On the Monodromy of the Double Affine Hecke Algebra

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Abstract

We study the monodromy of the double affine Hecke algebra (DAHA) $\ddot{\mathbf{H}}$, and use the q-Riemann-Hilbert correspondence of Sauloy in [Sau03] to produce representations of the *elliptic affine Hecke algebra* (ellAHA) $\mathcal{H}^{\mathrm{ell}}$ of Ginzburg-Kapranov Vasserot in [GKV95].

In the rank one case, we construct the *quantum Knizhnik-Zamolodchikov* (qKZ) functor $q\mathrm{KZ} : \mathcal{O}_{\ddot{\mathbf{H}}} \to \mathbf{Coh}^{\mathrm{flat}}(\mathcal{H}^{\mathrm{ell}})$ from Cherednik's [Che90] category $\mathcal{O}$ for DAHA to a representation category of ellAHA, giving a q-analogue of the trigonometric and rational Knizhnik-Zamolodchikov functors studied by Varagnolo-Vasserot in [VV04] and Ginzburg-Guay-Opdam-Rouquier in [GGOR03], respectively.

Our techniques rely on an analysis of the q-difference equations arising from standard modules of DAHA, and we expect that their monodromy yields modules over the ellAHA. Along the way, we find generalisations of results of [GGOR03] to $\mathcal{O}_{\ddot{\mathbf{H}}}$.

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Contents

Introduction

Hecke algebras are an important tool for understanding representations of algebraic groups. The first type of Hecke algebra that one encounters is the finite Hecke algebra $\mathbf{H}$, which arises as a q-deformation of the Weyl group W. Let $\mathbf{G}$ be a split, reductive group scheme with Borel subgroup $\mathbf{B}$. Let $\mathbb{F}_p$ be a finite field and let V be an irreducible, admissible representation of the algebraic group $\mathbf{G}(\mathbb{F}_p)$. Then, the subspace of $\mathbf{B}(\mathbb{F}_p)$-fixed points $V^{\mathbf{B}(\mathbb{F}_p)}$ can be equipped with the structure of a $\mathbf{H}$-module. In particular, there is an equivalence of categories

$$\mathbf{Rep}_{\mathbb{C}}(\mathbf{G}(\mathbb{F}_p)) \longrightarrow \mathbf{H}\text{-}\mathbf{Mod}, \quad V \longmapsto V^{\mathbf{B}(\mathbb{F}_p)}, \tag{1}$$

which sends irreducible objects to irreducible objects (see [Bum10]).

Passing to the setting of affine spaces, one extends the finite Weyl group by the weight or coweight lattice to produce the *affine Weyl group* $\widetilde{W}$. The *affine Hecke algebra* $\dot{\mathbf{H}}$ arises as a q-deformation of $\widetilde{W}$. The AHA was first described in [IM65], who constructed it as a convolution algebra over the algebraic group $\mathbf{G}(\mathbb{Q}_p)$ (see [HKP10], [IM65], [Cai22]). One may recover the finite Hecke algebra $\mathbf{H}$ from this convolution construction by considering a similar convolution algebra on the residue field $\mathbf{G}(\mathbb{F}_p)$ (see [Bum10]).

The affine Hecke algebra plays an important role in studying representations of p-adic groups. In [Bor76], Borel proved a p-adic analogue of (1) by replacing the Borel subgroup by the *Iwahori subgroup* $\mathbf{I}$, which was first introduced in [IM65], as the pre-image of $\mathbf{B}(\mathbb{F}_q)$ under the map $\mathbf{G}(\mathbb{Z}_p) \to \mathbf{G}(\mathbb{Q}_p)$, where $\mathbb{Z}_p$ are the p-adic integers. In this case, given any irreducible, admissible representation V of $\mathbf{G}(\mathbb{Q}_p)$, the subrepresentation $V^{\mathbf{I}}$ has the structure of a finite-dimensional $\dot{\mathbf{H}}$-module, and is irreducible if V is irreducible. Borel in [Bor76] showed that there is an equivalence of categories

$$\mathbf{Rep}_{\mathbb{C}}(\mathbf{G}(\mathbb{Q}_p)) \longrightarrow \dot{\mathbf{H}}\text{-}\mathbf{Mod}, \quad V \longmapsto V^{\mathbf{I}}.$$

The $\mathbf{G}(\mathbb{Q}_p)$-representation V is typically infinite-dimensional [Bum10], whereas $V^{\mathbf{I}}$ is finite-dimensional. By using the affine Hecke algebra, one reduces an infinite-dimensional problem to a finite-dimensional one.

Casselman, in[Cas80], showed that under the correspondence established by Borel, all irreducible admissible representations of $\mathbf{G}(\mathbb{Q}_p)$ arising from an irreducible $\dot{\mathbf{H}}$-module arise from a particular class of parabolically induced $\mathbf{G}(\mathbb{Q}_p)$-representations called *unramified principal series representations*, which play an important role in the local Langlands program (see [GH24], [BH06], [RW21]). All irreducible representations of $\dot{\mathbf{H}}$ have since been classified by Kazhdan and Lusztig in [KL87], using the ${}^{L}\mathbf{G}(\mathbb{C}) \times \mathbb{C}^{\times}$-equivariant K-theory of Springer fibres. We refer the interested reader to [CG09] for an exposition of this construction.

Hecke algebras appear in many other fields of mathematics as well. In [KL87], Kazhdan and Lusztig used affine Hecke algebras to define Kazhdan-Lusztig polynomials, which are combinatorial objects that encode deep representation theoretic information, and are still extensively studied to this day. In [Jon87], affine Hecke algebras were used to define Jones polynomials, which is also an object of extensive study in relation to quantum toplogy and knot theory.

The *double affine Hecke algebra* (DAHA) $\ddot{\mathbf{H}}$ was first introduced by Cherednik in [Che90], and arises as a non-trivial extension of the AHA by the (co)-weight lattice. The representations of $\ddot{\mathbf{H}}$ are generally not well-understood, though there is a classification of the simple, integrable modules of the DAHA using perverse sheaves given in [Vas05]. The usual representation-theoretic convolution construction that one has for AHA using an algebraic group, however, is not available in the DAHA case. This problem has been partially resolved by Braverman and Kazhdan in [BK11] for the spherical case. In their paper, the usual reductive p-adic group is replaced by an affine Kac-Moody group, to which they applied the convolution algebra construction for the spherical Hecke algebra.

The *elliptic affine Hecke algebra* (ellAHA) $\mathcal{H}^{\mathrm{ell}}$ — first introduced in [GKV95] — is an object that is closely related to the double affine Hecke algebra. A Deligne-Langlands classification of the irreducible representations of $\mathcal{H}^{\mathrm{ell}}$ was done by Zhao-Zhong in [ZZ15] using the ${}^{L}\mathbf{G}(\mathbb{C}) \times \mathbb{C}^{\times}$-equivariant elliptic cohomology of the Springer fibre.

One may obtain three more Hecke algebras through various degenerations of the DAHA. One may view the DAHA as an subalgebra of $\mathbb{C}[\mathbf{X}^{\pm}][\mathbf{Y}^{\pm}]$, where $\mathbf{X}$ and $\mathbf{Y}$ are the weight and coweight lattices, respectively. The trigonometric DAHA arises from the DAHA by degenerating one of the weight lattices, giving us two copies of the trigonometric DAHAs, denoted by $\ddot{\mathbf{H}}'_{\mathbf{X}}$ and $\ddot{\mathbf{H}}_{\mathbf{Y}}$, which are subalgebras of $\mathbb{C}[\mathbf{X}][\mathbf{Y}^{\pm}]$ and $\mathbb{C}[\mathbf{X}^{\pm}][\mathbf{Y}]$, respectively. The module categories of these $\ddot{\mathbf{H}}'_{\mathbf{X}}$ and $\ddot{\mathbf{H}}'_{\mathbf{Y}}$ are related by the Fourier transform of Evens-Mirkovic in [EM97]. Zhao-Zhong in [ZZ24] proved an elliptic analogue of this, and showed that the Fourier-Mukai transformation takes the elliptic affine Hecke algebra $\mathcal{H}^{\mathrm{ell}}$ to its Langlands dual ${}^{L}\mathcal{H}^{\mathrm{ell}}$.

A particularly interesting class of representations are known as *monodromy representations*, which are representations arising from the monodromy of systems of partial differential equations. Guay-Ginzburg-Opdam-Rouquier [GGOR03], and Vasserot-Varagnolo [VV04] studied systems of partial differential equations arising from the rational and trigonometric DAHA, respectively (see Appendix B). One then obtains the rational DAHA by degenerating the other weight lattice in the trigonometric DAHA. They found that the rational and trigonometric DAHA both give rise to a specific system of PDEs called the *Knizhnik-Zamolodchikov (KZ) connection*, which are a family of holonomic partial differential equations first introduced by Knizhnik-Zamolodchikov in [KZ84] in the study of two-dimensional quantum field theories.

In the trigonometric DAHA case, the monodromy representations of the KZ equations gives a functor from the category $\mathcal{O}$ of the trigonometric DAHA to the representation category of the affine braid group. As a byproduct, one obtains a functor from the category $\mathcal{O}$ to the representation category of the AHA by taking a categorical quotient of the representation category of the affine braid group. In the rational case, [GGOR03] showed that the monodromy factors through the representation category of the finite Hecke algebra. These functors were called the trigonometric and rational Knizhnik-Zamolodchikov functors, respectively.

Frenkel-Reshetikhin [FR92] later derived a q-analogue of the Knizhnik-Zamolodchikov equations associated to a quantum affine algebra, called the *quantum Knizhnik-Zamolodchiov* (qKZ) equations. Rather than PDEs, the qKZ equations are a system of linear q-difference equations, which are q-analogues of the usual partial derivative. Taking the formal limit $q \to 1$ recovers the usual PDEs defining the Knizhnik-Zamolodchikov equations of [KZ84].

Cherednik, in [Che92], constructed the quantum affine Knizhnik-Zamolodchikov equations associated to an affine Hecke algebra module and showed that it formed a holonomic system of q-difference equations. Stokman, in [Sto10], related the quantum affine KZ equations to the case when the AHA module is induced from a character of the parabolic subalgebra of the AHA. Moreover, [Sto10] also constructs a link between the double affine Hecke algebra and the qKZ equations in [Sto10, §3].

In this thesis, we study the monodromy of the qKZ equation arising from standard modules of the DAHA. The techniques used in [VV04] and [GGOR03] do not generalise immediately, as the mondromy of q-difference equations is less well-known. The main theorem employed in both the papers of [VV04] and [GGOR03] to compute monodromy is the *Riemann-Hilbert correspondence*. A q-analogue of the Riemann-Hilbert correspondence is known, but only for the case of functions of one variable. The proof of this result is due to Sauloy in [Sau03]. The q-Riemann-Hilbert for q-difference systems of more than one variable is still conjectural.

The rank one DAHA gives rise to q-difference equations of one variable, which allows us to apply the results of [Sau03] to study the monodromy of DAHA. We consider certain induced modules in the category $\mathcal{O}$ of DAHA, and show that these modules gives rise to the qKZ equations of [FR92] corresponding to $\mathcal{U}_q(\widehat{\mathfrak{sl}_2})$. The monodromy of the qKZ equations are well-studied in [FR92], [EFK98], and we use these results together with the q-Riemann-Hilbert correspondence of [Sau03] to outline how one may produce representations of the ellAHA. We notate the Hecke algebras we've described in the following way:

	Finite	Affine	Double Affine	Elliptic	Dynamical
Full	$\mathbf{H}$	$\dot{\mathbf{H}}$	$\ddot{\mathbf{H}}$	$\mathcal{H}^{\text{ell}}$	$\mathcal{H}^{\text{dyn}}$
Trigonometric		$\dot{\mathbf{H}}'$	$\ddot{\mathbf{H}}'$		
Rational			$\ddot{\mathbf{H}}''$		

This entire body of work can be summarised using the diagram:

$$
\begin{tikzcd}
\mathcal{H}^{\text{ell}} \arrow[rrrr, leftrightarrow, "\text{Fourier-Mukai Transform, [ZZ24]}"] & & & & {}^{L}\mathcal{H}^{\text{ell}} \\
\ddot{\mathbf{H}} \arrow[u, "q\text{KZ}"] \arrow[rrrr, leftrightarrow, "\text{Cherednik's } S\text{-Duality}"] \arrow[d, "\text{Weight Degeneration}"'] & & & & \ddot{\mathbf{H}} \arrow[u, "q\text{KZ}"'] \arrow[d, "\text{Coweight Degeneration}"] \\
\ddot{\mathbf{H}}'_{\mathbf{X}} \arrow[r, "\text{Trig KZ, [VV04]}"] \arrow[rrrr, leftrightarrow, bend right=40, "\text{Fourier Transform, [EM97]}"'] \arrow[drr, bend right, "\text{Coweight Degeneration}"'] & \dot{\mathbf{H}}^{\mathbf{X}} \arrow[rr, "\text{Not known}"] & & \dot{\mathbf{H}}^{\mathbf{Y}} & \ddot{\mathbf{H}}'_{\mathbf{Y}} \arrow[l, "\text{Trig KZ, [VV04]}"'] \arrow[dll, bend left, "\text{Weight Degeneration}"] \\
& & \ddot{\mathbf{H}}'' \arrow[d, "\text{Rational KZ, [GGOR03]}"] & & \\
& & \mathbf{H} & &
\end{tikzcd}
$$

The red arrows labelled "qKZ" denotes the portion of the literature advanced by this thesis.

Our results fit into a larger body of work concerning the monodromy of q-difference equations in relation to elliptic cohomology. Aganagic-Okounkov in [AO16] studied q-difference equations originating from the quantum K-theoretic counts of rational curves in a Nakajima variety X, and described its monodromy in terms of the equivariant elliptic cohomology of X. In the case for which $X = T^*\mathbb{P}^1$, the q-difference equations that arise are precisely the rank one quantum Knizhnik-Zamolodchikov equations, which we study extensively in this thesis.

In Okounkov-Smirnov's paper [OS22], the action of the quantum dynamical Weyl group on the equivariant quantum K-theory of a Nakajima variety X is considered. In the case for which $X = T^*\mathbb{P}^1$, we once again see the qKZ equations appearing once again in [OS22, (122)].

Our contributions in this direction can be summarised in the following diagram:

$$\begin{tikzcd}
 & & \text{DAHA} \arrow[dll, "\text{[BEG00]}"'] \arrow[drr, "\text{This Thesis}"] & & \\
\text{Quantum Torus} \arrow[d] & & & & \text{Elliptic AHA} \\
q\text{-Difference Equations} \arrow[drr, "\text{[Sau03]}"'] & & & & \begin{array}{c}\text{Elliptic Cohomology}\\ \text{of Nakajima Varieties}\end{array} \arrow[u] \arrow[dll, "\text{[AO16]}"] \\
 & & \text{Vector Bundles on } E & &
\end{tikzcd}$$

Moreover, [ZZ15] showed that the irreducible representations of $\mathcal{H}^{\text{ell}}$ are in one-to-one correspondence with certain nilpotent Higgs bundles on elliptic curves. As a corollary, our result will give a parametrisation of these Higgs bundles in terms of monodromy DAHA representations.

Statement of Main Theorem and Conjectures

We make substantial progress on the following conjecture:

Conjecture 1. *There exists a functor*

$$q\text{KZ} : \mathcal{O}_{\ddot{\mathbf{H}}} \longrightarrow \mathbf{Coh}^{\text{flat}}(\mathcal{H}^{\text{ell}}),$$

called the quantum Knizhnik-Zamolodchikov *(qKZ) functor. Moreover, the qKZ functor factors through the Serre quotient* $\mathcal{O}_{\ddot{\mathbf{H}}}/\mathcal{O}^{\text{tor}}_{\ddot{\mathbf{H}}}$*, and induces a map to the module category* $\mathbf{Coh}^{\text{flat}}(\mathcal{H}^{\text{ell}})$*:*

$$\begin{tikzcd}
\mathcal{O}_{\ddot{\mathbf{H}}} \arrow[rr, "q\text{KZ}"] \arrow[dr, two heads] & & \mathbf{Coh}^{\text{flat}}(\mathcal{H}^{\text{ell}}) \\
 & \mathcal{O}_{\ddot{\mathbf{H}}}/\mathcal{O}^{\text{tor}}_{\ddot{\mathbf{H}}} \arrow[ur, dotted] &
\end{tikzcd}$$

We have further conjectures about certain properties that the qKZ functor should satisfy:

Conjecture 2. *The restriction of* qKZ *to* $\mathcal{O}_{\ddot{\mathbf{H}}}/\mathcal{O}^{\text{tor}}_{\ddot{\mathbf{H}}}$ *is a fully faithful, essentially surjective functor.*

Additionally, we expect to be able to extend this work to representations of the *dynamical affine Hecke algebra* (see Appendix D, [ZZ15], [LZZ23], [ZZ24]).

Let ${}^L\mathcal{H}^{\text{ell}}$ be the Langlands dual of the elliptic affine Hecke algebra $\mathcal{H}^{\text{ell}}$. In [ZZ24, Corollary 5.1], it was shown that the Fourier-Mukai transformation gives a functor:

$$\mathcal{FM} : \mathbf{Coh}^{\mathrm{fin}}(\mathcal{H}^{\mathrm{ell}}) \longrightarrow \mathbf{Coh}^{\mathrm{flat}}({}^L\mathcal{H}^{\mathrm{ell}}).$$

Moreover, [ZZ24, Corollary 5.2] showed that there is an inverse Fourier-Mukai functor $\widehat{\mathcal{FM}}$.

Conjecture 3. *Let $\mathcal{H}^{\mathrm{dyn}}$ be the dynamical Hecke algebra. Then, there exists a functor*

$$\mathbb{C}[\mathbf{Y}]\text{-}\mathbf{Mod} \longrightarrow \mathcal{H}^{\mathrm{dyn}}\text{-}\mathbf{Mod},$$

such that the diagram

$$\begin{tikzcd}
\mathbb{C}[\mathbf{Y}]\text{-}\mathbf{Mod} \arrow[r, "\mathrm{Ind}_{\mathbb{C}[\mathbf{Y}]}^{\dot{\mathbf{H}}^Y}"] \arrow[d] & \dot{\mathbf{H}}^{\mathbf{Y}}\text{-}\mathbf{Mod} \arrow[r, "\mathrm{Ind}_{\dot{\mathbf{H}}^Y}^{\ddot{\mathbf{H}}}"] & \mathcal{O}_{\ddot{\mathbf{H}}} \arrow[d, "q\mathrm{KZ}"] \\
\mathbf{Coh}(\mathcal{H}^{\mathrm{dyn}}) \arrow[drr, "\mathrm{forget}"'] & & \mathbf{Coh}^{\mathrm{flat}}(\mathcal{H}^{\mathrm{ell}}) \arrow[d, bend right, "\widehat{\mathcal{FM}}"'] \\
& & \mathbf{Coh}^{\mathrm{fin}}({}^L\mathcal{H}^{\mathrm{ell}}) \arrow[u, bend right, "\mathcal{FM}"']
\end{tikzcd}$$

commutes.

Structure of this Thesis

Chapter 1 of this thesis introduces all the algebraic objects that we will require. A brief introduction to q-difference equations and their monodromy are given, and we introduce the DAHA and outline its relation to q-difference equations. Then, the elliptic affine Hecke algebra of [GKV95] is introduced. Following [GKV95], [LZZ23], [ZZ15], and [ZZ24], we detail the construction of the elliptic affine Hecke algebra, and construct its representation category $\mathbf{Coh}(\mathcal{H}^{\mathrm{ell}})$.

Chapter 2 details the construction of the category $\mathcal{O}$ of DAHA – which we denote by $\mathcal{O}_{\ddot{\mathbf{H}}}$. This definition was originally given by [Che95], and can also be found in [JV19]. We construct Δ-filtrations and standard modules in $\mathcal{O}_{\ddot{\mathbf{H}}}$, generalising results of [GGOR03] for the category $\mathcal{O}$ of the rational DAHA. We classify elements of $\mathcal{O}_{\ddot{\mathbf{H}}}$ in terms of modules admitting Δ-filtrations in Proposition 1. This is a modification of [GGOR03, Proposition 2.2]. We also briefly describe the torsion subcategory $\mathcal{O}_{\ddot{\mathbf{H}}}^{\mathrm{tor}}$ of $\mathcal{O}_{\ddot{\mathbf{H}}}$, and show that it is a Serre category.

Chapter 3 outlines the construction of the *quantum torus* of [BEG00], which is an algebra generated by q-difference equations acting on the algebra of meromorphic functions on the weight lattice. [BEG00, Theorem 7.2] proves that there is an isomorphism between the localised DAHA $\ddot{\mathbf{H}}_{\mathrm{loc}}$, and the quantum torus. In Proposition 2, we prove that the restriction of the isomorphism of [BEG00, Theorem 7.2] to $\mathcal{O}_{\ddot{\mathbf{H}}}$ lands in the Fuchsian quantum torus. This allows us to apply the q-Riemann-Hilbert functor of [Sau03] to the quantum torus. We also define the W*-equivariant connection category*, which is the connection category of [Sau03], but equipped with the action of a Weyl group W. The q-Riemann-Hilbert functor lands in this category when applied to the Fuchsian W-equivariant quantum torus.

In Chapter 4, we construct the quantum Knizhnik-Zamolodchikov (qKZ) functor for a class of induced DAHA modules called standard modules. A key result is Proposition 3, which shows that choosing an appropriate basis for the standard module gives the trigonometric R-matrix of the evaluation modules of $\mathcal{U}_q(\widehat{\mathfrak{sl}_2})$. This is the coefficient matrix for the qKZ equation of [FR92] corresponding to $\mathcal{U}_q(\widehat{\mathfrak{sl}}_2)$. Using techniques from [EFK98], [Sau03], we then compute the monodromy matrix of the qKZ equation. We discuss how this monodromy matrix can be used to produce an $\mathcal{H}^{\mathrm{ell}}$-module. The only technical condition that remains to be checked is Conjecture 4.

Chapter 1

Preliminaries

1.1. q-Difference Equations

Throughout, let us fix a complex number $q \in \mathbb{C}^\times$ such that $|q| < 1$, and let $E := \mathbb{C}^\times / q^{\mathbb{Z}}$ be an elliptic curve. Let $\mathbb{P}^1 = \operatorname{Proj} \mathbb{C}[x, y]$ be the complex projective space. Let $\mathcal{M}$ be the sheaf of meromorphic functions over $\mathbb{P}^1$. One may think of q-difference equations as q-deformations of the usual derivative. For some $q \neq 1$, one defines the *q-derivative operator* D_q by [Koe18, §3.1]:

$$D_q f(z) = \frac{f(z) - q^z f(qz)}{(1-q)z}, \quad z \neq 0,$$

where $f \in \mathbb{C}(z)$ (c.f. [AM10, (1.16)], [Koe18]). Observe that $D_q f(0) = f'(0)$, assuming the derivative exists. One readily checks that $D_q f(x) \to f'(x)$ as $q \to 1$. Generally, one can define q-difference operators on functions of multiple variables. However, we restrict ourselves to the case of one variable. The general theory for q-difference equations of multiple variables is not well-understood. To illustrate this idea of q-difference equations being q-analogues of differential equations, let us consider some q-analogues of classical special functions:

Example 1 (q-Hypergeometric Equation)**.** *The* hypergeometric equation *is given by:*

$${}_2F_1(a, b, c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

where $(a)_n$ *is the* Pochhammer symbol *defined by*

$$(a)_n := \begin{cases} 1 & \textit{if} \quad n = 0 \\ a(a+1)\cdots(a+n-1), & \textit{if} \quad n > 0 \end{cases}.$$

The hypergeometric equation satisfies the hypergeometric equation*, given by*

$$z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2} f(z) + (c - (a+b+1)z)\frac{\mathrm{d}}{\mathrm{d}z} f(z) - abf(z) = 0.$$

There is a well-known q-analogue of this special function. First, define the q-Pochhammer symbol *to be:*

$$(a; q)_n := \prod_{n \geq 0} (1 - q^n a).$$

Then, the q-hypergeometric equation *is defined by*

$$ {}_2\varphi_1\left[\begin{matrix} a & b \\ & c \end{matrix}; q; z\right] := \sum_{n\geq 0} \frac{(a;q)_n (b;q)_n}{(q;q)_n (c;q)_n} z^n, $$

and satisfies the second order q-difference equation:

$$ z(q^c - q^{a+b+1}z) D_q^2 f(z) + \left(\frac{1-q^c}{1-q} + \frac{(1-q^a)(1-q^b) - (1-q^{a+b+1})}{1-q} z \right) D_q f(z) - \frac{(1-q^a)(1-q^b)}{(1-q)^2} f(z) = 0. \tag{1.1} $$

One checks that the above equation tends to the hypergeometric differential equation as $q \to 1$. Substituing the formula for $D_q f(z)$ into (1.1) gives us the formula:

$$ (q^c - q^{a+b}z) f(q^2 z) + (-(q^c + q) + (q^a + q^b)z) f(qz) + (q - z) f(z) = 0. \tag{1.2} $$

Example 2 (q-Gamma Function)**.** *The usual Gamma function also has a q-analogue, called the* q-Gamma function*. It is defined to be:*

$$ \Gamma_q(z) = \frac{(q;q)_\infty}{q^z : q)_\infty} (1-q)^{1-z}, \quad 0 < |q| < 1. $$

One checks that for $n \in \mathbb{N}$,

$$ \Gamma_q(n) = \frac{(q;q)_{n-1}}{(1-q)^{n-1}}, $$

and

$$ \Gamma_q(x+1) = \frac{1-q^x}{1-q} \Gamma_q(x). $$

Again, taking the formal limit $q \to 1$ recovers the usual Gamma function $\Gamma(x)$. See [AM10, §1.6] for more details and historical remarks.

There are many more q-analogues of these kinds of special functions. We refer the reader to [AM10] for a comprehensive treatise of this subject. For the purpose of this text it is sufficient to understand the q-hypergeometric equation, and we will see that this equation arises as a solution to the rank one quantum Knizhnik-Zamolodchikov equation. This is the central object that we will be studying in relation to the double affine Hecke algebra (see 4).

Example 3 (q-Power Function)**.** *The power function $f(z) = (1-z)^{-a}$ can be defined as the solution of the differential equation*

$$ \frac{\mathrm{d}}{\mathrm{d}z} f(z) = \frac{a}{1-z} f(z). $$

q-Analogously, the q-power function

$$ g(z) := \prod_{n=0}^{\infty} \frac{1 - zq^{n+a}}{1 - zq^n}, $$

is a solution to the q-difference equation

$$ g(qz) = \frac{1-z}{1-zq^a} g(z). $$

Let us formally define the notion of a q-difference equation, and what it means to solve them.

Definition 1. Let K be a field, and σ an automorphism of K. Then, the pair (K, σ) is called a *difference field*. Moreover, if K' is a field extension of K, and σ' is an automorphism of K', then (K', σ') is an *extension* of (K, σ) if $\sigma'|_K = \sigma$.

As we saw in Example 1, any q-difference equation of the form

$$a_n(q,z)D_q^n f(z) + \cdots + a_0(q,z)f(z) = 0,$$

can be re-written into an equation of the form

$$A_n(q,z)f(q^n z) + \cdots + A_0(q,z)f(z) = 0.$$

Thus from now on, when we speak of q-difference equations, we will speak of the ones of the latter form. Let K be some function field (i.e. $\mathbb{C}(z)$, $\mathcal{M}(\mathbb{C}^\times)$), and let $\sigma_q \in \mathrm{End}(K)$ be the *q-difference* (or *q-shift*) operator acting by $\sigma_q : f(z) \mapsto f(qz)$. Then, the pair (K, σ_q) forms a difference field. In particular, note that $(\mathcal{M}(\mathbb{C}^\times), \sigma_q)$ is a field extension of $(\mathbb{C}(z), \sigma_q)$, where the operator σ_q is extended from $\mathbb{C}(z)$ to $\mathcal{M}(\mathbb{C}^\times)$ in the obvious way. More generally, one may consider q-difference equations of the form

$$\sigma_q X = AX, \tag{1.3}$$

where $X = (f_1, \cdots, f_n)^T$, $A \in \mathrm{GL}_n(\mathcal{M}(\mathbb{C}^\times))$, and the operator σ_q acts on X element-wise. We will call q-difference equations written in the form (1.3) a *q-difference system*, with *coefficient matrix* A. Furthermore, given a difference field (K, σ), one may form the category of q-difference equations, which we denote by $\mathbf{DiffEq}(K, \sigma)$. Its objects are pairs of the form (K^n, A), where $A \in \mathrm{GL}_n(K)$. A morphism $(K^n, A) \to (K^p, B)$ is a matrix $F \in \mathrm{Mat}_{p,n}(K)$ satisfying the relation $(\sigma F)A = BF$. One may take tensor products of objects by

$$(K^n, A) \otimes (K^p, B) = (K^{np}, A \otimes B),$$

where $\otimes$ in this case denotes the Kronecker product, and thus $A_1 \otimes A_2 \in \mathrm{Mat}_{np}(K)$. The tensor product equips $\mathbf{DiffEq}(K, \sigma)$ with the structure of a Tannakian category (see [Sau03, §1.1.2]). In particular, it is a neutral Tannakian category with a forgetful functor $(K^n, A) \mapsto K^n$.

Example 4 (Quantum Knizhnik-Zamolodchikov Equation)**.** *Let V and W be two weight modules of $\mathcal{U}_q(\widehat{\mathfrak{sl}_2})$ with weights m, n respectively. Then, when $m = n = 1$, the trigonometric R-matrix of the evaluation modules is given by*

$$R(z) = E_{11} \otimes E_{11} + E_{22} \otimes E_{22} + \frac{1-z}{q-q^{-1}z}(E_{22} \otimes E_{11} + E_{11} \otimes E_{22}) + \frac{q-q^{-1}}{q-q^{-1}z}(E_{12} \otimes E_{21} + zE_{21} \otimes E_{12}),$$

which is given in matrix form by:

$$R(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{1-z}{q-q^{-1}z} & \dfrac{z(q-q^{-1})}{q-q^{-1}z} & 0 \\ 0 & \dfrac{q-q^{-1}}{q-q^{-1}z} & \dfrac{1-z}{q-q^{-1}z} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then, the quantum Knizhnik-Zamolodchikov *equation is the q-difference system given by*

$$\begin{pmatrix} \Phi_1(qz_1, z_2) \\ \Phi_2(qz_1, z_2) \end{pmatrix} = R\left(\frac{z_1}{z_2}\right) \cdot \begin{pmatrix} \Phi(z_1, z_2) \\ \Phi(z_1, z_2) \end{pmatrix}.$$

(c.f. [EFK98], [FR92], [Sto10]). Solutions of this equation, as well as its relation to representations of double affine Hecke algebra and elliptic affine Hecke algebra will be explored in Chapter 4.

Definition 2. Let $\sigma_q X = AX$ be a q-difference system, and let $P \in \mathrm{GL}_n(K)$ be a matrix. Then, a *gauge transform* of A by the *gauge transformation matrix* P is the matrix

$$P \cdot [A]_q = (\sigma_q P) \cdot A \cdot P^{-1}.$$

A particularly nice property of q-difference systems is whether or not it is *Fuchsian*. The reasons for this will become clear when we begin explicitly building solutions over the field $\mathcal{M}(\mathbb{C}^\times)$ in the next section.

Definition 3. A q-difference system $\sigma_q X = AX$, with $A \in \mathrm{GL}_n(\mathbb{C}(z))$ is called *Fuchsian* at 0 (respectively, ∞) if $A(0) \in \mathrm{GL}_n(\mathbb{C})$ (respectively, $A(\infty) \in \mathrm{GL}_n(\mathbb{C})$), or if there exists a meromorphic gauge transformation $P \in \mathrm{GL}_n(\mathcal{M}(\mathbb{C}^\times))$ such that $(\sigma_q P) \cdot A \cdot P^{-1}$ is Fuchsian at 0 (resp. ∞).

Let **Fuch** be the category of Fuchsian equations. It is a full subcategory of $\mathbf{DiffEq}(\mathbb{C}(z), \sigma_q)$. (TODO: come back and elaborate more on this)

1.1.1. Solutions of q-difference Equations

So far, we have considered q-difference systems over $\mathbb{C}(z)$. However, we wish to build solutions in a field extension of $\mathbb{C}(z)$, specifically $\mathcal{M}(\mathbb{C}^\times)$. Both $\mathbb{C}(z)$ and $\mathcal{M}(\mathbb{C}^\times)$ come equipped with a q-shift operator σ_q that acts on elements in the same way. Thus, $(\mathcal{M}(\mathbb{C}^\times), \sigma_q)$ is a field extension of $(\mathbb{C}(z), \sigma_q)$. By abuse of notation, we use σ_q to denote the field automorphism in both categories. Observe first that

$$\mathcal{M}(\mathbb{C}^\times)^{\sigma_q} = \mathcal{M}(E),$$

the meromorphic sections of an elliptic curve $E = \mathbb{C}^\times / q^{\mathbb{Z}}$. Note that sections of the structure sheaf $\mathcal{O}_E$ consist of functions over $\mathbb{C}^\times$ for which $f(q^{\mathbb{Z}} u) = f(u)$. Such functions are thus called *elliptic* or *q-periodic*. Moreover, $\mathcal{M}(E)$ can be viewed as a localisation of the global sections of the structure sheaf $\mathcal{O}_E$. The solution functor is given by the fibre functor:

$$\mathrm{Sol} : \mathbf{DiffEq}(\mathcal{M}(\mathbb{C}^\times), \sigma_q) \longrightarrow \mathbf{Vect}_{\mathcal{M}(\mathbb{C}^\times)}.$$

Since the map $z \mapsto qz$ has only two fixed points on the Riemann sphere — specifically 0 and ∞ — solutions only exist in neighbourhoods of those fixed points [EFK98]. Moreover, another reason that solutions at other singular points are not considered is because if $f(z)$ is a solution of the equation $\sigma_q f(z) = A(z) f(z)$, with a singularity at $z_0 \neq 0, \infty$, then $f(z)$ has a singularity at any complex number $q^k z$ [RW22, Remark 2.7].

