Live-TeXed Lecture Notes

MAST90023 - Algebraic Topology

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Contents

Chapter 1

Week One

This week, we learned about... Something

1.1. Lecture 1, 28/02/2024

1.1.1. Toplogy and Topological Spaces

Definition 1. Let X be a set. Then, the *power set* of X – denoted by $\mathcal{P}(X)$ – is the set of all subsets of X. A subset $\tau \subset \mathcal{P}(X)$ is a *topology* if the following holds:

(i) $\emptyset, X \in \tau$,

(ii) (Closure under Arbitrary Union) $\bigcup_{\alpha\in\Lambda} U_\alpha \in \tau$ if $U_\alpha \in \tau$ for each $\alpha \in \Lambda$,

(iii) (Closure under Finite Intersections) $\bigcup_{i=1}^n U_i \in \tau$ if each $U_i \in \tau$.

The pair (X, τ) is called a *topological space.*

Remark 1. Often, we will just drop τ and say that X is a toplogical space, where there is no confusion.

Example 1. *The* one-point space *is given by* $\tau_{\text{in}} = \{X, \emptyset\}$. *The* discrete topology $\tau_{\text{discrete}} = \mathcal{P}(X)$ *is the smallest topology that can exist.*

Example 2 (Metric Spaces)**.** *A metric space* (X, d) *defines a topological space via open balls*

$$B_r(x) := \{y \in X : d(x, y) < r\}.$$

In particular, we say that $U \in \tau_d$ *if* $U = \bigcup_{x\in U} B_r(x)$. *This defines a topology on* (X, d)*, and is sometimes called the* induced topology*.*

Example 3 (Standard Euclidean Space)**.** *Let* $X = \mathbb{R}^n$*, and define*

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2},$$

to be the Euclidean metric*. The metric space* $(\mathbb{R}^n, d)$*, together with the induced topology* $(\mathbb{R}^n, \tau_d)$ *is called the* standard Euclidean space*.*

Example 4 (The n-Sphere)**.** *Let*

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : d(x, 0) = 1\},$$

be the n-sphere. *One can equip* $\mathbb{S}^n$ *with the structure of a topological space via the usual Euclidean metric, or we can take the one induced from the topological structure from the one on* $\mathbb{R}^{n+1}$.

1.1.2. Continuous Maps

Remark 2 (Diarmuid Wisdom). The point of topology is that it is the study of maps. Spaces are interesting, but maps are the stars of the show. In particular, they are at least as important as spaces (Diarmuid, 2024).

Definition 2. Given a map $f : X \to Y$ of topological spaces, then f is *continuous* if given any open set U of Y, the pre-image $f^{-1}(U)$ is open in X.

Exercise 1. *Prove that a map* $f : (X, d_X) \to (Y, d_Y)$ *is continuous (in the metric space sense) if and only if it is continuous (in the topology space sense).*

Recall:

Definition 3 (Metric Space Continuity)**.** A map of metric spaces $f : (X, d_X) \to (Y, d_Y)$ is *continuous* if for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $d_X(x, y) < \delta$, then $d_Y(f(x), f(y)) < \varepsilon$.

Example 5. *The map*

$$1_X : (X, \tau) \longrightarrow (X, \tau),$$

is continuous. The map

$$1_X : (X, \tau_{\text{in}}) \longrightarrow (X, \tau_{\text{discrete}}),$$

is not *continuous in general.*

Lemma 1. *If* $f : X \to Y$ *and* $g : Y \to Z$ *are both continuous, then* $f \circ g : X \to Z$ *is continuous.*

Proof. Let $U \subseteq Z$ be open. Then, $g^{-1}(U)$ is open in Y by continuity of g. By continuity of f, the set $(f \circ g)^{-1}(U)$ is open in X. □

Remark 3. Given any function $f : X \to Y$, and a subset $U \subseteq Y$, the *pre-image* of U is:

$$f^{-1}(U) := \{x \in X : f(x) \in U\}.$$

Example 6. *The addition function* $+ : \mathbb{R}^2 \to \mathbb{R}$ *and the multiplication function* $\cdot : \mathbb{R}^2 \to \mathbb{R}$ *are both continuous maps. This is an example of a* topological group *or* toplogical ring.

Definition 4. A bijection $f : X \to Y$ is a *homeomorphism* if f and f^{-1} are continuous.

Remark 4. Homeomorphisms serve as a topological analogue for isomorphisms.

Problem: Given X and Y, decide if they are homeomorphic or not. For instance, are $\mathbb{R}^2$ and $\mathbb{R}^3$ homeomorphic? The answer is no, thankfully, but this is quite difficult to prove. But it follows from invariance of domain. We will prove this later.

This is quite a difficult problem in general. When proving theorems like this, we need to be careful of space-filling curves — which are surjective maps $\gamma : I \to I^2$. This is quite shocking.

1.1.3. Examples of Spaces

Definition 5. Let (X, τ_X) be a topological space, and $A \subseteq X$ a subset. The *subspace topology* τ_A is a topology on A inherited from the one on X in the following way: $U \in \tau_A$ if and only if $U = A \cap V$ for some $V \in \tau_X$.

Exercise 2. *Show that τ_A is a toplogy on A.*

Exercise 3. *If $\tau_X = \tau_d$ for some metric d on X, then the restriction $d_A := d|_{A\times A}$ defines a metric on A. Show that $(A, \tau_A) = (A, \tau_{d_A})$.*

Example 7. *We can topologise $\mathbb{S}^n$ as a subspace of $\mathbb{R}^{n+1}$ via the subspace topology.*

Example 8 (Cantor Set)**.** *Start with $C_0 = [0, 1]$. Then, remove the middle third of the interval to get*

$$C_1 = [0, 1/3] \cup [2/3, 1],$$

and punch out the middle thirds in each of the disjoint intervals to get C_2. Then, repeat forever. It is a compact set.

Definition 6 (Neighbourhoods)**.** Given a point $x \in X$, then a subset $N \subset X$ is a *neighbourhood* of x if there exists a set $U \in \tau$ such that $x \in U \subseteq N$.

Remark 5. It is useful at times to think about neighbourhoods that are not open.

Definition 7 (Manifold)**.** M is an n-manifold if every $X \subseteq M$ has a neighbourhood homeomorphic to $\mathbb{R}^n$, or an open subset of $\mathbb{R}^n$. In other words, n-manifolds are *locally Euclidean of constant dimension.*

Remark 6. We technically also require that M be paracompact and Hausdorff. But, we will not define paracompactness. We will talk about the Hausdorff property next time. They are also covered in the notes.

Example 9. *$\mathbb{S}^1$ defines a 1-manifold. $\mathbb{S}^n$ defines an n-manifold.*

1.2. Lecture 2, 29/02/2024

1.2.1. Examples of Spaces Continued

Definition 8 (Product Topology)**.** Let X and Y be topological spaces. Then, define the topology $\tau_{X\times Y}$ to be the topology whose open sets are of the form $U = \bigcup_{\alpha,\beta\in\Lambda} U_\alpha \times V_\beta$, where U_α and V_β are open in X and Y, respectively.

Example 10. *The 2-torus is given by $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$. One may also think of T as a quotient of the space $I \times I$, where $I = [0, 1]$ is the unit interval.*

Moreover, one may also take the 3-torus by $T^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. More generally, the n-torus is given by $T^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$. We may topologise these spaces via the product topology, using the topology on $\mathbb{S}^1$, which is induced from the one on $\mathbb{R}^2$.

Another interesting topological space that we can consider is $M^4 := \mathbb{S}^2 \times \mathbb{S}^2$. Generally, $M = \mathbb{S}^{n_1} \times \mathbb{S}^{n_j}$, for $n_1, \cdots, n_j \in \mathbb{N}^+$. We make the identification $\mathbb{S}^0 = \{\pm 1\}$ as the two-point space.

Remark 7. The spaces $X \times (Y \times Z)$ and $(X \times Y) \times Z$ are canonically homeomorphic.

Lemma 2. *Let $f : W \to Y$ and $g : W \to Z$, and $h : X \to Z$ be maps of topological spaces. Then, the following are continous:*

(i) $f \times g : W \to Y \times Z$ defined by $w \mapsto (f(w), g(w))$,

(ii) $f \times h : W \times X \to Y \times Z$ defined by $(w, x) \mapsto (f(w), g(x))$.

Proof.

(i) Let $U_\alpha \in \tau_Y$, and $V_\beta \in \tau_Z$. Then,

$$(f \times g)^{-1}(U_\alpha \times V_\beta) = f^{-1}(U_\alpha) \cap g^{-1}(V_\beta),$$

which is open by continuity of f and g.

(ii) Similarly,

$$(f \times h)^{-1}(U_\alpha \times V_\beta) = f^{-1}(U_\alpha) \times h^{-1}(V_\beta),$$

which is open again by the continuity of f and h.

□

Remark 8. For any space W, we have

$$\Delta : W \longrightarrow W \times W, \quad w \longmapsto (w, w),$$

which is called the *diagonal map*. Let us write

$$f \overline{\times} g := (f \times g) \circ \Delta.$$

Exercise 4. *Show that Δ is continuous for all topological spaces.*

Lemma 3. *Consider the map*

$$\mathrm{pr}_j : \prod_{i=1}^{n} X_i \longrightarrow X_j,$$

which projects onto the i-th factor. Then, pr_j is continuous.

Proof. Exercise. □

Remark 9. $f : W \to Y \times Z$ is continuous if and only if $\mathrm{pr}_Y \circ f$ and $\mathrm{pr}_Z \circ f$ are continuous.

Recall from last time that we defined continuous maps

$$+ : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad \times : \mathbb{R}^2 \longrightarrow \mathbb{R}.$$

Lemma 4. *The map*

$$\mathrm{d}p : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}, \quad (\mathbf{x}, \mathbf{y}) \longrightarrow \sum_{i=1}^{n} x_i y_i,$$

is continuous.

Proof. Take projections $\mathrm{pr}_i \times \mathrm{pr}_j : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}$, and compose this with $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The projection maps are continuous by Lemma 3, and thus their products are continuous by Lemma 2[(i)]. The addition map is continuous, as aforementioned last lecture. □

1.2.2. The Quotient Topology

Let X be a topological space, and $\sim$ an equivalence relation on X.

Example 11. *On $\mathbb{R}$, we can define an equivalence relation stating that $x \sim y$ if and only if $x - y \in \mathbb{Z}$*

Example 12. *If a group G acts on X, then we set $x \sim y$ if and only if there exists $g \in G$ such that $x = yg$. We can quotient by the group action and obtain the space X/G.*

Let

$$\overline{X} := \{[x] : x \in X\},$$

where

$$[x] := \{y \in X : y \sim x\},$$

is the *equivalence class* of x. There is a quotient map

$$q : X \longmapsto \overline{X}, \quad x \longmapsto [x].$$

It is clear that q is surjective. We topologise $\overline{X}$ by defining open sets in $\overline{X}$ to be those sets U for which $q^{-1}(U)$ is open under the topology on X. Denote the resulting topology by $\overline{\tau}$.

Example 13. *Let $X = \mathbb{R}$, and define a $\mathbb{Z}$-action on $\mathbb{R}$ by $n \cdot x := x + n$. This is equivalent to defining an equivalence $x \sim y$ on $\mathbb{R}$ defined by the condition that $x - y \in \mathbb{Z}$. Then, the resulting quotient space is given by $\overline{\mathbb{R}} = \mathbb{R}/\mathbb{Z}$, which is homeomorphic to $\mathbb{S}^1$, since $0 \sim 1$.*

Lemma 5. *Given a continuous map $f : X \to Y$, and a quotient map $q : X \to \overline{X}$, there exists a continuous map $\overline{f} : \overline{X} \to Y$ such that the diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "q"'] & & Y \\ & \overline{X} \arrow[ur, dashed, "\overline{f}"'] & \end{tikzcd}$$

commutes.

Proof. Define $\overline{f}([x]) = f(x)$. Then, $\overline{f}$ is continuous and we are done. □

Lemma 6. *Any map $\overline{f} : \overline{X} \to Y$ is continuous if and only if $f := \overline{f} \circ q$ is continuous.*

Proof. Suppose $\overline{f}$ is continuous. Then, since q is continuous, then $f = \overline{f} \circ q$ is also continuous.

Suppose f is continuous. Let $U \in \tau_Y$, and consider the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "q"'] & & Y \\ & \overline{X} \arrow[ur, dashed, "\overline{f}"'] & \end{tikzcd}$$

Observe that $f^{-1}(U) = q^{-1}(\overline{f}^{-1}(U))$. Since f is continuous, it follows that $\overline{f}^{-1}(U)$ is also open. □

Example 14. *Consider the diagram*

$$\begin{tikzcd} \mathbb{R} \arrow[rr, "\exp"] \arrow[dr, "q"'] & & \mathbb{S}^1 \\ & \mathbb{R}/\mathbb{Z} \arrow[ur, dashed, "\overline{f}"'] & \end{tikzcd}$$

where $\exp : t \mapsto e^{2\pi i t}$. *Check that $\overline{f}$ is a continuous bijection. Since $\mathbb{R}/\mathbb{Z} = [0,1]/\sim$, it follows that $\mathbb{R}/\mathbb{Z}$ is compact. Moreover, since $\mathbb{S}^1$ is Hausdorff and compact, it follows then that $\overline{f}$ is a homeomorphism.*

1.2.3. Topological Equivalence

Lemma 7. *If $f : X \to Y$ and $g : Y \to Z$ are homeomorphisms, then so are $g \circ f : Y \to Z$ and $f^{-1} : Y \to X$.*

Corollary 1. *Homeomorphisms defines an equivalence relations on the set of all topological spaces.*

Exercise 5. *Show that compactness is preserved under homeomorphisms.*

Goal: Define a weaker equivalence relation on topological spaces.

In particular, we wish to say that $X \cong Y$ if and only if there exists $f : X \to Y$ and $g : Y \to X$ such that $f \circ g = 1_Y$, and $g \circ f = 1_X$.

1.2.4. Homotopy

Homotopy defines an equivalence relation on maps $X \to Y$. We say that $f, g : X \to Y$ are *homotopic* if there exists a $H : I \times X \to Y$ such that $f = H|_{\{0\}\times X}$, and $g = H|_{\{1\}\times X}$.

1.3. Lecture 3, 01/03/2024

Definition 9. Let $f, g : X \to Y$ be continuous maps. Then, f is *homotopic* to g – denoted $f \sim g$ if there exists a map $H : X \times I \to Y$ such that $f = H|_{X\times\{0\}}$, and $g = H|_{X\times\{1\}}$.

In effect, this allows us to "interpolate" between f and g.

Remark 10. One can topologise $\mathrm{Map}(X, Y)$.

Observe that we have a family of continuous maps $H_t := H|_{X\times\{t\}}$ for all $t \in [0, 1]$.

Definition 10. A *path* in a space Z is a map $\gamma : I \to Z$.

With this definition, one can view homotopies as paths in the space of continuous maps $\mathrm{Map}(X, Y)$. So, an alternate definition is to say that a homotopy is a path of continuous maps in $\mathrm{Map}(X, Y)$.

Example 15. *Let $X = \mathbb{S}^1$, and $Y = \mathbb{R}^2 \supset \mathbb{S}^1$. Then, consider the maps $f, g : \mathbb{S}^1 \to \mathbb{R}^2$ by $f(x, y) = (x, y)$, and $g(x, y) = (2x, 2y)$. The first map is the identity map, and the second map scales the radius of the circle by 2. Define*

$$H : \mathbb{S}^1 \times I \longrightarrow \mathbb{R}^2, \quad ((x, y), t) \longmapsto (1 + t)(x, y),$$

which defines a homotopy of f into g. Observe in particular that H lands in $\mathbb{R}^2 \setminus \{0\}$. In fact, all maps into any $\mathbb{R}^n$ are homotopic. Given any $f, g : X \to \mathbb{R}^n$, we may define a homotopy

$$H : X \times I \longrightarrow \mathbb{R}^n, \quad (x, t) \longmapsto (1 - t)f(x) + tg(x),$$

called the straight line homotopy*.*

Example 16. *Consider $f : \mathbb{S}^1 \to \mathbb{R}^2 \setminus \{0\}$ and $c : \mathbb{S}^1 \to \mathbb{R}^2 \setminus \{0\}$ mapping $c : (x, y) \longmapsto x$. These maps are* not *homotopic. We will see why this is the case later (in like, a month).*

1.3.1. Homotopy Equivalence of Spaces

Definition 11. A map $f : X \to Y$ is a *homotopy equivalence* if and only if there exists $g : Y \to X$ such that

$$g \circ f \sim \mathrm{id}_X, \quad f \circ g \sim \mathrm{id}_Y,$$

are homotopic. If this is the case, then we say that X and Y are *homotopy equivalent* – denoted by $X \simeq Y$.

Remark 11. Homeomorphisms are naturally homotopy equivalences. But this gives a much broader equivalence relation for topological spaces.

However, note that it remains to be shown that if $f : X \to Y$ and $h : Y \to Z$ are homotopy equivalences, then so is $h \circ f$.

To show this, we know that $g \circ f \sim \mathrm{id}_X$, $i \circ h \sim \mathrm{id}_Y$, where $i : Z \to Y$. Then, we wish to show that $(g \circ i)\circ)(h \circ f) \sim \mathrm{id}_X$. Observe that this can be re-written as $g \circ (i \circ h) \circ f \sim \mathrm{id}_Y$. So, now we may construct a composition of maps

$$I \times X \xrightarrow{\mathrm{id} \times f} I \times Y \xrightarrow{H} I \times Y \xrightarrow{\mathrm{id} \times g} I \times X.$$

It now remains to check that $(\mathrm{id} \times f) \circ H \circ (\mathrm{id} \times g)$ is a homotopy from $g \circ i \circ h \circ f$ to $g \circ f$. This follows as a result of the gluing lemma.

Example 17. *Let us show that $\mathbb{R}^n \simeq \mathrm{pt}$, where pt is the one-point space.*

Proof. Define $c : \mathbb{R}^n \to \mathrm{pt}$ and $i : \mathrm{pt} \to \mathbb{R}^n$. One checks that $c \circ i = \mathrm{id}_{\mathrm{pt}}$, and that $i \circ c$ is the constant map — that is, every point in $\mathbb{R}^n$ gets mapped to a point. Then just use the straight line homotopy.

But all maps to $\mathbb{R}^n$ from any space are homotopic. Hence, $i \circ c$ is homotopic to $1_{\mathbb{R}^n}$. We say that any space homotopy equivalen to pt is *contractible.* We say that a map homotopic to a constant map is *null-homotopic.* Further, a non-null-homotopic map is *essential.* $\square$

Example 18. *Consider the identity map $1_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{S}^1$ Is this essential or null-homotopic? The answer is that this map is essential.*

Example 19. *$1_X : X \to X$ is null-homotopic if and only if X is contractible.*

Let us consider the map:

$$f_d : \mathbb{S}^1 \longrightarrow \mathbb{S}^1, \quad z \longmapsto z^d, \quad d \in \mathbb{N}.$$

Question: Is f_d homotopic or null-homotopic?

Answer: The answer is no. We will explain why in two weeks.

Proposition 1 (Fascinating Fact)**.** *There exists essential aps $\mathbb{S}^{n+k} \to \mathbb{S}^k$ for n, k with $k > 1$.*

Example 20. *One such example of an essential map is the Hopf map from $\mathbb{S}^3 \to \mathbb{S}^2$.*

1.3.2. Cell Complexes

Cell Addition

Let $f : \mathbb{S}^{n-1} \to A$ be a map. Let D^n be the n0disk, and define

$$A \cup_f D^n := (A \sqcup D^n)/\sim_f,$$

where $A \sqcup D^n$ is topologise by defining the open sets to be those either coming from D^n or A, and the equivalence relation $\sim_f$ is given by $x \sim_f f(x)$ for all $x \in \mathbb{S}^{n-1}$. Check that $a \sim a'$ if and only if $a = a'$, for $a, a' \in A, D^n \setminus \mathbb{S}^{n-1}$, $x \sim x'$ if and only if $x = x'$ for $x, x' \in \mathrm{Int}(D^n) := D^n \setminus \mathbb{S}^{n-1}$, and $x \sim x'$ if and only if $f(x) = f(x')$ for $x, x' \in \mathbb{S}^{n-1}$.

Example 21. *We have that*

$$\mathbb{S}^1 = \mathrm{pt} \cup_{\mathrm{const}} D^1,$$

where const *is the constant map.*

Example 22. *Consider the 2-torus $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$. We may write this as*

$$T^2 = (\mathbb{S}^1 \vee \mathbb{S}^1) \cup_\varphi D^2,$$

where for (X, x) and (Y, x) pointed spaces, the product $\vee$ is the disjoint union

$$X \vee Y := X \sqcup Y,$$

obtained by setting $x = y$ and no other relations. Assume that $\mathbb{S}^1 \times \mathbb{S}^1 \cong (I \times I)/\sim$, where $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, y)$ is the equivalence relation. Visually, this is the square construction where opposite ends are identified and glued together. Note that $I \times I \cong D^2$.

Definition 12 (Informal Definition)**.** X is a *CW-complex* if there is a fitration

$$X^0 \subset X^1 \subset X^2 \subset \cdots \subset X,$$

such that

(i) X^0 is discrete,

(ii) (Inductive Step) $X^k = X^{k-1} \cup_{\varphi_i} D_i^k$, where D_i^k are *affine k-cells*,

(iii) $\dim X = \max\{k : X^k \neq X^{k-1}\}$. If $\dim X$ is infinite, then $U \subset X$ is open if and only if $U \cap X^k$ is open for all k.

Remark 12 (Diarmuid Wisdom)*.* If you think about everything just purely in terms of CW-complexes, you are guaranteed to get a high distinction in this class!

Remark 13 (Fun Fact)*.* Every manifold is a CW-complex except possibly in dimension 4.

Idea of the Class: Start with topology. From our point of view, they are given by continuous maps between topological spaces $f : X \to Y$.

