QWORLD

SOLVING MAX CUT VIA GROVER

Jibran Rashid

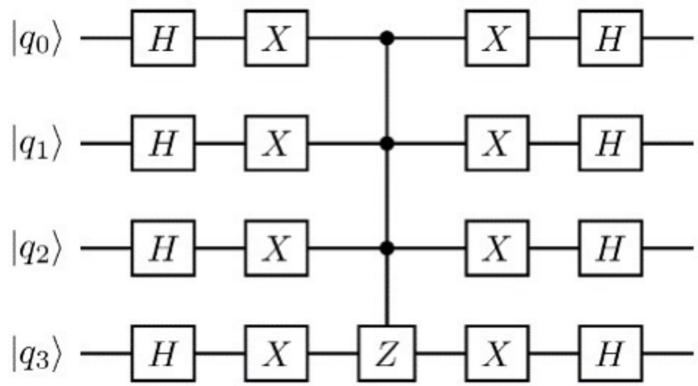
Implementing the Diffusion Operator

$$\underbrace{\mathbb{1} - 2|h\rangle\langle h|}$$

$$H^{\otimes n}\left(\mathbb{1} - 2|0^n\rangle\langle 0^n|\right)H^{\otimes n}$$

$$Z_0 = \mathbb{1} - 2|0^n\rangle\langle 0^n|$$

$$Z_0|x\rangle = \begin{cases} -|x\rangle & \text{if } x = 0^n \\ |x\rangle & \text{o/w} \end{cases}$$



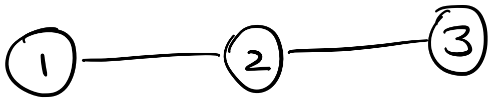
$$\underbrace{}_{\mathbb{1} - 2|0^n\rangle\langle 0^n|}$$

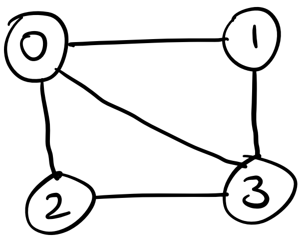
$$Z|0\rangle = |0\rangle$$

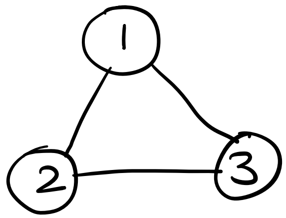
$$Z|1\rangle = -|1\rangle$$

Undirected Graph

A pair $G = (V, E)$ where $V = \{v_1, \ldots, v_n\}$ are the vertices & E contains unordered pairs of vertices, i.e., edges. $E = \{(v_i, v_j) \mid v_i, v_j \in V\}$







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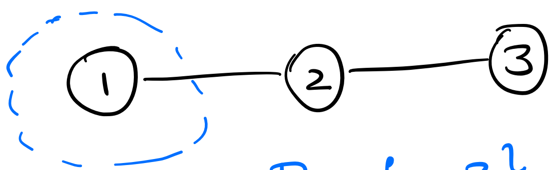
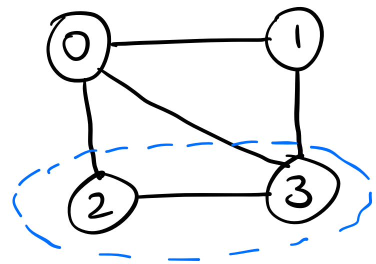
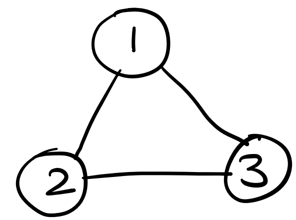


$S = \{1\}$ $R = \{2, 3\}$



$S = \{0, 1\}$, $R = \{2, 3\}$



Cut

A partition of the vertices into two non-empty sets S & $R = V - S$

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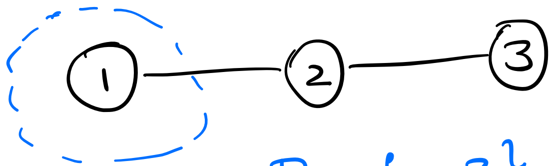

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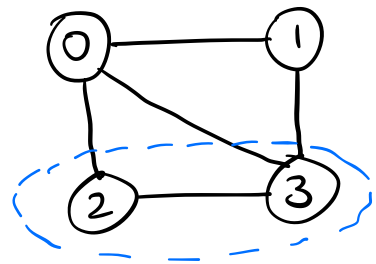

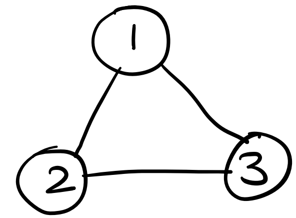

$S = \{0, 1\}$, $R = \{2, 3\}$, size of cut $= 3$

Cut

A partition of the vertices into two non-empty sets S & $R = V - S$

Size of a Cut

Number of edges b/w S & R.

