# Iwahori-Hecke Algebras and $p$-adic Groups

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# Contents

# Introduction

This thesis is concerned with studying the structure theory of the Iwahori-Hecke algebra $\mathcal{H}$ through the use of some representation theoretic tools — namely, principal series representations of $p$-adic groups. Throughout, we assume familiarity with abstract algebra, representation theory, and root systems. At times we also assume some category theory knowledge from the reader, though that is not as crucial to understanding the content of this thesis.

We follow [HKP10] very closely, with the aim of giving more accessible details to the proofs. In doing so, we hope to give the reader a glimpse into the rich theory surrounding the structure of the Iwahori-Hecke algebra and its representations. It is the hope of the author that this thesis will serve as a guide for the reader that is interested in learning more about $p$-adic theory and Hecke algebras. Indeed, these objects see use in many other fields of mathematics, and we also hope that the reader will find those fields more approachable once they are equipped with our exposition.

This chapter will begin by giving a brief overview of the structure of the Iwahori-Hecke algebra, and its relation to the affine Hecke algebra. Then, we will survey the representation theoretic literature concerning the Iwahori-Hecke algebra. In doing so, we hope to illustrate the importance of this object in modern representation theory. Following this, we then give a detailed outline of our thesis, and the key results that we will be covering.

## The Iwahori-Hecke Algebra

It is impossible to encapsulate all of the rich theory surrounding this algebra in one thesis. We restrict our attention to exploring connections between the Iwahori-Hecke algebra and representations of $p$-adic groups.

Some applications of the Iwahori-Hecke algebras lie in the theory of automorphic forms (see [Bum09] for an overview on this subject).

In [KL79], Kazhdan and Lusztig used this algebra to define Kazhdan-Lusztig polynomials, which are combinatorial objects that contain deep information about representation theory. Additionally, it is an object that is still extensively studied to this day.

And in [Jon87], Jones used the language of Iwahori-Hecke algebras to define Jones polynomials, which is an object that is studied extensively in quantum topology and knot theory.

Given an abstract root system $\Phi$, one may form the *finite Weyl group* $W$, generated by reflections about the simple roots of $\Phi$. It has the usual presentation:

$$W = \left\langle s_1, \cdots, s_r : \begin{array}{c} s_i^2 = 1 \\ (s_i s_j)^{m_{ij}} = 1 \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \end{array} \right\rangle,$$

where $m_{ij}$ is the *Coxeter matrix* [Hum90]. From this, one may define the *Hecke algebra* $\mathcal{H}_W$ of $W$ to be the algebra with underlying $\mathbb{C}$-vector space, with basis $\{T_w : w \in W\}$ subject to the relations:

$$T_{s_i}^2 = (q-1)T_{s_i} + qT_1,$$
$$T_{s_i}T_{s_{i+1}}T_{s_i} = T_{s_{i+1}}T_{s_i}T_{s_{i+1}}.$$

Extending $W$ by a group of translations, one then obtains the *affine Weyl group* $W_{\text{aff}}$ [Kir97]. Similarly, one can define a Hecke algebra of $W_{\text{aff}}$, called the *affine Hecke algebra* $\mathcal{H}_{\text{aff}}$. For a precise treatment on this subject, we will refer the reader to [Kir97, Lecture 3].

However, this thesis will primarily be concerned with the affine Hecke algebra as a convolution algebra. Given a subgroup $I$ (with certain properties to be defined in Chapter 2) of a split, reductive $p$-adic group $G$, the algebra of compactly-supported $I$-biinvariant functions on $G$ — denoted by $\mathcal{H} := C_c(I\backslash G/I)$ — is the *Iwahori-Hecke algebra*. This algebra was first described by Iwahori and Matsumoto's influential paper [IM65].

