Live-TeXed Lecture Notes for My Dumbass

MAST90017 - Representation Theory

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Contents

Chapter 1

Week One

Office Hours: Peter Hall 203 (starting week 3) — see Canvas
Assessment: 50% assignments, and 50% final exam (3 hours)

This week, we learned about . . . definition of a representation, subrepresentation. Examples of a representation of G are the regular representation (two definitions), and the permutation representation. These representations can be obtained given any group G.

Given two representations, we can make more representations by taking their direct sums, or taking their tensor product. The tensor product representation decomposes into two subrepresentations: the symmetric and alternating power representations.

A nice way of characterising isomorphic reprensentations is by studying their characters, since isomorphic representations have the same character.

More broadly, characters are examples of class functions, which are functions that take constant values on conjugacy classes.

1.1. Lecture 1, 24/07/2023

This course will mostly deal with the representation theory of finite groups, and of finite-dimensional algebras.

1.1.1. Basic on Linear Representations of Finite Groups

Definition 1. Let G be a finite group. A *linear representation* of G on a $\mathbb{C}$-vector space V is a group homomorphism

$$\rho : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V),$$

where $\mathrm{Aut}(V)$ is the group of invertible linear maps. If $\dim_{\mathbb{C}} V = n$, then we may choose a basis, and

$$\mathrm{Aut}(V) \cong \mathrm{GL}_n(\mathbb{C}).$$

We say that V is a *representation space* of G (or simply, a *representation of G*).

Exercise 1. *What happens if we replace $\mathbb{C}$ by some other field?*

Proof. It better be algebraically closed. $\square$

From now on, we assume that V is finite-dimensional. Serre explains that this is not a very severe condition. Often, it if possible to look at the finite points of V, and then study representations generated by those points. Further, since G is finite, this is even less of a problem.

The dimension of V — denoted $\dim_{\mathbb{C}} V$ — is called the *degree* of the representation V.

Definition 2. A homomorphism φ between two G-representations (ρ_1, V), and (ρ_2, W) is a linear map $\varphi : V \to W$ such that the diagram commutes for all $g \in G$:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho_1(g)"'] & W \arrow[d, "\rho_2(g)"] \\ V \arrow[r, "\varphi"] & W \end{tikzcd}$$

That is,

$$\varphi(\rho_1(g)v) = \rho_2(g)\varphi(v).$$

The two representations (ρ_1, V) and (ρ_2, W) are *isomorphic* if there exists a homomorphism of G-representations which is a linear isomorphism. We write $\mathrm{Hom}_G(V, W)$ to denote the set of homomomorphisms of G-representations (G-representations for short).

The above information allows us to define a category of finite-dimensional G-representations, denoted by $\mathbf{Rep}_G(\mathbb{C})$. It is an abelian category.

Exercise 2. *Think about what other ways you can define a morphism of representations.*

Proof. You can define a representation of G as a G-module, and then define morphisms as morphisms between G-modules. There is a bijection between G-representations and G-modules, anyway, since the map ρ can just be defined as

$$G \longrightarrow \mathrm{Aut}(V), \quad g \longmapsto (v \longmapsto gv).$$

$\square$

Example 1 (Examples of Representations of G)**.**

(a) ***One-Dimensional Representations*** *Any degree one representation of G, given by*

$$\rho : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}) \cong \mathbb{C}^{\times} = \mathrm{GL}_1(\mathbb{C}).$$

Question: *How do we find all degree one representations?*

Answer: *Since G is finite, for all $g \in G$, there exists some $m \geq 1$ for which $g^m = 1$, which implies that $\rho(g)$ is an m-th root of unity. This does not give us the whole answer, though. Think about it!*

Question: *What are degree 1 representations of $\mathbb{Z}/n\mathbb{Z}$?*

Answer: *Since $\mathbb{Z}/n\mathbb{Z}$ is a cyclic group, any generator gets mapped to an n-th root of unity under ρ. There are n such elements that we can map to an n-root of unity.*

(b) ***Regular Representation of*** *G Let*

$$V = \bigoplus_{g \in G} \mathbb{C}e_g,$$

be a vector space of dimension $\dim_{\mathbb{C}} V = |G|$. *Define maps:*

$$\rho_h : V \longrightarrow V, \quad e_g \longmapsto e_{hg},$$

and

$$R : G \longrightarrow \mathrm{GL}(V), \quad h \longmapsto \rho_h.$$

This representation R *is called the* regular representation *of* G.

(c) ***Another Definition of Regular Representation*** *Let* W *be the vector space of complex-valued functions on* G. *That is, the space of functions* $f : G \to \mathbb{C}$. *Define a linear map*

$$\tau : G \longmapsto \mathrm{GL}(W), \quad g \longmapsto (f \longmapsto gf),$$

where $f : G \to \mathbb{C}$, *and* $gf : G \to \mathbb{C}$. *In particular, the action* gf *is given by*

$$g \cdot f(h) = f(g^{-1}h),$$

for $g, h \in G$. *The factor of* g^{-1} *is added so that the map* $f \mapsto g \cdot f$ *is an isomorphism in* W.

Exercise 3.

(a) Check that this is a representation of G *(pay attention to the factor of* g^{-1} *instead of* g*).*

(b) Show that V *and* W *are isomorphic representations of* G.

Proof.

(a) Let $g, h, k \in G$, and $f \in W$. Then,

$$\tau(gh)(f)(k) = f(h^{-1}g^{-1}k) = \tau(h)(f)(g^{-1}k) = \tau(h)(\tau(g)(f))(k) = (\tau(h) \circ \tau(g))(k)),$$

and it follows that τ is a group homomorphism.

(b) Define a map $1_g : G \to \mathbb{C}$ by:

$$1_g(x) = \begin{cases} 1, & \text{if} \quad x = g, \\ 0, & \text{otherwise} \end{cases} .$$

We see from this that $W = \mathrm{Span}_{\mathbb{C}}\{1_g : g \in G\}$. For any $f = \sum_{g \in G} a_g 1_g \in W$, we observe that if $a_g \neq 0$ for all $g \in G$, then $f(G) \neq 0$. Thus, f defines the zero function if and only if $a_g = 0$ for all $g \in G$, and it follows that the 1_g's are linearly independent. It follows then that $\{1_g : g \in G\}$ form a basis for W, and $\dim_{\mathbb{C}} W = |G|$.

Define a map

$$\varphi : V \longrightarrow W, \quad e_g \longmapsto 1_g.$$

It is sufficient to show that φ gives a morphism of G-representations, as bijectivity is clear. We have a diagram:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "R(h)"'] & W \arrow[d, "\tau(h)"] \\ V \arrow[r, "\varphi"'] & W \end{tikzcd}$$

The composition gives:

$$\varphi(R(h)(e_g)) = \varphi(\rho_h(e_g) = \varphi(e_{hg}) = 1_{hg},$$

and the other composition gives:

$$\tau(h)(\varphi(e_g))(x) = \tau(h)1_g(x) = h \cdot 1_g(x) = 1_g(h^{-1}x) = 1_{hg}(x),$$

and thus the diagram commutes and we have

$$\varphi \circ R(h) = \tau(h) \circ \varphi.$$

Thus, φ gives a morphism of G-representations, and is in fact an isomorphism of G-representations since it maps basis vectors to basis vectors.

$\square$

(c) ***Permutation Representation*** *Suppose that G acts on a finite set X. Let*

$$V = \bigoplus_{x \in X} \mathbb{C}e_x,$$

be a finite-dimensional vector space with $\dim_{\mathbb{C}} V = |X|$. Then, there is a G-representation given by:

$$\rho : G \longrightarrow \mathrm{GL}(V), \quad g \longmapsto (e_x \longmapsto e_{gx}).$$

This representation is called the permutation representation *associated with X.*

1.2. Lecture 2, 27/07/2023

Question: Can you define a degree one representation ρ of G such that $\rho(G) = \{\pm 1\}$?

Answer: No! For instance, we cannot do this for $\mathbb{Z}/3\mathbb{Z}$.

Definition 3 (Subrepresentations)**.** Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G. A subspace W of V is *G-stable* if for any $w \in W$, and any $g \in G$, $\rho(g)(w) \in W$. A G-stable subspace W of V called a *subrepresentation* if there is a map $\rho_W : G \to \mathrm{GL}(W)$ given by $g \mapsto \rho(g)|_W$.

Example 2. *Consider the regular representation*

$$V = \bigoplus_{g \in G} \mathbb{C}e_g.$$

Then, the $\mathbb{C}$-span of the element $\sum_{g \in G} e_g$ is a G-stable subspace, and is in fact a degree 1 *trivial subrepresentation. Write this subrepresentation as*

$$W = \mathbb{C} \sum_{g \in G} e_g.$$

Question: Given a representation of G, what are all of its subrepresentations?

Before we answer this, we will recall some linear algebra.

1.2.1. Linear Algebra Recap

Let V be a $\mathbb{C}$-vector space, and $W \subset V$ a subspace. Then, we have the following bijection:

$$\{\text{Projections of } V \text{ onto } W\} \longleftrightarrow \{\text{Complements of } W \text{ in } V\}.$$

Recall that a *projection* of V onto W is a linear map $p : V \to W$ such that $p|_W = \mathrm{id}_W$. The bijection is given explicitly by the assignments

$$p \longmapsto \ker p,$$

$$V = W \oplus W' \longmapsto p(w + w') = w.$$

(TODO: make this look neater). The first map makes sense since $V = \ker p \oplus W$.

Exercise 4. *Check that this is a bijection.*

Proof. Given a projection $p : V \to W$, there is a short exact sequence

$$0 \longrightarrow \ker p \longrightarrow V \xrightarrow{p} W \longrightarrow 0.$$

Since, by definition $p|_W = \mathrm{id}_W$, it follows that the exact sequence is split, and thus $V \cong W \oplus \ker p$, and it follows that the map $p \longmapsto \ker p$ is a map from projections of V onto W to complements of W in V.

Conversely, given a decomposition $V = W \oplus W'$, there is a map $p : V \to W$ defined by $p(w + w') = w$, for $w \in W$, $w' \in W'$. This is clearly a projection map. $\square$

1.2.2. Back to Representation Theory

Theorem 1. *Let $\rho : G \to \mathrm{GL}(V)$ be a G-representation, and W a G-subrepresentation. There exists a complementary subrepresentation W' such that $V = W \oplus W'$.*

Proof. Let U be an arbitrary complementary subspace of W — that is, such that $V = W \oplus U$. Let $p_0 : V = W \oplus U \to W$ be the projection onto U. Howevef, this map is not necessarily a morphism of G-representations. Thus, we define an *averaging map*:

$$p := \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ p_0 \circ \rho(g)^{-1},$$

which defines a linear map $V \to W$. We now verify that p is a projection. Given some $w \in W$, then

$$p(w) = \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ \rho(g)^{-1}(w) = w,$$

and it follows that p is indeed a projection of V onto W. Using the aforementioned bijection, we obtain a complement of W given by $W' := \ker p$. It remains to show that W' is G-stable. Let $w' \in W'$. Then, $p(v) = 0$ by definition. Now, given any $h \in G$,

$$\begin{aligned}
p(\rho(h)(v)) &= \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ p_0 \circ \rho(g)^{-1}(\rho(h)(v)) \\
&= \frac{1}{|G|} \sum_{g \in G} \rho(h) \circ (\rho(h^{-1}g) \circ p_0 \circ \rho(g^{-1}h))(v) \\
&= \frac{1}{|G|} \rho(h) \sum_{g \in G} \rho_{h^{-1}g} \circ p_0 \circ \rho_{g^{-1}h}(v) \\
&= 0,
\end{aligned}$$

and it follows that $W' = \ker p$ is G-stable. $\square$

Another Proof. There exists a sesquilinear form $\langle -, - \rangle$ on V given by

$$\langle -, - \rangle : V \times V \longrightarrow \mathbb{C},$$

such that

$$\langle av, w\rangle = a\langle v, w\rangle, \quad \langle v, aw\rangle = \bar{a}\langle v, w\rangle, \quad \langle v, v\rangle > 0, \quad \langle v + v', w\rangle = \langle v, w\rangle + \langle v', w\rangle,$$

for all $v \neq 0, w \in V$, and $a \in \mathbb{C}$. Make it G-invariant by using the averaging trick from the previous proof. Let

$$\langle v, w\rangle_G := \sum_{g\in G} \langle gv, gw\rangle.$$

Let

$$W^c := \{v \in V : \langle v, w\rangle_G = 0 \text{ for all } w \in W\}.$$

Then, it follows that

$$V = W \oplus W^c.$$

Exercise 5. *Show that W^c is G-stable.*

Proof of Exercise 5. Let $v \in W^c$. Then, for some $x \in G$, and any $w \in W$:

$$\langle \rho(x)v, w\rangle_G = \sum_{g\in G} \langle \rho(x) \circ \rho(g)v, \rho(g)w\rangle,$$

and so $\rho(x)v \in W^c$, and thus W^c is G-stable. (TODO: finish this proof) $\square$

$\square$

Remark 1. What the above shows is that this decomposition is in general not unique.

Definition 4. A linear representation $\rho : G \to \mathrm{GL}(V)$ is irreducible (or simple) if $V \neq \{0\}$, and there exist no non-trivial subrepresentations — that is, the only subrepresentations are $\{0\}$ and V.

Corollary 1 (Complete Reducibility/Semisimplicity)**.** *Every $\mathbb{C}$-linear representation of G is a direct sum of irreducible representations.*

Proof. We proceed by inducting on the dimension of our representation. Given a G-representation V with $\dim_{\mathbb{C}} V = n$, if V is irreducible then we are done. If not, then there exists a subrepresentation W, and the Theorem tells us that $V = W \oplus W'$, for some other subrepresentation W'. We continue this process, finding smaler and smaller subrepresentations of W and W', until we the subrepresentation is irreducible. Eventually, we end up with a direct sum of irreducible representations. $\square$

Remark 2 (Caution!)*.* This may break down if we consider k-linear representations, when $\mathrm{char}\, k \neq 0$.

Exercise 6. *Can you think of an example?*

Example 3. *Consider the additive group of real numbers $(\mathbb{R}, +)$, and a group homomorphism*

$$\mathbb{R} \longrightarrow \mathrm{GL}_2(\mathbb{R}), \quad a \longmapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

There is a subrepresentation given by $\mathbb{R}e_1 \subset \mathbb{R}^2$.

Exercise 7. *Find the complement of this representation.*

Proof. Here, we have $V = \mathbb{R}$. For $v, w \in \mathbb{R}$, $\langle v, w\rangle = vw$, given by mutiplication in $\mathbb{R}$. $\square$

Question: Is this decomposition unique? (We will discuss more about this later).

1.3. Lecture 3, 28/07/2023

Recall from last time that we wanted to try and find ways of producing representations.

Definition 5. Let (V, ρ) and (V', ρ') be two representations of G. Then,

(a) $V \oplus V'$ is also a representation of G given by

$$\tau : G \longrightarrow \mathrm{GL}(V \oplus V'), \quad g \longmapsto \tau_g(v + v') = \rho(g)v + \rho'(g)v'.$$

(b) $V \otimes V'$ is also a representation of G given by:

$$\gamma : G \longrightarrow \mathrm{GL}(V \otimes V'), \quad g \longmapsto \gamma_g(v \otimes v') = \rho_g(v) \otimes \rho'_g(v').$$

(c) One can take a *dual representation* $V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$, given by:

$$\pi : G \longrightarrow \mathrm{GL}(V^*), \quad g \longmapsto \pi_g,$$

where $\pi_g : V^* \to V^*$ is given by $f \mapsto g \cdot f$, where g acts on f by $g \cdot f(v) = f(g^{-1}v)$.

(d) The space $\mathrm{Hom}_{\mathbb{C}}(V, V')$ defines a representation given by:

$$\pi : G \longrightarrow \mathrm{GL}(\mathrm{Hom}_{\mathbb{C}}(V, V')), \quad g \longmapsto \pi_g,$$

where

$$\pi_g : \mathrm{Hom}_{\mathbb{C}}(V, V') \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, V'), \quad f \longmapsto g \cdot f,$$

where the action is given by

$$g \cdot f(v) = \rho'_g f(\rho_{g^{-1}} v).$$

Remark 3. Note here that (c) is a special case of (d), given by replacing V' by the trivial representation.

Exercise 8. *Check that all of these defines a representation!*

Proof.

(a) Using the fact that ρ and ρ' are group homomorphisms for $v \in V$, $v' \in V'$, and $g, h \in G$:

$$\tau_{gh}(v + v') = \rho_{gh}(v) + \rho'_{gh}(v') = (\rho_g \circ \rho_h)(v) + (\rho'_g \circ \rho'_h)(v') = (\tau_g \circ \tau_h)(v + v').$$

(b) Once again, let $v \in V$, $v' \in V'$, and $g, h \in G$:

$$\gamma_{gh}(v \otimes v') = \rho_{gh}(v) \otimes \rho_{gh}(v') = (\rho_g \circ \rho_h)(v) \otimes (\rho'_g \circ \rho'_h)(v') = (\gamma_g \circ \gamma_h)(v \otimes v').$$

(c) Computing directly,

$$\tau_{gh}(f)(v) = (gh) \circ f(v) = f(h^{-1}g^{-1}v) = \pi_h(g\dot{f})(v) = \pi_h(\pi_g(f))(v) = (\pi_h \circ \pi_g)(f)(v).$$

(d) (TODO: check your previous calculation before attempting this one. I think you did something wrong)

$\square$

Exercise 9. *Show that*

$$\mathrm{Hom}_G(V, V') \cong \mathrm{Hom}_{\mathbb{C}}(V, V')^G,$$

where the left-hand side denotes the G-fixed points of $\mathrm{Hom}_{\mathbb{C}}(V, V')$.

Proof. Define a map

$$\Phi : \mathrm{Hom}_G(V, V') \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, V')^G, \quad \varphi \longmapsto g \cdot \varphi,$$

where $g \cdot \varphi(v) = \rho'_g \varphi(\rho_{g^{-1}} v)$. We show that $g \cdot \varphi$ is a G-invariant linear map. Since φ is a morphism of G-representations, $\rho'_g \varphi(v) = \varphi(\rho_g(v))$, and so

$$g \cdot \varphi(v) = \rho'_g \varphi(\rho_{g^{-1}}(v)) = \varphi(\rho_g \circ \rho_{g^{-1}}(v)) = \varphi(v),$$

and it follows then that $g \cdot \varphi \in \mathrm{Hom}_{\mathbb{C}}(V, V')^G$. It is clear that φ is a linear map. The map $\varphi \mapsto g^{-1} \cdot \varphi$ is a suitable choice of inverse for Φ, and thus Φ is an isomorphism of vector spaces. □

1.3.1. Symmetic Square/Alternating Square

Let (ρ, V) be a G-representation, and let $\{e_i\}_{i=1}^n$ be a basis for V. Then, $\{e_i \otimes e_j\}_{i,j=1}^n$ is a basis for $V \otimes V$. Then, one can decompose the tensor product into two subrepresentations in the following way:

$$V \otimes V \cong \mathrm{Sym}^2(V) \oplus \Lambda^2(V).$$

Define an *involution*:

$$\theta : V \otimes V \longrightarrow V \otimes V, \quad e_i \otimes e_j \longmapsto e_j \otimes e_i.$$

Extending by linearity, $\theta(v \otimes w) = w \otimes v$. Let

$$\mathrm{Sym}^2(V) := \{x \in V \otimes V : \theta(x) = x\},$$

$$\Lambda^2(V) := \{x \in V \otimes V : \theta(x) = -x\}.$$

A basis for $\mathrm{Sym}^2(V)$ (resp. $\Lambda^2(V)$) is then given by $\{e_i \otimes e_j + e_j \otimes e_i\}$ (resp. $\{e_i \otimes e_j - e_j \otimes e_i\}$). It has dimensions

$$\dim_{\mathbb{C}} \mathrm{Sym}^2(V) = \frac{n^2+n}{2}, \quad \dim_{\mathbb{C}} \Lambda^2(v) = \frac{n^2-n}{2}.$$

Exercise 10. *Show that* $\mathrm{Sym}^2(V)$ *and* $\Lambda^2(V)$ *are subrepresentations of* $V \otimes V$*.*

Proof. Indeed, $\mathrm{Sym}^2(V)$ is G-stable since for $v, w \in V$, and $vw \in \mathrm{Sym}^2(V)$, we have

$$\rho(g)(vw) = (gv)(gw),$$

and since each $gv \in V$, and $gw \in V$, it follows that $(gv)(gw) \in \mathrm{Sym}^2(V)$. The map $\rho(g)|_{\mathrm{Sym}^2(V)}$ which sends vw to $(gv)(gw)$ has an inverse given by $vw \mapsto (g^{-1}v)(g^{-1}w)$, and thus $\rho(g)|_{\mathrm{Sym}^2(V)} \in \mathrm{GL}(\mathrm{Sym}^2(V))$, and thus there is a map

$$\rho_{\mathrm{Sym}^2(V)} : G \longrightarrow \mathrm{GL}(\mathrm{Sym}^2(V)), \quad g \longmapsto (vw \longmapsto (gv)(gw)),$$

and thus $\mathrm{Sym}^2(V)$ is a subrepresentation of $V \otimes V$.

Similarly, G-stability of $\Lambda^2(V)$ follows from the fact that

$$\rho(g)(v \wedge w) = gv \wedge gw \in \Lambda^2(V).$$

The restriction $\rho(g)|_{\Lambda^2(V)}$ defines a map $v \wedge w \mapsto gv \wedge gw$, with inverse given by $v \wedge w \mapsto g^{-1}v \wedge g^{-1}w$, and so there is a map

$$\rho_{\Lambda^2(V)} : G \longrightarrow \mathrm{GL}(\Lambda^2(V)), \quad g \longmapsto (v \wedge w \longmapsto gv \wedge gw),$$

and so $\Lambda^2(V)$ is a subrepresentation of $V \otimes V$. □

Inductively, we can construct exterior powers $\Lambda^n(V)$ and symmetric powers $\mathrm{Sym}^n(V)$ (see Fulton and Harris Appendix B.1, B.2).

1.3.2. Characters

Definition 6. Let (ρ, V) be a representation of G. Then, a *character* of ρ is given by the $\mathbb{C}$-valued function

$$\chi_\rho : G \longrightarrow \mathbb{C}, \quad g \longmapsto \mathrm{tr}(\rho_g).$$

Recall from linear algebra that the trace is the sum of the eigenvalues of a matrix (with multiplicities). We will show that representations are isomorphic if and only if their characters are the same, and that characters play well with direct sums and tensor products.

Exercise 11. *Show that isomorphic representations have the same character.*

Proof. Let (ρ, V) and (τ, W) be two isomorphic G-representations — that is, there is a $\mathbb{C}$-linear isomorphism $\varphi : V \to W$ such that the diagram

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho_g"] & W \\ V \arrow[r, "\varphi"] & W \arrow[u, "\tau_g"'] \end{tikzcd}$$

commutes. Then,

$$\chi_\rho(g) = \mathrm{tr}(\rho_g) = \mathrm{tr}(\varphi^{-1} \circ \tau_g \circ \varphi) = \mathrm{tr}(\tau_g) = \chi_\tau,$$

where the second equality follows from the commutativity of the diagram, and the third equality follows by the invariance of trace. □

Basic Properties of Characters

Lemma 1. *Let (ρ, V) be a representation of G, and χ_ρ its character.*

(a) $\chi_\rho(1) = \dim_{\mathbb{C}} V$.

(b) $\chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$.

(c) $\chi_\rho(hgh^{-1}) = \chi_\rho(g)$, *for all* $h, g \in G$.

(d) $\chi_{\rho \oplus \rho'} = \chi_\rho + \chi_{\rho'}$.

Proof.

(a) Immediate.

(b) Suppose ρ_g has eigenvalues $\lambda_1, \cdots, \lambda_n$, where $n = \dim_{\mathbb{C}} V$. Then, $\rho_{g^{-1}}$ has eigenvalues $\lambda_1^{-1}, \cdots, \lambda_n^{-1}$, since ρ_g is a root of unity. Since $\lambda_i \in \mathbb{C}$, we have that $\lambda_i^{-1} = \overline{\lambda_i}$, and so

$$\chi_\rho(g^{-1}) = \lambda_1^{-1} + \cdots + \lambda_n^{-1} = \overline{\lambda_1} + \cdots + \overline{\lambda_n} = \overline{\chi_\rho(g)}.$$

Exercise 12. *Show that ρ_g is diagonalisable.*

(c) Immediate.

(d) Immediate.

□

Definition 7. A function $f : G \to \mathbb{C}$ is a *class function* if

$$f(ghg^{-1}) = f(h),$$

for all $h, g \in G$. That is, f takes constant values on conjugacy classes of G.

It follows then that the space of class functions has dimension equal to the number of conjugacy classes.

Remark 4. Characters of representations of G are an example of class functions.

Chapter 2

Week Two

This week, we learned about ... various properties of characters. In particular, properties of characters with respect to direct sum representations, and tensor representations. In particular, characters characterise (no pun intended) irreducible representations of G up to isomorphism.

An important result that we use all the time is Schur's lemma, which says that any morphism of two irreducible representations of G are either the zero map, or isomorphic. If the two representations are isomorphic, then $\dim_{\mathbb{C}} \operatorname{Hom}_G(V_1, V_2) = 1$.

We also learned that irreducible characters have norm 1, and are orthogonal to one another. In fact, they form an orthonormal basis for the space of class functions of G. A corollary of this fact is that — up to isomorphism — the number of irreducible representations of G is equal to the number of conjugacy classes.

2.1. Lecture 1, 31/07/2023

2.1.1. More Properties of Characters

Let (V, ρ) and (V', ρ') be two representations of G. We use the notations χ_V and χ_ρ interchangeably. Recall then that $\chi_{\rho\oplus\rho'} = \chi_\rho + \chi_{\rho'}$. Additionally, we have the following property:

$$\chi_{V\otimes V'} = \chi_V \chi_{V'},$$

and also we have that:

$$\chi_{V^*}(g) = \chi_V(g^{-1}) = \overline{\chi_V(g)}.$$

Exercise 13.

(a) Check these properties! Also, think about when $V \cong V^*$ as G-representations.

(b) Show that

$$\operatorname{Hom}_{\mathbb{C}}(V, V') \cong V^* \otimes V',$$

as G-representations. It follows then that

$$\chi_{\operatorname{Hom}_{\mathbb{C}}(V,V')}(g) = \overline{\chi_V(g)} \cdot \chi_{V'}(g).$$

(c) Express $\chi_{\operatorname{Sym}^2(V)}$ and $\chi_{\Lambda^2(V)}$ in terms of χ_V. (Hint: observe first that $\chi_{\operatorname{Sym}^2(V)} + \chi_{\Lambda^2(V)} = (\chi_V)^2$).

Proof.

(a) Let $\{v_i\}_{i=1}^k$, and $\{v_j\}_{j=1}^{\ell}$ be an eigenbasis for V and V', respectively. Then, the eigenbasis corresponding to $V \oplus V'$ and $V \otimes V'$ would be $\{v_i + v_j\}$ and $\{v_i \otimes v_j\}$. The dimension of the eigenbases are $k + \ell$ and $k\ell$, respectively. It follows then that $\chi_{\rho\oplus\rho'} = \chi_\rho + \chi_{\rho'}$, and $\chi_{\rho\otimes\rho'} = \chi_\rho \chi_{\rho'}$.

(b) Define a map
$$\mathrm{Hom}_{\mathbb{C}}(V, V') \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) \otimes_{\mathbb{C}} V', \quad f \longmapsto \varphi \otimes f.$$

(c)

□

2.1.2. Schur's Lemma

Proposition 1 (Schur's Lemma)**.** *Let (V_1, ρ_1) and (V_2, ρ_2) be two irreducible representations of G. Let $f \in \mathrm{Hom}_G(V_1, V_2)$. Then,*

(a) $f = 0$ or f is an isomorphism of representations.

(b) if $V_1 = V_2$, and $\rho_1 = \rho_2$, then $f = \lambda\, \mathrm{id}_{V_1}$, for some $\lambda \in \mathbb{C}$. That is, $\dim_{\mathbb{C}} \mathrm{Hom}_G(V_1, V_2) = 1$.

Proof.

(a) Assume $f \neq 0$. Then, $\mathrm{Im}(f) \neq 0$, and it follows then that $\mathrm{Im}(f)$ is a subrepresentation of V_2. Since V_2 is irreducible, $\mathrm{Im}(f) = V_2$ necessarily, and it follows that f is surjective. Similarly, $\ker f$ is a subrepresentation of V_1, and by irreducibility, and the fact that $f \neq 0$, we conclude that $\ker f = 0$. This shows that f is an isomorphism.

(b) Suppose that $V = V_1 = V_2$, and $\rho = \rho_1 = \rho_2$. Let $f : V \to V$ be a homomorphism of G-representations. Then, f has an eigenvalue $\lambda \in \mathbb{C}$ (this is guaranteed since $\mathbb{C}$ is algebraically closed). Consider the morphism of irreducible G-representations:
$$f' := (f - \lambda\, \mathrm{id}_V) : V \to V,$$
and we observe that there exists an eigenvector $0 \neq v \in V$ such that $f(v) = \lambda v$. Then, $v \in \ker(f')$, and it follows then that $\ker f' \neq 0$, and thus $f' = 0$ by part (a). It follows then that $f = \lambda\, \mathrm{id}_V$, as claimed.

□

2.1.3. Orthogonality Property of Characters

Definition 8. Let $\phi, \psi : G \to \mathbb{C}$ be $\mathbb{C}$-valued functions on G. Define an inner product given by:
$$(\phi|\psi) := \frac{1}{|G|} \sum_{g\in G} \phi(g)\overline{\psi(g)}.$$

This product is linear in ϕ, and sesqui-linear in ψ, and
$$(\phi|\phi) > 0, \quad \phi \neq 0$$

Remark 5. We may also define a bilinear form:
$$\langle \phi, \psi \rangle := \frac{1}{|G|} \sum_{g\in G} \phi(g)\psi(g^{-1}).$$

Now, this form is linear in both ϕ and ψ. Indeed,

$$\langle \phi, \psi \rangle = \langle \psi, \phi \rangle,$$

which one may show by making the variable change $g \mapsto g^{-1}$, making this a symmetric bilinear form.

In the case that ϕ and ψ are characters of G,

$$(\phi|\psi) = \langle \phi, \psi \rangle.$$

Lemma 2. *Let (ρ, V) be a G-representation, and V^G be the subrepresentation of G-fixed points of V. Then,*

$$\frac{1}{|G|} \sum_{g \in G} \chi_\rho(g) = \dim_{\mathbb{C}} V^G.$$

Proof. Consider a map

$$\varphi : V \longrightarrow V^G, \quad v \longmapsto \frac{1}{|G|} \sum_{g \in G} \rho(g)v.$$

The result follows by showing that $\operatorname{tr}(\varphi) = \dim_{\mathbb{C}} V^G$. We wish to show that φ is a projection onto V^G. First, we show projectivity. Let $h \in G$. Then,

$$\rho_h(\varphi(v)) = \frac{1}{|G|} \sum_{g \in G} \rho(hg)(v) = \varphi(v),$$

for all $v \in V$. Let $v \in V^G$. Then,

$$\varphi(v) = \frac{1}{|G|} \sum_{g \in G} \rho(g)(v) = \frac{1}{|G|} \sum_{g \in G} v = v,$$

and it follows then that φ is a projection. It follows then that $\operatorname{tr}(\varphi) = \dim_{\mathbb{C}} V^G$ — that is,

$$\frac{1}{|G|} \sum_{g \in G} \operatorname{tr}(\rho(g)) = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g) = \dim V^G.$$

□

Remark 6. See Serre's book for a more elementary, linear algebraic proof of this result.

Theorem 2 (Orthogonality of Characters)**.**

(a) If χ is the character of an irreducible G-representation, then $\langle \chi, \chi \rangle = 1$.

(b) If $\chi = \chi_\rho$, and $\chi' = \chi_{\rho'}$ for two non-isomorphic irreducible G-representations ρ and ρ', then

$$\langle \chi, \chi' \rangle = 0.$$

Proof.

(a), (b) Let (ρ, V) and (ρ', V') be two irreducible, G-representations. Recall that $\chi_{\mathrm{Hom}_{\mathbb{C}}(V,V')} = \overline{\chi_V(g)}\chi_{V'}(g)$. Then,

$$\langle \chi_V, \chi_{V'} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_{\mathrm{Hom}_{\mathbb{C}}(V,V')}(g).$$

By Schur's lemma,

$$\mathrm{Hom}_G(V, V') = \begin{cases} 0, & V \ncong V', \\ \mathbb{C}, & V \cong V' \end{cases}.$$

Recall that

$$\mathrm{Hom}_G(V, V') \cong \mathrm{Hom}_{\mathbb{C}}(V, V')^G.$$

The result follows by applying Lemma 2 to $\mathrm{Hom}_{\mathbb{C}}(V, V')$.

□

2.2. Lecture 2, 03/08/2023

2.2.1. Orthogonality of Irreducible Characters: Another Proof

Recall from last time:

Theorem 3. *If χ and χ' are both irreducible representations of G given by ρ and ρ', then*

$$\langle \chi, \chi \rangle = 1, \quad \langle \chi, \chi' \rangle = 0.$$

Our last proof used the fact that $\mathrm{Hom}_{\mathbb{C}}(V, V')^G \cong \mathrm{Hom}_G(V, V')$. We do something different this time.

Proof. Let (ρ, V) and (ρ', V') be irreducible representations. Consider a function $f : V \to V'$ be a $\mathbb{C}$-linear map, and define a G-invariant function by:

$$f' := \frac{1}{|G|} \sum_{g \in G} \rho'_g \circ f \circ \rho_{g^{-1}}.$$

It follows by construction that $f' \in \mathrm{Hom}_G(V, V')$. Then, by Schur's lemma, $f' = 0$ if $V \ncong V'$, and $f' = \lambda \,\mathrm{id}_V$ if $V \cong V'$. In particular, the constant λ is given by:

$$\mathrm{tr}(f') = \lambda \dim V = \mathrm{tr}(f).$$

So, $f' = \mathrm{tr}(f) \cdot \mathrm{id}_V$ if $V \cong V'$. Assume that $\dim V = n$, and $\dim V' = m$, and write everything in matrix form. It follows then that

$$f' = (f'_{ij})_{m \times n}, \quad f = (f_{ij})_{m \times n}, \quad \rho'_g = (\rho'_{ij}(g))_{m \times m}, \quad \rho_g = (\rho_{ij}(g))_{n \times n}.$$

Then, by construction we have:

$$f'_{ij} = \frac{1}{|G|} \sum_{g \in G} \sum_{k, \ell} \rho'_{ik}(g) f_{k\ell} \rho_{\ell j}(g^{-1}) = \begin{cases} 0, & \text{if} \quad \rho \ncong \rho' \\ \frac{\mathrm{tr}(f)}{n} \delta_{ij} & \text{if} \quad \rho \cong \rho' \end{cases}.$$

We may re-write the second case to be

$$\frac{\sum_k f_{kk}}{n} \delta_{ij},$$

and since it is equal f'_{ij} when $V \cong V'$, the coefficients are equal, and thus we have that

$$\frac{1}{|} G \sum_{g \in G} \rho_{ik}(g) \rho_{\ell j}(g^{-1}) = 0,$$

when $k \neq \ell$. Conversely, when $k = \ell$, we have

$$\frac{1}{|G|}\sum_{g\in G}\rho_{ik}(g)\rho_{kg}(g^{-1}) = \frac{\delta_{ij}}{n}.$$

This implies then that

$$\begin{aligned}\langle \chi_\rho, \chi_\rho\rangle &= \frac{1}{|G|}\sum_{g\in G}\chi_\rho(g)\chi_\rho(g^{-1}) \\ &= \frac{1}{|G|}\sum_{g\in G}\sum_{i=1}^{n}\rho_{ii}(g)\sum_{j=1}^{n}\rho_{jj}(g^{-1}) \\ &= \frac{1}{|G|}\sum_{g\in G}\sum_{i=1}^{n}\rho_{ii}(g)\rho_{ii}(g^{-1}) \\ &= \frac{1}{|G|}\sum_{g\in G}\sum_{i=1}^{n}\frac{1}{n} \\ &= 1\end{aligned}$$

where the third equality follows from the aforementioned fact that the sum vanishes when $i \neq j$. Suppose now that $V \ncong V'$. Then,

$$\frac{1}{|G|}\sum_{g\in G}\rho'_{ik}(g)\rho_{\ell j}(g^{-1}) = 0,$$

for all i, k, ℓ, j, from which it follows that $\langle \chi_\rho, \chi_{\rho'}\rangle = 0$. □

2.2.2. Characters of G Characterises G-Representations (no pun intended)

Theorem 4. *Suppose V is a representation of G with character ϕ. Let*

$$V = W_1 \oplus \cdots \oplus W_s,$$

be a decomposition of V into irreducible representations. Let W be an arbitrary irreducible representation of G, with character χ. Then, the inner product $\langle \phi, \chi\rangle$ gives the amount of W_i's that are isomorphic to W.

