QWORLD

INTRODUCTION TO QUANTUM SYSTEMS

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Quantum Bit (Qubit)

Modelling a quantum system X with $\Sigma = \{0, 1\}$

Central Claim of Quantum Physics

To describe an isolated quantum system we need to give an amplitude $(\alpha \in \mathbb{C})$ for each possible state $\sigma \in \Sigma$.

Classical

$$\hat{v} = \underbrace{p \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (1-p) \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{convex combination}}$$

$p \geq 0$

Quantum

$$\hat{v} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$\alpha, \beta \in \mathbb{C}$

What are the restrictions on α, β

Getting Probabilities from Amplitudes

Born Rule

Probability to observe a particular outcome,

e.g., Prob(X is in state 0) is given by

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \longrightarrow \begin{cases} \alpha^2 \text{ — Prob(X in state 0)} \\ \beta^2 \text{ — " (X " " 1)} \end{cases}$$

assuming $\alpha, \beta \in \mathbb{R}$

$$\hat{v} = \overbrace{\alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix}}^{\text{superposition}}$$

$$\alpha^2 + \beta^2 = 1$$

$$\alpha, \beta \in \mathbb{R}$$

Quantum Bit (Qubit)

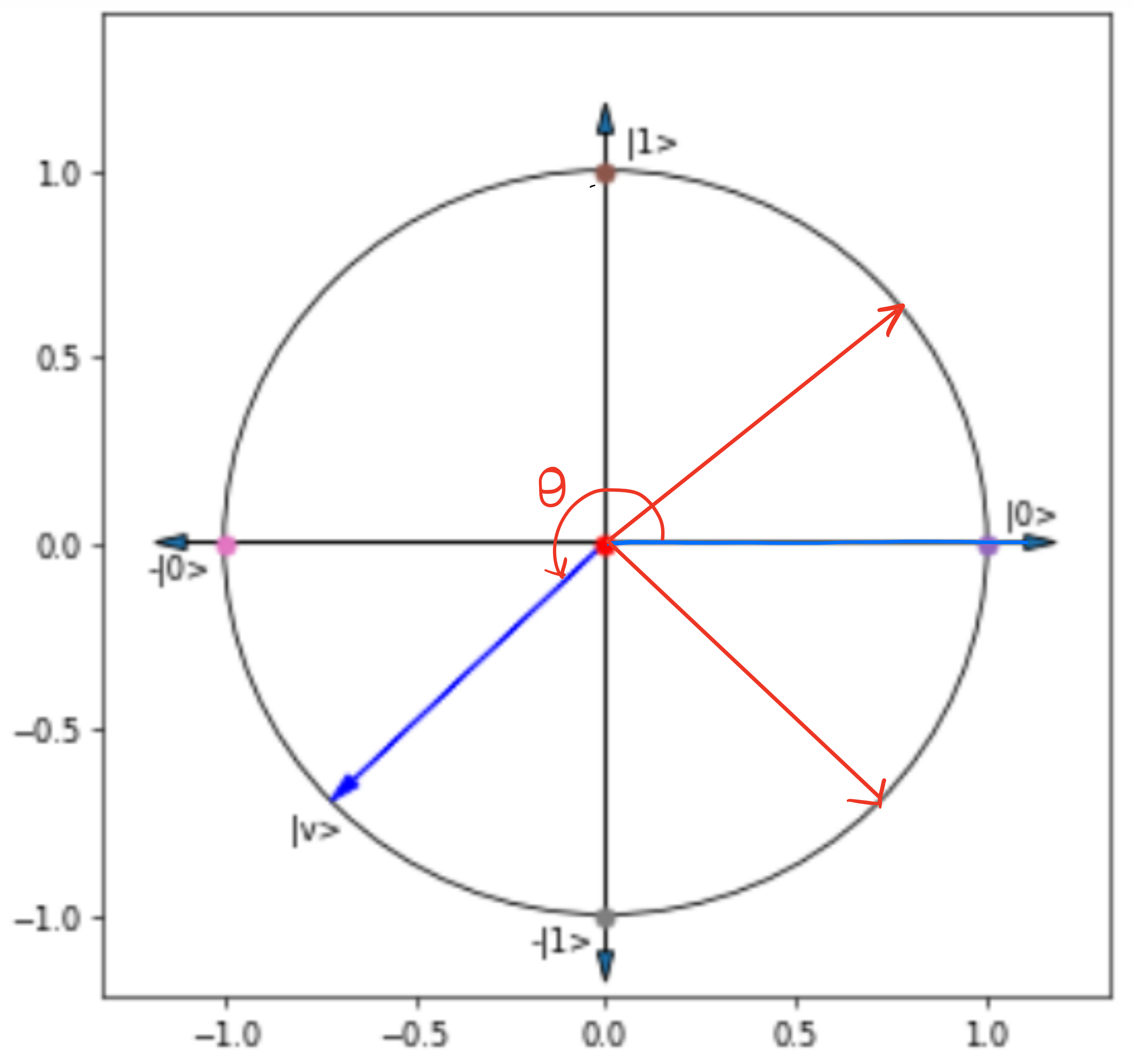
For real amplitudes, we can describe the state as