This subgroup $I$ plays a role in the structure theory of $G$. In [IM65], Iwahori and Matsumoto proved the *Iwahori-Bruhat decomposition*, a theorem that states that $G$ decomposes as a disjoint union of double cosets of $I$. [Bum10] describes this as result as a "completely unexpected instance of Bruhat's axioms that were found in a $p$-adic group".

From this, they showed that $\mathcal{H}$ obtains a basis of indicator functions over these double cosets, which they used to show that $\mathcal{H}$ is isomorphic to the affine Hecke algebra $\mathcal{H}_{\text{aff}}$ — an object that they were also the first to describe in [IM65, §1]. Furthermore, this basis of indicator functions also give rise to a presentation of $\mathcal{H}$ — the Iwahori-Matsumoto presentation (see [IM65, Theorem 3.3, Theorem 3.5], or Theorem 2.9 of this thesis).

Bernstein subsequently found a different presentation of $\mathcal{H}$ in terms of a different set of generators and relations. This result was never published by Bernstein himself, and was first presented by Lusztig in [Lus83]. Bernstein's first observation was that $\mathcal{H}_{\text{aff}}$ decomposed as a tensor product of $\mathcal{H}_W$ and some lattice of translations — which we denote this by $\mathcal{R}$. This result is showcased in [HKP10, Lemma 1.7.1], and we exposit the details of this proof in Theorem 3.16.

Further, in [KL87], Kazhdan and Lusztig constructed $\mathcal{H}_{\text{aff}}$ in a purely geometric way, as the ${}^LG \times \mathbb{C}^\times$-equivariant $K$-group of the Steinberg variety $\mathrm{St}$. The situation at hand can thus be summarised by the diagram:

$$\begin{tikzcd}
\mathcal{H}_{\text{aff}} \arrow[r, leftrightarrow, "\text{Bernstein}"] \arrow[d, leftrightarrow, "\text{Iwahori-Matsumoto}"'] & \mathcal{R} \otimes_{\mathbb{C}} \mathcal{H}_W \arrow[d, leftrightarrow, "\text{Kazhdan-Lusztig}"] \\
\mathcal{H} := C_c(I\backslash G/I) \arrow[r, leftrightarrow] & K^{{}^LG\times\mathbb{C}^\times}(\mathrm{St})
\end{tikzcd}$$

The red arrows represent what this thesis will seek to exposit, and can be regarded as the "algebraic" portion of the story surrounding the Iwahori-Hecke algebra. The rest of the arrows can be thought of as the "geometric" portion of the story.

## Applications in Representation Theory

The Iwahori-Hecke is of particular interest to representation theorists as it contains information about the category of $G$-representations generated by their $I$-fixed vectors. More precisely, given an irreducible, admissible representation $(\sigma, V)$ of $G$, the $G$-submodule of $I$-fixed vectors:

$$V^I := \{v \in V : \sigma(I)v = v\},$$

can be given an $\mathcal{H}$-module structure by convolution. By *admissible*, we mean that $V^I$ is a finite-dimensional $\mathcal{H}$-module for any compact open subgroup $H$.

In [Bor76], Borel constructed an exact functor from the category of finite-dimensional $\mathcal{H}$-modules to the category of admissible, smooth representations of $G$ which maps irreducible objects to irreducible objects. In other words, there is an equivalence of categories

$$\left\{\begin{matrix}\text{Admissible}\\ \text{Representations of } G\\ \text{with } I\text{-fixed vectors}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Finite-dimensional}\\ \text{Representations of } \mathcal{H}\end{matrix}\right\}.$$

This result was significant since admissible, smooth $G$-representations are typically of infinite dimension. In this way, the Iwahori-Hecke algebra reduces infinite-dimensional problems to finite-dimensional ones.

Casselman, in [Cas80], subsequently showed that under the correspondence established by Borel, all irreducible admissible representations of $G$ arising from irreducible $\mathcal{H}$-modules are precisely those that occur as quotients of the unramified principal series of $G$ [Cas80, Proposition 2.6].