So generally, given a q-difference system $(\mathbb{C}(z), \sigma_q) \in \mathbf{DiffEq}(\mathbb{C}(z), \sigma_q)$, there are three solution functors. There are two solution functors mapping to solutions around 0 and ∞, and one mapping to elliptic solutions, because solutions of the equation $\sigma_q f = f$ that are meromorphic at 0 and ∞ must be a constant. We denote these solution functors by

$$\mathrm{Sol}^{(0)}, \quad \mathrm{Sol}^{(\infty)}, \quad \mathrm{Sol}^{(\mathrm{ell})}.$$

In practise, we only wish to consider the functors $\mathrm{Sol}^{(0)}$ and $\mathrm{Sol}^{(\infty)}$. Given a q-difference equation, we wish to obtain a basis of solutions at 0 and ∞. The following result gives a way to check the linear independence of these solutions:

Lemma 1 (Lemma 2.3.3, [Sau16])**.** *Let $f_1, \cdots, f_n \in \mathcal{M}(\mathbb{C}^\times)$. Then, the* q-Wronskian *matrix is defined* *as:*

$$W_n(f_1, \cdots, f_n) := \begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ \sigma_q f_1 & \sigma_q f_2 & \cdots & \sigma_q f_n \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_q^{n-1} f_1 & \sigma_q^{n-1} f_2 & \cdots & \sigma_q^{n-1} f_n \end{pmatrix}.$$

Then, $f_1, \cdots, f_n$ are linearly dependent if and only if $\det W_n(f_1, \cdots, f_n) = 0$ over $\mathcal{M}(\mathbb{C}^\times)^{\sigma_q}$.

Definition 4 (Definition 2.10, [RW22])**.** A family of solutions $f_1, \cdots, f_n$ is a *fundamental solution* if $\det W_n(f_1, \cdots, f_n) \neq 0$.

When we are given a q-difference equation, our goal will be to find fundamental solutions in neighbourhoods of 0 and ∞. We now introduce some special functions that can be used to solve q-difference equations (c.f. [Sau03, §1.2.2], [RW22, §2]). The *Jacobi theta function* is defined as

$$\Theta_q(z) = \sum_{n\in\mathbb{Z}} q^{-\frac{n(n+1)}{2}} z^n,$$

which satisfies the q-difference relation $\sigma_q^n \Theta_q(z) = q^{\frac{1}{2}n(n+1)} z^n \Theta_q(z)$, and the famous *Jacobi triple product identity*:

$$\Theta_q(z) = (q^{-1}; q^{-1})_\infty (-q^{-1}z; q^{-1})_\infty (-z^{-1}; q^{-1})_\infty.$$

Remark 1. Another definition of the Jacobi theta function seen in the literature is given by:

$$\Theta_q(u) = \sum_{n\in\mathbb{Z}} q^{\frac{n(n+1)}{2}} z^n,$$

which satisfies the triple product identity

$$\Theta_q(u) = (q; q)_\infty (-z; q)_\infty (-qz^{-1}; q)_\infty,$$

and satisfies the q-difference relation $\Theta_q(q^n u) = q^{-\frac{1}{2}n(n+1)} u^{-n} \Theta_q(u)$.

Note that Θ_q has the property that

$$\frac{\Theta_q(q^{-1}u^{-1})}{\Theta_q(u^{-1})} = \frac{\Theta_q(au)}{\Theta_q(u)}.$$

For some $\lambda \in \mathbb{C}^\times$, the *$q$-character* associated to divisor λ is the function $e_{q,\lambda} \in \mathcal{M}(\mathbb{C}^\times)$ given by

$$e_{q,\lambda}(z) = \frac{\Theta_q(z)}{\Theta_q(z/\lambda)},$$

which satisfies the q-difference relation $\sigma_q e_{q,\lambda}(z) = \lambda \cdot e_q, \lambda(z)$. As aforementioned, q-periodic functions are sections of the structure sheaf, or some localisations of the structure sheaf. Let $\mathcal{O}(\lambda)$ denote the line bundle over E arising from a divisor λ. It follows then that the ratio $e_{q,\lambda}$ of theta functions defines a section of the line bundle $\mathcal{O}(\lambda)$. Generally, any function with q-period λ gives a section of $\mathcal{O}(\lambda)$.

The *q-logarithm* is the function $\ell_q \in \mathcal{M}(\mathbb{C}^\times)$ defined by

$$\ell_q(z) = z\frac{\Theta_q'(z)}{\Theta_q(z)},$$

which satisfies the q-difference relation $\ell_q(qz) = \ell(z) + 1$ ([RW22, Definition 2.5], [Sau03, §1.2.2]).

We can use this data to build a fundamental solution to q-difference equations of the form $\sigma_q X = AX$, given that the system satisfies the *non-resonance* condition:

Definition 5. Let $\sigma_q X = AX$ be a q-difference system defined by the matrix A. Let $\{\lambda_i\}_i$ be a collection of eigenvalues for $A(0)$. Then, this q-difference system is said to be *non-resonant* if if for every $i \neq j$, the ratio $\frac{\lambda_i}{\lambda_j} \notin q^{\mathbb{Z}\setminus\{0\}}$.

Let $\mathcal{M}_0$ and $\mathcal{M}_\infty$ denote the stalks of $\mathcal{M}$ at 0 and ∞. Given a non-resonant q-difference system, one can build a gauge transformation $M^{(0)} \in \mathrm{GL}_n(\mathcal{M}_0)$ (resp. $M^{(\infty)} \in \mathrm{GL}_n(\mathcal{M}_\infty)$) sending the matrix $A(0)$ (resp. $A(\infty)$) to the constant matrix $A(z)$ (see [Sau16]). Then, we take the Jordan-Chevalley decomposition of $A(0) = A_s A_u$, where A_s is the semisimple part, and A_u is the unipotent part. Since $A_s = Q\,\mathrm{diag}(\lambda_1, \cdots, \lambda_n)Q^{-1}$, we can define:

$$e_{q,A_s} := Q \cdot \begin{pmatrix} e_{q,\lambda_1} & & \\ & \ddots & \\ & & e_{q,\lambda_n} \end{pmatrix} \cdot Q^{-1}.$$

This satisfies the q-difference relation $\sigma_q e_{q,A_s} = A_s e_{q,A_s} = e_{q,A_s} A_s$. Identifying the q-characters as sections of a line bundle, we see that e_{q,A_s} is a section of the vector bundle over E:

$$\mathcal{F} = \mathcal{O}(\lambda_1) \oplus \cdots \oplus \mathcal{O}(\lambda_n).$$

For the nilpotent part, we first obeserve that since A_u is unipotent, $N = A_u - I_n$ is nilpotent. Thus, we may define:

$$A_u^{\ell_q} := \sum_{k \geq 0} \binom{\ell_q}{k} N^k,$$

where

$$\binom{\ell_q}{k} := \frac{\ell_q(\ell_q - 1)\cdots(\ell_q - (k-1))}{k!}.$$

Then, we set

$$e_{q,A_u} := A_u^{\ell_q}.$$

Let us write

$$e_{q,A(0)} := e_{q,A_s} \cdot e_{q,A_u}.$$

One checks then that $e_{q,A(0)}$ is a solution to the constant coefficient q-difference system:

$$\sigma_q X = A(0)X.$$

On constructs $e_{q,A(\infty)}$ analogously. Then, the *fundamental solution* around 0 is given by:

$$\mathbf{X}^{(0)} := M^{(0)} e_{q,A_s} \cdot e_{q,A_u}.$$

One constructs $\mathbf{X}^{(\infty)}$ analogously. We have proven the following:

Theorem 1 (Theorem 3.3.1, [Sau16])**.** *The q-difference system $\sigma_q X = AX$ admits a basis of fundamental solutions at* 0 *and* ∞ *given by:*

$$\mathbf{X}^{(0)} = M^{(0)} e_{q,A(0)}, \quad \mathbf{X}^{(\infty)} = M^{(\infty)} e_{q,A(\infty)}.$$

1.1.2. The Connection Data

We follow [EFK98, Chapter 12] for this section. Consider a q-difference system given by:

$$\Phi(qz) = A(z)\Phi(z), \tag{1.4}$$

where Φ is a column vector of meromorphic function. Then, we have the following result:

Theorem 2 (Theorem 12.1.1, [EFK98])**.** *Let $v_1, \cdots, v_n$ be an eigenbasis of $A(0)$, and $u_1, \cdots, u_n$ be an eigenbasis of $A(\infty)$. Fix complex numbers $a_1, \cdots, a_n$, and $b_1, \cdots, b_n$ such that $A(0)v_j = e^{ta_j}v_j$, and $u_j = e^{tb_j}u_j$. Then, there exists a unique basis of solutions $\Phi_1^{(0)}, \cdots, \Phi_n^{(0)}$, and $\Phi_1^{(\infty)}, \cdots, \Phi_n^{(\infty)}$ of (1.4) of the form*

$$\Phi_j^{(0)}(z) = z^{a_j}\varphi_j(z), \quad \Phi_j^{(\infty)} = z^{b_j}\psi(z),$$

where φ_j are regular at 0, and ψ_j are regular at ∞, and $\varphi_j(0) = v_j$, and $\psi(\infty) = u_j$.

Definition 6. Solutions of the form $\Phi_i^{(0)}$ and $\Phi^{(\infty)}$ seen in Theorem 2 will be called *asymptotic solutions* near 0 and ∞, respectively.

The *connection matrix* is an elliptic matrix that relates the two asymptotic solutions at 0 and ∞ in the following way:

Definition 7. Let $\Phi(qz) = A(z)\Phi(z)$ be a q-difference system, with asymptotic solutions $\Phi^{(0)}$ and $\Phi^{(\infty)}$ near 0 and ∞, respectively. Then, a *connection matrix* matrix $\mathbf{X}$ is an element of $\mathrm{GL}_n(\mathcal{M}(\mathbb{C}^\times)$ satisfying the relation:

$$\Phi^{(0)} = \mathbf{X} \cdot \Phi^{(\infty)},$$

and $\sigma_q\mathbf{X} = \mathbf{X}$.

The *Birkhoff connection matrix* of [Sau03] is defined to be:

$$\left(\mathbf{X}^{(\infty)}\right)^{-1} \cdot \mathbf{X}^{(0)},$$

where $\mathbf{X}^{(0)}$ and $\mathbf{X}^{(\infty)}$ are the fundamental solutions. Since $\mathbf{X}$ is q-periodic, one easily checks that

$$\left(\mathbf{X}^{(\infty)}\right)^{-1} \cdot \mathbf{X}^{(0)} = \mathbf{X}^T.$$

As aforementioned, one builds a meromorphic gauge transformation $M^{(0)} \in \mathrm{GL}_n(\mathcal{M}(\mathbb{C}^\times)$ (resp. $M^{(\infty)}$) that takes $A(0)$ (resp. $A(\infty)$) to $A(z)$. It follows then that the matrix

$$M := \left(M^{(\infty)}\right)^{-1} \cdot M^{(0)},$$

is a suitable meromorphic gauge transformation that takes $A(0)$ to $A(\infty)$. Using the formula for the fundamental solutions from Theorem 1, one deduces that M has the property that M gives a q-gauge transformation from $\mathbf{X}^{(0)}$ to $\mathbf{X}^{(\infty)}$:

$$(\sigma_q M) \cdot e_{q,A(0)} = e_{q,A(\infty)} \cdot M.$$

We thus call this matrix M the *monodromy matrix* of a q-difference system. Note that if we have an explicit formula for the connection matrix $\mathbf{X}$, then we can compute the monodromy matrix by the equation:

$$M = e_{q,A(\infty)} \cdot \mathbf{X}^T \cdot \left(e_{q,A(0)}\right)^{-1}.$$

1.1.3. The q-Riemann-Hilbert-Correspondence

As aforementioned in the introduction, the classical Knizhnik-Zamolodchikov equations of [KZ84], can be viewed as vector bundles over $\mathbb{P}^1$ equipped with a connection. Using this viewpoint, one may apply algebraic geometric techniques to study its properties. [Sau03] shows that something similar can be done for q-difference systems.

Associated to any q-difference system we may associate a triple $(e_{q,A(0)}, e_{q,A(\infty)}, M)$, which [Sau03] calls the *connection data*. A morphism between the connection data amounts to a change of basis for $\mathbf{X}^{(0)}$, $\mathbf{X}^{(\infty)}$, and M (see [Sau03, §3]). Thus, there is a category of connection data, which we denote by

$$\mathcal{C}onn.$$

[Sau03, Theorem 2.3.2.1] shows that there is a functor:

$$(e_{q,A(0)}, e_{q,A(\infty)}, M) \longmapsto (\mathcal{F}_0, \mathcal{F}_\infty, \varphi), \tag{1.5}$$

where $\mathcal{F}_0$ and $\mathcal{F}_\infty$ are locally free coherent sheaves (i.e. vector bundles) on an elliptic curve $E = \mathbb{C}^\times/q^{\mathbb{Z}}$, and $\varphi : \mathcal{F}_0 \dashrightarrow \mathcal{F}_\infty$ is a meromorphic map between the vector bundles. We will use dotted arrows to denote meromorphic maps from now on. Moreover, [Sau03, Theorem 2.3.2.1] proves that this is in fact an equivalence of monoidal categories. The rank of $\mathcal{F}_0$ and $\mathcal{F}_\infty$ corresponds to the amount of linearly independent asymptotic solutions at 0 and ∞, respectively. Recall that there is an isomorphism of coherent sheaves:

$$\mathscr{H}\mathrm{om}_{\mathcal{O}_E}(\mathcal{F}_0, \mathcal{F}_\infty) \cong \mathcal{F}_0 \otimes_{\mathcal{O}_E} \mathcal{F}_\infty^\vee.$$

In the case where $\mathcal{F}_0$ and $\mathcal{F}_\infty$ decompose as a direct sum of line bundles on E, one may identify local sections of φ with the matrix M in the connection data category of [Sau03], thus giving us a functor in the reverse direction. By abuse of notation, we will refer to the category of triples $(\mathcal{F}_0, \mathcal{F}_\infty, \varphi)$ as the connection category as well, and we will similarly denote it by $\mathcal{C}onn$. In particular, the functor

$$q\mathrm{RH} : \mathbf{DiffEq}(\mathbb{C}(z), \sigma_q) \longrightarrow \mathcal{C}onn,$$

gives a q analogue of the Riemann-Hilbert correspondence. According, we will call it the q-Riemann-Hilbert functor, and denote it by qRH.

Example 5. *As an explicit example, if we have a q-difference system $\sigma_q X = A(u)X$, such that $A(u)$ is semi-simple, then it follows that $A(0)$ and $A(\infty)$ are both diagonlisable. Let $\lambda_1, \cdots, \lambda_n$ be a set of eigenvalues for $A(0)$. Then, the matrix $e_{q,A(0)}$ gives the following vector bundle on E:*

$$\mathcal{F}_0 := \mathcal{O}(\lambda_1) \oplus \cdots \oplus \mathcal{O}(\lambda_n).$$

The procedure for $A(\infty)$ is similar, and we obtain a vector bundle $\mathcal{F}_\infty$ built from a direct sum of line bundles:

$$\mathcal{F}_\infty \cong \mathcal{O}(\mu_1) \oplus \cdots \oplus \mathcal{O}(\mu_n).$$

The monodromy map is then given by

$$\varphi : \mathcal{F}_0 \dashrightarrow \mathcal{F}_\infty.$$

Using the isomorphism $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \cong \mathcal{F}^\vee \otimes_{\mathcal{O}_X} \mathcal{G}$, we may identify the meromorphic map φ as a section of

$$\mathcal{F}_0^\vee \otimes_{\mathcal{O}_E} \mathcal{F}_\infty \cong \mathcal{O}(\lambda_1^{-1}\mu_1) \oplus \cdots \oplus \mathcal{O}(\lambda_n^{-1}\mu_n).$$

The monodromy matrix is then given by:

$$\varphi = \begin{pmatrix} M_{11} \cdot \dfrac{\Theta_q(u)}{\Theta_q(\lambda_1 \mu_1^{-1} u)} & \cdots & M_{1n} \cdot \dfrac{\Theta_q(u)}{\Theta_q(\lambda_n \mu_1^{-1} u)} \\ \vdots & \ddots & \vdots \\ M_{n1} \cdot \dfrac{\Theta_q(u)}{\Theta_q(\lambda_n \mu_1^{-1} u)} & \cdots & M_{nn} \cdot \dfrac{\Theta_q(u)}{\Theta_q(\lambda_n \mu_n^{-1} u)} \end{pmatrix},$$

where M_{ij} are q-periodic functions.

1.2. The Double Affine Hecke Algebra (DAHA)

Fix a root system $(\mathbf{X}, \Phi, \mathbf{Y}, \Phi^\vee)$, where $\mathbf{X}$ and $\mathbf{Y}$ are the weight and coweight lattices, respectively, and Φ and $\Phi^\vee$ denotes the roots and coroots. Given Φ, one may choose (in many different ways) a set of positive roots Φ^+. Then, the negative roots Φ^- are given by elements of the form $-\alpha$, where $\alpha \in \Phi^+$. One may similarly define positive and negative weights (resp. coweights), which we denote by $\mathbf{X}^+$ (resp. $\mathbf{Y}^+$). A root is *simple* if it cannot be written as a linear combination of positive roots. The *rank* r of a root system is given by the number of simple roots. Denote by $\mathbf{Q}$ the root lattice, and $\mathbf{P}$ the coroot lattice. They are sublattices of $\mathbf{X}$ and $\mathbf{Y}$, respectively. Let $\mathbb{C}[\mathbf{X}]$ and $\mathbb{C}[\mathbf{Y}]$ denote the group algebras of the weight and coweight lattices. We write their elements as $X^\mu \in \mathbb{C}[\mathbf{X}]$, and $Y^{\lambda^\vee} \in \mathbb{C}[\mathbf{Y}]$.

A weight λ (resp. coweight $\lambda^\vee$) is *minuscule* if $0 \le \langle \lambda, \alpha^\vee \rangle \le 1$ for each positive root $\alpha \in \Phi^+$ [Kir97, Definition 2]. From [Kir97, Lemma 2.2], we know that the minuscule weights (resp. coweights) form a set of representatives for $\mathbf{X}/\mathbf{Q}$ (resp. $\mathbf{Y}/\mathbf{P}$).

The ambient vector space that the roots lie in is given by $V := \mathbb{R}^r$. Let us define a new vector space $\widetilde{V} := V \oplus \mathbb{R}\delta$. Then, the *affine root system* is then given by $\widetilde{\Phi} := \Phi \times \mathbb{Z}\delta$. Then, the affine positive roots are defined to be those of the form $\alpha + k\delta$, for $\alpha \in \Phi^+$, and $k \ge 0$. A basis for $\widetilde{\Phi}^+$ is given by the simple roots $\alpha_0 := -\theta + \delta, \alpha_1, \cdots, \alpha_r$, where θ is the highest root (i.e. $\theta - \alpha \in \mathbb{Q}_+$ for all $\alpha \in \Phi$). The affine reflections are defined to be $s_{\widetilde{\alpha}} : \widetilde{\lambda} \mapsto \widetilde{\lambda} - \langle \lambda, \alpha^\vee \rangle \widetilde{\alpha}$, where $\langle -, - \rangle$ is the perfect pairing between $\mathbf{X}$ and $\mathbf{Y}$. Denote the *affine Weyl group* $\widetilde{W}$as the group generated by affine simple reflections. Then, [Kir97, Theorem 3.1] shows that $\widetilde{W} \cong W \ltimes \mathbf{X}$, where W acts on $\mathbf{X}$ in the usual way. We refer the reader to [Kir97, §2, §3] for more information on affine root systems.

The action of $\mu^\vee \in \mathbf{Y}$ on $\widetilde{V}$ is given by:

$$\tau(\mu^\vee) : \widetilde{\lambda} \longmapsto \widetilde{\lambda} - \langle \mu^\vee, \lambda \rangle \cdot \delta. \tag{1.6}$$

We make the following definitions, following [Kir97, §3]:

1. $Y^{\lambda^\vee} = T_{\tau(\lambda^\vee)}$,
2. if $\lambda^\vee = \mu^\vee - \nu^\vee$, with $\mu^\vee, \nu^\vee \in \mathbf{Y}^+$, then $Y^{\lambda^\vee} = Y^{\mu^\vee}(Y^{\nu^\vee})^{-1}$.

Theorem 3 (Theorem 3.7, [Kir97])**.** *(ii)*

(i) $Y^{\lambda^\vee}$ *is well-defined for all* λ, *and* $Y^{\lambda^\vee} \cdot Y^{\mu^\vee} = Y^{\lambda^\vee + \mu^\vee}$,

(ii) Let $\tau(\mu^\vee) = \pi_r s_{i_\ell} \cdots s_{i_1}$ *be a reduced expression, and let* $\alpha^{(1)}, \cdots, \alpha^{(\ell)}$ *be the associated sequence of affine roots. Then,*

$$Y^{\mu^\vee} = \pi_r T_{i_\ell}^{\varepsilon_\ell} \cdots T_{i_1}^{\varepsilon_1},$$

where $\varepsilon_i = 1$ *if the corresponding* $\alpha^{(i)} = \alpha + k\delta$*, for* $\alpha \in \Phi^+$*, and* $\varepsilon_i = -1$ *otherwise.*

Together, such elements generate the algebra of Laurent polynomials $\mathbb{C}[\mathbf{Y}]$ on the coweight lattice $\mathbf{Y}$. $\mathbb{C}[\mathbf{X}]$ is constructed similarly.

The *finite Hecke algebra* $\mathbf{H}$ arises as a q-deformation of the usual Weyl group. It is a $\mathbb{C}[q^{\pm}]$-algebra generated by elements of the form $\{T_w : w \in W\}$ satisfying the quadratic relation:

$$T_{s_\alpha}^2 = (q-1)T_{s_\alpha} + q,$$

where s_α is a simple reflection about a simple root α. Let $t_\alpha \in \mathbb{C}^\times$ be a parameter for each $\alpha \in \Phi$, with the property that $t_\alpha = t_{w(\alpha)}$ for each $w \in \widetilde{W}$. If α_i is a simple root, we write $t_i := t_{\alpha_i}$, and $T_i := T_{s_{\alpha_i}}$. Then, an alternative presentation of this algebra can be given as a $\mathbb{C}[t_\alpha^{\pm}]$-algebra with quadratic relation

$$T_i^2 = (t_i - t_i^{-1})T_i + 1.$$

Throughout this thesis, we will use the latter presentation. From this, one expects to obtain the *affine Hecke algebra* (AHA) $\dot{\mathbf{H}}$ as a q-deformation of the affine Weyl group $\widetilde{W} = W \rtimes \mathbf{X}$. Note that

$$W \rtimes \mathbf{X} \cong W \rtimes \mathbf{Y},$$

and thus deforming both of these groups should give rise to isomorphic AHAs. We will denote the resulting AHAs by $\dot{\mathbf{H}}^X$ and $\dot{\mathbf{H}}^Y$, respectively. Moreover, since there are two presentations for the affine Weyl group – one in terms of a lattice, and another in terms of the reflection about the affine simple root s_0 – one expects to obtain two different presentations of the AHA as well.

Remark 2. Some authors will refer to the *affine Weyl group* as the semidirect product $W \rtimes \mathbf{P}$, where $\mathbf{P}$ is the coroot lattice, and the *extended Weyl group* as the semidirect product of the affine Weyl group and the weight lattice $\mathbf{X}$. Other authors, on the other hand, will refer to the extended Weyl group simply as the affine Weyl group, and the distinction between the two affine Weyl groups is made relative to the reductive group which we are using to define the root system. This is because a simply-connected, split reductive group (e.g. SL_n), the affine root system constructed from the group will have no zero-length roots. Otherwise, for non-simply connected groups (e.g. GL_n), the affine root system will contain roots of zero length.

The two presentations of the affine Hecke algebra (AHA) are the *Iwahori-Matsumoto presentation* (first introduced in [IM65, Theorem 2.24]) and the *Bernstein presentation* (first introduced in [Lus83]). The Iwahori-Matsumoto presentation gives the AHA as an algebra generated by $T_0, \cdots, T_r$, where T_0 corresponds to the reflection about the affine root, and the $T_1, \cdots, T_r$ correspond to reflections about the simple roots. Further, there is an additional generator π that corresponds to roots of length zero. The generator π acts on the T_i's in the following way:

$$\pi T_i \pi^{-1} = T_j, \text{ if } \pi(\alpha_i) = \alpha_j,$$

where all indices are taken modulo r. Let Ω be the set of all roots of length zero.

Definition 8 (Iwahori-Matsumoto Presentation of AHA)**.** The *affine Hecke algebra* $\dot{\mathbf{H}}$ is a $\mathbb{C}[q^{\pm}, t_\alpha^{\pm}]$-algebra generated by the elements $\pi \in \Omega$, and the elements $T_0, T_1, \cdots, T_r$ subject to the following relations:

(i) $T_i^2 = (t_i - t_i^{-1})T_i + 1$,

(ii) For any $w, v \in \widetilde{W}$, $T_w T_v = T_{wv}$ if $\ell(wv) = \ell(w) + \ell(v)$.

The Bernstein presentation gives the AHA as a product of the affine Hecke algebra (generated by $T_1, \cdots, T_r$), and a lattice X^μ, where $\mu \in \mathbf{X}$ the coweight lattice. Alternatively, one may also commute the generators $T_1, \cdots, T_r$ with elements of the weight lattice. The T_i's commute in the usual way, and T_i and X^μ obey the Bernstein relation:

$$T_i X^\mu - X^{s_i(\mu)} T_i = (1-q)\frac{X^{s_i(\mu)} - X^\mu}{1 - X^{-\alpha_i}}. \tag{1.7}$$

In terms of the parameters t_i, the Bernstein relation is written:

$$T_i X^\mu - X^{s_i(\mu)} T_i = (t_i - t_i^{-1})\frac{X^{s_i(\mu)} - X^\mu}{1 - X^{-\alpha_i}}. \tag{1.8}$$

Then, we have the *Bernstein presentation* of the AHA:

Definition 9 (Bernstein Presentation of AHA)**.** The *affine Hecke algebra* is the $\mathbb{C}[q^\pm, t_\alpha^\pm]$-algebra (resp. $\mathbb{C}[q^\pm]$-algebra) generated by X^μ, for $\mu \in \mathbf{X}$, and $T_1, \cdots, T_r$ subject to the following relations:

(i) $T_i^2 = (t_i - t_i^{-1})T_i + 1$,

(ii) The Bernstein relations (1.8) (resp. (1.7)).

Now, one can define another affine Hecke algebra relative to the dual affine root system $\Phi^\vee$, from which one similarly affines T_i-terms, and elements X^μ for $\mu \in X$. The T_i's commute in the usual way, and commutes with X^μ according to the aforementioned Bernstein relation.

The double affine Hecke algebra can be defined as an algebra generated by T_i, X^μ, and $Y^{\mu^\vee}$. Then, the commutation relations between X^μ and Y^λ will be given by taking the product $X^\mu Y^\lambda$ and applying the Bernstein relation iteratively according to Theorem 3(ii). The resulting relations, however, turn out to be quite complicated. So, often it is best to choose an affine Hecke algebra — either $\dot{\mathbf{H}}^{\mathbf{X}}$ or $\dot{\mathbf{H}}^{\mathbf{Y}}$ — and then take the Iwahori-Matsumoto presentation of it. This gives us generators $\pi, T_0, T_1, \cdots, T_r$, which commute with the the elements in the other lattice via the Bernstein presentation.

Definition 10 (Definition 4.1, [Kir97])**.** The *double affine Hecke algebra* (DAHA) is the $\mathbb{C}[q^\pm, t_\alpha^\pm]$-algebra generated by the elements $\pi \in \Omega$, $T_0, \cdots, T_r$, and X^μ, where $\mu \in \mathbf{X}$ subject to the following relations:

(i) The Iwahori-Matsumoto relations for $\pi, T_0, \cdots, T_r$,

(ii) $X^\mu \cdot X^\nu = X^{\mu+\nu}$,

(iii)
$$T_i X^\mu = \begin{cases} X^\mu T_i & \text{if} \quad \langle \mu, \alpha_i^\vee \rangle = 0, \\ X^{s_i(\mu)} T_i + (t_i - t_i^{-1}) X^\mu & \text{if} \quad \langle \mu, \alpha_i^\vee \rangle = 1, \end{cases}$$

(iv) $\pi X^\mu \pi^{-1} = X^{\pi(\mu)}$.

Following [Kir97, pg. 274], we embed the element in X^δ in the AHA (and hence in the DAHA) in the following way:

$$X^\delta = q^{-2}. \tag{1.9}$$

Substituting this into (1.11), the generator T_0 corresponding to the affine simple reflection s_{α_0} becomes:

$$T_0 X^\mu - X^{\mu+\theta} q^2 T_0 = (t_0 - t_0^{-1}) X^\mu, \quad \langle \mu, \theta^\vee \rangle = 1.$$

We now record some useful properties of the DAHA that we will use:

Theorem 4 (Theorem 4.2, [Kir97]). *Every element $h \in \ddot{\mathbf{H}}$ can be written uniquely in the form:*

$$\sum_{\substack{\mu\in\mathbf{X}\\ \lambda\in\mathbf{Y}\\ w\in W}} a_{\mu\lambda w} X^{\mu} Y^{\lambda^\vee} T_w, \quad a_{\mu\lambda w} \in \mathbb{C}[q^{\pm}, t_{\alpha}^{\pm}].$$

Equivalently, Theorem 4 states that there is a triangular decomposition of $\ddot{\mathbf{H}}$:

$$\ddot{\mathbf{H}} \cong \mathbb{C}[\mathbf{X}] \otimes_{\mathbb{C}} \mathbf{H} \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{Y}]. \tag{1.10}$$

Note that the triangular decomposition is an isomorphism of $\mathbb{C}$-vector spaces, *not* algebras. Moreover, we also have $\mathbb{C}$-vector isomorphisms

$$\mathbb{C}[\mathbf{X}] \otimes_{\mathbb{C}} \mathbf{H} \cong \mathbf{H} \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{X}], \quad \mathbb{C}[\mathbf{Y}] \otimes_{\mathbb{C}} \mathbf{H} \cong \mathbf{H} \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{Y}],$$

by the Bernstein presentation of AHA.

Theorem 5 (Theorem 4.3, [Kir97]). *The following formulas give a faithful representation of $\ddot{\mathbf{H}}$ in $\mathbb{C}[q^{\pm}, t_{\alpha}^{\pm}][X^{\pm}]$:*

$$X^{\mu} \longmapsto X^{\mu},$$

$$\pi : X^{\mu} \longmapsto X^{\pi(\mu)}, \quad \pi \in \Omega,$$

$$T_i \longmapsto t_i s_i + (t_i - t_i^{-1}) \frac{s_i - 1}{X^{-\alpha_i} - 1}, \quad i = 0, \cdots, n. \tag{1.11}$$

The formula (1.11) is called the *Demazure-Lusztig operator* (c.f. [BEG00, (6.1)]). Given the group algebra $\mathbb{C}[\mathbf{X}]$, one may construct the algebraic torus

$$T = \operatorname{Spec} \mathbb{C}[\mathbf{X}],$$

which as a group scheme is isomorphic to $\mathbb{G}_{\mathrm{m}} \otimes_{\mathbb{Z}} \mathbf{X}$. If the root system has rank r, then $T \cong \mathbb{G}_{\mathrm{m}}^r$. One may identify T as the maximal torus of a suitable split, reductive algebraic group scheme.

Moreover, one may identify $\mathbb{C}[t_{\alpha}^{\pm}]$ as the $\mathbb{C}$-rational points of the group scheme T, and $\mathbb{C}[q^{\pm}]$ as the $\mathbb{C}$-rational points of the group scheme $\mathbb{G}_{\mathrm{m}}$. In this case, the torus $T \times \mathbb{G}_{\mathrm{m}}$ is the torus corresponding to the affine root system $\widetilde{\Phi}$. In particular, it is the maximal torus of a split, p-adic reductive group (see [Cai22], [HKP10]).

Then, with this formalism, one may view the Demazure-Lusztig operator as a rational function in $\mathbb{C}[T \times \mathbb{G}_{\mathrm{m}}]$. Moreover, one may view the elements X^{μ} as functions over $T \times \mathbb{G}_{\mathrm{m}}$ as well. This formalism will become more useful when we consider the quantum torus in Chapter 3. Alternatively, the interested reader can also consult [BEG00, §5, §6].

