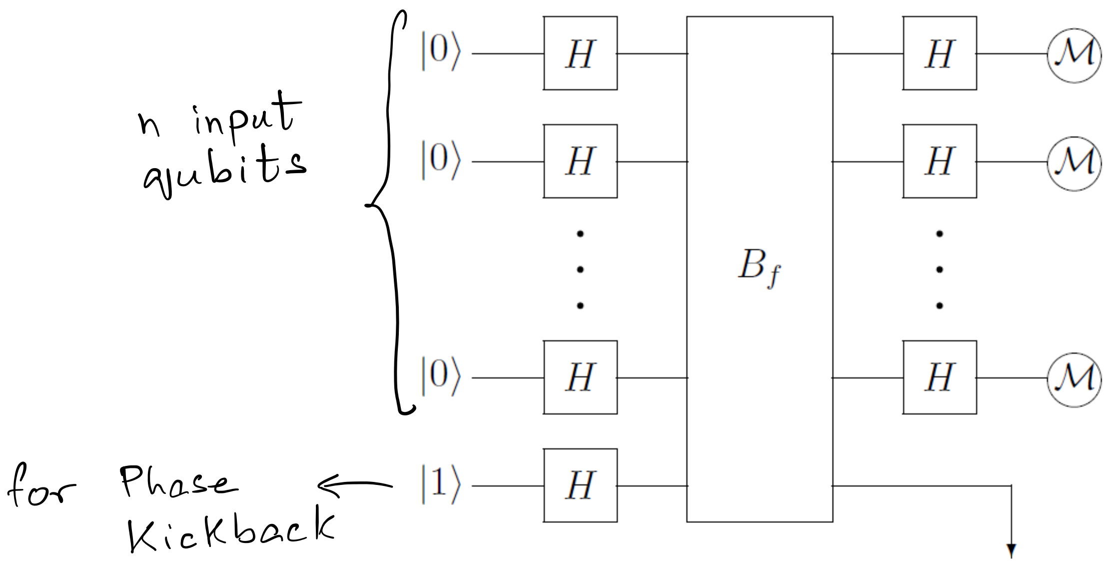
Deutsch Jozsa Algorithm

1. f is **constant**. In other words, either $f(x) = 0$ for all $x \in \{0,1\}^n$ or $f(x) = 1$ for all $x \in \{0,1\}^n$.
2. f is **balanced**. This means that the number of inputs $x \in \{0,1\}^n$ for which the function takes values 0 and 1 are the same:

$$|\{x \in \{0,1\}^n : f(x) = 0\}| = |\{x \in \{0,1\}^n : f(x) = 1\}| = 2^{n-1}.$$



Classical # of queries for zero error = $\frac{2^n}{2} + 1 = 2^{n-1} + 1$

Hadamard on n Qubits

over a single qubit

$$H|a\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{a}|1\rangle\right), \quad a \in \{0,1\}, \; b \in \{0,1\}$$

$$H|0\rangle = |+\rangle$$

$$H|1\rangle = |-\rangle$$

$$= \frac{1}{\sqrt{2}} \sum_{b\in\{0,1\}} (-1)^{a\cdot b}|b\rangle$$

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two qubits

$a_1, a_2, b_1, b_2 \in \{0,1\}$

$a, b \in \{0,1\}^2$

$$H\otimes H|\underbrace{a_1 a_2}_{a}\rangle = \underbrace{\left(\frac{1}{\sqrt{2}}\sum_{b_1}(-1)^{a_1\cdot b_1}|b_1\rangle\right)}_{H|a_1\rangle} \otimes \underbrace{\left(\frac{1}{\sqrt{2}}\sum_{b_2}(-1)^{a_2\cdot b_2}|b_2\rangle\right)}_{H|a_2\rangle}$$

$$= \frac{1}{2}\sum_{b}(-1)^{a\cdot b}|b\rangle \qquad a\cdot b = a_1\cdot b_1 \oplus a_2\cdot b_2$$

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for n qubits

$a, b \in \{0,1\}^n$

$a_i, b_i \in \{0,1\}$

$i \in \{1, 2, \ldots, n\}$

$$H^{\otimes n}|a\rangle = \frac{1}{\sqrt{2^n}}\sum_{b}(-1)^{a\cdot b}|b\rangle, \quad a\cdot b = a_1\cdot b_1 \oplus a_2\cdot b_2 \oplus \cdots \oplus a_n\cdot b_n$$

Deutsch Jozsa Algorithm

$$|\psi_1\rangle = \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle|-\rangle$$