Another important result that has far-reaching implications is the *Satake isomorphism*, first proved by Satake in [Sat63]. For a maximal compact subgroup $K$, the *spherical Hecke algebra* $\mathcal{H}_{\text{sph}} = C_c(K\backslash G/K)$ — a subring of $\mathcal{H}$ — is isomorphic to the Weyl group invariants of $\mathcal{R}$ [Sat63, Theorem 3]. We follow [HKP10, Section 4], and prove this in Theorem 5.7.

This provides a significant amount of information about representations of $G$ admitting $K$-fixed vectors — which are called *spherical representations*. These $K$-fixed vector spaces can be given a $\mathcal{H}_{\text{sph}}$-module structure. So, there is an exact functor

$$(-)^K : \mathbf{Rep}(G) \longrightarrow \mathbf{Rep}(\mathcal{H}_{\text{sph}}), \quad V \longmapsto V^K.$$

Unlike Borel's result, this is not an equivalence of categories. Despite this, there is still something that can be said.

The Weyl group invariants of $\mathcal{R}$ is abelian, and hence by the Satake isomorphism, $\mathcal{H}_{\text{sph}}$ is also abelian. That is, all irreducible $\mathcal{H}_{\text{sph}}$-modules are one-dimensional and can be identified with a character. After some work, one finds the bijection (seen in Chapter 6.3.1):

$$\left\{\begin{matrix}\text{Irreducible spherical}\\ \text{representations of } G\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Unramified characters}\\ \text{of } T \text{ up to the } W\text{-action}\end{matrix}\right\},$$

which gives a characterisation of all irreducible spherical $G$-representations. This result has important applications in the theory of automorphic representations by a theorem of [Fla79], which states that irreducible automorphic representations of the adelic group $G(\mathbb{A}_{\mathbb{Q}})$ are spherical for all but finitely many local components. That is, understanding spherical representations allows us to understand "most" automorphic representations of $G(\mathbb{A}_{\mathbb{Q}})$.

Another important application of the Satake isomorphism lies in the local Langlands correspondence for $\mathrm{GL}_n(\mathbb{C})$. Using highest weight theory, the Weyl invariants of $\mathcal{R}$ can be related to the representation ring of the Langlands dual group of $G$. Using this fact, and applying Satake isomorphism (see Chapter 6.3.4) eventually gives us the bijection:

$$\left\{\begin{matrix}\text{Irreducible, spherical}\\ \text{Representations of } \mathrm{GL}_n(\mathbb{C})\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Unramified Weil-Deligne}\\ \text{Representations}\end{matrix}\right\},$$

which is known as the *unramified local Langlands correspondence* for $\mathrm{GL}_n(\mathbb{Q}_p)$. Consequently, this makes the Satake isomorphism a crucial result for $p$-adic representation theory and local Langlands theory.

There is a sheaf-theoretic formulation of the Satake isomorphism, called the *geometric Satake equivalence* (see [Gin96, MV00, MV07]), which is used in the geometric Langlands program.

## Structure of the Thesis

The key results of this thesis are the following:

**Theorem 0.1.**

*(i) (Bernstein Presentation) The Iwahori-Hecke algebra has a basis* $\{T_w\Theta_\lambda : w \in W, \lambda \in X_*(T)\}$*, with the following presentation: for* $\lambda, \mu \in X_*(T)$*,* $s_\alpha$ *a simple reflection, and* $w, w \in W$ *such that* $\ell(ww') = \ell(w) + \ell(w')$*,*

$$\begin{aligned} T_{s_\alpha}^2 &= (q-1)T_{s_\alpha} + qT_1, \\ T_wT_{w'} &= T_{ww'}, \\ \Theta_\lambda\Theta_\mu &= \Theta_{\lambda+\mu}, \\ T_{s_\alpha}\Theta_\mu &= \Theta_{s_\alpha(\mu)}T_{s_\alpha} + (1-q)\frac{\Theta_{s_\alpha(\mu)} - \Theta_\mu}{1-\Theta_{-\alpha^\vee}}. \end{aligned}$$

*(ii) (Satake Isomorphism)* $\mathcal{H}_{\mathrm{sph}} \cong \mathcal{R}^W$.