Recall from (1.6) that operator $\tau(\mu^\vee) : \mathbf{X} \to \mathbf{X}$. One can extend this to a map on $\mathbb{C}[q^{\pm}, t_{\alpha}^{\pm}][\mathbf{X}] \to \mathbb{C}[q^{\pm}, t_{\alpha}^{\pm}][\mathbf{X}]$. Using the identification $X^{\delta} = q^{-2}$ in (1.9) , we obtain a *q-shift operator* given by

$$\tau(\mu^\vee) : \mathbb{C}[q^{\pm}, t_{\alpha}^{\pm}][\mathbf{X}] \longrightarrow \mathbb{C}[q^{\pm}, t_{\alpha}^{\pm}][\mathbf{X}], \quad X^{\lambda} \longmapsto q^{2\langle \mu^\vee, \lambda \rangle} X^{\lambda}, \tag{1.12}$$

for some weight $\lambda \in \mathbf{X}$. Recall that in terms of the Iwahori-Matsumoto presentation, we can write the lattice term $Y^{\mu^\vee} = T_{\tau(\mu^\vee)}$ where $\tau(\mu^\vee)$ acts on the affine weights by the formula: $\tau(\mu^\vee) : \lambda \mapsto \lambda - \langle \alpha^\vee, \lambda \rangle \delta$. We may consider meromorphic functions on the algebraic torus $T = \operatorname{Spec} \mathbb{C}[\mathbf{X}]$, which we will denote by $\mathbb{C}(T)$. By abuse of notation, we will also denote this by $\mathbb{C}(\mathbf{X})$ to emphasise that we are considering meromorphic functions coming from the group scheme Spec $\mathbb{C}[\mathbf{X}]$. The action of the q-shift operator is preserved under localisation. Moreover, $\mathbb{C}(\mathbf{X})$ can be equipped with a natural W-action, and thus one may form the *twisted tensor product*, which we denote as

$$\mathbb{C}(\mathbf{X}) \rtimes \mathbb{C}[W].$$

We elaborate more on this in Chapter 3. But this object is what is known as the *quantum torus*, and its construction is due to [BEG00]. The action of the q-shift operator naturally extends to elements in the quantum torus as well. These are examples of q-difference equations of multiple variables. As aforementioned, we wish to study q-difference equations of one variable, and we must therefore restrict ourselves to the rank one case.

In this case, there is only one fundamental weight and so $\mathbb{C}(\mathbf{X}) = \mathbb{C}(X)$ – the algebra of meromorphic functions in one variable X. As such, one may form the category

$$\mathbf{DiffEq}(\mathbb{C}(X) \rtimes \mathbb{C}[W], \tau(\mu^\vee)),$$

of q-difference equations in the sense of [Sau03]. We will explore this connection between the DAHA and q-difference equations in more detail in the later chapters.

1.3. The Elliptic Affine Hecke Algebra (EllAHA)

In this section, we briefly go through the construction of the elliptic affine Hecke algebra, which was originally introduced in [GKV95]. We follow [ZZ15, §4] and [GKV95, §4] for this construction. Let $T = \mathrm{Spec}(\mathbb{C}[\mathbf{X}])$ be an algebraic torus, and let $\mathbb{X}^*(T)$ be the character lattice, and $\mathbb{X}_*(T)$ be the co-character lattice. We have $\mathbb{Z}$-module isomorphisms $\mathbf{X} \cong \mathbb{X}^*(T)$ and $\mathbf{Y} \cong \mathbb{X}_*(T)$.

Fix an abelian variety $\mathfrak{A} := E \otimes_{\mathbb{Z}} \mathbb{X}^*(T)$, where $\mathbb{X}^*(T)$ is the weight lattice, and E is an elliptic curve. Note that $E \otimes_{\mathbb{Z}} \mathbb{X}^*(T) \cong E^r$, where r is the rank of the root system. Again, let Φ be the finite subset of X^* defining a root system. Let us fix an element $\hbar \in E$, and let

$$\chi_\alpha : E \otimes_{\mathbb{Z}} \mathbb{X}^*(T) \longrightarrow E, \quad \hbar \otimes \mu^\vee \longrightarrow \hbar^{\langle \mu^\vee, \alpha \rangle}.$$

Then, associated to each $\alpha \in \Phi$, let

$$T_\alpha := \ker \chi_\alpha,$$

be the kernel divisor of the map χ_α, and let

$$T_{\alpha,\hbar} := \ker(\chi_\alpha - \hbar),$$

for some $\hbar \in \mathfrak{A}$. The Weyl group acts on the kernel divisor by $wT_\alpha = T_{w^{-1}\alpha}$, and $T_{-\alpha} = T_\alpha$. One thinks of these divisors T_α as root hyperplanes.

The action of W on $\mathbb{X}^*(T)$ naturally extends to an action on $\mathfrak{A}$. Thus, let $\mathfrak{A}/W$ be the set of its W-orbits, which is is defined to be the affine GIT quotient $\mathfrak{A}/W = \mathrm{Spec}(\mathcal{O}_{\mathfrak{A}}(\mathfrak{A})^W)$. Let $\pi : \mathfrak{A} \to \mathfrak{A}/W$ be the natural map.

The sheaf $\pi_* \mathcal{O}_{\mathfrak{A}}$ is equipped with a natural action of W. Let

$$\mathcal{O}[W] := \pi_* \mathcal{O}_{\mathfrak{A}} \rtimes \mathbb{C}[W],$$

where $\rtimes$ denotes the twisted tensor product of $\pi_*\mathcal{O}_{\mathfrak{A}}$ and $\mathbb{C}[W]$. The algebra structure is given by

$$(f \otimes w)(g \otimes v) = (f \cdot {}^w g) \otimes (w \cdot v),$$

where ${}^w g$ is the image of g under the map

$$w^{-1} : \pi_*\mathcal{O}_{\mathfrak{A}}(U) \longrightarrow \pi_*\mathcal{O}_{\mathfrak{A}}(w^{-1}U).$$

This equips $\mathcal{O}[W]$ with the structure of a sheaf of associative algebras, Observe that by our construction, $\mathcal{S}$ is a sheaf of modules over $\mathcal{S}_W$. On any open subset, let us denote sections of $\mathcal{S}_W$ coming from $w \in W$ by δ_w. For any root $\alpha \in \Phi$, we will simply write $\delta_\alpha := \delta_{s_\alpha}$.

Now, let

$$\mathfrak{A}^{\mathrm{reg}} := \mathfrak{A} \setminus \left(\bigcup_{\alpha \in \Phi} T_\alpha \right),$$

so that W acts on $\mathfrak{A}^{\mathrm{reg}}$ freely. Indeed, a simple reflection s_α acts by $s_\alpha T_\alpha = T_{s_\alpha(\alpha)} = T_{-\alpha} = T_\alpha$, and thus the stabiliser is non-trivial on each divisor T_α. Removing these root hyperplanes from $\mathfrak{A}$ thus allows W to act freely. Let

$$j : \mathfrak{A}^{\mathrm{reg}}/W \longrightarrow \mathfrak{A}/W,$$

denote the inclusion. The action of $\mathcal{O}[W]$ on $\pi_*\mathcal{O}_{\mathfrak{A}}$ induces a morphism

$$\rho : \mathcal{O}[W]|_{\mathfrak{A}^{\mathrm{reg}}} \longrightarrow \mathscr{E}\mathrm{nd}(\pi_*\mathcal{O}_{\mathfrak{A}}|_{\mathfrak{A}^{\mathrm{reg}}}).$$

Pushing forward by j, we get a map:

$$j_*\rho : j_*\left(\mathcal{O}[W]|_{\mathfrak{A}^{\mathrm{reg}}}\right) \longrightarrow j_*\left(\mathscr{E}\mathrm{nd}(\pi_*\mathcal{O}_{\mathfrak{A}}|_{\mathfrak{A}^{\mathrm{reg}}}\right).$$

Let

$$p : \mathscr{E}\mathrm{nd}(\pi_*\mathcal{O}_{\mathfrak{A}}) \longrightarrow j_*\,\mathscr{E}\mathrm{nd}(\pi_*\mathcal{O}_{\mathfrak{A}}|_{\mathfrak{A}^{\mathrm{reg}}}),$$

be the canonical map.

Definition 11. Let f_w be local sections of $j_*(\pi_*\mathcal{O}_{\mathfrak{A}}|_{\mathfrak{A}^{\mathrm{reg}}})$. Then, following our notation from before, we will write sections of $j_*(\mathcal{O}[W]|_{\mathfrak{A}^{\mathrm{reg}}})$ as $\sum_{w \in W} f_w \delta_w$. The *elliptic affine Hecke algebra* $\mathcal{H}^{\mathrm{ell}}$ is the quasi-coherent subsheaf of

$$(j_*\rho)^{-1}\left(p\left(\mathscr{E}\mathrm{nd}(\pi_*\mathcal{O}_{\mathfrak{A}}|_{\mathfrak{A}^{\mathrm{reg}}})\right)\right),$$

whose local sections satisfy:

(a) each f_w has a pole of order at most one along T_α.

(b) $\mathrm{Res}_{T_\alpha}(f_w) + \mathrm{Res}_{T_\alpha}(f_{s_\alpha w}) = 0$.

(c) for any $\alpha \in \Phi(w) := \Phi^+ \cap w^{-1}\Phi^-$, f_w vanishes on $T_{\alpha,\hbar}$ for some $\hbar \in \mathfrak{A}$.

Theorem 6 (Proposition 4.3, [GKV95])**.** *$\mathcal{H}^{\mathrm{ell}}$ is a subsheaf of algebras in $j_*(\pi_*\mathcal{O}_{\mathfrak{A}}|_{\mathfrak{A}^{\mathrm{reg}}} \rtimes \mathbb{C}[W])$.*

1.4. The Module Category $\mathbf{Coh}(\mathcal{H}^{\mathrm{ell}})$

Consider the object

$$\mathcal{H}^{\mathrm{ell}} \in \mathbf{Coh}(\mathfrak{A}/W).$$

Over this category, the rational sections of $\mathcal{H}^{\mathrm{ell}}$ can be equipped with the structure of an algebra. This is only possible after taking the pushforward into $\mathbf{Coh}_W(\mathfrak{A}/W)$ (see [ZZ15, §4]). Over the category

$\mathbf{Coh}_W(\mathfrak{A})$, a convolution construction is needed in order to define this algebra structure (see [ZZ15, §5], [ZZ24, §2], Appendix C).

Definition 12. A $\mathcal{H}^{\mathrm{ell}}$*-module* $\mathcal{M}$ is an object in $\mathbf{Coh}(\mathfrak{A}/W)$, together with a multiplication map for which there exists a map:

$$\mathcal{H}^{\mathrm{ell}} \otimes_{\mathcal{O}_{\mathfrak{A}/W}} \mathcal{M} \longrightarrow \mathcal{M},$$

defined by multiplication, such that for each open set $U \subseteq \mathfrak{A}/W$, each regular sections $\mathcal{M}(U)$ has the structure of a $\mathcal{H}^{\mathrm{ell}}(U)$-module.

Let

$$\mathbf{Coh}(\mathcal{H}^{\mathrm{ell}}),$$

denote the category of $\mathcal{H}^{\mathrm{ell}}$-modules. Note that there is a subsheaf of algebras $\pi_*\mathcal{O}_{\mathfrak{A}} \subseteq \mathcal{H}^{\mathrm{ell}}$. Given some $\mathcal{M} \in \mathbf{Coh}(\mathfrak{A}/W)$, and a module action map

$$\pi_*\mathcal{O}_{\mathfrak{A}} \otimes_{\mathfrak{A}/W} \mathcal{M} \longrightarrow \mathcal{M},$$

there exists some sheaf $\widetilde{\mathcal{M}} \in \mathbf{Coh}(\mathfrak{A})$ such that

$$\mathcal{M} \cong \pi_*\widetilde{\mathcal{M}}.$$

We say that $\widetilde{\mathcal{M}}$ is the *underlying sheaf* of the $\pi_*\mathcal{O}_{\mathfrak{A}}$-module $\mathcal{M}$. By abuse of notation, we may consider $\mathcal{M}$ as a coherent sheaf on $\mathfrak{A}$. Let

$$\mathbf{Coh}^{\mathrm{fin}}(\mathcal{H}^{\mathrm{ell}}),$$

be the full subcategory of $\mathbf{Coh}(\mathcal{H}^{\mathrm{ell}})$ whose underlying coherent sheaf has zero-dimensional support in $\mathfrak{A}$, and let

$$\mathbf{Coh}^{\mathrm{flat}}(\mathcal{H}^{\mathrm{ell}}),$$

be the full subcategory of $\mathbf{Coh}(\mathcal{H}^{\mathrm{ell}})$ whose underlying coherent sheaf is a homogeneous vector bundle (i.e. locally free, coherent sheaf) on $\mathfrak{A}$. Homogeneous means that $\mathcal{M}$ is translation-invariant. That is, given left and right translation maps $L_g, R_g : \mathfrak{A} \to \mathfrak{A}$, we have isomorphisms $L_g^*\mathcal{M} \cong \mathcal{M} \cong R_g^*\mathcal{M}$.

Chapter 2

Category $\mathcal{O}$ of the Double Affine Hecke Algebra

In this chapter, we study a category $\mathcal{O}_{\ddot{\mathbf{H}}}$ of $\ddot{\mathbf{H}}$-modules called the *category* $\mathcal{O}$ of DAHA. This is a full subcategory of $\ddot{\mathbf{H}}$-**Mod** containing certain well-behaved $\ddot{\mathbf{H}}$-modules that we would like to consider. The definition of category $\mathcal{O}$ we use is due to [Che03], and [JV19]. In particular, we use the definition of [JV19], who defined $\mathcal{O}_{\ddot{\mathbf{H}}}$ for the type A case. [Che03] defines this category in more generality. The definition is similar to the one given in [GGOR03] for the rational DAHA, except the local nilpotency condition is replaced with a locally-finiteness condition.

The main result of this chapter is Proposition 1, which shows that every object in $\mathcal{O}_{\ddot{\mathbf{H}}}$ admits a particular filtration known as a Δ-*filtration*. Note that Proposition 1 is a modification of [GGOR03, Proposition 2.2] for the locally finite case.

Further, we also weaken [GGOR03]'s definition of a Δ-filtration – in particular, our Δ-filtrations can be of infinite length, and our composition factors do not have to be precisely isomorphic to DAHA modules induced from irreducible $\ddot{\mathbf{H}}^{\mathbf{Y}}$-modules (c.f. [GGOR03, Definition 2.3.3]).

However, in the next chapter we will construct the *localised* DAHA – denoted as $\ddot{\mathbf{H}}_{\mathrm{loc}}$. In this case, the category $\mathcal{O}_{\ddot{\mathbf{H}}}$ admits Δ-filtrations in the sense of [GGOR03]. That is, Δ-filtrations over the localised category $\mathcal{O}_{\ddot{\mathbf{H}}}$ all have finite length, and its composition factors are precisely isomorphic to DAHA modules induced from irreducible $\ddot{\mathbf{H}}^{\mathbf{Y}}$-modules (see Lemma 9).

2.1. Locally Finite Modules

We record the following definition of the category $\mathcal{O}$ of DAHA from [JV19]:

Definition 13 (Definition 5.1, [JV19])**.** The *category* $\mathcal{O}$ for DAHA — denoted $\mathcal{O}_{\ddot{\mathbf{H}}}$ — is the full subcategory of finitely-generated $\ddot{\mathbf{H}}$-modules such that for each finitely-generated module M, and $m \in M$, the its orbit $\mathbf{Y} \cdot m$ is finite-dimensional. We say that such a module M is $\mathbb{C}[\mathbf{Y}]$-*locally finite.*

This definition of category $\mathcal{O}$ is similar to the one given by Cherednik in [Che03, §6]. Further, Cherednik showed that the full subcategory of finite-generated $\mathbb{C}[\mathbf{X}]$ or $\mathbb{C}[\mathbf{Y}]$-locally finite $\ddot{\mathbf{H}}$-modules gives rise to the same category. For our purposes, we will work with $\mathbb{C}[\mathbf{Y}]$-locally finite modules.

Let $T^\vee = \operatorname{Spec}\mathbb{C}[\mathbf{Y}]$ be an algebraic torus. As a group scheme, it is isomorphic to $\mathbb{G}_{\mathrm{m}} \otimes_{\mathbb{Z}} \mathbf{Y}$. $T^\vee$ is the dual torus to $T = \operatorname{Spec}\mathbb{C}[\mathbf{X}] \cong \mathbb{G}_{\mathrm{m}} \otimes_{\mathbb{Z}} \mathbf{X}$. By construction, T is equipped with a Weyl group action. We follow [JV19, §5] for the definitions below. Let M be any $\ddot{\mathbf{H}}$-module. Then, for any $\lambda \in T$, its $\mathbb{C}[\mathbf{Y}]$-*weight space* is defined to be

$$M_\lambda := \{v \in M : Y^{\mu^\vee} v = \lambda(Y^{\mu^\vee})v, \text{ for } Y^{\mu^\vee} \in \mathbb{C}[\mathbf{Y}]\}.$$

We call a non-zero vector $v \in M_\lambda$ a *weight vector*. Its generalised weight space is given by:

$$M_\lambda^{\text{gen}} := \{v \in M : (Y^{\mu^\vee} - \lambda(Y^{\mu^\vee}))^m v = 0 \text{ for all } Y^{\mu^\vee} \in \mathbb{C}[\mathbf{Y}],\, m \geq 0\}.$$

Since we are working over $\mathbb{C}$, any $M \in \mathcal{O}_{\ddot{\mathbf{H}}}$ splits over $\mathbb{C}$, and thus there is a $\mathbb{C}$-vector space isomorphism:

$$M \cong \bigoplus_{\lambda \in T} M_\lambda^{\text{gen}}.$$

This is, in fact, an isomorphism of $\mathbb{C}[\mathbf{Y}]$-modules as well. Let $[\lambda] \in T/W$. Then, we may construct the weight space $M_{[\lambda]}^{\text{gen}}$, which obtains the structure of a $\ddot{\mathbf{H}}$-module. Then, for any $\ddot{\mathbf{H}}$-module M, we have a decomposition of $\ddot{\mathbf{H}}$-modules:

$$M \cong \bigoplus_{[\lambda] \in T/W} M_{[\lambda]}^{\text{gen}}.$$

2.2. Standard Modules

Here, we construct an appropriate filtration of $\ddot{\mathbf{H}}$-representations, similar to the ones seen in [GGOR03, §2.3] for the rational DAHA, but adapted to the DAHA case.

For each eigenvalue $\lambda \in T$ of some element in $\mathbb{C}[\mathbf{Y}]$, the ideal $(Y^{\mu^\vee} - \lambda)$ is maximal in $\mathbb{C}[\mathbf{Y}]$. Denote this maximal ideal by $\mathfrak{m}_\lambda := (Y^{\mu^\vee} - \lambda)$, and let us consider the weight space $\mathbb{C}[\mathbf{Y}]/\mathfrak{m}_\lambda$. Moreover, the generalised weight space $\mathbb{C}[\mathbf{Y}]/\mathfrak{m}_\lambda^k$ is finite-dimensional, and has the structure of a $\mathbb{C}[\mathbf{Y}]$-module. The *standard module* is the $\dot{\mathbf{H}}^{\mathbf{Y}}$-module given by:

$$\delta_\lambda := \operatorname{Ind}_{\mathbb{C}[\mathbf{Y}]}^{\dot{\mathbf{H}}^{\mathbf{Y}}} \left(\mathbb{C}[\mathbf{Y}]/\mathfrak{m}_\lambda\right).$$

Remark 3. Note that our definition of standard module here is a modification of one used in [GGOR03].

Lemma 2. *Each irreducible $\dot{\mathbf{H}}^{\mathbf{Y}}$-module V is a quotient of a standard module δ_λ for some λ.*

Proof. Let V be an irreducible $\dot{\mathbf{H}}^{\mathbf{Y}}$-module. Then, by Frobenius reciprocity, there is an isomorphism

$$\operatorname{Hom}_{\dot{\mathbf{H}}^{\mathbf{Y}}}(\delta_\lambda, V) \cong \operatorname{Hom}_{\mathbb{C}[\mathbf{Y}]}(\mathbb{C}[\mathbf{Y}]/\mathfrak{m}_\lambda, V).$$

As a $\mathbb{C}[\mathbf{Y}]$-module, V admits a weight space decomposition

$$V \cong \bigoplus_{\lambda \in T} V_\lambda,$$

and thus there exists a non-zero map $\mathbb{C}[\mathbf{X}]/\mathfrak{m}_\lambda \to V$. It follows therefore that there must exist non-trivial maps on the left-hand side. Moreover, by the irreducibility of V as a $\ddot{\mathbf{H}}^{\mathbf{Y}}$-module, all such $\ddot{\mathbf{H}}^{\mathbf{Y}}$-homeomorphisms $\delta_\lambda \to V$ must be surjective, and thus V arises as some quotient of δ_λ. $\square$

Induce the standard module to the DAHA,

$$\Delta(\delta_\lambda) := \operatorname{Ind}_{\dot{\mathbf{H}}^{\mathbf{Y}}}^{\ddot{\mathbf{H}}} \delta_\lambda,$$

and note the following property:

Lemma 3. *The functor* $\Delta = \mathrm{Ind}_{\dot{\mathbf{H}}^{\mathbf{Y}}}^{\ddot{\mathbf{H}}}$ *is exact.*

Proof. Let V be any finite-dimensional $\dot{\mathbf{H}}^{\mathbf{Y}}$-module. Then, by the PBW theorem of DAHA (1.10), there is an isomorphism of $\mathbb{C}$-vector spaces

$$\Delta(V) := \ddot{\mathbf{H}} \otimes_{\dot{\mathbf{H}}^{\mathbf{Y}}} V \cong \mathbb{C}[\mathbf{X}] \otimes_{\mathbb{C}} V.$$

It follows that $\Delta(V)$ is a free module over $\dot{\mathbf{H}}^{\mathbf{Y}}$. All free modules are flat, and thus Δ is exact. □

2.3. Δ-Filtrations of $\ddot{\mathbf{H}}$-modules

Definition 14. A Δ-*filtration* by a $\ddot{\mathbf{H}}$-module M is a (possibly infinite) filtration

$$0 = M_0 \subset M_1 \subset \cdots,$$

such that $\bigcup_i M_i = M$, and each composition factor M_{i+1}/M_i is a quotient of some $\Delta(E_i)$, where E_i is an irreducible $\dot{\mathbf{H}}^{\mathbf{Y}}$-module — that is, $M_i/M_{i-1} \twoheadrightarrow \Delta(E_i)$.

Remark 4. We remark that our definition of Δ-filtration here is a weaker version of the one given in [GGOR03, §2.3.3] — in particular, the authors in *loc. cit.* requires that the Δ-filtation is a finite filtration, and that each composition factor is precisely isomorphic to some $\Delta(V)$, for irreducible V. This is done in order to satisfy the axioms for a highest weight category in the sense of [CPS88]. However, we do not require such a strong condition in our case. In fact, it is likely that such Δ-filtrations do not exist in general in the category $\mathcal{O}$ of DAHA.

As a consequence of Lemma 2, we have the following:

Corollary 1. *A* $\ddot{\mathbf{H}}$*-module* M *admits a* Δ*-filtration if and only if it admits a filtration whose composition factors are quotients of* $\Delta(\delta_\lambda)$ *for some* λ*.*

We may define more general standard modules of the form:

$$\delta_\lambda^k := \mathrm{Ind}_{\mathbb{C}[\mathbf{Y}]}^{\dot{\mathbf{H}}^{\mathbf{Y}}} \left(\mathbb{C}[\mathbf{Y}]/\mathfrak{m}_\lambda^k\right),$$

which we then induce to the DAHA:

$$\Delta(\delta_\lambda^k) = \mathrm{Ind}_{\dot{\mathbf{H}}^{\mathbf{Y}}}^{\ddot{\mathbf{H}}} \delta_\lambda^k.$$

From this, we then have the following result:

Lemma 4. $\Delta(\delta_\lambda^k)$ *admits a* Δ*-filtration.*

Proof. Each generalised standard module δ_λ^k is filtered by the weight space $\mathbb{C}[\mathbf{Y}]/\mathfrak{m}_\lambda^k$ by

$$0 \subseteq \mathbb{C}[\mathbf{Y}]/\mathfrak{m}_\lambda \subseteq \mathbb{C}[\mathbf{Y}]/\mathfrak{m}_\lambda^2 \subseteq \cdots,$$

with composition factors of the form $\mathfrak{m}_\lambda^{i-1}/\mathfrak{m}_\lambda^i$ by the third isomorphism theorem for modules. Each composition factor $\mathfrak{m}_\lambda^{i-1}/\mathfrak{m}_\lambda^i$ has the structure of a vector space over $\mathbb{C}[\mathbf{Y}]/\mathfrak{m}_\lambda$, and thus is isomorphic to a direct sum of copies of $\mathbb{C}[\mathbf{Y}]/\mathfrak{m}_\lambda$.

$\Delta(\delta_\lambda^k)$ has a filtration of the form

$$0 \subseteq \Delta(\delta_\lambda) \subseteq \Delta(\delta_\lambda^2) \subseteq \cdots \subseteq \Delta(\delta_\lambda^k),$$

whose factors are

$$\Delta(\delta_\lambda^i)/\Delta(\delta_\lambda^{i-1}) \cong \frac{\mathbb{C}[\mathbf{X}] \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{Y}]/\mathfrak{m}_\lambda^i}{\mathbb{C}[\mathbf{X}] \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{Y}]/\mathfrak{m}_\lambda^{i-1}} \cong \mathbb{C}[\mathbf{X}] \otimes_{\mathbb{C}} (\mathfrak{m}_\lambda^{i-1}/\mathfrak{m}_\lambda^i) \cong \Delta(\mathfrak{m}_\lambda^{i-1}/\mathfrak{m}_\lambda^i),$$

where the first and last isomorphisms follows by the PBW theorem (1.10), and the second isomorphism follows by the exactness of Δ Lemma 3. As aforementioned, the composition factors $\mathfrak{m}_\lambda^{i-1}/\mathfrak{m}_\lambda^i$ is isomorphic to a direct sum of copies of $\mathbb{C}[\mathbf{Y}]/\mathfrak{m}_\lambda$. Let $n_{i,\lambda} := \dim_{\mathbb{C}[\mathbf{Y}]/\mathfrak{m}_\lambda}\left(\mathfrak{m}_\lambda^{i-1}/\mathfrak{m}_\lambda^i\right)$. Then,

$$\Delta(\mathfrak{m}_\lambda^{i-1}/\mathfrak{m}_\lambda^i) \cong \Delta(\delta_\lambda)^{\oplus n_{i,\lambda}},$$

which follows since the induction functor is additive. It thus follows that $\Delta(\delta_\lambda^k)$ is filtered by $\Delta(\delta_\lambda)$. Applying Corollary 1 proves the result. $\square$

Lemma 5. *Let $M \in \mathcal{O}_{\ddot{\mathbf{H}}}$. Then, for each $m \in M$, there exists $k \in \mathbb{N}$ and a map $\varphi_m \in \operatorname{Hom}_{\ddot{\mathbf{H}}}(\Delta(\delta_\lambda^k), M)$ such that $m \in \operatorname{Im}(\varphi_m)$.*

Proof. Applying Frobenius reciprocity twice gives us isomorphisms:

$$\operatorname{Hom}_{\ddot{\mathbf{H}}}(\Delta(\delta_\lambda^k), M) \cong \operatorname{Hom}_{\dot{\mathbf{H}}^{\mathbf{Y}}}(\delta_\lambda^k, M) \cong \operatorname{Hom}_{\mathbb{C}[\mathbf{Y}]}(\mathbb{C}[\mathbf{Y}]/\mathfrak{m}_\lambda^k, M).$$

As aforementioned, since M is $\mathbb{C}[\mathbf{Y}]$-semisimple, it follows that there exists a non-zero $\mathbb{C}[\mathbf{Y}]$-homomorphism $\widetilde{\varphi}_m : \mathbb{C}[\mathbf{Y}]/\mathfrak{m}_\lambda^k \to M \cong \bigoplus_\lambda M_\lambda^{\text{gen}}$ satisfying the required properties. Take the image of $\widetilde{\varphi}_m$ in $\operatorname{Hom}_{\ddot{\mathbf{H}}}(\Delta(\delta_\lambda^k), M)$ and we are done. $\square$

Lemma 6. *Let φ_m be defined as in Lemma 4. Then, each $\ddot{\mathbf{H}}$-submodule $\operatorname{Im}\varphi_m$ has a filtration by quotients of $\Delta(V_i)$, where V_i are irreducible $\dot{\mathbf{H}}^{\mathbf{Y}}$-modules.*

Proof. This follows from the fact that there exists a surjection $\Delta(\delta_\lambda^k) \twoheadrightarrow \operatorname{Im}\varphi_m$, and the fact that $\Delta(\delta_\lambda^k)$ admits a Δ-filtration. $\square$

Lemma 7. *Let $M \in \mathcal{O}_{\ddot{\mathbf{H}}}$, and let φ_m be defined as in Lemma 4. Then, for some $m \neq m' \in M$, there exists a a submodule N of M such that $N/\operatorname{Im}\varphi_m = \operatorname{Im}\varphi_{m'}$.*

Proof. For each non-zero $m' \in M$, there exists a map $\operatorname{Im}\varphi_{m'} \to M/\operatorname{Im}\varphi_m$. Define a map $\pi : M \twoheadrightarrow M/\operatorname{Im}\varphi_m$. Since $\operatorname{Im}\varphi_{m'} \subseteq M/\operatorname{Im}\varphi_{m'}$, we may consider the pre-image $\pi^{-1}(\operatorname{Im}\varphi_{m'})$, which is a submodule of M. From this, we also obtain a surjective map $\pi^{-1}(\operatorname{Im}\varphi_{m'}) \twoheadrightarrow \operatorname{Im}\varphi_{m'}$, whose kernel is $\operatorname{Im}\varphi_m$. Thus, by the first isomorphism theorem, we have that

$$\pi^{-1}(\operatorname{Im}(\varphi_{m'}))/\operatorname{Im}\varphi_m \cong \operatorname{Im}\varphi_{m'},$$

and we are done. $\square$

And we thus have the following characterisation of $\mathbb{C}[\mathbf{Y}]$-locally finite $\ddot{\mathbf{H}}$-modules:

Proposition 1. *Let M be a finitely-generated $\ddot{\mathbf{H}}$-module. Then, the following are equivalent:*

(i) $M \in \mathcal{O}_{\ddot{\mathbf{H}}}$,

(ii) M admits a Δ-filtration.

Proof. (i) $\implies$ (ii). Lemma 7 shows that M has a filtration by quotients of $\operatorname{Im}(\varphi_m)$. Then, Lemma 5 shows that each $\operatorname{Im}(\varphi_m)$ admits a Δ-filtration. Thus, M admits a Δ-filtration.

(ii) $\implies$ (i). Each composition factor of the filtration is of the form $\Delta(V)$, where V is an irreducible $\dot{\mathbf{H}}^{\mathbf{Y}}$-module. By the PBW theorem (1.10), there is an isomorphism of $\mathbb{C}$-vector spaces $\Delta(V) \cong \mathbb{C}[\mathbf{X}] \otimes_{\mathbb{C}} V$. Since M is finitely-generated, it follows that for any $m \in M$, the orbit m acts on $\Delta(V)$ from the right, and acts on V. $m \cdot \mathbf{Y}$ must also be finite. $\square$

2.4. Torsion Objects in $\mathcal{O}_{\ddot{\mathbf{H}}}$

In this section, we define the full subcategory $\mathcal{O}_{\ddot{\mathbf{H}}}^{\mathrm{tor}}$ of torsion objects in $\mathcal{O}_{\ddot{\mathbf{H}}}$. This will be important later when we construct the qKZ functor in the next chapter. In particular, one of our main theorems shows that the restriction of the qKZ functor to the subcategory of torsion-free objects $\mathcal{O}_{\ddot{\mathbf{H}}}/\mathcal{O}_{\ddot{\mathbf{H}}}^{\mathrm{tor}}$ is essentially surjective and fully faithful onto the representation category $\mathbf{Coh}(\mathcal{H}^{\mathrm{ell}})$ of the ellAHA.

By the $\mathbb{C}[\mathbf{Y}]$-locally finiteness condition of the category $\mathcal{O}$ of DAHA, it follows that given any $M \in \mathcal{O}_{\ddot{\mathbf{H}}}$, the action of any element of $\mathbb{C}[\mathbf{Y}]$ on M is torsion. As such, we instead wish to define the torsion objects of $\mathcal{O}_{\ddot{\mathbf{H}}}$ as those objects which are torsion with respect to $\mathbb{C}[\mathbf{X}]$. That is, elements $f(X) \in \mathbb{C}[\mathbf{X}]$ for which $f(X) \cdot M = 0$. Or equiavlently, treating M as a $\mathbb{C}[\mathbf{X}]$-module, M is torsion if $\mathrm{Ann}_{\mathbb{C}[\mathbf{X}]}(M) \neq 0$. Recall that the *support* of M – denoted $\mathrm{supp}(M)$ – is the non-vanishing locus of $\mathrm{Ann}_{\mathbb{C}[\mathbf{X}]}(M)$. We thus make the following definition:

Definition 15. An element $M \in \mathcal{O}_{\ddot{\mathbf{H}}}$ is *torsion* if its support $\mathrm{supp}(M)$ is a proper subvariety of the algebraic torus $\mathrm{Spec}\,\mathbb{C}[\mathbf{X}]$.

Let $\mathcal{O}_{\ddot{\mathbf{H}}}^{\mathrm{tor}}$ be the full subcategory of torsion objects in $\mathcal{O}_{\ddot{\mathbf{H}}}$. Moreover, we show that $\mathcal{O}_{\ddot{\mathbf{H}}}^{\mathrm{tor}}$ forms a Serre subcategory of $\mathcal{O}_{\ddot{\mathbf{H}}}$.

Definition 16. Let $\mathscr{A}$ be an abelian category. Then, a subcategory $\mathscr{C}$ is a *Serre subcategory* if for any short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

if $A, C \in \mathrm{Ob}(\mathscr{C})$, then $B \in \mathrm{Ob}(\mathscr{A})$.

