QWORLD

INTRODUCTION TO CLASSICAL SYSTEMS

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An engineer, a physicist and a mathematician…

What is Your Favourite Super Power?

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MANIPULATE PROBABILITY!!!

Deterministic Classical Information

A physical device X has some finite, non-empty set of states.

$\{☺, ☹\}$ $\quad \Sigma = \{0, 1\}$ $\quad \{H, T\}$ $\quad \{a, b, c, d\}$

How does the state of the system change?

$f : \{0, 1\} \rightarrow \{0, 1\}$

x	f_1	f_2	f_3	f_4
0	0	0	1	1
1	0	1	0	1

What about multiple such devices?

$\Sigma^2 = \{00, 01, 10, 11\}$

$|\Sigma^n| = 2^n$

$\underset{X_1}{2} \quad \underset{X_2}{2} \quad \cdots \quad \underset{X_n}{2} = 2^n$

x_1	x_2	$\ldots$	x_n	$f(x_1, \ldots x_n)$
0	0	$\ldots$	0	
0	0	$\ldots$	1	
		$\vdots$		
1	1	$\ldots$	1	

$f : \{0, 1\}^n \rightarrow \{0, 1\}$

$N = 2^n$

2^N such f.

What if we have Incomplete Information?

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Probability

Probability (State of $X=0$) $=$ p

Probability (State of $X=1$) $=$ $1-p$

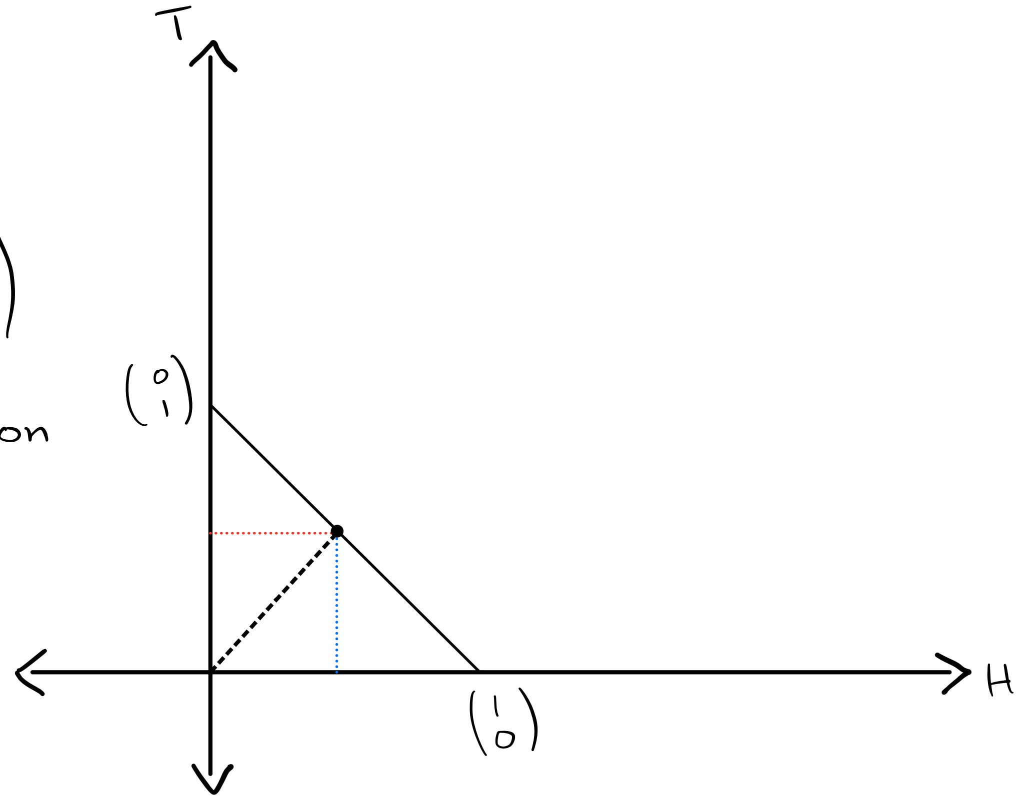
$$\hat{v} = \begin{pmatrix} p \\ 1-p \end{pmatrix} \begin{array}{l} \to v_0 \\ \to v_1 \end{array}$$

Note: When we look at X we do not see $\hat{v}$, p but rather some state $\sigma \in \Sigma$ with probability v_σ

What if we have Incomplete Information?

$$\hat{v} = \begin{pmatrix} p \\ 1-p \end{pmatrix}$$

$$= \underbrace{p\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (1-p)\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{convex combination}}$$



What if we have Incomplete Information?