From this, we can get the "algebra" part of "algebraic topology" by considering homology groups $H_i(X)$ and $H_i(Y)$. Then, any continuous map $f : X \to Y$ induces a map $f_* : H_i(X) \to H_i(Y)$ in homology. Moreover, if $f \sim g$ then $f_* \sim g_*$. From this, we can deduce that homotopy equivalent spaces have isomorphic homotopy groups.

Given a map of pointed spaces $f : (X, x) \to (Y, y)$, we can associate to them homotopy groups $\pi_i(X)$ and $\pi_i(Y)$, which are "similar" to homology groups. It induces a map $f_* : \pi_i(X) \to \pi_i(Y)$. When $i = 1$, we get an important class of groups called fundamental groups.

Chapter 2

Week Two

2.1. Lecture 1, 06/03/2024

2.1.1. Paths and Path Homotopy

Remark 14 (Unrecorded Diarmuid Wisdom). Should have at least 6 hours of study time – both prepping for lectures and working through exercises. Good idea to have a look at what lectures will be covering before the lectures.

Let X be a topological space. One simple way that we will make groups from spaces is the idea of a path homotopy.

Definition 13. A *path* in X is a continuous map $\gamma : I \to X$ such that $f(0) = x_0$, and $f(1) = x_1$.

Remark 15. The map does *not* have to be injective – i.e. it can have self-intersections.

Many paths can have the same endpoints. To capture this idea, we may define the idea of a *path homotopy*. In particular, we only consider paths up to homotopy.

Definition 14. A *path homotopy* f_t is a homotopy fixed on $I \times \{0, 1\}$. That is, it is a map $f_t : I \times I \to X$ defined by $(s, t) \mapsto f_t(s)$.

Definition 15. $\pi_0(X)$ is the set of all equivalence classes of points given by $x_0 \sim x_1$ if there is a path from x_0 to x_1.

Indeed, $x_0 \sim x_0$ via the constant path c_{x_0}. Given $x_0 \sim x_1$, we have $x_1 \sim x_0$ by taking the path going in the opposite direction. If f is a path from x_0 to x_1, we denote the path going in the opposite direction by $\overline{f}(s) = f(1-s)$. Given any a path f given by $x_0 \sim x_1$, and g given by $x_1 \sim x_2$, then we can glue the paths together the same way we glue continuous functions together. In particular, $x_0 \sim x_2$ via

$$f \cdot g(s) := \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases} .$$

Definition 16. X is *path-connected* if $|\pi_0(X)| = 1$. Equivalently, any pair of points on X has a path between them.

Example 23. *If $X = V$ is a topological vector space over $\mathbb{R}$ (i.e. $V = \mathbb{R}^n$). If $f_0(0) = f_1(0)$, and $f_0(1) = f_1(1)$, then f_1 and f_2 are path homotopic via the* straight line homotopy*:*

$$f_t(s) = (1-t)f_0(s) + tf_1(s).$$

Remark 16. The above construction does not make sense if we do not have a vector space structure on X.

Proposition 2. *Path homotopy defines an equivalence relation on the set of paths from x_0 to x_1.*

Proof. Exercise. Use the gluing lemma. $\square$

2.1.2. The Fundamental Groupoid

Recall that given $f_0(1) = f_1(0)$, we may *concatenate* the paths by:

$$f_0 \cdot f_1 = \begin{cases} f_0(2s) & 0 \le s \le 1/2 \\ f_1(2s-1) & 1/2 \le s \le 1 \end{cases} .$$

Definition 17. The *fundamental groupoid* – denoted by $\pi_1(X)$ – is the set

$$\pi_1(X) := \{[f] : \text{path homotopy classes of all paths in } X\}.$$

Given a map

$$\text{ep} : \pi_1(X) \longrightarrow X \times X, \quad [f] \longmapsto (f(0), f(1)),$$

then define

$$\pi_1(X, x_0, x_1) := \text{ep}^{-1}(x_0, x_1).$$

Then, we have maps

$$\pi_1(X, x_0, x_1) \times \pi_1(X, x_1, x_2) \longrightarrow \pi_1(X, x_0, x_2), \quad \longmapsto ([f], [g]) \longmapsto ([f \cdot g]).$$

One can use this to show associativity.

Definition 18. A path f for which $f(0) = f(1)$ is called a *loop*. Let $\pi_1(X, x_0) := \pi_1(X, x_0, x_0)$ be the set of path homotopy classes of loops at x_0. This is called the *fundamental group* of X.

We will now see why this is called a fundamental *group*:

Proposition 3. *$\pi_1(X, x_0)$ is a group, with identity given by the constant path $[c_{x_0}]$ at x_0. Given any $[f] \in \pi_1(X, x_0)$, its inverse is given by $[\overline{f}]$ – that is, $[f]^{-1} = [\overline{f}]$.*

Remark 17. Observe that $f \cdot (\cdot g \cdot h) \neq (f \cdot g) \cdot h$ in general, since f goes "twice as fast", but $g \cdot h$ goes four times as fast. On the other hand, $f \cdot g$ goes four times as fast, but h goes twice as fast. So they are not the same loop. Hatcher explains this idea much more nicely.

First, a lemma:

Lemma 8 (Reparametrisation)**.** *Let $\varphi : I \to I$ be a map such that $\varphi(0) = 0$ and $\varphi(1) = 1$. Then, $f \circ \varphi \sim f$.*

Proof. Consider the straight line homotopy $\varphi_t(s) = (1-t)\varphi(s) - st$. Note that $\varphi_t(s)$ lies between s and $\varphi(s)$ for all t. Then, $f_t = f \circ \varphi_t$ is a path homotopy from f to $f \circ \varphi$. $\square$

Proof of Proposition 3. $f \cdot (g \cdot h)$ is obtained by a reparametrisation of $(f \cdot g) \cdot h$ with a piecewise linear path (see Hatcher for the picture). Call this graph $\Gamma(\varphi)$. Hence, $[f \cdot (g \cdot h)] = [(f \cdot g) \cdot h]$. By definition, $([f] \cdot [g]) \cdot [h] = [f] \cdot ([g] \cdot [h])$, and this proves associativity.

$f \cdot c$ just makes f goes around twice as fast, and then does nothing for the remaining time. $c \cdot f$ does nothing for half the time, then makes f go aroudn twice as fast. To see this mathematically, one reparametrises $\overline{c}$ as graphs (see Hatcher again for the pictures).

To show that $[f]^{-1} = [\overline{f}]$, one needs to show that $\overline{f} \cdot f \sim c \sim f \cdot \overline{f}$. Set

$$f_t(s) = \begin{cases} f(t) & s \geq 1 - t \\ f(s) & 0 \leq s \leq 1 - t \end{cases}.$$

Then, set $h_t := \overline{f}_t \cdot f_t$, which is always a loop. Indeed, $h_0 = \overline{f}_0 \cdot f_0 = \overline{f} \cdot f$, and similarly $h_1 = \overline{f}_1 \cdot f = \overline{c} \cdot c = c$. It is an exercise to check that continuity, and that h_t defines a path homotopy. Similarly, check that $\overline{h}_t = f_t \cdot \overline{f}_t$ is a path homotopy from $f \cdot f$ to c. ☐

Proposition 4 (Basepoint Independence)**.** *For any two points x_0 and x_1, $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.*

Proof. If there exists a path h from x_0 to x_1, then we can define

$$\beta_h : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1), \quad [f] \longmapsto [\overline{h} \cdot f \cdot h].$$

If $[h] = [h'] \in \pi_1(X, x_0, x_1)$, then $\beta_h = \beta_{h'}$. This allows us to define $\beta_{[h]}$ one the equivalence classes of paths in $\pi_1(X, x_0, x_1)$. We show that β_h is an isomorphism. First, we see that

$$\beta_h([f] \cdot [g]) = [\overline{h} \cdot f \cdot h \cdot \overline{h} \cdot g \cdot h] = [\overline{h} \cdot f] \cdot [\overline{h} \cdot g \cdot h] = \beta_h([f]) \cdot \beta_h([g]).$$

We leave it as an exercise to check that $\beta_{\overline{h}}$ is a suitable choice of inverse. That is,

$$\beta_h \circ \beta_{\overline{h}} = \beta_{h \cdot \overline{h}} = \beta_{[c_{x_0}]} = \mathrm{id}_{\pi_1(X, x_0)}.$$

For the other direction, $\beta_{\overline{h}} \circ \beta_h = \beta_{\overline{h} \cdot h}\, \mathrm{id}_{\pi_1(X, x_1)}$. ☐

Remark 18 (Diarmuid Wisdom). Learning when you have to care about basepoints and ignore them is an important skill in algebraic topology.

2.2. Lecture 2, 07/03/2024

Definition 19. X is called *simply-connected* if

(i) X is path-connected,

(ii) $\pi_1(X, x_1, x_0) = 1$ – that is, all choices of paths are homotopic to one another

Proposition 5. *A path-connected space is simply-connected if and only if $\pi_1(X)$ is trivial.*

Proof. The $\Longleftarrow$ direction is clear. In particular, $\pi_1(X, x)$ acts freely and transitively on $\pi_1(X, x_0, x)$ by concatenation. The proof of this is left as an exercise. ☐

2.2.1. Fundamental Group of $\mathbb{S}^1$

We wish to consider $\pi_1(\mathbb{S}^1, 1)$.

For some $n \in \mathbb{Z}$, let $\omega_n(s) = e^{2\pi i n s}$, which is the loop that wraps around the circle n times. Intuitively, this tells us that $\pi_1(\mathbb{S}^1)$ should be isomorphic to $\mathbb{Z}$. This turns out to be the case.

Theorem 1. $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$.

The proof uses the idea of a covering space.

Definition 20. A *covering space* of X is a map $p : \widetilde{X} \to X$ such that for every $x \in X$, there exists an open set U containing x such that $p^{-1}(U)$ is homeomorphic to a disjoint union $\sqcup_{\alpha \in \Lambda} U_\alpha$ such that $p|_{U_\alpha} : U_\alpha \xrightarrow{\simeq} U$. Such a U is said to be *evenly covered*.

Exercise 6. *Show that $p : \mathbb{R} \to \mathbb{S}^1$ is a covering space.*

We have some properties of covering spaces:

(i) Given a map $f : I \to X$ given by $f(0) = x_0$, $f(1) = x_1$, then there exists a unique $\widetilde{f} : I \to \widetilde{X}$ lifting f – that is, the lift $\widetilde{f}$ starts at $\widetilde{f}(0) = \widetilde{x_0}$, and satisfies $p \circ \widetilde{f} = f$. That is, we have a commutative diagram:

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"'] \\ I \arrow[ur, dashed, "\exists!"] \arrow[r, "f"] & X \end{tikzcd}$$

(ii) For each homotopy $f_t : I \times I \to X$ of paths starting at x_0 and $\widetilde{x_0} \in p^{-1}(x_0)$, there is a unique lift $\widetilde{f_t} : I \to \widetilde{X}$ of paths starting at $\widetilde{x_0}$ such that the diagram commutes

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"'] \\ I \times I \arrow[ur, dashed, "\exists!"] \arrow[r, "f_t"'] & X \end{tikzcd}$$

Remark 19. The proof below is messily transcribed. You're much better off just going just reading page 30 of Hatcher, "Proof of Theorem 1.7".

Proof of Theorem 1. Define a map

$$\Phi : \pi_1(\mathbb{S}^1) \longrightarrow \mathbb{Z}, \quad [f] \longmapsto \widetilde{f}(1),$$

which is well-defined by Property (ii) of covering spaces. Further, if $f \sim f'$ via f_t, then we have $\widetilde{f}'_t(1) : I \to \mathbb{R}$, which is continuous, and I is path-connected. Therefore, $f'_t(I)$ is path-connected and constant. This implies that $\widetilde{f}'_t(1) = \widetilde{f}(1)$, and we thus have a well-defined map. By construction, we have

$$\Phi([\omega_n]) = \widetilde{\omega_n}(1) = n.$$

By Property (i), we then have that $\widetilde{\omega_n}(s) = ns$.

Suppose that $\Phi([f]) = 0$. Then, $\widetilde{f}(0) = 0 = \widetilde{f}(1)$. Take a straight line path homotopy from the loop $\widetilde{f}$ to the constant map c_0. Then, $p \circ \widetilde{f_t}$ is a path homotopy to c_1.

Exercise 7. *Show that if Φ is a homotopy, then Φ is a bijection.*

Proof. Just translate lifts along to show this. □

We have one more property of covering spaces:

(iii) Given a homotopy $F : Y \times I \to X$, and a covering space $p : \widetilde{X} \to X$, there exists a unique lift $\widetilde{F}$ of F wih $G : Y \times \{0\} \to \widetilde{X}$ such that $p \circ G = F|_{Y \times \{0\}}$, and such that $\widetilde{F}|_{Y \times \{0\}} = G$. That is, we have a commutative diagram:

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "G"] \arrow[d] & \widetilde{X} \arrow[d, "p"'] \\ Y \times I \arrow[ur, dashed, "\exists!"] \arrow[r, "F"'] & X \end{tikzcd}$$

Proof of Property (iii). Let $y \in Y$, and find a neighbourhood $y_0 \in N$ and construct $\widetilde{F}$ on $N \times I$. By continuity of F and definition of product topologyon $N \times I$, there exists neighbourhoods $N_t \times (a_t, b_t)$ containing (y_0, t) such that $F(N_t \times (a_t, b_t)) \subset U_t$ for some evenly covered U_t. Since every point in X has such a neighbourhood, we may take $F(y_0, t) = x_0$. By comppactness of $\{y_0\} \times I$, a finite number of these cover $y_0 \times I$. There thus exists a partitionof I $0 = t_0 < t_1 < \cdots < t_n = 1$ such that there exists a neighbourhood N of y_0 such that $F(N \times [t_i, t_{i+1}])$ is evenly covered.

Then, proceeding by induction, assume that $\widetilde{F}$ has been constructed on $N \times [0, t_1]$. Replace N with $\widetilde{F}^{-1}|_{N\times[0,t_1]}(\widetilde{U}_t)$ for $\widetilde{F}(t_0, t_i) \subset \widetilde{U}_t$. Then, from here define $\widetilde{F}$ on $N \times [0, t_{i+1}]$ by $\widetilde{F} := (p|_{\widetilde{U}_{t_{i+1}}})^{-1} \circ F$. Then, by gluing, $\widetilde{F}$ on $N \times [0, t_{i+1}]$ is continuous.

Exercise 8. *Show that $\widetilde{F}|_{y_0 \times I}$ is unique (or just look at Hatcher).*

Now, just glue all these solutions together to obtain the cover $Y = \bigcup_{y_0}(I \times N_{y_0})$. □

□

Observe that given the isomorphism $p : \mathbb{R} \to \mathbb{S}^1$, then observe that $\pi_1(\mathbb{S}^1) \cong \mathbb{Z} = p^{-1}(0)$. This is not a coincidence, and we ill come back in a month to see why this happens.

Theorem 2 (Gauss' Fundamental Theorem of Algebra)**.** *Let $p(e)$ be a non-constant polynomial $p : \mathbb{C} \to \mathbb{C}$. Then, there exists $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.*

Proof. Without loss of generality, let $p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$. For some $r \in [0, \infty)$, define

$$f_r(s) = \frac{p(re^{2\pi is})/p(r)}{|p(re^{2\pi is}/p(r)|}.$$

Note here that $f_0 = c_1$. As r varies continuously, f_r is path-homotopic to $f_0$for all r. Now, observe that $r > |a_1| + \cdots + |a_n|$, and $r > 1$, and that

$$|z^n| > (|a_1| + \cdots + |a_n|)|z|^{n-1} \geq |a_1 z^{n-1}| + \cdots + |a_n| \geq |a_1 z^{n+1} + \cdots + a_n|.$$

It follows then that $p_t(z) = z^n +_t (a_1 z^{n-1} + \cdots + a_n)$. It follows then that p_t has no roots in $\mathbb{S}^1_r$ for $0 \leq t \leq 1$. Then, $f_r = p_r(s)$, but $p_0(s) = \omega_n(s)$. It follows then that $0 = [f_0] \cdot [f_r] = [p_0] = [\omega_n] = n$. Thus $n = 0$, and we are done. □

2.3. Lecture 3, 08/03/2024

Today, we will consider maps $\mathbb{S}^2 \to \mathbb{R}^2$, and we will prove the Borsuk-Ulam theorem.

Theorem 3 (Borsuk-Ulam)**.** *Every continuous map $f : \mathbb{S}^1 \to \mathbb{R}^2$ maps pairs of antipodal points to the same point.*

Proof. Consider $g(x) = f(x) - f(-x)$. By contradiction, suppose that no such x exists for which $g(x) = 0$. Then,

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|} \in \mathbb{S}^1,$$

where by abuse of notation we denote the above function by g. set

$$\eta : \mathbb{S}^1 \to \mathbb{S}^1, \quad (x, y) \longmapsto (x, y, 0)$$

where the image of η is a copy of a great circle in $\mathbb{S}^2$. Let $\widetilde{g \circ \eta}$ be the unique lift of $g \circ \eta$ – then, $\widetilde{g \circ \eta}(t + 1/2) = \widetilde{g \circ \eta}(t) + \frac{2k\pi+1}{2}$, for some $k \in \mathbb{Z}$. Then, let us consider the continuous function

$$I \longrightarrow \frac{1}{2}\mathbb{Z}, \quad t \mapsto \widetilde{g \circ \eta}(t + 1/2) - \widetilde{g \circ \eta}(t)$$

. Then, since $\frac{1}{2}\mathbb{Z}$ is discrete, this function is constant. It follows then that $\widetilde{g \circ \eta}(1) = \widetilde{g \circ \eta}(1/2) + \frac{2k\pi+1}{2}$, and so $\widetilde{g \circ \eta}(0) = 0$, but also $\widetilde{g \circ \eta}(0) = \frac{2k\pi+1}{2}$, which is non-zero. It follows that $g \circ \eta : I \to \mathbb{S}^1$ represents $[g \circ \eta] = 2k + 1 \in \mathbb{Z} \cong \pi_1(\mathbb{S}^1)$, and moreover $2k + 1 \neq 0$. But $\eta : I \to \mathbb{S}^1$ is path homotopic to a constant map. This then implies that $g \circ \eta$ is homotopic to a constant, and so $[g \circ \eta] = 0$, which is a contradiction. $\square$

Theorem 4 (Brower's Fixed Point Theorem, Thm 1.10, Hatcher)**.** *Let $h : D^2 \to D^2$ be a map. Then, there exists $x \in D^2$ such that $h(x) = x$, where*

$$D^2 = \{x \in \mathbb{R}^2 : |x| \leq 1\}.$$

Definition 21. A *retraction* if a map $r : X \to A$ such that $r|_A = 1_A$.

Proof Idea. We wish to construct a retraction $D^2 \to \mathbb{S}^1 \subset D^2$. Use the fact that $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ to show this is impossible.

Define $r : D^2 \to \mathbb{S}^1$ by $r(x) = L_x \cap \mathbb{S}^1$, where L_x is the line passing through x. Then, we claim that if h is continuous, then so is r. If $x \in \mathbb{S}^1$, then $r(x) = X$ as required. See Hatcher for rest of the proof. $\square$

2.3.1. Induced Homomorphisms

Given a map $\varphi : X \to Y$, the *induced homomorphism* is the group homomorphism

$$\varphi_* : \pi_1(X, x) \longrightarrow \pi_1(Y, \varphi(x)), \quad [f] \longmapsto [\varphi \circ f].$$

One checks that φ_* is well-defined – in particular, one checks that composition of maps commutes with compositions of path homotopy.

Proposition 6 (Prop 1.12, Hatcher)**.** *Given two spaces X and Y, we have*

$$\pi_1(X \times Y, (x, y)) \cong \pi_1(X, x) \times \pi_1(Y, y).$$

Proof. Consider projections $\mathrm{pr}_X : X \times Y \to X$ and $\mathrm{pr}_Y : X \times Y \to Y$. Then, any map $f : Z \to X \times Y$ is continuous if and only if $\mathrm{pr}_\mathrm{X} \circ f$ and $f \circ \mathrm{pr}_Y$ are continuous. Define

$$\Phi : \pi_1(X \times Y, (x, y)) \longrightarrow \pi_1(X, x) \times \pi_1(Y, y), \quad [f] \longmapsto ([\mathrm{pr}_X \circ f], [\mathrm{pr}_Y \circ f]).$$

Φ defines a group homomorphism since $(\mathrm{pr}_X)_*$ and $(\mathrm{pr}_Y)_*$ define group homomorphisms. $\square$

Example 24. *Let $\mathbb{T}^2$ be the 2-torus. Then, $\pi_1(\mathbb{T}^2) = \pi_1(\mathbb{S}^1 \times \mathbb{S}^1) \cong \mathbb{Z} \times \mathbb{Z}$. Generally, $\pi_1(T^n) \cong \mathbb{Z}^n$.*

Definition 22. A map $r_t : I \times X \to A$ is a *deformation retraction* if

(i) $r_0 = A$,

(ii) $r_t|_A = 1_A$ for all t,

(iii) $r_1 = A$ – that is, r_1 is a retraction.

Definition 23. If φ_t is homotopic rel x_0, then $\varphi_t|_{I\times\{x_0\}} = c_{x_0}$. This implies then that $(\varphi_0)_* = (\varphi_1)_*$.

Proposition 7 (Prop 1.17, Hatcher)**.** *If $A \subset X$ is a retraction, $x_0 \in A$, then $\pi_1(A, x_0) \subseteq \pi_1(X, x_0)$ is a retraction. In particular, $i_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ is injective. Moreover, if $A \subseteq X$ is a deformation retract, then $i_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ is an isomorphism.*

Remark 20. Note that $\pi_1(-)$ defines a functor

$$\pi_1 : \mathbf{Top} \longrightarrow \mathbf{Ab},$$

where **Top** is the category of topological spaces, and **Ab** is the category of abelian groups.

Proof of Proposition 7. $r \circ i = 1_A$, and thus we have that $(r \circ i)_* = 1$. It follows then that $r_* \circ i_* = 1$, and thus i_* is injective.