Lemma 8. *The subcategory $\mathcal{O}_{\ddot{\mathbf{H}}}^{\mathrm{tor}}$ is a Serre subcategory.*

Proof. Consider a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

of objects in $\mathcal{O}_{\ddot{\mathbf{H}}}$. Suppose that M' and M'' are torsion. Then, M must be torsion since the union of two proper subvarieties is a proper variety. $\square$

As a result, one obtains the Serre quotient:

$$\mathcal{O}_{\ddot{\mathbf{H}}}/\mathcal{O}_{\ddot{\mathbf{H}}}^{\mathrm{tor}}.$$

In particular, objects of $\mathcal{O}_{\ddot{\mathbf{H}}}/\mathcal{O}_{\ddot{\mathbf{H}}}^{\mathrm{tor}}$ are objects of $\mathcal{O}_{\ddot{\mathbf{H}}}$, but the morphisms are given by:

$$\mathrm{Hom}_{\mathcal{O}_{\ddot{\mathbf{H}}}/\mathcal{O}_{\ddot{\mathbf{H}}}^{\mathrm{tor}}}(M, N) := \varinjlim_{M' \to M} \mathrm{Hom}_{\mathcal{O}_{\ddot{\mathbf{H}}}}(M', N),$$

where the limit is taken over "isomorphisms modulo $\mathcal{O}_{\ddot{\mathbf{H}}}^{\mathrm{tor}}$ $M' \to M$, meaning that its kernel and cokernel are both torsion.

Chapter 3

The Quantum Torus

This section will outline a key ingredient that we use in the construction of the qKZ functor: the *quantum torus* **qTor** of [BEG00]. We denote by $\mathbb{C}(\mathbf{X})$ the algebra of meromorphic functions on the algebraic torus $T = \operatorname{Spec}\mathbb{C}[\mathbf{X}]$. The quantum torus is a $\mathbb{C}(\mathbf{X})$-algebra generated by q-difference operators that is isomorphic to a localisation of the DAHA, which we denote by $\ddot{\mathbf{H}}_{\mathrm{loc}}$. In particular, there is an equivalence of categories

$$\ddot{\mathbf{H}}_{\mathrm{loc}}\text{-}\mathbf{Mod} \xrightarrow{\simeq} \mathbf{qTor}_W\text{-}\mathbf{Mod}, \tag{3.1}$$

where $\mathbf{qTor}_W$ is the twisted tensor product $\mathbf{qTor} \rtimes \mathbb{C}[W]$, called the *$W$-equivariant quantum torus.* The isomorphism is due to [BEG00, Theorem 7.2].

A special feature of the category $\mathcal{O}_{\ddot{\mathbf{H}}}$ is that all Δ-filtrations are of finite length, and each composition factor in the filtration is precisely isomorphic to a $\ddot{\mathbf{H}}$-module of the form $\Delta(V)$, where V is an irreducible $\dot{\mathbf{H}}^{\mathbf{Y}}$-module. This is proven in Lemma 9.

Another feature of $\mathcal{O}_{\ddot{\mathbf{H}}}$ is demonstrated in Proposition 2, which shows that the restriction of the functor (3.1) to the category $\mathcal{O}$ gives a functor

$$\mathcal{O}_{\ddot{\mathbf{H}}} \longrightarrow \mathbf{qTor}_W^{\mathrm{fuch}}\text{-}\mathbf{Mod},$$

into the Fuchsian subcategory of $\mathbf{qTor}_W$.

We give a definition of the qKZ functor, and outline the construction of this functor. In Chapter 4 we will outline the construction of this functor explicitly.

3.1. The Quantum Torus

As before, let $T = \operatorname{Spec}\mathbb{C}[\mathbf{X}]$, and $T^\vee = \operatorname{Spec}\mathbb{C}[\mathbf{Y}]$ be the algebraic dual torus of T. Let $\mathbb{C}(T)$ be the algebra of meromorphic functions on T. By abuse of notation, we will denote $\mathbb{C}(T)$ by $\mathbb{C}(\mathbf{X})$ to emphasise that we are considering meromorphic functions coming from the torus generated by the weight lattice. Similarly, for $\mathbb{C}(\mathbf{Y}) := \mathbb{C}(T^\vee)$. The *$q$-shift operator* acts on $\mathbb{C}(T)$ by:

$$\mathrm{D}_q^{\mu^\vee} f(t) = f(q^{2\mu}t),$$

which acts on meromorphic functions $f(X)$ on the torus T by

$$\mathrm{D}_q^{\mu^\vee} f(X) = f(q^{2\mu^\vee}X),$$

where $X = (X^{\omega_1}, \cdots, X^{\omega_i})$, where ω_i are the fundamental weights forming a basis of $\mathbf{X}$. Then, $q^{2\mu}X = (X^{\langle \mu^\vee, \omega_1\rangle}, \cdots, X^{\langle \mu^\vee, \omega_r\rangle})$. For some $X^\mu \in \mathbb{X}^*(T)$, we also have the following commutation relation:

$$\mathrm{D}_q^{\mu^\vee} \cdot X^\lambda \cdot (\mathrm{D}_q^{\mu^\vee})^{-1} = q^{2\langle \mu^\vee, \lambda\rangle} X^\lambda.$$

The algebra of meromorphic functions $\mathbb{C}(\mathbf{X})$, together with the q-differential operator $\mathrm{D}_q^{\mu^\vee}$, generate an algebra called the *quantum torus* – denoted by $\mathbf{qTor}$. It has an explicit presentation as a $\mathbb{C}(\mathbf{X})$-algebra by:

$$\mathbf{qTor} = \langle \mathrm{D}_q^{\mu^\vee} : \mathrm{D}_q^{\mu^\vee} X^\lambda (\mathrm{D}_q^{\mu^\vee})^{-1} = q^{2\langle \mu^\vee, \lambda\rangle} X^\lambda \text{ for } X^\lambda \in \mathbb{C}(\mathbf{X})\rangle.$$

For simplicity, let us write D_q for the q-shift operator when it is not necessary to specify $\mu^\vee$. One may define the module category

$$\mathbf{qTor}\text{-}\mathbf{Mod},$$

in two ways. One is to consider objects $M \in \mathbf{qTor}\text{-}\mathbf{Mod}$ as $\mathbb{C}(\mathbf{X})$-vector spaces, and morphisms as $\mathbb{C}(\mathbf{X})$-linear maps that respect the action by the q-shift operators. However, if we choose a basis $\{e_i\}_{i=1}^n$ for M, then elements of $\mathbf{qTor}\text{-}\mathbf{Mod}$ can be identified as pairs $(\mathbb{C}(\mathbf{X})^n, A)$, where $A \in \mathrm{GL}_n(\mathbb{C}(\mathbf{X}))$, and morphisms $(\mathbb{C}(\mathbf{X})^n, A) \to (\mathbb{C}(\mathbf{X})^p, B)$ are given by matrices $F \in \mathrm{Mat}_{p,n}(\mathbb{C}(\mathbf{X}))$ satisfying the gauge transformation relation $(\mathrm{D}_q F)A = BF$. In other words, objects of $\mathbf{qTor}\text{-}\mathbf{Mod}$ are q-difference systems

$$\mathrm{D}_q V(X) = A(X)V(X),$$

where $V(X) \in \mathbb{C}(\mathbf{X})^n$. This allows us to identify $\mathbf{qTor}\text{-}\mathbf{Mod}$ with the category of q-difference equations $\mathbf{DiffEq}(\mathbb{C}(\mathbf{X}), \mathrm{D}_q)$ of [Sau03].

This algebra inherits a Weyl group action, which allows us to define the twisted tensor product given by:

$$\mathbf{qTor} \rtimes \mathbb{C}[W],$$

with algebra structure given by

$$(f \otimes w) \cdot (g \otimes v) = (f \cdot {}^w g) \otimes (wv),$$

and ${}^w f(t) := f(w^{-1}t)$. In particular, the resulting twisted algebra has elements of the form:

$$\sum_{\substack{w\in W\\ \mu^\vee\in\mathbf{Y}}} h_{w,\mu^\vee} \mathrm{D}_q^{\mu}[w] : f \longmapsto \sum_{\substack{w\in W\\ \mu^\vee\in\mathbf{Y}}} h_{w,\mu^\vee} \mathrm{D}_q^{\mu}({}^w f), \quad h_{w,\mu^\vee} \in \mathbb{C}(T).$$

We call this resulting algebra the *W-equivariant quantum torus*, and we denote it by

$$\mathbf{qTor}_W.$$

Then, $\mathbf{qTor}_W\text{-}\mathbf{Mod}$ is the category of W-equivariant q-difference equations in $\mathbb{C}(\mathbf{X})$, which we will denote by $\mathbf{DiffEq}_W(\mathbb{C}(\mathbf{X}), \mathrm{D}_q)$.

3.2. The $\mathbb{Z}/2\mathbb{Z}$-equivariant Connection Category

We restrict ourselves to the rank one case for this section, as the connection category of [Sau03], only works for q-difference systems of one variable, which corresponds to the rank one localised DAHA. In this case, the Weyl group W is $\mathfrak{S}_2 \cong \mathbb{Z}/2\mathbb{Z}$.

As aforementioned in Chapter 1.1, there exists a connection category of $\mathcal{C}onn$ whose objects are triples

$(\mathcal{F}_0, \mathcal{F}_\infty, \varphi)$, where $\mathcal{F}_0$ and $\mathcal{F}_\infty$ are flat vector bundles on an elliptic curve E, and φ is the monodromy morphism $\varphi : \mathcal{F}_0 \dashrightarrow \mathcal{F}_\infty$. Since we are considering vector bundles $\mathcal{F}_0$ and $\mathcal{F}_\infty$ coming from fundamental solutions of equations from $\mathbf{qTor}_{\mathfrak{S}_2}$, one needs to take into the account the equivariance structure on the sheaves $\mathcal{F}_0$ and $\mathcal{F}_\infty$ as well.

Let

$$\mathcal{C}onn_W,$$

be the category of W-equivariant connection data, whose objects are triples $(\mathcal{F}_0, \mathcal{F}_\infty, \varphi)$, together with the action of the simple reflection $s \in \mathfrak{S}_2$ given by:

$$s^*(\mathcal{F}_0, \mathcal{F}_\infty, \varphi) := (s^*\mathcal{F}_\infty, s^*\mathcal{F}_0, w^*\varphi^{-1}),$$

and a morphism

$$B_s : (\mathcal{F}_0, \mathcal{F}_\infty, \varphi) \longrightarrow s^*(\mathcal{F}_0, \mathcal{F}_\infty, \varphi),$$

given by the commutative diagram

$$\begin{tikzcd}
\mathcal{F}_0 \arrow[r, dashed, "B_s^{(0)}"] \arrow[d, dashed, "\varphi"'] & s^*\mathcal{F}_\infty \arrow[d, dashed, "s^*\varphi^{-1}"] \\
\mathcal{F}_\infty \arrow[r, dashed, "B_s^{(\infty)}"'] & s^*\mathcal{F}_0
\end{tikzcd} \tag{3.2}$$

and with the property that

$$s^*B_s \circ B_s = B_e.$$

Taking the pushforwrad by $\pi : E \to E/\mathfrak{S}_2$, we have that $(\pi_*B_s)^2 = \mathrm{id}$. There are forgetful functors:

$$\mathcal{C}onn_W \longrightarrow \mathcal{C}onn, \quad \mathbf{qTor}_W^{\mathrm{fuch}} \longrightarrow \mathbf{qTor}^{\mathrm{fuch}}.$$

By construction, the following holds:

Corollary 2. *Applying the q-Riemann-Hilbert functor of [Sau03] to* $\mathbf{qTor}_W^{\mathrm{fuch}}$ *gives an equivalence of categories:*

$$\begin{tikzcd}
\mathbf{qTor}_W^{\mathrm{fuch}} \arrow[r, "\cong"] \arrow[d, "\mathrm{forget}"'] & \mathcal{C}onn_W \arrow[d, "\mathrm{forget}"] \\
\mathbf{qTor}^{\mathrm{fuch}} \arrow[r] & \mathcal{C}onn
\end{tikzcd}$$

3.3. Quantum Tori to DAHA

We follow [BEG00, §6] for this section. Recall that $T = \mathbb{G}_\mathrm{m} \otimes_\mathbb{Z} \mathbf{X}$. Since there is an isomorphism $\mathbb{X}^*(T) \cong \mathbf{X}$, we may identify elements $X^\mu \in \mathbb{C}[\mathbf{X}]$ with functions on T.

Let Φ be a root system of rank r, and let $\widehat{\Phi}$ denote the affine root system, and let $\widehat{T} = T \times \mathbb{G}_\mathrm{m}$ be the corresponding algebraic torus. We may identify the Laurent polynomial ring $\mathbb{C}[q^\pm]$ with the $\mathbb{C}$-points of the group scheme $\mathbb{G}_\mathrm{m}$, and $\mathbb{C}[t_\alpha^\pm]$ with the $\mathbb{C}$-points of T. The affine Hecke algebra associated to the affine Weyl group $W \ltimes \mathbf{X}$ has a faithful representation in $\mathbb{C}[\widehat{T}]$ via the Demazure-Lusztig operators (c.f. Theorem 5, [Kir97, Theorem 4.3], [BEG00, (6.1)]):

$$T_i = t_i s_i + (t_i - t_i^{-1})\frac{s_i - 1}{X^{\alpha_i} - 1}, \quad i = 0, 1, \cdots, r,$$

satisfying the Iwahori-Matsumoto relations for the AHA. Then, the double affine Hecke algebra $\ddot{\mathbf{H}}$ can be defined as the subalgebra of $\mathbb{C}[t_\alpha^{\pm}]$-linear endomorphisms of $\mathbb{C}[\widehat{T}][t_\alpha^{\pm}]$ generated by multiplication operators of the form T_i and X^μ, that commute via the Bernstein relation. The *localised DAHA* is then constructed in the following way:

$$\ddot{\mathbf{H}}_{\mathrm{loc}} := \ddot{\mathbf{H}} \otimes_{\mathbb{C}[\widehat{T}]} \mathbb{C}(\widehat{T}).$$

Recall that the DAHA itself comes equipped with a q-difference operator $\tau(\mu^\vee)$ (see (1.12)), that acts precisely the same way as $\mathrm{D}_q^{\mu^\vee} f(t)$. Thus, we have the following theorem from [BEG00]:

Theorem 7 (Theorem 7.2, [BEG00])**.** *There is an isomorphism:*

$$\ddot{\mathbf{H}}_{\mathrm{loc}} \cong \mathbf{qTor}_W\,.$$

Equivalently, there is an equivalence of categories

$$\ddot{\mathbf{H}}_{\mathrm{loc}}\,\text{-}\mathbf{Mod} \cong \mathbf{qTor}_W\,\text{-}\mathbf{Mod}\,.$$

Given any $\ddot{\mathbf{H}}$-module M, let

$$M_{\mathrm{loc}} := M \otimes_{\mathbb{C}[\widehat{T}]} \mathbb{C}(\widehat{T}),$$

denote the corresponding $\ddot{\mathbf{H}}_{\mathrm{loc}}$-module. Denote by $\mathrm{Loc} : \mathcal{O}_{\ddot{\mathbf{H}}} \to \mathbf{qTor}_W\,\text{-}\mathbf{Mod}$ the *localisation functor* given by the composition:

$$\begin{tikzcd} \mathcal{O}_{\ddot{\mathbf{H}}} \arrow[r] \arrow[rr, bend right, "\mathrm{Loc}"'] & \ddot{\mathbf{H}}_{\mathrm{loc}}\,\text{-}\mathbf{Mod} \arrow[r, "\cong"] & \mathbf{qTor}_W\,\text{-}\mathbf{Mod} \end{tikzcd}$$

Remark 5. Note that [BEG00, Theorem 7.2] is stated in terms of technical residue conditions, similar to the ones seen in [GKV95] for the defintion of ellAHA. However, for our purposes, we unpack [BEG00]'s theorems and definitions only to the extent that we require.

3.4. Recipe for a qKZ Functor

Since $\mathbf{qTor}_W$ is generated by meromorphic functions on T, and q-difference operators, it follows that the category $\mathbf{qTor}_W\,\text{-}\mathbf{Mod}$ can be identified with the category of q-difference equations $\mathbf{DiffEq}(\mathbb{C}(T), \mathrm{D}_q^{\mu^\vee})$ of [Sau03]. Let $\mathbf{qTor}_W^{\mathrm{fuch}}\,\text{-}\mathbf{Mod}$ be the full subcategory of (strictly) Fuchsian q-difference equations.

Lemma 9. *Let M_{loc} be the image of $M \in \mathcal{O}_{\ddot{\mathbf{H}}}$ under the localisation functor. Then, M_{loc} admits a finite Δ-filtration whose composition factors are isomorphic to $\Delta(V)$, where V is an irreducible $\ddot{\mathbf{H}}^{\mathbf{Y}}$-module.*

Proof. Let $M \in \mathcal{O}_{\ddot{\mathbf{H}}}$, and let

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M,$$

be a Δ-filtration. Then, by definition we have that $\Delta(V) \twoheadrightarrow M_i/M_{i-1}$. Observe that if V is an irreducible $\ddot{\mathbf{H}}^{\mathbf{Y}}$-module, then,

$$\Delta(V)_{\mathrm{loc}} = (\mathbb{C}[\mathbf{X}] \otimes_{\mathbb{C}} V)_{\mathrm{loc}} = \mathbb{C}[\mathbf{X}] \otimes_{\mathbb{C}} V \otimes_{\mathbb{C}(\widehat{T})} \mathbb{C}(\widehat{T}) \cong \mathbb{C}(\widehat{T}) \otimes_{\mathbb{C}} V = V_{\mathrm{loc}},$$

which is an irreducible $\ddot{\mathbf{H}}_{\mathrm{loc}}$-module. Since (M_i/M_{i-1}) is a quotient of $\Delta(V)$, we see that after localisation, either $(M_i/M_{i-1})_{\mathrm{loc}} = 0$, or $(M_i/M_{i-1})_{\mathrm{loc}} \cong (\Delta(V))_{\mathrm{loc}}$. Moreover, since M is finitely-generated, each generator can only lie in finitely many pieces in the filtration. Thus, all but finitely many composition factors are killed off after taking localisation. $\square$

Lemma 10. *The subcategory of Fuchsian equations is closed under extensions.*

Proof. It is sufficient to prove this result for strictly Fuchsian equations. By [Sau03, Lemma 2.1.2.1], the full subcategory of strictly Fuchsian equations is an essential subcategory. Thus, let N and N' be fuchsian **qTor**-modules fitting into a short exact sequence:

$$0 \longrightarrow N \longrightarrow M \longrightarrow N' \longrightarrow 0.$$

Choose a basis $\{e_1, \cdots, e_n\}$ for N. Then, $\mathrm{D}_q^{\omega_j^\vee} \cdot e_i = A_{ij} e_j$, which gives a matrix $A_{ij} \in \mathrm{GL}_n(\mathbb{C}(\mathbf{X}))$ by the Fuchsian condition on N. From this, one may extend this to a basis $\{e_1, \cdots, e_n, e_{n+1}, \cdots, e_m\}$ of M. As an N-module, we may identify the matrix $[A_{ij}]_{i,j=1}^n$ with the $m \times m$ matrix for which $A_{ij} = 0$ for all $j = n+1, \cdots, m$. This gives us an $m \times m$ block upper triangular matrix

$$A = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

The modules N and N' correspond to the diagonal blocks, and the upper triangular block corresponds to M. By the Fuchsian condition of N and N', it follows that the block diagonal matrices are invertible. Thus, the matrix A is invertible, since a block upper triangular matrix with invertible diagonals is invertible. M is therefore Fuchsian. $\square$

Lemma 11. *Any submodule of a Fuchsian q-difference module is Fuchsian.*

Proof. Repeat the same process as above to obtain a block upper triangular matrix. Taking quotients amounts to choosing one of of the block diagonal matrices, which, as aforementioned, is invertible. $\square$

Proposition 2. *The localisation functor lands in the full subcategory* $\mathbf{qTor}_W^{\mathrm{fuch}}$ *of* $\mathbf{qTor}_W$*. That is,* Loc *is a functor:*

$$\mathrm{Loc} : \mathcal{O}_{\ddot{\mathbf{H}}} \longrightarrow \mathbf{qTor}_W^{\mathrm{fuch}}\text{-}\mathbf{Mod}\,.$$

Proof. Given any $M \in \mathcal{O}_{\ddot{\mathbf{H}}}$, the $\mathbf{qTor}_W$-module $\mathbf{qTor}_W \otimes_{\mathbb{C}} M_{\mathrm{loc}}$ has a finite filtration with composition factors of the form $\Delta(V)_{\mathrm{loc}}$, where V is an irreducible $\dot{\mathbf{H}}^{\mathbf{Y}}$-module. Choose a basis for $\Delta(V)_{\mathrm{loc}}$, and write the q-differential operator $\mathrm{D}_q^{\mu^\vee}$ in this basis, which produces an invertible square matrix $A(X) \in \mathrm{GL}_n(\mathbb{C}(X))$. Then, for an arbitrary element $V \otimes_{\mathbb{C}} f(X) \in \Delta(V)_{\mathrm{loc}}$, we have:

$$V \otimes_{\mathbb{C}} \left(\mathrm{D}_q^{\mu^\vee} f(X)\right) = V \otimes_{\mathbb{C}} (A(X) f(X)),$$

using the fact that $\mathrm{D}_q^{\mu^\vee}$ acts on $\Delta(V)_{\mathrm{loc}}$ by the matrix $A(X)$. But also $\mathrm{D}_q^{\mu^\vee} f(X) = f(q^{2\mu} X)$, and thus we have a q-difference system

$$f(q^{2\mu} X) = A(X) f(X).$$

Since $A(X)$ is invertible, it follows then that if one takes the limit $X \to 0$ or $X \to \infty$, one obtains an invertible scalar matrix, and thus $\Delta(V)_{\mathrm{loc}}$ is Fuchsian. Further, $\Delta(V)_{\mathrm{loc}} \subseteq \Delta(\delta_\lambda^k)_{\mathrm{loc}}$ – that is, $\Delta(V)_{\mathrm{loc}}$ is a quotient of $\Delta(\delta_\lambda^k)_{\mathrm{loc}}$ Thus, by Lemma 10 and 11, $\Delta(\delta_\lambda^k)_{\mathrm{loc}}$ is also Fuchsian. Moreover, by Lemma 4 and 9, $\Delta(\delta_\lambda^k)_{\mathrm{loc}}$ admits a finite Δ-filtration. $\square$

As a consequence, we have the following:

Corollary 3. *The localisation functor factors through the Serre quotient $\mathcal{O}_{\ddot{\mathbf{H}}}/\mathcal{O}_{\ddot{\mathbf{H}}}^{\mathrm{tor}}$ to give a functor:*

$$\mathcal{O}_{\ddot{\mathbf{H}}}/\mathcal{O}_{\ddot{\mathbf{H}}}^{\mathrm{tor}} \longrightarrow \mathbf{qTor}_W^{\mathrm{fuch}}\text{-}\mathbf{Mod}\,.$$

Proof. Immediate consequence of Lemma 11, Proposition 2, and the universal property of Serre quotients. $\square$

Lemma 12. *Let $\mathcal{F}$ be a vector bundle over $\mathfrak{A}$. Then, for each $w \in W$, let Δ_w be a meromorphic morphism $\mathcal{F} \dashrightarrow w^*\mathcal{F}$. Then, $\pi_*\mathcal{F}$ has the structure of a sheaf of modules over $\mathcal{H}^{\mathrm{ell}}$ if the following hold:*

(i) $\pi_*\Delta_e = \mathrm{id}_{\pi_*\mathcal{F}}$, *and* $\{\pi_*\Delta_w\}_{w\in W}$ *satisfies the relations of the Weyl group* W,

(ii) $(\pi_*\Delta_w + \pi_*\Delta_{s_\alpha w})|_{T_\alpha}$ *vanishes,*

(iii) Δ_w *could have a pole of order* 1 *along the divisor* $T_{\alpha,\hbar}$ *for each* $\alpha \in \Phi(w)$.

Proof. Recall that a rational section of $\mathcal{H}^{\mathrm{ell}}$ are elements of the twisted group algebra $\mathcal{O}[W]$ of the form $T := \sum_{w\in W} f_w w$, for some $w \in W$. It is sufficient to show that the sum $\sum_{w\in W} \pi_*(f_w\Delta_w)$ defines a regular morphism $\pi_*\mathcal{F} \to \pi_*\mathcal{F}$, given that conditions (i) – (iii) hold.

One may identify each f_w as a rational morphism $f_w : \mathcal{F} \dashrightarrow \mathcal{F}$. For each $f_w w$, replace the Weyl group elements of with Δ_w to obtain $f_w\Delta_w$. These elements can be identified as morphisms $f_w\Delta_w : \mathcal{F} \dashrightarrow w^*\mathcal{F}$. Note that the sum $\sum_{w\in W} f_w\Delta_w$ is not well-defined since the target of each of the maps $f_w\Delta_w$ is different. However, if we take the pushforward, then we have

$$\pi_*(f_s\Delta_s) : \pi_*\mathcal{F} \dashrightarrow \pi_*s^*\mathcal{F}.$$

Observe that by the W-invariance of $\pi_*\mathcal{F}$, we have that $\pi_*s^*\mathcal{F} \cong \pi_*\mathcal{F}$. Thus, the sum $\sum_{w\in W} \pi_*(f_w\Delta_w)$ is a well-defined map $\pi_*\mathcal{F} \to \pi_*\mathcal{F}$. By abuse of notation, we will write $f_w w$ and $f_w\Delta_w$ for its image in the pushforward $\pi_*(f_w w)$ and $\pi_*(f_w\Delta_w)$. Given another element of the form $g_w\Delta_w$, we may use (i) to define their product is given by:

$$\left(\sum_{w\in W} f_w\Delta_w\right) \cdot \left(\sum_{v\in W} g_v\Delta_v\right) = \sum_{w,v\in W} f_w \cdot {}^w f_v \cdot \Delta_{wv},$$

where the coefficients of Δ_{wv} are multiplied according to the twisted group algebra structure on $\mathcal{O}[W]$. This gives the required multiplication map $\mathcal{H}^{\mathrm{ell}} \otimes_{\mathcal{O}_{\mathfrak{A}/W}} \pi_*\mathcal{F} \to \pi_*\mathcal{F}$ in $\mathbf{Coh}(\mathfrak{A}/W)$. Thus, condition (ii) implies then that the terms Δ_w has vanishing order at least 1 along T_α, and thus the terms of the form $f_w\Delta_w$ are regular.

Note that

$$(f_w\Delta_w + f_{s_\alpha w}\Delta_{s_\alpha w})\,|_{T_\alpha} = (\Delta_w|_{T_\alpha} + \Delta_{s_\alpha w}|_{T_\alpha}) \cdot f_w|_{T_\alpha}, \tag{3.3}$$

since $f_{s_\alpha w}|_{T_\alpha} = f_w|_{s\cdot T_\alpha} = f_w|_{T_{-\alpha}} = f_w|_{T_\alpha}$. By (3.3), and the residue condition from Definition 11(b):

$$\mathrm{Res}_{T_\alpha}(f_w) + \mathrm{Res}_{T_\alpha}(f_{s_\alpha w}) = 0,$$

it follows then that terms of the form $f_w\Delta_w + f_{s_\alpha w}\Delta_{s_\alpha w}$ are regular if $(\Delta_w + \Delta_{s_\alpha w})|_{T_\alpha} = 0$. Definition 11(c) states that f_w vanishes along the divisor $T_{\alpha,\hbar}$ for any $\alpha \in \Phi(w)$. In this case, the term $f_w\Delta_w$ will remain regular even if Δ_w has a pole of order at most 1 along $T_{\alpha,\hbar}$. Together, these all imply that we have a regular morphism:

$$\sum_{w\in W} f_w\Delta_w : \pi_*\mathcal{F} \longrightarrow \pi_*\mathcal{F}.$$

□

By Lemma 12,the problem of determining whether $\pi_*\mathcal{F}_0$ and $\pi_*\mathcal{F}_\infty$ are $\mathcal{H}^{\mathrm{ell}}$-modules will involve an analysis of the poles of the monodromy map $\varphi : \mathcal{F}_0 \to \mathcal{F}_\infty$. The maps $B_w^{(0)} : \mathcal{F}_0 \to w^*\mathcal{F}_\infty$, and $B_w^{(\infty)} : \mathcal{F}_\infty \to w^*\mathcal{F}_0$ will also need to be analysed in the general case.

As we will see in Chapter 4, in the rank one case there is only one morphism in the $\mathbb{Z}/2\mathbb{Z}$-equivariant connection category, given by B_s. Moreover, $\mathcal{F}_0 \cong \mathcal{F}_\infty^\vee \cong s^*\mathcal{F}_0$, and thus the morphism $B_s^{(0)}$ and $B_s^{(\infty)}$) are constant maps, hence regular. As such, the $\mathcal{H}^{\mathrm{ell}}$-module structure on $\pi_*\mathcal{F}_0$ and $\pi_*\mathcal{F}_\infty$ is controlled entirely by the poles of the monodromy map $\varphi : \mathcal{F}_0 \to \mathcal{F}_\infty$.

The *quantum Knizhnik-Zamolodchikov* (qKZ) functor is a functor

$$q\mathrm{KZ} : \mathcal{O}_{\ddot{\mathbf{H}}} \longrightarrow \mathbf{Coh}^{\mathrm{flat}}(\mathcal{H}^{\mathrm{ell}}),$$

that uniquely factors through the q-Riemann-Hilbert functor in the following way:

$$\begin{tikzcd} \mathbf{qTor}_W^{\mathrm{fuch}}\text{-}\mathbf{Mod} \arrow[rr, "q\mathrm{RH}"] \arrow[dr, "q\mathrm{KZ}"'] & & \mathcal{C}onn_W \\ & \mathbf{Coh}^{\mathrm{flat}}(\mathcal{H}^{\mathrm{ell}}) \arrow[ur, hook] & \end{tikzcd}$$

In particular, we will find that $\pi_*\mathcal{F}_0 \in \mathbf{Coh}(E/(\mathbb{Z}/2\mathbb{Z}))$ has the structure of a $\mathcal{H}^{\mathrm{ell}}$-module, and thus the qKZ maps an appropriate $\mathbf{qTor}_W^{\mathrm{fuch}}$-module to $\pi_*\mathcal{F}_0$, which is then embedded into $\mathcal{C}onn_{\mathbb{Z}/2\mathbb{Z}}$. Since $\mathcal{F}_\infty \cong \mathcal{F}_0^\vee$, it follows that $\pi_*\mathcal{F}_\infty$ also has the structure of a $\mathcal{H}^{\mathrm{ell}}$.

In Chapter 4, we outline construct vector bundles $\mathcal{F}_0$ and $\mathcal{F}_\infty$ by studying q-difference equations arising from standard modules in $\mathcal{O}_{\ddot{\mathbf{H}}}$. We study its monodromy, and we outline a proof of Conjecture 1. The only thing that remains to be to proven is a technical condition, outlined in Conjecture 4. Ultimately, we expect to obtain a diagram:

$$\begin{tikzcd} \mathcal{O}_{\ddot{\mathbf{H}}} \arrow[r, "\mathrm{Loc}"] \arrow[d, "q\mathrm{KZ}"'] & \mathbf{qTor}_W^{\mathrm{fuch}}\text{-}\mathbf{Mod} \arrow[d, "q\mathrm{RH}"] \\ \mathbf{Coh}(\mathcal{H}^{\mathrm{ell}}) \arrow[r, hook] & \mathcal{C}onn_W \end{tikzcd}$$

We observe that given some $M_{\mathrm{loc}} \in \mathcal{O}_{\ddot{\mathbf{H}}}$, $M_{\mathrm{loc}} = 0$ if and only if M is torsion. Therefore $q\mathrm{KZ}(M) \neq 0$ if and only if $M \in \mathcal{O}_{\ddot{\mathbf{H}}}/\mathcal{O}_{\ddot{\mathbf{H}}}^{\mathrm{tor}}$. This shows that the restriction of qKZ to the Serre quotient is faithful. Moreover, we expect that the restriction to this quotient gives a fully faithful, surjective functor – this is the content of Conjecture 2.

Chapter 4

Construction of the Rank One qKZ Functor

To begin, we derive some useful relations about the q-shift operator (1.12) in the first section. The first key result is Proposition 3, which gives an explicit link between the rank one DAHA representations admitting Δ-filtrations and the qKZ equation.

Sections 4.4 and 4.5 are dedicated to calculating solutions of the rank one qKZ equation, and calculating its monodromy matrix M. Using techniques of [Sau03], we can compute M up to some q-periodic constants. However, using [EFK98], we can compute these q-periodic constants explicitly. No originality is claimed in these sections.

In the last section, we construct the qKZ functor explicitly for the case of standard modules. Given a triple of connection data $(\mathcal{F}_0, \mathcal{F}_\infty, \varphi)$, we wish to construct morphisms $\Delta_s^{(0)} : \mathcal{F}_0 \to s^*\mathcal{F}_0$, and $\Delta_s^{(\infty)} : \mathcal{F}_\infty \to s^*\mathcal{F}_\infty$, and compute their sections explicitly, so that we can analyse its poles to show that the conditions outline in Lemma 12. Let $\pi : E \to E/(\mathbb{Z}/2\mathbb{Z})$. We then outline how $\pi_*\mathcal{F}_0$ and $\pi_*\mathcal{F}_\infty$ may be equipped with the structure of a module over the elliptic affine Hecke algebra $\mathcal{H}^{\text{ell}}$. The only technical condition that requires to be checked before we can prove that $\pi_*\mathcal{F}_0, \pi_*\mathcal{F}_\infty \in \mathbf{Coh}^{\text{flat}}(\mathcal{H}^{\text{ell}})$ is Conjecture 4.