Proof. By the decomposition of V, we have that

$$\phi = \sum_{i=1}^{s}\chi_{W_i}.$$

By linearity,

$$\langle \phi, \chi\rangle = \left\langle \sum_{i=1}^{s}\chi_{W_i}, \chi\right\rangle = \sum_{i=1}^{s}\langle \chi_{W_i}, \chi\rangle = \sum_{W_i\cong W}\langle \chi_{W_i}, \chi\rangle,$$

where the third equality follows by the orthogonality property of characters. □

Corollary 2. *$\#\{i : W_i \cong W\}$ does not depend on the decomposition of V.*

Corollary 3. *Two G-representations with the same character are isomorphic as G-representations.*

Proof. Let $\{V_i\}$ be a set of pairwise non-isomorphic irreducible representations of G, and let V be a representation with character ϕ. Then:

$$V \cong \bigoplus_i V_i^{\oplus \langle \phi, \chi_{V_i} \rangle}.$$

□

Corollary 4. *The number of non-isomorphic irreducible representations of G is equal to the number of irreducible characters of G, which is less than or equal to the number of conjugacy classes of G.*

Later, we will concern ourselves with finding all the irreducible representations of a given group G. But we now have an upper bound for the amount of them.

Theorem 5. *If ϕ is the character of a G-representation, then $\langle \phi, \phi \rangle \in \mathbb{Z}_{>0}$. Moreover, ϕ is irreducible if and only if $\langle \phi, \phi \rangle = 1$.*

Proof. If ϕ is irreducible, then we are done. Otherwise, every G-representation decomposes as a direct sum of irreducible representations — that is,

$$\phi = \sum_{i=1}^{k} m_i \chi_i,$$

where $\chi_i \neq \chi_j$ for $i \neq j$, and each χ_i is an irreducible character. Here, m_i are the multiplicities of V_i in the direct sum decomposition. It follows then that

$$\langle \chi, \chi \rangle = \left\langle \sum_{i=1}^{k} m_i \chi_i, \sum_{i=1}^{k} m_i \chi_i \right\rangle = \sum_{i=1}^{k} m_i^2 \langle \chi_i, \chi_i \rangle = \sum_{i=1}^{k} m_i^2,$$

where the second and third equality follows from the orthogonality of characters. At least one m_i is greater than 0, and thus $\langle \phi, \phi \rangle \in \mathbb{Z}_{>0}$. □

2.2.3. Characters of The Regular Representation

Recall that the regular representation R_G is given by the space:

$$R_G := \bigoplus_{g \in G} \mathbb{C} e_g,$$

and $\rho_h(e_g) = e_{hg}$. The character r_G is given by:

$$r_G(g) = \begin{cases} |G| & \text{if} \quad g = 1, \\ 0, & \text{if} \quad g \neq 1 \end{cases}.$$

This follows since the representation ρ takes the basis of R_G to itself. It follows then that the resulting matrix will have 0 on its diagonal entries.

Let χ be an irreducible character of G. Then,

$$\langle \chi, r_G \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) r_G(g^{-1}) = \chi(1) = \dim_{\mathbb{C}} V,$$

where V is the irreducible representation corresponding to χ.

Corollary 5. *The regular representation*

$$R_G \cong \bigoplus_{i=1}^{k} W_i^{\oplus \dim W_i},$$

where $W_1, \cdots, W_k$ are pairwise non-isomorphic irreducible representations of R_G.

Corollary 6. *Let us write $n_i := \dim W_i$, where W_i is as before, and $i = 1, \cdots, k$. Then,*

(a) $\sum_{i=1}^{k} n_i^2 = |G|$.

(b) $\sum_{i=1}^{k} n_i \chi_{W_i}(g) = 0$.

Proof. Use the fact that $r_G = \sum_i n_i \chi_i$. □

2.3. Lecture 3, 04/08/2023

2.3.1. Number of Irreducible Representations of G (up to isomorphism)

Let H be the space of class functions of G. We have seen that irreducible characters of G form a set of orthonormal vectors in H.

Exercise 14. *Check that distinct irreducible characters of G are linearly independent.*

Proof. Let G be finite group, and $V_1, \cdots, V_k$ distinct irreducible G-representations with characters $\chi_1, \cdots, \chi_k$. □

Proposition 2. *Let $f \in H$, and let (ρ, V) be a G-representation. Consider a map*

$$\rho_f := \sum_{g \in G} f(g)\rho_g : V \longrightarrow V.$$

If V is irreducible of degree n with character χ, $\rho_f = \lambda \operatorname{id}_V$, where

$$\lambda = \frac{|G|}{\dim_{\mathbb{C}} V}(f|\overline{\chi}),$$

where $\overline{\chi}(g) = \chi(g^{-1})$.

Proof. We first show that $\rho_f \in \operatorname{Hom}_G(V, V)$ — that is, for some $h \in G$, $\rho_h \circ \rho_f = \rho_f \circ \rho_h$. Let $h \in G$. Then,

$$\begin{aligned}
\rho_f \circ \rho_h(v) = \sum_{g \in G} f(g)\rho_g\rho_h(v) &= \sum_{g \in G} f(g)\rho_{gh} \\
&= \sum_{g \in G} f(gh^{-1})\rho_g(v) \\
&= \sum_{g \in G} f(h^{-1}gh^{-1})\rho_g(v) \\
&= \sum_{g \in G} f(h^{-1}g)\rho_g(v) \\
&= \rho_h \circ \rho_f(v).
\end{aligned}$$

Since V is irreducible, then this implies that that $\rho_f = \lambda \, \mathrm{id}_V$ by Schur's lemma, and further that

$$\lambda = \frac{\mathrm{tr}(\rho_f)}{\dim_{\mathbb{C}} V} = \frac{1}{\dim_{\mathbb{C}} V} \sum_{g \in G} f(g) \,\mathrm{tr}(\rho_g) = \frac{1}{\dim_{\mathbb{C}} V} \sum_{g \in G} f(g)\chi(g) = \frac{|G|}{n}(f|\overline{\chi})$$

□

Theorem 6. *The distinct irreducible characters of G, written as $\chi_1, \cdots, \chi_k$, form an orthonormal basis of H.*

Proof. We have shown that $\chi_1, \cdots, \chi_k$ are linearly independent and orthonormal, and we know that $\chi_1, \cdots, \chi_k$ are class functions. It remains to show that $\chi_1, \cdots, \chi_k$ span H. Let $f \in H$. Then, it suffices to show if $(f|\chi_i) = 0$, for all $i = 1, \cdots, k$, then $f = 0$.

Recall that

$$(f|\chi_i) = \frac{1}{|G|} \sum_{g \in G} f(g)\chi_i(g^{-1}).$$

Now, let (ρ, V) be any G-representation. Then, there is a direct sum decomposition of V into irreducibles: $V \cong W_1 \oplus \cdots \oplus W_m$, with $\rho_1, \cdots, \rho_m$ the representations corresponding to the components in the decomposition of V. Let $f \in H$ be such that $(f|\chi_i) = 0$ for all $i = 1, \cdots, k$. Define a map

$$\rho_f := \sum_{g \in G} \rho_g : V \to V,$$

It follows then that $\rho_f = (\rho_1)_f + \cdots + (\rho_m)_f$. Then, by Proposition 2, it follows then that $(\rho_i)_f = 0$ for each i, and so $\rho_f = 0$. It remains to show that this implies that $f = 0$.

Now, choose V to be the regular representation. Then, $V = \bigoplus_{g \in G} \mathbb{C}e_g$. It follows then that

$$\rho_f(e_1) = \sum_{g \in G} f(g)\rho_g(e_1) = \sum_{g \in G} f(g)e_g = 0,$$

and it follows then that $f = 0$, since $e_g \neq 0$ since it is a basis by construction. □

Corollary 7. *Let $f \in H$. Then,*

$$f = \sum_i (f|\chi_i)\chi_i.$$

Theorem 7. *The number of isomorphism classes of irreducible representations of G is equal to the number of conjugacy classes of G.*

Proof. There is an obvious basis of H given by characteristic functions of the form 1_{C_i}, where C_i denotes a conjugacy class of G. It follows then that $\dim_{\mathbb{C}} H$ is equal to the number of conjugacy classes of G, and by Theorem 6, it follows that the number of conjugacy classes is equal to the irreducible characters of G. Irreducible characters of G characterise irreducible G-representations up to isomorphism, and thus we have our result. □

Corollary 8. *Let $g \in G$ and $C(g)$ denote the conjugacy class of an element $g \in G$. Then,*

(a)

$$\sum_{i=1}^{k} \overline{\chi_i(g)}\chi_i(g) = \frac{|C(g)|}{|G|}.$$

(b) For h not in the conjugacy class of g, $\sum_{i=1}^{k} \overline{\chi_i(g)}\chi_i(h) = 0$.

Proof.

$$1_{C(g)}(h) = \begin{cases} 0, & \text{if} \quad h \notin C(g) \\ 1, & \text{if} \quad h \in C(g) \end{cases}.$$

Since $1_{C(g)}$ is a class function, we know that

$$1_{C(g)} = \sum_{i=1}^{k} (1_{C(g)}|\chi_i)\chi_i,$$

by Corollary 7. Computing the coefficients,

$$(1_{C(g)}|\chi_i) = \frac{1}{|G|} \sum_{h \in G} 1_{C(h)}\chi_i(h^{-1}) = \frac{1}{|G|} \sum_{h \in C(g)} \chi_i(h^{-1}) = \frac{1}{|G|} \sum_{h \in C(g)} \chi_i(g^{-1}) = \frac{|C(g)|}{|G|} \cdot \chi_i(g^{-1}),$$

where the third equality follows since h^{-1} is in the conjugacy class of g^{-1}, and the fact that χ_i is a class function. It follows then that

$$1_{C(g)} = \sum_{i=1}^{k} \frac{|C(g)|}{|G|} \cdot \chi_i(g^{-1})\chi_i.$$

Taking some $h \in G$,

$$1_{C(g)}(h) = \sum_{i=1}^{k} \frac{|C(g)|}{|G|} \chi_i(g^{-1})\chi_i(h) = \begin{cases} 1, & \text{if} \quad h \in C(g), \\ 0, & \text{if} \quad h \notin C(g) \end{cases}.$$

□

Chapter 3

Week Three

This week, we learned about ...

3.1. Lecture 1, 07/08/2023

3.1.1. Canonical Decomposition of a G-representation

Let $\chi_1, \cdots, \chi_k$ be the distinct irreducible characters of irreducible representations $W_1, \cdots, W_k$ of G, where k is the number of conjugacy classes of G. Let V be a representation of G, and

$$V = U_1 \oplus \cdots \oplus U_s,$$

a decomposition of V into irreducible subrepresentations. For $i = 1, \cdots, k$, let

$$V_i = \bigoplus_{j \text{ such that } U_j \cong_G W_i} U_j,$$

where $\cong_G$ denotes isomorphisms as G-representations. Then,

$$V \cong \bigoplus_{i=1}^{k} V_i.$$

Example 4. *If $W_1 = \mathbb{C}$ be the trivial representation, then, $V_1 = V^G$ since G acts trivially on V^G.*

Theorem 8.

(a) The decomposition $V = V_1 \oplus \cdots \oplus V_k$ does not depend on the decomposition $V \cong U_1 \oplus \cdots \oplus U_s$.

(b) The projection $p_i : V \to V_i$ is given by

$$p_i = \frac{n_i}{|G|} \sum_{g \in G} \overline{\chi_i(g)} \rho_g,$$

where $n_i = \dim_{\mathbb{C}} W_i$.

Remark 7. Recall from before, we used the fact that $V^G = \varphi(V)$, where

$$\varphi := \frac{1}{|G|} \sum_{g \in G} \rho_g,$$

to prove the orthogonality of the irreducible characters. The projection map seen in part (b) of the theorem is similar to this construction.

Proof. (b) $\implies$ (a), and thus it suffices to show (b). Let

$$q_i := \frac{n_i}{|G|} \sum_{g \in G} \overline{\chi_i(g)} \rho_g,$$

which defines a map $q_i : U_j \to U_j$. Since U_j is irreducible, it follows by Proposition 2 that

$$q_i = \frac{\dim_{\mathbb{C}} W_i}{|G|} (\overline{\chi_i} | \overline{\chi_{U_j}}) = \begin{cases} 1, & \text{if} \quad U_j \cong W_i, \\ 0, & \text{if} \quad U_j \ncong W_i \end{cases},$$

where the last equality follows by the orthogonality of characters. This tells us that $q_i|_{V_i} = 1$, and $q_i|_{V_j} = 0$, for $i \neq j$ — this follows by construction of V. It follows then that q_i is equal to the projection $p_i : V \to V_i$. □

The aforementioned decomposition

$$V \cong \bigoplus_{i=1}^{k} V_i,$$

is thus called the *canonical decomposition* of V. The V_i's appearing in the canonical decomposition are called the *isotypic components*.

Example 5. *Let $G = \{1, \sigma\} \cong \mathbb{Z}/2\mathbb{Z}$. There are only two irreducible representations, since there are only two conjugacy classes. Let $W^+ = \mathbb{C}$ be the trivial representation, with $\rho_\sigma^+ = 1$. The other representation is given by $W^- \cong \mathbb{C}$, where $\rho_\sigma^- = -1$.*

Given any $\mathbb{Z}/2\mathbb{Z}$-representation (ρ, V), the canonical decomposition is given by

$$V \cong V^+ \oplus V^-,$$

where V^+ collects all the trivial representations, and V^- collects the non-trivial ones. In particular,

$$V^+ = V^G = \{v \in V : \rho_\sigma v = v\},$$

and

$$V^- = \{v \in V : \rho_\sigma v = -v\}.$$

There are two projection maps, given by:

$$p^+ = \frac{1}{2}(\rho_1 + \rho_\sigma) : V \longrightarrow V^+,$$

$$p^- = \frac{1}{2}(\rho_1 - \rho_\sigma) : V \longrightarrow V^-.$$

It follows that V^+ and V^- both decompose into a direct sum of one-dimensional subspaces.

3.1.2. Explicit Decomposition of a Representation

Let (V, ρ) be a representation of G, and let $V = V_1 \oplus \cdots \oplus V_k$ be the canonical decomposition of V. We know that $V_i \cong W_i^{\oplus k_i}$, where $W_i \ncong W_j$ for $i \neq j$ is an irreducible G-representation. Fixing i, let $W := W_i$

be such that $\dim_{\mathbb{C}} W = n$, and consider a representation

$$\pi : G \longrightarrow \mathrm{GL}(W), \quad g \longmapsto (\Gamma_{\alpha\beta}(g))_{n\times n}.$$

Let $p_{\alpha\beta} : V \to V$, where

$$p_{\alpha\beta} := \frac{n}{|G|}\sum_{g\in G}\Gamma_{\beta\alpha}(g^{-1})\rho_g.$$

Proposition 3.

(a) $p_{\alpha\alpha}|_{V_j} = 0$ *for all* $j \neq i$*. Moreover,* $V_{i,\alpha} = \mathrm{Im}(p_{\alpha\alpha}) \subset V_i$*. Then,*

$$V_i = \bigoplus_{\alpha=1}^{n} V_{i,\alpha},$$

and $p_i = \sum_{\alpha=1}^{n} p_{\alpha\alpha}$.

(b) $p_{\alpha\beta}|_{V_j} = 0$ *for all* $j \neq i$*, and* $p_{\alpha\beta}|_{V_{i,\gamma}} = 0$ *for all* $\gamma \neq \beta$*. It follows then that there is a bijection*

$$p_{\alpha,\beta} : V_{i,\beta} \longrightarrow V_{i,\alpha}.$$

(c) *There is a chain of projections:*

$$V_{i,1} \xrightarrow{p_{12}} V_{i,2} \xrightarrow{p_{23}} V_{i,3} \xrightarrow{p_{34}} \cdots \xrightarrow{p_{n-1,n}} V_{i,n}.$$

Now, let $v_1 \in V$*,* $v_1 \neq 0$*, and let* $v_\alpha := p_{\alpha,1}(v_1) \in V_{i,\alpha}$*, and* $W(v_1) := \mathrm{Span}_{\mathbb{C}}(\{v_1, \cdots, v_n\})$*. Then,* $W(v_i) \subset V_i$ *is a subrepresentation, and* $\dim_{\mathbb{C}} W(v_1) = n$*, and* $W(v_1) \cong W_i$ *as G-representations.*

(d) *Let* $\{v_1^1, \cdots, v_1^m\}$ *be a basis of* $V_{i,1}$*, and* $m = \dim_{\mathbb{C}} V_{i,1}$*. Then,*

$$V_i = \bigoplus_{j=1}^{m} W(v_1^j),$$

gives an explicit decomposition of V_i*. (TODO: draw the tower picture that Ting drew in the lectures).*

Proof. Read Serre's book (or try it yourself!). Uses the relations we derived in the alternate proof of orthogonality of characeters. ☐

3.2. Lecture 2, 10/08/2023

3.2.1. Character Tables

Recall that given a group G, the number of irreducible characters are given by the amount of conjugacy classes up to isomorphism. A character table of G is a $k \times k$ table with conjugacy classes and irreducible characters of the form:

	C_1	C_2	$\cdots$	C_k
χ_1	$*$	$*$	$\cdots$	$*$
$\vdots$	$*$	$*$	$\cdots$	$*$
χ_k	$*$	$*$	$\cdots$	$*$

where C_i's are the conjugacy classes and χ_i's are the characters corresponding to that conjugacy class.

Example 6. *(TODO:) rewatch this example Let $G = \mathfrak{S}_3$. It has six elements given by:*

$$(1), \quad (12)(3), \quad (13)(2), \quad (1)(23), \quad (123), \quad (132),$$

and we observe that

$$C((12)) = \{(1), (12), (132), (13), (23)\},$$

and

$$C((123) = \{(1), (123), (132)\}.$$

$C(1)$ always defines a conjugacy class in any group. An example of a $\mathfrak{S}_3$-representation is the sign representation: given by

$$\mathrm{sgn} : G \longrightarrow \mathrm{GL}(\mathbb{C}^3) \xrightarrow{\det} \mathbb{C}^\times.$$

The mapping is given in the following way:

$$(12)(3) \longmapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longmapsto -1.$$

Similarly,

$$(123) \longmapsto \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \longmapsto 1.$$

Alternatively, we can define the sign *of an element $s \in \mathfrak{S}_3$ to be $\mathrm{sgn}(s) = (-1)^m$, where m is the number of adjacent transpositions appearing in its decomposition into generators (recall that symmetric groups are generated by its adjacent transpositions).*

It follows then that $\mathrm{sgn}((12))) = (-1)^1 = -1$, and $\mathrm{sgn}((123)) = \mathrm{sgn}((12)(13)) = (-1)^2 = 1$.

Suppose now that the last representation we are looking for has dimension n. Then, we know that

$$|G| = 1^2 + 1^2 + n^2 = 6,$$

and it follows that $n = 2$ — that is, $\chi_3(1) = 2$. Let us consider the regular representation — recall that the character is given by

$$r_{\mathfrak{S}_3} = \begin{cases} 6, & \text{if} \quad g = 1, \\ 0, & \text{otherwise} \end{cases}.$$

It follows then that $r_G = \chi_1 + \chi_2 + 2\chi_3$, and thus

$$\chi_3 = \frac{r_G - \chi_1 - \chi_2}{2}.$$

Let us verify that this is infact an irreducible character. By orthogonality, we know that $\langle \chi_3, \chi_3 \rangle = 1$ if it *is irreducible. So, computing directly:*

$$
\begin{aligned}
\langle \chi_3, \chi_3 \rangle &= \frac{1}{6} \sum_{s \in \mathfrak{S}_3} \chi_3(s)\overline{\chi_3(s)} \\
&= \frac{1}{6} \sum_{s \in \mathfrak{S}_3} \frac{r_G - \chi_1 - \chi_2}{2} \cdot \frac{\overline{r_G - \chi_1 - \chi_2}}{2} \\
&= \frac{(6-1-1)(6-1-1)}{24} + \frac{1}{6} \sum_{1 \neq s \in \mathfrak{S}_3} \frac{(\chi_1 + \chi_2)(\overline{\chi_1} + \overline{\chi_2})}{4} \\
&= \frac{2}{3} + \frac{1}{24} \sum_{1 \neq s \in \mathfrak{S}_3} \left(|\chi_1|^2 + \overline{\chi_1}\chi_2 + \chi_1\overline{\chi_2} + |\chi_2|^2\right) \\
&= \frac{2}{3} - \frac{1}{24} \left(|\chi_1|^2(1) + \overline{\chi_1}(1)\chi_2(1) + \chi_1(1)\overline{\chi_2}(1) + |\chi_2|^2(1)\right) + \frac{1}{24} \sum_{s \in \mathfrak{S}_3} (|\chi_1|^2 + \overline{\chi_1}\chi_2 + \chi_1\overline{\chi_2} + |\chi_2|^2) \\
&= \frac{2}{3} - \frac{1}{24}(1^2 + 1 \cdot 1 + 1 \cdot 1 + 1^2) + \frac{1}{24}(\langle \chi_1, \chi_1 \rangle + \langle \chi_2, \chi_1 \rangle + \langle \chi_1, \chi_2 \rangle + \langle \chi_2, \chi_2 \rangle) \\
&= \textit{not one!! } {>}{:}\,(
\end{aligned}
$$

(TODO: come back to this calculation and do it properly) We end up with the character table:

	$C((1)) = C_1$	$C((12)) = C_{\text{sgn}}$	$C((123)) = C_{\text{reg}}$
χ_1	1	1	1
$\chi_2 = \chi_{\text{sgn}}$	1	-1	1
$\chi_3 = \chi_{\text{std}}$	2	0	-1

Exercise 15. *Check orthogonality:* $(\chi_i|\chi_j) = \delta_{ij}$.

Proof. $\square$

In fact, it turns out that χ_3 *is a character of the* standard representation. *In particular, the representation space is given by*

$$\{(x_1, x_2, x_3) \in \mathbb{C}^3 : x_1 + x_2 + x_3 = 0\},$$

which we equip with a basis $v_1 = e_1 - e_2$*, and* $v_2 = e_2 - e_3$*. Then,* $\mathfrak{S}_3$ *acts by permuting the canonical basis vectors* e_1, e_2, e_3*. If* $s = (12)(3) \in \mathfrak{S}_3$*, then*

$$sv_1 = -v_1, \quad sv_2 = v_1 + v_2.$$

It follows then that

$$s \longmapsto \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix},$$

under this two dimensional representation. Similarly, for $c = (123)$*, we see that*

$$c \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Example 7. *Now, let* $G = \mathfrak{S}_4$*. Then, it has* 5 *conjugacy classes, given by*

$$1 = 1 = (1)(2)(3)(4), \quad t_1 = (12)(3)(4), \quad t_2 = (12)(34), \quad t_3 = (123)(4), \quad t_4 = (1234).$$

The sum of the lengths of each cycle gives a partition of 4*. In particular, these classes correspond to the* *partitions*

$$1+1+1+1, \quad 2+1+1, \quad 2+2, \quad 3+1, \quad 4.$$

Consider the standard representation given by $\mathfrak{S}_4$ *acting on*

$$\{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4 : x_1 + x_2 + x_3 + x_4 = 0\}.$$

We perform the same computation as before, and we find that

$$\rho_3 : \mathfrak{S}_4 \longrightarrow \mathrm{GL}(\mathbb{C}^3),$$

has trace given by

$$\mathrm{tr}(\rho_3(t_2)) = 1, \quad \mathrm{tr}(\rho_3(t_3)) = -1, \quad \mathrm{tr}(\rho_3(t_4)) = 0, \quad \mathrm{tr}(\rho_3(t_5)) = -1,$$

where the sizes of each conjugacy class is 6*,* 3*,* 8*, and* 6*, respectively. We verify irreducibility by checking that its inner product with itself is* 1*:*

$$\langle \chi_{\rho_3}, \chi_{\rho_3} \rangle = \frac{1}{24}\left(3^2 + 1^2 \cdot 6 + (-1)^2 \cdot 3 + (-1)^2 \cdot 6\right) = 1,$$

and it follows that this representation is irreducible.

There remain two representations, χ_4*, and* χ_5*, suppose they have dimensions* a *and* b*, respectively. We know that* $1^2 + 1^2 + 3^2 + a^2 + b^2 = 24$*, and thus the dimensions must satisfy* $a^2 + b^2 = 13$*. Let us consider the representation given by*

$$\chi_{\mathrm{sgn}} \otimes \chi_{\mathrm{std}},$$

which we can verify is an irreducible representation. This is given by χ_4*. The last representation* χ_5 *is given by the regular representation.*

	$C(1)$	$C(t_1)$	$C(t_2)$	$C(t_3)$	$C(t_4)$
χ_1	1	1	1	1	1
$\chi_2 = \chi_{\mathrm{sgn}}$	1	-1	1	1	-1
$\chi_{\mathrm{std}} = \chi_3$	3	1	-1	0	1
$\chi_{\mathrm{sgn}} \otimes \chi_{\mathrm{std}}$	3	-1	-1	0	1
χ_5	2	0	2	-1	0

3.3. Lecture 3, 11/08/2023

Recall from last time that we looked at the irreducible representations of $\mathfrak{S}_4$, whose irreducible characters are given by χ_{triv}, χ_{sgn}, χ_{std}, $\chi_{\mathrm{sgn}\otimes\mathrm{std}}$, and χ_5 corresponding to conjugacy classes (1), (12), (12)(34), (123), and (1234). We did not have time to determine what χ_5 was, and so we will do that today.

Let us consider the element $t_2 = (12)(34)$. We know that $\chi_5(t_2) = 2$. Since $t^2 = 1$, we know that $\rho(t_2)^2 = I_2$, the 2×2 identity matrix. From this, we can deduce that $\rho(t_2) = I_2$, since after diagonalisation, the trace must add up to 2, and has to have the property that $\rho(t_2)^2 = I_2$.

This then implies that ρ is the identity if $s \in \mathfrak{S}_4$ is a transposition. It follows thus that ρ is trivial on the subgroup with elements

$$G_0 = \{1, (12)(34), (13)(24), (14)(23)\}.$$

That is, $\rho|_{G_0} = 1$. In particular, G_0 is a normal subgroup.

Exercise 16. *Check that G_0 is a normal subgroup of $\mathfrak{S}_4$.*

Proof. We recall the fact that $\mathfrak{S}_4$ is generated by adjacent transpositions. That is,

$$\mathfrak{S}_4 = \langle (12), (13), (14), (23), (24), (34) : \text{some braid relations} \rangle .$$

Observe that elements G_0 is made up of products of adjacent transpositions. It follows then that G_0 is invariant under conjugation by all elements of $\mathfrak{S}_4$. $\square$

Then, one may check that

$$\mathfrak{S}_4/G_0 \cong \mathfrak{S}_3.$$

Exercise 17. *Check that this isomorphism is true. (Hint: $\mathfrak{S}_4$ acts on $(12)(34)$, $(13)(24)$, $(14)(23)$ by conjugation.)*

It follows then that a $\mathfrak{S}_4$-representation W factors through $\mathfrak{S}_4/G_0 \cong \mathfrak{S}_3$:

$$\rho : \xrightarrow{\simeq} \mathfrak{S}_4/G_0 \cong \mathfrak{S}_3 \longrightarrow \mathrm{GL}_2(W),$$

and it follows then that the representation χ_5 should correspond to the standard representation of $\mathfrak{S}_3$ that we deduced from the $\mathfrak{S}_3$ example.

3.3.1. Representations of Abelian Groups

Since G is abelian, it follows that each conjugacy class consists of one element.

Theorem 9. *The following properties are equivalent:*

(a) G is abelian

(b) all irreducible G-representations have dimension one

Proof. If $n_i = \dim_{\mathbb{C}} V_i$, for distinct, non-isomorphic, irreducible G-representations for $i = 1, \cdots, k$, then we know that $\sum n_i^2 = |G|$. It follows then that $n_i = 1$ if and only if the number of conjugacy classes of G is equal to $|G|$. This can only happen if G is abelian. $\square$

Corollary 9. *Let A be an abelian subgroup of a group G. Then, each irreducible representation of G has degree $\leq \frac{|G|}{|A|}$.*

Proof. Let (ρ, V) be an irreducible G-representation. Let

$$\rho_A : A \hookrightarrow G \xrightarrow{\rho} \mathrm{GL}(V).$$

Then, (ρ_A, V) defines a representation of A. Let $W \subset V$ be an irreducible subrepresentation. It follows then that $\dim_{\mathbb{C}} W = 1$ by Theorem 9. Now, let

$$V' := \sum_{g \in G} \rho_g W,$$

which is a G-stable subspace of V. Since $V' \neq \{0\}$, it follows then that $V' = V$. For any $h \in G$, and $a \in A$, we know that $\rho_{ha} W = \rho_h \rho_a W = \rho_h \rho_a W$, and it follows then that V' is a sum of A-cosets

$$V' = \sum_{\text{left cosets } gA,\ g \in G} \rho_{gA} W,$$

and it follows that
$$\dim_{\mathbb{C}} V = \dim_{\mathbb{C}} V' \leq \frac{|G|}{|A|}.$$
□

Remark 8. Note that we do not require A to be a normal subgroup, so we are not taking quotients. As such, we are simply taking left cosets.

3.3.2. Representations of Products of Groups

Let G_1 and G_2 be two finite groups, from which we may form the product of two groups given by $G_1 \times G_2$ with group structure given by applying group operation component-wise. Given two representations (ρ^1, V_1), and (ρ^2, V_2) of G_1 and G_2, respectively. Then, we define a $(G_1 \times G_2)$-module in the following way:
$$\rho : G_1 \times G_2 \longrightarrow \mathrm{GL}(V_1 \otimes V_2), \quad (g_1, g_2) \longmapsto \left(v_1 \otimes v_2 \longmapsto \rho^1_{g_1} v_1 \otimes \rho^2_{g_2} v_2\right).$$
The characters are thus given by:
$$\chi_{\rho^1 \otimes \rho^2}(g_1, g_2) = \chi_{\rho^1}(g_1)\chi_{\rho^2}(g_2).$$

Exercise 18. *Check that this holds.*

Proof. For each $g = (g_1, g_2) \in G$, let $\{w^1_i\}_{i=1}^k$, and $\{w^2_j\}_{j=1}^\ell$ be an eigenbasis of $\rho^1_{g_1}$, and $\rho^2_{g_2}$, respectively. Then, the representation ρ_g will have an eigenbasis given by $\{w^1_i \otimes w^2_j\}_{i,j}$, which will have dimension $k\ell$. It follows then that for each $g = (g_1, g_2)$, we have that
$$\mathrm{tr}(\rho_g) = \mathrm{tr}(\rho^1_{g_1} \otimes \rho^2_{g_2}) = \mathrm{tr}(\rho^1_{g_1})\,\mathrm{tr}(\rho^2_{g_2}),$$
and it thus follows that
$$\chi_{\rho^1 \otimes \rho^2}(g_1, g_2) = \chi_{\rho^1}(g_1)\chi_{\rho^2}(g_2),$$
as claimed. □

Remark 9. When $G_1 = G_2 = G$, we defined the tensor product representation as a G-representation, *not* a $(G \times G)$-representation. However, we may view G as a subgroup of $G \times G$ via the diagonal map, and view the tensor G-representation as the representation $\rho^1 \otimes \rho^2$ restricted to G — that is, $\rho^1 \otimes \rho^2|_G$.

Theorem 10.

(a) If ρ^1 is an irreducible G_1-representation, and ρ^2 is an irreducible G_2-representation, then $\rho^1 \otimes \rho^2$ is an irreducible $(G_1 \times G_2)$-representation.

(b) Each irreducible representation of $G_1 \times G_2$ is isomorphic to a representation $\rho^1 \otimes \rho^2$, where ρ^i is an irreducible G_i-representation for $i = 1, 2$.

Proof.

(a) Using the orthonormality of characters, it is sufficient to show that $\chi_{\rho^1 \otimes \rho^2}$ has norm 1. Thus, computing directly:

$$\begin{aligned}
\langle \chi_{\rho^1\otimes\rho^2}, \chi_{\rho^1\otimes\rho^2}\rangle &= \frac{1}{|G_1||G_2|}\sum_{\substack{g_1\in G_1\\ g_2\in G_2}} \chi_{\rho^1\otimes\rho^2}(g_1,g_2)\overline{\chi_{\rho^1\otimes\rho^2}(g_1,g_2)}\\
&= \frac{1}{|G_1||G_2|}\sum_{\substack{g_1\in G_1\\ g_2\in G_2}} \chi_{\rho^1}(g_1)\chi_{\rho^2}(g_2)\overline{\chi_{\rho^1}(g_1)\chi_{\rho^2}(g_2)}\\
&= \left(\frac{1}{|G_1|}\sum_{g_1\in G_2}\chi_{\rho^1}(g_1)\overline{\chi_{\rho^1}(g_1)}\right)\left(\frac{1}{|G_2|}\sum_{g_2\in G_2}\chi_{\rho^2}(g_2)\overline{\chi_{\rho^2}(g_2)}\right)\\
&= \langle \chi_{\rho^1}, \chi_{\rho^1}\rangle \cdot \langle \chi_{\rho^2}, \chi_{\rho^2}\rangle\\
&= 1.
\end{aligned}$$

(b) Suppose that the irreps of G_1 have dimensions n_i, and irreps of G_2 have dimensions m_j, for some $1 \le i \le k$, and $1 \le j \le \ell$. It follows then that $\rho_i^1 \otimes \rho_j^2$ gives an irreducible representation of dimension $n_i m_j$. Observe that

$$\sum_{i,j}(n_i m_j)^2 = \left(\sum_{i=1}^{k} n_i\right)\cdot\left(\sum_{j=1}^{\ell} m_j\right) = |G_1|\cdot|G_2| = |G_1\times G_2|,$$

and it follows that we have a collection of irreducible $(G_1\times G_2)$-representations of the form $\rho_i^1\otimes\rho_j^2$. It remains to show of these irreducible representations are distinct.