If r_t is a deformation retraction, then $i \circ r : X \to A$ is homotopic rel x_0 to 1_X. □

Corollary 2 (Corollary 1.16, Hatcher)**.** $\mathbb{R}^2 \ncong \mathbb{R}^n$*, for* $n \neq 2$*.*

Proof. By contradiction, suppose that

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^n$$

is a homeomorphism. Then,

$$f|_{\mathbb{R}^2\setminus\{0\}} : \mathbb{R}^2 \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{f(0)\},$$

is a homeomorphism. But then we have a map

$$(f|_{\mathbb{R}^2\setminus\{0\}})_* : \pi_1(\mathbb{S}^1) \longrightarrow \pi_1(\mathbb{S}^{n-1}),$$

and by Proposition 9, $\pi_1(\mathbb{S}^{n-1}) = 0$ for $n \geq 2$. But $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$, which is not isomorphic to 0. A contradiction. □

Chapter 3

Week Three

3.1. Lecture 1, 13/03/2024

3.1.1. Homotopy rel A

Maps of pairs are given by maps

$$f : (X, A) \longrightarrow (Y, B),$$

such that the map $f : X \to Y$ has the property that $f(A) \subseteq B$. If we specalise $A = \{x_0\}$ and $B = \{y_0\}$ to be singleton sets, then we get maps of pointed spaces, which are maps such that $f(x_0) = y_0$. Then, there is a notion of a *homotopy rel A*. There is a pointwise and a set-wise definition of this notion. Given maps

$$f, g : (X, A) \longrightarrow (Y, B),$$

we have maps

$$I \times A \longrightarrow I \times B, \quad H : I \times X \longrightarrow Y.$$

Typically, for a deformation retraction we require that $f|_A = g|_A$, and $H|_{I\times A} = f|_A \circ \mathrm{pr}_A$. We mention this because the definition of a deformation retraction is nothing more than a pointwise homotopy rel A. For a setwise homotopy rel A, we simply require that $H|_{\{t\}\times X}$ is a map of pairs.

3.1.2. Pointed Homotopy

Let

$$[X, Y] := \{[f] : f : X \to Y\},$$

where $[f]$ denotes the homotopy class of f. Given pointed spaces, we have:

$$[(X, x), (Y, y)]_* := \{f_* : f : (X, x) \longrightarrow (Y, y)\},$$

where $[f]_*$ denotes the *pointed homotopy class* of f.

The key point is that

$$\pi_1(X, x_0) = [(\mathbb{S}^1, 1), (X, x_0)]_*, [f] \longmapsto [\overline{f}],$$

where $\overline{f}$ is the induced map on $I \setminus \{0, 1\} \to \mathbb{S}^1$.

Remark 21. We have:

$$\pi_0(X) = [(\mathbb{S}^0, 1), (X, x_0)]_*,$$
$$\pi_1(X) = [(\mathbb{S}^1, 1), (X, x_0)]_*,$$

$$\pi_2(X) = [(\mathbb{S}^2, e_1), (X, x_0)]_*,$$

which is pointed-homotopy invariant. Moreover, $\pi_3(X) = [(\mathbb{S}^3, e_1)$, and all $\pi_i(X)$ are abelian groups. These are examples of *higher homotopy groups*, and they will be discussed more in Week 11.

Hatcher has some sneaky proofs for the following results:

Proposition 8 (Prop 1.14, Hatcher). *$\pi_1(\mathbb{S}^n, x_0)$ is trivial for all $n \geq 2$.*

Proposition 9 (Prop 1.18, Hatcher). *If $\varphi : X \to Y$ is a homotopy equivalence between path-connected spaces, then the induced map:*

$$\varphi_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, \varphi(x_0)),$$

is an isomorphism.

Before we can prove Prop 9 from Hatcher, we need the following lemma:

Lemma 9 (Lemma 1.19, Hatcher). *If $\varphi_t : X \to Y$ is a homotopy, then the diagram*

$$\begin{tikzcd}
 & \pi_1(Y, \varphi(x_0)) \arrow[dd, "\beta_h"] \\
\pi_1(X, x_0) \arrow[ur, "(\varphi_0)_*"] \arrow[dr, "(\varphi_1)_*"'] & \\
 & \pi_1(Y, \varphi(x_1))
\end{tikzcd}$$

commutes.

Proof. Go see Hatcher. □

Proof of Proposition 9. Consider the composition of maps $X \xrightarrow{\varphi} Y \xrightarrow{\psi} X$, and let ψ be a homotopy inverse to φ. That is, $\phi \circ \varphi \sim 1_X$, and $\psi \circ \varphi \sim 1_Y$. Thus, there are induced maps

$$\pi_1(X, x_0) \xrightarrow{\varphi_*} \pi_1(Y, \varphi(x_0)) \xrightarrow{\psi_*} \pi_1(X, \psi(\varphi(x_0))) \xrightarrow{\varphi_*} \pi_1(Y, \varphi(\psi(\varphi(x_0)))),$$

and by Lemma 9, $\psi_* \circ \varphi_*$ is an isomorphism, and $\varphi_* \circ \psi_*$ is an isomorphism. □

Final word on basepoints There are maps

$$\pi_1(X, x_0) = [(\mathbb{S}^1, 1), (X, x_0)]_* \longrightarrow [(\mathbb{S}^1, 1), (X, x_0)] \longrightarrow [\mathbb{S}^1, X],$$

obtained by forgetting the relevant structure with each sucessive map. In particular, we may show that the conjugacy classes of $\pi_1(X, x_0)$ is equal to $[\mathbb{S}^1, X]$.

Now we can prove our first negative result:

Proof of Proposition 8. We may cover $\mathbb{S}^n$ by

$$\mathbb{S}^n = A \cup B,$$

where $A = \mathbb{S}^n \setminus \{\text{north pole}\}$, and $B = \mathbb{S}^n \setminus \{\text{south pole}\}$. As spaces, A and B are homeomorphic to $\mathbb{R}^n$, and thus retractible – that is, $\pi_1(A) \cong \pi_1(B) \cong 1$. Given path f, if $f(I) \subseteq B$, then $[f]$ is in the image of the map $\pi_1(B) \to \pi_1(\mathbb{S}^n)$, and thus $[f]$ is trivial since the image of a trivial group is trivial. The same is true for A.

The goal is to show that f is path-homotopic to $[f_1] * \cdots * [f_n]$, where $f_i(I)$ lies either in A or B. To do this, we wish to cover the interval I by $f^{-1}(A)$ and $f^{-1}(B)$, and use the compactness of I to find a partition of I such that $f([t_i, t_{i+1}])$ is contained in either A or B. Suppose that $f_i([t_i, t_{i+1}])$ is contained in A, and assume inductively that $f(t_i) = x_0$. Since $A \cap B$ is path-connected, define a path h from $f(t_{i+1})$ to x_0. Then, add $h_i\overline{h_i}$ to f_i. Then, define

$$f_i := f|_{[t_i, t_{i+1}]},$$

from which we obtain:

$$f = f_0 * f_1 * \cdots * f_m = (f_0 * h_0) * (\overline{h_0} * f_1 * h_1) * (\overline{h_1}) * f * h_2) * \cdots * (\overline{h_m} * f_m),$$

and each $f_i * h_i$ has image in either A or B. □

3.2. The Vam Kampen Theorem

Goal: Suppose that $X = \bigcup_\alpha A_\alpha$ such that $x_0 \in A_\alpha$ is path-connected, and $A_\alpha \cap A_\beta$ is path-connected for all α, β, and $A_\alpha \cap A_\beta \cap A_\gamma$ is path connected for all α, β, γ. Then, we wish to determine $\pi_1(X, x_0)$ in terms of the components $\pi_1(A_\alpha, x_0)$.

3.2.1. Some Algebra

To achieve the goal, we need some algebra.

Definition 24. The *free product* of groups G_α is the group $*_\alpha G_\alpha$, whose underlying set is given by:

$$W = \{\text{reduced words from } G_\alpha : [\omega]\}.$$

In particular, a *reduced word* is a string $g_1 \cdots g_m$ where $g_i \in G_{\alpha_i} \setminus \{e_{\alpha_i}\}$, and if $g_i \in G_{\alpha_i}$, then $g_{i+1} \notin G_{\alpha_i}$. such An *alphabet* is the set

$$\sqcup_\alpha (G_\alpha \setminus \{e_\alpha\}).$$

In fact, reduced words can be thought of as equivalence classes of all words. The group operation $*$ is given by *juxtaposition* and *reduction*: given two reduced words $\omega_1 = g_1 \cdots g_m$ and $\omega_2 = h_1 \cdots h_n$, then

$$\omega_1\omega_2 = g_1 \cdots g_m h_1 \cdots h_n.$$

3.3. Lecture 2, 14/03/2024

Recall that the general setting of the van Kampen theorem is that we are trying to compute the fundamental group of a space after we have chopped it up into a bunch of pieces whose intersections are path-connected. The required algebra is the algebra of the free product.

The empty word ϕ denotes the identity in the free group $*_\alpha G_\alpha$. One readily checks that inverses hold. Indeed, associativity holds as well, and one can simply grind it out. But there is a much more clever way of proving this. Let $\mathfrak{S}_W$ denote the permutation group on the set W of reduced words, whose elements are bijections $K : W \to W$ under composition. Define a map

$$W \longrightarrow \mathfrak{S}_W, \quad g \longmapsto L_g,$$

where L_g is the left-multiplication map acting on a reduced ω by $L_g(\omega) = g\omega$. From this, one then checks that $L_{gg'} = L_g L_{g'}$. From this, one can then define the map

$$\Phi : W \longrightarrow \mathfrak{S}_W, \quad g_1, \cdots, g_m \longmapsto L_{g_1 \cdots g_m},$$

which is injective since $L_{g_1 \cdots g_m}(\phi) = g_1 \cdots g_m$. It follows then that if $L_{g_1 \cdots g_m} = L_{g'_1 \cdots g'_m}$, then $g_1 \cdots g_m = g'_1 \cdots g'_m$. By construction, Φ maps juxtaposition in W to composition in $\mathfrak{S}_W$. It is injective, and thus juxtaposition is associative since composition is associative in $\mathfrak{S}_W$. It thus follows that $*_\alpha G_\alpha$ is a group.

Example 25 (The Free Product on 2 Words)**.** *Let*

$$F_2 := \mathbb{Z} * \mathbb{Z} = \{a_1^{k_1} b_1^{k_2} \cdots\},$$

where each $k_i \in \mathbb{Z} \setminus \{0\}$ *for each* $i > 2$*. Then, we can define* $F_n := \mathbb{Z} * \cdots * \mathbb{Z}$*.*

3.3.1. Amalgamated Free Product

We have that $\pi_1(A_\alpha \cap A_\beta) = \{e\}$. Then, we have a commutative diagram

$$\begin{tikzcd} & \pi_1(A_\alpha, x_0) \arrow[dr] & \\ \pi_1(A_\alpha \cap A_\beta, x_0) \arrow[ur] \arrow[dr] & & \pi_1(X, x_0) \\ & \pi_1(A_\beta, x_0) \arrow[ur] & \end{tikzcd}$$

The maps going into $\pi_1(A_\alpha, x_0)$ and $\pi_1(A_\beta, x_0)$ land in completely different homotopy classes. So, in order to ensure that this does not happen, it is useful to define the *amalgamated free product.*

In general, suppose that we have

$$\begin{tikzcd} & G_\alpha \\ K \arrow[ur, "i_\alpha"] \arrow[dr, "i_\beta"'] & \\ & G_\beta \end{tikzcd}$$

Then, define

$$G_\alpha *_K G_\beta := G_\alpha * G_\beta / N,$$

where N is the smallest normal subgroup of $G_\alpha * G_\beta$ containing the elements of the form $i_\alpha(k) i_\beta^{-1}(k) = i_\alpha(k) i_\beta(k^{-1})$. That is, a typical element of N has the form

$$h_1 i_\alpha(k_1) i_\beta(k_1^{-1}) h_1^{-1} h_2 i_\alpha(k_2) i_\beta(k_2^{-1}) h_2^{-1} \cdots$$

The grup $G_\alpha *_K G_\beta$ is called the *amalgamated free product* of G_α and G_β.

More generally, we have

$$\begin{tikzcd}
 & G_\alpha \\
K_{\alpha,\beta} \arrow[ur, "i_{\alpha\beta}"] \arrow[dr, "i_{\beta\alpha}"'] & \\
 & G_\beta
\end{tikzcd}$$

for all $\alpha \neq \beta$. From this, we take the normal subgroup to be

$$N = \langle i_{\alpha\beta}(k) i_{\beta\alpha}(k^{-1}), i_{\beta\alpha}(k) i_{\alpha\beta}(k{-}1) \rangle.$$

This then defines an amalgamated free product of G_α over all α, β. We denote this by $*_{K_{\alpha,\beta}} G_\alpha$.

Remark 22. Can reduce this to

$$K_\alpha / \ker(i_{\alpha\beta} \cap \ker(i_{\beta\alpha}),$$

and we have maps induced maps $\overline{i_{\alpha\beta}}$, and $\overline{i_{\beta\alpha}}$.

Example 26. *There is an isomorphism*

$$\mathrm{SL}_2(\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}.$$

In this case, we have the diagram:

$$\begin{tikzcd}
 & \mathbb{Z}/6\mathbb{Z} \arrow[dr] & \\
\mathbb{Z}/2\mathbb{Z} \arrow[ur, "i_\alpha"] \arrow[dr, "i_\beta"'] & & \mathbb{Z}/6\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/4\mathbb{Z} \\
 & \mathbb{Z}/4\mathbb{Z} \arrow[ur] &
\end{tikzcd}$$

Example 27.

$$\begin{tikzcd}
\pi_1(A_\alpha \cap A_\beta) \arrow[r] \arrow[d] & \pi_1(A_\alpha) \arrow[d] & \\
\pi_1(A_\beta) \arrow[r] & \pi_1(A_\alpha) *_{\pi_1(A_\alpha \cap A_\beta)} \pi_1(A_\beta) \arrow[r, "\simeq"] & \pi_1(X)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d, "i"'] & \{e\} \arrow[d] \\
\pi_1(A_\beta) \arrow[r] & \pi_1(A_\beta)/\langle i(1) \rangle
\end{tikzcd}$$

$$\begin{tikzcd}
K \arrow[r, "\mathrm{id}_K"] \arrow[d] & K \arrow[d] \\
G \arrow[r] & G \cong G *_K K
\end{tikzcd}$$

3.3.2. Presentations of Groups

A *presentation* of a group is given by:

$$G = \langle g_1, \cdots, g_m : r_1, \cdots, r_k \rangle,$$

where $g_1, \cdots, g_m$ are the generators, and $r_1, \cdots, r_k$ are relations that the generators satisfied. One can re-write G in terms of a free product by:

$$G = *_{i=1}^{m}\mathbb{Z}/N\langle r_1, \cdots, r_k\rangle,$$

where $N\langle r_1, \cdots, r_k\rangle$ is the normal subgroup generated by the relations.

Example 28.

$$*_2\mathbb{Z} = \langle g_1, g_2 : \textit{no relations}\rangle.$$
$$\mathbb{Z}^2 = \langle g_1, g_2 : [g_1, g_2] = g_1 g_2 g_1^{-1} g_2\rangle,$$
$$\mathbb{Z}/d\mathbb{Z} = \langle g : g^d = 0\rangle.$$

3.3.3. Van Kampen's Theorem

Theorem 5 ((Vam Kampem) Thm 1.20, Hatcher)**.** *Let $X = \bigcup_\alpha A_\alpha$, where each A_α is open, and each $A_\alpha \cap A_\beta$ is path-connected. Let $x_0 \in A_\alpha$. Then, $i_\alpha : A_\alpha \hookrightarrow X$ be the injection map. Then, we have maps:*

$$*_\alpha(i_\alpha)_* : *_\alpha\pi_1(A_\alpha, x_0) \longrightarrow \pi_1(X, 0)$$

and

$$i_{\alpha\beta} : A_\alpha \cap A_\beta \longrightarrow A_\alpha.$$

Then, Φ is surjective.

Moreover, if $A_\alpha \cap A_\beta \cap A_\gamma$ are path-connected for all α, β, γ, then $\ker\Phi$ is the subgroup normally generated by

$$(i_{\alpha\beta})_*(k)(i_{\beta\alpha})_*(k^{-1}), \quad (i_{\beta\alpha})_*(k)(i_{\alpha\beta})_*(k^{-1}).$$

Example 29. *Recall that we proved that $\pi_1(\mathbb{S}^1, 1) = \{e\}$ follows from van Kampen by setting covering $\mathbb{S}^2$ by $U_N \cap U_S$, where U_N and U_S are the the sphere without the north and south poles. Then, we have a map*

$$\pi_1(U_N) *_{\pi_1(U_N \cap U_S)} \pi_1(U_S) \longrightarrow \pi_1(\mathbb{S}^2),$$

but since $U_N \cong U_S \cong \mathbb{R}^2$, it is thus retractible and has trivial fundamental group. Thus, $\pi_1(\mathbb{S}^2)$ is trivial.

Example 30. *Cover $\mathbb{S}^1 = U_N * U_S$ similarly as before. We can argue by a similar argument that U_N and U_S retractible, and thus conclude that $\pi_1(\mathbb{S}^1)$ is trivial. But this is not true, as we already know. What went wrong here is that $U_N \cap U_S = (0, 1) \sqcup (0, 1)$, which is not path-connected.*

3.3.4. Cell Addition and Cell Complexes

Define a map

$$\varphi : \mathbb{S}^{k-1} \longrightarrow X,$$

and define a path γ going from x_0 to $\varphi(*)$. Then, we can define a space

$$Y = X \cup_\varphi D^k.$$

Let us restrict ourselves to the case where $k = 2$. We wish to consider maps

$$\pi_1(X, x_0) \longrightarrow \pi_1(Y, x_0).$$

Our goal is to derive a van Kampen decomposition of $\pi_1(Y, x_0)$. In this case, consider:

$$X \cup_\varphi \underbrace{((1, 1-\varepsilon) \times \mathbb{S}^{k-1})}_{\subseteq D^k},$$

which deformation retracts back onto X by construction. Let us define

$$Y = X \cup_\varphi ((1, 1-\varepsilon) \times \mathbb{S}^{k-1}) \cup \mathrm{Int}(D^k),$$

Then, we have the van Kampen diagram:

$$\begin{tikzcd} \pi_1((-1, 1-\varepsilon) \times \mathbb{S}^{k-1}) \arrow[r] \arrow[d] & \pi_1(D^k) \cong \{e\} \\ \pi_1(X, x_0) \cong \pi_1(A_\alpha, x_0) \arrow[r] & \pi_1(X, x_0)/N\langle \mathbb{Z}(h\varphi\overline{h})\rangle \end{tikzcd}$$

The point is that if we understand van Kampen's theorem, then we understand how the fundamental group of cells change under cell addition.

3.4. Lecture 3, 03/15/2024

Given any map

$$f : X \longrightarrow Y,$$

then the *mapping cylinder* is

$$\mathrm{Cyl}(f) := ((I \times X) \sqcup Y)/\simeq,$$

where $(x, 1) = f(x)$. As Assignment One will show, the mapping cylinder is homotopy equivalent to Y. The *cone* CX of X is:

$$CX := (X \times I)/\sim,$$

where $(x, 0) \sim (x', 0)$. As an example,

$$C\mathbb{S}^{n-1} \cong D^n.$$

The *mapping cone* $\mathrm{Cone}(f)$ of f is given by:

$$\mathrm{Cone}(f) := (CX \sqcup Y)/\simeq,$$

where $[x, 1] \simeq f(x)$.

Example 31. *Cell attachment is equivalent to the mapping cone of a map $f : \mathbb{S}^{n-1} \to Y$. In particular,*

$$Y \cup_f D^n = \mathrm{Cone}(f).$$

3.4.1. Wedge Spaces

Definition 25. (X, x) is *well-pointed* if for each open neighbourhood U of x is contractible.

Example 32. *Manifolds M are well-pointed for all $m \in M$.*

Suppose that $X = \sqcup_\alpha X_\alpha$ is a wedge product of well-pointed base points $x_\alpha \in X_\alpha$. Then,

$$\pi_1(X) \cong *_\alpha \pi_1(X_\alpha, x_\alpha). \tag{3.1}$$

Proof. Set $A_\alpha = X_\alpha \bigcup_{\beta \neq \alpha} U_\beta$, of which x_β is an element. Then, $X_\alpha \to A_\alpha$ is homotopy equivalent. We have:

$$X_\alpha \cap X_\beta = U_\alpha \vee_{x_\alpha = x_\beta} U_\beta,$$

is path-connected. Then, the triple intersections are given by

$$X_\alpha \cap X_\beta \cap X_\gamma = U_\alpha \vee U_\beta \vee U_\gamma,$$

which is also path-connected. Moreover, they are path-connected and contractible. Therefore, by van Kampen's theorem,

$$\Phi : *_\alpha (X_\alpha, x_\alpha) \longrightarrow \pi_1(X),$$

is surjectivewith trivial kernel. Thus, the isomorphism (3.1) holds. □

Example 33. *Generally,*

$$\pi_1 \left(\vee_{i=1}^n \mathbb{S}^1 \right) \cong F_n,$$

where F_n is the free group on n generators.

3.4.2. Back to Cell Attachment

Let (X, x_0) be path-connected. Then, let

$$Y := X^{k+1} := X \cup_{\sqcup_i \varphi_i} \left(\bigsqcup_{i=1}^n D^k \right).$$

Proposition 10 (Prop 1.26, Hatcher). *In the situation above, suppose that $k \geq 2$, and consider the map*

$$i_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, x_0).$$

(a) If $k = 2$, then i_* is surjective, with kernel normally generated by the classes $[h_i \varphi \overline{h_i}]$, for $i = 1, \cdots, n$.

(b) If $k \geq 3$, then i_* is an isomorphism.

(c) The inclusion of the 2-skeleton X^2 in $X^2 \hookrightarrow X$ of a CW-complex induces an isomorphism $\pi_1(X^2, x_0) \to \pi_1(X, x_0)$.

Proof.