4.1. Rank One Reduction

For this chapter, we will restrict ourselves entirely to the rank one case. This section details some results that we will use in the remainder of our thesis. In particular, the q-shift operator arising in (1.12) is of great interest to us, as this is will play the role of the q-difference operator in Chapter 1.1.

In the rank one case, there is only one simple root – which we denote by α. We choose $\alpha = e_1 - e_2$ as our representative for the simple root, where e_i are the canonical basis vectors of $\mathbb{R}^2$. The fundamental weight is given by $\omega = \frac{1}{2}\alpha$, and generates $\mathbf{X} \cong \mathbb{Z}$.

We write $t := t_\alpha$, as there is now only one parameter. So, let us write $Y := Y^{\rho^\vee}$ and $X := X^\rho$. Moreover, the q-shift operator acts on X by $\tau(\rho)X = qX$. Let π_ρ denote the element of zero length. [Kir97, Corollary 3.3(i)] tells us that that $\ell(\tau(\rho)) = 2\langle\rho, \rho^\vee\rangle = 1$, and from [Kir97, Corollary 3.3(iv)], we know that $\ell(\tau(\rho)s) = \ell(\tau(\rho)) - 1 = 0$. Thus, it follows that $\pi_\rho := \tau(\rho)s$ is the element of zero length.

As aforementioned in Theorem 3(ii), Y can be written in as

$$Y = T_{\tau(\rho)} = \pi_\rho T_\alpha, \tag{4.1}$$

using the Iwahori-Matsumoto presentation. We write $t := t_\alpha$, as there is only one simple root. The following Lemma shows that the q-shift operator $\tau(\rho)$ satisfies the commutation relation outlined in the definition of the quantum torus of [BEG00]:

Lemma 13. *Let $f(X) \in \mathbb{C}[q^\pm, t^\pm][X^\pm]$ be a rational function in X. Then,*

$$\tau(\rho)f(X) = f(qX)\tau(\rho).$$

Proof. We have that $\tau(\rho)X = q^{2\langle\rho,\rho\rangle}X = qX$. Then,

$$\tau(\rho)X\tau(\rho)^{-1} = qX\tau(\rho)^{-1} = qXs\pi_\rho^{-1} = qs(sXs)\pi_\rho^{-1} = q^2sX^{-1}\pi_\rho^{-1} = q^2s\pi_\rho^{-1}(\pi_\rho X^{-1}\pi_\rho^{-1}).$$

Computing explicitly,

$$\pi_\rho X^{-1}\pi_\rho^{-1} = \pi_\rho X^{-\rho}\pi_\rho^{-1} = X^{-\alpha_0/2} = qX,$$

where the last equality follows from the fact that $\alpha = -\alpha + \delta$, and (1.9). Then, we have the relation:

$$\tau(\rho)X\tau(\rho)^{-1} = q\tau(\rho)^{-1}(qX) = qX,$$

and thus:

$$\tau(\rho)X = qX\tau(\rho). \tag{4.2}$$

Extending by linearity, it follows then that for any rational function $f(X) \in \mathbb{C}[q^\pm, t^\pm][X^\pm]$,

$$\tau(\rho)f(X) = f(qX)\tau(\rho),$$

as claimed. □

Lemma 14. *The q-shift operator $\tau(\rho)$ has a faithful representation in $\mathbb{C}[q^\pm, t_\alpha^\pm][X^\pm]$ given by:*

$$\tau(\rho)^{-1} = \left(\frac{1-X^2}{t-t^{-1}X^2} + \frac{t-t^{-1}}{t-t^{-1}X^2}\right)Y^{-1}.$$

Proof. Recall from Theorem 5 that T_α has a polynomial representation given by

$$T_\alpha = ts + (t-t^{-1})\frac{s-1}{X^{-2}-1}, \tag{4.3}$$

For simplicity, let us write $T := T_\alpha$. We may re-write (4.3) as

$$T - t = t(s-1) + (t-t^{-1})\frac{s-1}{X^{-2}-1} = \left(t + \frac{t-t^{-1}}{X^{-2}-1}\right)(s-1) = \frac{tX^{-2}-t^{-1}}{X^{-2}-1}(s-1),$$

and we obtain the formula

$$1 - s = \frac{1-X^{-2}}{tX^{-2}-t^{-1}}(T-t). \tag{4.4}$$

Now, using (4.1) and (4.3),

$$Y = \pi_\rho T = \tau(\rho)s\left(ts + (t-t^{-1})\frac{s-1}{X^{-2}-1}\right) = \tau(\rho)\left(t + (t-t^{-1})\frac{1-s}{X^2-1}\right). \tag{4.5}$$

Then, we substitute (4.4) into (4.5) to obtain:

$$\begin{aligned}
\tau(\rho)^{-1} &= \left(t + (t - t^{-1})\frac{1-s}{X^2-1}\right)Y^{-1} \\
&= \left(t + (t - t^{-1})\frac{X^{-2}(X^2-1)(T-t)}{(X^2-1)(tX^{-2}-t^{-1})}\right)Y^{-1} \\
&= \left(t + (t - t^{-1})\frac{X^{-2}(T-t)}{tX^{-2}-t^{-1}}\right)Y^{-1} \\
&= \left(\frac{t(tX^{-2}-t^{-1}) + (t-t^{-1})X^{-2}(T-t)}{tX^{-2}-t^{-1}}\right)Y^{-1} \\
&= \left(\frac{t^2X^{-2} - 1 + tX^{-2}T - t^2X^{-2} - T^{-1}X^{-2}T + X^{-2}}{tX^{-2}-t^{-1}}\right)Y^{-1} \\
&= \left(\frac{(t-t^{-1})X^{-2}T + (X^{-2}-1)}{tX^{-2}-t^{-1}}\right)Y^{-1} \\
&= \left(\frac{t-t^{-1}}{t-t^{-1}X^2}T + \frac{1-X}{t-t^{-1}X^2}\right)Y^{-1}
\end{aligned}$$

Thus,

$$\tau(\rho)^{-1} = \left(\frac{1-X^2}{t-t^{-1}X^2} + \frac{t-t^{-1}}{t-t^{-1}X^2}\right)Y^{-1}, \tag{4.6}$$

as claimed. □

Let us give a one-dimensional example of how one may monodromy of q-difference equations arising from DAHA. In this case, the matrix defining the q-difference system is just a scalar matrix – that is, it is a 1×1 matrix. From this, we wish to derive a gauge transformation that takes us from $A(0)$ to $A(\infty)$. This gauge transformation will serve as our monodromy matrix for this one-dimensional case. The logic is the same when we begin studying the monodromy of the qKZ equation in the next section for a two-dimensional representation.

Example 6 (One-Dimensional Monodromy Matrix)**.** *Let V be any finite-dimensional $\dot{\mathbf{H}}^{\mathbf{Y}}$-module. Then,*

$$M := \mathrm{Ind}_{\dot{\mathbf{H}}^{\mathbf{Y}}}^{\ddot{\mathbf{H}}} V = \ddot{\mathbf{H}} \otimes_{\dot{\mathbf{H}}^{\mathbf{Y}}} V \cong V \otimes_{\mathbb{C}} \mathbb{C}[q^{\pm}][X^{\pm}],$$

where the isomorphism follows from the PBW theorem for DAHA (1.10). As before, from (4.6), we have the polynomial representation given by:

$$\tau(\rho)^{-1} = \left(\frac{1-X^2}{t-t^{-1}X^2} + \frac{t-t^{-1}}{t-t^{-1}X^2}T\right)Y^{-1}.$$

We also have the polynomial representation for the simple reflection given by

$$s(X) = \frac{1-X^2}{t-t^{-1}X^2}(T-t) + 1.$$

As before, given some $v \otimes f(X) \in M$,

$$v \otimes \tau(\rho)^{-1} f(X) = v \otimes f(q^{-1}(X)),$$

but also

$$v \otimes \tau(\rho)^{-1} f(X) = v \otimes \left(\frac{1-X^2}{t-t^{-1}X^2} + \frac{t-t^{-1}}{t-t^{-1}X^2}T\right)Y^{-1} f(X).$$

where the first equality follows from Lemma 13. Apply $\tau(\rho)$ to both sides to obtain a second order q-difference equation

$$f(qX) = \frac{(t - t^{-1}X^2)Y}{(1 - X^2) + (t - t^{-1})T} f(X). \tag{4.7}$$

Let $A(X)$ be the coefficient of $f(X)$ in this q-difference equation. Suppose that the solution to (4.7) is of the form Then, computing directly, we see that:

$$A(0) = \frac{t}{1 + (t - t^{-1})T} Y = tT^{-2}Y,$$

using the quadratic relation for T, and

$$A(\infty) = t^{-1}Y.$$

The fundamental solutions around 0 and ∞ are then given by $e_{q,A(0)}$, and $e_{q,A(\infty)}$ in this case. The monodromy matrices $M^{(0)}$ and $M^{(\infty)}$ are trivial in this case. We observe that $s(0) = t^{-1}T$, and $s(\infty) = t(T - t) + 1$, and we claim that $s(0)$ gives a suitable gauge transform that takes us from $A(0)$ to $A(\infty)^{-1}$. Computing directly:

$$s(0)A(0)s(0)^{-1} = t^{-1}T(tT^{-2}Y)tT^{-1} = tT^{-1}YT^{-1} = tY^{-1} = A(\infty)^{-1},$$

where we use the relation $Y^{-1} = T^{-1}YT^{-1}$. Therefore, we see that $s(0)$ defines a gauge transformation from $A(0)$ to $A(\infty)$. What this shows is that $s(X)$ corresponds to a map of q-difference modules $M \to s^*M$. This property holds for the qKZ equations, as we shall see later.

In the sections to come, we will consider the case where the $\ddot{\mathbf{H}}$-module M is a standard module (see Chapter 2). In this case, calculating the fundamental solutions, and the monodromy matrix become a much more non-trivial task. Further, in this case the simple reflection s still plays the role of q-gauge transforming q-difference modules M to the module s^*M.

4.2. The Rank One qKZ Equation

The *quantum Knizhnik-Zalomodchikov* (qKZ) equation is given by [EFK98, Chapter 11.5, (11.27)]:

$$\Phi(z_1, \cdots, qz_j, \cdots, z_N) = R_{j,j-1}\left(\frac{qz_j}{z_{j-1}}\right) \cdots R_{j1}\left(\frac{qz_j}{z_1}\right) q^{(2\lambda)_j} R_{jN}\left(\frac{z_j}{z_N}\right) \cdots R_{j,j+1}\left(\frac{z_j}{z_{j+1}}\right) \Phi(z_1, \cdots, z_N), \tag{4.8}$$

This equation was first derived in [FR92, Theorem 5.3], as a q-analogue of the Knizhnik-Zamolodchikov equation for quantum affine algebras. Frenkel-Reshetikhin studied the solutions and monodromy of (4.8). Their solutions are given in [FR92, Proposition 6.1], and a general form for the connection matrix is given in [FR92, (6.32)].

We will consider consider the rank one case. We will follow [EFK98, §11, §12], and record the solutions of the qKZ equation, and its connection matrix. From [EFK98, Example 9.6.5], given V and W two representations of $\mathbf{U}_q(\widehat{\mathfrak{sl}_2})$, with weights m, n, respectively, we have in the case for $m = n = 1$, that the R-matrix is given by [EFK98, (9.41)]:

$$R(z) = E_{11} \otimes E_{11} + E_{22} \otimes E_{22} + \frac{1 - z}{q - q^{-1}z}(E_{22} \otimes E_{11} + E_{11} \otimes E_{22}) + \frac{q - q^{-1}}{q - q^{-1}z}(E_{12} \otimes E_{21} + zE_{21} \otimes E_{12}).$$

Written in matrix form, we get:

$$R(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{1-z}{q-q^{-1}z} & \dfrac{z(q-q^{-1})}{q-q^{-1}z} & 0 \\ 0 & \dfrac{q-q^{-1}}{q-q^{-1}z} & \dfrac{1-z}{q-q^{-1}z} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The quantum Knizhnik-Zamolodchikov equation then becomes:

$$\Phi(qz_1, z_2) = q^{(2\lambda)_1} R\left(\frac{z_1}{z_2}\right) \Phi(z_1, z_2),$$

$$\Phi(z_1, qz_2) = R\left(\frac{z_1}{qz_2}\right) q^{(2\lambda)_2} \Phi(z_1, z_2),$$

where Φ in this case takes values in $V \otimes W$, and $q^{(2\lambda)_1}$ is a diagonal matrix whose entries depend on the weights of the weight spaces. In particular, we restrict ourselves to the case where Φ takes values in the 2-dimensional weight space of $\mathcal{U}_q(\widehat{\mathfrak{sl}_2})$ — that is, the weight space corresponding to $m + n - 2$. This gives us the following matrix defining the quantum Knizhnik-Zamolodchikov equation:

$$\frac{q^{(2\lambda)_i}}{q - q^{-1}\frac{z_1}{z_2}} \begin{pmatrix} 1 - \frac{z_1}{z_2} & \frac{z_1}{z_2}(q - q^{-1}) \\ q - q^{-1} & 1 - \frac{z_1}{z_2} \end{pmatrix}, \quad i = 1, 2.$$

This corresponds to the coefficient matrix seen in [OS22, (69)], which is a modification of the quantum Knizhnik-Zamolodchikov equation given by $h^{\lambda}_{(1)} R(a_1/a_2)$, where — according to [OS22, (72)] — $h^{\lambda}_{(1)}$ should be given as a diagonal matrix whose entries are some monomials in *dynamical parameters* z (see [AO16]). Here, the variables a_1/a_2 are called *equivariant* parameters. In particular,

$$h^{\lambda}_{(1)} = \begin{pmatrix} z & \\ & z^{-1} \end{pmatrix}.$$

From [OS22, (122)], the R-matrix is given as:

$$R\left(\frac{a_1}{a_2}\right) = \begin{pmatrix} \dfrac{(1 - a_1/a_2)\hbar^{1/2}}{\hbar - a_1/a_2} & \dfrac{a_1/a_2(\hbar - 1)}{\hbar - a_1/a_2} \\ \dfrac{\hbar - 1}{\hbar - a_1/a_2} & \dfrac{(1 - a_1/a_2)\hbar^{1/2}}{\hbar - a_1/a_2} \end{pmatrix}.$$

So, the quantum Knizhnik-Zamolodchikov equations are given by:

$$\Phi(z, qa_1, a_2) = R(a_1/a_2) \begin{pmatrix} z & \\ & z^{-1} \end{pmatrix} \Phi(z, a_1, a_2).$$

To simplify our notation, we will write

$$u := a_1/a_2.$$

We keep in mind that Φ is a function of z, a_1, and a_2, where a_i are the equivariant parameters, and z is the dynamical parameter. But by abuse of notation we will write $\Phi(z,u)$. Thus we have:

$$\begin{pmatrix} \Phi_1(z,qu) \\ \Phi_2(z,qu) \end{pmatrix} = \begin{pmatrix} z\dfrac{(1-u)\hbar^{1/2}}{\hbar-u} & z\dfrac{u(\hbar-1)}{\hbar-u} \\ z^{-1}\dfrac{\hbar-1}{\hbar-u} & z^{-1}\dfrac{(1-u)\hbar^{1/2}}{\hbar-u} \end{pmatrix} \cdot \begin{pmatrix} \Phi_1(z,u) \\ \Phi_2(z,u) \end{pmatrix}, \tag{4.9}$$

Note from Proposition 3 that the dynamical parameters arising in the qKZ equation come from a non-trivial one-dimensional representation of $\mathbb{C}[\mathbf{Y}]$.

Remark 6. This is a q-analogue of the trigonometric Knizhnik-Zamolodchikov connection that is studied in [VV04] (∇_j in [VV04, Lemma 3.1(ii)]), in the case of the degenerate double affine Hecke algebra. The rational double affine Hecke algebra case is treated in [GGOR03], in which case the Knizhnik-Zamolodchikov equation arises. The affine Knizhnik-Zamolodchikov was studied by Cherednik in [Che90] — see in particular [Che90, (21)] in relation to affine Hecke algebra representations. The hypergeometric equation arises as a solution to these equations, and its monodromy has been extensively well-studied.

Remark 7. Moreover, these equations also appear when one considers of elliptic stable envelopes, which are objects that are related to the equivariant elliptic cohomology of Nakajima quiver varieties. These objects were considered in [AO16]. In particular, the rank one qKZ equations arise as vertex operators in some localisation of $K(T^*\mathbb{P}^1)$ (see [AO16, §6]). Under this identification, the variables z and u arise by considering the torus-equivariant elliptic cohomology of a Nakajima variety, and are called *dynamical* and *equivariant* variables, respectively ([AO16, (27)]).

4.3. From DAHA to the qKZ Equation

The following proposition shows that we can produce a two-dimensional DAHA representation of the q-shift operator $\tau(\rho)^{-1}$ that is precisely the matrix defining the qKZ equation. This is an explicit realisation of Proposition 2.

Proposition 3. *Let $\mathbb{C}_m$ be a one-dimensional $\mathbb{C}[Y^{\pm}]$-module acting by $m \in \mathbb{C}^\times$, and consider the standard module*

$$\delta_m = \mathrm{Ind}_{\mathbb{C}[Y^\pm]}^{\dot{\mathbf{H}}^{\mathbf{Y}}} \mathbb{C}_m,$$

and choose a standard ordered basis $\{1\otimes 1, T\otimes 1\}$. Then, the image of the q-shift operator $\tau(\rho)^{-1}$ in the induced module of the standard module:

$$\Delta(\delta_m) := \mathrm{Ind}_{\dot{\mathbf{H}}^{\mathbf{Y}}}^{\ddot{\mathbf{H}}} \delta_m,$$

is of the form:

$$\tau(\rho)^{-1} = R(X)\cdot \mathrm{diag}(m, m^{-1}),$$

where R is the R-matrix of the evaluation module of $\mathcal{U}_q(\widehat{\mathfrak{sl}_2})$ (see Example 4).

Proof. Fix some $m \in \mathbb{C}^\times$, and define a one-dimensional representation $\mathbb{C}_m$ where Y acts by m. Let us consider the induced representation

$$\delta_m := \mathrm{Ind}_{\mathbb{C}[Y^\pm]}^{\dot{\mathbf{H}}^Y} \mathbb{C}_m = \dot{\mathbf{H}}^Y \otimes_{\mathbb{C}[Y^\pm]} \mathbb{C}_m.$$

Then, by the PBW theorem (1.10), we have that

$$\delta_m \cong \mathbb{C}[t^\pm][T] \otimes_{\mathbb{C}} \mathbb{C}_m.$$

Choose a basis $\{1 \otimes 1, T \otimes 1\}$ for δ_m. Next, let us induce to the DAHA by taking

$$\Delta(\delta_m) = \mathrm{Ind}_{\dot{\mathbf{H}}^Y}^{\ddot{\mathbf{H}}} \delta_m = \ddot{\mathbf{H}} \otimes_{\dot{\mathbf{H}}^Y} \delta_m \cong \mathbb{C}[t^{\pm}, q^{\pm}][X^{\pm}] \otimes_{\mathbb{C}[t^{\pm}, q^{\pm}]} \delta_m,$$

where the isomorphism follows again from the PBW theorem for DAHA. Then, using our equation for $\tau(\rho)^{-1}$ from (4.6), and computing directly, we have:

$$\begin{aligned}
\tau(\rho)^{-1}(1 \otimes 1) &= \frac{t - t^{-1}}{t - t^{-1}X^2}(T \otimes m^{-1}) + \frac{1 - X^2}{t - t^{-1}X^2} \otimes m^{-1} \\
&= m^{-1}\left(\frac{t - t^{-1}}{t - t^{-1}X^2}\right)(T \otimes 1) + m^{-1}\left(\frac{1 - X^2}{t - t^{-1}X^2}\right)(1 \otimes 1),
\end{aligned}$$

and

$$\begin{aligned}
\tau(\rho)^{-1}(T \otimes 1) &= \left(\frac{1 - X^2}{t - t^{-1}X^2} + \frac{t - t^{-1}}{t - t^{-1}X^2}T\right) Y^{-1}(T \otimes 1) \\
&= \frac{1 - X^2}{t - t^{-1}X^2}(TY - (t - t^{-1})Y)(1 \otimes 1) + \frac{t - t^{-1}}{t - t^{-1}X^2}T(TY - (t - t^{-1})Y)(1 \otimes 1) \\
&= m\frac{1 - X^2}{t - t^{-1}X^2}(T \otimes 1) + m\frac{(X^2 - 1)(t - t^{-1})}{t - t^{-1}X^2}(1 \otimes 1) \\
&\quad + m\frac{(t - t^{-1})\left((t - t^{-1})T + 1 - (t - t^{-1})T\right)}{t - t^{-1}X^2}(1 \otimes 1) \\
&= m\frac{1 - X^2}{t - t^{-1}X^2}(T \otimes 1) + m\frac{1}{t - t^{-1}X^2}\left((X^2 - 1)(t - t^{-1}) + (t - t^{-1})\right) \\
&= \frac{m(1 - X^2)}{t - t^{-1}X^2}(T \otimes 1) + \frac{1}{t - t^{-1}X^2}(mX^2(t - t^{-1}))
\end{aligned}$$

and thus the representation of $\tau(\rho)^{-1}$ is given by

$$\tau(\rho)^{-1} = \begin{pmatrix} \dfrac{1 - X^2}{t - t^{-1}X^2} & \dfrac{X^2(t - t^{-1})}{t - t^{-1}X^2} \\[2ex] \dfrac{t - t^{-1}}{t - t^{-1}X^2} & \dfrac{1 - X^2}{t - t^{-1}X^2} \end{pmatrix} \begin{pmatrix} m^{-1} & \\ & m \end{pmatrix}, \tag{4.10}$$

as claimed. □

Indeed, setting $X^2 = u$, $t = \hbar^{1/2}$, and $m = z^{-1}$, we see that after multiplying each entry by $\hbar^{1/2}$ on the nominator and denominator we obtain:

$$\tau(\rho)^{-1} = \begin{pmatrix} z\dfrac{(1 - u)\hbar^{1/2}}{\hbar - u} & z\dfrac{u(\hbar - 1)}{\hbar - u} \\[2ex] z^{-1}\dfrac{\hbar - 1}{\hbar - u} & z^{-1}\dfrac{(1 - u)\hbar^{1/2}}{\hbar - u} \end{pmatrix} = R(u) \cdot \mathrm{diag}(z, z^{-1}),$$

which is precisely the coefficient matrix of the qKZ equation (4.9).

One may also compute the corresponding representation for $\tau(\rho)$. Note that this is not as simple as taking the inverse of $\tau(\rho)^{-1}$, since $\tau(\rho)$ is not a linear operator over $\ddot{\mathbf{H}}$. As such, one obtains a twist by a factor of q:

Lemma 15. *Let $A(X) = \tau(\rho)^{-1}$. Then, $\tau(\rho) = A(qX)^{-1}$.*

Proof. Let $C(X)$ denote the image of $\tau(\rho)$ under the $\ddot{\mathbf{H}}$-module $\Delta(\delta_m)$. Let $\mathbf{e}_1 = 1 \otimes 1$, and $\mathbf{e}_2 = T \otimes 1$. Then, computing directly,

$$\begin{aligned}
\mathbf{e}_1 &= \left(\tau(\rho)^{-1} \cdot \tau(\rho)\right)(e_1) \\
&= \tau(\rho)^{-1}\left(C_{11}(X)\mathbf{e}_1 + C_{21}(X)\mathbf{e}_2\right) \\
&= C_{11}(q^{-1}X)\tau(\rho)^{-1}\mathbf{e}_1 + C_{21}(q^{-1}X)\tau(\rho)^{-1}\mathbf{e}_2 \\
&= C_{11}(q^{-1}X)\left(A_{11}(X)\mathbf{e}_1 + A_{21}\mathbf{e}_2\right) + C_{21}(q^{-1}X)\left(A_{12}(X)e_1 + A_{22}(X)\mathbf{e}_2\right) \\
&= \left(C_{11}(q^{-1}X)A_{11}(X) + C_{21}(q^{-1}X)A_{12}(X)\right)\mathbf{e}_1 + \left(C_{11}(q^{-1}X)A_{21}(X) + C_{21}(q^{-1}X)A_{22}(X)\right)\mathbf{e}_2,
\end{aligned}$$

where the third equality follows from the commutation relation seen in Lemma 13. Similarly,

$$\begin{aligned}
\mathbf{e}_2 &= \left(\tau(\rho)^{-1} \cdot \tau(\rho)\right)(\mathbf{e}_2) \\
&= \tau(\rho)^{-1}\left(C_{12}(X)\mathbf{e}_1 + C_{22}(X)\mathbf{e}_2\right) \\
&= C_{12}(q^{-1}X)\tau(\rho)^{-1}\mathbf{e}_1 + C_{22}(q^{-1}X)\tau(\rho)^{-1}\mathbf{e}_2 \\
&= C_{12}(q^{-1}X)\left(A_{11}(X)\mathbf{e}_1 + A_{21}(X)\mathbf{e}_2\right) + C_{22}(q^{-1}X)\left(A_{12}(X)\mathbf{e}_1 + A_{22}(X)\right)\mathbf{e}_2 \\
&= \left(C_{12}(q^{-1}X)A_{11}(X) + C_{22}(q^{-1}X)A_{12}(X)\right)\mathbf{e}_1 + \left(C_{12}(q^{-1}X)A_{21}(X) + C_{22}(q^{-1}X)A_{22}(X)\right)\mathbf{e}_2
\end{aligned}$$

This calculations tells us that

$$C(q^{-1}X)^T \cdot A(X)^T = I,$$

where I is the identity matrix. Shifting both sides by q and then re-arranging shows us that:

$$C(X) = A(qX)^{-1},$$

as claimed. □

4.4. Solutions of the qKZ Equation

The solutions of the qKZ equation have been well-studied, so no originality is claimed here. We follow [EFK98, Chapter 11, 12] to produce solutions around 0 for the rank one qKZ equation.

The equation (4.9) then gives us two q-difference equations

$$\Phi_1(z, qu) = z\frac{(1-u)\hbar^{1/2}}{\hbar - u}\Phi_1(z,u) + z\frac{u(\hbar-1)}{\hbar-u}\Phi_2(z,u), \tag{4.11}$$

$$\Phi_2(z, qu) = z^{-1}\frac{\hbar - 1}{\hbar - u}\Phi_1(z,u) + z^{-1}\frac{(1-u)\hbar^{1/2}}{\hbar-u}\Phi_2(z,u). \tag{4.12}$$

Applying the q-difference operator to (4.9), we obtain a second-order q-difference equation:

$$\begin{pmatrix}\Phi_1(z,q^2u)\\ \Phi_2(z,q^2u)\end{pmatrix} = \begin{pmatrix} z\dfrac{(1-qu)\hbar^{1/2}}{\hbar-qu} & z\dfrac{qu(\hbar-1)}{\hbar-qu} \\ z^{-1}\dfrac{\hbar-1}{\hbar-qu} & z^{-1}\dfrac{(1-qu)\hbar^{1/2}}{\hbar-qu}\end{pmatrix} \cdot \begin{pmatrix}\Phi_1(z,qu)\\ \Phi_2(z,qu)\end{pmatrix},$$

which gives us:

$$\Phi_1(z,q^2u) = z\frac{(1-qu)\hbar^{1/2}}{\hbar-qu}\Phi_1(z,qu) + z\frac{qu(\hbar-1)}{\hbar-qu}\Phi_2(z,qu).$$

We wish to write this as a second order q-difference equation with only Φ_1. It is sufficient to only do this for Φ_1, since Φ_2 can be written as a linear combination of $\Phi_1(z, qu)$ and $\Phi_1(z, u)$. In particular, using (4.11), we can re-arrange this into:

$$\Phi_2(z,u) = E\left(\Phi_1(z,u)\right) := \frac{\hbar - u}{\hbar - 1}\Phi_1(z,qu) - \frac{\hbar - 1}{(1-u)\hbar^{1/2}}\Phi_1(z,u). \tag{4.13}$$

Let us drop the indices on Φ_1 and simply write $\Phi := \Phi_1$. This equation can be put into a second order q-difference equation (c.f. [EFK98, (11.33)]):

$$(qu - \hbar)\,\Phi(z, q^2u) + \hbar^{1/2}\left(-(qz + qz^{-1})u + qz + z^{-1}\right)\Phi(z,qu) + q\,(\hbar u - 1)\,\Phi(z,u) = 0. \tag{4.14}$$

The q-hypergeometric equation

$${}_2\varphi_1\left[\begin{matrix} q^a & q^b \\ & q^c \end{matrix}; q\,; u\right],$$

can be put into a second order q-difference equation by:

$$(q^{a+b}u - q^{c-1})f(q^2z) + (-(q^a + q^b)u + q^{c-1} + 1)f(qu) + (u-1)f(u) = 0, \tag{4.15}$$

(c.f. [Koe18], [EFK98, (11.21)]). Generally, one can show that any second order q-difference equation in the form:

$$(A_0z + B_0)f(q^2u) + (A_1z + B_1)f(qu) + (A_2z + B_2)f(u) = 0, \tag{4.16}$$

reduces to the q-hypergeometric equation (4.15). So, in the case of (4.14), we have:

$$A_0 = q, \quad B_0 = -\hbar, \quad A_1 = -\hbar^{1/2}q(z + z^{-1}), \quad B_1 = \hbar^{1/2}(qz + z^{-1}), \quad A_2 = q\hbar, \quad B_2 = -q.$$

4.4.1. Solutions Around 0

From this, [EFK98, Proposition 11.4.2] tells us how one may produce solutions for this equation in a neighbourhood of 0.

Theorem 8 (Proposition 11.4.2., [EFK98])**.** *Let s_i, a_i, b_i, c_i, where $i = 1, 2$ be the two distinct solutions of the system of equations*

$$q^{2s}B_0 + q^sB_1 + B_2 = 0, \tag{4.17}$$

$$q^{2s}\frac{A_0}{A_2} = q^{a+b}, \tag{4.18}$$

$$q^s\frac{A_1}{A_2} = -(q^a + q^b), \tag{4.19}$$

$$q^{2s}\frac{B_0}{B_2} = q^{c-1},. \tag{4.20}$$

Then, the functions

$$f(u) := u^{s_i}{}_2\varphi_1\left[\begin{matrix} q^{a_i} & q^{b_i} \\ & q^{c_j} \end{matrix}; q; -ua_2/b_2\right]$$

are a basis of fundamental solutions in a neighbourhood of 0.

It remains to solve the series of 4 equations listed in Theorem 8 for the coefficients a, b, c, and s to find solutions of the quantum Knizhnik-Zamolodchikov connection around 0 and ∞.

Substituting into (4.17), (4.18), (4.19) and(4.20), we get:

$$-\hbar q^{2s} + \hbar^{1/2}(qz + z^{-1})q^s - q = 0, \tag{4.21}$$
$$q^{2s}\hbar^{-1} = q^{a+b}, \tag{4.22}$$
$$q^s\hbar^{-1/2}(z + z^{-1}) = q^a + q^b, \tag{4.23}$$
$$q^{2s}\hbar = q^c. \tag{4.24}$$

Observe that the coefficients s, a, b, c are uniquely defined up to a permutation $(a, b) \mapsto (b, a)$. Solving the first quadratic equation (4.21) gives us:

$$s_1 = -\log_q z - \frac{1}{2}\log_q \hbar, \quad s_2 = 1 + \log_q z - \frac{1}{2}\log_q \hbar.$$

Using (4.22), (4.23), we find that there are two solutions for q^{a_1}, given by

$$q^{a_1} = \hbar^{-1}, \hbar^{-1}z^{-2}.$$

But $q^a + q^b = \hbar^{-1} + \hbar^{-1}z^{-2}$, corresponding to the two possible solutions of q^{a_1}. As aforementioned, since the solution is unique up to a permutation $(a, b) \mapsto (b, a)$, it is sufficient to choose just one of these solutions for a and b. Solving for c_1 then gives us: $c_1 = -2\log_q z$. Repeating the same for s_2, and a_2, b_2, c_2 then gives us solutions:

$$s_1 = -\log_q z - \frac{1}{2}\log_q \hbar, \quad a_1 = -\log_q \hbar, \quad b_1 = -2\log_q z - \log_q \hbar, \quad c_1 = -2\log_q z,$$
$$s_2 = 1 + \log_q z - \frac{1}{2}\log_q \hbar, \quad a_2 = 1 - \log_q \hbar, \quad b_2 = 1 - \log_q \hbar + 2\log_q z, \quad c_2 = 2(1 + \log_q z).$$

Given a set of solutions for Φ_1, one automatically obtains a set of solutions for Φ_2 since by (4.9), we may write Φ_2 as a linear combination of Φ_1. Thus, we have proved the following:

Proposition 4. *The equations*

$$\Phi_1^{(0)} := \begin{pmatrix} \Phi_{11}^{(0)} \\ \Phi_{12}^{(0)} \end{pmatrix}, \quad \Phi_2^{(0)} := \begin{pmatrix} \Phi_{21}^{(0)} \\ \Phi_{22}^{(0)} \end{pmatrix},$$

where

$$\Phi_{11}^{(0)}(z, u) = u^{-\log_q z - 1/2\log_q \hbar}\,{}_2\varphi_1\left[\begin{matrix} \hbar^{-1} \quad \hbar^{-1}z^{-2} \\ z^{-2} \end{matrix}; q; \hbar u\right],$$
$$\Phi_{12}^{(0)}(z, u) = u^{1+\log_q z - 1/2\log_q \hbar}\,{}_2\varphi_1\left[\begin{matrix} q\hbar^{-1} \quad q\hbar^{-1}z^2 \\ (qz)^2 \end{matrix}; q; \hbar u\right],$$

and

$$\Phi_{21}^{(0)}(z, u) = E\left(\Phi_{11}^{(0)}\right),$$
$$\Phi_{22}^{(0)}(z, u) = E\left(\Phi_{12}^{(0)}\right),$$

gives a basis of asymptotic solutions of the quantum Knizhnik-Zamolodchikov equation Φ in a neighbourhood of 0. Further,

$$\mathbf{X}^{(0)} = \begin{pmatrix} \Phi_{11}^{(0)}(z, u) & \Phi_{12}^{(0)}(z, u) \\ \Phi_{11}^{(0)}(z, qu) & \Phi_{12}^{(0)}(z, qu) \end{pmatrix},$$

is a fundamental solution of the quantum Knizhnik-Zamolodchikov equation in a neighbourhood of 0.