Chapter 1 sets up the required machinery needed for further chapters. Little proofs are given, but we attempt to give as many examples as possible. In particular, we introduce the notion of $p$-adic numbers, and study their structure. Some basic results on topological groups and Haar measures are stated. Most crucially, however, is our discussion of root systems within a $p$-adic group; these results will be employed extensively throughout the thesis.

Chapter 2 introduces the Iwahori-Hecke algebra $\mathcal{H}$. We use the Iwahori-Bruhat decomposition to study the structure of $\mathcal{H}$, and obtain a presentation for it in terms of double coset indicator functions. This is known as the *Iwahori-Matsumoto presentation*. Our sources for this are from [Bum10] and [IM65].

In Chapter 3, we begin to study principal series representations of $G$ and $\mathcal{H}$. We begin by studying universal principal series module $\mathcal{M}$ of $\mathcal{H}$ directly, and prove that it is a $(\mathcal{R}, \mathcal{H})$-bimodule. Then, we explicitly construct the principal series representation of $G$, from which we then recover $\mathcal{M}$ by taking its $I$-fixed vectors. From then on, we return to studying $\mathcal{M}$ and prove that it is a free, rank one $\mathcal{H}$-module.

Chapter 4 uses the machinery of the previous chapter to construct a family of intertwiners of $\mathcal{M}$ indexed by elements of the finite Weyl group. Much of the construction is involved, and are not employed for other results in the thesis. Thus, we refer the reader to [HKP10, Section 1.10-1.11] for the details of the construction. Using this, we are able to prove the Bernstein presentation of $\mathcal{H}$.

In Chapter 5, we construct the *Satake transform*, which is a map from the spherical Hecke algebra to the Weyl-invariants of $\mathcal{R}$. We then prove that the Satake transform is an isomorphism, which is the so-called *Satake isomorphism*. We then state some corollaries of this isomorphism and highlight its importance in modern mathematics.

In particular, we first highlight its applications in automorphic representations. The Satake isomorphism allows one to characterise irreducible spherical $G$-representations in terms of irreducible $\mathcal{H}_{\text{sph}}$-modules. And by a theorem of Flath in [Fla79], all but finitely many components of an irreducible automorphic representation of $G(\mathbb{A}_{\mathbb{Q}})$ are spherical.

Then, we highlight applications in the local Langland correspondences for $\mathrm{GL}_n$, which claims that there is a canonical bijection:

$$\left\{\begin{matrix}\text{Admissible, irreducible}\\ \text{complex representations}\\ \text{of } \mathrm{GL}_n(\mathbb{Q}_p)\end{matrix}\right\}_{/\cong} \longleftrightarrow \left\{\begin{matrix}\text{Finite-dimensional}\\ F\text{-semisimple}\\ \text{Weil-Deligne representations}\end{matrix}\right\}_{/\cong}$$

We do not showcase the proof of the general case (though [Wed00] gives an expository treatment on this subject). Rather, we specialise to the unramified case, and tamely ramified case, which are treated in Chapters 6.4.1 and Chapter 6.4.2, respectively.

# Chapter 1

# Preliminaries

We begin by introducing some basic objects and notions that we will require for the remainder of the thesis.