(a) Let 0 be the centre of D^2, and take

$$A_\alpha = (Y \setminus \{0\}) \cup_{\widetilde{h}} (I \times I),$$

where

$$\widetilde{h} : (I \times \{0\}) \cup (\{0\} \times I) \longrightarrow Y,$$

and let h be the map $h : \{0\} \times I \to Y$. The map $\widetilde{h}$ pushes into the disc $\varphi^{-1}(*) = h(1)$. It follows then that $A_\beta = \mathrm{Int}(D^2)$. First, let us define

$$\widetilde{Y} := Y \cup (I \times I),$$

and note that $Y \subseteq \widetilde{Y}$ is a deformation retract. Moreover, $X \subset A_\alpha$ as a deformation retraction. $A_\alpha \cap A_\beta$ contains $\mathbb{S}^1 \subseteq \mathrm{Int}(D^2)$ as a deformation retraction. In praticular, $A_\alpha \cap A_\beta$ is homotopy equivalent to $\mathbb{S}^1$. By construction, $A_\beta \cong \mathrm{Int}(D^2)$, which is contractible. From this, one then obtains the following van Kampen diagram:

$$\begin{tikzcd}
\pi_1(A_\alpha \cap A_\beta) \arrow[r] \arrow[d] & \pi_1(A_\alpha) \arrow[d] \\
\pi_1(A_\beta) \arrow[r] & \pi_1(\widetilde{Y})
\end{tikzcd}$$

which is, explicitly:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "i"] \arrow[d] & \pi_1(\widetilde{X}) \arrow[d] \\
\{e\} \arrow[r] & \pi_1(\widetilde{X})/N\langle i(1)\rangle
\end{tikzcd}$$

where we have that $\pi_1(A_\alpha \cap A_\beta) \cong \pi_1(\mathbb{S}^1) \cong \mathbb{Z}$, and $\pi_1(A_\beta) \cong 0$ since its retractible,

(b) The proof is similar to the $k = 2$ case. By Proposition 8, $\pi_1(\mathbb{S}^{k-1}) = \{e\}$. So, we obtain the van Kampen diagram:

$$\begin{tikzcd}
\{e\} \arrow[r, "i"] \arrow[d] & \pi_1(\widetilde{X}) \arrow[d] \\
\{e\} \arrow[r] & \pi_1(\widetilde{Y}) \cong \pi_1(Y)
\end{tikzcd}$$

(c) Follows from (b) via an inductive argument.

□

Example 34 (Closed Surfaces)**.** *Let M_g be a compact, connected surface of genus g, and let*

$$N_n := \#_n \mathbb{R}P^2,$$

where # is the connected sum of manifolds. Given two manifolds M_1 and M_2, then we take $M_1 \setminus \mathrm{Int}(D^2)$, and $M_2 \setminus \mathrm{Int}(D^2)$, into which we glue in an interval $\mathbb{S}^{n-1} \times I$ between the holes we cut out of the manifolds. This is the check sum, and is defined as:

$$M_1 \# M_2 := \big((M_1 \setminus \mathrm{Int}(D^2)) \cup (M_2 \setminus \mathrm{Int}(D^2)) \cup \mathbb{S}^{n-1} \times I\big) / \simeq .$$

Fact: *for $n = 2$, the homotopy type of $M_1 \# M_2$ is well-defined. For $n \geq 3$, one needs to take a choice of orientation.*
Recall that the 2-torus can be given by

$$T^2 = (I \times I)/\simeq \cong (\mathbb{S}^1_a \times \mathbb{S}^1_b) \cup_\varphi D^2,$$

where a and b are there to keep track of the two copies of the circle, and $[\varphi] \in \pi_1(\mathbb{S}^1 \vee \mathbb{S}^1) \cong F_2(a, b)$. If we travel once around the square defined by $(I \times I)/\simeq$, we find that $[\varphi] = aba^{-1}b^{-1} = [a, b]$. Then, we *have the van Kampen diagram:*

$$\begin{array}{c} \pi_1(T^2) \cong \langle a, b : [a,b] = 0 \rangle \cong \mathbb{Z}^2 \\ \big\downarrow \\ \pi_1(\mathbb{S}^1 \times \mathbb{S}^1) \\ \big\downarrow \\ \pi_1(\mathbb{S}^1) \times \pi_1(\mathbb{S}^1) \\ \big\downarrow \cong \\ \mathbb{Z} \times \mathbb{Z} \cong \mathbb{Z}^2 \end{array}$$

Example 35. *Let us consider*

$$\pi_1(\mathbb{R}P^2).$$

One finds that $\mathbb{R}P^2 \cong \mathbb{S}^1 \times_\varphi D^2$*. Take some* $[\varphi] \in \pi_1(\mathbb{S}^1)$*. Drawing the picture out, we find that* $[\varphi] = 2a$*, and thus by van Kampen, we have that*

$$\pi_1(\mathbb{R}P^2) = \langle a : 2a = 0 \rangle \cong \mathbb{Z}/2\mathbb{Z}.$$

Chapter 4

Week Four

4.1. Lecture 1, 20/03/2024

TODO: watch first 15 minutes of lecture

Proof Idea. An element of $G * H$ is a reduced word

$$g_1 \cdots g_m \longmapsto p_{\alpha_1}(g_1) \cdots p_{\alpha_m}(g_m),$$

where $g_i \in G_{\alpha_i}$, and $G = G_{\alpha_1}$, and $H = G_{\alpha_m}$. The exercise is to check that this map defines a homomorphism. $\square$

In the amalgamated case, if we are given maps $p : G \to U$ and $q : H \to U$, we have the commutative diagram:

$$\begin{tikzcd}
K \arrow[r, "i"] \arrow[d, "j"] & G \arrow[d, "j_G"] \arrow[ddr, bend left, "p"] & \\
H \arrow[r, "j_H"] \arrow[drr, bend right, "q"'] & G *_K H & \\
& & U
\end{tikzcd}$$

That is, we have the property that

$$p \circ i = q \circ j.$$

Then, there exists a unique map $\phi : G *_K H \to U$ such that

$$\phi \circ j_H = p, \quad \phi \circ j_H = q, \quad \phi \circ i_G \circ i = \phi \circ j_H \circ j$$

. That is, such that the diagram

$$\begin{tikzcd}
K \arrow[r, "i"] \arrow[d, "j"] & G \arrow[d, "j_G"] \arrow[ddr, bend left, "p"] & \\
H \arrow[r, "j_H"] \arrow[drr, bend right, "q"'] & G *_K H \arrow[dr, dashed, "\exists!\phi"] & \\
& & U
\end{tikzcd}$$

commutes.

Proof Idea. Observe that for $\phi : G * H \to U$, the maps $\phi(i(k)j(k^{-1})) = e$, and $\phi(j(k)i(k^{-1})) = e$. From this, it follows then that the restriction map $\phi|_N$ is trivial, where

$$N = \langle i(k)j(k^{-1}) : k \in K \rangle.$$

It follows then that ϕ descends to $\phi_\alpha : G *_K H \to U$. $\square$

4.1.1. Mapping Cylinders

We now talk about assignment problems! Given a continuous map $f : X \to Y$,

$$M_f := ((X \times I) \sqcup Y)/((x, 1) \simeq f(x).$$

In other words, we attach the endpoint of $x \in X$ to its image in f. Then, from this, one may deduce that there is a retraction

$$r : M_f \longrightarrow Y, \quad [x, t] \longmapsto f(x), \quad y \longmapsto y.$$

One may check using that using the universal property of quotient spaces, this is indeed a continuous map. From this, one may then put this into a deformation retraction. There is an obvious inclusion $i : X \hookrightarrow M_f$ given by $x \mapsto (x, 0)$. From this, we obtain two maps

$$r \circ i : X \longrightarrow Y.$$

Then, one observes that $r \circ i = f$ by construction.

Example 36. *Consider the winding map*

$$f_d : \mathbb{S}^1 \longrightarrow \mathbb{S}^1, \quad z \longmapsto z^d.$$

If we take M_{f_2}, then one may show that M_{f_2} is homeomorphic to the Möbius band. If we draw out the usual picture used to create the Möbius band, we see that we have one copy of $\mathbb{S}^1$ embedded in the edges with arrows, and another copy of $\mathbb{S}^1$ along the boundary of the Möbius band that is twice as large as the other copy.

Another hint! Problem 1.2 basically takes a bunch of these mapping cylinders and then attaches them together, denoted by $X_{d,\infty}$. You are not being asked to determine $\pi_1(X_{d,\infty})$ precisely, you just need to show that it is different from $\pi_1(\mathbb{S}^1)$. A key hint is to show that given the embedding $i : \mathbb{S}^1_0 \hookrightarrow X_{d,\infty}$, the induced homomorphism

$$i_* : \pi_1(\mathbb{S}^1_0) \longrightarrow \pi_1(X_{d,\infty}),$$

is non-zero. Once we show that, we are most of the way there, supposedly. Use the fact that D^2 is compact, and thus its image lies in some $X_{d,k}$ for some sufficiently large k, because it cannot "fit" into all the pieces of the mapping cone telescope because of its compactness.

A map

$$\mathbb{S}^1 \longrightarrow X,$$

is *nul-homotopic* if and only if f extends to $\overline{f} : D^2 \to X$.

Proof. There is a canonical identification

$$D^2 \cong C\mathbb{S}^1 \cong (\mathbb{S}^1 \times I)/(x_0 \sim (x', 0)).$$

Let $q : \mathbb{S}^1 \times I \to C\mathbb{S}^1$. Then, $\overline{f} \circ q|_{\mathbb{S}^1 \times \{1\}} = f$, and $\overline{f} \circ q|_{\mathbb{S}^1 \times \{0\}} = c_x$, and thus f is null-homotopic. $\square$

4.1.2. Discussion of van Kampen's Theorem (Thm 1.20, Hatcher)

Recall that we have a map
$$\Phi : *_\alpha \pi_1(A_\alpha, x_0) \longrightarrow \pi_1(X, x_0),$$
and we have shown that this is surjective. We wish to now analyse its kernel.

Definition 26. A *factorisation* of $[f] \in \pi_1(X, x_0)$ is a word $[f_1] \cdots [f_m]$, where $[f_i]$ belongs to $\pi_1(A_{\alpha_i}, x_0)$, and
$$[f] = (i_{\alpha_1})_*([f_1]) \cdot (i_{\alpha_2})_*([f_2]) \cdots (i_{\alpha_m})_*([f_m]).$$
This is an *unreduced* word for the groups $\pi_1(A_\alpha, x_0)$.

We consider the equivalence relation on factorisations generated by:

(i) multiplying adjacent elements in the same group,

(ii) if
$$[f_i] \in \mathrm{Im}((i_{\alpha,\beta})_* : \pi_1(A_\alpha \cap A_\beta, x_0) \to \pi_1(A_\alpha, x_0)),$$
then we regard $[f_i]$ as being in the image of the map
$$(i_{\beta\alpha})_* : \pi_1(A_\alpha \cap A_\beta, x_0) \longrightarrow \pi_1(A_\beta).$$

If two factorisations are equivalent, then they define the same element in the amalgamated product $*_\alpha \pi_1(A_\alpha, x_0)$. We must show that path homotopic factorisations are equivalent. We can define a map
$$f_t : I \times I \longrightarrow X,$$
and then "chop up" the square $I \times I$ into little rectangles $[a_i, b_i] \times [a_j, b_j]$. The point of this construction is that we can show the rectangles $R_i \subset I \times I$ such that $f(R_i) \subset A_{\alpha_i}$.

4.2. Lecture 2, 21/03/2024

4.2.1. Proof Idea of van Kampen's Theorem

Suppose that $[f]_1 * \cdots * [f_m]$ factorises $[f]$, and each $[f_i] \in \pi_1(A_{\alpha_i}, x_0)$. Suppose that $[f]'_1] * \cdots * [f'_n]$ is another factorisation of $[f]$, where $[f'_j] \in \pi_1(A_{\alpha_j}, x_0)$. Then, we chop up the square $I \times I$ into little rectangles such that we only have triple intersections (i.e. no quadruple intersections). We also pick the rectanlges in such a way that the rectangles R_i match up with the factorisations.

Claim: we obtain such rectangles R_i such that there is a homotopy $F : I \times I \to X$ such that $F(R_i) \subset A_{\alpha_i}$.

In particular, the homotopy maps $F(\{0,1\} \times I) = x_0$, so that two edges of the rectangle get mapped to a point, which gives us a loop.

Let $v \in I \times I$ be a vertex – i.e. a point where vertical and horizontal lines meet. We know that $F(v)$ belongs to $A_{\alpha_1} \cap A_{\alpha_2} \cap A_{\alpha_3}$ (it is possible that $|\{\alpha_1, \alpha_2, \alpha_3\}| < 3$). Now, take a path h_v from $F(v)$ To x_0. Then, insert copies of $h_v \overline{h_v}$ into paths arriving at v.

Hence, the loop $F \circ \gamma_r$ – where γ_r separates $R_1, \cdots, R_r$ from the other rectangles – defines a factorisation of $[f]$, at least once choices are made to place loops in A_{α_i}. Finally, passing from R_i to R_{i+1}, we obtain path homotopic loops.

Remark 23. We must decide to place all relevant loops in the same $\pi_1(A_{\alpha_i}, x_0)$.

Remark 24. **ALSO**, this is not examinable! We are only required to apply van Kampen's theorem, not to prove it.

4.2.2. Assignment Tips and Tricks: Mapping Tori

In the most general situation, we can consider an arbitrary continuous map $f : X \to X$. Then, define

$$T_f := \frac{X \times I}{(x,1) \sim (f(x),0)}.$$

By construction, $T_{\mathrm{id}} = X \times \mathbb{S}^1$. Observe that there is always a map

$$T_f \longrightarrow \mathbb{S}^1, \quad [x,t] \longmapsto [t],$$

under the identification $\mathbb{S}^1 \cong [0,1]/(0 \sim 1)$. As an example, let us consider a map $f : \mathbb{S}^1 \to \mathbb{S}^1$ defined by $z \mapsto e^{i\theta} z$, for some $\theta \in [0, 2\pi]$. Then, the mapping torus $T_f \cong \mathbb{S}^1 \times \mathbb{S}^1$, the 2-torus, since we can connect this to the identity map. Explicitly, the homeomorphism is given by $(z, [t]) \mapsto [ze^{i\theta t}, t]$.

Consider another map $f : \mathbb{S}^1 \to \mathbb{S}^1$ defined by $z \mapsto \overline{z}$, which is an antiholomorphic map that flips the circle 180 degrees around, and reverses the orientation. Then, T_f is homeomorphic to the Klein bottle K^2. If $f(x_0) = x_0$, then

$$\pi_1(T_f) \cong \pi_1(X, x_0) \ltimes \mathbb{Z},$$

where $\mathbb{Z}$ acts on $\pi_1(X, x_0)$ by 1 acting by f_*.

Definition 27. Let N and A be groups, and $\phi : A \to \mathrm{Aut}(N)$ be a homomorphism. Then,

$$N \rtimes_\varphi A,$$

is a group with underlying set $N \times A$, defined by

$$(n_1, a_1)(n_2, a_2) = (n_1 \varphi(a_1) n_2, a_1 a_2).$$

Note that if φ is non-trivial, then $N \ltimes A$ is *non-abelian.*

One checks that $N \rtimes_\varphi A$ defines a group. Using the isomorphism, we see then that $\pi_1(K^2) \cong \mathbb{Z} \ltimes \mathbb{Z}$, where the semidirect product is defined by the homomorphism $\varphi : \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. So, if we have the map $f_{-1} : \mathbb{S}^1 \to \mathbb{S}^1$ defined by $z \mapsto \overline{z}$, then the induced homomorphism

$$(f_{-1})_* : \pi_1(\mathbb{Z}) \longrightarrow \pi_1(\mathbb{Z}),$$

defined by multiplication by -1.

Moreover, another hint is that there is a canoincal relationship between T_f and T_{f^d}. More on that tomorrow! If you can answer that you're basically done with question 4.

4.2.3. Covering Spaces (Chapter 1.2, Hatcher)

Recall that $p : \widetilde{X} \to X$ is a *covering space* if every $x \in X$ has an open neighbourhood U such that $p^{-1}(U) \cong \sqcup_\alpha U_\alpha$, where each $p|_{U_\alpha}$ is a homeomorphism onto U. From now on, we assume that X is path-connected. We have the result that we've already proved:

Proposition 11 (Homotopy Lifting, Prop 1.30, Hatcher)**.** *Given a homotopy $F_t : Y \times I \to X$ and a* *covering map $p : \widetilde{X} \to X$, there exists a unique map $\widetilde{F}_t$ such that the diagram*

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "\widetilde{F}_0"] \arrow[d, hook] & \widetilde{X} \\ Y \times I \arrow[r, "F_t"] \arrow[ur, dashed, "\exists! \widetilde{F}_t"] & X \arrow[u, "p"'] \end{tikzcd}$$

That is, such that $p \circ \widetilde{F}_t = F_t$.

Proposition 12 (Prop 1.31, Hatcher)**.** *Let $\widetilde{x_0} \in p^{-1}(x_0)$. Then, the induced homomorphism*

$$p_* : \pi_1(\widetilde{X}, \widetilde{x_0}) \longrightarrow \pi_1(X, x_0),$$

is injective.

Proof. Suppose that $p_*([\gamma]) = e$. Then, $p \circ \gamma$ is path-homotopic to the constant loop c_{x_0} via a homotopy F_t. Then, by the lifting property, there exists a lift $\widetilde{F}_t$ using p as a lift. Then, we have the diagram:

$$\begin{tikzcd} I \times \{0\} \arrow[r, "\widetilde{F}_0"] \arrow[d, hook] & \widetilde{X} \\ I \times I \arrow[r, "F_t"] \arrow[ur, dashed, "\exists! \widetilde{F}_t"] & X \arrow[u, "p"'] \end{tikzcd}$$

We claim that $\widetilde{F}_t$ is a path homotopy to $c_{\widetilde{x_0}}$. Observe that

$$\widetilde{F}_t|_{\{1\}\times I} : I \longrightarrow p^{-1}(x_0),$$

where $p^{-1}(x_0)$ is discrete. It follows then that $\widetilde{F}_t|_{\{1\}\times I}$ is a path homotopy that lifts the constant loop. Thus, it lifts the constant loop. More on this next time. □

4.3. Lecture 3, 22/03/2024

Remark 25 (Diarmuid Wisdom)*.* The hardest thing about covering spaces is trying to remember that it's a map, not a space.

We will continue our discussion of covering spaces $p : (\widetilde{X}_0, \widetilde{x_0}) \to (X, x_0)$. Recall that we have Proposition 12, which is a useful fact to have.

Remark 26*.* Homotopy lifting means that path homotopies lift to path homotopies. Moreover, the uniqueness of lifting implies that constant maps lift to constant maps.

We want to relate the algebra of our fundamental group to the topology of our situation. Recall that if H is a subgroup of G, then we can always write

$$G = \sqcup_{g_\alpha} H g_\alpha,$$

where Hg is a right coset of G – that is, the right cosets of G give a partition of G. Recall the index $[H : G]$ of H in G is the size of H in G. In particular, $[H : G] = |\{g_\alpha\}|$. As an example, $3\mathbb{Z}$ has index 3 in $\mathbb{Z}$. The trivial subgroup $\{0\}$ has index ∞ in $\mathbb{Z}$.

Proposition 13 (Prop 1.32, Hatcher)**.** *Suppose that $\widetilde{X}$ and X are path-connected Then, the image of the induced homomorphism*

$$\left[p_* \left(\pi_1(\widetilde{X}, \widetilde{x_0})\right) : p_1(X, x_0)\right] = |p^{-1}(x_0)|,$$

which is called the degree *of the cover p. Hatcher calls it the "number of sheets".*

Proof. Consider a map

$$\Phi : \pi_1(X, x_0) \longrightarrow p^{-1}(x_0), \quad [\gamma] \longmapsto [\widetilde{\gamma}],$$

where $\widetilde{\gamma}$ is the lift of γ given by $\widetilde{\gamma}(0) = \widetilde{x_0}$. This is well-defined by the homotopy lifting property. In particular, we are applying homotopy lifting to the case where $Y = \mathrm{pt}$, and we thus can regard γ as a homotopy.

We now wish to show that Φ is constant on the right cosets of $p_*(\pi_1(\widetilde{X}, x_0))$. To see this, let $\gamma = \delta \cdot \gamma'$, where $\delta \in \mathrm{Im}(p_*)$. From this, we wish to determine $\widetilde{\gamma}(1)$. By our construction, we have $\widetilde{\gamma}(1) = (\widetilde{\delta} \cdot \widetilde{\gamma'})(1)$. But since $\widetilde{\delta}_{\widetilde{x_0}}(1) = \widetilde{x_0}$, it follows that $\widetilde{\gamma}_{\widetilde{x_0}}(1) = \widetilde{\gamma}_{x_0}(1)$. This uses the fact that $[\gamma'] \in \mathrm{Im}(p_*)$ if and only if $\widetilde{\gamma}_{\widetilde{x_0}} = \widetilde{x_0}$ (see Hatcher). It follows then that Φ descends to a map on the left coset:

$$\Phi : H \setminus G \longrightarrow p^{-1}(x_0),$$

where $G = \pi_1(X, x_0)$ and $H = p_*\left(\pi_1(\widetilde{X}, \widetilde{x_0})\right)$. Since $\widetilde{X}$ is path-connected, there exists $\widetilde{\gamma'}$ such that $\widetilde{\gamma'}(0) = \widetilde{x_0}$, and $\widetilde{\gamma}(1) = x \in p^{-1}(x_0)$. Now, $p \circ \gamma$ is a loop at x_0 such that $\widetilde{p \circ \gamma}_{x_0} = \widetilde{\gamma}$, amd thus $\Phi([p \circ \widetilde{\gamma}]) = x \in p^{-1}(x_0)$. This shows surjecitivty.