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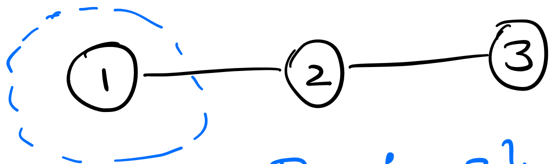

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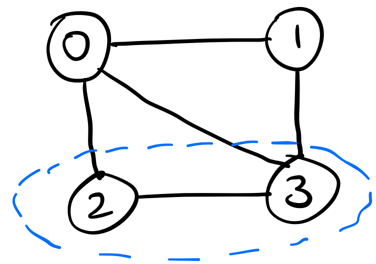

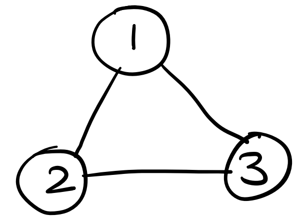

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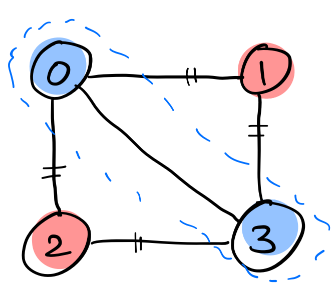
Cut

A partition of the vertices into two non-empty sets S & $R = V - S$

Size of a Cut

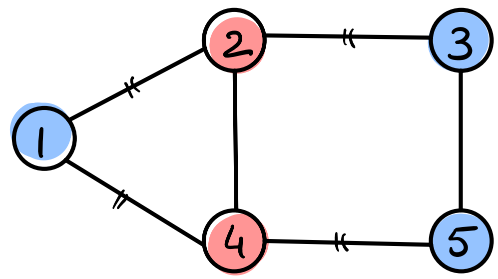
Number of edges b/w S & R.

Max-Cut — Find partitions that maximize the number of edges across S & R.



size of cut $= 4$

Max-Cut Example



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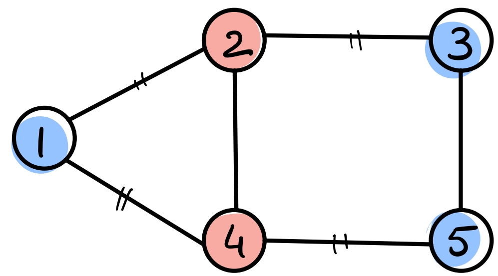
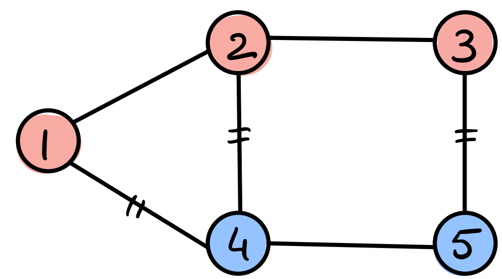
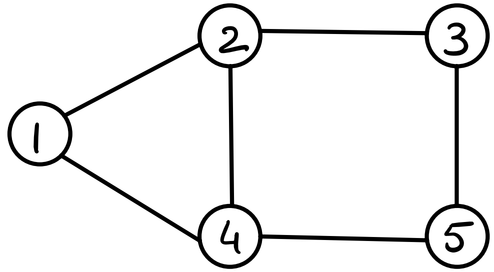
Max-Cut Example

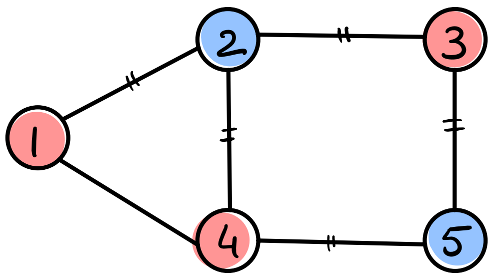




size of cut = 3

Max-Cut Example





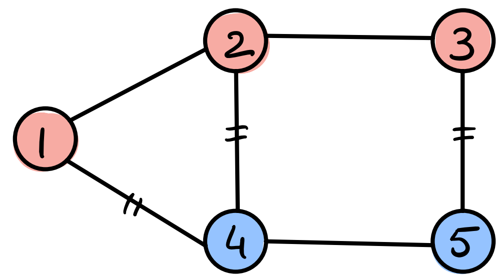
size of cut = 5

$R = \{1, 3, 4\}$

$S = \{2, 5\}$

$f(1) = f(3) = f(4) = 0$

$f(2) = f(5) = 1$



Decision Problem

Given G, does there exist a cut of size atleast k.

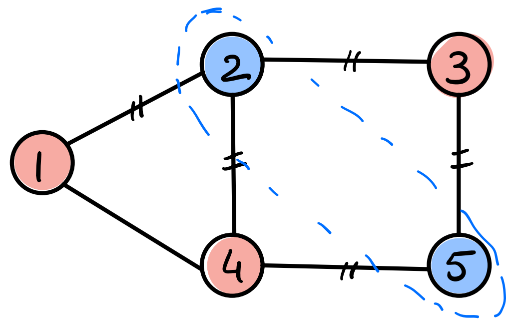
- Decision vs. Optimization — Logarithmic overhead from Binary Search.
- Problem is known to be NP-Complete.

  2^n possible partitions / colorings.

  - Generic Quantum speedup via Grover to get $O(\sqrt{2^n})$.