Exercise 19. *Show that these $(G_1\times G_2)$-irreps are distinct (Hint: use orthonormality property).*

Proof. Let $\rho_1\otimes\rho_2$ be an irreducible $(G_1\times G_2)$-representation that is distinct from $\rho^1\otimes\rho^2$. Then, computing directly:

$$\begin{aligned}
\langle \chi_{\rho^1\otimes\rho^2}, \chi_{\rho_1\otimes\rho_2}\rangle &= \frac{1}{|G_1|\cdot|G_2|}\sum_{\substack{g_1\in G_1\\ g_2\in G_2}} \chi_{\rho^1\otimes\rho^1}(g_1,g_2)\overline{\chi_{\rho_1\otimes\rho_2}(g_1,g_2)}\\
&= \frac{1}{|G_1|\cdot|G_2|}\sum_{\substack{g_1\in G_1\\ g_2\in G_2}} \chi_{\rho^1}(g_1)\chi_{\rho^2}(g_2)\overline{\chi_{\rho_1}(g_1)\chi_{\rho_2}(g_2)}\\
&= \left(\frac{1}{|G_1|}\sum_{g_1\in G_1}\chi_{\rho^1}(g_1)\overline{\chi_{\rho_1})(g_1)}\right)\left(\frac{1}{|G_2|}\sum_{g_2\in G_2}\chi_{\rho_2}(g_2)\overline{\chi_{\rho_2}(g_2)}\right)\\
&= \langle \chi_{\rho^1}, \chi_{\rho_1}\rangle \cdot \langle \chi_{\rho^2}, \chi_{\rho_2}\rangle\\
&= 0\cdot 0\\
&= 0,
\end{aligned}$$

where the second-last equality follows by orthogonality of irreducible characters. $\square$

$\square$

Chapter 4

Week Four

This week, we learned . . .

4.1. Lecture 1, 14/08/2023

We give an alternative proof this result from last time:

Theorem 11. *Every irreducible representation of* $G_1 \times G_2$ *is of the form* $\rho^1 \otimes \rho^2$*, where* ρ^i *is an irreducible representation of* G_i *for* $i = 1, 2$.

Proof. It suffices to show that for any class function f on $G_1 \times G_2$, if f is orthogonal to all characters of the form $\chi_{\rho_1 \otimes \rho_2}(g_1, g_2) = \chi_{\rho_1}(g_1)\chi_{\rho_2}(g_2)$, then $f = 0$. Let us write $\chi_1 = \chi_{\rho_1}$, and $\chi_2 = \chi_{\rho_2}$. Suppose that $\langle f, \chi_1 \otimes \chi_2 \rangle = 0$. Then,

$$0 = \frac{1}{|G|} \sum_{\substack{g_1 \in G_1 \\ g_2 \in G_2}} f(g_1, g_2)\overline{\chi_1(g_1)\chi_2(g_2)}. \tag{4.1}$$

Let

$$\widetilde{f}(g_1) = \sum_{g_2 \in G_2} f(g_1, g_2)\overline{\chi_2(g_2)},$$

which defines a class function on G_1. (4.1) then implies that $\langle \widetilde{f}, \chi_1 \rangle = 0$ for all irreducible characters χ_1 of G_1. It follows then that $\widetilde{f}(g_1) = 0$ for all $g_1 \in G_1$. That is,

$$\sum_{g_2 \in G_2} f(g_1, g_2)\overline{\chi_2(g_2)} = 0,$$

for all $g_1 \in G_1$. This thus implies then that $f(g_1, -)$ is a class function on G_2. It follows then that

$$\langle f(g_1, -), \chi_2 \rangle = 0,$$

for all irreducible characters χ_2 of G_2. It follows thus that $f(g_1, -) = 0$ for all $g_1 \in G_1$, and it thus follows that $f = 0$. $\square$

Theorem 10 thus tells us that the representations of $G_1 \times G_2$ can be reduced to the representation theory of G_1 and G_2 separately.

4.1.1. Induced Representations

This gives us a way to produce a representation of a group G from a known representation of a subgroup H. Let us first recall that given a subgroup $H \leq G$, the *left cosets* are given for $g \in G$ by:

$$gH := \{gh : h \in H\}.$$

Two cosets gH and $g'H$ are the same if and only if $g^{-1}g' \in H$. The set of left cosets is denoted by G/H.

Remark 10. This is *not* a quotient of G by H. In order for that to happen, H must be a *normal subgroup* — that is, a subgroup invariant under conjugation by G.

For R a set of representatives on G/H, we obtain a partition of G into left cosets:

$$G = \bigsqcup_{s \in R} sH.$$

Further,

$$|G/H| = \frac{|G|}{|H|} := [G : H],$$

the *index* of H in G. Now, let (ρ, V) be a G-representation, and define the *restriction representation* by:

$$\mathrm{Res}_H^G(V) := \rho|_H : H \hookrightarrow G \longrightarrow \mathrm{GL}(V).$$

Let $W \subset V$ be a subrepresentation of H, and let

$$\theta : H \longrightarrow \mathrm{GL}(W),$$

be the corresponding representation. Let $\sigma \in G/H$. Then, by the H-stability of W, it follows then that $\rho_s W = \rho_{s'} W$ if $s, s' \in \sigma$. So, for some $s \in \sigma$, let us define

$$W_\sigma := \rho_s(W).$$

Remark 11. A priori, we do not know if the W_σ's have trivial intersection. So, we cannot take direct sums.

It folows then that

$$\sum_{\sigma \in G/H} W_\sigma \subset V,$$

is a subrepresentation. It follows that G acts on W_σ by permuting the cosets. That is,

$$gW_\sigma = \rho_g \rho_s W = \rho_{gs} W,$$

which maps it to another coset, say $g\sigma = \sigma' \in G/H$

Definition 9. We say that the representation (ρ, V) of G is *induced* by the representation (W, θ) if:

$$V = \bigoplus_{\sigma \in G/H} W_\sigma = \bigoplus_{r \in R} \rho_r W.$$

We write

$$\mathrm{Ind}_H^G W := V.$$

We see from this definition that

$$\dim \mathrm{Ind}_H^G W = [G : H] \dim W.$$

Example 8.

(a) *Let $H \leq G$ a subgroup, and let $V = R_G$ and $W = R_H$, the regular representations of G and H, respectively. Then,*

$$R_G = \operatorname{Ind}_H^G R_H.$$

Exercise 20. *Verify this.*

Proof. Let $R_G : G \to \mathrm{GL}(V)$ and $R_H : H \to \mathrm{GL}(W)$ be the regular representations corresponding to G and H, respectively. Then, by definition,

$$V = \bigoplus_{g \in G} \mathbb{C}e_g, \quad W = \bigoplus_{h \in H} \mathbb{C}e_h.$$

It follows by construction that W is a H-subrepresentation of V (where V is identified as a H-representation by the restriction Res_H^G). Choosing a set of representatives R in the coset G/H, we recall that there is a partition of G:

$$G = \bigsqcup_{s \in R} sH.$$

Then, we may re-write V as:

$$V = \bigoplus_{g \in G} \mathbb{C}e_g = \bigoplus_{s \in R} \bigoplus_{\sigma \in sH} \mathbb{C}e_\sigma.$$

Define

$$W_\sigma := \bigoplus_{\sigma \in sH} \mathbb{C}e_\sigma = \bigoplus_{\{sh : h \in H\}} \mathbb{C}e_{sh} = \bigoplus_{h \in H} \mathbb{C}e_{sh} = \bigoplus_{h \in H} R_G(s)(\mathbb{C}e_h) = R_G(s)(W).$$

It follows then that for each $\sigma \in sH$,

$$V = \bigoplus_{s \in R} W_\sigma,$$

and it follows by definition then that $\operatorname{Ind}_H^G W = V$, and so

$$R_G = \operatorname{Ind}_H^G R_H,$$

as claimed. □

(b) *Let*

$$V = \bigoplus_{\sigma \in G/H} \mathbb{C}e_\sigma,$$

which we equip with a G-representation by

$$\rho : G \longrightarrow \mathrm{GL}(V), \quad g \longmapsto (e_\sigma \longmapsto e_{g\sigma}).$$

This action is called the permutation representation of G on G/H. *Observe here that H acts trivially on e_1, for $\sigma = 1H$. It follows then that*

$$V = \operatorname{Ind}_H^G(\mathbb{C}e_1).$$

(c) *Given two induced representations $\rho_1 = \operatorname{Ind}_H^G \theta_1$, and $\rho_2 = \operatorname{Ind}_H^G \theta_2$,*

$$\rho_1 \oplus \rho_2 = \operatorname{Ind}_H^G(\theta_1 \oplus \theta_2).$$

(d) *Let* $(V, \rho) = \mathrm{Ind}_H^G(W, \theta)$. *If* $W_1 \subset W$ *is a subrepresentation of* H, *then*

$$V_1 = \sum_{r \in R} \rho_r W_1 \subset V,$$

is a subrepresentation of G. *Further,*

$$V_1 = \mathrm{Ind}_H^G W_1.$$

(e) *If* $\rho = \mathrm{Ind}_H^G \theta$, *and* ρ' *is a representation of* G, *then*

$$\rho \otimes \rho' = (\mathrm{Ind}_H^G \theta) \otimes \rho' = \mathrm{Ind}_H^G(\theta \otimes \mathrm{Res}_H^G \rho').$$

It follows then that

$$\mathrm{Ind}_H^G(\mathrm{Res}_H^G) = \rho' \otimes \mathrm{Ind}_H^G \mathbb{C}_{\mathrm{triv}},$$

where $\mathbb{C}_{\mathrm{triv}}$ *is the trivial representation, as seen in part (b) of this example.*

4.1.2. Existence and Uniqueness of Induced Representation

We wish to show that given any subgroup H of G, and a representation of H, there always exists a unique G-representation given by the induced representation.

Lemma 3. *Suppose that* $(\rho, V) = \mathrm{Ind}_H^G(\theta, W)$. *Let* $\rho' : G = \mathrm{GL}(V')$ *be a representation of* G, *and let* $f : W \to V'$ *be an element of* $\mathrm{Hom}_H(W, \mathrm{Res}_H^G V')$. *Then,* f *can be extended uniquely to a linear map* $F : V \to V'$ *such that* $F|_W = f$, *and* $F \in \mathrm{Hom}_G(V, V')$. *In particular, the diagram*

$$\begin{tikzcd} V = \mathrm{Ind}_H^G(W, \theta) \arrow[dr, "\exists! F \in \mathrm{Hom}_G(V, V')"] & \\ W \arrow[u, hook, "i"] \arrow[r, "f"] & V' \end{tikzcd}$$

commutes

Proof. We will prove this later! □

Theorem 12. *Let* (W, θ) *be a* H*-representation. Then, there exists a unique (up to isomorphism)* G*-representation* (V, ρ) *which is induced by* H. *That is,* $V = \mathrm{Ind}_H^G W$.

Proof. (Existence) By additivity of Ind_H^G, as seen in Example 8(c), we can assume that (W, θ) is irreducible without loss of generality since we can extend linearly. It follows then that (W, θ) is a subrepresentation of R_H. Since $R_G = \mathrm{Ind}_H^G R_H$, it follows from Example 8(d) that V can be induced. This proves existence.

(Uniqueness) Suppose (V, ρ) and (V', ρ') are both G-representations induced from (W, θ). Let us consider a map $W \hookrightarrow V'$. It follows from Lemma 3 that there exists a map $F : V \to V'$ such that $F \in \mathrm{Hom}_G(V, V')$, and $F|_W = \mathrm{id}_W$. We will finish the proof of this next time! □

4.2. Lecture 2, 17/08/2023

4.2.1. Uniqueness of Induced Representation

Last time, we proved that the induced representation exists. Today, we will prove existence. We also has Lemma 3, which we did not prove.

Proof of Lemma 3. a By definition,
$$V = \bigoplus_{s \in R} \rho_s(W).$$
Then, define
$$F(\rho_s(W)) = \rho'_s F(w) = \rho'_s f(w),$$
for all $w \in W$ by property of a G-homomorphism.

Exercise 21. *Check that $F \in \operatorname{Hom}_G(V, V')$.*

□

Theorem 13. *Let (W, θ) be a H-representation. Then, there exists a unique (up to isomorphism) G-representation (V, ρ) which is induced by H. That is, $V = \operatorname{Ind}_H^G W$.*

Proof of Uniqueness. Suppose that (ρ, V) and (ρ', V') are both G-representations induced by representation of H. By Lemma 3, there exists a unique map F:

$$\begin{tikzcd}
V \arrow[dr, dashed, "\exists! F"] & \\
W \arrow[u, hook, "i"] \arrow[r, hook, "i'"'] & V'
\end{tikzcd}$$

such that $F|_W = \mathrm{id}$. By definition, we know that
$$V' = \bigoplus_{g \in G/H} \rho'_g W,$$
and since $F \in \operatorname{Hom}_G(V, V')$,
$$F(\rho_g W) = \rho'_g F(w)) = \rho'_g w,$$
and it thus follows that $\operatorname{Im} F \supset \bigoplus_{g \in G/H} \rho'_g W = V'$, and it follows that $\operatorname{Im} F = V'$. Since $\dim V = \dim V'$, we have that F is an isomorphism. This shows uniqueness.

□

4.2.2. Characters of Induced Representations

Theorem 14. *Let $(V, \rho) = \operatorname{Ind}_H^G(W, \theta)$. Then, for all $g \in G$,*
$$\chi_\rho(g) = \frac{1}{|H|} \sum_{\substack{s \in G \\ s^{-1}gs \in H}} \chi_\theta(s^{-1}gs).$$

Proof. We recall that by definition,
$$V = \bigoplus_{r \in R} \rho_r(W),$$
where R is a set of representatives of G/H. Then, the element $g \in G$ maps $g : \rho_r(W) \mapsto \rho_{gr} W$, for $gr \in r'H$. That is, elements of G act by permuting the cosets. It follows then that
$$\chi_\rho(g) = \operatorname{tr}(\rho_g) = \sum_{r'=r} \operatorname{tr}\left(\rho_g|_{\rho_r W}\right).$$

Recall that $r = r'$ if and only if $r^{-1}gr \in H$. So,

$$\chi_\rho(g) = \sum_{\substack{r\in R \\ r^{-1}gr\in H}} \operatorname{tr}\left(\rho_g|_{\rho_r(W)}\right).$$

Consider the composition

$$\rho_r(W) \xrightarrow{\rho_r^{-1}} W \xrightarrow{\theta_{r^{-1}gr}} W \xrightarrow{\rho_r} \rho_r(W)$$

which evaluates to

$$\rho_r \circ \theta_{r^{-1}gr} \circ \rho_r^{-1} = \rho_g,$$

since θ is just the restriction of ρ to H. It follows then that

$$\operatorname{tr}(\rho_g|_{\rho_r(W)}) = \operatorname{tr}(\theta_{r^{-1}gr}) = \chi_\theta(r^{-1}gr).$$

Then,

$$\chi_\rho(g) = \sum_{\substack{r\in R \\ r^{-1}gr\in H}} \chi_\theta(r^{-1}gr).$$

Now, if $s \in rH$, and $r^{-1}gr \in H$, then

$$\chi_\theta(s^{-1}gs) = \chi_\theta\left((r^{-1}s(s^{-1}gs)s^{-1}r)\right) = \chi_\theta(r^{-1}gr),$$

where the last equality follows since χ is a class function. It follows then that this funtion is independent of our choice of representative. We may thus re-write the equation for $\chi_\rho(g)$ in the following way:

$$\chi_\rho(g) = \frac{1}{|H|} \sum_{\substack{s\in G \\ s^{-1}gs\in H}} \chi_\theta(s^{-1}gs).$$

□

4.2.3. Representations of the Dihedral Group

Let us consider the dihedral group D_n, which is the group of rotations and reflecitons of the plane that preserves a regular n-gon. The group has order $2n$. It is generated by a rotation element r that rotates the n-gon by $\frac{2\pi}{n}$, and a relfection element s of order 2. That is,

$$D_n = \left\langle r, s : r^n = 1,\, s^2 = 1,\, rs = sr^{-1} \right\rangle.$$

Conjugacy classes of D_n

From this, we see that r^ℓ is conjugate to $r^{n-\ell}$, for $1 \le \ell \le \frac{n}{2}$, and sr^k is conjugate to $sr^{k-2\ell}$. It follows then that

$$|C(r^\ell)| = \begin{cases} \frac{n}{2}+1 \text{ classes if } n \text{ even} \\ \frac{n+1}{2} \text{ classes if } n \text{ odd} \end{cases},$$

and

$$|C(sr^k)| = \begin{cases} 2 \text{ classes if } n \text{ even} \\ 1 \text{ class if } n \text{ odd} \end{cases}.$$

It follows then that the total number of conjugacy classes is given by

$$\begin{cases} \frac{n}{2}+3 \text{ classes if } n \text{ even} \\ \frac{n+3}{2} \text{ classes if } n \text{ odd} \end{cases}.$$

Irreducible Representations of D_n

Let $C_n = \langle r \rangle \cong \mathbb{Z}/n\mathbb{Z} \hookrightarrow D_n$ be the subgroup of index 2. Since C_n is abelian, it follows that all irreducible representations of D_n have degree ≤ 2.

Case One: n even We begin by finding all one-dimensional representations. We have:

$$\chi(r^n) = 1 = \chi(r)^n,$$

$$\chi(s^2) = 1 = \chi(s)^2 \implies \chi(s) = \pm 1,$$
$$\chi(rs) = \chi(sr^{-1}) \implies \chi(r)\chi(s) = \chi(s)\chi(r^{-1}).$$

Generally, given a one-dimensional representation $\rho : G \to \mathbb{C}^\times$, then $\rho(gh) = \rho(hg)$ implies that $\rho(ghg^{-1}h^{-1}) = 1$, which implies that $\rho|_{[G,G]} = 1$, where $[G,G]$ is the commutator (or derived) subgroup. From this, we obtain an abelian group representation

$$\rho : G/[G,G] \longrightarrow \mathbb{C}^\times.$$

This implies then that

$$\chi(r) = \chi(r^{-1}) \implies \chi(r) = \pm 1.$$

From this, we obtain 4 irreducible characters of dimension one.

	r^k	s	sr
χ_1	1	1	-1
χ_2	1	-1	-1
χ_3	$(-1)^k$	1	-1
χ_4	$(-1)^k$	-1	1

It follows then that there are

$$\frac{n}{2} + 3 - 4 = \frac{n}{2} - 1,$$

irreducible representations of degree 2. Now, let

$$\phi_m = \operatorname{Ind}_{C_n}^{D_n} \chi_m,$$

where

$$\chi_m : C_n \longrightarrow \mathbb{C}^\times, \quad r \longmapsto \zeta_n^m,$$

where

$$\zeta_n = \exp\left(\frac{2\pi i}{n}\right).$$

Computing directly, then

$$\phi_m(r^k) = \frac{1}{n} \sum_{\substack{g \in D_n \\ g^{-1} r^k g \in C_n}} \chi_m(g^{-1} r^k g) = \zeta_n^{km} + \zeta_n^{-km}.$$

This is because for $g = sr^\ell$, we have:

$$g^{-1}r^k g = r^{-\ell}sr^k sr^\ell = r^{-\ell}ssr^{-k}r^\ell = r^{-k},$$

but if $g = r^\ell$, then $g^{-1}r^k g = r^k$.

Exercise 22. *Show that $\phi_m(sr^k) = 0$.*

Proof. If $g = sr^\ell$, then

$$(r^{-\ell}s)sr^k sr^\ell = r^{-2\ell+k}s = sr^{2\ell-k},$$

and if $g = r^\ell$, then

$$r^{-\ell}sr^k r^\ell = sr^{2\ell+k}.$$

That is, $g^{-1}sr^k g$ is never in C_n, and thus $\phi_m(sr^k) = 0$. □

We will finish this next time.

4.3. Lecture 3, 18/08/2023

Recall from last time that we attempted to classify the irreducible representations of the dihedral group D_n. It has a presentation given by

$$D_n = \left\langle s, r : r^n = 1,\, s^2 = 1,\, rs = sr^{-1} \right\rangle.$$

If n is even, then

$$\phi_m = \operatorname{Ind}_{C_n}^{D^n} \chi_m,$$

where $C_n := \langle r \rangle \cong \mathbb{Z}/n\mathbb{Z}$, and $\chi_m : C_n \to \mathbb{C}^\times$ is the character defined by $r \mapsto \zeta_n^m$, for $0 \leq m \leq n-1$. On the generators,

$$\phi_m(r^k) = \zeta_n^{km} + \zeta_n^{-km},$$

$$\phi_m(sr^k) = 0,$$

since for any $g \in D_n$, $gsr^k g^{-1} \notin C_n$. Further, note that

$$\phi_m = \phi_{n-m}.$$

We wish to show that the ϕ_m's are irreducible representations of D_n. We do this using the inner product. Computing directly:

$$\begin{aligned}
\langle \phi_m, \phi_m \rangle &= \frac{1}{2n} \sum_{k=0}^{n-1} \phi_m(r^k)\phi_m(r^{-k}) \\
&= \frac{1}{2n} \sum_{k=0}^{n-1} \left(\zeta_n^{km} + \zeta_n^{-km}\right)^2 \\
&= \frac{1}{2n} \sum_{k=0}^{n-1} \left(\zeta_n^{2km} + 2 + \zeta_n^{-2km}\right) \\
&= \begin{cases} \sum_{k=0}^{n-1} \zeta_n^{2km} = 1 & \text{if} \quad m \neq 0, \frac{n}{2} \\ 2 & \text{if} \quad m = 0, \frac{n}{2} \end{cases}.
\end{aligned}$$

Exercise 23. *Check that the last equality in the above calculation is true.*

So, $\phi_1, \cdots, \phi_{\frac{n}{2}-1}$ are all irreducible, distinct two-dimensional representations of D_n.

Exercise 24. *Check that these characters are distinct.*

Proof. Computing directly,

$$\begin{aligned}\langle \phi_k, \phi_\ell \rangle &= \frac{1}{2n}\sum_{m=0}^{n-1} \phi_k(r^m)\phi_\ell(r^{-m}) \\ &= \frac{1}{2n}\sum_{m=0}^{n-1} (\zeta_n^{mk} + \zeta_n^{-mk})(\zeta_n^{m\ell} + \zeta_n^{-m\ell}) \\ &= \frac{1}{2n}\sum_{m=0}^{n-1} (\zeta_n^{m(k+\ell)} + \overline{\zeta_n^{m(k+\ell)}} + \zeta_n^{m(k-\ell)} + \overline{\zeta_n^{m(k-\ell)}}) \\ &= 0,\end{aligned}$$

and thus the representations are disctinct by orthogonality. □

So, now we have the following character table:

	r^k	s	sr
χ_1	1	1	-1
χ_2	1	-1	-1
χ_3	$(-1)^k$	1	-1
χ_4	$(-1)^k$	-1	1
ϕ_m $(1 \leq m \leq \frac{n}{2}-1)$	$\zeta_n^{km} + \zeta_n^{-km}$	0	0

In particular, the two-dimensional representation given by ϕ_m is the representation given by

$$\rho_m : r \longmapsto \begin{pmatrix} \zeta_n^m & \\ & \zeta_n^{-m} \end{pmatrix}, \quad s \longmapsto \begin{pmatrix} & \zeta_n^{-m} \\ \zeta_n^m & \end{pmatrix}.$$

Exercise 25. *Check that $\chi_{\rho_m} = \phi_m$ (in particular, check that the ρ_m's are irreducible). Revisit this calculation once more after we discuss more about induced representations.*

Proof. Indeed, by definition

$$\chi_{\rho_m}(r^k) = \operatorname{tr}\left(\rho(r)^k\right) = \zeta_n^{mk} + \zeta_n^{-mk} = \phi_m(r^k).$$

Further, we have:

$$\rho(sr^k) = \rho(s)\rho(r)^k = \begin{pmatrix} & \zeta_n^{-m} \\ \zeta_n^m & \end{pmatrix} \cdot \begin{pmatrix} \zeta_n^{mk} & \\ & \zeta_n^{-mk} \end{pmatrix} = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix},$$

and thus $\rho_m(sr^k) = 0 = \phi_m(sr^k)$. The irreducibility of ρ_m follows from the irreducibility of ϕ_m. □

Exercise 26. *Show that for n odd, there are two irreducible one-dimensional irreducible representations, and $\frac{n-1}{2}$ irreducible two-dimensional representations. In particular,*

	r^k	s
χ_1	1	1
χ_2	1	-1

Proof. We recall that $C(r^\ell)$ has $\frac{n+1}{2}$ classes if n is odd, and $C(sr^k)$ has 1 class for n odd. □

4.3.1. Representations of the Alternating Group A_4

Recall that the alternating group is a subgroup A_4 of $\mathfrak{S}_4$ of index 2. Specifically, it is the subgroup of even permutations. As such, it has order $|A_4| = 12$, and is generated by elements

$$x = (12)(34), \quad y = (13)(24), \quad z = (14)(23), \quad t = (123), \quad \cdots,$$

which gives us four conjugacy classes in A_4,

$$\{1\}, \quad \{x, y, z\}, \quad \{t, tx, ty, tz\}, \quad \{t^2, t^2x, t^2y, t^2z\}.$$

Recall from our classification of $\mathfrak{S}_4$ representations, that we had a normal subgroup given by

$$H = \{1, x, y, z\}.$$

Indeed, one verifies that

$$txt^{-1} = z, \quad tzt^{-1} = y, \quad tyt^{-1} = x.$$

Its quotient $\mathfrak{S}_4/H$ is isomorphic to $\mathfrak{S}_3$. Further, one has another subgroup given by

$$K = \{1, t, t^2\} \cong \mathbb{Z}/3\mathbb{Z}.$$

Since $K \cap H = \{1\}$, the above properties above thus imply that there is a well-defined semidirect product:

$$A_4 \cong K \rtimes H,$$

which means that every element in A_4 can be written uniquely as kh, where $k \in K$ and $h \in H$. Closure under the group operation follows from the following fact:

$$(k_1h_1)(k_2h_2) = k_1k_2 \underbrace{(k_2^{-1}h_1k_2)}_{\in H} h_2,$$

which means that the product of two elements k_1h_1 and k_2h_2, produces another element of the form kh. In this way, K *acts on* H by conjugation — this, together with the fact that their intersection is trivial, allows us to define this semidirect product,

Now, for some $k \in K$, and $h \in H$, define a character

$$\chi : K \longrightarrow \mathbb{C}^\times, \quad t \longmapsto \zeta_3^i,$$

and observe that

$$\chi_i(kh) = \chi_i(k),$$

which follows since χ_i is a class function. From before, we know that

$$\chi_i((k_1h_1)(k_2h_2)) = \chi_i(k_1h_1) \cdot \chi_i(k_2h_2),$$

which implies that

$$\chi_i(k_1k_2) = \chi_i(k_1) \cdot \chi_i(k_2).$$

From this, we obtain the following character table:

	1	x	t	t^2
χ_0	1	1	1	1
χ_1	1	1	ζ_3	ζ_3^2
χ_2	1	1	χ_3^2	χ_3

There remains one more representation — call it ψ — which we have to find. The formula $\sum n_i^2 = |A_4|$ (n_i's are dimensions of all irreps) tells us that this last representation has dimension 3.

Exercise 27. *Show that*

$$\psi = \operatorname{Ind}_H^{A_4} \theta,$$

where θ is the representation defined by

$$1 \longmapsto 1, \quad x \longmapsto 1, \quad y \longmapsto -1, \quad z \longmapsto -1.$$

Putting this all together, we have the following complete character table for A_4.

	1	x	t	t^2
χ_0	1	1	1	1
χ_1	1	1	ζ_3	ζ_3^2
χ_2	1	1	χ_3^2	χ_3
ψ	3	-1	0	0

Let us compare this with the character table for $\mathfrak{S}_4$:

	1	$t_1 = (12)(34)$	$t_2 = (123)$	$t_3 = (12)$	$t_4 = (1234)$
$\widetilde{\chi}_{\mathrm{triv}}$	1	1	1	1	1
$\widetilde{\chi}_1$	1	1	1	-1	-1
$\widetilde{\chi}_2$	2	2	-1	0	0
$\widetilde{\chi}_3$	3	-1	0	1	-1
$\widetilde{\chi}_4$	3	-1	0	-1	1

Observe here that

$$\widetilde{\chi}_3|_{A_4} = \widetilde{\chi}_4|_{A_4},$$

$$\widetilde{\chi_2}|_{A_4} = \chi_1 + \chi_2,$$

$$\chi|_{A_4} = \widetilde{\chi}_1|_{A_4} = \chi_0.$$

4.3.2. Brief Remark on Compact Groups

Definition 10. A *topological group* G is a group with a topology such that the multiplication $m : G \times G \to G$ and the inversion map $\iota : G \to G$ mapping $g \mapsto g^{-1}$ are both continuous maps. The group G is called *compact* if it is compact with respect to its topology.

In the case of compact groups, we replace the averaging maps by maps of the form

$$\int_G f(g)\,\mathrm{d}g,$$

where $\mathrm{d}g$ is the Haar measure on G, and f is a function on G. It satisfies the following properties:

1. (Right-invariance)

$$\int_G f(g)\,\mathrm{d}g = \int_G f(gh)\,\mathrm{d}g,$$

for all $h \in G$.

2.
$$\int_G \mathrm{d}g = 1,$$

Example 9. *$G = \mathbb{S}^1$, where group elements are given by $g = e^{i\theta}$, where $0 \leq \theta \leq \pi$, and the Haar measure is given by*
$$\mathrm{d}g = \frac{1}{2\pi}\,\mathrm{d}\theta.$$

Example 10 (Non-example)**.** *Let G be the group that preserves the form $(t, x, y, z) \in \mathbb{R}^4$ given by $x^2 + y^2 + z^2 - t^2$ — called the* Lorentz group*.*

Definition 11. A linear representation of a compact group G in a finite-dimensional $\mathbb{C}$-vector space is a group homomorphism
$$\rho : G \longrightarrow \mathrm{GL}(V),$$
which is continuous — that is, the map
$$G \times V \longrightarrow V, \quad (g, v) \longmapsto \rho_g(v),$$
is continuous.

In this case, we may define a scalar product of representations given by:
$$(\phi|\psi) = \int_G \phi(g)\overline{\psi(g)}\,\mathrm{d}g.$$

From this, one may similarly show that complete reducibility, orthogonality of characters, and irreducibility criterion hold.

Further, there exists a notion of a regular representation in this case too. As before, the regular representation ρ_s acts on functions $f \in L^2(G)$ by
$$\rho_g f(t) = f(g^{-1}t).$$

There is a theorem that says that this contains, in an appropriate sense, all the irreducible finite-dimensional representations of G. This is something called the Peter-Weyl theorem.

Example 11. *Let $G = \mathbb{S}^1$. Then, since G is abelian, all irreducible representations have dimension one, and are given by*
$$\chi_n(e^{i\theta}) = e^{in\theta}.$$

The orthogonality of each of these characters follows by a direct computation. The fact that these characters span the space of functions on G follows from the fact that periodic has a Fourier series decomposition.

Chapter 5

Week Five

This week, we learned . . .

5.1. Lecture 1, 21/08/2023

Let $\mathbf{G}$ be a split, linear reductive group scheme over $\mathbb{F}_q$. Since irreducible characters characterise irreducible $\mathbf{G}$-representations, it follows that there is a ring isomorphism

$$K(\mathbf{Rep}_{\mathbb{C}}(\mathbf{G})) \xrightarrow{\simeq} \mathrm{ClFun}(\mathbf{G}, \overline{\mathbb{Q}_\ell}).$$

We will now devote the remaining eight weeks to classifying all the finite groups of Lie type using Deligne-Lusztig theory. Our first step will be to develop the theory of character sheaves, and ℓ-adic cohomology blah blah blah blah blah

5.1.1. The Group Algebra

Just kidding. Let G be a finite group, and K be a commutative ring of characteristic 0. Then, the *group ring*

$$K[G],$$

is a ring where every element $f \in K[G]$ has the form

$$f = \sum_{g \in G} a_g g.$$

Addition is given by

$$\sum_{g \in G} a_g g + \sum_{g \in G} b_g g = \sum_{g \in G} (a_g + b_g) g,$$

and multiplication is given by

$$\left(\sum_{g \in G} a_g g \right) \left(\sum_{g \in G} b_h h \right) = \sum_{\substack{g \in G \\ h \in G}} a_g b_h g h.$$

Now, let k be a field, and V a k-vector space, and let

$$\rho : G \longrightarrow \mathrm{GL}(V),$$

be a linear representation of G. Then, V obtains the structure of a *left $k[G]$-module*. For all $f = \sum_{g\in G} a_g g$, f acts on an element $v \in V$ by:

$$f \cdot v = \sum_{g\in G} a_g \rho_g(v).$$

Conversely, given a left $k[G]$-module, there is a linear G-representation given by

$$\rho : G \longrightarrow \mathrm{GL}(V), \quad g \longmapsto (v \longmapsto g \cdot v).$$

It follows then that there is a one-to-one correspondence between linear representations of G over k, and left $k[G]$-modules.

Proposition 4 (Complete Reducibility of Modules)**.** *If k is a field of characteristic 0, then the algebra $k[G]$ is semisimple — that is, every $k[G]$-module is semisimple. This means that every $k[G]$-submodule admits a complementary submodule.*

Proof. To make our lives easier, we will do this in the case for $k = \mathbb{C}$. Let $\rho^i : G \to \mathrm{GL}(W_i)$ be pairwise non-isomorphic irreducible representations of G, for $i = 1, \cdots, m$. Extending, by linearity, we obtain a $\mathbb{C}[G]$-representation:

$$\widetilde{\rho^i} : \mathbb{C}[G] \longrightarrow \mathrm{End}(W_i).$$

Choosing a basis for W_i, we may identify $\mathrm{End}(W_i)$ with $\mathrm{Mat}_{n_i}(\mathbb{C})$, where $n_i = \dim W_i$. From this, one obtains a homomorphism of algebras

$$\widetilde{\rho} : \mathbb{C}[G] \longrightarrow \prod_{i=1}^{m} \mathrm{End}(W_i) = \prod_{i=1}^{m} \mathrm{Mat}_{n_i}(\mathbb{C}), \quad g \longmapsto (\widetilde{\rho^1}(g), \cdots, \widetilde{\rho^m}(g)) = (\rho^1(g), \cdots, \rho^m(g)).$$

That is,

$$\widetilde{\rho}\left(\sum_{g\in G} a_g g\right) := \sum_{g\in G} a_g \widetilde{\rho}(g).$$

Exercise 28. *Check that this defines a $\mathbb{C}$-algebra homomorphism.*

Proof. It suffices to check this for $a_g g \in \mathbb{C}[G]$, and $b_h h \in \mathbb{C}[G]$, and then extend by linearity. We have:

$$\begin{aligned}
\widetilde{\rho}((a_g g) \cdot (b_h h)) &= a_g b_h \widetilde{\rho}(gh) \\
&= a_g b_h \cdot (\rho^1(gh), \cdots, \rho^m(gh)) \\
&= a_g b_h \cdot (\rho^1(g)\rho^1(h), \cdots, \rho^m(g)\rho^m(h)) \\
&= a_g b_h \cdot (\rho^1(g), \cdots, \rho^m(g)) \cdot (\rho^1(h), \cdots, \rho^m(h)) \\
&= a_g b_h \widetilde{\rho}(g)\widetilde{\rho}(h) \\
&= \widetilde{\rho(a_g g)} \cdot \widetilde{\rho}(b_h h),
\end{aligned}$$

as claimed. ☐

We now wish to show that $\widetilde{\rho}$ is an algebra isomorphism, from which our result will follow. Let us first show that $\widetilde{\rho}$ is injective. Suppose otherwise — then, there exists some element $\sum_{g\in G} a_g g \in \mathbb{C}[G]$ such that $\sum_{g\in G} a_g \rho^i_g = 0$, for all $i = 1, \cdots, m$. Recall that the regular representation has a decomposition

$$R_G \cong \bigoplus_{i=1}^{m} W_i^{\oplus n_i},$$

which implies that $\sum_{g\in G} a_g R_G(g) = 0$. Applying this to the element e_1, we thus have that

$$\sum_{g\in G} a_g R_G(g) e_1 = \sum_{g\in G} a_g e_g = 0,$$

which thus implies that $e_g = 0$ for all $g \in G$, since e_g is a basis of the regular representation by construction.