Next, we wish to show injectivity. Suppose that $\Phi(H[\gamma_1]) = \Phi(H[\gamma_2])$. Then, $\widetilde{\gamma_1\overline{\gamma_2}}(1) = \widetilde{x_0}$. It follows then that $[\gamma_1\overline{\gamma_2}] \in \mathrm{Im}(p_*)$, and therefore $H[\gamma_1] = H[\gamma_2]$ by property of cosets. $\square$

Given a map $f : Y \to X$, and a covering space $p : \widetilde{X} \to X$, a natural question to ask is if there exists a lift $Y \to \widetilde{X}$ such that the diagram

$$\begin{tikzcd}
 & \widetilde{X} \arrow[d, "p"] \\
Y \arrow[ur, dashed] \arrow[r, "f"'] & X
\end{tikzcd}$$

commutes. Observe that if such a lift does exist, then the induced homomorphisms give a commutative diagram:

$$\begin{tikzcd}
 & \pi_1(\widetilde{X}, \widetilde{x_0}) \arrow[d, "p_*"] \\
\pi_1(Y, y_0) \arrow[ur, dashed, "\widetilde{f_*}"] \arrow[r, "f_*"'] & \pi_1(X, x_0)
\end{tikzcd}$$

That is, $f_* = p_* \circ \widetilde{f_*}$. It follows from this then that $\mathrm{Im}(f_*) \subseteq \mathrm{Im}(p_*)$. However, this turns out to be an if and only if, as we will show later.

Definition 28 (Template of a Definition)**.** Let P be a topological property (i.e. path-connectedness, compactness etc.), and X a space. Then, we say that X is *locally* P if for all $x \in X$, and for all neighbourhoods U of x, there exists a neighbourhood $x \in V \subset U$ such that V is P.

Remark 27. You don't always want open neighbourhoods for the above definition – i.e. if we are concerned with local compacntess then we will require closed neighbourhoods contained in an open neighbourhood of x.

Remark 28. For some reason path-connected *doesn't* imply locally path-connected??? The topolgoists sine curve is an example of this.

Proposition 14 (Prop 1.33, Hatcher)**.** *If X is path-connected and locally path-connected, then any map $f : Y \to X$ can be lifted to one on $\widetilde{X}$ if and only if*

$$f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x_0})).$$

Proof. The $\Longrightarrow$ direction is clear, as we have already done that. Let us consider the other direction. Then, we have maps:

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

Let $y \in Y$. Then, since X is path-connected, then there exists a path $\gamma : I \to Y$ such that $\gamma(0) = y_0$ and $\gamma(1) = y$. Then, the path $f \circ \gamma$ in X can be lifted to a path $\widetilde{f \circ \gamma}$ starting at $x - 0$. Then, we can define

$$\widetilde{f}(y) := \widetilde{f \circ \gamma}_{x_0}(1).$$

Let us check that $\widetilde{f}$ is well-defined. If γ' is path-homotopic to γ, then

$$\widetilde{f \circ \gamma}_{x_0}(1) = \widetilde{f \circ \gamma'}_{x_0}(1),$$

by path-lifting. Now, we wish to check this for path-homotopy classes. In general, let $\gamma'(0) = y_0$, and $\gamma'(1) = y$. Then, $\gamma \simeq \gamma'\overline{\gamma}\gamma$. Observe that $[\gamma'\overline{\gamma}] \in \pi_1(Y, y_0)$. Then, $f \circ \gamma' \simeq f \circ (\gamma'\overline{\gamma}\gamma) = f \circ (\gamma'\overline{\gamma}) \cdot f \circ \gamma$. Then, we see that

$$\widetilde{f \circ \gamma'}(1) = \widetilde{f \circ \gamma}(1),$$

since $[f \circ \gamma'] \in f_*(Y, y_0) \subset p_*(\widetilde{X}, \widetilde{x_0})$, and thus the induced homomorphism is $\widetilde{f}_*$ is well-defined.

We now show that $\widetilde{f}$ is continuous. Let $N \subseteq Y$ be a neighbourhood of $y \in Y$ such that $f(N) \subset U$, where U is evenly covered by p. (we're running out of time so just go see Hatcher) $\square$

Remark 29. Local path-connectedness is actually a stronger condition than path-connectedness, since you can choose as small of a neighbourhood as you want and then prove your result on that. Which is wild.

4.3.1. More Assignemnt Tips on Mapping Tori

Suppose that we have a homeomorphism $f : X \to X$, we can raise to the d-th power by f^d. Then, we have T_{f^d} and T_f. Now that we know some more things about covering spaces, we can say more about how they may be related.

The answer is that they are related by a covering space. Suppose that $d = 3$. Then, we can take three copies of the interval, and glue each copy of the interval to f. One can check that this is a three-fold covering. Call this thing $T_{f,f,f}$. This construction is modelled on the $z \mapsto z^d$ map. We want to show that

$$T_{f^3} \cong T_{f,f,f}.$$

This is what we have to show. If you glue two cylinders via a homeomorphism, it is still a homeomorphism. And you can always untwist it. Moreover, you can say that you can assume that the homeomorphism has a fixed point.

Chapter 5

Week Five

5.1. Lecture 1, 27/03/2024

Let us recall one of the key results that we unfortunately skipped.

Proposition 15 (Prop 1.33, Hatcher)**.** *Let Y be path-connected, and locally path-connected, and consider a covering space $p : \widetilde{X} \to X$ fitting into a diagram:*

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\widetilde{f}"] \arrow[r, "f"] & X \end{tikzcd}$$

Then, a lift $\widetilde{f}$ exists if and only if $f_*(\pi_1(Y, y)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$.

Example 37. *As a nice application of Proposition 1.33, let us look at the covering space*

$$p : \mathbb{R} \longrightarrow \mathbb{S}^1.$$

Then, for any simply-connected space Y, any map $Y \to \mathbb{S}^1$ is null-homotopic, since the fundamental group of Y is trivial. Since $\mathbb{R}$ is contractible, it follows that every map into $\mathbb{R}$ is null-homotopic via the straight line homotopy. Then, apply Proposition 1.33 to show that we have a lift $\widetilde{f} : Y \to \mathbb{R}$, which factors through the null-homotopic map $Y \to \mathbb{S}^1$. Thus, the lift $\widetilde{f}$ is also null-homotopic.

Proposition 16 (Hatcher, Prop 1.34)**.** *Consider maps*

$$\begin{tikzcd} & (\widetilde{X}, \widetilde{x_0}) \arrow[d, "p"] \\ (Y, y) \arrow[r, "f"] & (X, x) \end{tikzcd}$$

where p is a covering space, and f has lifts $\widetilde{f}_1, \widetilde{f}_2 : Y \to \widetilde{X}$. If Y is path-connected, then $\widetilde{f}_1 = \widetilde{f}_2$ if and only if $\widetilde{f}_1(y_0) = \widetilde{f}_2(y_0)$.

Proof. $\Longleftarrow$ Let

$$Z = \{y : \widetilde{f}_1(y) = \widetilde{f}_2(y)\}.$$

Let $f(y) \in U$, where U is evenly covered. Then, by continuity, there exists a neighbourhood N such that $\widetilde{f}_1(N) \subset \widetilde{U_1}$, and $\widetilde{f}_2(N) \subset \widetilde{U_2}$ If $\widetilde{U_1} \neq \widetilde{U_2}$, then we may conclude that N is not contained in Z, and thus

Z is closed. Conversely, if the two agree, then we can agree that Z is open, since $N \subset Z$. If $Z \subset Y$ is non-empty, open and closed, then $Z = Y$ since Y is connected. $\square$

5.1.1. Classification of Covering Spaces

Our goal now is to classify all pointed covering spaces of (X, x), where X is path-connected and locally path-connected. By a classification, we mean the following. Given two covering spaces p_1, and p_2 over X, there exists a pointed homeomorphism h such that that the diagram

$$\begin{tikzcd}
(\widetilde{X}_1, \widetilde{x}_1) \arrow[rr, "h"] \arrow[dr, "p_1"'] & & (\widetilde{X}_2, \widetilde{x}_2) \arrow[dl, "p_2"] \\
& (X, x) &
\end{tikzcd}$$

commutes.

Observe that if $p_1 \cong p_2$, then since $p_1 = p_2 \circ h$, it follows then that

$$(p_1)_*(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = (p_2)_*(f_*(\pi_1(\widetilde{X}_1, \widetilde{x}_1))) = (p_2)_*(\pi_1(\widetilde{X}_2, \widetilde{x}_2)).$$

That is, $\operatorname{Im}(p_1)_* = \operatorname{Im}(p_2)_*$, and so $H = \operatorname{Im}(p_1)_*$ is an invariant of covering spaces up to isommorphism.

Example 38. *Recall that for $X = \mathbb{S}^1 \vee \mathbb{S}^1$, $\pi_1(X) = \mathbb{Z} * \mathbb{Z} =: F_2$, which is the free group on two letters. In this case, $\widetilde{X} = \mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^1$, and $\pi_1(\widetilde{X}) = F_3$, and we have $F_3 \hookrightarrow F_2 = \pi_1(X)$. The group H here is generated by $\langle a^2, b^2, ab \rangle$, which is a subgroup of index 2 in F_2. In particular, note that the index of the subgroup is equal to how many sheets the cover has. Indeed, in this case, $\widetilde{X} \to X$ is a two-sheeted cover.*

5.1.2. Existence

We want, for every subgroup H of $\pi_1(X, x)$, a covering space $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x)$ such that $\operatorname{Im} p_* = H$. The covering space corresponding to the case for which $H = \{e\}$ will be called the *universal cover*

$$X_{\text{univ}},$$

since every other covering space will arise as a quotient of the universal cover. If such a cover exists, then any point $x \in X$, and evenly covered set U containing x, the homomorphism

$$\pi_1(U, x) \longrightarrow \pi_1(X, x),$$

is trivial. Choose any one of the even covers $\widetilde{U}_i$ in $\widetilde{X}$. Then, $p|_{\widetilde{U}_i} : \widetilde{U}_i \to U$ is a homeomorphism, and thus induces a group isomorphism. We have the commutative diagram:

$$\begin{tikzcd}
\pi_1(\widetilde{U}_i, \widetilde{x}_i) \arrow[r, "i"] \arrow[d, "(p|_{\widetilde{U}_i})_*"] & \pi_1(\widetilde{X}, \widetilde{x}_0) \arrow[d, "p"] \\
\pi_1(U, x) \arrow[r] & \pi_1(X, x)
\end{tikzcd}$$

However, the composition $p \circ i$ is trivial, and since $(p|_{\widetilde{U}_i})_*$ is an isomorphism, it follows that the map $\pi_1(U, x) \to \pi_1(X, x)$ must be trivial.

Definition 29. A space X is *semi-locally simply-connected* if for all $x \in X$, there exists a neighbourhood U of x such that for all $x' \in U$, the map $\pi_1(U, x') \to \pi_1(X, x')$ is trivial – that is, every loop in U must be contractible.

Example 39 (A non-semi-locally simply-connected space)**.** *Consider an infinite shrinking wedge of circles (a.k.a a Haiwaiian earring), given by*

$$X = \bigcup_n \mathbb{S}^1_{1/n} \subseteq \mathbb{R}^2,$$

It is not well-behaved at the point where all the circles intersect, and it is not semi-locally simply-connected.

5.2. Lecture 2, 28/03/2024

Theorem 6 (Classification of Covering Spaces, Thm 1.38, Hatcher)**.** *Let X be path-connected, locally path-connected, and semi-locally simply-connected. Then, there are bijections*

$$\{\textit{isomorphism classes of pointed covering spaces}\} \xleftrightarrow{p_*} \{\textit{Subgroups of } \pi_1(X,x),$$

where by $\xleftrightarrow{p}$ we mean that the bijection is given by the induced map p_* of the covering space.

If we forget the basepoint of the covering spaces, then the bijection becomes

$$\{\text{isomorphism classes of covering spaces}\} \longleftrightarrow \{\text{conjugacy classes } [H] \text{ of subgroups of } \pi_1(X,x_0)\}.$$

Proof Idea. Let us first begin by showing that existence of the covering space corresponding to the group $H = \{e\}$. The resulting covering space will be called the *universal covering space*, and be denoted by $\widetilde{X}_{\text{univ}}$. All other covering spaces will arise as a quotient of this covering space.

Recall that $\pi_1(X,x_0)$ acts on $\Pi_1(X,x_0,x)$ freely and transitively. Thus, let us set

$$\widetilde{X} := \bigcup_{x\in X} \Pi_1(X,x_0,x),$$

for some fixed $x_0 \in X$. Then, we take a map

$$p : \bigcup_{x\in X} \Pi_1(X,x_0,x) \longrightarrow X, \quad [\gamma] \longmapsto \gamma(1),$$

where $\gamma(1) = x$. This is called the *universal covering* of X. The next step is to topologise $\widetilde{X}$ – keep in mind that the whole idea of covering spaces is that their local topology is the same as the topology on the base space X.

Define $\mathcal{U} \subseteq \tau_X$, where

$$\mathcal{U} = \{U \in \tau_X : U \text{ is path-connected and } \pi_1(U) \to \pi_1(X) \text{ is trivial}\},$$

which are basepoint independent because if the map is trivial, then it is trivial on every basepoint. Note that as a consequence, two paths that are path-homotopic in U is also path-homotopic in X by path-connectedness. Thus,

$$i_* : \Pi_1(U,x_1,x_2) \longrightarrow \Pi_1(X,x_1,x_2),$$

then $|\operatorname{Im} i_*| = 1$.

Note that if $V \subset U$, and V is path-connected, and $U \in \mathcal{U}$, then $V \in \mathcal{U}$, since we the induced map of the inclusion $\pi_1(V,x) \to \pi_1(X,x)$ is trivial, and the map $\pi_1(V,x) \to \pi_1(X,x)$ is also trivial, since X is locally path-connected. This implies that $\mathcal{U}$ is a *basis* for τ_X – that is, every $U \in \tau_X$ can be written as a union of elements in $\mathcal{U}$.

Example 40. *Let X be a metric space. Then, the set*

$$\mathcal{B} := \{B_\varepsilon(x) : x \in X, \varepsilon > 0\},$$

forms a basis for τ_d, where $d : X \times X \to \mathbb{R}_{\geq 0}$ is a metric on X.

A set $\mathcal{B} \in \mathcal{P}(X)$ defines $\tau_{\mathcal{B}}$ for which $\mathcal{B} \subset \tau_{\mathcal{B}}$ is a basis if

(i) for all $X = \bigcup_\beta U_\beta$, $U_\beta \in \mathcal{B}$,

(ii) for all $U_{\beta_1}, U_{\beta_2} \in \mathcal{B}$ and $x \in U_{\beta_1} \cap U_{\beta_2}$, there exists $U_{\beta_3} \in \mathcal{B}$ such that $x \in U_{\beta_3} \subset U_{\beta_1} \cap U_{\beta_2}$.

Define $\tau_{\mathcal{B}}$ by $U \in \tau_{\mathcal{B}}$ if and only if

$$U = \bigcup_\beta U_\beta,$$

for $U_\beta \in \mathcal{B}$.

To summarise, $\mathcal{U}$ is a basis for τ_X, and we define $\widetilde{\mathcal{U}}$ satisfying (i) and (ii) from $\mathcal{U}$, and use $\widetilde{\mathcal{U}}$ to topologise $\widetilde{X}$, which we recall is

$$\widetilde{X} = \bigcup_{x \in X} \Pi_1(X, x_0, x).$$

Define

$$\widetilde{\mathcal{U}} := \{U_{[\gamma]} : U \in \mathcal{U}\ ,\ [\gamma] \in \widetilde{X},\ \gamma(1) \in U\},$$

and

$$U_{[\gamma]} := \{[\gamma \cdot \eta] : \eta \in \Pi_1(U, \gamma(1), u),\ u \in U\}.$$

Remark 30. $U_{[\gamma]}$ depends only on U and $[\gamma]$.

Observe that the map

$$p : U_{[\gamma]} \longrightarrow U, \quad [\gamma \cdot \eta] \longmapsto (\gamma \cdot \eta)(1),$$

is surjective, since U is path-connected. Moreover, p is injective, since different choices of η are path-homotopic in X.

Claim: $U_{[\gamma]} = U_{[\gamma']}$ if and only if $[\gamma'] \in U_{[\gamma]}$.

Proof. Suppose that $\gamma' = \gamma$. Then, elements of $U_{[\gamma']}$ have the form $[\gamma' \cdot \eta] = [\gamma \cdot \eta \cdot \mu]$. This proves the $\subseteq$ direction. The other direction is analogous. ☐

Claim: $\widetilde{\mathcal{U}}$ is a basis for a topology.

Proof. Certainly, every $[\gamma] \in \widetilde{X}$ lies in some $U_{[\gamma]}$, just choose some $\gamma(1) \in U$. ☐

Now, let us consider $U_{[\gamma]}$, and $V_{[\gamma']}$, and suppose that $[\gamma''] \in U_{[\gamma]} \cap V_{[\gamma']}$. Then, $U_{[\gamma']} = U_{[\gamma'']}$, and similarly $V_{[\gamma']} = V_{[\gamma'']}$ by our previous claim. Now, let us take an open, path-connected subset $W \subset U \cap V$ such that $\gamma''(1) \in W$. Then, $W_{[\gamma'']} \subset U_{[\gamma']} = U_{[\gamma]}$, and $W_{[\gamma'']} \subset V_{[\gamma'']} = V_{[\gamma]}$. That is,

$$W_{[\gamma'']} \subset U_{[\gamma]} \cap V_{[\gamma']}.$$

It follows then that $(\widetilde{X}, \tau_{\widetilde{\mathcal{U}}})$ is a topological space. Now, we need to show the two facts:

(i) p is continuous,

(ii) p is a covering space.

Recall the bijection

$$p : U_{[\gamma]} \longrightarrow U.$$

Then, one can check that p defines a bijection to

$$\{V_{[\gamma']}\} \subseteq \widetilde{\mathcal{U}},$$

where $[\gamma'] \in U_{[\gamma']}$. This implies that p is continuous. To check that p is a covering space, it remains to show that $U_{[\gamma]} \cap U_{[\gamma']} = \emptyset$ if $[\gamma] \neq [\gamma']$, and $\gamma(1) = \gamma'(1)$. But, this holds since $U_{[\gamma]} = U_{[\gamma']}$ if and only if $[\gamma'] \in U_{[\gamma']}$ by our previous claim. All $[\gamma''] \in U_{[\gamma]}$ such that $\gamma''(1) = x$ are path-homotopic.

We must show now that X is path-connected, and simply-connected. Recall that

$$\widetilde{X} = \{[\gamma] : \text{base point } \widetilde{x} = [c_{x_0}]\}.$$

We wish to construct a path γ_t going from γ to c_{x_0}. So, let us set

$$\gamma_t(s) = \begin{cases} s & \text{if} \quad s \leq 1-t, \\ 1-t & \text{if} \quad s \geq 1-t, \end{cases}$$

One can check that $\gamma_t(s)$ is continuous. But since $p_* : \pi_1(\widetilde{X}, \widetilde{x}) \to \pi_1(X, x_0)$, is injective, it suffices to show that the image of p_* is trivial. Let $\widetilde{\gamma}$ be a path on $\widetilde{X}$. Observe that $\widetilde{\gamma} \sim \widetilde{\gamma}_1$. But $\widetilde{\gamma}_1 = c_{x_0}$, and so $[\widetilde{\gamma}] = [c_{\widetilde{x}}]$. The map $p \circ \widetilde{\gamma}$ has a lift, but since $\widetilde{\gamma}$ is the constant path, it gets lifted to the constant path in X. Thus, the image of p_* is trivial, since our choice of γ was arbitrary. Just go read Hatcher. $\square$

5.3. Lecture 3, 29/03/2024

Good Friday. No lecture.

Chapter 6

Week Six

6.1. 10/04/2024

Before the easter break, we proved the following. Let X be a path-connected, locally path-connnected, and semi-locally simply connected (p.c., l.p.c., s.l.s.c.). Then, there exists a covering space

$$p : \widetilde{X} \longrightarrow X,$$

such that $\pi_1(\widetilde{X}) = 0$ – that is, $\widetilde{X}$ is simply-connected. It follows then that p_* is the trivial homomorphism.

Proposition 17 (Prop 1.36, Hatcher)**.** *If X is p.c., l.p.c., and s.l.s.c., then for all subgroups $H < \pi_1(X, x)$, there exists a path-connected covering space*

$$p_H : (\widetilde{X}_H, \widetilde{x}_0) \longrightarrow (X, x_0),$$

such that the image of $(p_H)_*$ is H.

Proof. Recall that

$$\widetilde{X}_{univ} := \bigcup_{x \in X} \Pi(X, x_0) = \{[\gamma] : \gamma(0) = x_0\},$$

such that there is a sequence of maps

$$\widetilde{X}_{univ} \longrightarrow \widetilde{X}_H \xrightarrow{p_H} X.$$

Then, define an equivalence relation on $\widetilde{X}_{univ}$ by $[\gamma] \sim [\gamma']$ if and only if $\gamma(1) = \gamma'(1)$ and $[\gamma \cdot \bar{\gamma}'] \in H$. One checks that this defines an equivalence relation. Define

$$\widetilde{X}_H := \widetilde{X}_{univ} / \sim,$$

and define the map p_H by:

$$\widetilde{X}_H \longrightarrow X, \quad [[\gamma]] \longmapsto \gamma(1),$$

which makes sense since the equivalence relation on $\widetilde{X}_{univ}$ does nothing to the end points of the path by construction. Further, note that p_H is induced from the map $p_{univ} : \widetilde{X}_{univ} \to \widetilde{X}$. One readily checks that p_H indeed defines a covering Space. In particular, $\pi_1(X, x)$ labels the $U_{[\gamma]}$'s in $\widetilde{X}$, each of which is a disjoint copy of the space U in X.

The last thing that remains to be checked is the fact the image of $(p_H)_*$ is equal to H. We may argue this by lifting. This follows essentially by construction, using the fact that there exists a path (that is not a loop) from a point in $U_{[\gamma]}$ to $U_{[\gamma']}$. But under the quotient, all such points are identified, and each point corresponds to a loop in H. $\square$

Example 41. *Consider a map*

$$p : \mathbb{R} \longrightarrow \mathbb{S}^1,$$

and let $H = d\mathbb{Z}$. We know that $\pi_1(\mathbb{S}^1, 1) \cong \mathbb{Z}$. Then, the map p factors through a map

$$\mathbb{R} \longrightarrow \mathbb{R}/d\mathbb{Z} \xrightarrow{p} \underbrace{\mathbb{R}/\mathbb{Z}}_{\cong \mathbb{S}^1}.$$

Proposition 18 (Prop 1.37, Hatcher)**.** *If X is a path-connected, locally path-connected, then two covering spaces $p_1 : (\widetilde{X}_1, \widetilde{x}_1) \to (X, x)$, and $p_2 : (\widetilde{X}_2, \widetilde{x}_2) \to (X, x)$ are isomorphic if and only if* $\mathrm{Im}((p_1)_*) = \mathrm{Im}((p_2)_*)$.

