine if a point is inside or outside of the surface. This results in following classification

$$f(x) \begin{cases} < C, & \text{x is } \textbf{inside} \text{ of the surface} \\ > C, & \text{x is } \textbf{outside} \text{ of the surface} \\ = C, & \text{x is } \textbf{on} \text{ the surface} \end{cases} \tag{3}$$

which separates the scalar field into an inside and outside region and therefore the function represent an interface. This function or interface is then sampled in a grid to render the implicit surface.

For overall modelling, the surface normals are of high importance. The surface normal of a implicit surface is the normalized gradient

$$\nabla f(x) = [\frac{\partial f(x)}{\partial x}, \frac{\partial f(x)}{\partial y}, \frac{\partial f(x)}{\partial z}]^T. \tag{4}$$

Since the surface normal and the gradient are parallel to each other, we can use this property to derive also an expression for the *mean curvature* K

$$K = \frac{1}{2}\nabla \times \vec{n} = \frac{1}{2}(\frac{\partial n_1}{\partial x} + \frac{\partial n_2}{\partial y} + \frac{\partial n_3}{\partial z}). \tag{5}$$

Both equations consist of partial derivatives. With respect to x at point x_0 it is defined for the gradient or normal as

$$\frac{\partial f(x)}{\partial x} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}. \tag{6}$$

When calculating, h can not be arbitrarily small due to the limited precision of the computer's floating point. However, it is possible to approximate the derivative by a sufficiently small value ε instead of using $h \to 0$. Unfortunately, this equation would be asymmetric and gives more weight towards values in the positive x direction. Hence, it is better to evaluate the derivative by using the central difference approximation:

$$D_x(x_0) \approx \frac{f(x_0 + \varepsilon) - f(x_0 - \varepsilon)}{2\varepsilon} \tag{7}$$

For the curvature we need to discretize equation 5. We can also simplify the equation by using a simpler approximation

$$K \approx \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{8}$$

where the second partial derivatives can be calculated as following:

$$\frac{\partial^2 f}{\partial x^2} = D_{xx}(x_0) \approx \frac{f(x_0 + \varepsilon) - 2f(x_0) + f(x_0 - \varepsilon)}{\varepsilon^2} \tag{9}$$

Equation 7 and 9 are identical for the other dimensions y and z.

One type of implicit surfaces are implicit quadric surfaces which are defined by following function:

$$\begin{aligned} f(x,y,z) = {} & Ax^2 + 2Bxy + 2Cxz \\ & +2Dx + Ey^2 + 2Fyz \\ & +2Gy + Hz^2 + 2Iz \\ & +J \end{aligned} \tag{10}$$

It can also be written in matrix form:

$$p^T Q p = \begin{bmatrix} x & y & z & 1 \end{bmatrix} \begin{bmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \tag{11}$$

The quadric is quite flexible and can be used to describe different shapes by assigning values of Q. In the lab it is necessary to implement following surfaces in a quadric representation:

- Planes

$$f(x,y,z) = ax + by + cz = 0 \tag{12}$$

- Cylinders

$$f(x,y,z) = x^2 + y^2 - 1 = 0 \tag{13}$$

- Ellipsoids

$$f(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \tag{14}$$

- Cones

$$f(x,y,z) = x^2 + y^2 - z^2 = 0 \tag{15}$$

- Paraboloids

$$f(x,y,z) = x^2 \pm y^2 - z = 0 \tag{16}$$

- Hyperboloids

$$f(x,y,z) = x^2 + y^2 - z^2 \pm 1 = 0 \tag{17}$$

In order to map these representations to Q, it is necessary to compare them with equation 10. As an example the representation for the cylinder (equation 13) looks like following:

$$Q_{\text{cylinder}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{18}$$

When setting A = 1, E = 1, and J = -1 equation 13 is received.

The gradient of a quadric surface can be written as

$$\nabla f(x,y,z) = 2Q_{\text{sub}}P \tag{19}$$

where Q_{sub} is the first three rows of the quadric matrix Q.