Surjectivity follows since

$$\dim_{\mathbb{C}} \mathbb{C}[G] = |G| = \sum_{i=1}^{m} n_i^2 = \dim_{\mathbb{C}} \left(\mathrm{Mat}_{n_i}(\mathbb{C})\right).$$

It follows thus that $\widetilde{\rho}$ is an isomorphism of algebras. □

Alternative Proof of Surjectivity for $\widetilde{\rho}$. We may also show directly that $\widetilde{\rho}$ is surjective. Proceed by contradiction, and suppose otherwise. Then, $\mathrm{Im}\,\widetilde{\rho}$ is a strict subset of $\prod_{i=1}^{m} \mathrm{Mat}_{n_i}(\mathbb{C})$. That is, there exists a non-zero linear form f on $\prod_{i=1}^{m} \mathrm{Mat}_{n_i}(\mathbb{C})$ such that $f|_{\mathrm{Im}(\widetilde{\rho})} = 0$. This means that

$$\sum_{i_k, j_k} a^k_{i_k, j_k} (\rho^k_g)_{i_k, j_k} = 0,$$

for $1 \leq i_k, j_k \leq n_k$, for $1 \leq k \leq m$. The above notation basically just means that we are taking entries of elements of $\prod_{i=1}^{m} \mathrm{Mat}_{n_i}(\mathbb{C})$. This implies that

$$\sum_{i_k, j_k} a^k_{i_k, j_k} \rho^k_{i_k, j_k} = 0,$$

as a function $G \to \mathbb{C}$, where in particular

$$\rho^k_{i_k, j_k} : G \longrightarrow \mathbb{C}, \quad g \longmapsto (\rho^k_g)_{i_k, j_k}.$$

We now wish to show that

$$(\rho^k_{i_k, j_k}) : G \to \mathbb{C}, \quad 1 \leq k \leq m, \quad 1 \leq i_k, j_k \leq n_k,$$

form a basis of $\mathbb{C}$-valued functions on G, since this would imply then that $a^k_{i_k, j_k} = 0$. Recall the inner product given by

$$\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g\in G} \varphi(t)\psi(t^{-1}).$$

Then, by orthogonality of irreducible characters, it would then follow that

$$\left\langle \rho^a_{ik}, \rho^b_{j\ell} \right\rangle = \frac{\delta_{k\ell}\delta_{ij}}{n_a}.$$

The required result thus follows. □

5.2. Lecture 2, 24/08/2023

Proposition 5 (Fourier Inversion Formula)**.** *Let $(u_1, \cdots, u_m) \in \prod_{i=1}^{k} \mathrm{End}(W_i)$, and let*

$$u = \sum_{g\in G} u(g) g \in \mathbb{C}[G],$$

be such that $\widetilde{\rho}(u) = (u_1, \cdots, u_k)$, where $\widetilde{\rho}$ is the $\mathbb{C}$-algebra isomorphism

$$\widetilde{\rho} : \mathbb{C}[G] \longrightarrow \prod_{i=1}^{k} \mathrm{End}(W_i).$$

Then,

$$u(g) = \frac{1}{|G|} \sum_{i=1}^{m} n_i \operatorname{tr}_{W_i}\left(\rho^i(g^{-1})u_i\right).$$

Proof. By definition, we know that

$$\widetilde{\rho}(u) = \sum_{g \in G} u(g)\widetilde{\rho}(g) = \left(\sum_{g \in G} u(g)\rho^1(g), \cdots, \sum_{g \in G} u(g)\rho^k(g) \right) = (u_1, \cdots, u_k).$$

It follows then that

$$u_i = \sum_{g \in G} u(g)\rho^i(g).$$

Plugging this into the formula for the Fourier inversion formula, we see that

$$\begin{aligned}
&\frac{1}{|G|} \sum_{j=1}^{k} n_j \operatorname{tr}_{W_j}(\rho^j(g^{-1}u_j) \\
&= \frac{1}{|G|} \sum_{j=1}^{k} n_j \operatorname{tr}_{W_j}(\rho^j(g^{-1})) \sum_{h \in G} u(h)\rho^j(h) \\
&= \frac{1}{|G|} \sum_{j=1}^{k} n_j \sum_{h \in G} \operatorname{tr}_{W_j}(\rho^j(g^{-1}) \cdot \rho^j(h)) \\
&= \frac{1}{|G|} \sum_{h \in G} u(h) \sum_{j=1}^{k} n_j \operatorname{tr}_{W_j}(\rho^j(g^{-1}) \cdot \rho^j(h)) \\
&= \frac{1}{|G|} \sum_{h \in G} u(h) \sum_{j=1}^{k} \chi_j(1)\chi_j(g^{-1}h) \\
&= \frac{1}{|G|} \sum_{h \in G} u(h)\delta_{gh}|G| \\
&= u(g),
\end{aligned}$$

where the second-last equality follows by Corollary 8. □

5.2.1. The Center of $\mathbb{C}[G]$

By definition,

$$\mathrm{Cent}(\mathbb{C}[G]) := \{x \in \mathbb{C}[G] : xy = yx \text{ for all } y \in \mathbb{C}[G]\} = \{x \in \mathbb{C}[G] : xg = gx \text{ for all } g \in G\},$$

where the second equality follows by linearly extending. Let $C \subset G$ be a conjugacy class. Define $Z_C := \sum_{g\in C} g \in \mathbb{C}[G]$. Then, for any $h \in G$,

$$hZ_C h^{-1} = h\left(\sum_{g\in C} g\right)h^{-1} = \sum_{g\in C} hgh^{-1} = \sum_{h\in C} h = Z_C,$$

and so $Z_C \in \mathrm{Cent}(\mathbb{C}[G])$.

Exercise 29. *$\{Z_C : C$ a conjugacy class of $G\}$ is a basis for $\mathrm{Cent}(\mathbb{C}[G])$. It follows then that*

$$\dim_{\mathbb{C}}(\mathrm{Cent}(\mathbb{C}[G])) = \textit{no. of conjugacy classes of } G.$$

Proof. Let $C_1, \cdots, C_k$ be a complete list of conjugacy classes for G. Then, it is clear that the Z_{C_i}'s are linearly independent, since conjugacy classes give a partition of G. For some central element $x \in \mathbb{C}[G]$, we have that $x = g^{-1}xg$, for any $g \in \mathbb{C}[G]$. Thus, writing

$$x = \sum_{h\in G} a_h x = \sum_{h\in G} a_h g^{-1} x g,$$

which implies that $a_h = a_{g^{-1}hg}$ for all g — that is, a_h is constant on conjugacy classes. It follows then that we may write any central element as a linear combination of the form $\sum_{i=1}^k \lambda_i Z_{C_i}$, and it follows that the Z_{C_i}'s spans $\mathrm{Cent}(\mathbb{C}[G])$. □

Proposition 6. *The homomorphism $\widetilde{\rho}_i : \mathbb{C}[G] \to \mathrm{End}(W_i)$ gives rise to an algebra homomorphism*

$$\omega_i = \widetilde{\rho}_i|_{\mathrm{Cent}\,\mathbb{C}[G]} : \mathrm{Cent}\,\mathbb{C}[G] \longrightarrow \mathrm{Cent}(\mathrm{End}(W_i)) \cong \mathbb{C}.$$

In particular, $\mathrm{Cent}(\mathrm{End}(W_i))$ is isomorphic to the space of scalar matrices — that is, scalar multiples of id_{W_i}. Further, the family of ω_i's defines an isomorphism

$$(\omega_i)_{1\le i\le k} : \mathrm{Cent}\,\mathbb{C}[G] \longrightarrow \mathbb{C}^k \subset \prod_{i=1}^k \mathrm{End}(W_i).$$

Moreover, if $u = \sum_{g\in G} u(g)g \in \mathrm{Cent}\,\mathbb{C}[G]$, then

$$\omega_i(u) = \frac{1}{n_i}\sum_{g\in G} u(g)\chi_i(g)$$

Proof. Let $z \in \mathrm{Cent}\,\mathbb{C}[G]$. Then, from this we obtain a morphism of G-representations given by

$$\widetilde{\rho}_i(z) : W_i \longrightarrow W_i.$$

Since $z = \sum_{g\in G} a_g g$, we may extend linearly and write

$$\widetilde{\rho}_i(z) = \sum_{g\in G} a_g \rho_i(g).$$

We wish to check that

$$\widetilde{\rho}_i(z) = h(\rho_i(h)w) = \rho_i(h)\widetilde{\rho}_i(h)w,$$

for all $h \in H$. But since z is in the center, we thus have that

$$\widetilde{\rho}_i(zh) = \widetilde{\rho}_i(hz),$$

and by Schur's lemma (Proposition 1), we deduce that $\widetilde{\rho}_i(z) = \lambda_z \operatorname{id}_{W_i}$. If $u \in \operatorname{Cent} \mathbb{C}[G]$, then

$$\begin{aligned}\omega_i(u) &= \frac{1}{n_i} \operatorname{tr}_{W_i}(\widetilde{\rho}_i(u)) \\ &= \frac{1}{n_i} \operatorname{tr}_{W_i}\left(\sum_{g \in G} u(g)\rho_i(g)\right) = \frac{1}{n_i} \sum_{g \in G} u(g)\chi_i(g)\end{aligned}$$

□

Next, our goal is to show that the dimensions of irreducible representations divides the order of G. More generally, they divide $\frac{|G|}{|Z(G)|}$, where $Z(G)$ is the center of G. The proof of this will use the theory of algebraic integers, which we define below.

5.2.2. Algebraic Integers

Let R be a commutative ring of characteristic 0. Let $x \in R$. We say that x is *integral* over $\mathbb{Z}$ if there exists an integer $n \geq 1$ and $a_1, \cdots, a_n \in \mathbb{Z}$ such that x is the root of a monic polynomial with $\mathbb{Z}$-coefficients.

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0.$$

Definition 12. A complex number which is integral over $\mathbb{Z}$ is called an *algebraic integer.*

Example 12. *The roots of integers are algebraic integers.*

Exercise 30. *If $x \in \mathbb{Q}$, and x is an algebraic integers, then $x \in \mathbb{Z}$.*

Proof. Any polynomial $x^n + a_1 x^{n-1} + \cdots + a_n = 0$ over $\mathbb{Q}$ can be made into a polynomial over $\mathbb{Z}$ by multiplying out its denominators, thus making it a polynomial over $\mathbb{Z}$. □

5.3. Lecture 3, 25/08/2023

Proposition 7. *Let R be a commutative ring (of characteristic 0). Given some $x \in R$, the following statements are equivalent:*

(i) x is integral over $\mathbb{Z}$,

(ii) The subring $\mathbb{Z}[x]$ of R generated by x is finitely-generated as a $\mathbb{Z}$-module,

(iii) there exists a finitely generated $\mathbb{Z}$-submodule of R which contains $\mathbb{Z}[x]$.

Proof. (ii) $\iff$ (iii). $\mathbb{Z}$ is a Noetherian ring, since it is a principal ideal domain. It follows then that every submodule of a finitely generated $\mathbb{Z}$-module is finitely generated.

(i) $\implies$ (ii). If x is integral, then it always has the form $x^n + a_{n-1}x^{n-1} + a_0 = 0$ for some $a_i \in \mathbb{Z}$. And so $\mathbb{Z}[x] = \mathbb{Z}\langle 1, x, \cdots, x^{n-1}\rangle$.

(ii) $\implies$ (i) Suppose $\mathbb{Z}[x]$ is finitely generated. Let $R_n = \mathbb{Z}\langle 1, x, \cdots, x^{n-1}\rangle$. We thus have a chain of ideals

$$\cdots \subseteq R_n \subseteq R_{n+1} \subseteq \cdots,$$

for which there exists some k such that $R_k \cong \mathbb{Z}[x]$. □

Corollary 10. *If R is a finitely-generated $\mathbb{Z}$-module, then every element of R is integral over $\mathbb{Z}$.*

Corollary 11. *The elements of a commutative ring R of characteristic zero which are integral over $\mathbb{Z}$ form a subring of R.*

Proof. Let $x, y \in R$ integral elements over $\mathbb{Z}$. Then, it follows that $\mathbb{Z}[x] \otimes \mathbb{Z}[y]$ is finitely generated over $\mathbb{Z}$. It follows then that $\mathbb{Z}[x, y]$ is finitely generated over $\mathbb{Z}$. □

5.3.1. Integrality of Characters

Proposition 8 (Integrality of Characters)**.** *Let ρ be a representation of G, and χ_ρ its character. Then, $\chi_\rho(g)$ is an algebraic integer over $\mathbb{C}$ for any $g \in G$.*

Proof. By definition, $\chi_\rho(g) = \mathrm{tr}(\rho(g))$, which are the sum of eigenvalues of $\rho(g)$, which are roots of unity. Since roots of unity are algebraic integers over $\mathbb{C}$, the result follows. □

Proposition 9. *Let $u = \sum_{g\in G} u(s)s \in \mathrm{Cent}\,\mathbb{C}[G]$ such that the coefficients $u(s)$ are algebraic integers. Then, u is integral over $\mathbb{Z}$.*

Proof. Since $\mathrm{Cent}\,\mathbb{C}[G]$ is a commutative ring, it has the structure of a $\mathbb{Z}$-module. Let $e_i = \sum_{g\in C_i} g$, where C_i is a conjugacy classs of G, and $i = 1, \cdots, k$. These form a basis for $\mathrm{Cent}\,\mathbb{C}[G]$ as a $\mathbb{C}$-vector space. Thus, we may write u as

$$u = \sum_{i=1}^{k} u_i e_i,$$

where $u_i \in \mathbb{C}$ is an algebraic integer. By Corollary 10, it suffices to show that the e_i's are integral over $\mathbb{Z}$. Since $e_i \cdot e_j$ is an integral combination of the basis elements $\{e_j\}_{j=1}^k$. It follows then that

$$\mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_k,$$

is a subring of $\mathrm{Cent}\,\mathbb{C}[G]$, and is finitely generated over $\mathbb{Z}$. It follows thus by Corollary 10 that e_i is integral for each $i = 1, \cdots, k$.

□

Corollary 12. *Let (ρ, W) be an irreducible representation of G of dimension n, and character χ. If $u = \sum_{s\in G} u(s)s \in \mathrm{Cent}\,\mathbb{C}[G]$ such that $u(s)$ are algebraic integers, then*

$$\frac{1}{n}\sum_{s\in G} u(s)\chi(s),$$

is an algebraic integer.

Proof. Recall that there is an algebra isomorphism

$$\tilde{\rho} : \mathbb{C}[G] \longrightarrow \mathrm{End}(W),$$

which induces an algebra isomorphism (as in Proposition 6)

$$\omega : \mathrm{Cent}\,\mathbb{C}[G] \longrightarrow \mathbb{C}, \quad \sum_{g\in G} a_g g \longmapsto \frac{1}{n}\sum_{g\in G} a_g \chi(g),$$

where $\mathbb{C}$ is identified with the scalar matrix subring in $\mathrm{End}(W)$. Since ω is a homomorphism of algebras, the fact that $u = \sum_{s\in G} u(s)s$ is integral over $\mathbb{Z}$ (by Corollary 10), implies the fact that $\omega(u) = \frac{1}{n}\sum_{s\in G} u(s)\chi(s)$ is integral over $\mathbb{Z}$.

□

Corollary 13. *The dimension of irreducible representations of G divides $|G|$.*

Proof. We want $u = \sum_{s\in G} u(s)s \in \operatorname{Cent} \mathbb{C}[G]$ such that $u(s)$ are algebraic integers, and $\frac{1}{n}\sum_{s\in G} u(s)\chi(s) = \frac{|G|}{n}$. Let us try $u(s) = \chi(s^{-1})$, which is an algebraic integer. Then,

$$\frac{1}{n}\sum_{s\in G} \chi(s^{-1})\chi(s) = (\chi|\chi) = \frac{|G|}{n},$$

but also $(\chi|\chi) = 1$ by the irreducibility of χ. By Corollary 12, it follows that $\frac{|G|}{n}$, is an algebraic integer. Further, $\frac{|G|}{n} \in \mathbb{Q}$, which implies that $\frac{|G|}{n} \in \mathbb{Z}$.

$\square$

Exercise 31. *Let $a = \frac{\lambda_1+\cdots+\lambda_n}{n}$ and algebraic integer such that the λ_i's are integer. Then, $a = 0$, or $\lambda_i = \lambda_j$.*

Proof. $\square$

Chapter 6

Week Six

This week, we learned . . .

6.1. Lecture 1, 28/08/2023

Proposition 10. *Let $Z(G)$ be the centre of G. Then, the dimension of an irreducible representation of G divides $[G : Z(G)] = \frac{|G|}{|Z(G)|}$.*

Proof. Let $\rho : G \to \mathrm{GL}(W)$ be an irreducible representation of G of degree n. Let $s \in Z(G)$. Then, by Schur's Lemma $\rho(s) = \lambda(s)\,\mathrm{id}_W \in \mathbb{C}^\times$, which gives rise to a group homomorphism $\lambda : Z(G) \to \mathbb{C}^\times$, defined by $s \mapsto \lambda(z)$.

Let $m \geq 0$ be an integer, and consider the representation

$$\rho^m : G^m := \prod_{i=1}^{m} G \longrightarrow \mathrm{GL}(V^{\otimes m}),$$

which is irreducible by the irreduciblity of ρ. Consider the subgroup $Z(G)^m \leq G^m$, and $(s_1, \cdots, s_m) \in Z(G)^m$. Then,

$$\rho^m(s_1, \cdots, s_m) = \lambda(s_1) \cdots \lambda(s_m)\,\mathrm{id}_{W^{\otimes m}} = \lambda(s_1 \cdots s_m)\,\mathrm{id}_{W^{\otimes m}}.$$

Let $H \leq Z^m$ be a subgroup defined by

$$H = \{(s_1, \cdots, s_m) \in Z(G)^m : s_1 \cdots s_m = 1\},$$

so that ρ^m restricts to the identity on $W^{\otimes m}$ — that is, $\rho^m|_H = \mathrm{id}_{W^{\otimes m}}$. Equivalently, one says that H acts trivially on $W^{\otimes m}$. Further, H is normal in G^m since it is in the centre of G, and thus the quotient subgroup G^m/H is well-defined. It follows then that the representation ρ^m factors through the quotient G^m/H:

$$\rho^m : G^m \longrightarrow G^m/H \xrightarrow{\overline{\rho}} \mathrm{GL}(W^{\otimes m}).$$

This means that $W^{\otimes m}$ is irreducible as a G^m/H-representation, since by construction there exists no non-trivial stable under G^m/H, otherwise it would also be stable under G. This implies that $\dim_{\mathbb{C}} W^{\otimes m} = n^m$, which divides $\frac{|G^m|}{|H|} = \frac{|G|^m}{|Z(G)|^{-1}}$, and it follows thus that

$$\left(\frac{|G|}{|Z(G)| \cdot n}\right)^m \in \frac{1}{|Z(G)|}\mathbb{Z},$$

for all $m \geq 0$. Thus, $\mathbb{Z}\left[\frac{|G|}{|Z(G)|n}\right]$ is a finitely-generated $\mathbb{Z}$-module — specifically, it is generated by $\frac{1}{|Z(G)|}$. It follows then that $\frac{|G|}{|Z(G)|\cdot n}$ is integral over $\mathbb{Z}$, and since it is rational, it follows that $\frac{|G|}{|Z(G)|\cdot n} \in \mathbb{Z}$, and it follows therefore that n divides $\frac{|G|}{|Z(G)|}$, as claimed.

$\square$

6.1.1. Induced Representations, Re-visited

Let $H \leq G$ be a subgroup, and R a system of left coset representatives of the coset G/H. We now interchange between the language of G-representations and $\mathbb{C}[G]$-modules. Let V be a $\mathbb{C}[G]$-module, which also inherits the structure of a $\mathbb{C}[H]$-module by restriction. Let W be a $\mathbb{C}[H]$-submodule of V.

Recall that we said that V is *induced* by W if

$$V = \operatorname{Ind}_H^G W := \bigoplus_{s \in R} s \cdot W.$$

However, we want a more canonical construction of the induced module, which we will outline now.

Recall that given a k-algebra A, and V a right A-module, and W a left A-module, the *tensor product over A* — denoted by $\otimes_A$ — is given by

$$V \otimes_A W := V \otimes W / \langle (v \cdot a) \otimes w - v \otimes (a \cdot w) : v \in V, w \in W, a \in A \rangle.$$

Now, consider $W' = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$. Then, W' is a $\mathbb{C}[G]$-module: with action given by

$$\mathbb{C}[G] \times W' \longrightarrow W', \quad (g, x \otimes w) \longmapsto (g \cdot x) \otimes w.$$

Exercise 32. *Check that this is well-defined.*

Proof. Let $g, g' \in \mathbb{C}[G]$ be such that $gx \otimes w = g'x \otimes w$. Then,

$$(gx \otimes w) \cdot (g^{-1}x \otimes w) = (g'x \otimes w)(g^{-1}x \otimes w),$$

gives $x \otimes w = g'g^{-1}x \otimes w$, which implies that $g'g^{-1} = 1$, and thus $g = g'$. $\square$

From this, we now have a map

$$\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W \longrightarrow V,$$

defined by $x \otimes w \mapsto x \cdot w$, and extending linearly.

Proposition 11. *V is induced by W if and only there is an isomorphism*

$$\varphi : \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W \xrightarrow{\simeq} V.$$

Proof. Observe that $\dim_{\mathbb{C}} \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W = \frac{|G|}{|H|}$. A basis of $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$ is given by elements of the form $r \otimes w_i$, where $r \in R$, and $\{w_i\}$ is a basis of W. Then, the map φ is defined by $r \otimes w \mapsto r \cdot w$.

Exercise 33. *Convince yourself that this proves the statement.*

Proof. The proof works because φ maps basis elements to basis elements, hence is an isomorphism. $\square$

$\square$

Exercise 34. *Show that*

$$\operatorname{Ind}_H^G W \cong \operatorname{Hom}_H(\mathbb{C}[G], W) = \{f : G \to W : f(gh) = h \cdot f(g), \text{ for all } h \in H,\ g \in G\}.$$

$\operatorname{Hom}_H(\mathbb{C}[G], W)$ *obtains the structure of a* $\mathbb{C}[G]$*-module by the action*

$$g' \cdot f(g) = f(gg'),$$

for all $g, g' \in G$.

Proof. □

Properties of Induction

(i) If $V = \operatorname{Ind}_H^G W$, and E is a $\mathbb{C}[G]$-module, then there is a canonical isomorphism

$$\operatorname{Hom}_H(W, \operatorname{Res}_H^G E) \cong \operatorname{Hom}_G(\operatorname{Ind}_H^G W, E).$$

That is, Res_H^G, and Ind_H^G are adjoint to each other as functors in the category of G-representations.

Exercise 35. *Produce an isomorphism*

$$\operatorname{Hom}_{\mathbb{C}[H]}(W, E) \cong \operatorname{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W, E),$$

which is equivalent to proving the above property.

Proof. Define a map

$$\Phi : \operatorname{Hom}_{\mathbb{C}[H]}(W, E) \longrightarrow \operatorname{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W, E),$$

by

$$f \longmapsto (g \otimes w \longmapsto g \cdot f(w)).$$

Then, we claim that the map in the other direction given by

$$\Phi^{-1} : \varphi \longmapsto (w \longmapsto \varphi(1 \otimes w)),$$

is a suitable choice of inverse for Φ. Indeed,

$$(\Phi \circ \Phi^{-1})(\varphi)(w) = \Phi(\varphi(1 \otimes w)) = 1 \cdot \varphi(w) = \varphi(w),$$

and

$$(\Phi^{-1} \circ \Phi)(f)(g \otimes w) = \Phi^{-1}(g \cdot f(w)) = g \cdot f(1 \otimes w) g \cdot f(w).$$

□

(ii) There is an isomorphism of $\mathbb{C}[G]$-modules:

$$V \otimes_{\mathbb{C}} \operatorname{Ind}_H^G W \cong \operatorname{Ind}_H^G(\operatorname{Res}_H^G V \otimes_{\mathbb{C}} W).$$

Exercise 36. *Show that there is a* $\mathbb{C}[G]$*-module isomorphism*

$$V \otimes_{\mathbb{C}} (\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W) \cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} (V \otimes W),$$

and deduce the above property from this.

Proof. It is clear by construction that $\mathbb{C}[G]$ is a $(\mathbb{C}[G], \mathbb{C}[H])$-bimodule. Let us equip W with a right H-action by:

$$w \cdot h = h^{-1} w.$$

Thus, W is a $\mathbb{C}[H]$-bimodule. Similarly, one makes V a $(\mathbb{C}[H], \mathbb{C}[G])$=bimodule. Then, by associativity of tensor products, we thus have an isomorphism:

$$V \otimes_{\mathbb{C}} (\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W) \cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} (V \otimes_{\mathbb{C}} W).$$

□

(iii) (Transitivity of Induction) Let $H \leq K \leq G$. Then, there is an isomorphism of G-representations:

$$\operatorname{Ind}_K^G(\operatorname{Ind}_H^G W) \cong \operatorname{Ind}_H^G W.$$

Exercise 37. *Show that*

$$\mathbb{C}[G] \otimes_{\mathbb{C}[K]} (\mathbb{C}[K] \otimes_{\mathbb{C}[H]} W) \cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W.$$

Proof. Since $\mathbb{C}[G]$ is a $(\mathbb{C}[G], \mathbb{C}[K])$-bimodule, we have:

$$\mathbb{C}[G] \otimes_{\mathbb{C}[K]} (\mathbb{C}[K] \otimes_{\mathbb{C}[H]} W) = (\mathbb{C}[G] \otimes_{\mathbb{C}[K]} \mathbb{C}[K]) \otimes_{\mathbb{C}[H]} W = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W.$$

□

6.1.2. Frobenius Reciprocity

Let $H \leq G$ be a subgroup, and let f be a class function on H. Let $f' : G \to \mathbb{C}$ be defined by

$$f'(g) = \frac{1}{|H|} \sum_{\substack{g \in G \\ s^{-1} g s \in H}} f(s^{-1} g s).$$

We say that f' is *induced* by f and we write

$$f' = \operatorname{Ind}_H^G(f).$$

Proposition 12.

(i) $\operatorname{Ind}_H^G f$ *is a class function,*

(ii) If $f = \chi_W$, *a character of a* H*-representation* W, *then* $\operatorname{Ind}_H^G f = \chi_{\operatorname{Ind}_H^G W}$.

Proof.

(i) Every class function is a linear combination of characters. So, (ii) $\implies$ (i).

(ii) Already proved.

□

Let V_1, V_2 be two $\mathbb{C}[G]$-modules. Set

$$\dim_{\mathbb{C}} \operatorname{Hom}_G(V_1, V_2) =: \langle V_1, V_2 \rangle_G.$$

Lemma 4.
$$\langle V_1, V_2\rangle_G = \langle \chi_{V_1}, \chi_{V_2}\rangle_G,$$
where
$$\langle \varphi, \psi\rangle_G := \frac{1}{|G|}\sum_{g\in G}\varphi(g^{-1})\psi(g).$$

Proof. Each $\mathbb{C}[G]$-module admits a decomposition into irreducible $\mathbb{C}[G]$-modules by:
$$V_1 \cong \bigoplus_{i=1}^{k} W_i^{\oplus a_i}, \quad V_2 = \bigoplus_{i=1}^{k} W_i^{\oplus b_i},$$
where W_i are pairwise non-isomorphic irreducible representations. Then, it follows by Schur's Lemma that
$$\langle V_1, V_2\rangle_G = \sum_{i=1}^{k} a_i b_i.$$
But we also have that
$$\langle \chi_{V_1}, \chi_{V_2}\rangle_G = \left\langle \sum_i a_i\chi_i, \sum_i b_i\chi_i \right\rangle = \sum_i a_i b_i,$$
and the result follows. ☐

Theorem 15 (Frobenius Reciprocity)**.** *If ψ is a class function on H, and φ is a class function on G, then*
$$\left\langle \psi, \operatorname{Res}_H^G \varphi\right\rangle_H = \left\langle \operatorname{Ind}_H^G \psi, \varphi\right\rangle.$$

Proof. As before. ☐

Corollary 14. *If W is an irrep of H, and E is an irrep of G, then the number of times that W occurs in $\operatorname{Res}_H^G E$ is equal to the number of times that occurs in $\operatorname{Ind}_H^G W$.*

6.2. Lecture 2, 31/08/2023

6.2.1. Restriction to Subgroups

Let $H \leq G$ and $K \leq G$ subgroups. If $\rho : H \to \mathrm{GL}(W)$ is a representation of G, and $V = \operatorname{Ind}_H^G W$, then how does one define $\operatorname{Res}_K^G V$ from this?

Consider double cosets $K\backslash G/H$, and choose a set of representatives such that $G = \sqcup_{s\in S} KsH$, where
$$KsH = \{ksh : k \in K, h \in H\}.$$
For $s \in S$, define a subgroup given by
$$Hs := sHs^{-1} \cap K \leq K.$$
From this, consider the representation given by
$$\rho^s : Hs \longrightarrow \mathrm{GL}(W), \quad x \longmapsto \rho(s^{-1}xs).$$
Let $W_s := (W, \rho^s)$ be the corresponding representation of Hs.

Proposition 13.

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G W = \bigoplus_{s\in S\cong K\backslash G/H} \operatorname{Ind}_{Hs}^K(W_s).$$

In particular,

$$\frac{|G|}{|H|}\dim_{\mathbb{C}} W = \sum_{s\in S}\frac{|K|}{|Hs|}\dim_{\mathbb{C}} W.$$

Proof. By definition,

$$V = \operatorname{Ind}_H^G W = \bigoplus_{r\in G/H} \rho_r(W).$$

Let $s \in S$, and let

$$V(s) := \sum_{x\in KsH} \rho_x(W),$$

which is a K-stable K-representation under the left action of K. We now wish show that there is an isomorphism of K-representations:

$$V(s) \cong \operatorname{Ind}_{Hs}^K W_s.$$

Let $x_1 \in KsH$, and $x_2 \in KsH$. Then, $\rho_{x_1}(W) = \rho_{x_2}(W)$ if and only if $x_2^{-1}x_1 \in H$. Suppose that $x_1 = k_1sh_1$, and $x_2 = k_2sh_2$. Then, $x_2^{-1}x_1 = h_2^{-1}s^{-1}k_2^{-1}k_1sh_1$, is an element of H if and only if $k_2^{-1}k_1 \in sHs^{-1}\cap K = Hs$. This implies that

$$V(s) = \bigoplus_{z\in K\backslash Hs} z(sW) \cong \operatorname{Ind}_{Hs}^K W_s,$$

where the isomorphism follows since sW is Hs-stable, and the definition of an induced representation. Since $V = \bigoplus_{s\in S} V(s)$, it now remains to show that $sW \cong Ws$ as a Hs-representation. Indeed, there is a map of Hs-representations

$$Ws \longrightarrow sW, \quad w \longmapsto sw = \rho^s(w).$$

Let $h_s \in sHs^{-1}\cap K$, so that $h_s = shs^{-1}$. Then, $h_s\cdot w = h\cdot w$, by construction of Ws. This gets mapped to $sh\cdot w = h_s s\cdot w$ under the above map, and it thus follows that the map is a homomorphism of Hs-modules. Bijectivity follows since $w \mapsto s^{-1}w$ is a suitable inverse of the map.

□

6.2.2. Mackey's Irreducibility Criterion

Consider $K = H$, and $Hs = sHs^{-1}\cap H$, where $s \in G$. Then, $Hs \le H$. Given a representation $\rho : H \to \mathrm{GL}(W)$, we have two representations of Hs, given by

$$\rho^s : Hs \longrightarrow \mathrm{GL}(W), \quad x \longmapsto \rho(s^{-1}xs),$$

and

$$\operatorname{Res}_{H_s}^H \rho : Hs \longrightarrow \mathrm{GL}(W), \quad x \longmapsto \rho(x).$$

Proposition 14 (Mackey's Irreducibility Criterion)**.** *The induced representation $V = \operatorname{Ind}_H^G W$ is irreducible if and only if the following conditions are satisfied:*

(i) W is an irreducible H-representation

(ii) for all $s \in G - H$ (this is the set-theoretic minus), ρ^s, and $\operatorname{Res}_{Hs}^H \rho$ are disjoint *as Hs-representations (two representations V_1, V_2 of G are* disjoint *if they have no irreducible components in common — i.e. $\langle V_1, V_2\rangle_G = 0$)*

Proof. V is irreducible if and only if $\langle V, V\rangle_G = 1$. By Frobenius reciprocity, and applying Proposition 13:

$$\begin{aligned}\left\langle \operatorname{Ind}_H^G W, V\right\rangle_G &= \left\langle W, \operatorname{Res}_H^G V\right\rangle_H \\ &= \left\langle W, \bigoplus_{s\in H\backslash G/H} \operatorname{Ind}_{H_s}^H \rho^s\right\rangle_H \\ &= \sum_{s\in H\backslash G/H} \left\langle W, \operatorname{Ind}_{Hs}^H \rho^s\right\rangle_H \\ &= \sum_{s\in H\backslash G/H} \left\langle \operatorname{Res}_{Hs}^H W, \rho^s\right\rangle_{Hs}.\end{aligned}$$

Let $s = 1$, then $HsH = H$. Then,

$$\left\langle \operatorname{Res}_{Hs}^H W, \rho^s\right\rangle_{Hs} = \langle W, W\rangle_H = \langle \rho, \rho\rangle_H \geq 1.$$

So, $\langle V, V\rangle_G = 1$ if and only if $\langle \rho, \rho\rangle_H = 1$ and all other $\left\langle \operatorname{Res}_{Hs}^H W, \rho^s\right\rangle_{Hs} = 0$ for all $s \neq 1$ in $H\backslash G/H$. □

Corollary 15. *Suppose that H is a normal subgroup of G. Then, $\operatorname{Ind}_H^G \rho$ is irreducible if and only if ρ is irreducible and $\rho \not\cong \rho^s$ for all $s \notin H$.*

6.3. Lecture 3, 01/09/2023

6.3.1. Revisiting Characters of $A_4 \leq \mathfrak{S}_4$

In light of what we learned last time, we want to re-visit the representations of the alternating group A_4 in $\mathfrak{S}_4$. Recall that the character table of $\mathfrak{S}_4$ is given by

	1	(12)	(12)(34)	(123)(4)	(1234)
χ_{triv}	1	1	1	1	1
χ_{sgn}	1	-1	1	1	-1
χ_{std}	3	1	-1	0	-1
$\chi_{\text{std}\otimes\text{sgn}}$	3	-1	-1	0	1
χ_5	2	0	2	-1	0

Recall that $A_4 \cong K \ltimes H$, where $K = \{1, t, t^2\}$, for $t = (123)$, and $H = \{1, x, y, z\}$. From this, we obtain the character table:

	1	$x = (12)(34)$	$t = (123)$	t^2
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	-1	0	0

where ω is the third root of unity. Note here in particular that $\rho_{\text{triv}}|_{A_4} \cong \rho_{\text{sgn}}|_{A_4}$, and $\rho_{\text{std}}|_{A_4} \cong \rho_{\text{std}\otimes\text{sgn}}|_{A_4}$, just by inspecting the respective character tables.

Question: Is the restriction $\operatorname{Res}_{A_4}^{\mathfrak{S}_4} \rho_{\text{std}}$ irreducible?

Let ρ be an irreducible representation of $\mathfrak{S}_4$. Then, consider $\mathrm{Res}^{\mathfrak{S}_4}_{A_4}\rho$. Computing directly,

$$\left\langle \mathrm{Res}^{\mathfrak{S}_4}_{A_4}\rho, \mathrm{Res}^{\mathfrak{S}_4}_{A_4}\rho \right\rangle_{A_4} = \frac{1}{|A_4|}\sum_{x\in A_4}\chi_\rho(x^{-1})\chi_\rho(x) = \frac{2}{|\mathfrak{S}_4|}\sum_{x\in A_4}\chi(x^{-1})\chi(x) \leq 2\langle \chi_\rho, \chi_\rho\rangle_{\mathfrak{S}_4} = 2,$$

where the second equality follows since A_4 is a subgroup of index 2 in $\mathfrak{S}_4$. From this, we conclude that either $\mathrm{Res}^{\mathfrak{S}_4}_{A_4}$ is irreducible, or it is the direct sum of two non-isomorphic irreducible representations (have to be non-isomorphic otherwise the inner product evaluates to 4)— that is, $\mathrm{Res}^{\mathfrak{S}_4}_{A_4}\rho \cong \rho_1 \oplus \rho_2$, where $\rho_1 \not\cong \rho_2$.