Note: Min-Cut $\in P$ via maxflow algorithm

Max-Cut Example



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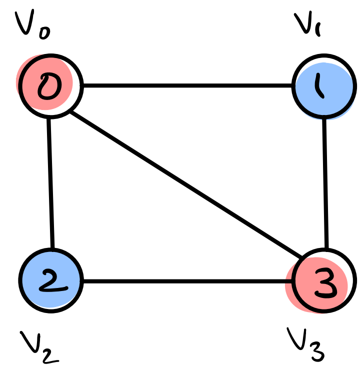
How do we construct the oracle?

Representing Vertices

n vertices $\longrightarrow$ n qubits

$|0\rangle \longrightarrow$ colored Red

$|1\rangle \longrightarrow$ colored Blue

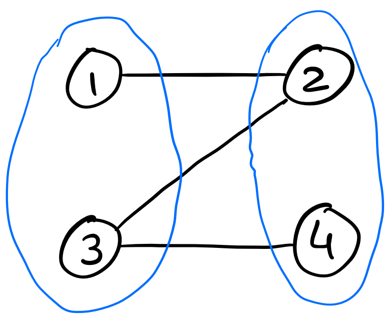


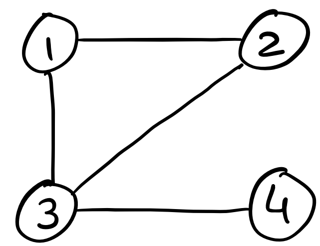
$$|V_0\, V_1\, V_2\, V_3\rangle = |0\,1\,1\,0\rangle$$

Special Case: Bipartite Graphs

V can be partitioned/colored into two disjoint sets S & R, such that no edge exist between vertices in S & no edge exists b/w vertices in R.

Example





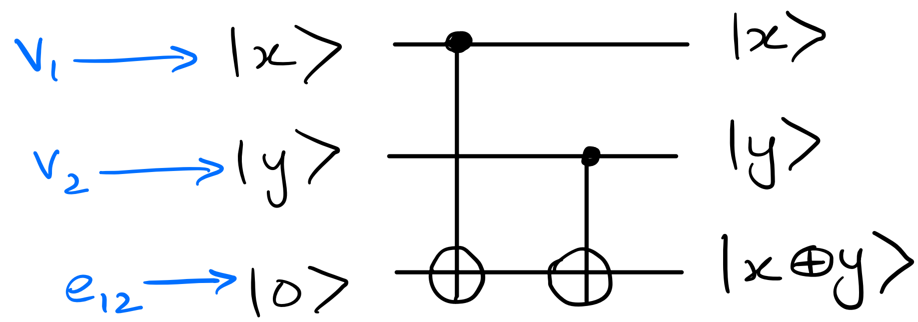
Not bipartite

Bipartite graphs admit two coloring. (One color for each set)

So, max-cut solution is just total # of edges.
(Since all edges are cross-connecting)

Edge Checking

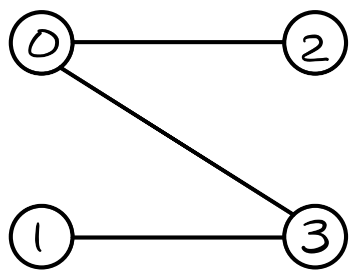
Does each end of an edge have vertices with different colors?