$P_{0|0}$ $\qquad$ $P_{0|1}$

$$\begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} \boxed{a} & \boxed{b} \\ c & d \end{pmatrix} \end{array} \begin{pmatrix} p \\ 1-p \end{pmatrix} \begin{array}{c} 0 \\ 1 \end{array} = \underbrace{\begin{pmatrix} \_\_\_\_ \\ \_\_\_\_ \end{pmatrix}}_{\text{should also be a probability vector}}$$

$P_{1|0}$ $\qquad$ $P_{1|1}$

All classical transformations are stochastic matrices:

i) all entries of matrix are ≥ 0

ii) columns sum to 1.

What if we have Incomplete Information?

Deterministic ops. in matrix form

$$A_1 = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \end{array} \qquad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad A_4 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

Take a convex combination of these 4.

<u>Probabilistic Transf.</u>

$$B = \sum_{i=1}^{4} p_i A_i , \quad p_i \geq 0 , \quad \Sigma p_i = 1$$

$$F = \frac{1}{2} A_2 + \frac{1}{2} A_3 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$F \begin{pmatrix} p \\ 1-p \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

How does the State of the System change?

Qs. We flip a coin $\begin{array}{cc} T & \\ H & \end{array}\begin{pmatrix} p \\ q \end{pmatrix}$. $\begin{cases} \text{if we get a head, we flip again.} \\ \text{if we get a tail, we turn the coin over, i.e., we make it a head.} \end{cases}$

$$\begin{array}{ccc} & T & H \\ T & \multicolumn{2}{c}{\multirow{2}{*}{}} \\ H & & \end{array}$$

$$\begin{array}{c} T \\ H \end{array}\begin{pmatrix} \overset{T}{0} & \overset{H}{p} \\ 1 & q \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} pq \\ p+q^2 \end{pmatrix}$$

Multiple Devices with Incomplete Information

Coin X_1 $\begin{array}{c} \\ \\ \end{array}\begin{pmatrix} p \\ q \end{pmatrix}\begin{array}{c} H \\ T \end{array}$ Coin X_2 $\begin{pmatrix} r \\ s \end{pmatrix}\begin{array}{c} H \\ T \end{array}$

<u>Tensor Product</u>

$$\begin{pmatrix} p \\ q \end{pmatrix} \underset{\text{Tensor Product}}{\otimes} \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} pr \\ ps \\ qr \\ qs \end{pmatrix}\begin{array}{c} HH \\ HT \\ TH \\ TT \end{array}$$

<u>Note</u>: Not all 4-dim prob. vectors can be written as a tensor product of two 2-dim prob. vectors.

$$\frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\begin{array}{c} HH \\ HT \\ TH \\ TT \end{array} = \begin{pmatrix} p \\ q \end{pmatrix} \otimes \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} pr \\ ps \\ qr \\ qs \end{pmatrix}$$

$$p(X_1, X_2) = p(X_1)\, p(X_2)$$

Multiple Devices with Incomplete Information

$$A \otimes B = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \otimes \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} \boxed{a_1 B} & a_2 B \\ a_3 B & a_4 B \end{pmatrix}$$

$$= \begin{pmatrix} \begin{matrix} a_1 b_1 & a_1 b_2 \\ a_1 b_3 & a_1 b_4 \end{matrix} & \\ & \end{pmatrix}$$

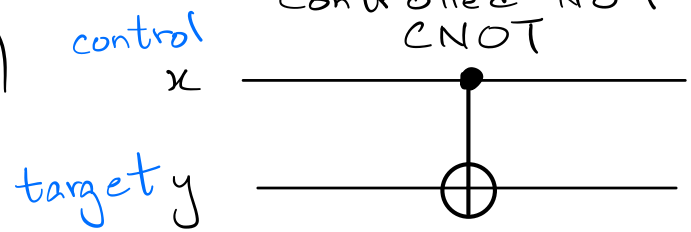
There exist 4×4 transf. that cannot be decomposed into $A \otimes B$, where A & B are 2×2 matrices.

Multiple Devices with Incomplete Information

NOT $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Controlled NOT

CNOT



(control)
When x is 0
do nothing to target

When x is 1
Apply NOT gate
to target

x	y	$x \oplus y$
0	0	0
0	1	1
1	0	1
1	1	0

$$\begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array} \begin{array}{c} xy \\ \left(\begin{array}{cc|cc} 00 & 01 & 10 & 11 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) \end{array} \neq A \otimes B$$

How can a Biased Coin Simulate a Fair Coin?

Assume we have a coin $\begin{pmatrix} p \\ 1-p \end{pmatrix}$, $p \neq 0$, $p \neq 1/2$.

How can we simulate a fair coin?