From the above calculation, we see that

$$\left\langle \chi_{\mathrm{Res}^{\mathfrak{S}_4}_{A_4}\rho}, \chi_{\mathrm{Res}^{\mathfrak{S}_4}_{A_4}\rho} \right\rangle_{A_4} = 2,$$

if and only if $\chi_\rho(x) = 0$ for all $x \notin A_4$. So, looking at the table, we see that $\mathrm{Res}^{\mathfrak{S}_4}_{A_4}\rho_{\mathrm{std}}$ is therefore irreducible. Further, $\mathrm{Res}^{\mathfrak{S}_4}_{A_4}\rho_5$ decomposes as the sum of two irreducible A_4-representations. Looking at the character table, we see that

$$\mathrm{Res}^{\mathfrak{S}_4}_{A_4}\rho_5 \cong \phi_1 \oplus \phi_2,$$

where ϕ_1 and ϕ_2 are the representations corresponding to χ_1, χ_2, respectively.

Question: How do we find 3-dimensional irreducible representations of A_4? One way is by the restriction functor $\mathrm{Res}^{\mathfrak{S}_4}_{A_4}\rho_{\mathrm{std}}$.

The second way is by inducing from H to A_4. $H = \{1, x, y, z\} = \{1, (12)(34), (13)(24), (14)(23)\}$. Since every element has at most order 2, we see that this is isomorphic to the Klein group — that is,

$$H \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

This is a subgroup of index 3 in A_4. It follows thus that H has 4 irreducible characters given by

	1	x	y	z
ψ_1	1	1	1	1
ψ_2	1	-1	1	-1
ψ_3	1	1	-1	-1
ψ_4	1	-1	-1	1

Let us consider the inner product and apply Frobenius reciprocity, we obtain:

$$\left\langle \mathrm{Ind}^{A_4}_H \psi_i, \chi_4 \right\rangle_{A_4} = \left\langle \psi_i, \mathrm{Res}^{A_4}_H \chi_4 \right\rangle_H = \begin{cases} 0, & \text{if } \ i = 1 \\ 1, & \text{if } \ i = 2, 3, 4 \end{cases}.$$

Exercise 38. *Verify this calculation.*

This shows us that

$$\mathrm{Ind}^{A_4}_H \psi_2 \cong \mathrm{Ind}^{A_4}_H \psi_3 \cong \mathrm{Ind}^{A_4}_H \psi_4 \cong \phi_4.$$

Exercise 39. *Calculate* $\mathrm{Ind}^{A_4}_H \psi_1$*, and compute its decomposition into irreducibles.*

Let us compute these induced representations explicitly — that is, explicitly compute the map $\rho : A_4 \to \mathrm{GL}_3(W)$.

Question: What is $\operatorname{Ind}_H^{A_4} \psi_2$?

By definition,

$$\operatorname{Ind}_H^{A_4} \psi_2 \cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} \mathbb{C}_{\psi_2}.$$

Let us choose a basis of W. Recall that there is an isomorphism $A_4 \cong K \rtimes H$ — that is, any element $x \in A_4$ can be written uniquely as $x = kh$, for some $k \in K$, and $h \in H$. Let e_1 be the basis for the one-dimensional space $\mathbb{C}_{\psi_2}$, and let us choose a basis given by:

$$\{1 \otimes e_1, t \otimes e_1, t^2 \otimes e_1\} =: \{v_1, v_2, v_3\}.$$

From this, we see that

$$tv_1 = v_2, \quad tv_2 = v_3, \quad tv_3 = v_1.$$

Under this map,

$$t \longmapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Similarly,

$$xv_1 = x \otimes e_1 = 1 \otimes xe_1 = -1 \otimes e_1 = -v_1,$$

where we can move x across the tensor product since $x \in \mathbb{C}[H]$.

$$xv_2 = xt \otimes e_1 = t(t^{-1}xt) \otimes e_1 = ty \otimes e_1 = t \otimes ye_1 = t \otimes e_1 = v_2.$$

$$xv_3 = xt^2 \otimes e_1 = t^2 z \otimes e_1 = t^2 \otimes ze_1 = -t^2 \otimes e_1 = -v_3.$$

It follows thus that

$$x \longmapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Exercise 40. *Complete this computation and check directly that* $\chi_W = \chi_4$.

Exercise 41. *Use Mackey's irreducibility criterion to deduce that* $\operatorname{Ind}_H^{A_4} \phi_2$ *is irreducible.*

6.3.2. More on Dimensions of Irreps of G

Proposition 15. *Let $H \leq G$ be a normal subgroup, and let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation, as before. Then,*

(a) Either $\rho|_H$ is isotypic — that is, a direct sum of isomorphic irreducible representations (i.e. the canonical decomposition only has one summand),

(b) or, there exists a proper subgroup K of G containing H, and there exists an irreducible representation σ of K such that $\rho = \operatorname{Ind}_K^G \sigma$.

Proof. Suppose that

$$\operatorname{Res}_H^G(\rho, V) = \bigoplus_i V_i,$$

where V_i are the isotypic components — i.e. $\operatorname{Hom}_H(V_i, V_j) = 0$ for all $i \neq j$. Then, for all $g \in G$, $g \cdot V_i$ is H-stable since $h(g \cdot V_i) = g(g^{-1}hgV_i) = gV_i$, by the normality of H. Further, we see that $gV_i \subseteq V_j$ for some j. Moreover, $V_i \subseteq g^{-1}V_i \subseteq V_k$ for some k. This implies that $i = k$ by property of isotypic components. It follows that g permutes the V_i's.

Fixing i_0, if $V_{i_0} = V$, then (a) is fine — that is, $\mathrm{Res}_H^G = V_{i_0}$. Now, fixing i_0, if $V_{i_0} = V$, then (a) is true. Otherwise, consider $K = \{g \in G : gV_{i_0} = V_{i_0}\} \supseteq H$. It follows then that $K \neq G$, because V is irreducible, and $V_{i_0} \neq V$. Now, it follows then that

$$V = \bigoplus_{g \in G/K} gV_{i_0} =: \mathrm{Ind}_K^G V_{i_0}.$$

This tells us then that V_{i_0} must be irreducible as a K-representation, otherwise V would not be irreducible. □

Chapter 7

Week Seven

7.1. Lecture 1, 04/09/2023

We prove a consequence of Proposition 15 from last time:

Corollary 16. *If A is an abelian and normal subgroup of G, then the dimension of each irreducible G-representation divides $[G : A] = \frac{|G|}{|A|}$.*

Proof. We proceed by inducting on $|G|$. Let (ρ, V) be an irreducible G-representation. Suppose we are in (a) of Proposition 15 — that is, $\mathrm{Res}_A^G \rho$ is isotypic — and so $\mathrm{Res}_A^G \rho cong \chi^{\oplus N}$, where χ is a 1-dimensional representation of A. This then implies that $\rho(A) \subset Z(\rho(G))$. Consider $\rho : G/A \to \rho(G)/\rho(A)$, which is a surjection. We thus have $\frac{|\rho(G)|}{|\rho(A)|}$. We think of V as an irreducible representation of $\rho(G)$. Since $\rho(A) \subset Z(\rho(G))$, we have shown before that $\dim_{\mathbb{C}} \rho = \dim V$, which divides $[\rho(G) : \rho(A)]$. It follows then that $\dim_{\mathbb{C}} \rho$ divides $[G : A]$. Suppose we are in case (b) of Proposition 15. Then, $\rho = \mathrm{Ind}_K^G \sigma$, and $A \subseteq K \subset G$, and σ is an irreducible K-representation. Then, inducting on the order of G, $\dim_{\mathbb{C}} \sigma$ didivdes $[K : A]$, and thus $\dim_{\mathbb{C}} \rho = [G : K] \cdot \dim_{\mathbb{C}} \sigma$, which then divides $[G : K][K : A] = [G : A]$. □

7.1.1. Semidirect Product by an Abelian Subgroup

Suppose that $G = H \ltimes A$, where A is an abelian normal subgroup of G. Using a method of Wigner and Mackey, one can construct irreps of G from certain subgroups of H. Recall that if A is abelian, then all of its irreducible representations are one-dimensional. All such represenions form a group, denoted by

$$X = \mathrm{Hom}_{\mathbb{C}}(A, \mathbb{C}^\times),$$

with the group structure given by

$$(\chi_1 \chi_2)(a) = \chi_1(a) \cdot \chi_2(a),$$

for some $a \in A$. Note that G acts on X — for $\chi \in X$, and $g \in G$,

$$g \cdot \chi(a) = \chi(g^{-1} a g).$$

Let $\{\chi_i\}_{i \in X/H}$ be a system of representatives for H-orbits on X. For each $i \in X/H$, let $H_i := \{h \in H : h \cdot \chi_i = \chi_i\}$ be the stabiliser subgroup. Define

$$G_i := H_i \ltimes A,$$

which is a subgroup of G. For each G_i, define a one-dimensional representation given by

$$\chi_i : G_i \to \mathbb{C}^\times,$$

given by $\chi_i(ha) = \chi_i(a)$, for $h \in H$, and $a \in A$. We now have to check that this is a group homomorphism. Indeed,

$$\begin{aligned}\chi_i(h_1a_1h_2a_2) &= \chi_i(h_1h_2h_2^{-1}a_1h_2a_2)\\ &= \chi_i(h_2^{-1}a_1h_2a_2)\\ &= \chi_i(h_2^{-1}a_1h_2)\chi_i(a_2)\\ &= h_2 \cdot \chi_i(a_1)\chi_i(a_2)\\ &= \chi_i(a_1)\chi_i(a_2) = \chi\\ &= \chi_i(h_1a_1)\chi_i(h_2a_2),\end{aligned}$$

using the fact that $h_2 \in H_i$, and $h_2 \cdot \chi_i = \chi_i$. Let (ρ, W) be an irreducible representation of H_i, and and consider the composition:

$$\widetilde{\rho} : G_i \longrightarrow G_i/A \cong H_i \xrightarrow{\rho} \mathrm{GL}(W),$$

and we have $\widetilde{\rho}(ha) = \rho(h)$, for each $h \in H$, $a \in A$. We now have an irreducible representation of G_i given by $\widetilde{\rho}$. Let

$$Q_{i,\rho} := \mathrm{Ind}_{G_i}^{G}(\chi_i \otimes \widetilde{\rho}).$$

7.2. Lecture 2, 07/09/2023

Proposition 16.

(i) $Q_{i,\rho}$ *is an irreducible* G*-representation,*

(ii) $Q_{i,\rho} \cong Q_{i',\rho'}$ *as* G*-representations if and only if* $i = i'$*, and* $\rho \cong \rho'$*, as* H_i*-representations,*

(iii) Every irreducible representation of G *is isomorphic to one of* $Q_{i,\rho}$*.*

Proof.

(i) Using Mackey's irreduciblity criterion. Let $s \notin G_i = H_i \ltimes A$. Let $K_s = G_i \cap sG_is^{-1}$. Consider the two representations, given by $\mathrm{Res}_{K_s}^{G_i}(\chi_i \otimes \widetilde{\rho})$, and $(\chi_i \otimes \widetilde{\rho})^s : K_s \to \mathrm{GL}(\widetilde{W})$, where the latter denotes a twisted action, given by $x \mapsto \chi_i(s^{-1}xs)\widetilde{\rho}(s^{-1}xs)$. We wish to show that these two representations contain no common factors — that is, they are disjoint representations. It is enough to consider their restriction to $A \subset K_s$. Then,

$$\mathrm{Res}_A^{K_s} \mathrm{Res}_{K_s}^{G_i}(\chi_i \otimes \widetilde{\rho}) = \chi_i^{\oplus \dim_{\mathbb{C}} W},$$

and

$$\mathrm{Res}_A^{K_s}(\chi_i \otimes \widetilde{\rho})^s = s \cdot \chi_i^{\dim_{\mathbb{C}} W}.$$

But $s \cdot \chi_i \neq \chi_i$, since $s \notin G_i$. Thus, they are disjoint.

(ii) Observe that

$$\mathrm{Res}_A^G Q_{i,\rho} = \mathrm{Res}_A^G \mathrm{Ind}_{G_i}^G(\chi_i \otimes \widetilde{\rho}) = \bigoplus_{s \in A\backslash G/G_i \cong G/G_i} s \cdot \chi_i^{\oplus \dim_{\mathbb{C}} \rho},$$

where the isomorphism $A\backslash G/G_i \cong G/G_i$ follows since $AgG_i = gAG_i = gG_i$. Note also that $G/G_i \cong H/H_i$ by the same argument.

Exercise 42. *Check the above identity.*

Proof. We have:

$$\operatorname{Res}_A^G \operatorname{Ind}_{G_i}^G(\chi_i \otimes \widetilde{\rho}) = \operatorname{Res}_A^G \bigoplus_{s \in G/G_i} s \cdot \chi^{\oplus \dim_{\mathbb{C}} \rho} = \bigoplus_{s \in A \backslash G/G_i} s \cdot \chi^{\oplus \dim_{\mathbb{C}} \rho},$$

as claimed. □

Observe that $A = sG_is^{-1} \cap A$, for each $s \in G$. It follows then that $\operatorname{Res}_A^G Q_{i,\rho}$ depends only on each H-orbit of χ_i. It follows then that $Q_{i,\rho}$ determines i. Now, we wish to show that $Q_{i,\rho}$ determines ρ. Let V be the representation space corresponding to $Q_{i,\rho}$, and define

$$V_i := \{v \in V : Q_{i,\rho}(a)v = \chi_i(a)v, \text{ for all } a \in A\}.$$

We claim that V_i is H_i-stable. Indeed,

$$Q_{i,\rho}(a)h_i \cdot v = Q_{i,\rho}(a)\theta_{i,\rho}(h_i)v = Q_{i,\rho}(h_i)\theta_{i,\rho}(h_i^{-1}ah_i)v = Q_{i,\rho}(h_i)\underbrace{\chi_i(h_i^{-1}ah_i)}_{=h_i\cdot\chi_i(a)=\chi_i(a)}v = \chi_i(a)h_iv.$$

It follows then that $V_i \cong \rho$ as H_i-representations, and it thus follows that $Q_{i,\rho}$ determines ρ — this is because $\chi_i \otimes \widetilde{\rho} \cong V_i$ as H_i-representations.

Exercise 43. *Prove this isomorphism.*

Proof. □

(iii) Let $\sigma : G \to \mathrm{GL}(V)$ be an irreducible G-representation. Suppose

$$\operatorname{Res}_A^G V = \bigoplus_{\chi \in X = \operatorname{Hom}(A, \mathbb{C}^\times)} V_\chi,$$

where

$$V_\chi = \{v \in V : av = \chi(a)v\},$$

is the canonical decomposition of $\operatorname{Res}_A^G V$. Then, there exists some $\chi \in X$ such that $V_\chi \neq \{0\}$. Any element $s \in G$ acts by permuting the isotypic components (recall this from our discussion about normal subgroups)— that is,

$$\sigma(s) : V_\chi \longrightarrow V_{s\cdot\chi}.$$

Suppose now that χ is in the H-orbit of χ_i. Then, $V_{\chi_i} \neq 0$, and V_{χ_i} is H_i-stable. Let W_i be an irreducible $\mathbb{C}[H_i]$-module of V_{χ_i}, and $\rho : H_i \to \mathrm{GL}(W_i)$ the corresponding irreducible representation. Then, as a representation of $G_i = H_i \ltimes A$,

$$W_i \cong \chi_i \otimes \widetilde{\rho}.$$

Then, computing, and using Frobenius reciprocity:

$$\left\langle \sigma, \underbrace{\operatorname{Ind}_{G_i}^G(\chi_i \otimes \widetilde{\rho})}_{=Q_{i,\rho}} \right\rangle_G = \left\langle \sigma, \operatorname{Ind}_{G_i}^G W_i \right\rangle_G = \left\langle \operatorname{Res}_{G_i}^G \sigma, W_i \right\rangle_{G_i} \neq 0,$$

which is non-zero, since W_i is a subspace of σ. It follows then that $\sigma \cong Q_{i,\rho}$, since both σ and $Q_{i,\rho}$ are irreducible.

□

Exercise 44. *Classify all irreducible representations of the group of signed permutations. This is the Weyl group of type B.*

7.2.1. Group Theory Review: p-Groups, Nilpotent, (Super)solvable

Definition 13. A group G is *solvable* if there exists a sequence

$$\{1\} \leq G_0 \leq G_1 \leq \cdots \leq G_n = G,$$

such that $G_{i-1} \trianglelefteq G_i$, and G_i/G_{i-1} is abelian. A group G is *supersolvable* of it is solvable, and moreover $G_i \trianglelefteq G$, and G_i/G_{i-1} is cyclic. A group G is *nilpotent* if it is solvable, and $G_i/G_{i-1} \subset Z(G/G_{i-1})$.

We note here that nilpotency is the strongest condition here. That is, any nilpotent group is supersolvable.

Example 13.

1. *A_4 is solvable, but not supersolvable,*

2. *The dihedral group D_n is supersolvable, and nilpotent if and only if $n = 2^m$ for some m,*

Exercise 45. *Prove the above statements.*

Definition 14. If p is a prime number, then a *p-group* is a group G whose order is a power of p.

There is the following theorem, which we will state without proof:

Theorem 16. *Every p-group is nilpotent.*

Lemma 5. *Let G be a p-group acting on a finite set X, and let X^G be the set of fixed points of the G-action. Then,*

$$|X| \equiv |X^G| \pmod{p}.$$

Proof. The set $X \setminus X^G$ is a union of non-trivial orbits of G — that is, $Z_G(x) \neq G$, for x in the orbit of G. Each of these orbits have size p^a, for some $a \geq 1$. It follows then that $|X - X^G| = |X| - |X^G| \equiv 0 \pmod{p}$. □

Proposition 17. *Let V be a non-zero vector space over a field k of characteristic p, for p a prime number. Assume that*

$$\rho : G \longrightarrow \mathrm{GL}(V),$$

is a linear representation of G, where G is a p-group. Then there exists, $0 \neq v \in V$ such that

$$\rho(g)v = v,$$

for all $g \in G$.

Proof. Let $0 \neq x \in V$, and let

$$X := \mathrm{Span}_{\mathbb{F}_p}(\rho(g)x : g \in G\}.$$

Then, G acts on X. It follows then that $|X| = p^\alpha$, for some $\alpha \geq 1$. By Lemma 5, it follows then that $|X^G| \equiv 0 \pmod{p}$, from which it follows then that there exists some $v \in X^G$, as claimed. □

7.3. Lecture 3, 08/09/2023

Recall from last time that if V if a vector space over $\mathbb{F}_p$, and G is a p-group, then there exists a fixed point of the representation $\rho : G \to \mathrm{GL}(V)$. We obtain the following corollary:

Corollary 17. *The only irreducible representation of a p-group G in characteristic p is the trivial reprensentation.*

Let p be a prime number, and G a group of order $p^n m$, with $\gcd(m, p) = 1$. A subgroup of order p^n is called a *Sylow p-subgroup.* That is, Sylow p-subgroups are the largest p-subgroups of a p-grop. From this, we have the following theorem, known as Sylow's theorem:

Theorem 17 (Sylow's Theorem)**.**

(a) There exists Sylow p-subgroups,

(b) The Sylow p-subgroups are conjugate by inner automorphisms of G — i.e. if P and Q are Sylow subgroups, then there exists $g \in G$ so that $gPg^{-1} = Q$,

(c) Each p-subgroup of G is contained in a Sylow p-subgroup.

Proof.

(a) Let $Z(G)$ be the center of G. Then, if $|Z| \equiv 0 \pmod p$, then $Z(G)$ contains a cyclic group of order p, denoted by $D \cong \mathbb{Z}/p\mathbb{Z}$. By induction on the order of G, G/D has a Sylow p-subgroup of order p^{n-1}. Consider the map $G \to G/D$. The pre-image of H in G is a Sylow p-subgroup of G.

We now consider the case when $|Z| \not\equiv \pmod p$. Consider the set $G \setminus Z$ (we are minusing the sets, not taking quotients). Then,

$$G \setminus Z = \coprod_{C_i \text{ conjugacy class of } G} C_i.$$

It follows then that $|G \setminus Z| \not\equiv \pmod 0$, and thus p does not divide $|C_i|$, for some i. Let $x \in C_i$. Recall that

$$|C_i| = \frac{|G|}{|\operatorname{Stab}_x|},$$

and thus $|\operatorname{Stab}_x| = p^n m'$, for some $m' < m$. So, inducting on the order of G, we see that $H = \operatorname{Stab}_x$ has a Sylow p-subgroup.

(b), (c) Let P be a Sylow p-subgroup, and let Q be a p-subgroup of G. Let Q act on the coset $X := G/P$. Recall from last time that

$$|X| = |X^Q| \pmod p,$$

by a Lemma from last time (TODO: add reference). Then, $|X^Q| \not\equiv 0 \pmod p$. It follows therefore that X^Q is non-empty, and there thus exists some $g \in G$, for which $qgp_1 = gp_2$, and thus $q = gp_2 p_1^{-1} g^{-1} \in gPg^{-1}$, for any $q \in Q$, and $p_1, p_2 \in P$. This shows that $Q \subset gPg^{-1}$. It follows thus that any p-subgroup is contained in a Sylow p-subgroup.

If Q is a Sylow p-subgroup, then $|Q| = |gPg^{-1}| = p^n$, and therefore $Q = gPg^{-1}$, as claimed.

∎

Lemma 6. *Let G be a non-abelian supersolvable group. Then, there exists a normal abelian subgroup of G which is not contained the centre of G.*

Proof. Let $H := G/Z(G)$, which is still supersolvable — that is, there is a descending series of strict inclusions

$$0 \subset H_1 \subset H_2 \subset \cdots \subset H,$$

where each H_1 is a cyclic normal subgroup of H. Let $G_1 := \pi^{-1}(H_1)$, where $\pi : G \to H$. Then, G_1 is as desired.

Exercise 46. *Show this.*

□

Theorem 18. *Let G be a supersolvable group. Then, every irreducible representation of G is induced by a one-dimensional representation of a subgroup of G.*

Proof. By induction on the order of G, let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation. We can assume that ρ is faithful — if not, we can mod out by the kernel and argue again by induction. If G is abelian, then we are done.

Thus, suppose that G is non-abelian. Then, by Lemma 6, there exists a normal abelian subgroup A of G that is not contained in $Z(G)$. Since ρ is faithful, $\rho(A)$ is not contained in $Z(\rho(G))$ — the centre of $\rho(G)$. This thus implies that $\rho|_A$ is not isotypic. As such, it follows that ρ is of the form $\rho = \mathrm{Ind}_H^G \sigma$, for some proper subgroup H of G, and σ an irreducible H-representation.

Inducting on the order of G, $\sigma = \mathrm{Ind}_K^H \mathbb{C}_\chi$, where K is a proper subgroup of H, and $\mathbb{C}_\chi$ is a one-dimensional representation defined by a character χ of K. □

We state some important results without proof:

Theorem 19 (Artin's Theorem)**.** *Each character of G is a $\mathbb{Q}$-linear combination of characters induced by characters of cyclic groups.*

Definition 15. A group H is *p-elementary* if $H = C \times P$, where C is cyclic of order prime up to p, and P is a p-group. A subgroup of G is *elementary* if it is p-elementary for at least one prime number p.

Definition 16. A character of G is *monomial* if it is induced from a character of some subgroup.

Theorem 20 (Brauer's Theorem)**.**

(i) Each character of G is a $\mathbb{Z}$-linear combination of characters of elementary subgroups.

(ii) Each character of G is a $\mathbb{Z}$-linear combination of monomial characters of.

7.3.1. Representations of $\mathfrak{S}_n$

We begin by recalling some facts about the symmetric group. In particular, the number of conjugacy classes are given by the number of partitions of n. The correspondence is given by the cycle types of elements of $\mathfrak{S}_n$.

Write $p(n)$ for the number of partitions of n. It has a generating function given by

$$\sum_{n=1}^{\infty} p(n)t^n = \frac{1}{\prod_{s=1}^{\infty}(1-t)^s} = (1+t+t^2+\cdots)(1+t^2+t^4+\cdots)(\cdots .$$

This function converges exactly in $|t| < 1$. Moreover,

$$p(n) \sim \left(\frac{1}{\alpha n}\right) e^{\beta\sqrt{n}},$$

where $\alpha = 4\sqrt{3}$, and $\beta = \pi\sqrt{2/3}$.

Chapter 8

Week Eight

8.1. Lecture 1, 11/09/2023

8.1.1. Young Diagrams

Recall from last time that conjugacy classes of $\mathfrak{S}_n$ are in bijection with partitions of a positive integer n. To a partition, one may associate a *Young diagram.* For instance, given a partition $\lambda = 3+2+1+1$ of 7, the Young diagram is given by The *conjugate partition* of λ — denoted λ' — is given by the Young diagram A *Young tableau* is a Young diagram with numbers $1, \cdots, n$, with each number appearing exactly once. As an example,

$$t_\lambda := \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} 6 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} 7 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array},$$

is a Young tableau corresponding to the partition $\lambda = 3+2+1+1$.

Fixing a tableau of shape λ, define

$$P := P_{t_\lambda} = \{g \in \mathfrak{S}_n : g \text{ preserves each row}\},$$

$$Q = Q_{t_\lambda} = \{g \in \mathfrak{S}_n : g \text{ preserves each column}\}.$$

For $\lambda = \lambda_1 + \cdots + \lambda_k$, for which $\lambda_1 \geq \cdots \geq \lambda_k \geq 0$, we see that

$$P \cong \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_k},$$

$$Q \cong \mathfrak{S}_{\lambda'_1} \times \cdots \times \mathfrak{S}_{\lambda'_k},$$

where λ'_i correspond to the conjugate partition of λ_i. These are called the *Young subgroups* of $\mathfrak{S}_n$.

Let us now set

$$a_\lambda = \sum_{g \in P} e_g \in \mathbb{C}[\mathfrak{S}_n], \quad b_\lambda := \sum_{g \in Q} \operatorname{sgn}(g) e_g \in \mathbb{C}[\mathfrak{S}_n].$$

Let us also set

$$c_\lambda := a_\lambda b_\lambda \in \mathbb{C}[\mathfrak{S}_n].$$

This element c_λ is called the *Young symmetriser*, and it turns out that it will give us all the irreducible representations of $\mathfrak{S}_n$.

Theorem 21.

(i) Some scalar multiple of c_λ is idempotent — that is, $c_\lambda^2 = n_\lambda c_\lambda$,

(ii) $\mathbb{C}[\mathfrak{S}_n]c_\lambda$ is an irreducible representation of $\mathfrak{S}_n$,

(iii) Every irreducible representation of $\mathfrak{S}_n$ can be obtained this way.

Let V be any finite-dimensional $\mathbb{C}$-vector space. Then, $\mathfrak{S}_n$ acts on $V^{\otimes n}$ by permuting the factors of $V^{\otimes n}$. It follows then that there is a map

$$\mathbb{C}[\mathfrak{S}_n] \longrightarrow \mathrm{End}(V^{\otimes n}).$$

It follows then that a_λ defines a map $a_\lambda : V^{\otimes n} \to V^{\otimes n}$. In particular,

$$\mathrm{Im}(a_\lambda) \cong \mathrm{Sym}^{\lambda_1} V \otimes \mathrm{Sym}^{\lambda_2} V \otimes \cdots \times \mathrm{Sym}^{\lambda_k} V \subset V^{\otimes n}, \tag{8.1}$$

$$\mathrm{Im}(b_\lambda) \cong \Lambda^{\lambda_1} V \otimes \cdots \otimes \Lambda^{\lambda_k} V. \tag{8.2}$$

To see why (8.1) and (8.2) hold, consider the example:

Example 14. *If we let $\lambda = n$, then we have a Young tableau given by*

$$\begin{array}{|c|c|c|}\hline 1 & \cdots & n \\ \hline\end{array}$$

Then, $a_\lambda = \sum_{g\in\mathfrak{S}_n} e_g$, $b_\lambda = e_1$ since Q is trivial. We can think of

$$\mathrm{Sym}^n V = \{T \in V^{\otimes n} : s(T) = T, \textit{ where } s \in \mathfrak{S}_n \textit{ is a transposition.}\}.$$

It follows then that $\mathrm{Im}\, a_\lambda = \mathrm{Sym}^n V$. Then, $\mathbb{C}[\mathfrak{S}_n]c_\lambda = \mathbb{C}c_\lambda$, and corresponds to the trivial $\mathbb{C}[\mathfrak{S}_n]$-module.

Let us consider the other extreme example:

Example 15. *$\lambda = (1 + \cdots + 1)$, with tableau given by:*

$$\begin{array}{|c|}\hline 1 \\ \hline \vdots \\ \hline 1 \\ \hline\end{array}.$$

Then, $a_\lambda = e_1$, and $b_\lambda = \sum_{g\in\mathfrak{S}_n} \mathrm{sgn}(g) e_g$. It follows then that

$$\mathrm{Im}(b_\lambda) = \Lambda^n V = \{T \in V^{\otimes n} : s(T) = -T, \textit{ where } s \in \mathfrak{S}_n \textit{ is a transposition.}\}.$$

Then, $\mathbb{C}[\mathfrak{S}_n]c_\lambda = \mathbb{C}c_\lambda$, corresponding to the sign representation. This is one-dimensional, hence irreducible.

Let us look at a non-trivial example:

Example 16. *$G = \mathfrak{S}_3$, and $\lambda = (2 + 1)$, with Young tableau*

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}.$$

Then, $a_\lambda = e_1 + e_{(12)}$, and $b_\lambda = e_1 - e_{(13)}$. It follows then that $c_\lambda = e_1 - e_{(13)} + e_{(12)} - e_{(132)}$.

Exercise 47. *Calculate this at home. The answer is* $\mathbb{C}[\mathfrak{S}_3]c_\lambda = \mathrm{Span}_{\mathbb{C}}\{c_\lambda, e_{(13)}c_\lambda\}$*, which is a two-dimensional representation corresponding to the standard representation.*

Example 17.

$$\rho_{\mathrm{triv}} = \begin{array}{|c|c|c|c|}\hline * & * & * & * \\ \hline\end{array},$$

$$\rho_{\mathrm{sgn}} = \begin{array}{|c|}\hline * \\ \hline * \\ \hline * \\ \hline * \\ \hline\end{array},$$

$$\rho_{\mathrm{std}} = \begin{array}{|c|c|c|}\hline * & * & * \\ \hline * \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline * & * \\ \hline * & * \\ \hline\end{array}, \quad \textit{(2-dimensional)},$$

$$\rho_{\mathrm{std}} \otimes \rho_{\mathrm{sgn}} = \begin{array}{|c|c|}\hline * & * \\ \hline * \\ \cline{1-1} * \\ \cline{1-1}\end{array}.$$

Exercise 48.

(i) Show that the standard representation

$$V = \{(x_1, \cdots, x_n) \in \mathbb{C}^n : x_1 + \cdots + x_n = 0\} \cong \mathbb{C}[\mathfrak{S}_n]c_\lambda,$$

where $\lambda = ((n-1)+1)$*.*

(ii) Show that $\Lambda^k V$ *is irreducible if and only if its corresponding Young tableau is of the form*

$$\begin{array}{|c|c|c|}\hline * & * & * \\ \hline * \\ \cline{1-1} \vdots \\ \cline{1-1} * \\ \cline{1-1}\end{array},$$

where the first column has $k+1$ *entries.*

Observe that $P \cap Q = \{1\}$. This means that every $g \in \mathfrak{S}_n$ can be written in at most one way — as pq, for $p \in P$, and $q \in Q$. This implies that

$$c_\lambda = \left(\sum_{g \in P} e_g\right)\left(\sum_{h \in Q} \mathrm{sgn}(h) e_{gh}\right) = \sum_{g \in PQ} (\pm) e_g,$$

where the sign of e_g in the last equality depends on the sign of q in the decomposition $g = pq$.

Lemma 7. *Let* T *be a tableau of shape* λ $g \in \mathfrak{S}_n$*, and let* $T' = gT$ *— that is, we replace* i *by* $g(i)$ *in* T*. Suppose there are no pair of distinct integers so that they appear in the same row of* T*, and in the same column of* T'*. Then, there exists* $p \in P$*,* $q \in Q$ *such that* $g = pq$*.*

Proof. First note that $Q' = Q_{gT} = gQ_Tg^{-1}$ — that is, the columns of gT are stabilised. We now show that there exists some $p_1 \in P$, $q_1' \in gQg^{-1}$ such that p_1T, and $q_1'T'$ have the same first row. Performing a column operation on $gT = T'$, we can move the n_i's to the first column of T' such that $q_1'T'$ has first row entries given by $n_1, \cdots, n_k$ in some order. Then, perform row operations on T so that p_1T and $q_1'T'$ have the same first row. Repeating this procedure on the rest of the tableau, we can find $p \in P$, and $q' \in Q'$ such that $pT = q'T'$. Since $pT = gqg^{-1}gT = gqT$, this implies that $p = gq$, and thus $g = pq^{-1}$.

□

8.2. Lecture 2, 14/09/2023

Lemma 8. *Let $a = \sum_{g \in P} e_g$, $b = \sum_{g \in Q} \operatorname{sgn}(g) e_g$, and $c = ab$. Then,*

(i) for all $p \in P$, $pa = a = ap$,

(ii) for all $q \in Q$, $(\operatorname{sgn}(q)q)b = b = b(\operatorname{sgn}(q)q)$,

(iii) for all $p \in P$, $q \in Q$, $p \cdot c \operatorname{sgn}(q) q = c$,

(iv) if $x \in \mathbb{C}[\mathfrak{S}_n]$ satisfies $px \operatorname{sgn}(q) q = x$, then for all $p \in P$, $q \in Q$, then x is a scalar multiple of c.

Proof. (i), (ii), and (iii) are clear. For (iv), let $x = \sum_{g \in \mathfrak{S}_n} m_g e_g$, where $m_g \in \mathbb{C}$. Then,

$$px \operatorname{sgn}(q) q = \sum m_g \operatorname{sgn}(q) e_{pgq} = \sum_{g \in \mathfrak{S}_n} m_g e_g,$$

for any $p \in P$, $q \in Q$. This is true if and only if $m_g \operatorname{sgn}(q) = m_{pgq}$, for any $p \in P$, and $q \in Q$. Recall from last time, we proved:

$$c = \sum_{\substack{p \in P \\ q \in Q}} \operatorname{sgn}(q) e_{pq}.$$

It thus suffices to show that $m_g = 0$ if $g \notin P \cdot Q$. Suppose $g \notin PQ$. Then, using Lemma 7, there exists two distinct integers $i \neq j$ such that i, j are in the same row of T and same column of $gT = T'$. Let $t = (ij)$ be the transposition — then, $t \in P_T$, and $t \in Q_{T'} = gQ_T g^{-1}$. That is, $t = gqg^{-1}$ for some $q \in Q$, and $t \in P$. This then implies that $g = tgq^{-1}$, and htus

$$m_g = m_{tgq^{-1}} = m_g \operatorname{sgn}(q^{-1}) = m_g \operatorname{sgn}(q) = m_g \operatorname{sgn}(t) = m_g \cdot (-1),$$

and thus $m_g = 0$.