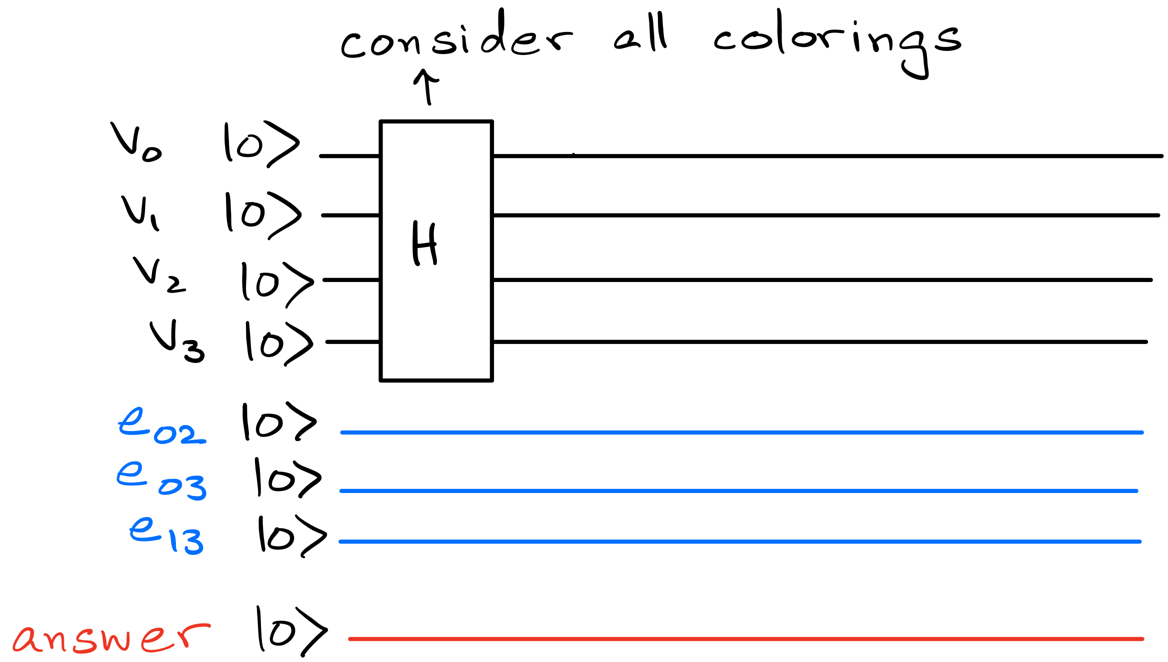


x	y	$x \oplus y$
0	0	0
0	1	1
1	0	1
1	1	0

Answer to whether both vertices have different colors

Oracle Construction





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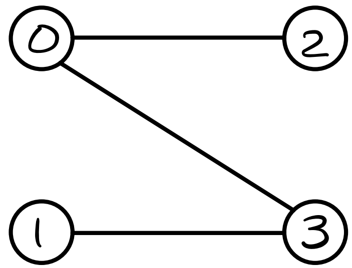