$$\hat{v} = \cos\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \theta \in [0, 2\pi)$$

$$= \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\cos^2\theta + \sin^2\theta = 1$$

Qubit on a Unit Circle



Quantum Operations

$H^T = H$

Symmetric

Hermitian
(for complex entries)

$U \hat{v} = \hat{w}$

$$U \begin{pmatrix} \vdots \\ \alpha_i \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \beta_i \\ \vdots \end{pmatrix}$$

$$\sum_i \alpha_i^2 = 1 = \sum_i \beta_i^2$$

i) Unitary matrices $(\overline{U})^T U = \mathbb{1}$ identity

ii) $U^T = U^{-1}$

iii) U is reversible

Quantum Operations

Ket Notation

$|\cdot\rangle \equiv$ column vector

$\underbrace{|\cdot\rangle}_{\text{ket}}$

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hat{i}$

$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\underbrace{\langle\cdot|}_{\text{bra}} \equiv$ row vector

$\langle 0| = \begin{pmatrix} 1 & 0 \end{pmatrix}$

$\langle 1| = \begin{pmatrix} 0 & 1 \end{pmatrix}$

$\alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow \alpha|0\rangle + \beta|1\rangle = |\psi\rangle$

$\gamma|0\rangle + \delta|1\rangle = |\varphi\rangle$

$\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \longrightarrow \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle$

dot product $\langle\varphi||\psi\rangle \longrightarrow \langle\varphi|\psi\rangle$

$= (\gamma\langle 0| + \delta\langle 1|)(\alpha|0\rangle + \beta|1\rangle)$

$= \alpha\gamma \underbrace{\langle 0|0\rangle}_{1} + \beta\gamma \underbrace{\langle 0|1\rangle}_{0} + \alpha\delta \underbrace{\langle 1|0\rangle}_{0} + \beta\delta \underbrace{\langle 1|1\rangle}_{1}$