Proof. Use the unique lifting property of Proposition 1.34. Given two covering spaces p_1 and p_2, there exists a lift $\widetilde{h}_1 : \widetilde{X}_2 \to \widetilde{X}_1$, by the existence of the lifting property. Conversely, there exists another lift $\widetilde{h}_2 : \widetilde{X}_1 \to \widetilde{X}_2$.

$$\begin{tikzcd}
\widetilde{X}_1 \arrow[rr, bend left, "h_1"] \arrow[dr, "p_1"'] & & \widetilde{X}_2 \arrow[ll, bend left, "h_2"] \arrow[dl, "p_2"] \\
& X &
\end{tikzcd}$$

But by uniqueness of lifts, it follows that $\widetilde{h}_1 \circ \widetilde{h}_2$ is the identity – that is, h_1 and h_2 are inverses of one another. $\square$

This proves the pointed classification of covering spaces in Theorem 1.38.

6.1.1. Classification of Pointed Covering Spaces

Consider two covering spaces with different choices of base points:

$$\begin{tikzcd}
(\widetilde{X}, \widetilde{x}_1) \arrow[r, "p"] & (X, x_0) \\
(\widetilde{X}, \widetilde{x}_2) \arrow[ur, "p"] &
\end{tikzcd}$$

Question: How are $p_*(\pi_1(\widetilde{X}, \widetilde{x}_1))$ and $p_*(\pi_1(\widetilde{X}, \widetilde{x}_2))$ related?

Let $\gamma : I \to \widetilde{X}$ be such that $\gamma(0) = \widetilde{x}_1$, and $\gamma(1) = \widetilde{x}_2$. Then, define a map

$$\beta_\gamma : \pi_1(\widetilde{X}, \widetilde{x}_1) \longrightarrow \pi_1(\widetilde{X}, \widetilde{x}_2), \quad [\delta] \longmapsto [\overline{\gamma} \cdot \delta \cdot \gamma].$$

Under p_*, we conjugate by $[p \circ \gamma]$, and we obtain:

$$\mathrm{Im}(p_1)_* = [p \circ \overline{\gamma}]\, \mathrm{Im}(p_2)_* [p \circ \gamma].$$

6.1.2. Deck Transformations and Group Actions

Definition 30. Let $p : \widetilde{X} \to X$ be a covering space. Then,

$$\mathcal{G}(\widetilde{X}) := \{h : X \xrightarrow{\simeq} X : p \circ h = p\} = \mathrm{Aut}(p).$$

By construction, $\mathrm{Aut}(p)$ is a subgroup of the group of homeomorphisms on $\widetilde{X}$. The gruop $\mathcal{G}(\widetilde{X})$ (or $\mathrm{Aut}(p)$) is called the group of *deck transformations* of p.

Example 42. *Consider the covering space*

$$p : \mathbb{R} \to \mathbb{S}^1, \quad t \longmapsto e^{2\pi it} = e^{2\pi i(t+n)}, \quad n \in \mathbb{Z}.$$

Then,

$$\mathcal{G}(\mathbb{R}) = \mathbb{Z},$$

by the $\mathbb{Z}$-invariance of p. Note that $\mathcal{G}(\mathbb{R}) \cong \pi_1(\mathbb{S}^1, 1)$. Later, we will learn that this is not a coincidence.

Example 43. *Consider the d-fold cover of $\mathbb{S}^1$, given by*

$$p_d : \mathbb{S}^1 \longrightarrow \mathbb{S}^1, \quad z \longmapsto z^d.$$

Then,

$$\mathcal{G}(p_d) \cong \mathbb{Z}/d\mathbb{Z}.$$

Observe that

$$\mathcal{G}(p_d) \cong \pi_1(\mathbb{S}^1, 1)/\operatorname{Im}(p_d)_*.$$

Definition 31 (Normal Covering Space)**.** Let X be path-connected. A map $p : \widetilde{X} \to X$ is a *normal covering space* if $\mathcal{G}(\widetilde{X})$ acts transitively on $p^{-1}(x)$, for all $x \in X$.

Remark 31. If X is path-connected, there is an embedding $\mathcal{G}(\widetilde{X}) \hookrightarrow p^{-1}(x)$, given by $h \mapsto h(\widetilde{x}_0)$.

6.2. Lecture 2, 11/04/2024

Recall for each covering space, $p : \widetilde{X} \to X$, that $\mathcal{G}(\widetilde{X}) := \mathrm{Aut}(p)$, the automorphism group of p. We say that p is *normal* if $\mathcal{G}(\widetilde{X})$ acts transitively on the fibres $p^{-1}(x)$.

Proposition 19 (Prop 1.39, Hatcher)**.** *Let $p : (\widetilde{X}, \widetilde{x}) \to (X, x)$ be a covering space with $\widetilde{X}$ path-connected, and X path-connected, and locally path-connected. Then,*

(i) p is normal if and only if $H := p_*(\pi_1(\widetilde{X}, \widetilde{x}))$ is normal in $\pi_1(X, x)$,

(ii) There is a group isomorphism:

$$\mathcal{G}(\widetilde{X}) \cong N(H)/H,$$

where

$$N(H) := \{[\gamma] \in \pi_1(X, x) : [\gamma]H[\overline{\gamma}] = H\},$$

is the normaliser *of H.*

Proof.

(i) Base-point change in $\widetilde{X}$ conjugates H by the image of a path between the basepoints. So, $[p \circ \gamma] \in N(H)$ if and only if $p_*(\pi_1(\widetilde{X}, \widetilde{x}_2)) = \pi_*(\pi_1(\widetilde{X}, \widetilde{x}_2))$. Then, by lifting, there exists some

$$h : (\widetilde{X}, \widetilde{x}_1) \longrightarrow (\widetilde{X}, \widetilde{x}_2).$$

Then, using the uniqueness of liftings, we may find an inverse for h. Since every loop in X based at x lifts to such a γ, we see that p is normal if and only if $N(H) = \pi_1(X, x)$ – that is, H is normal in $\pi_1(X, x)$.

(ii) Define
$$\varphi : N(H) \longrightarrow \mathcal{G}(\widetilde{X}), \quad [\gamma] \longmapsto h_{[\gamma]},$$
where $h_{[p\circ\gamma]}$. By the path-connectivedness of $\widetilde{X}$, φ is surjective. Given some $h \in \mathcal{G}(\widetilde{X})$, take γ to be a path from $\widetilde{x}_1$ to $\widetilde{x}_2 = h(\widetilde{x}_1)$. As an exercise, check that φ is a homomorphism and $\ker\varphi = H$.

□

6.2.1. Group Actions

Given a group G, and a space Y, an *action* of G on Y is a group homomorphism
$$\rho : G \longrightarrow \mathrm{Homeo}(Y).$$
We will write group actions as follows:
$$G \times Y \longrightarrow Y, \quad (g, y) \longmapsto g \cdot y := \rho(g)y.$$
With this definition, for $g_1, g_2 \in G$, we have $g_1 \cdot (g_2 \cdot y) = (g_1 g_2) \cdot y$.

Remark 32. If the multiplication and inverse maps $m : G \times G \to G$ and $i : G \to G$ are continuous, then G is called a *topological group*. Moreover, if the group $G \times Y \to \mathrm{Homeo}(Y)$ is continuous, then we have a *continuous group action*.

Definition 32 (Covering Space Action)**.** A space Y is equipped with a *covering space action* if for all $y \in Y$, there exists a neighbhourhood U containing y such that for all $g_1 \neq g_2 z \in G$, $g_1 U \cap g_2 U = \emptyset$.

A group action defines an equivalence relation on X by setting $x \sim x'$ if and only if $x = gx'$, for some $g \in G$. The equivalence classes under this equivalence relation are called the *orbits* of G:
$$X/G := [x] = \{x' : x' \sim x\}.$$
Then, we wish to determine when the map
$$q : X \longrightarrow X/G,$$
is a covering space.

Definition 33. An action is *free* if for all $g \in G$, $gx = x$ implies that $g = e$ the identity element – that is, the only fixed point is the identity. An action is *transitive* if X/G is a singleton set.

Note that if $\widetilde{G}(\widetilde{X})$ acts on $\widetilde{X}$, then this action is a covering space action. To see this, take an evenly covered subset $U \subset X$, and let $\widetilde{x} \in \widetilde{U}_i \subset \widetilde{X}$. Then,
$$h(\widetilde{x}) = \begin{cases} \widetilde{x} \iff h = \mathrm{id} \\ \widetilde{x} \in \widetilde{U}_i, \quad \widetilde{U}_i \cap \widetilde{U}_j = \emptyset \end{cases}$$

Proposition 20 (Prop 1.40, Hatcher)**.** *(i) If G acts on Y as a covering space action, then $Y \to Y/G$ is a covering*

(ii) If Y is path-connected, then $G = \mathcal{G}(Y)$.

(iii) If Y is path-connecte and locally path-connected, then $G \cong \pi_1(Y/G)/p_*(\pi_1(Y))$.
space.

Proof Idea. (i) We wish to show that Y/G is evenly covered. Given $[y]$, take $Y \subset U$, where U is as in the covering space action. Then, $[y] \in p(U)$ is evenly covered by p. One checks that $p|_U : U \to p(U)$ is a homeomorphism and $p^{-1}(p(U)) \cong \bigsqcup U$.

(ii) Evidenlty, $G \subseteq \mathcal{G}(Y)$ via the group action $G \times Y \to Y$. Each $g \in G$ defines a homeomorphism $g : Y \to Y$ that commutes with the projection $Y \to Y/G$. In particular, the diagram:

$$\begin{tikzcd} Y \arrow[rr, "g"] \arrow[dr] & & Y \arrow[dl] \\ & Y/G & \end{tikzcd}$$

commutes. Then, if Y is path-connected, $G \to \mathcal{G}(Y)$ is surjective (exercise).

(iii) This is an immediate consequence of Proposition 1.31(ii) from Hatcher, which we proved.

□

Example 44. *Consider maps*

$$\pi : \mathbb{S}^n \longrightarrow \mathbb{RP}^n,$$

where $\mathbb{RP}^n$ *is the space of lines pasing through the origin in* $\mathbb{R}^n$*. There is a* $\mathbb{Z}/2\mathbb{Z} \cong \{\pm 1\}$ *action on* $\mathbb{S}^1$ *given by*

$$\mathbb{S}^1 \times \mathbb{Z}/2 \longrightarrow \mathbb{S}^1, \quad (x, -1) \longmapsto -x.$$

One readily checks that this defines a covering space action, and thus $\mathbb{S}^n$ *defines a covering space of* $\mathbb{RP}^n$*. For* $n \geq 2$*, we know that* $\pi_1(\mathbb{S}^n)$ *is trivial, and thus* $\mathbb{S}^n \to \mathbb{RP}^n$ *is the universal covering. In particular,* $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2\mathbb{Z}$ *by Proposition 1.40(iii).*

In fact, this is a general phenomenon. If we are given a universal covering

$$\widetilde{X}_{\text{univ}} \longrightarrow X,$$

then

$$\mathcal{G}(\widetilde{X}_{\text{univ}}) \cong \pi_1(X, x).$$

To see this, let us consider

$$p : \widetilde{X}_{\text{univ}} \longrightarrow X.$$

Then, the fibres of this map are given by

$$p^{-1}(x) = \Pi_1(X, x, x_0) \equiv \pi_1(X, x_0).$$

Since $\widetilde{X}_{\text{univ}} \to X$ is normal, it follows then that

$$\widetilde{G}(\widetilde{X}_{\text{univ}}) \longrightarrow p^{-1}(x_0),$$

is a bijection (of sets). It thus follows that $\mathcal{G}(\widetilde{X}_{\text{univ}}) \equiv \pi_1(X, x_0)$.

Example 45. *Given the covering space* $p : \mathbb{R} \to \mathbb{S}^1$*, we have that*

$$\pi_1(\mathbb{S}^1, 1) \cong \mathbb{Z} \subseteq \text{Homeo}(\mathbb{R}).$$

Moreover, the map

$$\mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{S}^1 \times \mathbb{S}^1 = T^2.$$

Then, $\pi_1(T^2) = \mathbb{Z}^2 \subseteq \text{Homeo}(\mathbb{R}^2)$.

6.2.2. Cayley Graphs and Cayley Complexes

Consider a group G, together with a set of generators $g_1, \cdots, g_n$. That is,

$$G = \langle g_1, \cdots, g_n : r_1, \cdots, r_m \rangle.$$

Let S be the generating set of G. Then, the Cayley graph is constructed as follows:

- each element $g \in G$ is assigned to a vertex,
- each element $s \in S$ is assigned a colour c_s,
- for every $g \in G$ and $s \in S$, there is a directed edge of colour c_s from the vertex g to the vertex gs

6.3. Lecture 3, 12/04/2024

6.3.1. Eilenberg-Maclane Spaces

Definition 34. A path-connected space X is an *Eilenberg-Maclane space* (or $K(G,1)$-space) if $\pi_1(K) \cong G$, and its universal covering $\widetilde{K}_{\text{univ}} \simeq \text{pt}$ – i.e. $\widetilde{K}$ is contractible. Recall that this means that $\pi_i(K) = 0$ for all $i \geq 2$.

Example 46. *$\mathbb{S}^1$, T^2, K^2, and F_g^2 for $g \geq 1$ are all $K(G,1)$-spaces.*

$K(G,1)$ always exists, and we can build it by taking

$$G = \langle g_1, \cdots, g_n : r_1, \cdots, r_m \rangle.$$

From this, we can then build a space whose fundamental group is G by taking

$$\mathbb{S}^1_{g_1} \vee \cdots \vee \mathbb{S}^1_{g_n},$$

where the subscript is used to label each copy of $\mathbb{S}^1$ in the wedge product. The fundamental group of this is F_n, which is G but if the relations r_i were all empty. To obtain the relations, we attach 2-cells to $\vee_{i=1}^m \mathbb{S}^1_{g_i}$ along maps generating the relation r_j.

Recall that $\pi_1(T^2) \cong \mathbb{Z}^2 = \langle a, b : [a,b] = 0 \rangle$. We may construct

$$X_G := \left(\vee_{i=1}^m \mathbb{S}^1_{g_i}\right) \cup \left(\bigcup_{i=1}^n D^2_{ij}\right),$$

whici is called the *presentation* 2*-complex.* By construction, one may see that

$$\pi_1(X_G) \cong G.$$

From here, we can then attach 3-cells to X_G to kill π_2. Then, we can attach 4-cells to kill π_3, and so on.

6.3.2. Homology

Goal: Given a space X, we wish to associate to it a family of abelian groups $H_i(X)$ that define functors $H(-) : \textbf{Top} \to \textbf{Ab}$. More generally, one can also define $H_i(X)$ to be R-modules. Then, one recovers abelian groups by taking $R = \mathbb{Z}$. To specialise what coefficients we are using, we will write

$$H_i(X; \mathbb{Z}) = H_i(X).$$

If we are choosing coefficients from an R-module M, we write

$$H_i(X; R).$$

Definition 35. The *standard n-simplex* is given by

$$\Delta^n = \{t = (t_0, \cdots, t_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} t_i = 1, \quad t_i \geq 0\}.$$

Example 47. $\Delta^0 = \{1\}$. $\Delta^1 \cong [0,1]$, *and is given by the line going fro* $(1,0)$ *to* $(0,1)$. Δ^2 *can be thought of as a triangle with vertices given by the canonical basis vectors of* $\mathbb{R}^3$.

More generally, given points $v_0, \cdots, v_n$ linearly independent in $\mathbb{R}^{n+1}$ (or any $(n+1)$-dimensional vector space), set

$$[v_0, \cdots, v_n] := \left\{ \sum_{i=0}^{n} t_i v_i : t \in \Delta^n \right\}.$$

The set $[v_0, \cdots, v_n]$ is called an *affine n-simplex*. Moreover, there is a map

$$\Delta^n \longrightarrow [v_0, \cdots, v_n], \quad t \longmapsto \sum_{i=0}^{n} t_i v_i.$$

If we specify an ordering of vertices, then an affine n-simplex is canonically homeomorphic to Δ^n. The image of the map $\Delta^n \to [v_0, \cdots, v_n]$ are called *barycentric coordinates.*

Definition 36. The *i-th face* of $[v_0, \cdots, v_n]$ is the $(n-1)$-simplex

$$[v_0, \cdots, \widehat{v_i}, \cdots, v_n],$$

where the $\widehat{(-)}$ denotes omission. For Δ^n, define a map:

$$\partial_i \Delta^n = [e_0, \cdots, \widehat{e_i}, \cdots, e_n],$$

called the *boundary map.*

Remark 33. $\Delta^n \cong D^n \cong B^n \cong \underbrace{I \times \cdots \times I}_{n \text{ times}}$.

Define:

$$\partial \Delta^n := \bigcup_{i=0}^{n} \partial_i \Delta^n,$$

$$\partial [v_0, \cdots, v_n] = \bigcup_{i=0}^{n} [v_0, \cdots, \widehat{v_i}, \cdots, v_n],$$

$$\mathrm{Int}(\Delta^n) = \Delta^n \setminus \partial \Delta^n,$$

hence why ∂ is the called the *boundary operator.*

6.3.3. Simplicial Homology for Δ-Complexes

Definition 37. A *Δ-complex* structure on a space X consists of a collection of maps

$$\sigma_\alpha : \Delta^n \longrightarrow X,$$

where $n = n(\alpha)$, such that:

(i)

$$\sigma_\alpha|_{\mathrm{Int}(\Delta^n)} : \mathrm{Int}(\Delta^n) \longrightarrow X,$$

is injective, and for all $x \in X$, there exists a unique α such that $x \in \mathrm{Im}(\sigma_\alpha|_{\mathrm{Int}(\Delta^n)})$,

(ii)

$$\sigma_\alpha|_{\partial_i \Delta^n} \in \{\sigma_\alpha : \text{identify } \partial_i \Delta^n \text{ with } \Delta^{n-1}\},$$

(iii) $U \subset X$ is *open* if and only if $\sigma_\alpha^{-1}(U) \subset \Delta^n$ is open in $\mathbb{R}^{n+1}$ for all α.

Remark 34 (Diarmuid Wisdom). Δ-complexes are important, but they are not the main game. They're just a useful tool for getting us into homology. However, this is not an excuse to fall asleep for the next 15 minutes.

Fact:

$$X \cong \left(\bigsqcup_\alpha \Delta_\alpha^{n(\alpha)} \right) / \sim,$$

where $\sim$ is defined via canonical homeomorphisms between faces.

Example 48. *The Δ-complex structure on $\mathbb{S}^1$ is given by*

$$\{\sigma_\alpha\} = \{\sigma^0, \sigma^1\},$$

using the fact that $\mathbb{S}^1 \cong I/(0 \sim 1)$. In this case, we have one 0-simplex labelled by v_0, which is the image of σ^0, and an edge from v_0 to itself, which is the image of σ^2.

Example 49. *For $T^2 \cong (I \times I)/\sim$, we have 1-simplices labelled by v, and 2-simplices labelled by a and b, and 3-simplices labelled by U and L (given by drawing a diagonal line c – which is a 1-simplex – across the square, and labelling the two partitioned areas by U and L). Together, this is written to be:*

$$\{\sigma_v^0, \sigma_a^1, \sigma_b^1, \sigma_c^1, \sigma_U^2, \sigma_L^2\}.$$

Definition 38 (Homology of Δ-complex Structure)**.** Set

$$\Delta_n(X) := \left\{ \sum_\alpha n_\alpha \sigma_\alpha^n \right\},$$

the *free abelian group on the n-simplices*, which consists of all formal linear combinations of the n-simplices. Let $\Delta_n(X; M)$ be the free abelian group for which $n_\alpha \in M$, where M is an R-module. That is, $\Delta_n(X; M)$ is a free R-module in σ_α^n, tensored over R with respect to M. It follows then that

$$\Delta_n(X; M) \cong \bigoplus_{i=1}^{r_n} M,$$

where r_n is the number of n-simplices σ_α^n. Define the *boundary homomorphism* to be the map

$$\partial_n : \Delta_n(X) \longrightarrow \Delta_{n-1}(X), \quad \sigma_\alpha^n \longmapsto \sum_{i=0}^n (-1)^i \sigma_\alpha|_{\partial_i \Delta^n}.$$

Remark 35. The factor of $(-1)^i$ is just a miraculous addition that works. We will see in a bit why we need this factor.

Next, we wish to show that the boundary operator has the property that $\partial^2 = 0$, which we need to define homology.

Example 50. *Using the simplex* $[v_0, v_1]$*, we have:*

$$\partial_1 \circ \partial_2 = \partial_1([v_0, v_1] - [v_0, v_2] + [v_1, v_2]) = -v_1 + v_2 - (-v_2 + v_2) + (-v_0 + v_1) = 0.$$

Indeed, we see that this is the case in general:

Lemma 10 (Lemma 2.1, Hatcher)**.** $\partial_{i-1} \circ \partial_i = 0.$

Proof.

$$\begin{aligned}\partial_{n-1} \circ \partial_n(\sigma_\alpha^n) &= \partial_{n-1}\left(\sum_{i=0}^{n}(-1)^i \sigma_\alpha|_{[v_0,\cdots,\widehat{v_i},\cdots,v_n]}\right) \\ &= \sum_{j<i}(-1)^{i+j}\sigma_\alpha|_{[v_0,\cdots,\widehat{v_j},\cdots,\widehat{v_i},\cdots,v_n]} + \sum_{j>i}(-1)^{i+j}\sigma_\alpha|_{[v_0,\cdots,\widehat{v_i},\cdots,\widehat{v_j},\cdots,v_n]} \\ &= 0,\end{aligned}$$

since $[v_0, \cdots, \widehat{v_i}, \cdots, \widehat{v_j}, \cdots, v_n] = -[v_0, \cdots, \widehat{v_j}, \cdots, \widehat{v_i}, \cdots, v_n]$. $\square$

Chapter 7

Week Seven

7.1. Lecture 1, 17/04/2024

Definition 39. A *chain complex* (over $R = \mathbb{Z}$) is a sequence of free R-modules

$$\cdots \longrightarrow C_{i+1} \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_i \longrightarrow \cdots,$$

and homomorphisms $\partial_i : C_i \to C_{i-1}$ such that $\partial_{i-1} \circ \partial_i = 0$, for all $i \in \mathbb{Z}$. Usually, as convention we set $C_i = 0$ for all $i < 0$.