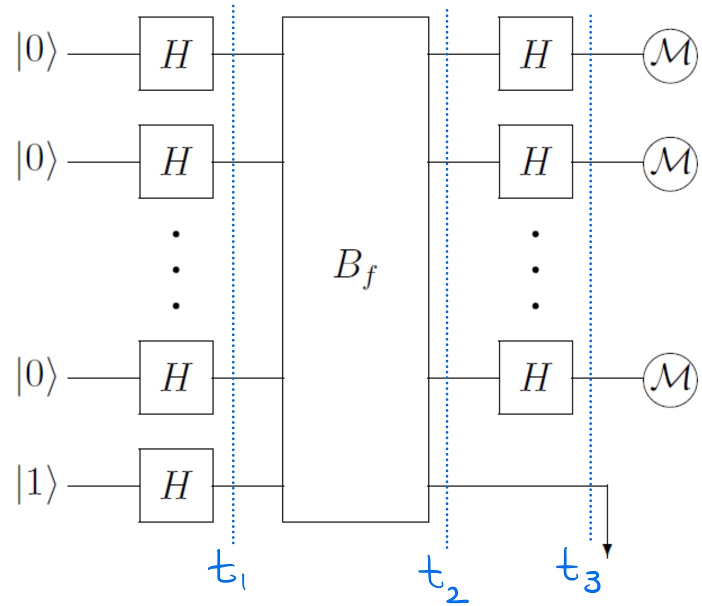
$$|\psi_2\rangle = \frac{1}{\sqrt{2^n}} \sum_{x} (-1)^{f(x)} |x\rangle|-\rangle$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2^n}} \sum_{x} (-1)^{f(x)} \left(H^{\otimes n}|x\rangle\right) |-\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x} (-1)^{f(x)} \left(\frac{1}{\sqrt{2^n}} \sum_{y} (-1)^{x\cdot y} |y\rangle\right) |-\rangle$$

$$= \left[\sum_{y} \left(\frac{1}{2^n} \sum_{x} (-1)^{f(x)\oplus x\cdot y} |y\rangle\right)\right] |-\rangle$$

$y \in \{0,1\}^n$



$$H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y\in\{0,1\}^n} (-1)^{x_1y_1+\cdots+x_ny_n} |y\rangle$$

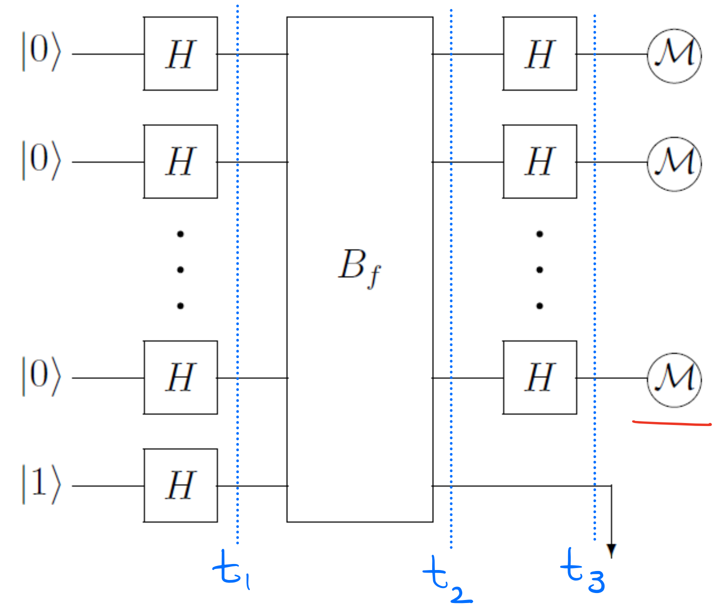
Deutsch Jozsa Algorithm

$$\sum_y \underbrace{\left(\frac{1}{2^n} \sum_x (-1)^{f(x) \oplus x \cdot y} \right)}_{\text{Gives us the probability to observe a given } y.} |y\rangle$$

What is the probability to observe $y = 0^n = \underbrace{00\ldots0}_{n \text{ times}}$

$$\left(\frac{1}{2^n} \sum_x (-1)^{f(x) \oplus \underbrace{x \cdot y}_{=0}} \right)^2 \quad \text{for } y = 0^n \implies |y\rangle = |0^n\rangle$$

$$= \left(\frac{1}{2^n} \sum_x (-1)^{f(x)} \right)^2 = \begin{cases} +1 & f \text{ is constant} \\ 0 & f \text{ is balanced} \end{cases}$$



$$H^{\otimes n} |x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x_1 y_1 + \cdots + x_n y_n} |y\rangle$$

So a single oracle call suffices!