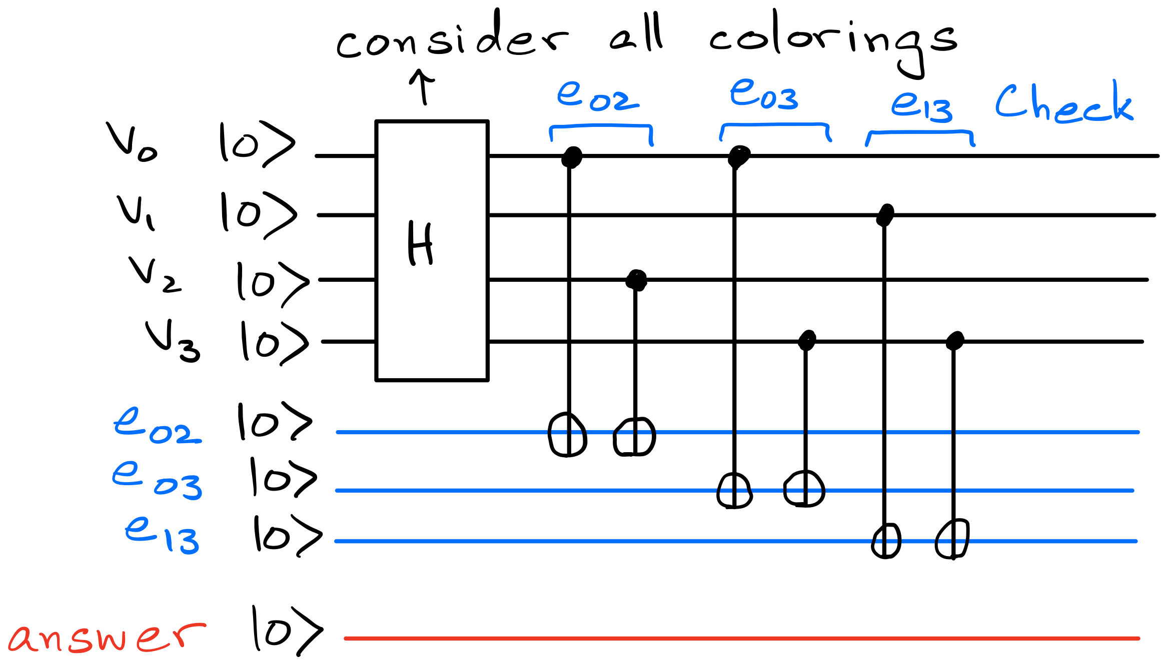

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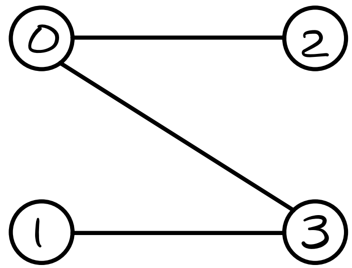
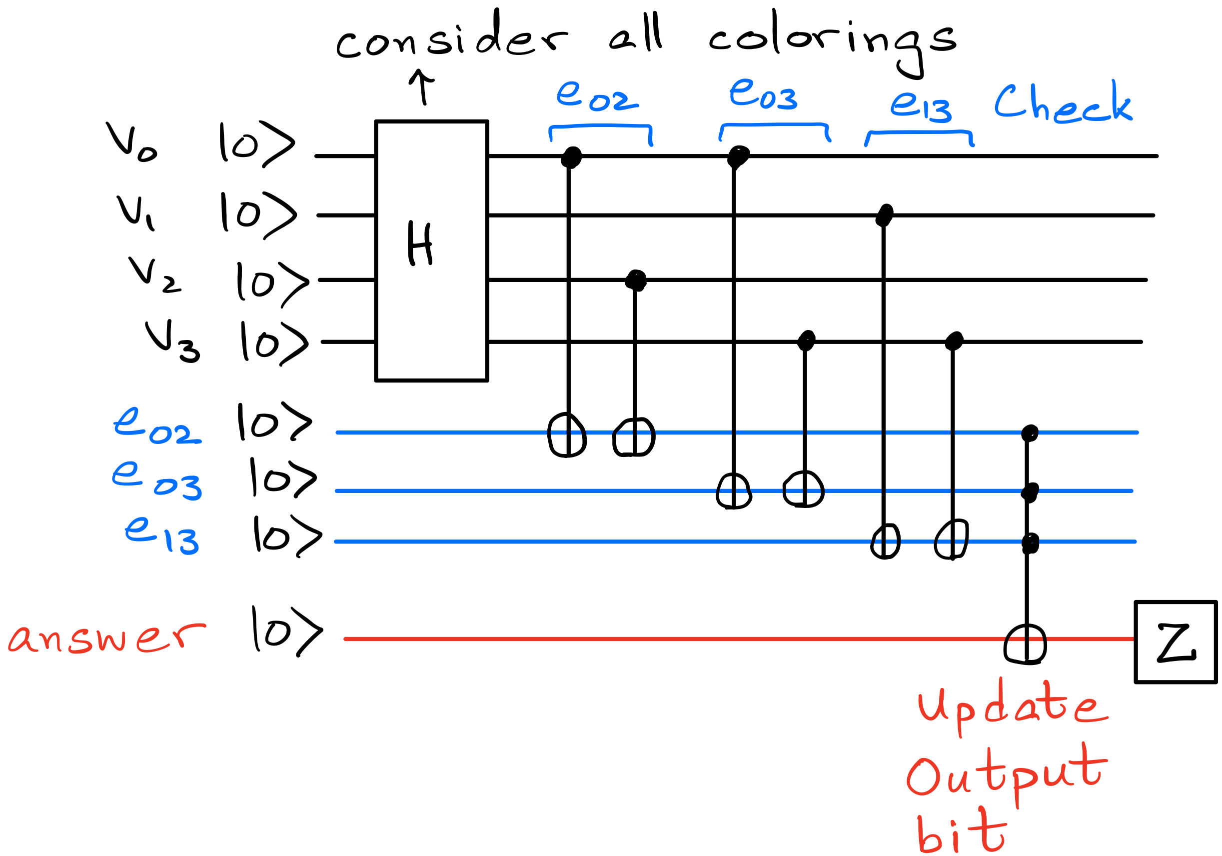
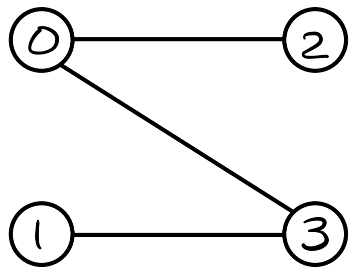
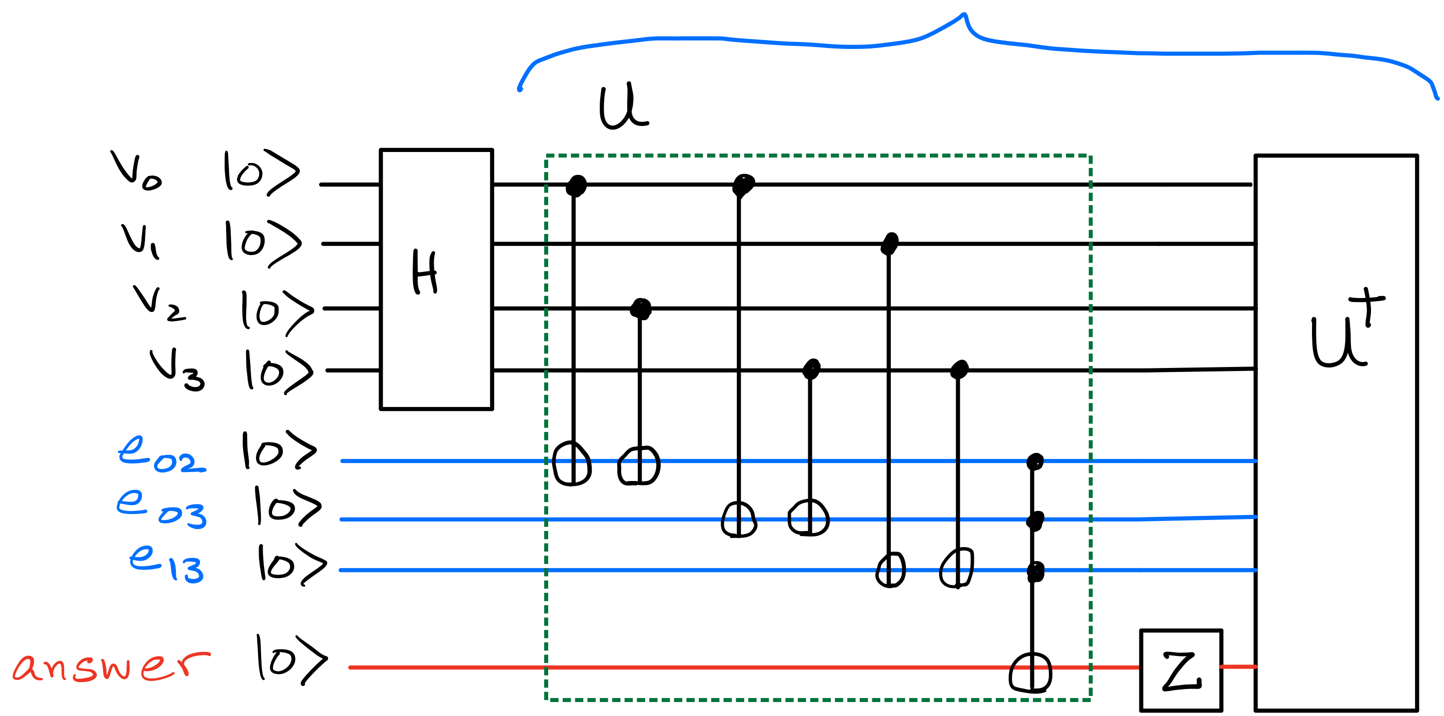