Definition 40. The i-th *homology* of the complex C_* is defined to be:

$$H_i(C_*, \partial) := \frac{\ker \partial_i}{\operatorname{Im} \partial_{i+1}}.$$

Example 51. *If X is a Δ-complex, then*

$$H_i(X) = H_i(\Delta_*(X)).$$

Example 52. *If $X = \mathbb{S}^1$, then $\Delta^1 = [v_0 v_1]$. This is formed from $\{\sigma_0, \sigma_1\}$. $f : \Delta^1 \to \mathbb{S}^1$ is the surjection. Then, we have a boundary map*

$$C_1 \xrightarrow{\partial_1} C_0,$$

which is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z}.$$

Then, $\partial \delta_1 = f(v_1) - f(v_0) = v - v = 0$. Thus,

$$H_1(\mathbb{S}^1, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0, 1, \\ 0, & otherwise. \end{cases}$$

Example 53. *Another Δ^1-structure can be defined on $\mathbb{S}^1$ by taking $\{\sigma_a^1, \sigma_b^1, \sigma_v^0, \sigma_w^0\}$. Then, we have a chain complex*

$$\begin{gathered} 0 \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2 \longrightarrow 0 \\ a \longmapsto w - v \\ b \longmapsto v - w \end{gathered}$$

Then,

$$H_0(\mathbb{S}^1, \mathbb{Z}) \cong \mathbb{Z}^2(v, w)/(v - w) \cong \mathbb{Z},$$
$$H_1(\mathbb{S}^1, \mathbb{Z}) \cong \ker \partial_1 = \{ma + mb : m \in \mathbb{Z}\} \cong \mathbb{Z},$$

where the last isomorphism in $H_1(\mathbb{S}^1, \mathbb{Z})$ follows from the fact that $\partial_1(a) = -\partial_1(b)$.

Example 54. *Consider the 2-torus T^2. Recall that we have simplices*

$$\{\sigma_U^2, \sigma_L^2, \sigma_a^1, \sigma_b^1, \sigma_c^1\}.$$

Then, we have a chain complex

$$\begin{aligned} 0 \longrightarrow \mathbb{Z}^2 = \langle U, L\rangle \longrightarrow \mathbb{Z}^3 = \langle a, b, c\rangle &\longrightarrow \mathbb{Z} \longrightarrow 0 \\ a &\longmapsto v - v = 0 \\ b &\longmapsto 0 \\ c &\longmapsto 0 \end{aligned}$$

The map $\mathbb{Z}^2 \to \mathbb{Z}^3$ is given by:

$$U \longmapsto c - a - b, \quad L \longmapsto c - a - b.$$

Then, we have

$$H_2(T^2, \mathbb{Z}) \cong \mathbb{Z}([U - L]),$$
$$H_1(T^2, \mathbb{Z}) \cong \mathbb{Z}^2([a], [b]),$$

where $[c] = [a] + [b]$.

$$H_0(T^2, \mathbb{Z}) \cong \mathbb{Z}([v]).$$

7.1.1. Remarks on Simplicial Complex and Homology

It is important to know the difference between simplicial complex and singular complex.

Remark 36 (Historical Remark). A *simplicial complex* is a Δ-complex, where each simplex embeds into X. That is, in any a simplicial complex, we know identify faces from different simplices. A *finite simplicial complex* is homeomorphic to a subcomplex of Δ^n for some n.

Remark 37. The minimal simplicial structure on T^2 has 18 2-simplices. Chop up $T^2 = (I \times I)/\sim$ into 9 squares, and divide each square in half along the diagonal to get two triangles. All together, we will have 18 triangles, corresponding to 18 simplices. So, we have a chain complex

$$0 \longrightarrow \mathbb{Z}^{18} \longrightarrow \mathbb{Z}^{27} \longrightarrow \mathbb{Z}^9 \longrightarrow 0.$$

Later we will see that the alternating sum of the dimensions of the complexes – in this case given by $18 - 27 + 9 = 0$ – is equal to the Euler characteristic of the space. A priori, in this case the Euler characteristic of T^2 is 0.

7.1.2. Singular Homology

Definition 41. Let X be a space. A *singular n-simplex* in X is a map

$$\sigma : \Delta^n \longrightarrow X.$$

Definition 42. The *n-th chain group* $C_n(X)$ is the free abelian group on the singular n-simplices in X.

Remark 38. Note that $C_n(X)$ is a large group – in particular, it is uncountably generated.

In particular, any $\zeta \in C_n(X)$ is a finite formal sum

$$\zeta = \sum_i n_i \delta_i,$$

for some $i \in \mathbb{Z}$, and $n_i \in M$, for some R-module M.

Definition 43. The *boundary map* is defined to be:

$$\partial_n : C_n(X) \longrightarrow C_{n-1}(X), \quad \sigma \longmapsto \sum_{i=0}^{n} \sigma|_{[v_0, \cdots, \widehat{v_i}, \cdots, v_n]},$$

and extending linearly.

The same proof seen in Lemma 2.1 shows that the boundary map has the property that $\partial^2 = 0$.

Let $C_*(X) = (C_*(X), \partial)$ be the singular chain complex of X. Then, the n-th singular homology group of X is

$$H_i(X, \mathbb{Z}) = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}.$$

If $\partial_\zeta = 0$, we call ζ a *cycle*, and we write

$$[\zeta] \in H_n(X, \mathbb{Z}),$$

and call it the *homology class* represented by ζ.

Remark 39 (Relation to Geomtery). Given a cycle $\zeta = \sum_i n_i \sigma_i$, re-write this as $\sum_j \varepsilon_j \sigma_j$, where $\varepsilon_j \in \{\pm 1\}$. Since $\partial(\zeta) = 0$, the $(n-1)$-dimensional faces, can be paired and identified to give a Δ-complex,

$$f_\zeta : K_\zeta \longrightarrow X.$$

K_ζ has a lot of manifold points, and only fails to be a manifold around the $(n-1)$-skeleton. In particular, this is a finite Δ-complex, and one can show that $H_n(K_\zeta, \mathbb{Z}) \cong \mathbb{Z}$, and $[\zeta] = (f_\zeta)_*(1)$, where

$$(f_\zeta)_* : H_i(K_\zeta) \longrightarrow H_i(X).$$

Slogan: cycles give geometric carriers of homology classes.

So, for instance, we have $H_1(T^2, \mathbb{Z}) \cong \mathbb{Z}(a, b)$, where the generators correspond to the two copies of the circle embedded in T^2 given by $\mathbb{S}^1_a \hookrightarrow T^2$, and $\mathbb{S}^1_b \hookrightarrow T^2$, where the subscript denotes the copy of $\mathbb{S}^1$ corresponding to one of the generators of the 1-st homology.

Proposition 21 (Prop 2.6, Hatcher)**.** *Write as a disjoint union of path-connected components:*

$$X = \sqcup_\alpha X_\alpha.$$

Then,

$$H_n(X) = \bigoplus_\alpha H_n(X_\alpha).$$

Proof. Each $\delta : \Delta^n \to X$ has path-connected image in some X_α, and $\partial\sigma$ has image in X_α. So,

$$C_*(X, \partial) = \bigoplus_\alpha C_*(X_\alpha, \partial_\alpha).$$

Thus,

$$H_*(X) \cong \bigoplus_\alpha H_*(X_\alpha),$$

where we can simply write $*$ as a variable for i, or we can take the *total homology*:

$$H_*(X) := \bigoplus_{i=0}^{\infty} H_i(X).$$

The result works for both definitions of $H_*(X)$. □

Proposition 22 (Prop 2.7, Hatcher)**.** *If $X \neq$ is path-connected, then*

$$H_0(X) \cong \mathbb{Z}.$$

Proof. By definition. $H_0(X) = C_0(X)/\operatorname{Im}\partial_1$. But by definition, the boundary of a 0-simplex is always 0, and so

$$C_1(X) \xrightarrow{\partial_1} C_0(X) \longrightarrow 0.$$

Define

$$\varepsilon : C_0(X) \longrightarrow \mathbb{Z}, \quad \sum_i n_i \varepsilon_i \longmapsto \sum_i n_i.$$

Since $X \neq \emptyset$, ε is onto.

Claim: $\operatorname{Im}\partial_1 = \ker \varepsilon$, which would imply that ε induces a pushforward map

$$\varepsilon_* : H_0(X) \longrightarrow \mathbb{Z},$$

which is an isomorphism by the claim. We will finish the proof of this next time. □

Remark 40. In general,

$$H_0(X) \cong \bigoplus_\alpha H_0(X_\alpha) = \bigoplus_\alpha \mathbb{Z},$$

where the number of copies of $\mathbb{Z}$ is equal to the number of path-connected components of X.

7.2. Lecture 2, 18/04/2024

Proposition 23 (Prop 2.7, Hatcher)**.** *If $X \neq \emptyset$ is path-connected, then $H_0(X) \cong \mathbb{Z}$.*

Proof. Define

$$\varepsilon : C_0(X) \longrightarrow \mathbb{Z}, \quad \sum_i n_i \delta_i \longmapsto \sum_i n_i.$$

Claim: $\operatorname{Im}\partial_1 = \ker \varepsilon$.

Observe that $\operatorname{Im}\partial_1 \subset \ker \varepsilon$, since if $\delta' : \Delta' \to X$ is a singular 1-simplex, then

$$\varepsilon(\partial(\sigma')) = \varepsilon(\sigma(v_1) - \sigma(v_0)) = 1 - 1 = 0.$$

Suppose now that

$$\varepsilon\left(\sum_i n_i \sigma_i^0\right) = 0.$$

Let $x_i = \sigma_i^0(*)$. Let σ_i^1 be paths from a basepoint x_0 to x_i.

Claim: $\partial_1\left(\sum_i n_i\sigma_i^1\right) = \sum_i n_i\delta_i^0$.

To see this, observe that

$$\partial\left(\sum_i \sigma_i^1\right) = \sum_i n_i\sigma_i^0 - \underbrace{\sum_i n_i x_0}_{=0}. = \sum_i n_i\sigma_i^0,$$

where the first equality follows from the fact that $\sigma_i^1(v_0) = x_0$.

This claim thus implies the converse inclusion, and thus shows that $\ker\varepsilon = \operatorname{Im}\partial_1$. □

Definition 44. The *reduced homology* of X is given by:

$$\widetilde{H}_0(X) := H_0(C_*^{\text{aug}}),$$

where C_*^{aug} is the *augmented chain complex*, which is given by:

$$\cdots \longrightarrow C_n(X) \longrightarrow C_{n-1}(X) \longrightarrow \cdots \longrightarrow C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

This define a chain complex since $\varepsilon\circ\partial_1 = 0$.

Remark 41.

$$\widetilde{H}_i(X) = \begin{cases} H_i(X), & i > 0 \\ H_0(X)/\mathbb{Z}, & i = 0 \end{cases}.$$

Example 55.

$$\widetilde{H}_i(\mathbb{S}^0) = \begin{cases} 0, & i > 0, \\ \mathbb{Z}, & i = 0. \end{cases}$$

Then,

$$H_1(\mathbb{S}^0) \cong \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z}.$$

Then, $\widetilde{H}_0(\mathbb{S}^0) \cong \mathbb{Z}^{|\alpha|-1}$*, where* $|\alpha|$ *denotes the number of path connected components of* X*.*

Proposition 24 (Prop 2.8, Hatcher)**.** *If* $X = \{\text{pt}\}$*, then*

$$H_i(\text{pt}) = 0, \quad \textit{for all} \quad i > 0,$$

$$\widetilde{H}_i(\text{pt}) = 0, \quad \textit{for all} \quad i \geq 0.$$

Proof. $\Delta_n(\text{pt}) = \mathbb{Z}(c_{\text{pt}})$. This gives us an n-simplex

$$\sigma^n : \Delta^n \longrightarrow \text{pt}\,.$$

Recall that an n-simplex has $(n+1)$-faces. Thus, there is precisely one simplex here, given by

$$C_n(\text{pt}) \longrightarrow C_{n-1}(\text{pt}), \quad \sigma \longmapsto \sum_i (-1)^i\sigma^{n-1}.$$

It follows then that

$$C_{2k+1}(\text{pt}) \longrightarrow C_{2k}(\text{pt}),$$

is the zero map, and the next map

$$C_{2k}(\text{pt}) \longrightarrow C_{2k-1}(\text{pt}),$$

is an isomorphism. This is because if we have a 2-simplex $\sigma^2 : \Delta^2 \to \mathrm{pt}$, then

$$\partial\sigma^2 = \underbrace{\sigma^2|_{[v_1 v_2]} - \sigma^2|_{[v_0 v_2]}}_{=0} + \sigma^2|_{v_0 v_1}.$$

More generally,

$$\sigma^n|_{[v_0 \cdots \widehat{v_i} \cdots v_n]} = \sigma^n|_{[v_0 \cdots \widehat{v_j} \cdots v_n]},$$

for all i, j. So, the chain complex is given by

$$C_4 \xrightarrow{\simeq} C_3 \xrightarrow{0} C_2 \xrightarrow{\simeq} C_1 \xrightarrow{0} C_0 \longrightarrow 0.$$

Thus, by inspection,

$$H_i(C_*(\mathrm{pt})) = \begin{cases} 0 & i > 0 \\ \mathbb{Z}, & i = 0 \end{cases}.$$

□

Remark 42. Let X be path-connected, and consider $\pi_1(X, x_0)$. Observe that $\gamma : I \to X$, for $\gamma(0) = \gamma(1) = x_0$, can be identified with a singular 1-cycle Δ^1. So, there is a natural map

$$\rho_1 : \pi_1(X, x_0) \longrightarrow H_1(X), \quad [\gamma] \longmapsto [\sigma^1_\gamma],$$

mapping a path homotopy class to a homology class. As we will see, $H_1(-)$ defines a functor as well. As such, what we see here is an example of a natural transformation.

One may show that the natural map ρ_1 is a surjective homomorphism, with kernel given by the normal subgroup generated by commutators:

$$\langle [\gamma_1], [\gamma_2] \rangle.$$

That is, one may identify

$$H_1(X) = \pi_1(X, x_0)_{\mathrm{ab}},$$

where $\pi_1(X, x_0)_{\mathrm{ab}}$ denotes the *abelianisation* of $\pi_1(X, x_0)$. In particular, given any group G,

$$G_{\mathrm{ab}} := G / \langle [g, h] \rangle.$$

(see Hatcher A.2 for more details). More generally, there exists maps

$$\rho_n : \pi_n(X, x_0) \longrightarrow H_n(X),$$

called *Hurewitz homomorphisms* (look up Hurewitz's theorem for more information, or wait until week 11).

7.2.1. Homotopy Invariants of Homology

Definition 45. Let $f : X \to Y$ be a continuous map. Then, f induces a map of chain complexes:

$$f_\# : C_n(X) \longrightarrow C_n(X), \quad \sigma_\alpha \longmapsto f \circ \sigma_\alpha.$$

Extending linearly, the map is defined to be:

$$\sum_\alpha n_\alpha \sigma_\alpha \longmapsto \sum_\alpha n_\alpha \left(f \circ \sigma_\alpha \right).$$

The composition $f \circ \sigma : \Delta^n \to Y$ is continuous, and thus defines a singular complex in Y. Further, a *chain map* is given by:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f_\#} & & \downarrow{\scriptstyle f_\#} & & \\ \cdots & \longrightarrow & C_n(Y) & \xrightarrow{\partial} & C_{n-1}(Y) & \longrightarrow & \cdots \end{array}$$

such that each square commutes. Indeed, by construction the squares commute since:

$$\partial(f_\#\sigma) = \partial(f \circ \sigma) = \sum_i f \circ \partial_i \sigma = f_\# \left(\sum_i \partial_i \sigma \right) = f_\#(\partial\sigma).$$

Hence,

$$\partial f_\# = f_\# \partial.$$

In general, a collection of maps $f_i : C_i \to D_i$, for $i \in \mathbb{Z}$ is called a *chain map* if $f\partial f_i = f_{i-1}\partial$, for any abstract chain complexes C_* and D_*.

Remark 43. The study of chain complexes and their properties is called *homological algebra.*

Lemma 11. *Any chain map induces a homomorphism*

$$H_i(C_*) \longrightarrow H_i(D_*), \quad [c] \longmapsto [f_i(c)].$$

Proof. $[c] \in H_i(C_*)$ if and only if $\partial c = 0$. Since $\partial f_i(c) = f_i(\partial c) = 0$, it follows that $[f_i(c)] \in H_i(D_*)$. Moreover, $[c] = c + \partial_{i+1}(C_{i+1})$, which gets mapped to $f_i([c]) + \partial_{i-1}(D_{i+1})$. $\square$

It follows then that $f : X \to Y$ induces a map in homology:

$$f_* : H_i(X) \longrightarrow H_i(Y).$$

Basic Properties:

1. The composition $X \xrightarrow{f} Y \xrightarrow{g} Z$, induces a map

$$(g \circ f)_* : H_i(X) \xrightarrow{f_*} H_i(Y) \xrightarrow{g_*} H_i(Z).$$

2. The identity 1_X induces the identity map $1_{H_i(X)}$ in homology.

The following is the non-trivial property:

Theorem 7 (Thm 2.10, Hatcher, Homotopy Invariance)**.** *Let $f, g : X \to Y$ be homotopic maps.*

$$f_* = g_* : H_i(X) \longrightarrow H_i(Y).$$

Corollary 3 (Cor 2.11, Hatcher)**.** *If $f : X \to Y$ is a homotopy equivalence, with homotopy inverse $g : Y \to X$, then, $f_* : H_i(X) \to H_i(Y)$ is an isomorphism with inverse $g_* : H_i(Y) \to H_i(X)$.*

Proof. $g \circ f : X \to X$ is homotopic to the identity 1_X. Hence, $(g \circ f)_* = (1_X)_* = 1_{H_i(X)}$. But on the other hand, $g_* \circ f_* = 1_{H_i(X)}$. Similarly, one shows that $f_* \circ g_* = 1_{H_i(Y)}$. $\square$

Proof of Theorem 7. Let $H : X \times I \to Y$ be a homotopy from f to g. That is, we have maps:

$$g : X \times \{1\} \to Y, \quad f : X \times \{0\} \longrightarrow Y.$$

We may define a simplex:

$$\sigma \times \mathrm{id}_I : \Delta^n \times I \longrightarrow X \times I,$$

with

$$\sigma^n : \Delta^n \longrightarrow X \times \{0\}.$$

For $C_0(X)$, $\mathrm{pt} \times X \cong \Delta^1$. For $C_1(X)$, $\Delta^1 \times I \cong \Delta^1 \cup \Delta^1$. This is a square with vertices given by v_0, v_1, w_0, w_1. Thus, we have

$$\Delta^1 \cup \Delta^1 \xrightarrow{g \circ \sigma^1} X \times I \xrightarrow{H} Y,$$

the composition of which is $f \circ \sigma^1$. We will finish this proof of tomorrow. Or you can go read Hatcher. □

7.3. Lecture 3, 19/04/2024

7.3.1. Assignment Tips

Q 2.4 Uses cellular homology and tensor products. In the case of the 2-torus, recall that we have a Δ-complex given by $\{\sigma_v^0, \sigma_a^1, \sigma_b^1, \sigma_c^1, \sigma_U^2, \sigma_L^2\}$. This gives us a chain complex

$$\mathbb{Z}^2 \longrightarrow \mathbb{Z}^3 \longrightarrow \mathbb{Z}.$$

The Klein bottle K has a chain complex given by

$$\mathbb{Z} \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z}.$$

The second map will be zero no matter how we define these maps. Recall that $H_1(X) \cong \pi_1(X, x)_{\mathrm{ab}}$. The cell attaching maps $\mathbb{S}^1 \to \mathbb{S}_a^1 \vee \mathbb{S}_b^1$, we induce a map

$$\underbrace{H_1(\mathbb{S}^1)}_{\cong \mathbb{Z}} \longrightarrow \underbrace{H_1(\mathbb{S}_a^1 \vee \mathbb{S}_b^\vee)_{\cong \mathbb{Z} \oplus \mathbb{Z}}}, \quad z \longmapsto aba^{-1}b, \quad 1 \longmapsto (0, 2).$$

One checks that $H_1(K^2) \cong \pi_1(K^2) \cong (\mathbb{Z} \rtimes \mathbb{Z})_{\mathrm{ab}}$.

Q 2.2 This is probably the hardest question on the assignment. A *manifold with boundary* is a topological space M such that $x \in \partial M$ is locally homeomorphic to the upper half plane $\mathbb{R}^{\geq 0} \times \mathbb{R}^{n-1}$. For every point in the interior of M is locally homeomorphic to $\mathbb{R}^n$.

One can glue two copies of the manifold M together via the antipodal map by

$$\widetilde{X} = M \cup_a M,$$

where $a : \mathbb{S}^n \to \mathbb{S}^n$ is the antipodal map. Then, $\mathbb{Z}/2$ acts on $\widetilde{X}$ by the twisting around the area where we glue. We actually end up with a free $\mathbb{Z}/2$ action, and free $\mathbb{Z}/2$ actions are (inaudible).