□

8.2.1. Lexicographical Order on Partitions of n

Given $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$, and $\mu = (\mu_1 \geq \cdots \geq \mu_\ell)$, then $\lambda > \mu$ if $\lambda_1 > \mu_1$, or $\lambda_i = \mu_i$ for $1 \leq i \leq j$, and $\lambda_{j+1} > \mu_{j+1}$ for some j. This is a *lexicographical ordering on partitions of* n.

Lemma 9. *Let T is a tableau of shape λ, and T' a tableau of shape μ. Suppose that $\lambda > \mu$. Then, there exists $i \neq j$ such that i, j are in the same row of T, and the same column of T'.*

Proof. Assume otherwise. Suppose that the first row of T is $\{n_1, \cdots, n_s\}$, and that n_a's are in different columns of T'. Then, the n_a's are in different columsn of T', and it follows then that $\mu_1 \geq \lambda_1$. Since $\lambda > \mu$, this implies that $\mu_1 = \lambda_1$. Repeating this, we obtain $\lambda = \mu$, which is a contradiction. □

Lemma 10.

(i) If $\lambda > \mu$, then for all $x \in \mathbb{C}[\mathfrak{S}_n]$,

$$a_\lambda x b_\mu = 0.$$

In particular, if $\lambda > \mu$, then

$$c_\lambda \cdot c_\mu = 0.$$

(ii) For all $x \in \mathbb{C}[\mathfrak{S}_n]$,

$$c_\lambda x c_\lambda.$$

is a scalar multiple of c_λ. In particular, $c_\lambda^2 = n_\lambda c_\lambda$, for some $n_\lambda \in \mathbb{C}$.

Proof.

(i) It suffices to show that $a_\lambda e_g b_\mu = 0$, for all $g \in \mathfrak{S}_n$. Since $g b_{T_\mu} g^{-1} = b_{gT_\mu}$, and it thus suffices to show that $a_{T_\lambda} b_{T_\mu} = 0$, for any tableau T_λ of shape λ, and T_μ of shape μ.

Then, by Lemma 9, there exists $i \neq j$ such that i and j are in the same row of T_λ, and the same column of T_μ. Let $t = (ij)$, then $t \in P_\lambda$, and $t \in Q_\mu$. It follows then that $a_\lambda t = a_\lambda$, and $t b_\mu = -b_\mu$. It follows then that $a_\lambda t \cdot t b_\mu = a_\lambda b_\mu = a_\lambda \cdot (-b_\mu)$, and it thus follows that $a_\lambda b_\mu = 0$.

(ii) By Lemma 8(iv), for all $p \in P$, and $q \in Q$, we have that $p c_\lambda x c_\lambda \operatorname{sgn}(q) q = c_\lambda x c_\lambda$, and thus $c_\lambda x c_\lambda$ is a scalar multiple of c_λ.

□

Lemma 11.

(i) Each $V_\lambda = \mathbb{C}[\mathfrak{S}_n] c_\lambda$ is an irreducible representation of $\mathfrak{S}_n$.

(ii) If $\lambda \neq \mu$, then V_λ is not isomorphic to V_μ.

Proof.

(i) Let $W \subset V_\lambda$ be a subrepresentation of $\mathfrak{S}_n$. Then, $c_\lambda W \subset c_\lambda V_\lambda \subset \mathbb{C} c_\lambda$, by Lemma 10. It follows then that either $c_\lambda W = 0$, or $c_\lambda W = \mathbb{C} c_\lambda$. Suppose that $c_\lambda W = \mathbb{C} c_\lambda$. Then, $V_\lambda = \mathbb{C}[\mathfrak{S}_n] c_\lambda \subseteq \mathbb{C}[\mathfrak{S}_n] c_\lambda W \subseteq \mathbb{C}[\mathfrak{S}_n] W = W$, and thus $V_\lambda = W$.

For the second case, suppose that $c_\lambda W = 0$. Then, $W \cdot W \subseteq \mathbb{C}[\mathfrak{S}_n] c_\lambda W = 0$. From this, we wish to claim that $W = 0$. Recall that there is an orthogonal decomposition given by $\mathbb{C}[\mathfrak{S}_n] = W \oplus W'$, where W' is a complementary subrepresentation. Consider the projection map $p : \mathbb{C}[\mathfrak{S}_n] \to W$. It is a homomorphism of $\mathfrak{S}_n$-representations mapping $1 \mapsto w \in W$, and $w \mapsto w \cdot w = w$, since w must map to w. But $w^2 = 0$, and thus $w = 0$. It follows then that $W = 0$.

Remark 12. It follows from this proof that $c_\lambda V_\lambda \neq 0$, and $c_\lambda \cdot c_\lambda \neq 0$.

(ii) We can assume that $\lambda > \mu$. Then, $c_\lambda V_\lambda = \mathbb{C} c_\lambda \neq 0$. Further, $c_\lambda V_\mu = \mathbb{C}[\mathfrak{S}_n] c_\mu = 0$ by Lemma 10(i). It follows then that $V_\mu \ncong V_\lambda$.

□

8.3. Lecture 3, 15/09/2023

Recall from last time that $\mathbb{C}[\mathfrak{S}_n] c_\lambda$ is an irreducible $\mathfrak{S}_n$-representation.

Lemma 12. *For any λ,*

$$c_\lambda \cdot c_\lambda = n_\lambda c_\lambda,$$

where

$$n_\lambda = \frac{n!}{\dim V_\lambda}.$$

Proof. Consider a map

$$F : \mathbb{C}[\mathfrak{S}_n] \longrightarrow \mathbb{C}[\mathfrak{S}_n], \quad x \longmapsto x \cdot c_\lambda.$$

Then, $\operatorname{Im} F = V_\lambda$, and it follows then that

$$\mathbb{C}[\mathfrak{S}_\lambda] \cong \ker F \oplus \operatorname{Im} F \cong \ker F \oplus V_\lambda.$$

As such, we may now view F as a projection $F : \ker F \oplus V_\lambda \to V_\lambda$. For some $x \in V_\lambda$, it follows then that $F(x) = x \cdot c_\lambda = n_\lambda x$. It follows thus that

$$\operatorname{tr}(F) = n_\lambda \dim V_\lambda.$$

Now, for some basis element $e_g \in \mathbb{C}[\mathfrak{S}_n]$, we have that

$$F(e_g) = e_g \cdot \sum_{\substack{p \in P \\ q \in Q}} \operatorname{sgn}(q) e_{pq} = e_g + \text{things not containing } e_g,$$

which thus tells us that $\operatorname{tr}(F) = |\mathfrak{S}_n| = n!$, and the result follows.

$\square$

8.3.1. Frobenius Formula for Irreducible Characters of $\mathfrak{S}_n$

Let $\widehat{i} := (i_1, \cdots, i_n)$, with $i_k \geq 0$ for each $1 \leq k \leq n$. The components of $\widehat{i}$ are the *parts* of the partition. Then, let

$$\lambda_{\widehat{i}} := n^{i_n}(n-1)^{i_{n-1}} \cdots 1^{i_1},$$

called the *multiplicity* of the partition. As an example, the partition of n given by $(n) = \lambda_{(0,\cdots,0,1)}$, and $(1 + \cdots + 1) = \lambda_{(n,0,\cdots,0)}$. Generally, for some $\sum_{j=1}^n i_j j = n$, where j appears i_j times in $\lambda_{\widehat{i}}$. Let $C_{\widehat{i}}$ be the conjugacy class corresponding to a partition.

Let $x_1, \cdots, x_k$ be independent variables, and let k be a number that is greater than or equal than the number of rows in $\lambda_{\widehat{i}}$. Define a polynomial

$$p_j(x) := x_1^j + \cdots + x_k^j, \quad j = 1, \cdots, n,$$

called the *power sum*, which is a symmetric polynomial in these variables. The *discriminant* or the *van der Monde determinant* is given by:

$$\Delta(x) := \prod_{1 \leq i < j \leq k} (x_i - x_j).$$

If $f(x) = f(x_1, \cdots, x_k)$ is a formal power series, and $(\ell_1, \cdots, \ell_k)$ a k-tuple of elements in $\mathbb{Z}_{\geq 0}$, then denote

$$[f(x)]_{(\ell_1,\cdots,\ell_k)} := \text{coefficient of } x_1^{\ell_1} \cdots x_k^{\ell_k} \text{ in } f.$$

Let $\lambda = (\lambda_1, \cdots, \lambda_k)$, where in this case we now allow λ_k to be zero. Assume that $\lambda_1 \geq \cdots \geq \lambda_k \geq 0$ be a partition of n. Set $\ell_1 := \lambda_1 + k - 1$, $\ell_2 := \lambda_2 + k - 2$, and $\ell_k = \lambda_k$. Then, $\ell_1 > \ell_2 > \cdots > \ell_k$.

Theorem 22 (Frobenius Formula)**.**

$$\chi_\lambda(C_{\widehat{i}}) = \left[\Delta(x) \cdot \prod_{j=1}^n p_j(x)^{i_j}\right]_{(\ell_1,\cdots,\ell_k)}, \quad \ell_i = \lambda_i + k - i.$$

Given this, let us try to deduce a formula for the dimension of V_λ. The conjugacy class corresponding the identity is $\widehat{i} = (n, 0, \cdots, 0)$. It follows then that

$$\dim V_\lambda = \chi_\lambda(C_{(n,0,\cdots,0)}) = [\Delta(x) \cdot p_1(x)^n]_{(\ell_1,\cdots,\ell_k)}.$$

$\Delta(x)$ is called the van der Monde determinant because it is the determinant of the matrix:

$$\Delta(x) = \det \begin{pmatrix} 1 & x_k & \cdots & x_k^{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_1 & \cdots & x_1^{k-1} \end{pmatrix} = \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}(\sigma) x_k^{\sigma(1)-1} \cdots x_1^{\sigma(k)-1}.$$

We have that:

$$p_1(x)^n = (x_1 + \cdots + x_k)^n = \sum_{r_1+\cdots+r_k=n} \frac{n!}{r_1! \cdots r_k!} x_1^{r_1} \cdots x_k^{r_k}.$$

We have the relation that $\sigma(k) - 1 + r_1 = \ell_1$, which implies that $r_1 = \ell_1 + 1 - \sigma(k) \geq 0$. Iterating this process over all the r_i's, we get:

$$\begin{aligned} \dim V_\lambda &= \sum_{\substack{\sigma \in \mathfrak{S}_k \\ \sigma(i) \leq \ell_{k+1-i}+1 \\ 1 \leq i \leq k}} \operatorname{sgn}(\sigma) \frac{n!}{(\ell_1 - \sigma(k) + 1)! \cdots (\ell_k + 1 - \sigma(1))!} \\ &= \frac{n!}{\ell_1! \cdots \ell_k!} \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}(\sigma) \prod_{j=1}^{k} \ell_j(\ell_j - 1) \cdots (\ell_j - \sigma(k-j+1) + 2) \\ &= \frac{n!}{\ell_1! \cdots \ell_k!} \cdot \det \begin{pmatrix} 1 & \ell_k & \ell_k(\ell_k - 1) & \cdots & (\ell_k) \cdots (\ell_k - k + 2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \ell_1 & \ell_1(\ell_1 - 1) & \cdots & (\ell_1) \cdots (\ell_1 - k + 2) \end{pmatrix} \\ &= \frac{n!}{\ell_1! \cdots \ell_k!} \prod_{i<j} (\ell_i - \ell_j), \end{aligned}$$

where the second equality follows from multiplying the numerator and denominator by $\frac{n!}{\ell_1! \cdots \ell_k!}$. The last equality follows by using the van der Monde determinant.

The above does not depend on the choice of k, as long as it is bigger than or equal to the number of rows in $\lambda_{\widehat{i}}$.

Exercise 49. *Convince yourself of this.*

Theorem 23 (Hook-Length Formula)**.**

$$\dim V_\lambda = \frac{n!}{\prod \textit{hook lenghts in lambda}}.$$

Example 18. *Given the Young diagram*

$$\begin{array}{|c|c|c|c|} \hline \times & \times & \times & \times \\ \hline \times & * & * \\ \cline{1-3} \times & * & * \\ \cline{1-3} \times \\ \cline{1-1} \end{array}.$$

The $\times$ *symbols give a* hook *of length* 7.

$$\begin{array}{|c|c|c|c|}\hline 7 & 5 & 4 & 1 \\ \hline 5 & 3 & 2 \\ \cline{1-3} 4 & 2 & 1 \\ \cline{1-3} 1 \\ \cline{1-1}\end{array}.$$

Labelling the entries, we get

$$\dim V_\lambda = \frac{11!}{7 \cdot 5^2 \cdot 4^2 \cdot 1 \cdot 3 \cdot 2^2},$$

using the hook length formula.

Exercise 50. *Deduce the hook length formula.*

Proof. The proof is on wikipedia, and it looks long :(□

8.3.2. Sketch of Proof of Frobenius' Formula

Given a partition λ of n, $\lambda = (\lambda_1, \cdots, \lambda_k)$, define

$$U_\lambda := \mathrm{Ind}_{\mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_k}}^{\mathfrak{S}_n} \mathbb{C}_{\mathrm{triv}}.$$

That is, we are inducing from the Young subgroup P to $\mathfrak{S}_n$.

Claim:

$$U_\lambda \cong \mathbb{C}[\mathfrak{S}_n] \cdot a_\lambda.$$

Define a map:

$$U_\lambda = \mathbb{C}[\mathfrak{S}_n] \otimes_{\mathbb{C}[P_\lambda]} \mathbb{C}_{\mathrm{triv}} \longrightarrow \mathbb{C}[\mathfrak{S}_n] \cdot a_\lambda, \quad g \otimes v_0 \longmapsto g \cdot a_\lambda.$$

Exercise 51. *Show that this map is well-defined. That is,* $gh \otimes v_0 \mapsto gha_\lambda$ *for some* $h \in P$ *is equal to the map* $g \otimes v_0 \mapsto ga_\lambda$.

Further, show that this map is an isomorphism.

Proof. By properties of tensor products over modules, we have that for some $h \in P_\lambda$,

$$gh \otimes v_0 = g \otimes h \cdot v_0 = g \otimes v_0,$$

which gets mapped to $g \cdot a_\lambda$. The second equality follows because h acts by 1 on v_0, because $v_0 \in \mathbb{C}_{\mathrm{triv}}$ is a trivial $\mathbb{C}[P_\lambda]$-module. The map

$$\mathbb{C}[\mathfrak{S}_n] \cdot a_\lambda \longrightarrow \mathbb{C}[\mathfrak{S}_n] \otimes_{\mathbb{C}[P_\lambda]} \mathbb{C}_{\mathrm{triv}}, \quad g^{-1} \cdot a_\lambda \longmapsto g^{-1} \cdot a_\lambda \otimes v_0 = g^{-1} \otimes a_\lambda \cdot v_0 = g^{-1} \cdot v_0,$$

is an inverse to the map $g \otimes v_0 \mapsto g \cdot a_\lambda$, and thus the map is an isomorphism. □

Exercise 52. *Let* $\lambda = ((n-1)+1)$. *Show that*

$$U_{(n-1,1)} \cong V_{(n-1,1)} \oplus V_{(n)}.$$

Proof. □

Chapter 9

Week Nine

9.1. Lecture 1, 18/09/2023

9.1.1. Actual Proof of Frobenius' Formula

Recall from last time that we defined an induced representation given by

$$U_\lambda := \operatorname{Ind}_{P_\lambda}^{\mathfrak{S}_n} \mathbb{C}_{\text{triv}},$$

where P_λ is the Young subgroup of $\mathfrak{S}_n$ corresponding to a partition λ. Further, there is an isomorphism $U_\lambda \cong \mathbb{C}[\mathfrak{S}_n]a_\lambda$, and a surjective homomorphism

$$\mathbb{C}[\mathfrak{S}_n] \cdot a_\lambda \longrightarrow V_\lambda = \mathbb{C}[\mathfrak{S}_n]c_\lambda.$$

Exercise 53. *Show that $V_\lambda \cong \mathbb{C}[\mathfrak{S}_n] \cdot a_\lambda b_\lambda \cong \mathbb{C}[\mathfrak{S}_n] b_\lambda a_\lambda$, and thus V_λ defines a U_λ-subrepresentation.*

Now, let $\psi_\lambda = \chi_{U_\lambda}$ be a character of U_λ. Recall that for any subgroup H of G, and $\rho = \operatorname{Ind}_H^G \sigma$,

$$\chi_\rho(g) = \frac{1}{|H|} \sum_{\substack{s \in G \\ s^{-1}gs \in H}} \chi_\sigma(s^{-1}gs).$$

If σ is trivial, then

$$\chi_\rho(g) = \frac{1}{|H|} |C(g) \cap H| \frac{|G|}{|C(g)|}.$$

Now, let $G = \mathfrak{S}_n$, and $H = P_\lambda$. Then, for some $g \in C_{\vec{i}} = C(g)$, then

$$|C(g)| = \frac{n!}{i_1! \cdot i_2! \cdots i_n!}.$$

Thus,

$$\begin{aligned}
\psi_\lambda(g) &= \frac{|\mathfrak{S}_n|}{|\mathfrak{S}_{\lambda_1}\times\cdots\times\mathfrak{S}_{\lambda_k}|\cdot|C_{\widehat{i}}|}\cdot|C_{\widehat{i}}\cap\mathfrak{S}_{(\lambda)}| \\
&= \frac{i_1!\cdots i_n! 1^{i_1}\cdot 2^{i_2}\cdots n^{i_n}}{\lambda_1!\cdots\lambda_k!}\sum_{r_{pq}\in R_{p,q}}\prod_{p=1}^{k}\frac{\lambda_p!}{r_{p_1}!\cdots r_{pq}! 1^{r_{p1}}\cdots n^{r_{p_n}}} \\
&= \sum_{r_{pq}\in R_{p,q}}\frac{\prod_{i=1}^n q^{i_q} i_q!}{\prod_{p=1}^k\prod_{q=1}^n q^{r_{pq}} r_{pq}!} \\
&= \sum_{r_{pq}\in R_{p,q}}\frac{\prod_{q=1}^n q^{i_q} i_q!}{\prod_{q=1}^n q^{\sum_{p=1}^k r_{pq}}\prod_{p=1}^k\prod_{q=1}^n r_{pq}!} \\
&= \sum_{r_{pq}\in R_{p,q}}\prod_{q=1}^{n}\frac{i_q!}{r_{1q}!\cdot r_{2q}!\cdots r_{kq}!} \\
&= \text{coefficient of } x_1^{\lambda_1}\cdots x_k^{\lambda_k} \text{ in } P^{(\widehat{i})} := (x_1+\cdots+x_k)^{i_1}(x_1^2+\cdots x_k^2)^{i_2}\cdots(x_1^n+\cdots+x_k^n)^{i_n}.
\end{aligned}$$

where $R_{p,q}$ has elements r_{pq} such that $1\le p\le k, 1\le q\le n$, with $i_q = r_{1q}+\cdots+r_{kq}$, and $\lambda_p = r_{p_1}+2r_{p2}+\cdots+nr_{p_n}$.

Exercise 54. *Go home and think about this calculation.*

It follows then that

$$\psi_\lambda(C_{\widehat{i}}) = \left[P^{(\widehat{i})}\right]_{\lambda=(\lambda_1\ge\cdots\ge\lambda_k)}.$$

To prove Frobenius' formula, let us define

$$\omega_\lambda(\widehat{i}) := \left[\Delta\cdot P^{(\widehat{i})}\right]_{\ell_1,\cdots,\ell_k}.$$

From this, we wish to show that:

$$\chi_\lambda(C_{\widehat{i}}) = \omega_\lambda(\widehat{i}),$$

where χ_λ is an irreducible character of $\mathfrak{S}_n$.

General Identity of Symmetric Polynomials

For any symmetric polynomial P. Then,

$$[P]_{\lambda_1,\cdots,\lambda_k} = \sum_{\mu \text{ a partition of } n} k_{\mu\lambda}[\Delta\cdot P]_{\mu_1+k-1,\mu_2+k-2,\cdots,\mu_k}. \tag{9.1}$$

We will not give a proof of this fact, but it can be found in Fulton-Harris. The coefficients $k_{\mu\lambda}$ admit an interesting combinatorial interpretation.

They are called *Kostka numbers*, which are defined as the number of ways to fill the boxes of the Young diagram for μ with λ_1 many 1's, λ_2 many 2's, $\cdots$, and λ_k many k's in such a way that the entries in each row are non-decreasing, and in each column strictly increasing.

Such tableaux are called *semi-standard Young tableaux on μ of type λ*. In particular, $k_{\lambda\lambda}=1$. Thus, if $\mu<\lambda$, then $k_{\mu\lambda}=0$. That is, $k_{\mu\lambda}$ is an upper-triangular matrix.

By (9.1), it follows thus that
$$\psi_\lambda(C_{\widehat{i}}) = \omega_\lambda(\widehat{i}) + \sum_{\mu>\lambda} k_{\mu\lambda}\omega_\mu(\widehat{i}).$$

Lemma 13 (Fulton-Harris, Lemma A.28)**.**
$$\frac{1}{n!}\sum_{\widehat{i}} |C_{\widehat{i}}| \cdot \omega_\lambda(\widehat{i})\omega_\mu(\widehat{i}) = \delta_{\lambda\mu}.$$

That is, $\{\omega_\lambda\}$ satisfies some orthogonal relations on $\{\chi_\lambda\}$.

Proposition 18. $\chi_\lambda(C_{\widehat{i}}) = \omega_\lambda(\widehat{i})$*. That is,* $\chi_\lambda = \chi_{V_\lambda}$*.*

Proof. Since V_λ is a subrepresentation of U_λ, it follows that
$$\psi_\lambda = \sum_{\mu \text{ a partition of } n} n_{\lambda\mu}\chi_\mu, \quad n_{\lambda\lambda} \geq 1, \quad n_{\lambda\mu} \geq 0.$$

Thus, we may write
$$\omega_\lambda = \sum m_{\lambda\mu}\chi_\mu,$$
and
$$\psi_\lambda = \omega_\lambda + \sum_{\mu>\lambda} k_{\mu\lambda}\omega_\mu = \sum_{\mu \text{ partition of } n} n_{\lambda\mu}\chi_\mu.$$
Then, by inducting on order of λ, we can show that $m_{\lambda\mu} \in \mathbb{Z}$. But
$$(\omega_\lambda, \omega_\lambda) = \sum m_{\lambda\mu}^2 = 1,$$
and thus
$$\omega_\lambda = \pm\chi,$$
for some irreducible character χ. We know that $\omega_{(n)} = \chi_{(n)}$. Now, if we fix λ, then by induction $\chi_\mu = \omega_\mu$, for any $\mu > \lambda$. Then,
$$\begin{aligned}\psi_\lambda &= \omega_\lambda + \sum_{\mu>\lambda} k_{\mu\lambda}\chi_\mu \\ &= n_{\lambda\lambda}\chi_\lambda + \sum_{\mu\neq\lambda} n_{\lambda\mu}\chi_\mu,\end{aligned}$$
which implies that $\omega_\lambda = n_{\lambda\lambda}\chi_\lambda + \cdots$, for $n_{\lambda\lambda} \geq 1$, and so $\omega_\lambda = \chi_\lambda$. ☐

Corollary 18 (Young's Rule)**.**
$$U_\lambda \cong V_\lambda \oplus \bigoplus_{\mu>\lambda} V_\mu^{\oplus k_{\mu\lambda}}.$$

9.2. Lecture 2, 21/09/2023

9.2.1. More Properties of Irreducible $\mathfrak{S}_n$-representations

Corollary 19. $\dim_{\mathbb{C}} V_\lambda$ *is equal to the number of standard Young tableaux on λ — that is, the number of ways to fill the Young diagram λ with numbers $1, \cdots, n$ so that all rows and columsn are increasing.*

Proof. Let $\lambda = \underbrace{(1, \cdots, 1)}_{n \text{ times}}$. Then,
$$U_\lambda = \mathbb{C}[\mathfrak{S}_n],$$

and using Young's rule,

$$U_{(1^n)} \cong V_{(1^n)} \oplus \bigoplus_{\mu > 1^n} V_\mu^{k_{\mu,(1^n)}}.$$

Since every partition is bigger than 1^n, it thus follows that

$$k_{\mu,(1^n)} = \dim_{\mathbb{C}} V_\mu,$$

which by construction of $k_{\mu,(1^n)}$ is the number of ways of filling boxes in μ with $1, \cdots, n$ such that the rows are non-decreasing, and the columns are increasing, which is precisely equal to the number of standard Young tableaux of shape μ.

$\square$

What this tells us is that there is a bijection

$$\mathfrak{S}_n \longleftrightarrow \{(A_\lambda, B_\lambda) : \text{both } A_\lambda \text{ and } B_\lambda \text{ are Young tableaux}\}.$$

Given any element of $\mathfrak{S}_n$, one can produce such a pair of tableaux using the Robinson-Schensted algorithm.

Recall that we have a Young symmetriser c_λ, which a priori is an element of $\mathbb{C}[\mathfrak{S}_\lambda]$, but induces an action on $V^{\otimes n}$. So,

$$c_\lambda : V^{\otimes n} \longrightarrow V^{\otimes n}.$$

Suppose that $N = \dim_{\mathbb{C}} V$.

Definition 17. The *Schur polynomial* $s_\lambda(x_1, \cdots, x_N)$ is given by:

$$s_\lambda(x_1, \cdots, x_N) := \frac{\det\left(x_j^{\lambda_i + N - i}\right)}{\Delta},$$

where Δ is the van der Monde determinant.

Remark 13. This is related to the Weyl character formula from Lie algebras.

Further, $\mathrm{GL}(V)$ acts on $V^{\otimes n}$ by $g(v_1 \otimes \cdots \otimes v_n) = gv_1 \otimes \cdots \otimes gv_n$. Define a *Schur functor*

$$\mathbb{S}_\lambda(V) := \mathrm{Im}(c_\lambda).$$

Then,

1. $\mathbb{S}_\lambda(V) = 0$ if $\lambda_{N+1} = 0$,
2. otherwise,

$$\dim_{\mathbb{C}} \mathbb{S}_\lambda(V) = \prod_{1 \leq i \leq j \leq k} \frac{\lambda_i - \lambda_j + j - i}{j - i},$$

if $\lambda = (\lambda_1 \geq \cdots \geq \lambda_N \geq 0)$. Moreover,

$$\chi_{\mathbb{S}_\lambda}(g) = s_\lambda(x_1, \cdots, x_N),$$

where $x_1, \cdots, x_N$ are eigenvalues of $g \in \mathrm{GL}(V)$ on V, and s_λ is a *Schur polynomial.*

9.2.2. Schur-Weyl Duality

As a representation of $\mathfrak{S}_n \times \mathrm{GL}(V)$,

$$V^{\otimes n} \cong \bigoplus_\lambda V_\lambda \otimes L_\lambda,$$

where L_λ is a $\mathrm{GL}(V)$-representation that is irreudcible when $\lambda \neq 0$, and V_λ is the aforementioned irreducible $\mathfrak{S}_n$-representation that we have constructed. In particular,

$$L_\lambda = \mathrm{Hom}_{\mathfrak{S}_n}(V_\lambda, V^{\otimes n}),$$

which are distinct irreps of $\mathrm{GL}(V)$ if $\lambda \neq 0$. Varying n, we thus obtain infinitely many irreducible $\mathrm{GL}(V)$-representations.

Example 19. *$L_{(n)} = \mathrm{Sym}^n V$, and $L_{(1^n)} = \Lambda^n V$, both of which vanishes when $n > \dim_{\mathbb{C}} V$.*

Remark 14 (Littlewood-Richardson). The representation

$$\mathrm{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_k}^{\mathfrak{S}_{m+k}} V_\lambda \otimes V_\mu = \sum_{\nu \text{ partition of } m+k} N_{\lambda\mu}^{\nu} \nu,$$

where λ is a partition of m, and μ a partition of k. The coefficients $N_{\lambda\mu}^{\nu}$ can be determined using the *Littlewood-Richardson rule.*

Remark 15 (Pieri's Formula).

$$\mathrm{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} V_\lambda = \sum_{\mu \,:\, \text{one box removed from } \lambda} V_\mu.$$

Remark 16. For λ, μ both partitions of n,

$$V_\lambda \otimes V_\mu = \bigoplus_{\nu \text{ partitions } n} c_{\lambda\mu}^{\nu} V_\nu.$$

Remark 17. Let $R_n := \mathrm{Rep}(\mathfrak{S}_n)$ be the representation ring of $\mathfrak{S}_n$. Then,

$$R = \bigoplus_{n=0}^{\infty} R_n,$$

has the structure of a graded Hopf algebra.

Exercise 55. *Think about this.*

9.2.3. Alternating Groups

For $n \geq 5$, the alternating group A_n is a simple group. Recall that A_n is a subgroup of $\mathfrak{S}_n$ of index 2. Generally, let H be an index 2 subgroup of G. Then, H is normal, since $G/H = \{1, r\}$ such that $rh = h'r$. G acts on G/H, and the corresponding permutation representation is given by

$$\mathbb{C}_{\text{triv}} \oplus \mathbb{C}_{\text{non-triv}}.$$

We wish to use this fact to deduce some facts about A_n-representations.

Proposition 19. *Let V be an irreducible representation of G and $W = \mathrm{Res}_H^G V$. Then, one of the following holds:*

(i) If $V' = V \otimes \mathbb{C}_{\text{non-triv}}$, then $V \cong V'$, W is irreducible, and

$$\mathrm{Ind}_H^G W \cong V \oplus V'.$$

(ii) If $V \cong V'$, then $W = W' \oplus W''$, where W', and W'' are irreducible, non-isomorphic H-representations. Moreover,

$$\mathrm{Ind}_H^G W' \cong V \cong \mathrm{Ind}_H^G W''.$$

(iii) Every irreducible representation of H arises uniquely this way.

Proof. Let $\chi = \chi_V$ be the character of V. Then, $\langle \chi | \chi \rangle = 1$. Thus,

$$\sum_{h \in H} |\chi(h)|^2 + \sum_{g \notin H} |\chi(g)|^2 = |G| = 2|H|,$$

which implies that

$$|H| \cdot \langle \chi, \chi \rangle_H + \sum_{g \notin H} |\chi(g)|^2 = 2|H|.$$

It follows then that $\langle \chi, \chi \rangle_H$ is either 1 or 2. If $\langle \chi, \chi \rangle_H = 1$, then W is irreducible. Then,

$$\operatorname{Hom}_G(V, \operatorname{Ind}_H^G W) = \operatorname{Hom}_H(W, W) = 1,$$

by hom-tensor adjunction. It follows then that

$$\operatorname{Ind}_H^G W \cong \operatorname{Ind}_H^G \operatorname{Res}_H^G V \cong V \oplus V',$$

by a previous result.

If $\langle \chi, \chi \rangle_H = 2$, then W is not irreducible, and and has two distinct components in its decomposition: $W \cong W' \oplus W''$, where $W' \not\cong W''$ are irreducible. Then, $\sum_{g \notin H} |\chi(g)|^2 = 0$, which implies that $\chi(g) = 0$ for all $g \notin H$. We thus have that

$$\chi_{V'}(g) = \begin{cases} \chi_V(g), & \text{if} \quad g \in H, \\ -\chi_V(g), & \text{if} \quad g \notin H. \end{cases},$$

and

$$\chi_{\mathbb{C}_{\text{non-triv}}}(g) = \begin{cases} 1 & \text{if} \quad g \in H, \\ -1 & \text{if} \quad g \notin H. \end{cases}.$$

It thus follows that $\chi_{V'}(g) = \chi_V(g)$, and so $V \cong V'$. Then,

$$\operatorname{Hom}_G(V, \operatorname{Ind}_H^G W) = \operatorname{Hom}_H(W, W') = 1 = \operatorname{Hom}_G(V, \operatorname{Ind}_H^G W'').$$

Now, let W be any irreducible representation. Then,

$$\operatorname{Res}_H^G \operatorname{Ind}_H^G W \cong W \oplus W^r,$$

where for each $r \in G/H$, W^r is the representation defined by

$$\rho^r : H \longrightarrow \mathrm{GL}(W), \quad h \longmapsto \rho(r^{-1}hr).$$

□

9.3. Lecture 3, 22/09/2023

9.3.1. Conjugacy Classes in Subgroups of Index 2

There are two types: for C a conjugacy class in H, we have $C \subset G$ a conjugacy class in G, or $C \cup C' \subset H \subset G$ a conjugacy class in G. The latter type of conjugacy class are called *split conjugacy classes*. In particular, $C = rC'r^{-1}$, for $1 \neq r \in G/H$. There is a way to detect which conjugacy class splits, which we will cover more next time. In particular, the conjugacy class of an element $x \in C$ splits if $Z_G(x) \subset H$. More on this next time.

For $G = \mathfrak{S}_n$, we any conjugacy class C_λ — where λ is a partition of n — splits into two conjugacy classes in A_n if and only if $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_k$, where $\lambda_i \neq \lambda_j$, $i \neq j$, and λ_i's are odd.

Let V_λ be an irreducible representation of $\mathfrak{S}_n$, and $U' = \mathbb{C}_{\text{sgn}}$ — the sign representation — which is the the representation obtained from the action of $\mathfrak{S}_n$ on $\mathfrak{S}_n/A_n$. Then, there is an isomorphism

$$V_\lambda \otimes \mathbb{C}_{\text{sgn}} \cong V_{\lambda'},$$

where λ' is the conjugate of λ — that is, it is the *dual partition* of λ.

In case (i) of Proposition 19, if $\lambda \neq \lambda'$, then

$$W_\lambda = \operatorname{Res}^{\mathfrak{S}_n}_{A_n} V_\lambda \cong \operatorname{Res}^{\mathfrak{S}_n}_{A_n} V_{\lambda'},$$

is an irreducible representation of A_n. Then,

$$\chi_{W_\lambda} = \chi_{V_\lambda}|_{A_n}.$$

In case (ii) of Proposition 19, then

$$\operatorname{Res}^{\mathfrak{S}_n}_{A_n} V_\lambda \cong \operatorname{Res}^{\mathfrak{S}_n}_{A_n} V'_\lambda \cong W'_\lambda \oplus W''_\lambda,$$

where $W'_\lambda \ncong W''_\lambda$, are non-isomorphic irreducible A_n-representations. Let us write,

$$\chi'_\lambda := \chi_{W'_\lambda}, \quad \chi''_\lambda := \chi_{W''_\lambda}.$$

The partitions in case (i) are called *self-dual partitions of* n. What we have showed is that there is a bijection

$$\{\text{self-conjugate partitions of } n\} \longleftrightarrow \{\text{partitions of } n \text{ into sum of distinct odd parts}\}. \tag{9.2}$$

Example 20. *The partition given by:*

$$\lambda = \begin{array}{|c|c|c|c|c|}\hline * & * & * & * & * \\ \hline * & * & * & * & \\ \cline{1-4} * & * & * & & \\ \cline{1-3} * & * & & & \\ \cline{1-2} * & & & & \\ \cline{1-1}\end{array},$$

is a self-conjugate partition. The mapping of the bijection is given by:

$$(\lambda_1, \cdots, \lambda_k) \longmapsto (2\lambda_1 - 1, 2\lambda_2 - 3, \cdots, 2\lambda_i - (2i-1), \cdots).$$

Proposition 20. *Let C, C' be a pair of split conjugacy classes obtained from $q_1 > \cdots > q_r$, for q_i odd. Suppose that $\lambda = \lambda'$. Then,*

(i) If C and C' do not correspond to λ under the bijection (9.2), then

$$\chi'_\lambda(C) = \chi'_\lambda(C') = \chi''_\lambda(C') = \chi''_\lambda(C) = \frac{1}{2}\chi_\lambda(C \cup C').$$

(ii) If C and C' correspond to λ under (9.2), then

$$\chi'_\lambda(C) = \chi''_\lambda(C') = a, \quad \chi'_\lambda(C') = \chi''_\lambda(C) = b,$$

where a, b are

$$\frac{1}{2}\left((-1)^m \pm \sqrt{(-1)^m q_1 \cdots q_r}\right), \quad m = \frac{1}{2}(n-r).$$

Exercise 56. *Prove this proposition using the following steps:*

(i) Determine χ'_λ and χ''_λ on a non-split conjugacy class.