For simplicity, let us write $\Phi_1^{(0)} := \Phi_{11}^{(0)}$, and $\Phi_{12}^{(0)} := \Phi_2^{(0)}$. Observe that functions of the form $u^{\log_q \hbar}$ has the same periodicity as

$$e_{q,\hbar} = \frac{\Theta_q(u)}{\Theta_q(\hbar^{-1}u)},$$

and thus the following equations:

$$\Phi_1^{(0)} = \frac{\Theta_q(u)}{\Theta_q(z\hbar^{1/2}u)}{}_2\varphi_1\left[\begin{matrix}\hbar^{-1} & \hbar^{-1}z^{-2} \\ & z^{-2}\end{matrix}\,;q\,;\hbar u\right],$$

$$\Phi_2^{(0)} = \frac{\Theta_q(u)}{\Theta_q(z^{-1}\hbar^{1/2}u)}{}_2\varphi_1\left[\begin{matrix}q\hbar^{-1} & q\hbar^{-1}z^{2} \\ & (qz)^{2}\end{matrix}\,;q\,;\hbar u\right],$$

also give a basis of solutions of the qKZ equation in a neighbourhood of 0.

[EFK98, §12.2] gives a procedure for how one can produce a basis of fundamental solutions in a neighbourhood of ∞. The procedure is similar to the one detailed above. As before, we obtain q-hypergeometric equations, whose coefficients are given by u raised the power of $\log_q$ applied to the eigenvalues of $A(\infty)$:

Proposition 5 (Proposition 12.2.1, [EFK98])**.** *The functions*

$$u^{\log_q \lambda_1}\cdot {}_2\varphi_1\left[\begin{matrix}q^a & q^{a-c+1} \\ & q^{a-b+1}\end{matrix}\,;q\,;q^{c+1-a-b}z^{-1}\right],$$

$$u^{\log_q \lambda_2}\cdot {}_2\varphi_1\left[\begin{matrix}q^b & q^{b-c+1} \\ & q^{b-a+1}\end{matrix}\,;q\,;q^{c+1-a-b}z^{-1}\right],$$

where λ_i are the eigenvalues of the matrix $A(\infty)$, are a basis of solutions of the quantum Knizhnik-Zamolodchikov equation Φ in a neighbourhood of ∞.

Here, the constants a, b, c are the same ones seen in the q-hypergeometric equations appearing in Proposition 4.

4.5. Monodromy of the qKZ Equation

We begin by first studying the monodromy of the second order q-difference equation (4.14), following the techniques outlined in [Sau03]. In particular, our technique here uses [Sau03, Theorem 2.3.2.1] to derive the monodromy matrix M for the qKZ equation. This matrix can be derived explicitly using more classical results in the theory of q-difference equations (in particular, from [EFK98, Chapter 11-12]), as we will see in the next section.

Lemma 16. *The quantum Knizhnik-Zamolodchikov equation is strictly Fuchsian at* 0 *and* ∞ *in the sense of [Sau03, §1.2.1].*

Proof. Computing directly:

$$A(u) := R(u)\cdot \mathrm{diag}(z, z^{-1}) = \begin{pmatrix} z\dfrac{(1-u)\hbar^{1/2}}{\hbar-u} & z\dfrac{u(\hbar-1)}{\hbar-u} \\ z^{-1}\dfrac{\hbar-1}{\hbar-u} & z^{-1}\dfrac{(1-u)\hbar^{1/2}}{\hbar-u}\end{pmatrix}.$$

Then, we see that

$$A(0) = \begin{pmatrix} z\hbar^{-1/2} & 0 \\ z^{-1}\dfrac{\hbar-1}{\hbar} & z^{-1}\hbar^{-1/2} \end{pmatrix},$$

which has determinant $\hbar^{-1} \neq 0$, and thus $A(0) \in \mathrm{GL}_2(\mathbb{C})$. Similarly,

$$A(\infty) = \begin{pmatrix} z\hbar^{1/2} & z(1-\hbar) \\ 0 & z^{-1}\hbar^{1/2} \end{pmatrix},$$

which has determinant $\hbar \neq 0$, and thus $A(\infty) \in \mathrm{GL}_2(\mathbb{C})$. $\square$

Let

$$\mathbf{X}^{(0)} = \begin{pmatrix} \Phi_{11}^{(0)}(z,u) & \Phi_{12}^{(0)}(z,u) \\ \Phi_{11}^{(0)}(z,qu) & \Phi_{12}^{(0)}(z,qu) \end{pmatrix}, \quad \mathbf{X}^{(\infty)} = \begin{pmatrix} \Phi_{11}^{(\infty)}(z,u) & \Phi_{12}^{(\infty)}(z,u) \\ \Phi_{11}^{(\infty)}(z,qu) & \Phi_{12}^{(\infty)}(z,qu) \end{pmatrix},$$

denote the fundamental solutions around 0 and ∞, respectively. For simplicity, we will write $\mathcal{T}_u(-)$ for the operator that shifts a variable u by q. By Theorem 1, we know that these constant coefficients can be written in the form

$$\mathbf{X}^{(0)} = M^{(0)} e_{q,A(0)}, \quad \mathbf{X}^{(\infty)} = M^{(\infty)} e_{q,A(\infty)},$$

where $M^{(0)}$ and $M^{(\infty)}$ are the monodromy matrices satisfying a gauge transform relation. We wish to obtain an expression for these matrices. Observe that since $A(0)$ is lower triangular, and has two distinct entries on its diagonal, it is diagonalisable, and thus semi-simple. Thus, its Jordan-Chevalley decomposition consists of just the semi-simple portion. Thus,

$$A(0) = S \begin{pmatrix} z^{-1}\hbar^{-1/2} & 0 \\ 0 & z\hbar^{-1/2} \end{pmatrix} S^{-1},$$

is the Jordan-Chevalley decomposition for $A(0)$, where

$$S = \begin{pmatrix} 0 & \dfrac{\hbar^{1/2}(z^2-1)}{\hbar-1} \\ 1 & 1 \end{pmatrix}.$$

Remark 8. Observe that the powers of u $\Phi_1^{(0)}$ and $\Phi_2^{(0)}$ in Proposition 4 correspond to the diagonal entries of the diagonalisation of $A(0)$. In particular, $q^{-\log_q z - 1/2\log_q \hbar} = z^{-1}\hbar^{-1/2}$, and $q^{\log_q z - 1/2\log_q \hbar} = z\hbar^{-1/2}$.

Then, applying the recipe from Section 1.1.1 gives us:

$$e_{q,A(0)} = S \begin{pmatrix} \dfrac{\Theta_q(u)}{\Theta_q(z\hbar^{1/2}u)} & 0 \\ 0 & \dfrac{\Theta_q(u)}{\Theta_q(z^{-1}\hbar^{1/2}u)} \end{pmatrix} S^{-1}$$

Similarly, for $A(\infty)$,

$$A(\infty) = Q \begin{pmatrix} z^{-1}\hbar^{1/2} & 0 \\ 0 & z\hbar^{1/2} \end{pmatrix} Q^{-1},$$

where

$$Q = \begin{pmatrix} \dfrac{(\hbar-1)z^2}{\hbar^{1/2}(z^2-1)} & 1 \\ & \\ 1 & 0 \end{pmatrix}.$$

So,

$$e_{q,A(\infty)} = Q \begin{pmatrix} \dfrac{\Theta_q(u)}{\Theta_q(z\hbar^{-1/2}u)} & 0 \\ & \\ 0 & \dfrac{\Theta_q(u)}{\Theta_q(z^{-1}\hbar^{-1/2}u)} \end{pmatrix} Q^{-1}.$$

From [Sau03], we have a monodromy matrix

$$M = \left(M^{(\infty)}\right)^{-1} M^{(0)} = e_{q,A(\infty)} \left(\mathbf{X}^{(\infty)}\right)^{-1} \mathbf{X}^{(0)} \left(e_{q,A(0)}\right)^{-1}, \tag{4.25}$$

where the matrix $\left(\mathbf{X}^{(\infty)}\right)^{-1}\mathbf{X}^{(0)}$ is called *Birkhoff's connection matrix* [Sau03, §1.2.3]. The matrix $e_{q,A(0)}$ is a section of the following rank two vector bundle on the elliptic curve $E = \mathbb{C}^\times/q^{\mathbb{Z}}$:

$$\mathcal{F}_0 := \mathcal{L}_1^{(0)} \oplus \mathcal{L}_2^{(0)} := \mathcal{O}(z^{-1}\hbar^{-1/2}) \oplus \mathcal{O}(z\hbar^{-1/2}).$$

Similarly, $e_{q,A(\infty)}$ is a section of the rank two bundle on E:

$$\mathcal{F}_\infty := \mathcal{L}_1^{(\infty)} \oplus \mathcal{L}_2^{(\infty)} := \mathcal{O}(z^{-1}\hbar^{1/2}) \oplus \mathcal{O}(z\hbar^{1/2}).$$

The monodromy matrix M then corresponds to a meromorphic map of sheaves $\varphi : \mathcal{F}_0 \dashrightarrow \mathcal{F}_\infty$.

Proposition 6 (Proposition 12.2.2, [EFK98]). *The q-hypergeometric equation can be written in the form:*

$${}_2\varphi_1\left[\begin{matrix} q^a & q^b \\ & q^c \end{matrix} ; q ; u\right] = \alpha(u)\Phi_{11}^{(\infty)}(u) + \beta(u)\Phi_{12}^{(\infty)}(u),$$

where

$$\alpha(u) = \frac{\Gamma_q(c)\Gamma_q(b-a)}{\Gamma_q(b)\Gamma_q(c-a)} \cdot \frac{u^{-\log_q \lambda_1}\Theta_q(q^{-a}u)}{\Theta_q(u)},$$

and

$$\beta(u) = \frac{\Gamma_q(c)\Gamma_q(a-b)}{\Gamma_q(a)\Gamma_q(c-b)} \cdot \frac{u^{-\log_q \lambda_2}\Theta_q(q^{-b}u)}{\Theta_q(u)},$$

*where $\Phi_{11}^{(\infty)}$ and $\Phi_{12}^{(\infty)}$ are a basis of solutions in a neighbourhood of ∞. $\Gamma_q(a)$ is the q-*Gamma function.

Proof. The *q-Gamma function* is given by:

$$\Gamma_q(x) = (1-q)^{1-x}\frac{(q;q)_\infty}{(q^x;q)_\infty}. \tag{4.26}$$

Then, [GR09, (4.3.2)], [Koe18, (2.30)] has the following formula:

$$
\begin{aligned}
{}_2\varphi_1\left[\begin{matrix} q^a & q^b \\ & q^b \end{matrix} ; q ; u\right] &= \frac{(q^b;q)_\infty (q^{c-a};q)_\infty}{(q^c;q)_\infty (q^{b-a};q)_\infty} \cdot \frac{(q^a u;q)_\infty (q^{1-a}u^{-1};q)_\infty}{(u;q)_\infty (qu^{-1};q)_\infty} \cdot {}_2\varphi_1\left[\begin{matrix} q^a & q^{a-c+1} \\ & q^{a-b+1} \end{matrix} ; q ; q^{c+1-a-b}u^{-1}\right] \\
&+ \frac{(q^a;q)_\infty (q^{c-b};q)_\infty}{(q^c;q)_\infty (q^{a-b};q)_\infty} \cdot \frac{(q^b u;q)_\infty (q^{1-b}u^{-1};q)_\infty}{(u;q)_\infty (qu^{-1};q)_\infty} \cdot {}_2\varphi_1\left[\begin{matrix} q^b & q^{b-c+1} \\ & q^{b-a+1} \end{matrix} ; q ; q^{c+1-a-b}u^{-1}\right].
\end{aligned}
$$

Then, using Proposition 5, and the Jacobi triple product identity, and (4.26):

$$
\begin{aligned}
{}_2\varphi_1\left[\begin{matrix} q^a & q^b \\ & q^b \end{matrix} ; q ; u\right] &= \frac{\Gamma_q(c)\Gamma_q(b-a)}{\Gamma_q(b)\Gamma_q(c-a)} \cdot u^{-\log_q \lambda_1} \frac{\Theta_q(q^a u^{-1})}{\Theta_q(u^{-1})} \cdot \Phi_1^{(\infty)} \\
&+ \frac{\Gamma_q(c)\Gamma_q(a-b)}{\Gamma_q(a)\Gamma_q(c-b)} \cdot u^{-\log_q \lambda_2} \frac{\Theta_q(q^b u^{-1})}{\Theta_q(u^{-1})} \cdot \Phi_2^{(\infty)},
\end{aligned}
$$

and the result follows from using the fact that

$$
\frac{\Theta_q(a^{-1}u^{-1})}{\Theta_q(u^{-1})} = \frac{\Theta_q(au)}{\Theta_q(u)}.
$$

□

Let $\mathbf{X}$ be the connection matrix. Then, we have:

$$
\Phi_i^{(0)} = \sum_j \mathbf{X}_{ij} \Phi_j^{(\infty)}.
$$

We may replace $u^{-\log_q \lambda_i}$ with the ratio $\frac{\Theta_q(\lambda_i^{-1}u)}{\Theta_q(u)}$. Then, using Proposition 4, Proposition 5, and Proposition 6, we obtain the connection matrix:

$$
\mathbf{X} = \begin{pmatrix} X_{11} \cdot \dfrac{\Theta_q(z\hbar^{-1/2}u)}{\Theta_q(z\hbar^{1/2}u)} \cdot \dfrac{\Theta_q(\hbar^2 u)}{\Theta_q(\hbar u)} & X_{12} \cdot \dfrac{\Theta_q(z^{-1}\hbar^{-1/2}u)}{\Theta_q(z\hbar^{1/2}u)} \cdot \dfrac{\Theta_q(\hbar^2 z^2 u)}{\Theta_q(\hbar u)} \\ X_{21} \cdot \dfrac{\Theta_q(z\hbar^{-1/2}u)}{\Theta_q(z^{-1}\hbar^{1/2}u)} \cdot \dfrac{\Theta_q(q\hbar^2 z^{-2} u)}{\Theta_q(\hbar u)} & X_{22} \cdot \dfrac{\Theta_q(z^{-1}\hbar^{-1/2}u)}{\Theta_q(z^{-1}\hbar^{1/2}u)} \cdot \dfrac{\Theta_q(q\hbar^2 u)}{\Theta_q(\hbar u)} \end{pmatrix},
$$

where

$$
\begin{aligned}
X_{11} &= \frac{\left(\Gamma_q(-2\log_q z)\right)^2}{\Gamma_q(-\log_q \hbar - 2\log_q z)\Gamma_q(\log_q \hbar)}, \\
X_{12} &= \frac{\Gamma_q(-2\log_q z)\Gamma_q(2\log_q z)}{\Gamma_q(-\log_q \hbar)\Gamma_q(\log_q \hbar)}, \\
X_{21} &= \frac{\Gamma_q(2+2\log_q z)\Gamma_q(2\log_q z)}{\Gamma_q(1-\log_q \hbar - 2\log_q z)\Gamma_q(1+\log_q \hbar + 2\log_q z)}, \\
X_{22} &= \frac{\Gamma_q(2+2\log_q z)\Gamma_q(-2\log_q z)}{\Gamma_q(1-\log_q \hbar)\Gamma_q(1+\log_q \hbar)},
\end{aligned}
$$

and are treated as constants, since they only depend on z and $\hbar$. Choose a basis such that the $e_{q,A(0)}$ and

$e_{q,A(\infty)}$ are diagonal – that is:

$$e_{q,A(0)} = \begin{pmatrix} \frac{\Theta_q(u)}{\Theta_q(z\hbar^{1/2}u)} & 0 \\ 0 & \frac{\Theta_q(u)}{\Theta_q(z^{-1}\hbar^{1/2}u)} \end{pmatrix}$$

and

$$e_{q,A(\infty)} = \begin{pmatrix} \frac{\Theta_q(u)}{\Theta_q(z\hbar^{-1/2}u)} & 0 \\ 0 & \frac{\Theta_q(u)}{\Theta_q(z^{-1}\hbar^{-1/2}u)} \end{pmatrix}.$$

We can determine the monodromy matrix M explicitly using the formula

$$M = e_{q,A(\infty)} \cdot \mathbf{X}^T \cdot \left(e_{q,A(0)}\right)^{-1},$$

from which we obtain:

$$M = \begin{pmatrix} X_{11} \cdot \frac{\Theta_q(\hbar^2 u)}{\Theta_q(\hbar u)} & X_{21} \cdot \frac{\Theta_q(q\hbar^2 z^{-2}u)}{\Theta_q(\hbar u)} \\ X_{12}\frac{\Theta_q(\hbar^2 z^2 u)}{\Theta_q(\hbar u)} & X_{22}\frac{\Theta_q(q\hbar^2 u)}{\Theta_q(\hbar u)} \end{pmatrix}, \tag{4.27}$$

as the monodromy matrix for the qKZ equation.

Remark 9. Note that the theta terms containing the dynamical parameters z on the denominator are cancelled out. This tells us that the only poles of the monodromy matrix M are $\hbar^{-1}$. As we will see in the next section, the morphism $\Delta_s : \mathcal{F} \to s^*\mathcal{F}$ from Lemma 12 can be expressed as the product of a constant matrix and the monodromy matrix – that is, the zeroes and poles of Δ_s are controlled entirely by M. It follows readily that the pole condition Lemma 12(iii) is satisfied.

Generally, it is possible that the monodromy of any arbitrary DAHA module will contain poles at dynamical parameters, which could give rise to representations of the dynamical AHA. We expect that it is possible to recover an ellAHA module from this dynAHA module by "forgetting" the dynamical parameter. This informs our formulation of Conjecture 3.

4.6. Rank One qKZ Functor

Let $s \in W \cong \mathbb{Z}/2\mathbb{Z}$ be the only non-trivial simple reflection in the rank one Weyl group. As aforementioned, we obtain two flat vector bundles $\mathcal{F}_0$ and $\mathcal{F}_\infty$ that are dual to one another. Let $\mathbf{qTor}^{\mathrm{fuch}}_{\mathfrak{S}_2}$ be the $\mathfrak{S}_2$-equivariant Fuchsian quantum torus. Then, the only non-trivial generator $s \in \mathfrak{S}_2$ acts on the equivariant variables by $s : u \mapsto u^{-1}$, and acts on the q-difference operator $\sigma_q := \mathrm{D}_q^{\rho^\vee}$ by $s(\sigma_q) : x \mapsto q^{-1}x$.

Moreover, given a line bundle $\mathcal{O}(z)$ over E for some divisor z, a section of $\mathcal{O}(z)$ is given by Jacobi theta functions of the form $\frac{\Theta_q(u)}{\Theta_q(zu)}$. But since Θ_q is an odd function, it satisfies the property that

$$\frac{\Theta_q(u)}{\Theta_q(zu)} = \frac{\Theta_q(u^{-1})}{\Theta_q(z^{-1}u^{-1})}. \tag{4.28}$$

It follows then that there is an isomorphism

$$\mathcal{O}(z^{-1}) \cong s^*\mathcal{O}(z).$$

Thus, relative to a choice of basis, there are isomorphisms of vector bundles:

$$\mathcal{F}_0^\vee \cong s^*\mathcal{F}_0, \quad \mathcal{F}_\infty \cong s^*\mathcal{F}_\infty.$$

Let M be the $\mathbf{qTor}_{\mathfrak{S}_2}^{\text{fuch}}$-module coming from the $\ddot{\mathbf{H}}_{\text{loc}}$-module in Proposition 3. Given such a module, we wish to explicitly describe the module s^*M. Then, given a basis for M, we can write an element as a q-difference system $\sigma_q V(x)A(x) = V(x)$, where $A \in \mathrm{GL}_2(\mathbb{C}[X^\pm])$, and V is a column vector. Then, it follows that there exists a module s^*M corresponding to the q-difference equation:

$$s(\sigma_q)V(u^{-1}) = A(qu^{-1})^{-1}V(u^{-1}), \tag{4.29}$$

where $A(qu^{-1})^{-1}$ comes from the relation in DAHA:

$$s\tau(\rho)^{-1}s^{-1} = \tau(\rho),$$

(c.f. Lemma 15). We also use the fact that s acts on $V(u))$ by sending $V(u) \mapsto V(u^{-1})$. Taking the limit $x \to 0$ and $x \to \infty$, and using the theory of [Sau03] gives us vector bundles $\mathcal{F}_0^\vee$ and $\mathcal{F}_\infty$ over E associated to the q-difference system (4.29). Taking monodromy gives us a morphism $\varphi^\vee : \mathcal{F}_0^\vee \to \mathcal{F}_\infty^\vee$. Equivalently,

$$\varphi^\vee = s^*\varphi : s^*\mathcal{F}_0 \to s^*\mathcal{F}_\infty,$$

and since $\mathcal{F}_0$ and $\mathcal{F}_\infty$ are dual to one another, the q-monodromy morphism $s^*\varphi$ just gives the monodromy going from local solutions around ∞ to local solutions around 0. It thus follows that there is an isomorphism:

$$B_s : M \xrightarrow{\simeq} s^*M.$$

This corresponds to the morphism on the level of the $\mathfrak{S}_2$-equivariant connection category:

$$B_s : (\mathcal{F}_0, \mathcal{F}_\infty, \varphi) \longrightarrow s^*(\mathcal{F}_0, \mathcal{F}_\infty, \varphi),$$

where

$$s^*(\mathcal{F}_0, \mathcal{F}_\infty, \varphi) = (s^*\mathcal{F}_\infty, s^*\mathcal{F}_0, s^*\varphi^{-1}),$$

and the map B_s is given by the commutative diagram:

$$\begin{tikzcd} \mathcal{F}_0 \arrow[r, dashed, "B_s^{(0)}"] \arrow[d, dashed, "\varphi"'] & s^*\mathcal{F}_\infty \arrow[d, dashed, "s^*\varphi^{-1}"] \\ \mathcal{F}_\infty \arrow[r, dashed, "B_s^{(\infty)}"'] & s^*\mathcal{F}_0 \end{tikzcd} \tag{4.30}$$

Let us define the maps Δ_s to be the following composition:

$$\begin{tikzcd} \mathcal{F}_0 \arrow[r, "\varphi"] \arrow[rr, bend right, "\Delta_s^{(0)}"'] & \mathcal{F}_\infty \arrow[r, "B_s^{(\infty)}"] & s^*\mathcal{F}_0 \end{tikzcd}$$

and

$$\begin{tikzcd} \mathcal{F}_\infty \arrow[r, "B_s^{(\infty)}"] \arrow[rr, bend right, "\Delta_s^{(\infty)}"'] & s^*\mathcal{F}_0 \arrow[r, "s^*\varphi"] & s^*\mathcal{F}_\infty \end{tikzcd}$$

Our goal now is to construct a suitable map $B_s^{(\infty)}$ that takes us from $\mathcal{F}_0$ to $s^*\mathcal{F}_\infty$. Concretely, the existence of such a map is equivalent to the existence of some some gauge transformation B taking us from $A(u)$ to $A(qu^{-1})^{-1}$, which corresponds to the map of $\mathbf{qTor}^{\text{fuch}}_{\mathfrak{S}_2}$-modules $M \to s^*M$. This will then elucidate $\Delta_s^{(0)}$, which will be given by $M \cdot B_s^{(\infty)}$, where by abuse of notation we take $B_s^{(\infty)}$ to be a section of the sheaf morphism $\mathcal{F}_\infty \to s^*\mathcal{F}_0$. We will then analyse the poles of $\Delta_s^{(0)}$, and use this to prove Lemma 12, which will show that $\mathcal{F}_0$ is a module over $\mathcal{H}^{\text{ell}}$.

Lemma 17. *Let $B(X)$ be the image of the simple reflection s in the localised standard module $\Delta(\delta_m)_{\text{loc}}$, written with respect to the ordered basis $\{1\otimes 1, T\otimes 1\}$. Then, $B(X)$ is of the form:*

$$B(X) = R(X)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $R(X)$ is the trigonometric R-matrix.

Proof. Using (1.11), we have:

$$s(X) = \frac{1-X^2}{t-t^{-1}X^2}(T-t)+1. \tag{4.31}$$

Then, computing directly, we have:

$$\begin{aligned} \left(\frac{1-X^2}{t-t^{-1}X^2}(T-t)+1\right)\otimes 1 &= \frac{1-X^2}{t-t^{-1}X^2}T\otimes 1 + \left(1-\frac{t(1-X^2)}{t-t^{-1}X^2}\right)1\otimes 1 \\ &= \frac{(t-t^{-1})X^2}{t-t^{-1}X^2}1\otimes 1 + \frac{1-X^2}{t-t^{-1}}T\otimes 1, \end{aligned}$$

and

$$\begin{aligned} \left(\frac{1-X^2}{t-t^{-1}X^2}(T-t)+1\right) &= \frac{1-X^2}{t-t^{-1}X^2}T^2\otimes 1 + \left(1-\frac{t(1-X^2)}{t-t^{-1}X^2}\right)T\otimes 1 \\ &= \frac{1-X^2}{t-t^{-1}X^2}((t-t^{-1})T\otimes 1 + 1\otimes 1) + \frac{(t-t^{-1})X^2}{t-t^{-1}X^2}T\otimes 1 \\ &= \frac{1-X^2}{t-t^{-1}X^2}1\otimes 1 + \frac{t-t^{-1}}{t-t^{-1}X^2}T\otimes 1, \end{aligned}$$

and so we obtain a matrix:

$$B(X) = \begin{pmatrix} \dfrac{(t-t^{-1})X^2}{t-t^{-1}X^2} & \dfrac{1-X^2}{t-t^{-1}X^2} \\ \dfrac{1-X^2}{t-t^{-1}X^2} & \dfrac{t-t^{-1}}{t-t^{-1}X^2} \end{pmatrix}, \tag{4.32}$$

which we observe is equal to $R(X)\cdot\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. □

Written using our notation for the qKZ equation, this is given by

$$B(u) = \begin{pmatrix} \dfrac{(h-1)u}{h-u} & \dfrac{(1-u)\hbar^{1/2}}{h-u} \\ \dfrac{(1-u)\hbar^{1/2}}{\hbar-u} & \dfrac{\hbar-1}{\hbar-u} \end{pmatrix}.$$

One readily checks that $R(u)^{-1} = R(u^{-1})^T$. So, we have that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot B(u^{-1}) = R(u)^{-1}$. The following Lemma will show that $B(u)$ is a gauge transform that takes us from $A(u)$ to $A(qu^{-1})^{-1}$. That is, $B(u)$ corresponds precisely to the $\mathbf{qTor}^{\mathrm{fuch}}_{\mathfrak{S}_2}$-module isomorphism $B_s : M \to s^*M$.

Corollary 4. *The matrix $B(u)$ gives a q-gauge transformation of $A(u)$ to $A(qu^{-1})^{-1}$. That is,*

$$B(u)^{-1} \cdot A(u)B(q^{-1}u) = A(qu^{-1})^{-1}.$$

Proof. Computing directly,

$$\begin{aligned} B(u)^{-1}A(u)B(q^{-1}u) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot R(u)^{-1} \cdot R(u) \cdot \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \cdot R(q^{-1}u) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \left(R(q^{-1}u) \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \right) \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= R(q^{-1}u)^T \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix} \\ &= R(qu^{-1})^{-1} \cdot \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix} \\ &= A(qu^{-1})^{-1}, \end{aligned}$$

where the third equality follows from the fact that:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}.$$

The diagonal entries of $R(q^{-1}u)$ are equal, and so conjugating by this matrix gives us preicsely $R(q^{-1}u)^T$. Conjugating $\mathrm{diag}(z, z^{-1})$ by this matrix gives us $\mathrm{diag}(z^{-1}, z)$, which is its inverse. The fourth equality follows from the equality $R(u)^{-1} = R(u^{-1})^T$. The last equality is by definition of $A(u)$ as $A(u) = R(u) \cdot \mathrm{diag}(z, z^{-1})$. $\square$

Corollary 4 shows that $B(u)$ is a q-gauge transformation in the category of $\mathbf{qTor}^{\mathrm{fuch}}_{\mathfrak{S}_2}$-modules. However, the following lemma will additionally show that this gauge transformation relation also holds in the category of $\ddot{\mathbf{H}}_{\mathrm{loc}}$-modules:

Lemma 18. *Let $B(X)$ be the 2×2 matrix given by the image of the simple reflection s in the $\ddot{\mathbf{H}}_{\mathrm{loc}}$-module $\Delta(\delta_m)_{\mathrm{loc}}$. Then,*

$$B(X)^{-1}A(X)B(q^{-1}X) = A(q^{-1}X^{-1})^{-1}.$$

That is, B gives a gauge transformation from $A(X)$ to $A(qX^{-1})^{-1}$.

Proof. We use the commutation relation from the localised DAHA:

$$\tau(\rho)^{-1}s = s\tau(\rho),$$

together with the equation for $\tau(\rho)$ seen in Lemma 15. We know that $\tau(\rho) = A(qX)^{-1}$, but let us write $C(X) = A(qX)^{-1}$ for simplicity. Let $\mathbf{e}_1 := 1 \otimes 1$, and $\mathbf{e}_2 := T \otimes 1$. Then, computing directly,

$$\begin{aligned}(s\cdot\tau(\rho))(\mathbf{e}_1) &= s(C_{11}(X)\mathbf{e}_1 + C_{21}(X)\mathbf{e}_2)\\ &= C_{11}(X^{-1})s\mathbf{e}_1 + C_{21}(X^{-1})s\mathbf{e}_2\\ &= C_{11}(X^{-1})\left(B_{11}(X)\mathbf{e}_1 + B_{21}(X)\mathbf{e}_2\right) + C_{21}(X^{-1})\left(B_{12}(X)\mathbf{e}_1 + B_{22}(X)\mathbf{e}_2\right)\\ &= \left(C_{11}(X^{-1})B_{11}(X) + C_{21}(X^{-1})B_{12}(X)\right)\mathbf{e}_1 + \left(C_{11}(X^{-1})B_{21}(X) + C_{21}(X^{-1})B_{22}(X)\right)\mathbf{e}_2,\end{aligned}$$

$$\begin{aligned}(s\cdot\tau(\rho))(\mathbf{e}_2) &= s(C_{12}(X)\mathbf{e}_1 + C_{22}(X)\mathbf{e}_2)\\ &= C_{12}(X^{-1})s\mathbf{e}_1 + C_{22}(X^{-1})s\mathbf{e}_2\\ &= C_{12}(X^{-1})\left(B_{11}(X)\mathbf{e}_1 + B_{21}(X)\mathbf{e}_2\right) + C_{22}(X^{-1})\left(B_{12}(X)\mathbf{e}_1 + B_{22}(X)\mathbf{e}_2\right)\\ &= \left(C_{12}(X^{-1})B_{11}(X) + C_{22}(X^{-1})B_{12}(X)\right)\mathbf{e}_1 + \left(C_{12}(X^{-1})B_{21}(X) + C_{22}(X^{-1})B_{22}(X)\right)\mathbf{e}_2,\end{aligned}$$

$$\begin{aligned}(\tau(\rho)^{-1}\cdot s)(\mathbf{e}_1) &= \tau(\rho)^{-1}(B_{11}(X)\mathbf{e}_1 + B_{21}(X)\mathbf{e}_2)\\ &= B_{11}(q^{-1}X)\tau(\rho)^{-1}\mathbf{e}_1 + B_{21}(q^{-1}X)\tau(\rho)^{-1}\mathbf{e}_2\\ &= B_{11}(q^{-1}X)\left(A_{11}(X)\mathbf{e}_1 + A_{21}(X)\mathbf{e}_2\right) + B_{21}(q^{-1}X)\left(A_{12}(X)\mathbf{e}_1 + A_{22}(X)\mathbf{e}_2\right)\\ &= \left(B_{11}(q^{-1}X)A_{11}(X) + B_{21}(q^{-1}X)A_{12}(X)\right)\mathbf{e}_1 + \left(B_{11}(q^{-1}X)A_{21}(X) + B_{21}(q^{-1}X)A_{22}(X)\right)\mathbf{e}_2\end{aligned}$$

$$\begin{aligned}(\tau(\rho)^{-1}\cdot s)(\mathbf{e}_2) &= \tau(\rho)^{-1}(B_{12}(X)\mathbf{e}_1 + B_{22}(X)\mathbf{e}_2)\\ &= B_{12}(q^{-1}X)\tau(\rho)^{-1}\mathbf{e}_1 + B_{22}(q^{-1}X)\tau(\rho)^{-1}\mathbf{e}_2\\ &= B_{12}(q^{-1}X)\left(A_{11}(X)\mathbf{e}_1 + A_{21}(X)\mathbf{e}_2\right) + B_{22}(q^{-1}X)\left(A_{12}(X)\mathbf{e}_1 + A_{22}(X)\mathbf{e}_2\right)\\ &= \left(B_{12}(q^{-1}X)A_{11}(X) + B_{22}(q^{-1}X)A_{12}(X)\right)\mathbf{e}_1 + \left(B_{12}(q^{-1}X)A_{21}(X) + B_{21}(q^{-1}X)A_{22}(X)\right)\mathbf{e}_2\end{aligned}$$

Then, we see that we have relations $C(u^{-1})^T B(u)^T = B(q^{-1}u)^T A(u)^T$. From Lemma 15, we know that $C(X) = A(qX)^{-1}$, and so we have the relation:

$$B(u)A(qu^{-1})^{-1} = A(u)B(q^{-1}u).$$

□

Computing directly,

$$B(0) = \begin{pmatrix} 0 & \hbar^{-1/2} \\ \hbar^{-1/2} & \dfrac{\hbar - 1}{\hbar} \end{pmatrix}, \quad B(\infty) = \begin{pmatrix} 1-\hbar & \hbar^{1/2} \\ \hbar^{1/2} & 0 \end{pmatrix}.$$

As a consequence of Corollary 4 or Lemma 18, we have the relations:

$$B(0)^{-1}A(0)B(0) = A(\infty)^{-1}, \quad B(\infty)^{-1}A(\infty)B(\infty) = A(0)^{-1}.$$

Recall that $A(0)$ and $A(\infty)$ give rise to divisors (and thus line bundles) on E by diagonalising them. In particular, $A(0) = S \cdot E_0 \cdot S^{-1}$, and $A(\infty) = Q \cdot E_\infty \cdot Q^{-1}$, where E_0 and E_∞ are the diagonal entries obtained from diagonlising $A(0)$ and $A(\infty)$, respectively. Then, using the gauge transformation relation, we have:

$$\left(Q^{-1}B(0)^{-1}S\right) \cdot E_0 \cdot \left(Q^{-1}B(0)^{-1}S\right)^{-1} = E_\infty^{-1},$$

and thus $QB(0)^{-1}S$ corresponds to a section of a map $\mathcal{F}_0 \to s^*\mathcal{F}_\infty$. Thus,

$$B_s^{(0)} = Q^{-1}B(0)^{-1}S = \begin{pmatrix} 0 & \dfrac{\hbar(z^2-1)}{\hbar-1} \\ \hbar^{1/2} & 2\hbar^{1/2}(1-z^2) \end{pmatrix}. \tag{4.33}$$

Analogously, one sees that

$$B_s^{(\infty)} = S^{-1}B(\infty)^{-1}Q = \begin{pmatrix} \dfrac{2(\hbar-1)}{\hbar} & \hbar^{-1/2} \\ \dfrac{\hbar-1}{\hbar(z^2-1)} & 0 \end{pmatrix}. \tag{4.34}$$

Indeed, one expects these matrices to be constant matrices, since $s^*\mathcal{F}_\infty \cong \mathcal{F}_\infty^\vee \cong \mathcal{F}_0$. So, $B_s^{(0)}$ is actually a map $B_s^{(0)} : \mathcal{F}_0 \to \mathcal{F}_0$. However, these two matrices are inverses of one another, which implies that $B_s^{(0)} = \left(B_s^{(\infty)}\right)^{-1}$, which is an unexpected relation. Then, it follows then that

$$\Delta_s^{(0)} = M(u) \cdot B_s^{(\infty)}. \tag{4.35}$$

This gives us:

$$\Delta_s^{(0)} = \begin{pmatrix} X_{11} \cdot \dfrac{2(\hbar-1)}{\hbar} \cdot \dfrac{\Theta_q(\hbar^2 u)}{\Theta_q(\hbar u)} + X_{21} \cdot \dfrac{\hbar-1}{\hbar(z^2-1)} \cdot \dfrac{\Theta_q(q\hbar^2 z^{-2}u)}{\Theta_q(\hbar u)} & X_{11} \cdot \hbar^{-1/2} \cdot \dfrac{\Theta_q(\hbar^2 u)}{\Theta_q(\hbar u)} + X_{21} \\ X_{12} \cdot \dfrac{2(\hbar-1)}{\hbar} \cdot \dfrac{\Theta_q(\hbar^2 z^2 u)}{\Theta_q(\hbar u)} + X_{22} \cdot \dfrac{\hbar-1}{\hbar(z^2-1)} \cdot \dfrac{\Theta_q(\hbar^2 u)}{\Theta_q(\hbar u)} & X_{12} \cdot \hbar^{-1/2} \cdot \dfrac{\Theta_q(\hbar^2 z^2 u)}{\Theta_q(\hbar u)} \end{pmatrix}.$$

Proposition 7.