7.3.2. Homotopy Invariants

Theorem 8. *If $f, g : X \to Y$ are homotopic maps, then $f_* = g_* : H_i(X) \to H_i(Y)$.*

Proof. Let $H : X \times I \to Y$ be the homotopy. Then, we have induced chain maps:

$$f_*, g_* : C_*(X) \longrightarrow C_*(Y).$$

Then, we have a map

$$\Delta^n \times I \overset{\sigma\times 1\times I}{\longrightarrow} X \times I \overset{H}{\longrightarrow} Y.$$

Let us first restrict to the case where $n = 1$. In this case, we have $\Delta^1 \times I$, which is a square whose vertices are labelled by v_0, v_1, w_0, w_1. We have maps:

$$g : \Delta^1 \times \{1\} \longrightarrow Y, \quad f : \Delta^1 \times lbrace0\} \longrightarrow Y.$$

Then, this induces maps

$$\begin{aligned} f_\#(\sigma^1) &= f \circ \sigma|_{[v_0,v_1]}, \\ g_\#(\sigma^1) &= g \circ \sigma|_{[w_0,w_1]}. \end{aligned}$$

Define

$$P : C_1(X) \longrightarrow C_2(X), \quad \sigma \longmapsto H \circ \sigma^2|_{[v_0,w_0,w_1]} - H \circ \sigma|_{[v_0,v_1,w_1]},$$

called the *prism operator*. Then, applying the boundary map, we obtain:

$$\begin{aligned} \partial([v_0,w_0,w_1] - [v_0,v_1,w_1]) &= ([v_0,w_0] - [v_0,w_1] + [w_0,w_1]) - ([v_0,v_1] - [v_0,w_1] + [v_1,w_1]) \\ &= [w_0,w_1] - [v_0,v_1] + [v_0,w_1] - [v_1,w_1] \\ &= -P\partial \end{aligned}$$

What this shows us is that $\partial P = g_\# - f_\# = -P\partial$. Indeed, we can chop $\Delta^n \times I$ into $(n-1)$ n-simplices via

$$\bigcup_{i=0}^{n} [v_0, \cdots, v_i, w_i, \cdots, w_m].$$

□

Remark 44. More explicitly, in the example above, what we really have are maps:

$$\begin{tikzcd} \Delta^2_{[v_0,v_1,v_2]} \arrow[r] & Y \arrow[rr, "g"] \arrow[dr] & & Y \arrow[dl] \\ \Delta^2_{[w_0,w_1,w_2]} \arrow[ur] & & Y/G & \end{tikzcd}$$

and the prism operator just takes the difference betwen the two 2-simplices that are being mapped in.

Definition 46. Generally, a *prism operator* is a map

$$P_H : C_n(X) \longrightarrow C_{n+1}(Y),$$

such that $\partial P = g_\# - f_\# - P\partial$.

Chapter 8

Week Eight

8.1. Lecture 1, 24/04/2024

8.1.1. Relative Homology

For any pair (X, A), define:

$$C_n(X, A) := \frac{C_n(X)}{C_n(A)}.$$

Recall that a singular pair is a map $\Delta^n \to X$, which might factor through A, in which case one may consider a singular pair in A. This leads us to consider pairs (X, A). Since we have boundary maps:

$$\begin{tikzcd}
C_n(X) \arrow[r, "\partial"] & C_{n-1}(X) \\
C_n(A) \arrow[u] \arrow[r, "\partial"] & C_{n-1}(A) \arrow[u]
\end{tikzcd}$$

we thus have an induced boundary map

$$\partial : \frac{C_n(X)}{C_n(A)} \longrightarrow \frac{C_{n-1}(X)}{C_{n-1}(A)} =: C_{n-1}(X, A).$$

Lemma 12. *There exists a long exact sequence*

$$H_{k-1}(X, A) \xrightarrow{\partial} H_k(A) \xrightarrow{i_*} H_k(X) \longrightarrow H_k(X, A) \xrightarrow{\partial} H_{k-1}(A).$$

Remark 45. By abuse of notation, we are denoting all the boundary maps to be ∂, but in practise they are different maps.

Proof Sketch. We have a level-wise short exact sequence of chain complexes and chain maps, given by the chain map

$$0 \to C_*(A) \xrightarrow{i_*} C_*(X) \xrightarrow{j_*} C_*(X, A) \to 0.$$

Whenever we have a short exact sequence of chain complexes,

$$0 \to A_* \to B_* \to C_* \to 0,$$

we obtain a long exact sequence in homology

$$\cdots \to H_{i+1}(C) \xrightarrow{\partial} H_i(A) \xrightarrow{i_\#} H_i(B) \xrightarrow{j_\#} H_i(C) \to \cdots .$$

The key point of the proof is the definition of $\partial : H_{i+1}(C) \to H_i(A)$. We have chain maps:

$$\begin{array}{ccccccc}
A_{n+1} & \longrightarrow & A_n & \longrightarrow & A_{n-1} & \longrightarrow & A_{n-2} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
B_{n+1} & \longrightarrow & B_n & \longrightarrow & B_{n-1} & \longrightarrow & B_{n-2} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C_{n+1} & \longrightarrow & C_n & \longrightarrow & C_{n-1} & \longrightarrow & C_{n-2}
\end{array}$$

Using the fact that each vertical map is short exact, we may use a diagram chasing argument to conclude that $\partial\widetilde{a} = 0$. $\square$

8.2. Lecture 2, 25/04/2024

8.3. Lecture 3, 26/04/2024

Chapter 9

Week Nine

9.1. Lecture 1, 01/05/2024

9.2. Lecture 2, 02/05/2024

9.3. Lecture 3, 03/05/2024

Chapter 10

Week Ten

10.1. Lecture 1, 08/05/2024

10.1.1. Equivalence of Simplicial and Singular Homology

We wish to prove naturality: that is, a diagram as below induces a commutative diagram of H_* long exact sequences:

$$\begin{tikzcd}
\cdots \arrow[r] & H_{k+1}^{\Delta}(X^{i+1}, X^i) \arrow[r] \arrow[d] & H_k^{\Delta}(X^i) \arrow[r] \arrow[d] & H_k^{\Delta}(X^{i+1}) \arrow[r] \arrow[d] & H_k^{\Delta}(X^i) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_{k+1}(X^{i+1}, X^i) \arrow[r] & H_k(X^i) \arrow[r] & H_k(X^{i+1}) \arrow[r] & H_k(X^{i+1}, X^i) \arrow[r] & \cdots
\end{tikzcd}$$

Strategy: induction on $\dim X$ for $H_k^{\Delta}(X^i) \to H_k(X^i)$. Our basic input is something of the form $H_k^{\Delta}(X^{i+1}, X^i) \to H_k(X^{i+1}, X^i)$. These are isomorphisms:

$$H_k^{\Delta}(X^{i+1}, X^i) \cong \bigoplus_{\alpha} \mathbb{Z}(\Delta_\alpha^k), \quad H_k(X^{i+1}, X^i) \cong H_k(\vee_\alpha \mathbb{S}^k) \cong \bigoplus_{\alpha} \mathbb{Z}_\alpha.$$

Lemma 13. *Singular homology $H_n(\Delta^n, \partial\Delta^n)$ is generated by $[\mathrm{id} : \Delta^n \to \Delta^n]$, where*

$$\partial \,\mathrm{id} = \sum_{i=0}^{n} \mathrm{id}\,|_{\partial_i \Delta^n} : \partial_i \Delta^n \longrightarrow \partial\Delta^n \subset \Delta^n.$$

Proof. The proof is by induction. Let $\Lambda \supset \Delta^n$ be the union of the last n faces. That is,

$$\Lambda = \bigcup_{i=1}^{n} [v_0, \cdots, \widehat{v_i}, \cdots, v_n].$$

Example 56. *For $n = 2$, we have a triangle with vertices labelled by v_0, v_1, v_2 with that orientation. Then, Λ is given by omiiting $v_0 v_1$, and $v_0 v_2$. In fact, Λ is a cone on the simplex of the boundary of the simplex. Hence, Λ is contractible, since all cones are contractible, and Λ has the homology of a point.*

Generally, Λ is a cone, or a "horn" as Diarmuid calls it because of its shape. Thus, let us consider the triple $(\Delta^n, \partial\Delta^n, \Lambda)$. Consider the map on relative homology:

$$H_n(\Delta^n, \partial\Delta^n) \longrightarrow H_{n-1}(\partial\Delta^n, \Lambda) \longrightarrow \underbrace{H_{n-1}(\Delta^n, \Lambda)}_{=0},$$

where the last homology group goes to 0 since Δ^n and Λ have the same homotopy type. Thus, $H_n(\Delta^n, \partial\Delta^n) \cong H_{n-1}(\partial\Delta^n, \Lambda)$. Additionally, we have a map:

$$\begin{tikzcd}
0 \arrow[r] & H_n(\Delta^n, \partial\Delta^n) \arrow[r, "\partial"] & H_{n-1}(\partial\Delta^n, \Lambda) \arrow[r] & 0 \\
& & H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}) \arrow[u] &
\end{tikzcd}$$

Since there are relative homeomorphisms:

$$\begin{tikzcd}
(\partial\Delta^n, \Lambda) \arrow[r] & (\partial\Delta^n/\Lambda) \\
(\Delta^{n-1}, \partial\Delta^{n-1}) \arrow[u] \arrow[r] & \Delta^{n-1}/\partial\Delta^{n-1} \arrow[u, "\text{homeo}"']
\end{tikzcd}$$

there are additions that we can make to the sequence:

$$\begin{tikzcd}
0 \arrow[r] & H_n(\Delta^n, \partial\Delta^n) \arrow[r, "\partial"] & H_{n-1}(\partial\Delta^n, \Lambda) \arrow[rr] \arrow[dr, Rightarrow] & & 0 \\
& & H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}) \arrow[u] \arrow[d] & H_{n-1}(\partial\Delta^n/\Lambda) & \\
& & H_{n-1}(\Delta^{n-1}/\partial\Delta^{n-1}) \arrow[ur, "\cong"'] & &
\end{tikzcd}$$

This will be the inductive step that we will perform. Reduce to the $n = 1$ case. Then, by Lemma 13, we have that there is a diagram:

$$\begin{tikzcd}
H_i^{\Delta}(X^i, X^i) \cong \mathbb{Z}([\mathrm{id} : \Delta^i_\alpha \to \Delta^i_\beta]) \arrow[r, "\cong"] \arrow[ddr, bend right] & H_i(X^i, X^{i-1}) \arrow[d] \\
& H_i(X^i, X^{i-1}) \arrow[d] \\
& H_i(\vee_\alpha \mathbb{S}^i)
\end{tikzcd}$$

By induction on the 5-lemma, it follows that

$$H_k^{\Delta}(X) \longrightarrow H_k(X),$$

is an isomorphism for all k. If (X, A) is a Δ-complex pair, then $H_k^{\Delta}(X, A) \to H_k(X, A)$ is an isomorphism for all k. Then, by the 5-lemma:

$$\begin{tikzcd}
H_i^{\Delta}(A) \arrow[r] \arrow[d, "\cong"] & H_i^{\Delta}(X) \arrow[r] \arrow[d, "\cong"] & H_i^{\Delta}(X, A) \arrow[d, "\cong \text{ by 5-lemma}"] \\
H_i(A) \arrow[r] & H_i(X) \arrow[r] & H_i(X, A)
\end{tikzcd}$$

$\square$

10.1.2. CW-Homology

A CW-complex of X is a chain complex

$$C_*^{\mathrm{CW}}(X) = \left(\bigoplus_{i=0}^{\infty} C_i^{\mathrm{CW}}(X), \partial_i\right),$$

where each

$$C_i(X) = \mathbb{Z}(e_\alpha^i).$$

We are going to make the identifications

$$C_i(X) \cong H_i(X^i, X^{i-1}) \cong H_i(X^i/X^{i-1}).$$

Theorem 9. *There exists a natural isomorphism*

$$H_i^{\mathrm{CW}}(X) \longrightarrow H_i(X).$$

Lemma 14 (Lemma 2.34, Hatcher)**.**

(i)

$$H_k(X^{\cdot} X^{i-1}) = \begin{cases} 0, & k \neq n \\ \mathbb{Z}|^{c_n}, & k = n \end{cases}$$

(b) $H_k(X^n) = 0$ *for all* $k > n$*,*

(c) $X^n \hookrightarrow X$ *induces* $H_i(X^n) \to H_i(X)$*, which is an isomorphism for all* $i < n$*, and surjection for all* $i = n$*.*

Proof. (i) follows by construction. For (ii) and (iii), consider a long exact sequence:

$$\cdots \longrightarrow H_{k+1}(X^n, X^{n-1}) \longrightarrow H_k(X^{n-1}) \longrightarrow H_k(X^n) \longrightarrow H_k(X^n, X^{n-1}) \longrightarrow H_{k-1}(X^{n-1}) \longrightarrow \cdots$$

If $k > n$,

$$H_k(X^n) \cong H_k(X^{n-1}) \cong H_k(X^{n-2}) \cong \cdots \cong H_k(X^0) \cong 0,$$

where $X^0 = \mathrm{pt}$. This proves (ii). Note that if $k < n$, we have an isomorphism

$$H_k(X^{n-1}) \to H_k(X^n) \to H_k(X^{n+1}) \to \cdots,$$

unless $\dim X$ is infinite, in which case just go see Hatcher. For $k = n$, we argue similarly. $\square$

Now, we want to construct a map between cellular homology, and argue that it is an isomorphism.

Proof of Theorem 9. Could repeat H_*^{Δ}-argument. Assume that

$$H_n(X^{n+1}) \longrightarrow H_n(X),$$

is an isomorphism. Then, consider the following diagram:

$$\begin{tikzcd}
H_n(X^{n-1}) \arrow[drr, "= 0 \text{ by Lemma 14(b)}"] & & & H_n(X^{n+1}) \cong H_n(X) \arrow[r] & H_n(X^{n+1}, X^n) = 0 \\
& & H_n(X^n) \arrow[ur, two heads] \arrow[dr] & & \\
H_{n+1}(X^{n+1}, X^n) \arrow[urr, "\partial"] \arrow[rrr, "\partial_{n+1}^{\mathrm{CW}}"] & & & H_n(X^n, X^{n-1}) \arrow[r, "\partial_n^{\mathrm{CW}}"] \arrow[d, "\partial"] & H_{n-1}(X^{n-1}, X^{n-2}) \\
& & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"'] & \\
& 0 \arrow[r] & H_{n-1}(X^{n-2}) \arrow[ur] & &
\end{tikzcd}$$

By a diagram chase argument, we have isomorphisms:

$$H_n(X) \cong H_n(X^{n+1}) \cong H_n(X^n)/\operatorname{Im}(\partial_{n+1}) \cong j(H_n(X^n))/\operatorname{Im}\partial_n^{\mathrm{CW}} \cong \ker(\partial_n)/\operatorname{Im}(\partial_n^{\mathrm{CW}}).$$

But since j_{n-1} is injective, we thus have that

$$\frac{\ker \partial_n}{\operatorname{Im}\partial_{n+1}^{\mathrm{CW}}} \cong \frac{\ker \partial_n^{\mathrm{CW}}}{\operatorname{Im}(\partial_{n+1}^{\mathrm{CW}})} = H_n^{\mathrm{CW}}(X^n, X^{n-1}).$$

□

It follows that map $H_i^{\mathrm{CW}}(X) \to H_i(X)$ is a zig-zag:

$$\begin{tikzcd}
& H_i(X) & \\
H_i(X^i) \arrow[ur, "i"] \arrow[rr] & & H_i(X^i, X^{i-1}) \\
{[c]} \arrow[rr, mapsto] & & {[c]_{\mathrm{CW}}}
\end{tikzcd}$$

It is an exercise to check that this is well-defined.

10.1.3. Cellular Boundary Map

We have:

$$H_i(X^i, X^{i-1}) \longrightarrow H_{i-1}(X^{i-1}) \xrightarrow{j} H_{i-1}(X^{i-1}, X^{i-2}),$$

for an i-cell $e_\alpha^i = \psi(D_\alpha^i)$. Using the fact that $(D^i, \mathbb{S}^{i-1}) \cong (\Delta^i, \Delta^{i-1})$ and $[\mathrm{id} : (\Delta^i, \partial\Delta^i) \to (\Delta^i, \partial\Delta^i)]$ generates $H_i(\Delta^i, \partial\Delta^i) \cong H_i(D^i, \mathbb{S}^{i-1})$, we see that $\partial_{\mathrm{CW}}(e_\alpha^i)$ is given by taking

$$H_{i-1}(\mathbb{S}_\alpha^{i-1}) \xrightarrow{\varphi_\alpha} H_{i-1}(X^{i-1}) \xrightarrow{q} H_{i-1}(X^{i-1}/X^{i-2}) \cong H_i(\vee_\beta \mathbb{S}_\beta^i) \xrightarrow{\mathrm{pr}_\beta} \mathbb{S}_\beta^{i-1},$$

where pr_β is the projection map. Recall that $\partial(e_\alpha^i) = \sum n_{\alpha\beta} e_\beta^{i-1}$, where

$$n_{\alpha\beta} = \deg\left(\varphi_\alpha \circ q \circ \mathrm{pr}_\beta\right).$$

10.2. Lecture 2, 09/05/2024

10.2.1. Degree and Local Degree

Suppose that $y \in \mathbb{S}^n$ is such that
$$|f^{-1}(y)| < \infty.$$

That is, we are supposing that y is a regular value of a smooth map. We wish to show that for each $x_i \in f^{-1}(y)$, we may assign a degree $\deg_{x_i}(f) \in \{\pm 1\}$ such that

$$\deg(f) = \sum_i \deg_{x_i}(f).$$

Let $V \subset \mathbb{S}^n$ be a neighbourhood of y such that $V \cong \mathbb{R}^n$, and $x_i \in U_i$ neighourhoods such that $f|_{U_i} : U_i \to V_i$. On homology, we have:

$$H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{\mathrm{pt}\}) \xrightarrow{\partial} H_n(\mathbb{R}^n \setminus \{\mathrm{pt}\}) \cong H_{n-1}(\mathbb{S}^{n-1}) \cong \mathbb{Z},$$

where the isomorphism follows since $\mathbb{R}^n \setminus \{\mathrm{pt}\}$ deformation retracts to $\mathbb{S}^{n-1}$. Thus, for our choice of open sets, we have:

$$\begin{tikzcd}
 & H_n(U_i, U_i \setminus \{x_i\}) \arrow[r, "f_*"] \arrow[dl, "\cong"'] \arrow[d] & H_n(V, V \setminus \{y\}) \arrow[d] \\
H_n(\mathbb{S}^n, \mathbb{S}^n \setminus \{x_i\}) \arrow[r] & H_n(\mathbb{S}^n, \mathbb{S}^n \setminus f^{-1}(y)) \arrow[r] & H_n(\mathbb{S}^n, \mathbb{S}^n \setminus \{y\}) \\
 & H_n(\mathbb{S}^n) \arrow[r] \arrow[ul, "\cong"] \arrow[u] & H_n(\mathbb{S}^n) \arrow[u, "\cong"']
\end{tikzcd}$$

The left two isomorphisms follow by excision. Explicitly, we have:

$$H_n(\mathbb{S}^n, \mathbb{S}^n \setminus f^{-1}(y)) \cong \bigoplus_{|f^{-1}(y)|} \mathbb{Z}.$$

Let $C = \mathbb{S}^n / mH(U_i)$, which is homotopy equivalent to $\mathbb{S}^n \setminus f^{-1}(y)$. Thus, on homology, we have:

$$H_n(\mathbb{S}^n, \mathbb{S}^n \setminus f^{-1}(y)) \cong H_n(\mathbb{S}^n, C) \cong H_n(\mathbb{S}^n / C) \cong H_n(V\mathbb{S}^n).$$

We have a map: $H_n(\mathbb{S}^n) \to H_n(\mathbb{S}^n / C)$, which becomes

$$\Delta : \mathbb{Z} \longrightarrow \mathbb{Z}^{|f^{-1}(y)|},$$

and
$$H_n(\mathbb{S}^n, \mathbb{S}^n \setminus f^{-1}(y)) \longrightarrow H_n(\mathbb{S}^n, \mathbb{S}^n \setminus \{y\}),$$

whichb ecomes a map
$$\mathbb{Z}^{|f^{-1}(y)|} \longrightarrow \mathbb{Z}, \quad (d_1, \cdots, d_m) \longmapsto d_1 + \cdots + d_m,$$

where for simplicity we are writing $m = |f^{-1}(y)|$. This fact follows from the commutativity of the diagram above.

Question: Is it true that $\deg_{x_i}(f) \in \{\pm 1\}$? This would imply that if y is regular, then $f|_{U_i} : U_i \to V$ is a diffeomorphism.

Example 57. *Consider the map $\mathbb{C} \to \mathbb{C}$ defined by $z \mapsto z^2$. We have that $f^{-1}(0) = 0$, thus regular. Then,* $\deg_0(f) = 2$.

Example 58. *Consider $f_d : \mathbb{S}^1 \to \mathbb{S}^1$ given by $z \mapsto z^d$. Choose an orientation for the sphere. We have that* $\deg(f_d) = d$. *By construction, $f_{-1} : z \mapsto \overline{z}$, and we have that* $\deg(f_{-1}) = -1$.

Properties of Degrees

1. $\deg(R) = -1$ for R a reflection,
2. If f has no fixed points, then $\deg(f) = \deg(A) = (-1)^{n+1}$

Chapter 11

Week Eleven

11.1. Lecture 1, 15/05/2024

Proposition 25 (Prop 2.30 (extended)). *Let $f : M \to N$ be a map between closed, connected, oriented n-manifolds (or a self-map of a closed, connected, orientable n-manifold $h : M \to M$), then*

$$H_n(M) \cong \mathbb{Z}([M]), \quad H_n(N) \cong \mathbb{Z}([N]),$$

and define $\deg(f)$ *by* $f_*[M] \deg(f) \cdot [N]$. *Then,*

$$\deg(f) = \sum_{x_i \in f^{-1}(y)} \deg_{x_i}(f),$$

where $\deg_{x_i}(f)$ *is the local degree of* f *at* x_i.

Proof. Take $M / (M \setminus \bigcup_{i=1}^n U_i) \cong \vee \mathbb{S}^n$. □