(ii) Prove the above proposition (follow the steps in Fulton-Harris).

9.3.2. Aside: Nilpotent Orbits in $\mathfrak{gl}_n(\mathbb{C})$

The group $G = \mathrm{GL}_n(\mathbb{C})$ acts on the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ by conjugation. These are in bijection with partitions of n. From this, one obtains *Springer representations*: given a nilpotent element $x \in \mathfrak{gl}_n(\mathbb{C})$, we consider the a subvariety $B_x \subset G/B$, where B is the Borel subgroup of G. The subvariety B_x is called the *Springer fibre*, which under this correspondence will be labelled by Young tableaux.

9.3.3. Representations of Algebras

Let k be an algebraically closed field.

Definition 18. An *associative algebra* over k is a k-vector space A with an associative bilinear multiplication

$$A \times A \longrightarrow A, \quad (a, b) \longmapsto ab.$$

Further, we will always assume that A contains a unit — that is, A is *unital*.

Example 21. *Let V be a k-vector space. Then,* $\mathrm{End}(V)$ *is a k-algebra with respect to composition of linear maps. Equivalently,* $\mathrm{End}(V) \cong \mathrm{Mat}_{\dim_k V}(k)$ *is an algebra with respect to matrix multiplication.*

For a finite group, the group algebra $k[G]$ is a k-algebra if $\operatorname{char} k$ *does not divide $|G|$.*

The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a unital, associative algebra.

$k[x_1, \cdots, x_n]$ is a k-algebra.

For a finite-dimensional k-vector space, the tensor algebra $T(V)$ is a k-algebra.

The Weyl algebra *is a k-algebra of differential operators on a polynomial ring $k[x_1, \cdots, x_n]$.*

One can also have an algebra defined using generators and relations. That is,

$$A = k\langle x_1, \cdots, x_n\rangle / \langle f_1, \cdots, f_m\rangle.$$

Hecke algebras are the best *examples of this. Diagram algebras, Temperley-Lieb algebras, and Brauer algebras are other examples of such algebras.*

Definition 19. A *left A-module* (or, a *representation*) of an associative algebra A is a k-vector space V equipped with a homomorphism

$$\rho : A \longrightarrow \mathrm{End}_k(V).$$

That is, it is a linear map preserving multiplication and unit. Similarly a right A-module is a k-vector space equipped with a right action, and a homomorphism

$$\rho : A \longrightarrow \mathrm{End}_k(V), \quad \rho(ab) = \rho(b)\rho(a), \quad \rho(1) = 1.$$

An A-*submodule* (or A-subrepresentation) of V is a subspace $U \subset V$ such that $\rho(a)U \subset U$ — that is, it is stable under the action of any $a \in A$.

Remark 18. If A is commutative, then V clearly defines an A-bimodule.

Definition 20. A non-zero A-module V is *irreducible* (or *simple*) if its only submodules are $\{0\}$ and V. V is *indecomposable* if it cannot be written as a direct sum of two non-zero submodules.

Remark 19. Irreducibility implies indecomposability, but the converse is not true in general. That is, irreducibility is a much stronger condition than indecomposability.

Example 22. *Consider $A = \mathbb{C}[t^{\pm}]$. Then, the two-dimensional representation given by*

$$t \longmapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

defines an indecomposable representation that is not irreducible, since $\mathbb{C}[t^{\pm}] \cong \mathbb{C}[\mathbb{Z}]$, and we know that all irreducible representations of $\mathbb{Z}$ are one-dimensional.

Example 23. *Let $A = k[x]$. Then, the representation given by sending x to its Jordan decomposition given by block diagonal matrices of the form $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. This is an example of an indecomposable representation. Calling this representation $V_{\lambda,n}$, we see that $V_{\lambda,n} \cong k^n$.*

Chapter 10

Week Ten

10.1. Lecture 1, 02/10/2023

Example 24. *Let A be a k-algebra, where k is algebraically closed.*

1. *(Regular Representation) Let $V = A$, which defines an A-module structure by $\rho : A \to \mathrm{End}_k(A)$, given by $\rho(a)b = ab$.*

2. *Let $A = k$. Then, a k-module is given by a vector space over k.*

3. *Let $A = k\langle x_1, \cdots, x_n \rangle$, a free algebra generated by n elements. Then, an A-module is a vector space over k, with $\rho(x_i) : V \to V$ for each generator x_i.*

Definition 21. Let V_1, and V_2 be representations of A. Then, A acts on the direct sum $V_1 \oplus V_2$ by $a(v_1 + v_2) = av_1 + av_2$, for $a \in A$, $v_1 \in V_1$, and $v_2 \in V_2$. An *A-homomorphism* $\phi : V_1 \to V_2$ is a linear map such that $\phi(av_1) = a\phi(v_1)$ for $a \in A$, $v_1 \in V_1$ — that is, ϕ *intertwines* the A-action. The map ϕ is an *isomorphism* if it is also an isomorphism of k-vector spaces.

Then, as before, we may write

$$\mathrm{Hom}_A(V_1, V_2) := \{\text{homomorphisms of } A\text{-modules } V_1 \to V_2\}.$$

10.1.1. Schur's Lemma for A-modules

Theorem 24 (Schur's Lemma)**.** *Let V_1, V_2 be A-modules, and $\phi : V_1 \to V_2$ a non-trivial A-homomorphism. Then,*

(i) If V_1 is irreducible, then ϕ is injective,

(ii) If V_2 is irreducible, then ϕ is surjective,

(iii) If V_1 and V_2 are irreducible, then ϕ is an isomorphism,

Moreover, if V is a finite-dimensional irreducible A-module, then for any A-homomorphism $\phi : V \to V$, we have that $\phi = \lambda\, \mathrm{id}_V$, where $\lambda \in k$ (note that k is algebraically closed).

Proof. Basically same as proof of Schur's lemma for finite groups. □

Corollary 20. *Let A be a commutative algebra. Then, every irreducible finite-dimensional A-module is one-dimensional.*

Proof. Let V be a finite-dimensional irreducible A-module, and for each $a \in A$, define $\phi_a : V \to V$ by $v \mapsto av$. Then, ϕ_a is a homomorphism, and by Schur's lemma, we have that $\rho_a = \lambda_a \operatorname{id}_V$ for all $a \in A$. $\square$

Example 25. *Let $A = k[x]$. This is a commutative k-algebra. Let $V = k$, and define a map $\rho_\lambda : A \to k$ mapping $x \mapsto \lambda$, for some $\lambda \in k$. Then, the indecomposable representations of A are given by*

$$V_{\lambda,n} = k^n,$$

following the theory of Jordan normal forms. The indecomposables then take the form $\rho(x)$, which is a matrix with λ on the diagonals, and 1's on the superdiagonals.

Exercise 57. *Show this.*

Proof. The representation mapping x to the $n \times n$ matrix with λ on the diagonals, and 1 on the superdiagonals is indecomposable, but not irreducible, because all irreps of $k[x]$ should be one-dimensional. Call this representation $\rho : k[x] \to \operatorname{End}_k(V_{\lambda,n})$. It follows from the representation that we defined that $V_{\lambda,n} \cong k^n$. $\square$

10.1.2. Ideals

Definition 22. A *left (resp. right) ideal* of a k-algebra A is a subspace $I \subseteq A$ such that $aI \subseteq I$ for all $a \in I$. A *two-sided ideal* is a subspace that is both a left and right ideal.

Example 26.

1. *Given a subset $S \subset A$, we can define a two-sided ideal generated by S given by*

$$\langle S \rangle = \operatorname{Span}_k\{asb : a \in A, s \in S, b \in A\}.$$

2. *For any homomorphism $\phi : A \to B$ of k-algebras, the kernel $\ker \phi$ form a two-sided ideal of A.*

Definition 23. An algebra $A \neq 0$ is *simple* if the only ideals are 0 and A — that is, its ideals are the trivial ideals.

Exercise 58. *Show that $\operatorname{Mat}_n(k)$ is simple.*

Proof. Choose a basis of matrices $\{E_{ij}\}_{i,j=1}^n$, where E_{ij} is the $n \times n$ matrix with 1 in the (i, j) position, and 0 elsewhere. Let I be a non-zero ideal of $\operatorname{Mat}_n(k)$, and let $A \in I$. Then, there exists indicies $1 \leq s, t \leq n$ such that $A_{st} \neq 0$. Then, $E_{ss}A$ produces a matrix that kills all the rows except for the s-th row. Multiplying by E_{tt} on the right kills everything except for the t. Thus, $E_{ss} \cdot A \cdot E_{tt} = A_{st} \cdot E_{st}$, which is in I. It follows then that I must necessarily contain all the matrices in $\operatorname{Mat}_n(k)$, since the matrices E_{ij} form a basis for $\operatorname{Mat}_n(k)$. $\square$

10.1.3. Quotients

Let A be a k-algebra, and I a two-sided ideal. Then, show that A/I is a k-algebra, with elements given by $a + I$. Given an A-module V, and a subrepresentation $W \subset V$. Then, we may form an A-module given by V/W.

Example 27. *There is a bijection*

$$\{\textit{subrepresentations of regular } A\textit{-modules} \longleftrightarrow \{\textit{left ideals of } A\}.$$

Exercise 59. *Show that if $A = k[x_1, \cdots, x_n]$, and let I be the ideal containing all homogeneous polynomials of degree $\geq N$. Show that A/I is an indecomposable A-module. (If you find this difficult, start with one variable first).*

Proof. We can induct on the number of variables in A. Let us first consider $A = k[x]$. Let I be the ideal containing all homogeneous polynomials of degree $\geq N$. Then, we can decompose $k[x]/I$ by degree in the following way:

$$A/I = \bigoplus_{n \geq N} k[x]/\langle x^n \rangle.$$

The result follows from showing that each $k[x]/\langle x^n \rangle$ is indecomposable. We will show that each ideal of $k[x]/\langle x^n \rangle$ intersects non-trivially. Suppose that N is a non-trivial $k[x]$-module. Then, there exists some

$$n := \alpha_j x^j + \cdots + \alpha_{n-1} x^{n-1} + \langle x^n \rangle \in N.$$

But

$$n \alpha_j^{-1} x^{n-1-j} = x^{n-1} + \langle x^n \rangle,$$

which implies that every ideal N of $k[x]/\langle x^n \rangle$ contains $x^{n-1} + \langle x^n \rangle$. Thus, each $k[x]/\langle x^n \rangle$ is indecomposable.

Now, let us induct on the number of variables, let $I_n := k[x_1, \cdots, x_n]_{\deg = n}$ be the $k[x]$-submodule of all polynomials of degree n. Then,

$$A/I = \bigoplus_{\ell \geq N} k[x_1, \cdots, x_n]/I_\ell.$$

The same argument as before shows that A/I is indecomposable, by replacing x^ℓ with $(x_1^{i_1} \cdots x_n^{i_n})^\ell$, with the condition that $i_1 + \cdots + i_n = \ell$ in the previous argument. □

10.1.4. Algebras defined by generators and relations

These are k-algebras A of the form

$$A = \frac{k\langle x_1, \cdots, x_n \rangle}{\langle f_1, \cdots, f_m \rangle}.$$

Example 28. *The Weyl algebra is the algebra given by*

$$A = \frac{k\langle x, y \rangle}{\langle yx - xy - 1 \rangle}.$$

Proposition 21. *A basis for the Weyl algebra is given by*

$$\{x^i y^j : i, j \geq 0\}.$$

Proof. It is clear that elements of this form span the Weyl algebra. It thus suffices to show linear independence. Let

$$E := t^a k[a][t, t^{-1}],$$

where a is a variable. Then, for any $f \in E$, we set the relation

$$xf = tf, \quad yf = \frac{\partial f}{\partial t}.$$

One checks that $(xy - yx)f = f$, and thus E defines a module over the Weyl algebra. Suppose that

$$\sum_{i,j} G_j x^i y^j = 0.$$

Then,

$$L = \sum_{i,j} C_{ij} t^i \left(\frac{\partial}{\partial t} \right)^j,$$

acts by 0 on E. Write

$$L = \sum_{j=0}^{r} Q_j(t) \left(\frac{\partial}{\partial t} \right)^j,$$

where $Q_r(t) \neq 0$. It follows then that $Lt^a = \sum_{j=0}^{r} Q_j(t) a(a-1) \cdots (a-j+1) t^{a-j} = 0$, which implies then that $a^r Q_r(t) t^{a-r} + \cdots = 0$, and so $Q_r(t) = 0$, which is a contradiction. $\square$

Definition 24. A representation $\rho : A \to \mathrm{End}_k(A)$ of A is *faithful* if ρ is injective.

Example 29. *Let $A = k[t]$. Let $\mathrm{char}\, k = 0$. Then, E is faithful. But if $\mathrm{char}\, k = p$ for p a prime, then E is not faithful.*

Exercise 60. *Show this.*

Proof. In characteristic p, if 0 is in the kernel, then any integral multiple of p is also in the kernel, and thus E is not faithful. $\square$

10.2. Lecture 2, 05/10/2023

10.2.1. Semisimple A-modules

As before, A is a k-algebra, where k is an algebraically closed field.

Definition 25. A *semisimple* (or *completely reducible*) A-module is an A-module that admits a direct sum decomposition into irreducible A-modules:

$$A = \bigoplus_i A_i,$$

where each A_i is an irreducible A-module.

Example 30. *Let V be an irreducible A-module of dimension n over k. Then, there is a k-algebra homomorphism $\rho : A \to \mathrm{End}_k(V)$. Moreover, there is an A-module isomorphism:*

$$\mathrm{End}_k(V) \cong V^{\oplus n}, \quad x \longmapsto (xv_1, \cdots, xv_n),$$

where $\{v_1, \cdots, v_n\}$ is a basis of V. It follows then that $\mathrm{End}_k(V)$ is a semisimple A-module.

Exercise 61. *Check that this is an A-module isomorphism.*

Proof. Injectivity is clear, unless k has prime characteristic, in which case it is not injective. We can write V as a direct sum

$$\mathrm{End}_k(V) = \mathrm{Hom}_k(V, V) = \mathrm{Hom}_k\left(\bigoplus_i \mathbb{C}v_i, \bigoplus_j \mathbb{C}v_j \right) = \bigoplus_{i,j} \mathrm{Hom}_k(\mathbb{C}v_i, \mathbb{C}v_j),$$

and thus $\mathrm{End}_k(V)$ takes basis elements of V to basis elements of V. It follows then that the map is surjective. Thus, $\mathrm{End}_k(V) \to V^{\oplus n}$ is an isomorphism. $\square$

Exercise 62. *If V is a semisimple, finite-dimensional A-module, show that there is an isomorphism of A-modules:*

$$V \cong \bigoplus_{W_i \in \mathrm{Irr}(A)} \mathrm{Hom}_A(W_i, V) \otimes_k W_i.$$

Sketch Outlined in Class. Define a map

$$\sum_i f_i \otimes w_i \longmapsto \sum_i f_i(w_i),$$

and apply Schur's lemma. Observe that $\bigoplus_{W_i \in \mathrm{Irr}(A)} \mathrm{Hom}_A(W_i, V)$ is the multiplicity space — that is,

$$\bigoplus_{W_i \in \mathrm{Irr}(A)} \mathrm{Hom}_A(W_i, V) \otimes_k W_i \cong \bigoplus_{W_i \in \mathrm{Irr}(A)} W_i^{\oplus \dim \mathrm{Hom}_A(W_i, V)}.$$

$\square$

Proof. Previously, we proved that

$$\mathrm{Hom}_k(V, V') \cong V^* \otimes_k V,$$

in an exercise. Using this fact, we have:

$$\mathrm{Hom}_A(W_i, V) \otimes_k W_i \cong W_i^* \otimes_A V \otimes_k W_i \cong \mathrm{Hom}_k(V^* \otimes_A W_i, W_i),$$

and then by hom-tensor adjunction, we have:

$$\mathrm{Hom}_k(V^* \otimes_A W_i, W_i) \cong \mathrm{Hom}_k(V, \mathrm{Hom}_A(W_i, W_i)) \cong \mathrm{Hom}_k(V, k) = V^* \cong V,$$

where the second isomorphism follows by Schur's lemma, and the last isomorphism follows by choosing a basis v^i for V^* acting on V by $v^i(v_j) = \delta_{ij}$ — it is a non-canonical isomorphism. Since V^* can be equipped with the structure of an A-module, it follows that $V^* \cong V$ is also an A-module isomorphism. $\square$

Let

$$V = \bigoplus_{W_i \in \mathrm{Irr}(A)} V_{W_i} \otimes_k W_i,$$

where V_{W_i} is the multiplicity space as in the above exercise, and $U = \bigoplus_{W_i \in \mathrm{Irr}(A)} U_{W_i} \otimes_k W_i$, then

$$\mathrm{Hom}_A(V, U) \cong \bigoplus_{W_i \in \mathrm{Irr}(A)} \mathrm{Hom}_k(V_{W_i}, U_{W_i}), \quad f \longmapsto (f_i).$$

Then, f is injective, surjective, or an isomorphism if and only if all f_i are also injective, surjective, or an isomorphism, respectively.

Lemma 14. *Suppose that $V = \bigoplus_{i \in I} V_i$, where the V_i's are irreducible A-modules, and assume that we have a surjective A-module homomorphism $f : V \to U$. Then, there exists a subset $J \subset I$ such that $V_J := \bigoplus_{j \in J} V_j$ is mapped isomorphically onto U by f.*

Proof. Let $J \subseteq I$ be a maximal subset such that $f|_{V_J}$ is injective. If $f(V_J) \neq U$, then there exists $i \notin J$ such that $f(V_i) \not\subseteq f(V_J)$. Then,

$$\bar{f} : V_i \longrightarrow U/f(V_J),$$

is such that $\bar{f} \neq 0$. Then, by Schur's lemma, it follows that $\bar{f}$ is injective, and thus f is injective on $\bigoplus_{j \in J \cup \{i\}} V_j$, which is a contradiction. $\square$

Proposition 22. *Let V_i, for $1 \leq i \leq n$ be irreducible, finite-dimensional A-modules that are pairwise non-isomorphic, and let $W \subset V = \bigoplus_{i=1}^m V_i^{\oplus n_i}$, be a A-submodule. Then, W is semisimple and $W \cong \bigoplus_{i=1}^m V_i^{\oplus r_i}$. Moreover, the embedding $\phi : W \hookrightarrow V$ is given by*

$$\phi_i : V^{\oplus r_i} \longrightarrow V_i^{\oplus n_i}, \quad (v_1, \cdots, v_{r_i}) \longmapsto (v_1, \cdots, v_{r_i})X_i,$$

where X_i is a $r_i \times n_i$ matrix with linearly independent rows. That is, ϕ restricts isotypic components.

Proof. Let W be an A-submodule of $V \cong \bigoplus_{W_i \in \mathrm{Irr}(A)} V_{W_i} \otimes_k W_i$. Consider

$$f : V \longrightarrow V/W.$$

Then, by Lemma 15,

$$V/W \cong \bigoplus_{W_i} X_{W_i} \otimes_k W_i,$$

which implies that

$$W = \ker f \cong \bigoplus_{W_i} \ker(V_{W_i} \to X_{W_i}) \otimes_k W_i,$$

by the above discussion about the addivity of $\mathrm{Hom}_k(-,-)$. □

Corollary 21. *Let V be a finite-dimensional, irreducible A-module, and let $v_1, \cdots, v_n \in V$ be linearly independent. Then, for all $w_1, \cdots, w_n \in V$ that are also linearly independent, there exists $a \in A$ such that $av_i = w_i$ for each i.*

Proof. Suppose otherwise. Consider an A-module homomorphism

$$f : A \longrightarrow V^{\oplus n}, \quad a \longmapsto (av_1, \cdots, av_n).$$

It follows then that $\mathrm{Im}(f) \cong V^{\oplus r}$, where $r < n$. By Proposition 22, there exists some $u_1, \cdots, u_r \in V$ such that $(u_1, \cdots, u_r)X = (v_1, \cdots, v_n)$, and X is a $r \times n$ matrix. Since $r < n$, it follows that there is some $(q_1, \cdots, q_n)$ such that $X(q_1, \cdots, q_n)^t = 0$, since the rank of X is at most r, and so $(v_1, \cdots, v_n)(q_1, \cdots, q_n)^t = 0$, which is a contradiction. □

Theorem 25 (Density Theorem)**.**

(i) Let V be a finite-dimensional irreducible A-module. Then,

$$\rho : A \longrightarrow \mathrm{End}_k(V),$$

is surjective.

(ii) Let $V_1, \cdots, V_r$ be pairwise non-isomorphic finite-dimensional irreducible A-modules. Then,

$$\rho = (\rho_1, \cdots, \rho_r) : A \longrightarrow \prod_{i=1}^r \mathrm{End}_k(V_i),$$

is surjective.

Proof.

(i) Let $x \in \mathrm{End}_k(V)$, and $v_1, \cdots, v_n$ a basis of V. Then, by Corollary 21, there exists $a \in A$ such that $\rho(a)v_i = w_i$. It follows then that $\rho(a) = x$, and thus ρ is surjective.

(ii) Let $B_i := \mathrm{Im}(\rho_i) = \mathrm{End}_k(V_i)$ by part (i). Let $B = \mathrm{Im}\,\rho$. Then, it follows that $B \subset \prod_{i=1}^r B_i$. But as an A-module,

$$\prod_{i=1}^r \mathrm{End}_k(V_i) \cong \bigoplus_{i=1}^r V_i^{\oplus d_i}.$$

Thus, we may compose ρ with a projection,

$$A \longrightarrow \prod_{i=1}^r \mathrm{End}_k(V_i) \longrightarrow \mathrm{End}_k(V_i),$$

so that $B = \prod_{i=1}^r B_i$, and thus ρ is surjective.

□

10.3. Lecture 3, 06/10/2023

Recall from last time, we defined representations of algebras.

Definition 26 (Dual Representations)**.** Given a representation V of a unital k-algebra A, the *dual representation* is given by $V^* := \mathrm{Hom}_k(V, k)$ has the structure of a right A-module structure given by

$$(f \cdot a)(v) = f(a \cdot v), \quad f \in V^*, \ a \in A, \ v \in V.$$

Alternatively, one may think of V^* as a representation of the opposite algebra A^{op}, which has opposite multiplication $a * b = ba$.

Theorem 26. *Let $A = \prod_{i=1}^r \mathrm{Mat}_{d_i}(k)$. Then, the irreducible representations of A are*

$$V_1 = k^{d_1}, \cdots, V_r = k^{d_r},$$

each of which obtains an A-module structure by multiplication. Moreover, any finite-dimensional representation of A is isomorphic to $\bigoplus V_i^{\oplus n_i}$.

Proof. First, we show that the V_i's are irreducible. It suffices to show that for all $0 \neq v \in V_i$, we have $Av = V_i$.

Exercise 63. *Show that for all $w \in V_i$, there exists $a \in A$ such that $av = w$.*

Now, let us suppose that V is an n-dimensional representation of A. Then, V^* is an n-dimensional representation of A^{op}. Note that there is a k-algebra isomorphism

$$\mathrm{Mat}_{d_i}(k)^{\mathrm{op}} \xrightarrow{\simeq} \mathrm{Mat}_{d_i}(k), \quad X \longmapsto X^T.$$

It follows then that $A^{\mathrm{op}} \cong A$. Thus, we may think of V^* as an A-representation. Define a surjection

$$\phi : \underbrace{A \oplus \cdots \oplus A}_{n \text{ times}} \longrightarrow V^*, \quad (a_1, \cdots, a_n) \longmapsto a_1 e_1 + \cdots + a_n e_n,$$

where $\{e_1, \cdots, e_n\}$ is a basis of V^*. Indeed, ϕ is a surjection since $k \subset A$. Moreover, the map ϕ induces an injective morphism of A-representations:

$$\phi^* : V \longrightarrow (A \oplus \cdots \oplus A)^*.$$

Exercise 64. *Show that $A^n \cong (A^n)^*$ as A-representations.*

It follows then that $\phi^*(V) \cong V \cong \bigoplus_{i=1}^r V_i^{\oplus m_i}$, since $A^n \cong \bigoplus_{i=1}^r V_i^{\oplus nd_i}$.

□

10.3.1. Filtrations

We have seen in our study of algebra representations that it is not necessarily the case that every algebra representation is completely reducible. So, another way to study their representations is to consider filtrations of representations.

Definition 27. Let A be a k-algebra. A *(finite) filtration* of an A-representation V is a sequence of subrepresentations

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V.$$

Lemma 15. *Every finite-dimensional representation V of a k-algebra A admits a finite filtration*

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V,$$

such that V_i/V_{i-1} is irreducible for any i.

Proof. We induct on $\dim_k V$. Let $V_1 \subset V$ be an irreducible A-representation. Consider $U = V/V_1$, which defines an A-representation. Then, by induction, we obtain a filtration for U given by

$$0 \subset U_1 \subset \cdots \subset U_m \subset U,$$

such that U_i/U_{i-1} are irreducible. There is a projection $\pi : V \to V/V_1 = U$. From this, one may produce a filtration for V by setting $V_i := \pi^{-1}(U_i)$. Note here that $V_i/V_{i-1} \cong U_i/U_{i-1}$.

Exercise 65. *Fill in the details.*

Proof. We induct on U. If U is simple, then we are done. Suppose that U is non-simple. Then, we have $0 \subset U_1 \subset U$. If U/U_1 is not irreducible, then put another submodule $0 \subset U_1 \subset U_2 \subset U$ until we have a filtration

$$0 \subset U_1 \subset \cdots \subset U_m \subset U,$$

such that U_i/U_{i-1} is simple. Now, let us set $V_i = \pi^{-1}(U_i)$, where $\pi : V \to V/V_1 = U$ is the projection map. Then, we have that

$$V_i/V_{i-1} = \pi^{-1}(U_i)/\pi^{-1}(U_{i-1}) \cong U_i/U_{i-1},$$

and thus we have a filtration

$$0 \subset V_1 \subset \cdots \subset V_m \subset V,$$

such that the factors V_i/V_{i-1} are irreducible. □

□

10.3.2. Finite-Dimensional Algebras

Let A be a finite-dimensional k-algebra.

Definition 28. The *radical* of a finite-dimensional k-algebra A is the set of all $a \in A$ such that $aV_i = 0$, for any irreducible A-representation V_i. Denote this by $\mathrm{Rad}(A)$.

Proposition 23. $\mathrm{Rad}(A)$ *is a two-sided ideal of A.*

Proposition 24.

(i) Let I be a nilpotent two-sided ideal in A — i.e. $I^n = 0$ for some n. Then, $I \subset \mathrm{Rad}(A)$.

(ii) Rad(A) *is the maximal two-sided nilpotent ideal in* A*.*

Proof.

(i) Let V be an irreducible A-representation. Let $0 \neq v \in V$. Then, Iv us a subrepresentation of V. But by irreducibility of V, we have that $Iv = V$, since $v \neq 0$. Thus, there exists some $x \in I$ such that $xv = v$, which implies that $x^n v = v = 0$, which is a contradiction.

(ii) Let

$$0 = A_0 \subset A_1 \subset \cdots \subset A_n = A,$$

be a filtration of the regular representation of A, such that A_i/A_{i-1} is irreducible for $1 \leq i \leq n$. Such a filtration exists by Lemma 15. Then, by definition of the radical of A, we have:

$$\operatorname{Rad}(A) A_i/A_{i-1} = 0.$$

It therefore follows that $(\operatorname{Rad}(A))^n A = 0$. But since A has an identity, this then implies that $(\operatorname{Rad}(A))^n = 0$.

□

Example 31. *Let* $A := k[x]/(x^n)$*, which is a finite-dimensional commutative* k*-algebra. Then,* A *has a unique irreducible representation given by* $xv = 0$*, for all* $v \in k$*. Here, we have that*

$$\operatorname{Rad}(A) = (x).$$

Thus,

$$A/\operatorname{Rad}(A) \cong k.$$

Chapter 11

Week Eleven

11.1. Lecture 1, 09/10/2023

Today, we shall prove the following theorem:

Theorem 27. *A finite-dimensional k-algebra A has only finitely many irreducible representations V_i (up to isomorphism). Moreover, these irreducible representations are finite-dimensional, and*

$$A/\operatorname{Rad}(A) \cong \prod_i \operatorname{End}_k(V_i) \cong \prod_i \operatorname{Mat}_{\dim_k V_i}(k).$$

Proof. We begin by first showing that every irreducible A-module is finite-dimensional. Let V be an irreducible A-module. Let $0 \neq v \in V$. Then, Av is a subrepresentation of V. But by the irreducibility of V, then it follows necessarily that $Av = V$, since $Av \neq 0$ since $v \neq 0$. Since A is finite-dimensional, it follows that V is finite-dimensional.

We now wish to show that there are only finitely many irreducible A-modules. Suppose that $V_1, \cdots, V_r$ are non-isomorphic irreducible A-modules. Then, by the density theorem, the map $A \to \prod_{i=1}^r \operatorname{End}(V_i)$ is surjective, and it thus follows that

$$r \leq \sum_{i=1}^r \dim_k \operatorname{End}(V_i) \leq \dim_k A,$$

where the inequaltiy follows by the surjectivity of the map. It thus follows that there are at most $\dim_k A$ many irreducible A-modules.

Let $V_1, \cdots, V_k$ be all non-isomorphic representations of A. Then, again by the density theorem there is a surjective k-algebra homomorphism

$$A \longrightarrow \prod_{i=1}^k \operatorname{End}(V_i),$$

whose kernel is $\operatorname{Rad}(A)$. It follows by the first isomorphism theorem that

$$A/\operatorname{Rad}(A) \cong \prod_{i=1}^k \operatorname{End}_k(V_i).$$

$\square$

Corollary 22. *Let $V_1, \cdots, V_k$ be all the irreducible A-modules up to isomorphism. Then,*

$$\sum_{k=1}^{k} (\dim_k V_i)^2 \leq \dim_k A.$$

Example 32. *Let A be the subalgebra of $\mathrm{Mat}_n(k)$ consisting of the upper triangular $n \times n$ matrices. Then, we claim that $\mathrm{Rad}(A)$ is the subalgebra of strictly upper triangular matrices with 0's on the diagonal. An irreducible A-module is given by*

$$\rho : A \longrightarrow \mathrm{End}_k(k) \cong k,$$

sending any element of A to any entry on its diagonal. That is $\rho : (x_{ij})_{i \geq j} \mapsto x_{kk}$, $1 \leq k \leq n$. These give all the irreducible representations. It follows then that

$$A/\operatorname{Rad}(A) \cong k^n.$$

Definition 29. A finite-dimensional k-algebra A is *semisimple* if $\mathrm{Rad}(A) = 0$.

Proposition 25. *For a finite-dimensional k-algebra A, the following are equivalent:*

(i) *A is semisimple,*

(ii) *Let $V_1, \cdots, V_k$ be all the irreducible A-modules up to isomorphism. Then,*

$$\sum_{i=1}^{k} (\dim_k V_i)^2 = \dim_k A,$$

(iii) *Let $d_i := \dim_k V_i$. Then,*

$$A \cong \prod_{i=1}^{k} \mathrm{Mat}_{d_i}(k),$$

(iv) *Every finite-dimensional A-module is completely reducible — that is, isomorphic to a direct sum of irreducible representations,*

(v) *Every A-submodule U of a finite-dimensional A-module V has a complementary A-submodule W — i.e. $V \cong U \oplus W$,*

(vi) *A is a completely reducible A-module.*

Proof. The (i) $\Longleftrightarrow$ (ii) direction follows from Theorem 27. Theorem 27, 26 also implies (ii) $\Longleftrightarrow$ (iii). (iii) $\Longrightarrow$ (iv) follows also from Theorem 26.

(iv) $\Longrightarrow$ (v). Let V be a finite-dimensional representation of A. Let $U \subset V$ be a strict A-submodule. Let

$$M := \{\text{subrepresentations } X \subseteq V \text{ such that } X \cap U = \{0\}\}.$$

Since $\{0\} \in M$, it follows that $M \neq \emptyset$. Let $W \in M$ be of largest possible dimension.

We now claim that $V = U \oplus W$. Indeed, $U \oplus W \subseteq V$ by construction. Suppose that $U \oplus W \neq V$. Then, by the complete reducibility of V, there exists an element a simple A-submodule $S \subseteq V$ such that $S \nsubseteq U \oplus W$. It follows then that $S \cap (U \oplus W) = \{0\}$, and thus $S \oplus W \in M$, which is a contradiction, since W is of maximal dimension by assumption.

Exercise 66. *Show that (v) $\Longrightarrow$ (iv).*

Proof. Suppose that every A-submodule U of a finite-dimensional A-module V has a complementary A-submodule W such that $V = U \oplus W$. Then, if U and W are simple we are done.

Thus, suppose that U is not simple. Then, every submodule of U has a complementary submodule as well. That is $U = U_1 \oplus U_2$. Repeat this process for each of the factors in the decomposition of U until we can no longer find non-trivial submodules — that is, we have only irreducible factors. Repeat this for W as well, and we find that V decomposes as a direct sum of irreducible A-submodules. □

(vi) $\implies$ (v) is trivial. It thus remains to show (vi) $\implies$ (iii). Suppose that

$$A \cong \bigoplus_{i=1}^{m} V_i^{\oplus n_i},$$

where the V_i's are irreducible A-modules. Note that a priori, we do not know if the factors appearing in the decomposition of A is a complete list of irreducible A-representations.

Since A is itself an A-module, we may consider

$$\mathrm{End}_A(A) \cong \mathrm{Hom}_A(A, A) \cong \prod_{i=1}^{m} \mathrm{Mat}_{n_i}(k),$$

which follows from Schur's lemma, which says that we can only map isotypic components to isotypic components. We note that there is an isomorphism

$$\mathrm{End}_A(A) \cong A^{\mathrm{op}} \cong A,$$

and thus $A \cong \prod_{i=1}^{m} \mathrm{Mat}_{n_i}(k)$, which thus proves the result.

Exercise 67. *Show that* $\mathrm{End}_A(A) \cong A^{\mathrm{op}}$.

Proof. Every A-module endomorphism of A is given by right-multiplication of elements of A. Moreover, for any map $\varphi_a \in \mathrm{End}_A(A)$ mapping $\varphi_a : x \mapsto xa$, we have

$$(\varphi_a \circ \varphi_b)(x) = \varphi_a(xb) = xba = \varphi_{ba}(x).$$

Defining a product

$$a *_{\mathrm{End}_A(A)} b := \varphi_a \circ \varphi_b,$$

we thus have the relation:

$$a *_{\mathrm{End}_A(A)} b = ba,$$

which is precisely the way that the opposite algebra is defined, and thus $\mathrm{End}_A(A) = A^{\mathrm{op}}$. □

□

11.1.1. Characters of A-modules

Let A be a k-algebra, and V a finite-dimensional representation of A with $\rho : A \to \mathrm{End}_k(V)$. The *character* of V is

$$\chi_V : A \longrightarrow k,$$

defined by

$$\chi_V(a) = \mathrm{tr}(\rho(a)).$$

Let
$$[A, A] := \mathrm{Span}_k\{[x, y] = xy - yx\},$$
be the *commutator* of A. This induces a map
$$\chi_V : A/[A, A] \longrightarrow k.$$
That is, χ_V factors through $A/[A, A]$.