$$Z_f|x\rangle = (-1)^{f(x)}|x\rangle$$





Net result is flip of sign for qubits representing correct colorings.

This can be used as an oracle for Grover!

Checking a Graph is Bipartite

Our Quantum Algorithm: $O\left(\sqrt{2^n}\right)$

Best Classical Algorithm: $O\left(n^2\right)$

General Construction

Oracle Construction:

1. Edge checking for each edge
2. Sum the outputs of edge checking
3. Check whether sum $\geq k$ & store result
4. Apply Z gate on result

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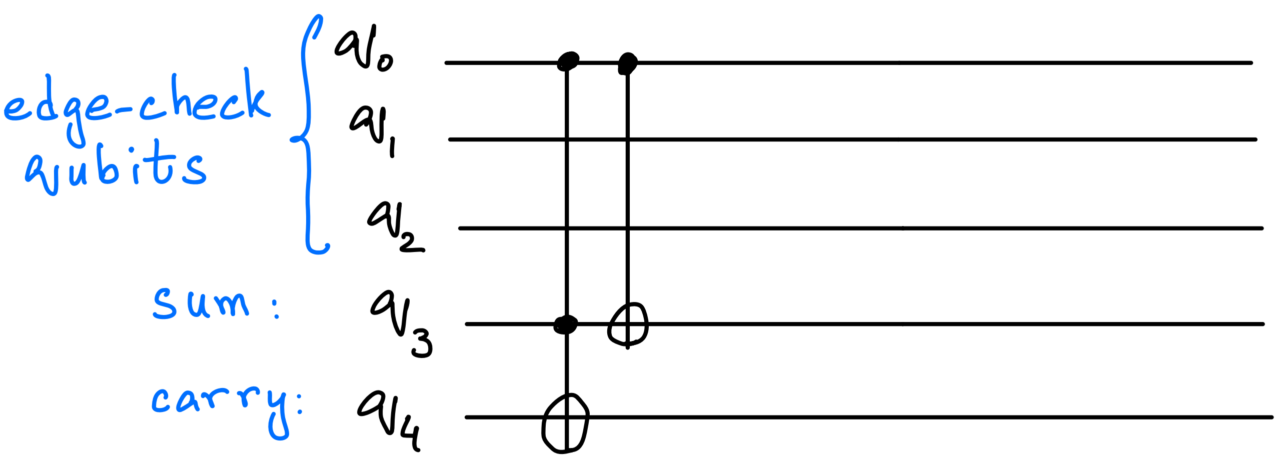
In-place Addition: Given 3 bits, how many are set to 1.



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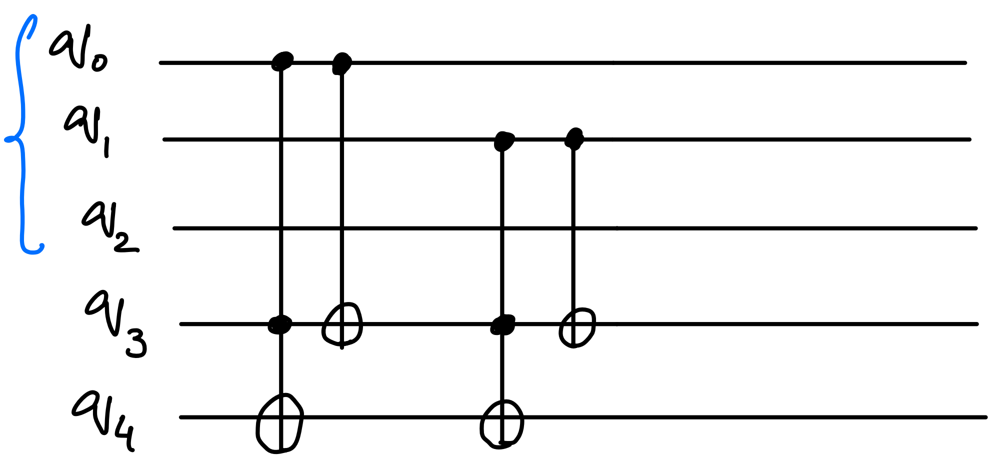