(i) $\left(\pi_*\Delta_s^{(0)}\right)^2 = \mathrm{id}_{\pi_*\mathcal{F}_0}$,

(iii) $\Delta_s^{(0)}$ *has a pole of order at most* 1 *along* $T_{\alpha,\hbar}$.

Proof.

(i) By construction, we have that $s^*\Delta_s^{(0)} \circ \Delta_s^{(0)} = \mathrm{id}_{s^*\mathcal{F}_0}$. Then, $\pi_*(s^*\Delta_s^{(0)} \circ \Delta_s^{(0)}) = \left(\pi_*\Delta_s^{(0)}\right)^2 = \mathrm{id}_{\pi_*\mathcal{F}_0}$.

(ii) From (4.27), the only poles of M are $\hbar^{-1}$, which is of order 1. Recall that we have a map

$$\chi_\alpha : E \otimes \mathbf{Y} \longrightarrow E, \quad u \otimes \mu^\vee \longmapsto u^{\langle \mu^\vee, \alpha \rangle}.$$

The element $\hbar^{-1} \otimes (-\rho^\vee)$ is in $T_{\alpha,\hbar}$. Then, $\Delta_s^{(0)}$ can only have poles of order at most 1 along the divisor $T_{\alpha,\hbar}$. Note that the element $u \in \mathfrak{A}$ is identified to be $u \otimes \rho^\vee$, since $\rho^\vee$ is the only fundamental coweight, and thus a basis of the coweight lattice $\mathbf{Y}$.

$\square$

Conjecture 4. $\pi_*\Delta_s^{(0)}|_{T_\alpha} = -\pi_*\Delta_e^{(0)}|_{T_\alpha}$.

We briefly outline a strategy that one may use to prove Conjecture 4. Recall from (4.30) that we have a commutative diagram:

$$\begin{tikzcd}
\mathcal{F}_0 \arrow[r, dashed, "B_s^{(0)}"] \arrow[d, dashed, "\varphi"'] & s^*\mathcal{F}_\infty \arrow[d, dashed, "s^*\varphi^{-1}"] \\
\mathcal{F}_\infty \arrow[r, dashed, "B_s^{(\infty)}"'] & s^*\mathcal{F}_0
\end{tikzcd}$$

Note that from (4.33) and (4.34) that $B_s^{(0)} = \left(B_s^{(\infty)}\right)^{-1}$. Using the commutativity of the diagram (4.30), we see that

$$\Delta_s^{(0)} = B_s^{(0)} \circ s^*\varphi^{-1} = B_s^{(\infty)} \circ \varphi.$$

Note that

$$\Delta_s^{(0)}|_{T_\alpha} = \left(B_s^{(0)} \circ s^*\varphi^{-1}\right)|_{T_\alpha} = B_s^{(0)} \circ s^*\varphi^{-1}|_{T_\alpha} = B_s^{(0)} \circ \varphi^{-1}|_{s\cdot T_\alpha} = \left(B_s^{(\infty)}\right)^{-1} \circ \varphi^{-1}|_{T_\alpha}.$$

Using the fact that $B_s^{(0)} = \left(B_s^{(\infty)}\right)^{-1}$, we have:

$$\Delta_s^{(0)}|_{T_\alpha} = \left(\varphi|_{T_\alpha} \circ B_s^{(\infty)}\right)^{-1} = \left(\varphi|_{T_\alpha} \circ B_s^{(\infty)}\right),$$

and thus $(\Delta_s^{(0)}|_{T_\alpha})^2 = \mathrm{id}$. Thus, we know that the eigenvalues of $\Delta_s^{(0)}|_{T_\alpha}$ are given by ± 1.

The issue now is to determine these eigenvalues, but this turns out to be a non-trivial computation, and we leave it as a conjecture for the timebeing. We expect that there should be some way to simplify the connection matrix so the eigenvalues can be easily calculated. In particular, these connection and monodromy matrices are known to be related to elliptic R-matrices. Felder's R-matrix is given in [AO16, (85)] as:

$$R_{\text{standard}}(u) = \begin{pmatrix} \dfrac{\Theta_q(z\hbar)\Theta_q(z\hbar^{-1})\Theta_q(u)}{\Theta_q(z)^2\Theta_q(\hbar^{-1}u)} & -\dfrac{\Theta_q(\hbar)\Theta_q(zu)}{\Theta_q(z)\Theta_q(\hbar^{-1}u)} \\ -\dfrac{\Theta_q(\hbar)\Theta_q(zu^{-1})}{\Theta_q(z)\Theta_q(\hbar^{-1}u)} & \dfrac{\Theta_q(u)}{\Theta_q(\hbar^{-1}u)} \end{pmatrix}.$$

The 2-torsion points of the elliptic curve $E = \mathbb{C}^\times/q^{\mathbb{Z}}$ are given by ± 1, and $\pm q^{-1/2}$. However, note that $M(u)$ is really a matrix $M(X^2)$, where $X = X^\rho$ in the fuchsian quantum torus $\mathbf{qTor}^{\text{fuch}}_{\mathfrak{S}_2}$ (or, the rank one localised DAHA). Thus, choosing $X = \pm q^{1/2}$, we see that $X^2 = q$, which is identified with 1 in E. It follows then that the only 2-torsion points we need to consider for $M(u)$ is the case for which $u = 1$. Indeed, substituting $u = 1$ in R_{standard} gives us:

$$R_{\text{standard}}(1) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

which has two distinct eigenvalues, given by $1, -1$. Conjecture 4 will imply that $\pi_*\mathcal{F}_0$ has the structure of a $\mathcal{H}^{\text{ell}}$-module by Lemma 12. Since $\mathcal{F}_\infty \cong \mathcal{F}_0^\vee$, it follows that $\pi_*\mathcal{F}_\infty \cong \pi_*\mathcal{F}_0^\vee$ also has the structure of a $\mathcal{H}^{\text{ell}}$-module. Thus, we have maps:

$$\begin{tikzcd}
\Delta(\delta_m)_{\text{loc}} \arrow[rr, mapsto, "q\text{RH}"] \arrow[dr, "q\text{KZ}^{(0)}"'] & & (\mathcal{F}_0, \mathcal{F}_\infty, \varphi) \\
& \pi_*\mathcal{F}_0 \arrow[ur, hook] &
\end{tikzcd}$$

This will show the following:

Conjecture 5. *Given a localised standard module* $\Delta(\delta_m)_{\mathrm{loc}} \in \mathcal{O}_{\ddot{\mathbf{H}}}$*, let* $(\mathcal{F}_0, \mathcal{F}_\infty, \varphi)$ *be its image under the q-Riemann-Hilbert functor* $q\mathrm{RH}$*. Then,* $\pi_*\mathcal{F}_0$ *and* $\pi_*\mathcal{F}_\infty$ *have the structure of a flat module over* $\mathcal{H}^{\mathrm{ell}}$*. Moreover, there exists unique functors* $q\mathrm{KZ}^{(0)}$ *and* $q\mathrm{KZ}^{(\infty)}$ *factoring through* $q\mathrm{RH}$ *uniquely.*

Chapter 5

Further Directions

We expect to be able to prove Conjecture 4, given slightly more time. From this, Conjecture 5 would immediately follow, which is a weaker version of Conjecture 1. The expectation is that Conjecture 1 should be easily deducable from Conjecture 5 by applying a dévissage argument using the Δ-filtration structure on $\mathcal{O}_{\ddot{\mathbf{H}}}$. Following this, we expect that the restriction of qKZ to the categorical quotient $\mathcal{O}_{\ddot{\mathbf{H}}}/\mathcal{O}_{\ddot{\mathbf{H}}}^{\mathrm{tor}}$ should give a fully faithful and essentially surjective functor into the module category $\mathbf{Coh}^{\mathrm{flat}}(\mathcal{H}^{\mathrm{ell}})$ – this is outlined in Conjecture 2. By construction, we already know that qKZ restricted to $\mathcal{O}_{\ddot{\mathbf{H}}}/\mathcal{O}_{\ddot{\mathbf{H}}}^{\mathrm{tor}}$ is faithful.

The full conjecture we have relates this idea of monodromy to the setting of the dynamical affine Hecke algebra, which is an object in $\mathbf{Coh}(\mathfrak{A} \times \mathfrak{A}^{\vee})$. One constructs this object by considering the elliptic affine Hecke algebra equipped with a *dynamical parameter* arising from $\mathfrak{A}^{\vee}$ (see Appendix D). In the case of the standard module, one observes these dynamical parameters z arising from the character of the non-trivial one-dimensional representation of $\mathbb{C}[\mathbf{Y}]$.

These dynamical parameters vanish when we compute the monodromy matrix M, which allows us to cleanly apply Lemma 12 in order to obtain an $\mathcal{H}^{\mathrm{ell}}$-module. Generally, one should observe these dynamical parameters appearing in the poles of the monodromy matrix, in which case we should be able to obtain a $\mathcal{H}^{\mathrm{dyn}}$-module structure. Forgetting the dynamical parameter should then give us a $\mathcal{H}^{\mathrm{ell}}$-module with zero-dimensional support – this is outlined in Conjecture 3. From this, we expect to obtain a commutative diagram:

$$\begin{tikzcd}
\mathbb{C}[\mathbf{Y}]\text{-}\mathbf{Mod} \arrow[r, "\text{Induction}"] \arrow[d] & \mathcal{O}_{\ddot{\mathbf{H}}} \arrow[d, "q\mathrm{KZ}"] \\
\mathbf{Coh}(\mathcal{H}^{\mathrm{dyn}}) \arrow[dr, bend right, "\text{forget}"'] & \mathbf{Coh}^{\mathrm{flat}}(\mathcal{H}^{\mathrm{ell}}) \arrow[d, bend right, "\widehat{\mathcal{FM}}"'] \\
 & \mathbf{Coh}^{\mathrm{fin}}(\mathcal{H}^{\mathrm{ell}}) \arrow[u, bend right, "\mathcal{FM}"']
\end{tikzcd}$$

where ${}^{L}\mathcal{H}^{\mathrm{ell}}$ is the Langlands dual of $\mathcal{H}^{\mathrm{ell}}$, and $\mathcal{FM}$ is the Fourier-Mukai transform constructed in [LZZ23]. There is also an inverse Fourier-Mukai functor $\widehat{\mathcal{FM}}$ mapping in the other direction.

Indeed, the expectation is that the ideas in this thesis should be generalisable beyond the rank one DAHA. The primary limitation that we experienced was the fact that the q-Riemann-Hilbert correspondence is only known for q-difference systems of one variable; the multivariate case is still conjectural. Otherwise, the techniques employed in this thesis would generalise easily to the general rank case.

Vasserot in [Vas05] classified the irreducible, integrable representations of the DAHA using perverse sheaves. The irreducible modules of $\mathcal{H}^{\mathrm{ell}}$ have also been classified using the equivariant elliptic cohomology of the Springer resolution in [ZZ15]. Using these classifications, we expect to be able to relate irreducible objects in $\mathcal{H}^{\mathrm{ell}}$ to non-torsion irreducible objects in $\mathcal{O}_{\ddot{\mathbf{H}}}$ using the qKZ functor. Conjecture 2 posits that the restriction of $q\mathrm{KZ}$ to the Serre quotient should be fully faithful and essentially surjective, in which case the composition $\widehat{\mathcal{FM}} \circ q\mathrm{KZ}\,|_{\mathcal{O}_{\ddot{\mathbf{H}}}/\mathcal{O}_{\ddot{\mathbf{H}}}^{\mathrm{tor}}} : \mathcal{O}_{\ddot{\mathbf{H}}}/\mathcal{O}_{\ddot{\mathbf{H}}}^{\mathrm{tor}} \to \mathbf{Coh}^{\mathrm{fin}}({}^L\mathcal{H}^{\mathrm{ell}})$ should give a one-to-one correspondence between non-torsion irreducible objects in $\mathcal{O}_{\ddot{\mathbf{H}}}$ and irreducible ${}^L\mathcal{H}^{\mathrm{ell}}$-modules. [ZZ15] showed that the irreducible representations of $\mathcal{H}^{\mathrm{ell}}$ may be parametrised by certain nilpotent Higgs bundles. As a corollary, one would obtain a similar parametrisation of irreducible non-torsion DAHA-modules by these nilpotent Higgs bundles.

Our work also ties in to a larger body of work around q-difference equations and the elliptic cohomology of Nakajima varieties. Aganagic-Okounkov in [AO16] constructed q-difference equations arising from the elliptic cohomology of a Nakajima variety X. In the case for which $X = T^*\mathbb{P}^1$, one obtains the qKZ equations. The solutions of the qKZ equations – the q-hypergeometric equations – then admit an interpretation via vertex operators (see [AO16, §6]). The monodromy of these equations are constructed as a morphism on elliptic cohomology:

$$\varphi : \mathcal{E}\ell\ell(T^*\mathbb{P}^1) \longrightarrow \mathcal{E}\ell\ell((T^*\mathbb{P}^1)_{\mathrm{flop}}).$$

Bibliography

[AM10] M.H. Annaby and Z.S. Mansour. *q-Fractional Calculus and Equations*, volume Lecture Notes in Mathematics. Springer Verlag, 2010.

[AO16] M. Aganagic and A. Okounkov. Elliptic Stable Envelopes. *Journal of the American Mathematical Society*, 2016.

[BEG00] V. Baranovsky, S. Evens, and V. Ginzburg. Representations of quantum tori and double-affine Hecke algebras. 2000.

[BH06] C. Bushnell and G. Henniart. *The Local Langlands Conjecture for* GL(2), volume Grundlehren der mathematischen Wissenschaften, volume 335. Springer Verlag, 2006.

[BK11] A. Braverman and D. Kazhdan. The spherical Hecke algebra for affine Kac-Moody groups I. *Ann. Math.*, 174:pg. 1603–1642, 2011.

[Bor76] A. Borel. Admissible Representations of a Semi-Simple Group over a Local Field with Vectors Fixed under an Iwahori subgroup. *Invent. Math.*, 35:pg. 233–256, 1976.

[Bum10] D. Bump. Hecke Algebras. 2010.

[Cai22] J. Cai. Iwahori-Hecke Algebras and p-adic Groups. *Honour's Thesis, University of Queensland*, 2022.

[Cas80] W. Casselman. The Unramified Principal Series of p-adic Groups I: The Spherical Function. *Comp. Math.*, 40(3), 1980.

[CG09] N. Chriss and V. Ginzburg. *Representation Theory and Complex Geometry.* Springer Science and Business Media, 2009.

[Che90] Ivan Cherednik. Affine extensions of Knizhnik-Zamolodchikov equations and Lusztig's isomorphisms. *ICM-90 Satell. Conf. Proc.*, pages pg. 63–77, 1990.

[Che92] I. Cherednik. Quantum Knizhnik-Zamolodchikov Equations and Affine Root Systems. *Commun. Math. Phys*, (150):pg. 109–136, 1992.

[Che95] I. Cherednik. Double affine Hecke algebras and Macdonald's conjectures. *Ann. of Math.*, 141(1):pg. 191–216, 1995.

[Che03] I. Cherednik. Double affine Hecke algebras and difference Fourier transforms. *Invent. Math.*, 152:pg. 213–303, 2003.

[CPS88] E. Cline, B. Parshall, and L. Scott. Finite dimensional algebras and highest weight categories. *J. reine angew. Math.*, 391:pg. 85–99, 1988.

[EFK98] P. Etingof, I. Frenkel, and A. Kirillov. *Lectures on Representation Theory and Knizhnik-Zalomodchikov Equations.* Number 58. Mathematical Surveys and Monographs, 1998.

[EM97] S. Evens and I. Mirkovic. Fourier Transform and the Iwahori-Matsumoto Involution. *Duke Mathematical Journal*, 86(3):pg. 435–464, 1997.

[Eti17] P. Etingof. Introduction to Algebraic D-modules. 2017.

[FR92] I.B. Frenkel and N.Y. Reshetikhin. Quantum Affine Algebras and Holonomic Difference Equations. *Commun. Math. Phys.*, (146):pg. 1–60, 1992.

[GGOR03] V. Ginzburg, N. Guay, E. Opdam, and R. Rouquier. On the category $\mathcal{O}$ for rational Cherednik algebras. *Invent. Math.*, 153(3):pg. 617–651, 2003.

[GH24] J.R. Getz and H. Hahn. *An Introduction to Automorphic Representations: with a view toward trace formulae*, volume Graduate Texts in Mathematics, volume 300. Springer Verlag, 2024.

[GKV95] V. Ginzburg, Kapranov, and Vasserot. Residue construction of Hecke algebras. *Adv. Math.*, 128:pg. 1–19, 1995.

[GR09] G. Gasper and M. Rahman. *Basic Hypergeometric Series.* Cambridge University Press, 2009.

[Har20] Y. Haraoka. *Linear Differential Equations in the Complex Domain*, volume 2271 of *Lecture Notes in Mathematics.* Springer Verlag, 2020.

[HKP10] T.J. Haines, R.E. Kottwitz, and A. Prasad. Iwahori-Hecke Algebras. *Journal of the Ramanujuan Mathematical Society*, (25):pg. 113–145, 2010.

[Hot98] R. Hotta. Equivariant D-modules. 1998.

[IM65] N. Iwahori and H. Matsumoto. On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups. *Pub. Math. de l'I.H.É.S*, 25:pg. 5–48, 1965.

[Jon87] V. Jones. Hecke algebra representations of braid groups and link polynomials. *Ann. Math.*, 126:pg. 335–388, 1987.

[JV19] D. Jordan and M. Vazirani. The Rectangular Representation of the Double Affine Hecke Algebra via Elliptic Schur-Weyl Duality. *Int. Math. and Research Note.*, 2021(8):pg. 5698–6019, 2019.

[Kir97] Alexander Kirillov. Lectures on affine Hecke algebras and Macdonald's conjectures. *Bull. Amer. Math. Soc. (N.S.)*, (3):pg. 251–292, 1997.

[KL87] K. Kazhdan and G. Lusztig. Proof of the Deligne-Langlands Conjecture for Hecke algebras. *Invent. Math.*, 87(1):pg. 153 — 215, 1987.

[Koe18] Erik Koelnik. q-special functions, basic hypergeometric series and operators. *Lecture notes for course at the OPSFA Summer school in Sousse, Tunisia*, 2018.

[KZ84] V.G. Knizhnik and A.B. Zamolodchikov. Current algebra and Wess-Zumino model in two dimensions. *Nucl. Phys. B*, (247):pg. 83–103, 1984.

[Lus83] G. Lusztig. Singularities, character formulas, and a q-analog of weight multiplicities. *Analyse et topologie sur les espaces singuliers, II, III*, 101:pg. 208–229, 1983.

[LZZ23] C. Lenart, G. Zhao, and C. Zhong. Elliptic classes via the periodic Hecke module and its Langlands dual. *Pre-print*, 2023.

[OS22] A. Okounkov and A. Smirnov. Quantum difference equation for Nakajima varieties. *Invent. Math.*, 229:pg. 1203–1299, 2022.

[Pol09] A. Polishchuk. *Abelian Varieties, Theta Functions and the Fourier Transform.* Cambridge University Press, 2009.

[RW21] A. Romanov and G. Williamson. Langlands correspondence and Bezrukavnikov's correspondence. 2021.

[RW22] Yong Bin Ruan and Yao Xiong Wen. Quantum K-theory and q-difference equations. *Acta Mathematica Sinica, English Series volume*, 38:pg. 1677–1704, 2022.

[Sau03] Jacques Sauloy. Galois theory of Fuchsian q-difference equations. *Ann. Sci. Ecole. Norm. Sup.*, 4(36), 2003.

[Sau16] J. Sauloy. Analytic study of q-difference equations. *Galois Theories of Linear Difference Equations: An Introduction*, Math. Surveys Monogr.(211), 2016.

[Sie80] C.L. Siegel. Advanced Analytic Number Theory. *Tata Inst. Fundam. Res. Stud. Math.*, (9), 1980.

[Sto10] J.V. Stokman. Quantum Affine Knizhnik-Zamolodchikov Equations and Quantum Spherical Functions, I. *Int. Math. Res.*, (5), 2010.

[Vas05] E. Vasserot. Induced and simple modules of double affine Hecke algebras. *Duke Math. J.*, 125(2):pg. 251–323, 2005.

[VV04] M. Varagnolo and E. Vasserot. From double affine Hecke algebras to quantized affine Schur algebras. *Int. Math. Research Notices*, 2004(26):pg. 1299–1333, 2004.

[ZZ15] G. Zhao and C. Zhong. Representations of the Elliptic Affine Hecke Algebras. *Adv. Math.*, 2015.

[ZZ24] G. Zhao and C. Zhong. A Langlands Duality of Elliptic Hecke Algebras. *In preparation.*, 2024.

Appendix A

Monodromy of Differential Equations

Let $\mathbb{S} := \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. Consider a linear differential equation of the form

$$\frac{dY}{dx} = A(x)Y, \tag{A.1}$$

of holomorphic functions on some open domain D of $\mathbb{S}$, and a fundamental system of solutions $Y(x)$. Then, one may define a loop γ in D with basepoint x. Analytically continuing the family of solutions along γ gives rise to a new family of solutions — call it $F(x)$ — satisfying $Y(x) = M_\gamma(x)F(x)$, where $M_\gamma \in \mathrm{GL}_n(\mathbb{C})$. Each homotopy class $[\gamma]$ uniquely determines such a M_γ, see [Har20, Theorem 3.3]. The map $\gamma \mapsto M_\gamma$ defines a representation of the fundamental group $\pi_1(D, x)$, called the *monodromy representation* associated to (A.1) [Har20, Definition 5.1].

Let X be a complex manifold, or a variety (that is, reduced, irreducible scheme of finite type) over $\mathbb{C}$. Let $\mathcal{O}_X$ be its structure sheaf, and $\mathcal{F}$ be a vector bundle (that is, a locally free, coherent $\mathcal{O}_X$-module) on X. Let $\mathbf{Vect}(X)^\nabla$ denote the category of vector bundles on X equipped with a flat connection ∇ (see Stacks Project 3.60.15).

Given any local system $\mathcal{L}$ on $\mathbb{C}$, the tensor product $\mathcal{L} \otimes_\mathbb{C} \mathcal{O}_X$ defines a vector bundle on X. The $\mathbb{C}$-linear differential $d : \mathcal{O}_X \to \Omega^1_{X/\mathbb{C}}$ then gives a flat connection $d \otimes 1$ on $\mathcal{L} \otimes_\mathbb{C} \mathcal{O}_X$. Then, the *Riemann-Hilbert correspondence* posits the following equivalence of categories:

$$\mathbf{LocSys}(X) \longrightarrow \mathbf{Vect}(X)^\nabla, \quad \mathcal{L} \longmapsto (\mathcal{L} \otimes_\mathbb{C} \mathcal{O}_X, 1 \otimes d).$$

There is a quasi-inverse given by by the *solution functor* $(\mathcal{F}, \nabla) \mapsto \ker \nabla$, where $\ker \nabla$ is the sheaf of horizontal sections on $\mathcal{F}$. Further, there is a *monodromy functor*

$$\mathbf{LocSys}(X) \longrightarrow \pi_1(X, x), \quad \mathcal{L} \longmapsto \mathcal{L}_x,$$

obtained by analytically continuing the stalks of $\mathcal{L}$ along a loop in X.

Appendix B

Monodromy of Degenerate Double Affine Hecke Algebra Representations

This idea of monodromy turns out to be an important tool in studying the representations of various Hecke algebras. Affine Hecke algebras [Kir97, Definition 3] and degenerate affine Hecke algebras ([Kir97, Definition 1], [Che90, Definition 4]) both give rise to systems of differential equations, whose monodromy can be studied.

The case of the degenerate affine Hecke algebra case was studied by [Che90], who studied the monodromy of the affine Knizhnik-Zamolodchikov connection [Che90, (21)], and showed that the aforementioned monodromy functor factors through representations of the affine Hecke algebra [Che90, Theorem 10].

The double affine Hecke algebra (DAHA) $\ddot{\mathbf{H}}$ [Kir97, Definition 4], was first introduced by [Che95]. From this, there are two degenerations that one can obtain. One level of degeneration gives us the degenerate (or trigonometric) DAHA [VV04, (2.1.1)], and another level of degeneration gives us the rational DAHA [GGOR03, Section 3.1]. Denote these resulting algebras by $\ddot{\mathbf{H}}'$ and $\ddot{\mathbf{H}}''$, respectively.

The $\ddot{\mathbf{H}}'$ case is treated in [VV04] Given some affine root datum [Kir97, Section 3] $(X, X^\vee, \Phi, \Phi^\vee)$, define the torus $T := \mathbb{G}_{\mathrm{m}} \otimes_{\mathbb{Z}} X^\vee$. Following [GKV95, Section 1], for each $\lambda \in X$, there is a morphism of group schemes $T \to \mathbb{G}_{\mathrm{m}}$ defined by $q \otimes x \mapsto q^{\langle \lambda, x \rangle}$. For some $\lambda \in X$, we denote by T_λ the kernel divisor of this map. Then, define

$$T^{\mathrm{reg}} := T \setminus \left(\bigcup_{\alpha \in \Phi} T_\alpha \right).$$

We may localise the degenerate DAHA by taking $\ddot{\mathbf{H}}'_{\mathrm{reg}} := \ddot{\mathbf{H}}' \otimes_{\mathbb{C}[X]} \mathbb{C}[T^{\mathrm{reg}}]$. Modules over $\ddot{\mathbf{H}}$ are localised in the same way: given a $\ddot{\mathbf{H}}'$- module M, we define $M_{\mathrm{reg}} := M \otimes_{\mathbb{C}[X]} \mathbb{C}[T^{\mathrm{reg}}]$. Let $\mathcal{D}_{T^{\mathrm{reg}}}$ denote the sheaf of differential operators on T^{reg},whose sections are the usual $D_{T^{\mathrm{reg}}}$-modules (see [Hot98], [Eti17]) . There is a natural W-action on $D_{T^{\mathrm{reg}}}$, which allows us to form the semi-direct product $D_{T^{\mathrm{reg}}} \rtimes \mathbb{C}[W]$. Then, there is a unique ring isomoprhism [VV04, Lemma 3.1(ii)]:

$$D_{T^{\mathrm{reg}}} \rtimes \mathbb{C}[W] \xrightarrow{\simeq} \ddot{\mathbf{H}}'_{\mathrm{reg}}, \tag{B.1}$$

which maps the differential operator to the *trigonometric Knizhnik-Zamolodchikov connection* (∇_j in [VV04, Lemma 3.1(ii)]). Composing the the localisation functor $\mathcal{O}' \to \ddot{\mathbf{H}}'_{\mathrm{reg}}$-Mod, the isomorphism (B.1), and the sheafification functor $D_{T^{\mathrm{reg}}}$-**Mod** $\to \mathcal{D}_{T^{\mathrm{reg}}}$-**Mod**, we obtain a functor $\mathcal{L} : \mathcal{O}' \to \mathcal{D}_{T^{\mathrm{reg}}} \rtimes \mathbb{C}[W]$-**Mod**. We compose the solution functor from the Riemann-Hilbert correspondence with the monodromy functor from before to obtain:

$$\mathcal{D}_{T^{\mathrm{reg}}} \rtimes \mathbb{C}[W]\textbf{-Mod} \longrightarrow \mathbf{LocSys}(T^{\mathrm{reg}}/W) \longrightarrow \pi_1(T^{\mathrm{reg}}/W)\textbf{-Mod} \cong \mathrm{Br}_{\mathrm{aff}}\textbf{-Mod},$$

where $\mathrm{Br}_{\mathrm{aff}}$ is the affine Braid group. Composing $\mathcal{L}$ together with the above functor then gives us the *monodromy functor* of a $\ddot{\mathbf{H}}'$-module:

$$\mathcal{M}on : \mathcal{O}' \longrightarrow \mathrm{Br}_{\mathrm{aff}}\textbf{-Mod}.$$

The $\ddot{\mathbf{H}}''$ case is treated by [GGOR03]. Define a new torus $T_{\mathrm{a}} := \mathbb{G}_{\mathrm{a}} \otimes_{\mathbb{Z}} X^{\vee}$. The scheme $T_{\mathrm{a}}^{\mathrm{reg}}$ is defined using kernel divisors T_α in the same way as before. As before, there is a unique ring isomorphism [GGOR03, Theorem 5.6]

$$D_{T_{\mathrm{a}}^{\mathrm{reg}}} \rtimes W \xrightarrow{\simeq} \ddot{\mathbf{H}}_{\mathrm{reg}}.$$

mapping differential operators on $T_{\mathrm{a}}^{\mathrm{reg}}$ to a W-equivariant connection on a $\ddot{\mathbf{H}}''$-module M [GGOR03, Proposition 5.7]. The same argument from the prior section is used to produce a monodromy functor

$$\mathcal{O}'' \longrightarrow \mathrm{Br}\textbf{-Mod},$$

where Br is now the usual braid group. In [GGOR03], this functor is called the *Knizhnik-Zamolodchikov* functor, and is denoted by KZ.

Appendix C

Convolution Structure on $\mathbf{Coh}(\mathfrak{A} \times_{\mathfrak{A}/W} \mathfrak{A})$

Let $\pi : \mathfrak{A} \to \mathfrak{A}/W$, and $\pi_{\text{dyn}} : \mathfrak{A}^\vee \to \mathfrak{A}^\vee/W^{\text{dyn}}$ be the projection maps. We wish to define a new category:

$$\mathbf{Coh}(\mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A}^\vee),$$

in which the dynamic elliptic Hecke algebra lives. This will have the benefit of the pushforward map π_* having some nicer properties. In particular, this category will be equipped with a monoidal structure that is similar to the convolution algebra construction originally given in the paper of Kazhdan-Lusztig's proof of the Deligne-Langlands conjecture [KL87] (see also [CG09]).