Exercise 68. *Show that if $W \subset V$ are finite-dimensional representations of A, then*
$$\chi_V = \chi_W + \chi_{V/W}.$$

Proof. Choose a basis for V such that the first $n = \dim_k W$ vectors form a basis for W. In this basis, every element of $\rho(a)$ has the form
$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$
where A and C are restrictions of $\rho(a)$ to W and V/W, respectively, and B is a projection of W. Indeed, we have:
$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}\begin{pmatrix} w \\ 0 \end{pmatrix} = \begin{pmatrix} Aw \\ 0 \end{pmatrix}.$$
For some element $u \in V/W$, we may write $u = v + w$, from which we obtain:
$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}\begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} Aw + Bv \\ Cv \end{pmatrix},$$
which implies that $Aw + Bv \in W$. Thus, we have that
$$\mathrm{tr}(\rho(a)) = \mathrm{tr}(\rho(a)|_W) + \mathrm{tr}(\rho(a)|_{V/W}).$$
That is, $\chi_V = \chi_W + \chi_{V/W}$, since the trace is independent of our choice of basis. □

11.2. Lecture 2, 12/10/2023

11.3. Lecture 3, 13/10/2023

Lemma 16. *Let W be a finite-dimensional A-module. Then:*

(i) Any homomorphism $\theta : W \to W$ is either an isomorphism or nilpotent.

(ii) If $\theta_s : W \to W$ for $s = 1, \cdots, n$ is a nilpotent homomorphism, then so is $\theta = \theta_1 + \cdots + \theta_n$.

Proof.

(i)

(ii) By inducting on n, if $n = 1$, then there is nothing to prove. Suppose that θ is not nilpotent. Then, by part (i), θ is an isomorphism. It follows then that $1 = \theta^{-1}\theta_1 + \cdots + \theta^{-1}\theta_n$, and thus $\theta^{-1}\theta_i$ is nilpotent, because it is not an isomorphism for each $i = 1, \cdots, n$. Then,
$$1 - \theta^{-1}\theta_1 = \theta^{-1}\theta_2 + \cdots \theta^{-1}\theta_n,$$
and on the left hand side we have an isomorphism, but on the right hand side we have a nilpotent element by the inductive step, which is a contradiction.

$\square$

Back to Proof of Theorem. Recall that $\theta_s : V_1 \to V_1$, for $s = 1, \cdots, n$, and $\sum_{s=1}^n \theta_s = 1$. Then, by Lemma 16, it follows that there exists some s such that θ_s is an isomorphism. We can assume that $s = 1$. Recall that θ_s is defined by the composition

$$V_1 \xrightarrow{i_1} V \xrightarrow{p'_1} V'_1 \xrightarrow{i'_1} V \xrightarrow{p_1} V_1,$$

where i_1, i'_1 are injections, and p_1, p'_1 are projection maps. Sinec θ_1 is an isomorphism, then this implies that $p_1 \circ i'_1$ is surjective, and further $p'_1 \circ i_1$ is injective. From this, we may conclude that $V'_1 \cong \ker(p_1 \circ i'_1) \oplus \mathrm{Im}(p'_1 \circ i_1)$. But since V'_1 is indecomposable, it follows then that $V_1 \cong \mathrm{Im}(p'_1 \circ i_1) \cong V_1$. It thus follows that both of the maps $p'_1 \circ i_1 : V_1 \to V'_1$ and $p_1 \circ i'_1 : V'_1 \to V_1$ are isomorphisms. Now, let

$$W = \bigoplus_{j>1} V_j, \quad W' = \bigoplus_{j>1} V'_j.$$

Then, $V = V_1 \oplus W = V'_1 \oplus W'$. Define a map

$$h : W \xrightarrow{i} V = V'_1 \oplus W' \xrightarrow{p} W',$$

where i is an injection, and p is a projection map.

Claim: h is an isomorphism.

Suppose that $h(w) = 0$, for some $w \in W$. Then, $p(w) = 0$, and so $w \in V'_1$. Consider the projection $p_1 : V_1 \oplus W \to V_1$. Then, if $p(w) = 0$, then $p_1 \circ i'_1(w) = 0$, and since $p_1 \circ i'_1$ is an isomorphism, it follows that $w = 0$. Thus, $\ker h = \{0\}$, and thus h is injective. Since $\dim W = \dim W'$, h is an isomorphism. Then, by induction, we are done. $\square$

Remark 20. In general, Krull-Schmidt might fail for infinite-dimensional representations. See exercises in (Etingof et al.)

11.3.1. Representations of Tensor Products of Algebras

Let A, B, be two k-algebras. Then, their tensor product over k is given by:

$$A \otimes_k B,$$

obtains a k-algebra structure by

$$(a_1 \otimes_k b_1) \cdot (a_2 \otimes_k v_2) = a_1 a_2 \otimes_k b_1 b_2,$$

for $a_1, a_2 \in A$, and $b_1, b_2 \in B$. It is a unital k-algebra, with unit given by $1 \otimes_k 1$.

Exercise 69. *Show that there is a k-algebra isomorphism:*

$$\mathrm{Mat}_m(k) \otimes_k \mathrm{Mat}_n(k) \cong \mathrm{Mat}_{mn}(k).$$

Proof. For $M \in \mathrm{Mat}_m(k)$, and $N \in \mathrm{Mat}_n(k)$, define

$$M \otimes N := \begin{pmatrix} M_{11}N & \cdots & M_{1m}N \\ \vdots & \ddots & \vdots \\ M_{m1}N & \cdots & M_{mm}N \end{pmatrix},$$

and the isomorphism follows. □

Let V be an A-module, and W a B-module. Then, $V \otimes_k W$ has the structure of an $A \otimes_k B$-module. The module structure is given by:

$$(a \otimes b)(v \otimes w) = av \otimes bw,$$

for $v \in V$, $w \in W$, $a \in A$, $b \in B$.

Theorem 28.

(i) Let V be an irreducible finite-dimensional A-module, and W an irreducible finite-dimensional B-module. Then, $V \otimes_k W$ is an irreducible finite-dimensional representation of $A \otimes_k B$.

(ii) Any irreducible finite-dimensional $A \otimes_k B$-module is of the form $V \otimes_k W$, for a unique irreducible A-module V, and irreducible B-module W.

Proof.

(i) By the density theorem, the maps $A \to \mathrm{End}_k(V)$ and $B \to \mathrm{End}_k(W)$ are surjective. It follows therefore that

$$A \otimes_k B \to \mathrm{End}_k(V) \otimes_k \mathrm{End}_k(W) \cong \mathrm{End}_k(V \otimes_k W),$$

is surjective. Thus, $V \otimes_k W$ is irreducible.

(ii) Let M be a finite-dimensional irreducible of $A \otimes_k B$. Then, we have maps $\varphi_A :: A \to \mathrm{End}_k(M)$, and $\varphi_B : B \to \mathrm{End}_k(M)$. Let $A' := \mathrm{Im}(\varphi_A)$, and $B' := \mathrm{Im}(\varphi_B)$, both of which are finite-dimensional k-algebras. It follows then that M is a representation of $A' \otimes B'$. So, we can assume that A and B are both finite-dimensional. Thus, $A/\operatorname{Rad}(A)$ and $B/\operatorname{Rad}(B)$ are both isomorphic to matrix algebras.

Claim: $\operatorname{Rad}(A \otimes_k B) = \operatorname{Rad}(A) \otimes_k B + A \otimes_k \operatorname{Rad}(B) =: J$.

First J is a nilpotent two-sided ideal in $A \otimes_k B$ (Exercise). Thus, $J \subset \operatorname{Rad}(A \otimes_k B)$. It is left as an exercise to show that there is an isomorphism:

$$(A \otimes_k B)/J \cong A/\operatorname{Rad}(A) \otimes_k B/\operatorname{Rad}(B).$$

Proof of Exercise. □

Both factors in the right hand side of the isomorphism are semisimple. Thus, its tensor product is semisimple, and $(A \otimes_k B)/J$ is semisimple, and thus $\operatorname{Rad}(A \otimes_k B) \subseteq J$. Since M is a finite-dimensional, irreducible representation of the semisimple algebra $(A \otimes_k B)/\operatorname{Rad}(A \otimes_k B)$, it follows then that $M \cong V \otimes_k W$, where V is an irreducible $A/\operatorname{Rad}(A)$-module, and W an irreducible $B/\operatorname{Rad}(B)$-module.

□

Remark 21. The above theorem may fail for infinite-dimensional representations. For instance, consider $A = B = \mathbb{C}(x)$, which is an infinite-dimensional$\mathbb{C}$-algebra. Then, $V = W = \mathbb{C}(x)$ is an irreducible module over A, and B, respectively. Then, $V \otimes_k W = \mathbb{C}(x) \otimes_{\mathbb{C}} \mathbb{C}(y)$ is not irreducible. See (Etingof et al.)

Chapter 12

Week Twelve

12.1. Lecture 1, 16/10/2023

12.1.1. Double Centraliser Theorem

Theorem 29 (Double Centraliser Theorem). *Let A and B be two k-subalgebras of the algebra* $\mathrm{End}_k(E)$*, were E is a finite-dimensional k-vector space, such that A is semisimple, and $B = \mathrm{End}_A(E)$. Then,*

(i) $A = \mathrm{End}_E(B)$. That is, the centraliser of A in A is A.

(ii) B is also semisimple.

(iii) As an $A \otimes_k B$-module, E decomposes as

$$E \cong \bigoplus_{i \in I} V_i \otimes W_i,$$

where the V_i's are all irreducible A-modules, and W_i's are all irreducible B-modules. In particular, there is a natural bijection between irreducible A-modules and irreducible B-modules.

Proof. Suppose that A is semisimple. Then,

$$E \cong \bigoplus_{i \in I} V_i \otimes W_i,$$

for V_i an irrep of A, and $W_i = \mathrm{Hom}_A(V_i, E)$, which is the multiplcity space of V_i in E Since $A \subset \mathrm{End}_k(E)$, we have that $W_i \neq 0$. It follows then that I indexes the set of all irreps of A. This implies that

$$A \cong \prod_{i \in I} \mathrm{End}_k(V_i).$$

Now,

$$B = \mathrm{Hom}_A(E, E) \cong \prod_{i \in I} \mathrm{End}_k(W_i),$$

by Schur's lemma. So, B is semisimple, and

$$\mathrm{End}_B(E) = \prod_{i \in I} \mathrm{End}_k(V_i).$$

□

12.1.2. Schur-Weyl Duality

Suppose that $E = V^{\otimes n}$, where V is a finite-dimensional $\mathbb{C}$-vector space. We wish to apply the double centraliser theorem to this. Let

$$A := \mathrm{Im}\left(\mathbb{C}[\mathfrak{S}_n] \to \mathrm{End}_{\mathbb{C}}(E)\right).$$

Then, A is semisimple. Let $B := \mathrm{End}_A(E)$. We have the following theorem:

Theorem 30. *B is the image of $\mathcal{U}(\mathfrak{gl}(V))$ under its natural action on $E = V^{\otimes n}$, where*

$$\mathcal{U}(\mathfrak{gl}(V)) := T(\mathfrak{gl}(V))/\langle X \otimes Y - Y \otimes X - XY + YX \rangle,$$

is the universal enveloping algebra *of the Lie algebra $\mathfrak{gl}(V)$. That is, B is generated by elements of the form*

$$\Delta_n(b) := b \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes b \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes b,$$

where each of the summands in $\Delta_n(b)$ have n factors.

Lemma 17.

(i) For all finite-dimensional vector spaces U over $\mathbb{C}$, $\mathrm{Sym}^n U$ is spanned by elements of the form $u \otimes \cdots \otimes u$, for some $u \in U$.

(ii) For any $\mathbb{C}$-algebra A, its symmetric powers $\mathrm{Sym}^n A$ is generated by elements of the form $\Delta_n(a)$, for $a \in A$.

Proof.

(i) This follows from the fact that $\mathrm{Sym}^n(U)$ is an irreducible $\mathrm{GL}(U)$-representation (we will not prove this). It is left as an exercise to show this, if you are willing (see Etingof et al., Problem 4.12.3). It follows then that $\mathrm{Span}_{\mathbb{C}}(\{u \otimes \cdots \otimes u : u \in U\})$ is a non-zero subrepresentation.

(ii) By the fundamental theorem on symmetric functions, there exists a polynomial p with rational coefficients such that $p(H_1(x), \cdots, H_n(x)) = x_1 \cdots x_n$, where $x = (x_1, \cdots, x_n)$, and $H_m(x) := \sum_i x_i^m$. This tells us then that

$$p(\Delta_n(a), \Delta_n(a^2), \cdots, \Delta_n(a^n)) = a \otimes \cdots \otimes a, \quad a \in A.$$

Then, using (i), we are done.

□

The above lemma then implies the theorem. We have:

$$B = \mathrm{End}_A(V^{\otimes n}) = \mathrm{Sym}^n \mathrm{End}_k(V).$$

We may now apply the double centraliser theorem, which implies the following:

Theorem 31 (Schur-Weyl Duality)**.**

(i) The image A of $\mathbb{C}[\mathfrak{S}_n]$ in $\mathrm{End}(V^{\otimes n}$, and the image of B of $\mathcal{U}(\mathfrak{gl}(V))$ in $\mathrm{End}(V^{\otimes n})$ are centralisers of each other.

(ii) Both A and B are semisimple. In particular, $V^{\otimes n}$ is a semisimple $\mathcal{U}(\mathfrak{gl}(V))$-module.

(iii) We have a decomposition of $A \otimes B$-representations by:

$$V^{\otimes n} = \bigoplus_{\lambda \text{ a partition of } n} V_\lambda \otimes L_\lambda,$$

where V_λ are the irreducible representations of $\mathfrak{S}_n$ that we have studied before, and L_λ is either 0 *or distinct (i.e. non-isomorphic) irreps of $\mathcal{U}(\mathfrak{gl}(V))$.*

Remark 22. We may replace $\mathcal{U}(\mathfrak{gl}(V))$ by $\mathrm{GL}(V)$. Then, Schur-Weyl duality as stated still holds. We can do this because of the following proposition below.

Proposition 26. *The image of* $\mathrm{GL}(V)$ *in* $\mathrm{End}_{\mathbb{C}}(V^{\otimes n})$ *spans* B.

Proof. Let $B' := \mathrm{Span}(\{g^{\otimes n} : g \in \mathrm{GL}(V)\})$. Let $b \in \mathrm{End}_{\mathbb{C}}(V)$, then we claim that $b^{\otimes n} \in B'$. If we can show this then we are done by Lemma 17. Observe that $t\,\mathrm{Id} + b \in \mathrm{GL}(V)$ for all but finitely many t. Consider a linear functional $f : \mathrm{End}_{\mathbb{C}}(V^{\otimes n}) \to \mathbb{C}$ such that $f|_{B'} = 0$. This implies that

$$f((t\,\mathrm{Id} + b)^{\otimes n}) = 0,$$

for all but finitely many t, where f is a polynomial in t. Therefore, $f = 0$, since a polynomial can only have finitely many zeroes. $\square$

12.1.3. The Characters of L_λ

Recall that L_λ are the irreducible representations of $\mathrm{GL}(V)$. Let $\dim V = n$, and $g \in \mathrm{GL}(V)$, and that $x_1, \cdots, x_n$ are eigenvalues of g on V.

Theorem 32 (Weyl Character Formula)**.** *The representations $L_\lambda = 0$ if and only if $N < k$, where k is the number of parts of $\lambda = L(\lambda)$. If $N \geq k$, then the character of L_λ is the Schur polynomial*

$$s_\lambda(x) := \frac{\det(x_i^{\lambda_j + N - j})}{\Delta(x)}, \quad \Delta(x) = \prod_{1 \leq i < j \leq N} (x_i - x_j).$$

In particular,

$$\dim_{\mathbb{C}} L_\lambda = \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + (j-1)}{j - i}.$$

Exercise 70. *For those who have learned the Weyl character formula before, think about why this is equivalent to the Weyl character formula from the Lie algebra course.*

Proposition 27.

$$\prod_m (x_1^m + \cdots + x_N^m)^{i_m} = \sum_{\lambda \text{ partitions } n \leq N} \chi_\lambda(C_{\hat{i}}) s_\lambda(x),$$

where $\chi_\lambda(C_{\hat{i}})$ are the characters of the irreducible $\mathbb{C}[\mathfrak{S}_n]$-modules V_λ.

Idea. Given $s \in C_i$,

$$\mathrm{tr}_{V^{\otimes n}}(g^{\otimes n} s) = \prod_m (x_1^m + \cdots + x_N^m)^{i_m}.$$

$\square$

Next time, we will talk more about finite-dimensional algebras. For now, we give some examples. Consider the universal enveloping algebra of $\mathfrak{sl}_2$, given by:

$$\mathcal{U}(\mathfrak{sl}_2) = \frac{\mathbb{C}[e, h, f]}{\langle he - eh - 2e, hf - fh + 2f, ef - fe - h \rangle}.$$

Then:

Theorem 33. *$\mathcal{U}(\mathfrak{sl}_2)$ has one irreducible representation V_d of dimension d (up to isomorphism), where V_d is the space of homogeneous polynomials of two variables x, y of degree $d-1$. That is,*

$$V_d = \{x^{d-1}, x^{d-2}y, \cdots, y^{d-1}\}.$$

The generators act by differential operators:

$$\rho(h) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \quad \rho(e) = x\frac{\partial}{\partial y}, \quad \rho(f) = y\frac{\partial}{\partial x}.$$

12.2. Lecture 2, 19/10/2023

We wish to study the structure of finite-dimensional algebras.

12.2.1. Lifting of Idempotents

Let k be an algebraically closed field. Recall the notion of an idempotent. Let A be a k-algebra and $I \subset A$ a nilpotent two-sided ideal.

Proposition 28. *Let $e_0 \in A/I$ be an idempotent — i.e. $e_0^2 = e_0$. Then, there exists an idempotent $e \in A$ such that the image of e in A/I is e_0. This idempotent e is called a* lift *of e_0 to A, and it unique up to conjugacy by an element of $1 + I$.*

Remark 23. The last sentence makes sense, since I is nilpoten, and thus $1 + I$ is invertible, which allows us to conjugate the idempotent.

Proof. Suppose first that $I^2 = 0$. Let

$$\pi : A \longrightarrow A/I,$$

and let $e_1 \in \pi^{-1}(e_0)$, which a priori is not necessarily an idempotent. By definition $e_1 - e_0 \in I$. Now, we can define a new element

$$a = e_1^2 - e_1.$$

There is some $x \in I$ such that $e_1 = e_0 + x$. Then, $e_1^2 - e_1 = e_0^2 + e_0x + xe_0 + x^2 - e_0 - x \in I$. Now, we have

$$\begin{aligned}
e_0a - ae_0 &= e_0(e_1^2 - e_1) - (e_1^2 - e_1)e_0 \\
&= e_0(e_0^2 + e_0x + xe_0 + x^2 - e_0 - x) - (e_0^2 + e_0x + xe_0 + x^2 - e_0 - x)e_0 \\
&= e_0^2x + e_0xe_0 - e_0^2 - e_0x - e - 0xe_0 - xe_0^2 + e_0^2 + xe_0 \\
&= (e_0^2 - e_0)x - x(e_0^2 - e_0) \\
&= \underbrace{(e_0 - e_0)}_{\in I}\underbrace{x}_{\in I} - \underbrace{x}_{\in I}\underbrace{(e_0 - e_0)}_{\in I} \\
&= 0.
\end{aligned}$$

Now, let $e := e_1 + b$, for some $b \in I$. We want to show that $e^2 = e$, if and only if $a = b - be_0 - e_0b$. Let us set $b = (1 - 2e_0)a$. Then, one checks that $a = b - be_0 - e_0b$.

Exercise 71. *Verify this in your own time. Just remember that our k-algebra is not necessarily commutative.*

So, we have $e = e_1 + (1 - 2e_0) \cdot (e_1^2 - e_1)$, which is an idempotent. Now, we wish to show that this idempotent is unique up to conjugacy. Suppose that $e' = e + c$, for some $c \in I$ is another idempotent. Then,

$$(e')^2 - e' = e^2 + ec + ce - e - c = 0,$$

if and only if $ec + ce = c$. This is equivalent to the condition that $ece = 0$, and $(1-e)c(1-e) = 0$. That is, given another idempotent e', we get these two relations. So, we have that:

$$c = ec(1-e) + (1-e)ce = e[e,c] - [e,c]e,$$

where $[-,-]$ is the commutator. It follows thus that

$$e + c = (1 - [e,c])e(1 + [e,c]) = (1 - [e,c])e(1 - [e,c])^{-1},$$

which proves the proposition when $I^2 = 0$. Now, we will show by induction on k that there exists a lift e_k of $e_{k-1} \in A/I^k$ to A/I^{k+1}, which is unique up to conjugation by an element in $1 + I^k$ (because $I^N = 0$ for some N). Suppose that this is true for $k = m-1$. Then, for $k = m$, consider a map $A/I^{m+1} \to A/I^m$, and observe that $(I^m)^2 = 0$ in A/I^{m+1}. This completes the proof. □

Definition 30. A *complete system of orthogonal idempotents* in a unital k-algebra B is a collection of idempotents $e_1, \cdots, e_n \in B$ such that $e_i \cdot e_j = 0$ if $i \neq j$, and $e_1 + \cdots + e_n = 1$.

Corollary 23. *Let $e_{01}, \cdots, e_{0m}$ be a complete system of orthogonal idempotents of A/I. Then, there exists a lift of a complete system of orthogonal idempotents $e_1, \cdots, e_m$ such that each e_i is equal to e_{0i} in the image of A/I.*

Proof. We induct on m. Suppose $m = 2$. Then, by Proposition 28, we can find $e_1 \in A$ such that $e_1^2 = e_1$ such that the image of e_1 in A/I is e_{01}. Let $e_2 = 1 - e_1$. Then, $e_2^2 = e_2$, and $e_1 e_2 = 0$, and $e_1 + e_2 = 1$. Suppose now that $m > 2$. Let $e_1 \in A$ be such that $e_1^2 = e_1$, and the image of e_1 in A/I is e_{01}. Let us consider

$$A' := (1-e_1)A(1-e_1) \subset A.$$

This is a subalgebra, with unit given by $1 - e_1$ in A'. We are given that $e_{02}, \cdots, e_{0m}$ is a complete system of orthogonal idempotents in for $A'/(1-e_1)I(1-e_1)$. By induction, there exists idempotents $\widetilde{e_2}, \cdots, \widetilde{e_m}$ in A' such that $\widetilde{e_2} + \cdots + \widetilde{e_m} = 1 - e_1$, and $\widetilde{e_i}\widetilde{e_j} = 0$ for $i \neq j$. So, it follows then that $e_1, \widetilde{e_2}, \cdots, \widetilde{e_m}$ is a complete system of orthogonal idempotents for A. □

12.2.2. Projective Covers

Theorem 34. *Let A be a k-algebra and P a left A-module. The following properties of P are equivalent:*

(i) Consider the morphisms of A-modules:

$$\begin{tikzcd} & P \arrow[d, "\nu"] \\ M \arrow[r, "\alpha"] & N \end{tikzcd}$$

Then, there exists a map $u : P \to M$ such that $\alpha \circ u = \nu$.

(ii) Any surjective homomorphism $\alpha : M \to P$ splits. That is, there exists $u : P \to M$ such that $\alpha \circ u = \mathrm{id}$.

(iii) There exists another A-module Q such that $P \oplus Q$ is a free A-module.

(iv) The functor $\mathrm{Hom}_A(P, -)$ is exact.

Definition 31. A module satisfying the above conditions is called a *projective module.*

Theorem 35. *Let A be a finite-dimensional k-algebra with simple A-modules $M_1, \cdots, M_n$. Then,*

(i) *For each $i = 1, \cdots, n$, there exists a unique indecomposable finitely generated projective A-module P_i such that*

$$\dim \operatorname{Hom}_A(P_i, M_j) = \delta_{ij}.$$

(ii) *There is an A-module isomorphism:*

$$A \cong \bigoplus_{i=1}^{n} P_i^{\oplus \dim M_i}.$$

(iii) *Any indecomposable finitely generated projective A-module is isomorphic to P_i for some i.*

Definition 32. These projective modules P_i, as seen in the above theorem, is called a *projective cover* of M_i.

12.2.3. Lecture 3, 20/10/2023

Proof of Theorem 35. Recall that

$$A/\operatorname{Rad}(A) \cong \prod_{i=1}^{n} \operatorname{End}_k(M_i),$$

and $\operatorname{Rad}(A)$ is a nilpotent two-sided ideal. Let $e_{ij}^0 := E_{jj}^i$, for $1 \leq i \leq n$, and $1 \leq j \leq \dim M_i$ be a complete system of orthogonal idempotents of the matrix algebra $A/\operatorname{Rad}(A)$. Then, by Corollary 23, we can left $\{e_{ij}^0\}$ to a complete set of orthogonal idempotents e_{ij} to A. Now, let $P_{ij} := Ae_{ij}$, which gives subalgebras of A.

Exercise 72. *Show that this implies that*

$$A = \bigoplus_{i,j} Ae_{ij}.$$

Thus, we have that

$$A = \bigoplus_{i,j} Ae_{ij} = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{\dim M_i} P_{ij},$$

as A-modules. It follows then that the P_{ij} are projective. So, we have:

$$\operatorname{Hom}_A(P_{ij}, M_k) = \operatorname{Hom}_A(Ae_{ij}, M_k) \cong e_{ij} M_k.$$

The last isomorphism is left as an exercise. In particular, defining a homomorphism $\operatorname{Hom}_A(Ae_{ij}, M_k) \to e_{ij} M_k$ is the same as finding an element of $e_{ij} M_k$. This then implies that

$$\dim \operatorname{Hom}_A(P_{ij}, M_k) = \delta_{ik}.$$

Now, $P_{ij} \cong P'_{ij}$, because e_{ij} is conjugate to e'_{ij} by an element in $A^\times$ — the invertible elements of A, by Proposition 28.

We write $P_i := P_{ij}$. It now remains to show that the P_i's are indecomposable. Suppose otherwise. Then, if $P_i = Q_1 \oplus Q_2$, then either $\operatorname{Hom}_A(Q_1, M_k) = 0$ for all k, or $\operatorname{Hom}_A(Q_2, M_k) = 0$ for all k. Otherwise, the dimension of the hom space will be at least 2. It follows then that $Q_1 = 0$, or $Q_2 = 0$. So, P_i is actually indecomposable.

Now, any indecomposable projective module occurs in the decomposition of A. □

Proposition 29. *Let N be any finite-dimensional representation of A. Then,*

$$\dim \mathrm{Hom}_A(P_i, N) = [N : M_i],$$

where the right hand side denotes the multiplicity of occurrence of M_i in the Jordan-Holder series of N. That is, for a series

$$0 \subset N_1 \subset \cdots \subset N_m \subset N,$$

such that N_i/N_{i+1} are irreducible, then

$$[N : M_i] := |\{j : N_j/N_{j+1} \cong M_i\}|.$$

Proof. If $N = M_j$, then $[M_j : M_i] = \delta_{ij} = \dim \mathrm{Hom}_A(P_i, M_j)$. If the sequence

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0,$$

is exact, then

$$\mathrm{Hom}_A(P_i, N_1) \longrightarrow \mathrm{Hom}_A(P_i, N_2) \longrightarrow \mathrm{Hom}_A(P_i, N_3) \longrightarrow 0,$$

is also exact, by definition of projectivity of P_i. □

12.2.4. The Cartan Matrix of a Finite-Dimensional Algebra

Let A be a finite-dimensional k-algebra, and M_i and P_i a collection of irreducible A-modules, and projective A-modules, respectively, with $i = 1, \cdots, n$. Let

$$C_{ij} := \dim \mathrm{Hom}_A(P_i, P_j) = [P_j : M_i].$$

Definition 33. The matix $C = (C_{ij})$ is the *Cartan matrix* of A.

Example 33. *Let A be the k-algebra of 2×2 upper triangular matrices. Observe that*

$$\begin{pmatrix} a & b \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 & b' \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & ab' \\ 0 & 0 \end{pmatrix}.$$

Similarly,

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix} = \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}.$$

So, we have two composition series of A given by:

$$0 \subset \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \right\} \subset \left\{ \begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} \right\} \subset A,$$

and

$$0 \subset \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \right\} \subset \left\{ \begin{pmatrix} b & c \\ 0 & 0 \end{pmatrix} \right\} \subset A.$$

The idempotents are then given by

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

which forms a system of orthogonal idempotents. Further, we have

$$A = Ae_1 \oplus Ae_2 = \left\{ \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} \right\}.$$

In this case, we have $M_1 = \mathbb{C}$*, given by*

$$A \longrightarrow \mathbb{C}, \quad \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \longmapsto a,$$

and $M_2 = \mathbb{C}$*, with*

$$A \longrightarrow \mathbb{C}, \quad \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \longmapsto c.$$

So, here we have:

$$P_1 := Ae_1 = M_1, \quad Ae_2 = P_2.$$

Let us now look at the composition series of P_2*. Then, we have:*

$$0 \subset \left\{ \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix} \right\} \subset P_2.$$

Thus, we obtain the entries of the Cartan matrix:

$$[P_1 : M_1] = 1, \quad [P_1 : M_2] = 0, \quad [P_2 : M_1] = 1, \quad [P_2 : M_2] = 1.$$

So,

$$C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Thus, we may conclude that

$$\dim \operatorname{Hom}_A(P_1, P_2) = [P_2 : M_1] = 1.$$

This is left as an exercise for us to think about.

Chapter 13

Textbook exercises I went and did

13.1. Serre, Chapter 2

(2.1) *Proof.* We recall the character formula for $\mathrm{Sym}^2(V)$:

$$\chi_{\mathrm{Sym}^2(V)}(g) = \frac{1}{2}(\chi(g)^2 + \chi(g^2)).$$

Then, computing directly,

$$\begin{aligned}(\chi + \chi')_{\mathrm{Sym}^2(V)}(g) &= \frac{1}{2}((\chi + \chi')^2 + (\chi + \chi')(g^2)) \\ &= \frac{1}{2}(\chi^2 + \chi(g^2)) + \frac{1}{2}(\chi'^2 + \chi'(g^2)) + \chi\chi' \\ &= \chi^2_{\mathrm{Sym}^2(V)} + \chi'^2_{\mathrm{Sym}^2(V)} + \chi\chi'.\end{aligned}$$

The calculation for $(\chi + \chi')_{\Lambda^2(V)}$ is analogous, using

$$\chi_{\Lambda^2(V)} = \frac{1}{2}(\chi(g)^2 - \chi(g^2)).$$

□

(2.2) Recall that the permutation representation is given by

$$V = \bigoplus_{x \in X} \mathbb{C}e_x,$$

with

$$\rho : G \longrightarrow \mathrm{GL}(V), g \longmapsto (e_x \longmapsto e_{gx}).$$

The matrix representation of the map $e_x \mapsto e_{gx}$ going to be a $|X| \times |X|$ matrix, with an entry of 1 at the (x, gx) position. Thus, the diagonal entries correspond to elements for which $gx = x$ — that is, they are given by the points which are fixed by g. Then, by definition,

$$\chi_\rho(g) = \mathrm{tr}(e_x \mapsto e_{gx}),$$

and thus the $\chi_\rho(g)$ counts the number of elements of X fixed by g.

(2.3) Define a G-representation on V^* by:

$$\rho : G \longrightarrow \mathrm{GL}(V^*), \quad g \longmapsto (f \longmapsto g \cdot f),$$

where $g \cdot f(v) = f(g^{-1}v)$, for $v \in V$, $f \in V^*$. Then, we have that:

$$\langle g^{-1}v, f \rangle = \langle v, g \cdot f \rangle,$$

and thus it satisfies the property that

$$\langle gv, gf \rangle = \langle v, f \rangle.$$

(2.4) This was one of the exercises above. We have proven this.

(2.5) From the lectures, we know that if χ is an irreducible character, and ψ is any character of G, then the inner product $\langle \chi, \psi \rangle$ counts the number of times that χ appears in the decomposition of ψ. Thus, the amount of times that the trivial representation 1 appears in χ is given by

$$\langle \chi, 1 \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g).$$

(2.6) (a)

(b) The character of the corresponding representation is the $|X|^2 \times |X|^2$ matrix with 1's at the $((x, y), (gx, gy))$ entry. It follows then that the diagonal entries are given by g such that $gx = x$, and $gy = y$, and so its corresponding character is given by χ_ρ^2.

(2.7) Let r_G denote the character of the regular representation of G. Then, $r_G(1) = |G|$, and $r_G(g) = 0$ for $g \neq 1$. Now, let $\rho : G \to \mathrm{GL}(V)$ be a linear G-representation such that $\chi_\rho(g) = \mathrm{tr}(\rho(g)) = 0$ for $g \neq 1$.

We have that $\chi_\rho(1) = \dim_{\mathbb{C}} V$. If ρ is irreducible, then we know from the lectures that its dimension must divide $|G|$. Otherwise, V decomposes into a direct sum of irreducible representations, each of which has dimensions that divide $|G|$. Thus, $\dim_{\mathbb{C}} V$ divides $|G|$.

(2.8) We recall that $W_1, \cdots, W_s$ are the irreducible representations of G (up to isomorphism), and let $V = U_1 \oplus \cdots \oplus U_m$ be the decomposition of V into a direct sum of irreps. Define

$$V_i = \bigoplus_{j \text{ such that } U_j \cong_G W_i} U_j.$$

Then, $V = V_1 \oplus \cdots \oplus V_s$ is the canonical decomposition.

(a) Let $H_i := \mathrm{Hom}_{\mathbb{C}}(W_i, V)$, where each $h \in H_i$ maps W_i into V_i. We proceed inductively. Let $V = V_i = W_i$. Then, $\dim_{\mathbb{C}} H_i = 1 = \dim_{\mathbb{C}} V_i / \dim_{\mathbb{C}} W_i$ by Schur's lemma. Inducting on the number of copies of isomorphic copies of W_i in V_i then gives the result.

(b) Define a map

$$F : H_i \otimes W_i \longrightarrow V_i, \quad \sum_\alpha h_\alpha \otimes w_\alpha \longmapsto \sum_\alpha h_\alpha(w_\alpha).$$

Reduce to the case where $V_i = W_i$. Then, the map is given by $h \otimes w \mapsto h(w)$. Since h is a scalar multiple of the identity map by Schur's lemma, it follows that F defines an isomorphism. Inducting on the number of isomorphic copies of W_i in V_i gives the result.

(c) This is basically an immediate consequence of (b).

(3.1) Let $\rho : A \to \mathrm{GL}(V)$ be any irreducible representation of an abelian group A. Then, since A is abelian, for any $g, h \in A$, we have $\rho(h)^{-1}\rho(g)\rho(h) = \rho(g)$, and thus by Schur's lemma $\rho(g) = \lambda \operatorname{id}_V$ for all $g \in G$, where $\lambda \in \mathbb{C}$.

(3.2) (a) Consider the restriction representation of the irreducible representation ρ, given by $\rho|_{Z(G)} : Z(G) \to \mathrm{GL}(V)$.

(b)

(3.3)

(3.4) Let $V = \bigoplus_{g\in G} \mathbb{C}e_g$ be the regular representation of G. Then, $\operatorname{Res}_H^G V = \bigoplus_{h\in H} \mathbb{C}e_h$, which is stable since clearly for any $h' \in H$,

$$h' \cdot \operatorname{Res}_H^G V h' \cdot \bigoplus_{h\in H} \mathbb{C}e_h = \bigoplus_{h\in H} \mathbb{C}e_{h'h} = \operatorname{Res}_H^G V,$$

and thus the restriction is H-stable. Then,

$$\operatorname{Ind}_H^G \operatorname{Res}_H^G V = \operatorname{Res}_H^G V \otimes_{\mathbb{C}[H]} \mathbb{C}[G] = \bigoplus_{h\in H} \mathbb{C}e_h \otimes_{\mathbb{C}[H]} \mathbb{C}[G] = V.$$

Since every irreducible G-representation is contained in the regular representation, the result follows.

(3.5) We have shown this before in an exercise given in class.

(3.6) Let $G = H \times K$, and θ a H-representation, and $\rho = \operatorname{Ind}_H^G \theta$. Let r_K be the regular representation of K.