$= \alpha\gamma + \beta\delta$

Standard Basis / Computational Basis $\{|0\rangle, |1\rangle\}$

Quantum Operations

Hadamard Matrix

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad H^T H = \mathbb{1}$$

$$H^T = H$$

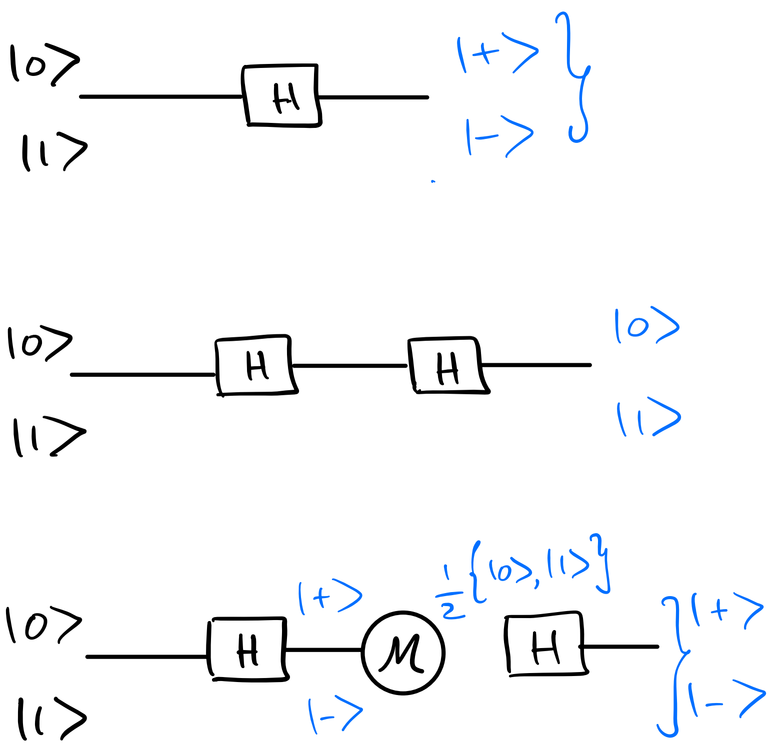
$$F = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

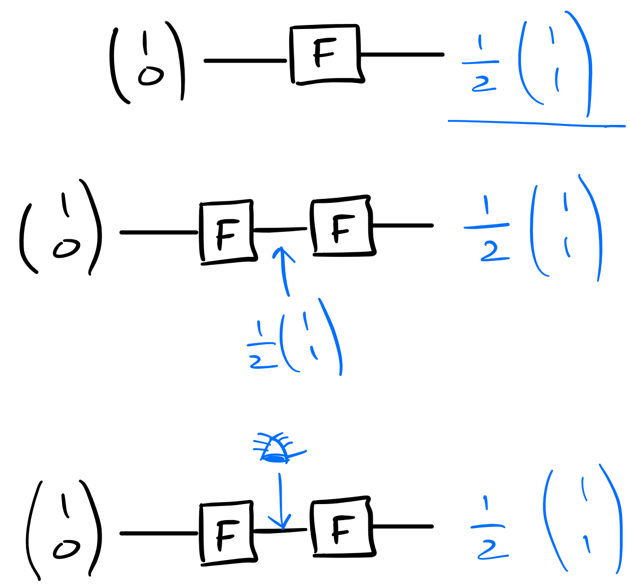
$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) = |+\rangle$$

$$H|1\rangle = H\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) = |-\rangle$$