Implicit surfaces can also be used to create complex objects ans shapes by combining simpler objects, such as spheres and cubes. In order to achieve this, Boolean operations like union, intersection, and difference between implicit surfaces are used.

$$f(x)_{A\cup B} = min(f_A(x), f_b(x)) \tag{20}$$

$$f(x)_{A\cap B} = max(f_A(x), f_b(x)) \tag{21}$$

$$f(x)_{A-B} = max(f_A(x), -f_b(x)) \tag{22}$$

To make it more feasible, as an example by taking the maximum of f_A and f_B, the result is negative only at points where both functions are negative which is inside of both surfaces. The result turns positive for points where at least one of the functions are positive. The surfaces are defined as interfaces between these regions, hence the result at the location where they meet, the value is zero.

2 Results

The results of the lab implementations are presented in following sub chapters.

2.1 CSG Operators

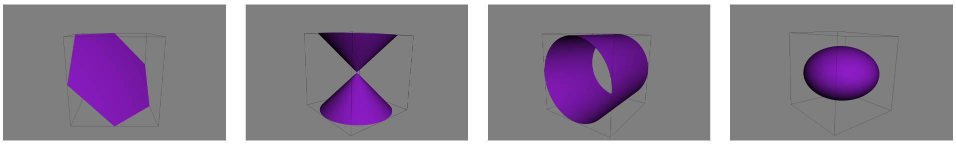
The results of the three CSG operators can be found in figure 1.

(a) Seperate sphere objects (b) Union operation (c) Intersection operation (d) Difference operation

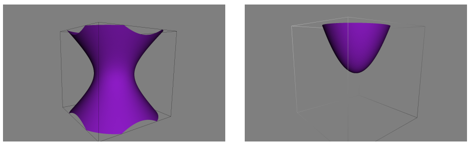
Figure 1: Results of the CSG operations on two seperate spheres

2.2 Quadric surfaces

The different quadric surfaces can be found in figure 2. Note, that the rendered surfaces can differ for Paraboloids and Hyperboloids depending on the chosen sign in equation 16 and 17.



(a) Plane (b) Cone (c) Cylinder (d) Ellipsoid



(e) Hyperboloid (f) Paraboloid

Figure 2: Representation of the different implemented quadric surfaces.

2.3 Discrete gradient operator

The result of the discrete gradient operator implementation can be found in figure 3.

2.4 Discrete curvature operator

In figure 4 the curvature approximation for different ε can be found. The range was chosen close to the automatic set values.

(a) $\varepsilon = 0.01$ (b) $\varepsilon = 1$

Figure 3: Results of the gradient operator with different values for ε.

(a) $\varepsilon = 0.01$ (b) $\varepsilon = 0.50$

Figure 4: Results of the curvature operator using HSV colormap with range of [-6, 250] with different values for ε.

3 Conclusion

Looking at the results of implicit functions and surfaces special properties can be observed. They are rather computational fast and can be computed to an infinite resolution without the need of creating a complex mesh. However, the calculated surface still needs to be triangulated before the¡ are able to be rendered which, at the end, also results in a mesh but without the need of creating one manually.

With CSG operators it is possible to create more complex objects with simple implicit objects/surfaces. The problem here is that sharp edges can occur which then need to be smoothed either manually or by an algorithm.

With the help of quadric surfaces a variety of surfaces can be created. They also provide unique properties which allows an easy calculation of the gradient(normal) and surface curvature.

The gradient, also called discrete gradient operator, can be varied with the selection of ε. Smaller values overall result in better approximation which corresponds to the initial equation 6 where h should be arbitrarily small.

For the discrete curvature operator, the same rule applies. The smaller ε is, the better the results of the curvature approximation. This can be especially seen in figure 4, where $\varepsilon = 0.01$ produces a good approximation of the curvature and also at the generated sharp edges. Once ε is increased the local approximation gets worse and even the sharp edges are displayed as curved.

4 Lab partner and grade

The lab and all tasks were solved together with William Toft(wilto938). This report aims for grade 5.