Let us also consider projection maps:

$$\pi \times \pi^\vee : \mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A}^\vee \longrightarrow \mathfrak{A}/W \times \mathfrak{A}^\vee,$$

and

$$\pi \times \pi_{\text{dyn}} : \mathfrak{A} \underset{\mathfrak{A}^\vee/W^{\text{dyn}}}{\times} \mathfrak{A}^\vee \longrightarrow \mathfrak{A} \times \mathfrak{A}^\vee/W^{\text{dyn}}.$$

Now, we equip the category of coherent sheaves

$$\mathbf{Coh}(\mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A}),$$

with a monoidal structure in the following way. Let

$$p_{ij} : \mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A} \longrightarrow \mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A},$$

denote the projection onto the (i, j)-th factor. We also define a map:

$$\Delta_2 : \mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A} \longrightarrow \mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A}, \quad (z_1, z_2, z_3) \longmapsto (z_1, z_2, z_2, z_3),$$

where the subscript in Δ_2 reminds us that the coordinate z_2 is repeated. Then, the convolution product in $\mathbf{Coh}(\mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A})$ is defined to be:

$$\mathcal{F} \star \mathcal{G} := (p_{13})_* \, \Delta_2^*(\mathcal{F} \boxtimes \mathcal{G}).$$

Lemma 19. *The functor*

$$\phi : \mathbf{Coh}_W(\mathfrak{A}) \longrightarrow \mathbf{Coh}(\mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A}), \quad \mathcal{F}_w \longmapsto \widetilde{w}_* \mathcal{F}_w,$$

is a monoidal functor. Here, $\widetilde{w}_*$ is the pushforward along the map

$$\widetilde{w} : \mathfrak{A} \longrightarrow \mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A}, \quad z \longmapsto (z, w^{-1}(z)).$$

Proof. Let $\mathcal{F}_w, \mathcal{G}_v \in \mathbf{Coh}_W(\mathfrak{A})$. We wish to show that

$$\phi(\mathcal{F}_w \star \mathcal{G}_v) = \phi(\mathcal{F}_w) \star \phi(\mathcal{G}_v).$$

On the left-hand side, we have:

$$\phi(\mathcal{F}_w \star \mathcal{G}_v) = \phi(\mathcal{F}_w \otimes (w^{-1})^* \mathcal{G}_v) = (\widetilde{wv})_* \left(\mathcal{F}_w \otimes (w^{-1})^* \mathcal{G}_v\right) = (\widetilde{wv})_* \widetilde{w}^* (\mathcal{F}_w \boxtimes \mathcal{G}_v),$$

and on the right-hand side,

$$\phi(\mathcal{F}_w) \star \phi(\mathcal{G}_v) = \widetilde{w}_* \mathcal{F}_w \star \widetilde{v}_* \mathcal{G}_v = (p_{13})_* \, \Delta_2^*(\widetilde{w}_* \mathcal{F}_w \boxtimes \widetilde{v}_* \mathcal{G}_v).$$

We now show that these two relations coincide.

$$\begin{tikzcd}
 & \mathfrak{A} \arrow[dl, "\widetilde{wv}"'] \arrow[d, "p"] \arrow[r, "\widetilde{w}"] & \mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A} \arrow[d, "\widetilde{w} \times \widetilde{v}"] \\
\mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A} & \mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A} \arrow[l, "p_{13}"] \arrow[r, "\Delta_2"] & \mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A}
\end{tikzcd}$$

Here, we defined

$$p(z) := (z, w^{-1}(z), (wv)^{-1}(z)),$$

so $(\widetilde{wv})_* = (p_{13})_* p_*$, and $p_* \widetilde{v}^* = \Delta_2^*(\widetilde{w} \times \widetilde{v})$. Thus,

$$\begin{aligned}
(\widetilde{wv})_* \widetilde{w}^* (\mathcal{F}_w \boxtimes \mathcal{G}_v) &= (p_{13})_* p_* \widetilde{w}^* (\mathcal{F}_w \boxtimes \mathcal{G}_v) \\
&= (p_{13})_* \Delta_2^* (\widetilde{w} \times \widetilde{v})_* (\mathcal{F}_w \boxtimes \mathcal{G}_v) \\
&= (p_{13})_* \Delta_2^* (\widetilde{w}_* \mathcal{F}_w \boxtimes \widetilde{v}_* \mathcal{G}_v),
\end{aligned}$$

as claimed. □

The proof of the following is similar, and we omit the proofs:

Lemma 20. *The following functors are monoidal, where the monoidal structures on the domains and target are similar to the ones seen in Lemma 19. By abuse of notation, we call all of these functors ϕ.*

(i)

$$\phi : \mathbf{Coh}_{W \times W^{\mathrm{dyn}}}(\mathfrak{A} \times \mathfrak{A}^\vee) \longrightarrow \mathbf{Coh}_{W^{\mathrm{dyn}}}(\mathfrak{A}_{\mathfrak{A}/W}\mathfrak{A} \times \mathfrak{A}^\vee), \quad \mathcal{F}_{w,v^{\mathrm{dyn}}} \longmapsto \widetilde{w}_* \mathcal{F}_{w,v^{\mathrm{dyn}}},$$

(ii)

$$\phi : \mathbf{Coh}_{W^{\mathrm{dyn}}}(\mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A} \times \mathfrak{A}^\vee) \longrightarrow \mathbf{Coh}(\mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A} \times \mathfrak{A}^\vee \underset{\mathfrak{A}^\vee/W^{\mathrm{dyn}}}{\times} \mathfrak{A}^\vee), \quad \mathcal{F}_v \longmapsto (\widetilde{v^{\mathrm{dyn}}})_* \mathcal{F}_v.$$

Lemma 21. *The pushforward functors are lax monoidal, with the target equipped with the usual tensor structure of coherent sheaves:*

(i) $\mathbf{Coh}(\mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A}) \longrightarrow \mathbf{Coh}(\mathfrak{A}/W)$,

(ii) $\mathbf{Coh}(\mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A} \times \mathfrak{A}^\vee \underset{\mathfrak{A}^\vee/W^{\mathrm{dyn}}}{\times} \mathfrak{A}^\vee) \longrightarrow \mathbf{Coh}(\mathfrak{A}/W \times \mathfrak{A}^\vee \times_{\mathfrak{A}/W^{\mathrm{dyn}}} \mathfrak{A}^\vee)$,

(iii) $\mathbf{Coh}_{W^{\mathrm{dyn}}}(\mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A} \times \mathfrak{A}^\vee) \longrightarrow \mathbf{Coh}_{W^{\mathrm{dyn}}}(\mathfrak{A}/W \times \mathfrak{A}^\vee)$.

C.1. Localisation to $\mathfrak{A}^{\mathrm{reg}}$

Construct the root hyperplanes T_α, divisors $T_{\alpha,\hbar}$, and $\mathfrak{A}^{\mathrm{reg}}$ as before. Let $j : \mathfrak{A}^{\mathrm{reg}} \to \mathfrak{A}$ be the embedding. Similarly, we also have co-root hyperplanes given by $T_{\alpha^\vee}$, and divisors $T_{\alpha^\vee,\hbar}$ of $\mathfrak{A}^\vee$. Define $(\mathfrak{A}^\vee)^{\mathrm{reg}}$ similarly, and let $j^\vee : (\mathfrak{A}^\vee)^{\mathrm{reg}} \to \mathfrak{A}^\vee$ denote the embedding. The open subsets $\mathfrak{A}^{\mathrm{reg}}$ and $(\mathfrak{A}^\vee)^{\mathrm{reg}}$ are invariant under the actions of W and W^{dyn}, respectively. The categories $\mathbf{Coh}_W(\mathfrak{A}^{\mathrm{reg}})$ and $\mathbf{Coh}(\mathfrak{A}^{\mathrm{reg}} \underset{\mathfrak{A}/W}{\times} \mathfrak{A}^{\mathrm{reg}})$ are equipped with induced monoidal structures, and are furthermore categorically equivalent by the freeness of W and W^{dyn}:

$$\mathbf{Coh}_W(\mathfrak{A}^{\mathrm{reg}}) \cong \mathbf{Coh}(\mathfrak{A}^{\mathrm{reg}} \underset{\mathfrak{A}/W}{\times} \mathfrak{A}^{\mathrm{reg}}).$$

The induced monoidal structures make the localisation functors

$$\mathbf{Coh}_W(\mathfrak{A}) \longrightarrow \mathbf{Coh}_W(\mathfrak{A}^{\mathrm{reg}}),$$

and

$$\mathbf{Coh}(\mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A}) \longrightarrow \mathbf{Coh}(\mathfrak{A}^{\mathrm{reg}} \underset{\mathfrak{A}/W}{\times} \mathfrak{A}^{\mathrm{reg}}),$$

monoidal. Together, we have a commutative diagram:

$$\begin{tikzcd}
\mathbf{Coh}_W(\mathfrak{A}^{\mathrm{reg}}) \arrow[r, "\cong"] & \mathbf{Coh}(\mathfrak{A}^{\mathrm{reg}} \underset{\mathfrak{A}/W}{\times} \mathfrak{A}^{\mathrm{reg}}) \\
\mathbf{Coh}_W(\mathfrak{A}) \arrow[u, "\text{localise}"] \arrow[r, "\phi"] & \mathbf{Coh}(\mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A}) \arrow[u, "\text{localise}"']
\end{tikzcd}$$

We also have the following commutative diagram:

$$\begin{tikzcd}
\mathbf{Coh}_W(\mathfrak{A}^{\mathrm{reg}} \times (\mathfrak{A}^\vee)^{\mathrm{reg}}) \arrow[r, "\cong"] & \mathbf{Coh}(\mathfrak{A}^{\mathrm{reg}} \underset{\mathfrak{A}/W}{\times} \mathfrak{A}^{\mathrm{reg}} \times (\mathfrak{A}^\vee)^{\mathrm{reg}} \underset{\mathfrak{A}^\vee/W^{\mathrm{dyn}}}{\times} (\mathfrak{A}^\vee)^{\mathrm{reg}}) \\
\mathbf{Coh}_{W \times W^{\mathrm{dyn}}}(\mathfrak{A} \times \mathfrak{A}^\vee) \arrow[u, "\text{localise}"] \arrow[r, "\phi"] & \mathbf{Coh}(\mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A} \times \mathfrak{A}^\vee \underset{\mathfrak{A}^\vee/W^{\mathrm{dyn}}}{\times} \mathfrak{A}^\vee) \arrow[u, "\text{localise}"']
\end{tikzcd}$$

From this, one may recover the elliptic twisted group algebra $\mathcal{O}[W]$ of [GKV95] by

$$\pi_* \phi(\mathcal{O}_{\mathfrak{A}^{\mathrm{reg}} \underset{\mathfrak{A}/W}{\times} \mathfrak{A}^{\mathrm{reg}}}) \cong \pi_* \mathcal{O}_{\mathfrak{A}^{\mathrm{reg}}} \rtimes \mathbb{C}[W] = \mathcal{O}[W],$$

from which we construct the ellAHA $\mathcal{H}^{\mathrm{ell}}$ as as subsheaf of $j_*\mathcal{O}[W]$ satisfying some residue conditions.

C.2. Convolution Construction of $\mathbf{Coh}(\mathcal{H}^{\mathrm{dyn}})$ and $\mathbf{Coh}(\mathcal{H}^{\mathrm{ell}})$

The monoidal category $\mathbf{Coh}(\mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A})$ acts on $\mathbf{Coh}(\mathfrak{A})$ via convolution, similar to the aforementioned convolution category construction. We outline this construction here. Let $p_i : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ denote the projection map onto the i-th factor, and $\Delta_2 : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A} \times \mathfrak{A} \times \mathfrak{A}$ by $(u_1, u_2) \mapsto (u_1, u_2, u_2)$. Then, given a coherent sheaf $\mathcal{F}$ on $\mathfrak{A}$, and $\mathcal{G}$ on $\mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A}$, define

$$\mathcal{G} \star \mathcal{F} := (p_1)_* \Delta_2^* \left(\mathcal{G} \boxtimes \mathcal{F}\right).$$

In this case, we say that $\mathbf{Coh}(\mathfrak{A})$ has the structure of a *(categorical)* $\mathbf{Coh}(\mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A})$*-module*. Recall that the ellAHA $\mathcal{H}^{\mathrm{ell}}$ is an object in $\mathbf{Coh}(\mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A})$. We define

$$\mathbf{Coh}(\mathcal{H}^{\mathrm{ell}}),$$

to be the category of (categorical) $\mathcal{H}^{\mathrm{ell}}$-modules in $\mathbf{Coh}(\mathfrak{A})$. We observe that this notion is quivalent to the notion of coherent sheaves of $\mathcal{H}^{\mathrm{ell}}$ on $\mathfrak{A}/W$. Indeed, using the lax monoidal property of the pushforward π_* in Lemma 21, any objects in $\mathbf{Coh}(\mathfrak{A})$ gives rise to an object in $\mathbf{Coh}(\mathfrak{A}/W)$ via the pushforward π_*. Conversely, any $\mathcal{H}^{\mathrm{ell}}$-module has the action of the subsheaf of algebras $\pi_*\mathcal{O}_{\mathfrak{A}} \hookrightarrow \mathcal{H}^{\mathrm{ell}}$, which equips it with the structure of a coherent sheaf on $\mathfrak{A}$. Indeed, the category $\mathbf{Coh}(\mathcal{H}^{\mathrm{ell}})$ is the appropriate representation category that we wish to utilise for the ellAHA.

Let

$$\mathbf{Coh}^{\mathrm{fin}}(\mathcal{H}^{\mathrm{ell}}),$$

be the full subcategory of $\mathbf{Coh}(\mathcal{H}^{\mathrm{ell}})$ whose underlying coherent sheaf has zero-dimensional support in $\mathfrak{A}$. Let

$$\mathbf{Coh}^{\mathrm{flat}}(\mathcal{H}^{\mathrm{ell}}),$$

be the full subcategory of $\mathbf{Coh}(\mathcal{H}^{\mathrm{ell}})$ whose underlying coherent sheaf is a homogeneous vector bundle (i.e. locally free, coherent sheaf) on $\mathfrak{A}$.

Similarly, the monoidal category $\mathbf{Coh}(\mathfrak{A} \times \mathfrak{A}^\vee)$ obtains the structure of a $\mathbf{Coh}(\mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A} \times \mathfrak{A}^\vee \underset{\mathfrak{A}^\vee/W^{\mathrm{dyn}}}{\times} \mathfrak{A}^\vee)$-module via convolution. Let us define a projection map

$$p_{ij} : (\mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A}) \times (\mathfrak{A}^\vee \underset{\mathfrak{A}^\vee/W^{\mathrm{dyn}}}{\times} \mathfrak{A}^\vee) \longrightarrow \mathfrak{A} \times \mathfrak{A}^\vee, \quad (u_1, u_2, z_1, z_2) \longmapsto (u_i, z_j),$$

and

$$\begin{aligned} \Delta_{ij} : (\mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A}) \times (\mathfrak{A}^\vee \underset{\mathfrak{A}^\vee/W^{\mathrm{dyn}}}{\times} \mathfrak{A}^\vee) &\longrightarrow (\mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A}) \times (\mathfrak{A}^\vee \underset{\mathfrak{A}^\vee/W^{\mathrm{dyn}}}{\times} \mathfrak{A}^\vee) \times (\mathfrak{A} \times \mathfrak{A}^\vee) \\ (u_1, u_2, z_1, z_2) &\longmapsto (u_1, u_2, z_1, z_2, u_i, z_j). \end{aligned}$$

Then, the convolution action is given by:

$$\mathcal{G} \star \mathcal{F} := (p_{11})_* \Delta_{22}^* (\mathcal{G} \boxtimes \mathcal{F}).$$

Since $\mathcal{H}^{\mathrm{dyn}}$ is an object in $\mathbf{Coh}(\mathfrak{A} \underset{\mathfrak{A}/W}{\times} \mathfrak{A} \times \mathfrak{A}^\vee \underset{\mathfrak{A}^\vee/W^{\mathrm{dyn}}}{\times} \mathfrak{A}^\vee)$, let us define

$$\mathbf{Coh}(\mathcal{H}^{\mathrm{dyn}}),$$

to be the category of $\mathcal{H}^{\mathrm{dyn}}$-modules in $\mathbf{Coh}(\mathfrak{A} \times \mathfrak{A}^\vee)$.

Appendix D

The Dynamical Elliptic Affine Hecke Algebra

In this section, we follow the construction outlined in [ZZ15, §4.3], [ZZ24, §1, §3], and [LZZ23]. The idea is that instead of defining a Hecke algebra on an elliptic curve E, we wish to take the product $E \times E^\vee$ instead. Here, $E^\vee$ denotes the dual abelian variety, and is define to be the Picard group of degree zero line bundles on E — denoted $E^\vee := \mathrm{Pic}^0(E)$ [Pol09, §9.3].

A key benefit of the dynamical ellAHA is that one can define elliptic analogues of Demazure-Lusztig operators that satisfy the usual braid relations (see [ZZ15, (16) and Proposition 4.11]). This is not a feature that is available in the usual ellAHA.

In this case, E is self-dual — that is, $E \cong \mathrm{Pic}^0(E)$ [Pol09, §9.4]. Then, the dual abelian variety of $\mathfrak{A}$ is given by

$$\mathfrak{A}^\vee := \mathrm{Pic}^0(\mathfrak{A}) \cong E \otimes \mathbb{X}^*(T),$$

and similarly we have that $\mathfrak{A} \cong \mathfrak{A}^\vee$ by the self-duality of E. We use u to denote elements in $\mathfrak{A}$, and z to denote elements in $\mathfrak{A}^\vee$. The degree zero line bundle determined by the divisor $z \in \mathfrak{A}^\vee$ is denoted by $\mathcal{O}(z)$. On the product $\mathfrak{A} \times \mathfrak{A}^\vee$, there exists an universal line bundle $\mathcal{P}$ called the *Poincaré line bundle* [Pol09, §9.4], which is characterised by the property that for any $z \in \mathfrak{A}^\vee$, the restriction sheaves have the property:

$$\mathcal{P}|_{\mathfrak{A}\times\{z\}} \cong \mathcal{O}(z), \quad \mathcal{P}|_{\{u\}\times\mathfrak{A}^\vee} \cong \mathcal{O}(u).$$

Consider the origin $0 \in E$. Then, the line bundle $\mathcal{O}(0)$ is lifted to the cover $\mathbb{C}^\times$ of E, it becomes trivial. Given any line bundle $\mathcal{L}$ over E, denote the lift by $\mathcal{L} \to \mathbb{C}^\times$, and let $\widetilde{s} : \mathbb{C}^\times \to \mathcal{L}$ be a section of the line bundle. Then, the local trivialisation

$$\begin{tikzcd}
\mathcal{L} \arrow[rr, "\varphi"] \arrow[dr, bend left] & & \mathbb{C}^\times \times \mathbb{C} \arrow[dl] \\
& \mathbb{C}^\times \arrow[ul, bend left, "\widetilde{s}"] &
\end{tikzcd}$$

is uniquely determined by two conditions:

(i) the isomorphism φ must commute with multiplication by $q^{\mathbb{Z}}$,

(ii) the derivative of $\varphi \circ \widetilde{s}$ should be equal to 1 at $1 \in \mathbb{C}^\times$ [Sie80, pg. 38].

With this in mind, the q-Jacobi theta function is given by:

$$\Theta_q(u) = (q^{-1}; q^{-1})_\infty(-q^{-1}u; q^{-1})\infty(-u^{-1}; q^{-1})\infty, \quad u \in \mathbb{C}^\times, \quad |q| < 1,$$

defines a holomorphic function on a double cover of $\mathbb{C}^\times$. Let us denote the coordinates of $\mathfrak{A}$ and $\mathfrak{A}^\vee$ as u and z, respectively. Then, the ratio of Jacobi theta functions:

$$\frac{\Theta_q(z^{-1}u))}{\Theta_q(u)},$$

defines a rational section of $\mathcal{P}$.

The Weyl group actions on $\mathbb{X}_*(T)$ and $\mathbb{X}^*(T)$ induce actions on $\mathfrak{A}$ and $\mathfrak{A}^\vee$, respectively. To distinguish between the Weyl group actions, let us write W for the Weyl group acting on $\mathfrak{A}$, and W^{dyn} for the Weyl group acting on $\mathfrak{A}^\vee$, called the *dynamic Weyl group*. Let us write $w^{-1} : \mathfrak{A} \to \mathfrak{A}$. Then, for any $z \in \mathfrak{A}^\vee$,

$$\mathcal{O}(w^{\text{dyn}} \cdot z) = (w^{-1})^*\mathcal{O}(z),$$

which is to say that the Poincaré line bundle $\mathcal{P}$ is preserved by the diagonal action of $W \times W^{\text{dyn}}$. Moreover, $\mathcal{P}$ is a W-equivariant line bundle.

Recall from our discussion of the ellAHA that there is a map

$$\chi_\alpha : \mathfrak{A} \longrightarrow E, \quad u \otimes \mu^\vee \longmapsto u^{\langle \mu^\vee, \alpha\rangle}.$$

Dually, we may define a map

$$\chi_{\alpha^\vee} : \mathfrak{A}^\vee \longrightarrow E, \quad z \otimes \mu \longmapsto z^{\langle \alpha^\vee, \mu\rangle}.$$

Let $T_\alpha = \ker \chi_\alpha$, and $T_{\alpha^\vee} := \ker \chi_{\alpha^\vee}$. As before, $T_{\alpha,\hbar} = \ker(\chi_\alpha - \hbar)$, and $T_{\alpha^\vee,z} = \ker(\chi_{\alpha^\vee} - z)$.

D.1. The Dynamical Elliptic Twisted Group Algebra

In this section, we construct a dynamic version of the sheaf of algebras $\mathcal{O}[W]$ from the previous section. Let $\mathbf{Coh}_{W\times W^{\text{dyn}}}(\mathfrak{A} \times \mathfrak{A}^\vee)$ to be the category of $W \times W^{\text{dyn}}$-equivariant coherent sheaves on $\mathfrak{A} \times \mathfrak{A}^\vee$. Objects in this category are coherent sheaves with a direct sum decomposition

$$\mathcal{F} = \bigoplus_{\substack{w\in W \\ v^{\text{dyn}}\in W^{\text{dyn}}}} \mathcal{F}_{w,v^{\text{dyn}}},$$

where each component $\mathcal{F}_{w,v^{\text{dyn}}}$ is a coherent sheaf called the degree (w, v^{dyn}) component of $\mathcal{F}$. Morphisms in this category are such that the grading is respected. One may equip this category with a monoidal structure by defining:

$$\mathcal{F}_{w_1,v_1^{\text{dyn}}} \star \mathcal{G}_{w_2,v_2^{\text{dyn}}} := \mathcal{F}_{w_1,v_1^{\text{dyn}}} \otimes (w_1^{-1})^*((v_1^{\text{dyn}})^{-1})^*\mathcal{G}_{w_2,v_2^{\text{dyn}}},$$

which by construction is a coherent sheaf of degree $(w_1w_2, v_1^{\text{dyn}}v_2^{\text{dyn}})$. The object $\mathcal{O}_{\mathfrak{A}\times\mathfrak{A}^\vee}$ concentrated in degree 0 is treated as the unit object. Recall that given two schemes X and Y over a scheme S, and $\mathcal{F}$ an $\mathcal{O}_X$-module, and $\mathcal{G}$ an $\mathcal{O}_Y$-module, the *external tensor product* is defined to be the $\mathcal{O}_{X\times_S Y}$-module:

$$\mathcal{F} \boxtimes_{\mathcal{O}_S} \mathcal{G} := \mathrm{pr}_X^*(\mathcal{F}) \otimes_{\mathcal{O}_{X\times_S Y}} \mathrm{pr}_Y^*(\mathcal{G}),$$

where pr_X and pr_Y denote projections onto the first and second factor of $X \times_S Y$, respectively.

Now, let us define a line bundle

$$\mathcal{L} := \mathcal{P} \otimes (\mathcal{O}(\hbar \otimes \rho) \boxtimes \mathcal{O}(-\hbar \otimes \rho^\vee)) .$$

One regards the line bundle $\mathcal{L}$ as a certain ρ-shift of the Poincaré line bundle $\mathcal{P}$. Using the (dynamic) Weyl group actions, one obtains a new line bundle:

$$\mathcal{O}_{w,v^{\mathrm{dyn}}} := \mathcal{L} \otimes (w^{-1})^*((v^{\mathrm{dyn}})^{-1})^* \mathcal{L}^{-1}.$$

Definition 17. The object

$$\mathcal{O}[W \times W^{\mathrm{dyn}}] := \bigoplus_{\substack{w \in W \\ v^{\mathrm{dyn}} \in W^{\mathrm{dyn}}}} \mathcal{O}_{w,v^{\mathrm{dyn}}},$$

is called the *dynamical elliptic twisted group algebra.*

As the name suggests, $\mathcal{O}[W \times W^{\mathrm{dyn}}]$ in $\mathbf{Coh}_{W \times W^{\mathrm{dyn}}}(\mathfrak{A} \times \mathfrak{A}^\vee)$ defines an algebra (a.k.a. monad object) in a suitable category of coherent sheaves. In particular, it is an algebra object in the category

$$\mathbf{Coh}_{W \times W^{\mathrm{dyn}}}(\mathfrak{A} \times \mathfrak{A}^\vee).$$

The algebra structure is given by the following:

Lemma 22 (Lemma 2.2, [LZZ23])**.**

(i) We have the following composition properties:

$$\mathcal{O}_{w_1 w_2, v_1^{\mathrm{dyn}} v_2^{\mathrm{dyn}}} = \mathcal{O}_{w_1, v_1^{\mathrm{dyn}}} \otimes (w_1^{-1})^* (v_1^{-1})^{\mathrm{dyn}*} \mathcal{O}_{w_2, v_2^{\mathrm{dyn}}},$$
$$\mathcal{O}^{-1}_{w,v^{\mathrm{dyn}}} = (w^{-1})^* (v^{-1})^{\mathrm{dyn}*} \mathcal{O}_{w^{-1},(v^{-1})^{\mathrm{dyn}}}.$$

(ii) The objects $\mathcal{O}[W \times W^{\mathrm{dyn}}]$ and $\mathcal{O}[W \times W^{\mathrm{dyn}}]^{-1} := \bigoplus_{\substack{w \in W \\ v^{\mathrm{dyn}} \in W^{\mathrm{dyn}}}} \mathcal{O}^{-1}_{w,v^{\mathrm{dyn}}}$ are monoidal objects, where the degrees of $\mathcal{O}_{w,v^{\mathrm{dyn}}}$ and $\mathcal{O}^{-1}_{w,v^{\mathrm{dyn}}}$ are both (w, v).

Remark 10. [LZZ23] calls this the "Kostant-Kumer" twisted product, but augmented by the additional dynamic Weyl group action (see [LZZ23, §2.6]).

Similarly, we also have the monoidal categories $\mathbf{Coh}_W(\mathfrak{A})$, and $\mathbf{Coh}_{W^{\mathrm{dyn}}}(\mathfrak{A}^\vee)$. The twisted group algebra of [GKV95]:

$$\mathcal{O}_{\mathfrak{A}} \rtimes \mathbb{C}[W] := \bigoplus_{w \in W} \mathcal{O}_{\mathfrak{A}},$$

is an algebra object of $\mathbf{Coh}_W(\mathfrak{A})$. The dynamic version $\mathcal{O}_{\mathfrak{A}^\vee} \rtimes \mathbb{C}[W^{\mathrm{dyn}}]$ is similarly also an algebra object of $\mathbf{Coh}_{W^{\mathrm{dyn}}}(\mathfrak{A}^\vee)$.

D.2. Residue Construction of the Dynamical Hecke Algebra

Definition 18. Let $\mathcal{L}$ be a line bundle over the abelian variety $\mathfrak{A} \times \mathfrak{A}^\vee$, and let f be a rational section of $\mathcal{L}$, with a pole along T_α of order at most 1. Then, the *residue* of f at T_α is given by:

$$\mathrm{Res}_\alpha(f) := (\Theta_q(u) f) \,|_{T_\alpha},$$

where Θ_q is the Jacobi q-theta function.

This definition says that taking the residue is equivalent to evaluating $\Theta_q(u)f$ at $u = 0$. More explicitly, for an open set $U \subseteq \mathfrak{A} \times \mathfrak{A}^\vee$, and $f \in \mathcal{L}(U)$, then $\mathrm{Res}_\alpha(f)$ is a rational section of

$$(\mathcal{L} \otimes \mathcal{O}(-T_\alpha))\,|_{T_\alpha},$$

on $T_\alpha \cap U$. Moreover, if f is a rational section of $\mathcal{O}_{w,v^{\mathrm{dyn}}}$, and g is a rational section of $\mathcal{O}_{s_\alpha w, v^{\mathrm{dyn}}}$, then

$$\mathrm{Res}_\alpha(f+g) = \mathrm{Res}_\alpha(f) + \mathrm{Res}_\alpha(g),$$

(see [ZZ15, Lemma 2.1]). Consider the map

$$j \times j^\vee : \mathfrak{A}^{\mathrm{reg}} \times (\mathfrak{A}^\vee)^{\mathrm{reg}} \longrightarrow \mathfrak{A} \times \mathfrak{A}^\vee,$$

and the map

$$\pi \times \pi_{\mathrm{dyn}} : \mathfrak{A} \times \mathfrak{A}^\vee \longrightarrow \mathfrak{A}/W \times \mathfrak{A}^\vee/W^{\mathrm{dyn}}.$$

Let us write rational sections of $\mathcal{O}[W \times W^{\mathrm{dyn}}]$ as

$$\sum_{\substack{w \in W \\ v^{\mathrm{dyn}} \in W^{\mathrm{dyn}}}} a_{w,v^{\mathrm{dyn}}} \delta_w \delta_v^{\mathrm{dyn}},$$

where $a_{w,v^{\mathrm{dyn}}}$ is a rational section of $\mathcal{O}_{w,v^{\mathrm{dyn}}}$, and δ_w, and δ_v^{dyn} denotes its degree. We can now define the dynamical Hecke algebra:

Definition 19. Let $\mathcal{H}^{\mathrm{dyn}}$ be the subsheaf of $(j \times j^\vee)_*(j \times j^\vee)^*\mathcal{O}[W \times W^{\mathrm{dyn}}]$, so that on any open subset $U \subseteq \mathfrak{A} \times \mathfrak{A}^\vee$, sections of $\mathcal{H}^{\mathrm{dyn}}(U)$ are of the form

$$\sum_{\substack{w \in W \\ v^{\mathrm{dyn}} \in W^{\mathrm{dyn}}}} a_{w,v^{\mathrm{dyn}}} \delta_w \delta_v^{\mathrm{dyn}},$$

satisfying the following conditions:

(i) $a_{w,v^{\mathrm{dyn}}}$ only has poles at T_α or $T_{\alpha^\vee}$ of at most order 1, for finitely many $\alpha \in \Phi^+$,

(ii) $\mathrm{Res}_\alpha(a_{w,v^{\mathrm{dyn}}} + a_{s_\alpha w, v^{\mathrm{dyn}}}) = 0$, and $\mathrm{Res}_{\alpha^\vee}(a_{w,v^{\mathrm{dyn}}} + a_{w, s_\alpha v^{\mathrm{dyn}}}) = 0$,

(iii) For any $\alpha \in \Phi(w) = \Phi^+ \cap w^{-1}\Phi^-$, as a rational section of $\mathcal{O}_{w,v} \otimes \widetilde{\mathcal{O}}(-\hbar \otimes \alpha)$, the function

$$a_{w,v} \frac{\Theta_q(z_\alpha)}{\Theta_q(\hbar z_\alpha^{-1})},$$

is regular at $T_{\alpha,\hbar}$.

D.3. The Module Category Coh$(\mathcal{H}^{\mathrm{dyn}})$

Definition 20. Let $\pi^\vee : \mathfrak{A}^\vee \to \mathfrak{A}^\vee/W^{\mathrm{dyn}}$ be the dual natural map. A $\mathcal{H}^{\mathrm{dyn}}$*-module* $\mathcal{F}$ is an object in **Coh**$(\mathfrak{A}/W \times_{\mathrm{Spec}\,\mathbb{C}} \mathfrak{A}^\vee/W^{\mathrm{dyn}})$ for which there exists a multiplication map:

$$(\pi \times \pi^\vee)_*\mathcal{H}^{\mathrm{dyn}} \otimes_{\mathcal{O}_{\mathfrak{A}/W \times \mathfrak{A}^\vee/W^{\mathrm{dyn}}}} \mathcal{F} \longrightarrow \mathcal{F},$$

such that for each $V \subseteq \mathfrak{A}/W \times_{\mathrm{Spec}\,\mathbb{C}} \mathfrak{A}^\vee/W^{\mathrm{dyn}}$, each regular section $\mathcal{F}(U)$ has the structure of a $(\pi \times \pi^\vee)_*\mathcal{H}^{\mathrm{dyn}}(U)$-module.

Let $\mathbf{Coh}(\mathcal{H}^{\mathrm{dyn}})$ denote the category of $\mathcal{H}^{\mathrm{dyn}}$-modules. Define the full subcategories:

$$\mathbf{Coh}^{\mathrm{flat}}(\mathcal{H}^{\mathrm{dyn}}), \quad \mathbf{Coh}^{\mathrm{fin}}(\mathcal{H}^{\mathrm{dyn}}),$$

analogously.