$\{|+\rangle, |-\rangle\}$ Hadamard Basis

Classical vs Quantum Coin Flipping



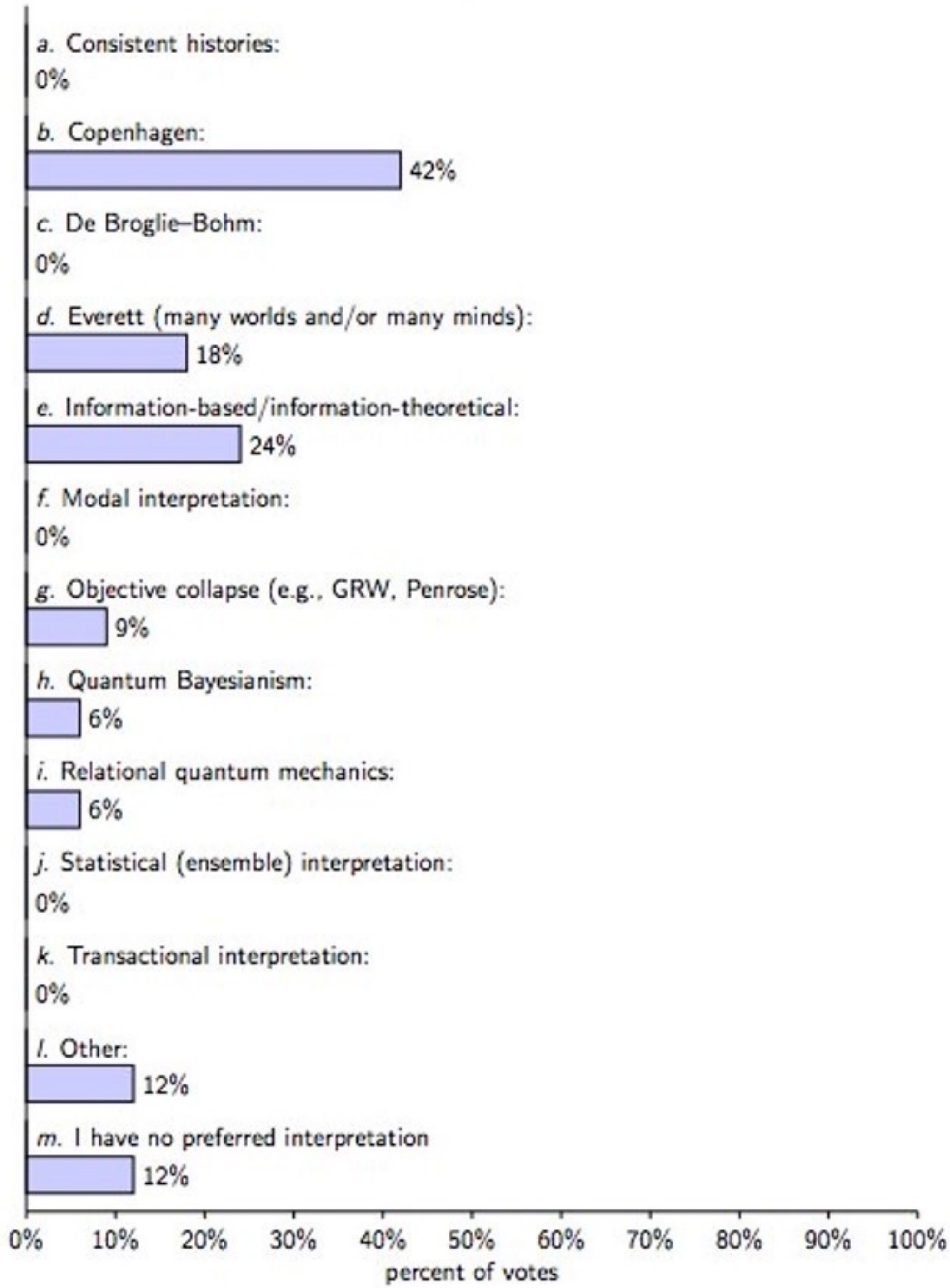


$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$F = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$F\begin{pmatrix} p \\ 1-p \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Question 12: What is your favorite interpretation of quantum mechanics?



Multiple Qubits

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\varphi\rangle = \gamma|0\rangle + \delta|1\rangle$

$$
\begin{aligned}
|\psi\rangle \otimes |\varphi\rangle &= |\psi\rangle|\varphi\rangle = |\psi\varphi\rangle = |\chi\rangle \\
&= (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle) \\
&= \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle =
\end{aligned}
$$

$$
\begin{array}{cc}
\begin{pmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{pmatrix} &
\begin{array}{l} 00 \\ 01 \\ 10 \\ 11 \end{array}
\end{array}
$$

When can n quantum bits be simulated efficiently on a classical computer?

$$|\psi\rangle = \underbrace{(\alpha_1|0\rangle + \beta_1|1\rangle) \otimes (\alpha_2|0\rangle + \beta_2|1\rangle) \otimes \cdots \otimes (\alpha_n|0\rangle + \beta_n|1\rangle)}_{\text{Separable}}$$

Just store the α_i's and update them as state evolves. Can compute β_i from α_i.

Needs $O(n)$ memory.

QBronze Summary

n-qubit Quantum State $\quad |\psi\rangle = \sum_{i=0}^{2^n-1} \alpha_i |i\rangle$, $\sum_{i=0}^{2^n-1} \alpha_i^2 = 1$, $\alpha_i \in \mathbb{R}$

Unitary Evolution $\quad U|\psi\rangle = |\varphi\rangle = \sum_{i=0}^{2^n-1} \beta_i |i\rangle$, $\sum_{i=0}^{2^n-1} \beta_i^2 = 1$, $\beta_i \in \mathbb{R}$ $\quad U^T U = \mathbb{1}$

Measurement $\quad$ Probability to observe particular outcome i on measuring $|\psi\rangle$ is given by α_i^2