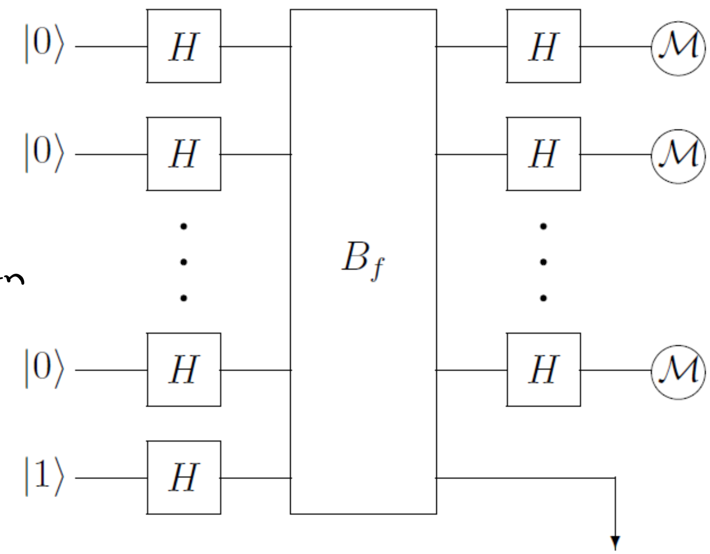
Bernstein-Vazirani Problem

Suppose a function $f : \{0,1\}^n \to \{0,1\}$ is given as a black-box in the usual way, i.e., as a unitary transformation B_f that acts as follows for all $x \in \{0,1\}^n$ and $y \in \{0,1\}$:

$$B_f : |x\rangle\,|y\rangle \mapsto |x\rangle\,|y \oplus f(x)\rangle\,.$$

This time you are promised that there exists some string $s \in \{0,1\}^n$ such that $f(x) = s \cdot x$ for all $x \in \{0,1\}^n$, where

$$s \cdot x = \sum_{i=1}^{n} s_i x_i \pmod 2). = s_1 x_1 \oplus s_2 x_2 \oplus \cdots \oplus s_n x_n$$



$n = 3, \quad s \cdot x = s_1 x_1 \oplus s_2 x_2 \oplus s_3 x_3$

s_1 s_2 s_3	$f(x)$
0 0 0	0
0 0 1	x_3
0 1 0	x_2
0 1 1	$x_2 \oplus x_3$
⋮	⋮
1 1 1	$x_1 \oplus x_2 \oplus x_3$

Classical Query Complexity

x_1 x_2 x_3	$f(x)$
1 0 0	s_1
0 1 0	s_2
0 0 1	s_3

$\Big\}\ s = s_1 s_2 s_3$

$O(n)$ classical oracle calls

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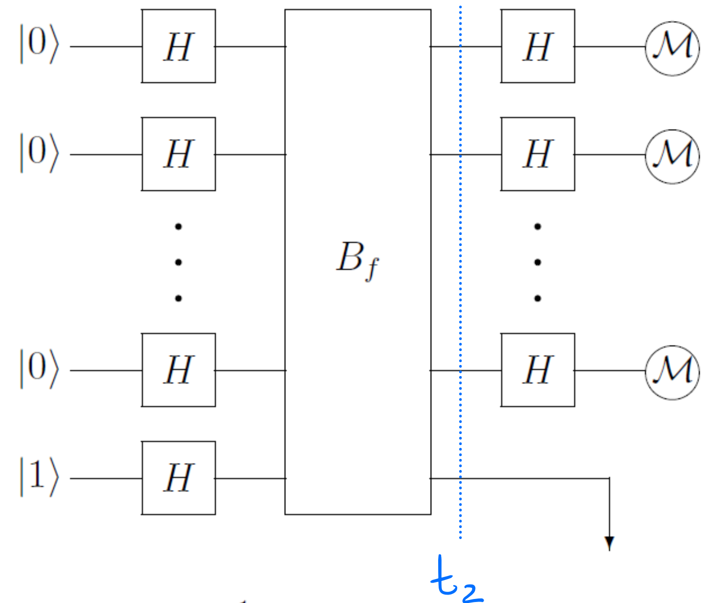
$$s \cdot x = \sum_{i=1}^{n} s_i x_i \pmod 2).$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2^n}} \sum_x (-1)^{f(x)} |x\rangle |-\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_x (-1)^{s \cdot x} |x\rangle |-\rangle$$

$$H^{\otimes n} \left(\frac{1}{\sqrt{2^n}} \sum_x (-1)^{s \cdot x} |x\rangle \right) = |s\rangle$$



$$H^{\otimes n} |x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x_1 y_1 + \cdots + x_n y_n} |y\rangle$$

$$H^{\otimes n} |x\rangle = \frac{1}{\sqrt{2^n}} \sum_y (-1)^{x \cdot y} |y\rangle$$