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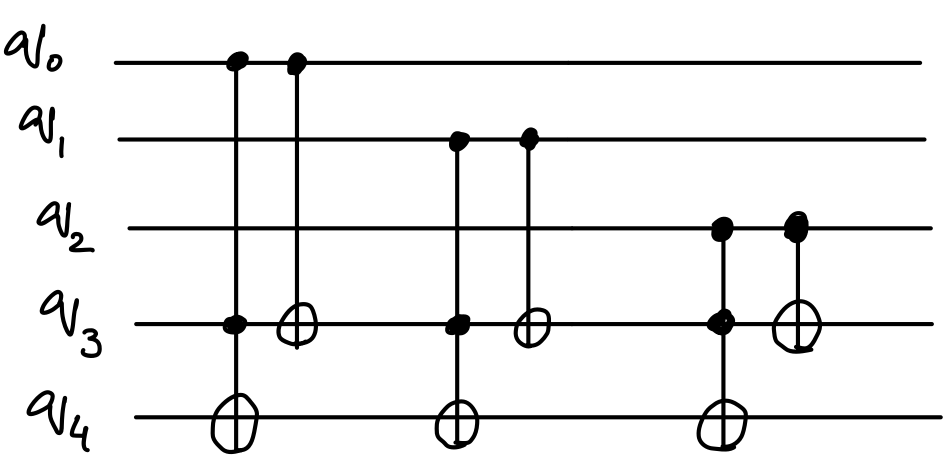
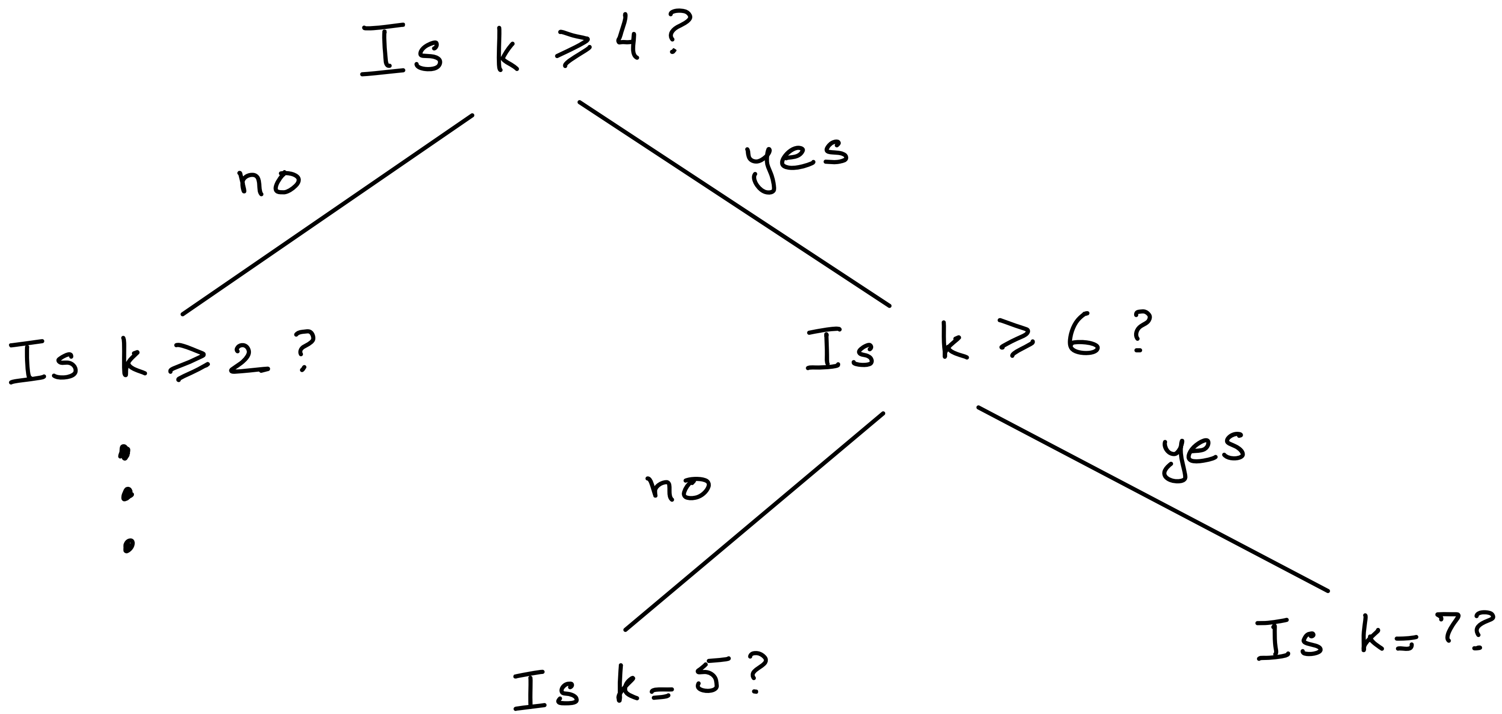


Number-Checking



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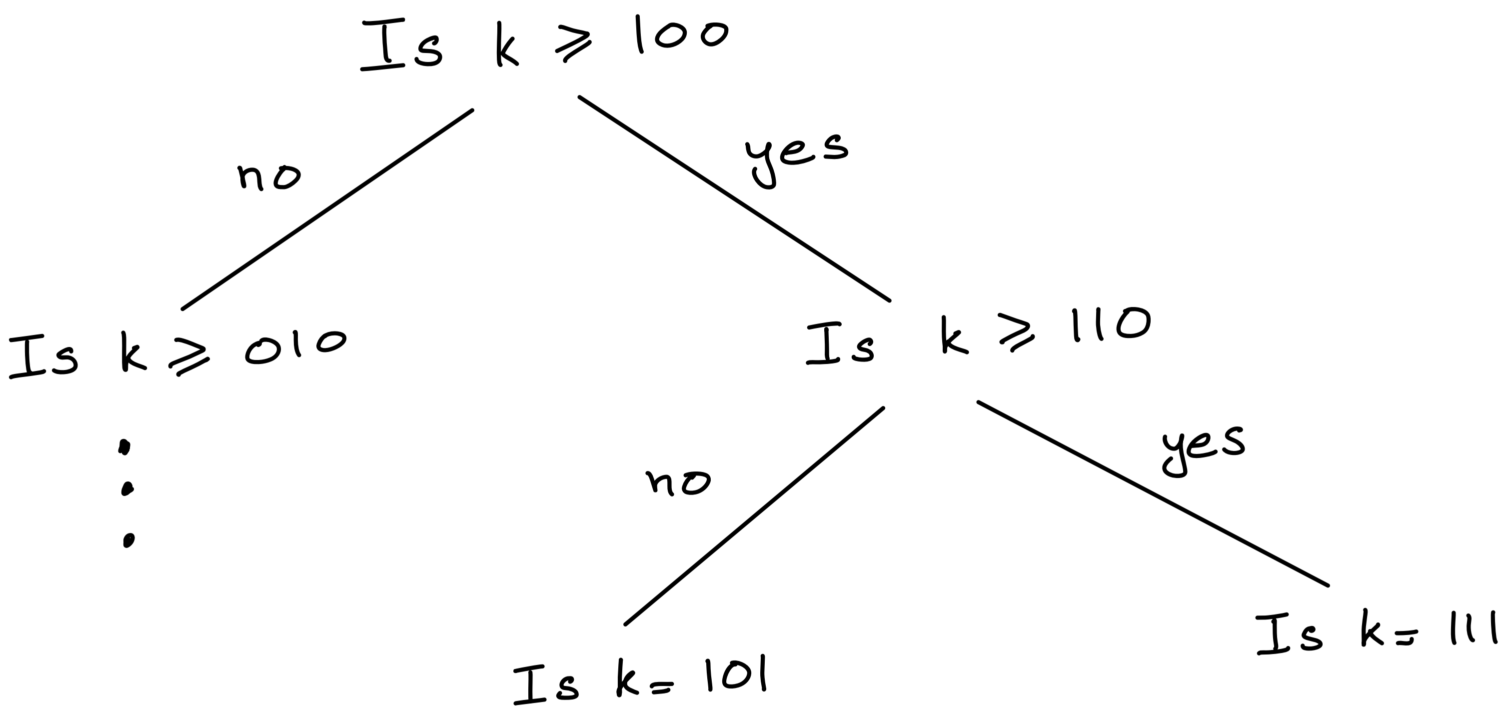
(3 bits)



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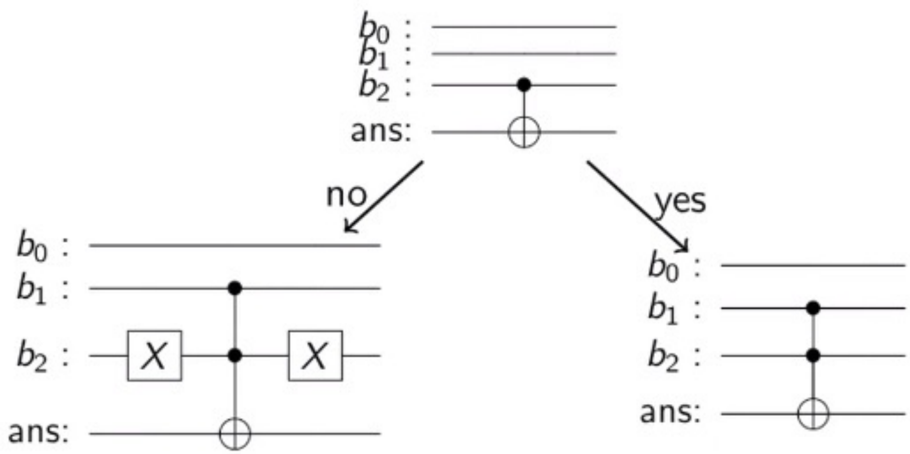

Note: Each case will require a separate Grover search!

At most lg (# of edges)

What is Your Favourite Super Power?

MANIPULATE PROBABILITY!!!