## 1.1 The $p$-adic Numbers

For this section, we are following [Iwa86, Chapter 1]. Throughout, we fix a prime number $p$. We define a norm on $\mathbb{Q}$ as follows: for any non-zero $x \in \mathbb{Q}$, there is a unique integer $n$ such that $x = p^n \left(\frac{a}{b}\right)$, where both $a \in \mathbb{Z}$ and $0 \neq b \in \mathbb{Z}$ are not divisible by $p$. The *p-adic valuation* of $x$ is a map $v_p : \mathbb{Q} \to \mathbb{R}$ defined by

$$v_p(x) = n,$$

and the *p-adic norm* is given by

$$|x| = p^{-v_p(x)} = p^{-n}.$$

By convention we set $|0|_p = 0$. It follows by our construction that $v_p(\mathbb{Q}) = \mathbb{Z}$.

The *p-adic expansion* of $x$ is of the form:

$$x = \sum_{i=k}^{\infty} a_i p^i,$$

where $a_i \in \{0, \cdots, p-1\}$. The norm induces a metric $d_p(x, y) := |x - y|_p$, and induces the metric topology on $\mathbb{Q}$.

The completion of $\mathbb{Q}$ under $|\cdot|_p$ then produces the *p-adic numbers* $\mathbb{Q}_p$. By definition of a completion, elements of $\mathbb{Q}_p$ are equivalence classes of Cauchy sequences [**?**, Theorem

4.1]. Thus, each element $x \in \mathbb{Q}_p$ corresponds to a series $\sum_{i=k}^{\infty} a_i p^i$ converging to $x$, where $0 \le a_i < p$.

The *p-adic integers* — denoted $\mathbb{Z}_p$ — are those elements for which $k$ is non-negative. In terms of the valuation $v_p$, this means that

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : v_p(x) \ge 0\}.$$

It is easy to see that $p\mathbb{Z}_p$ is the unique maximal ideal of $\mathbb{Z}_p$. Since the image of $v_p$ is in $\mathbb{Z}$, every ideal of $\mathbb{Q}_p$ has the form $p^j\mathbb{Z}_p$ for some $j \ge 0$. In particular:

$$\mathbb{Z}_p \supseteq p\mathbb{Z}_p \supseteq p^2\mathbb{Z}_p \supseteq \cdots$$

Thus, $\mathbb{Z}_p$ is a principal ideal domain, and each ideal is generated by a unique element $p$. Such an element $p$ is called the *uniformiser*.

### 1.1.1 Non-Archimedean Local Fields

Generally, one does not work with $\mathbb{Q}_p$ directly, but rather with finite-degree extensions $F$ of $\mathbb{Q}_p$. As the following theorem shows, the structure of $F$ is similar to that of $\mathbb{Q}_p$.

**Theorem 1.1** (Iwa86, Theorem 2.4). *Any non-Archimedean local field $F$ of characteristic zero is a finite-degree extension of $\mathbb{Q}_p$. Further, $F$ has discrete valuation ring $\mathcal{O}$, a unique maximal ideal $\pi\mathcal{O}$ generated by a uniformiser $\pi$, and finite residue field $\mathcal{O}/\pi\mathcal{O} \cong \mathbb{F}_q$.*

Let $v_F$ be a discrete valuation on $F$ corresponding to $\mathcal{O}$. As before, one may define a norm on $F$ by

$$\|x\| = q^{-v_F(x)},$$

giving a metric on $F$, relative to which $F$ is complete. This metric induces the metric topology, and thus $F$ is a topological field. As we observed in the case where $F = \mathbb{Q}_p$, there is an descending filtration of ideals:

$$\mathcal{O} \supset \pi\mathcal{O} \supset \pi^2\mathcal{O} \supset \cdots$$

where for some $j \in \mathbb{N}$,

$$\pi^j\mathcal{O} = \{x \in F : \|x\| \le q^{-j}\}.$$

Since the residue field is finite, it follows then that each $\mathcal{O}/\pi^j\mathcal{O}$ is finite, with cardinality $q^j$. This, together with the completeness of $F$, we have that

$$\mathcal{O} \cong \varprojlim_{n\ge 1} \mathcal{O}/\pi^n\mathcal{O}.$$

## 1.2 Topological Groups

A group $G$ is a *topological group* if it is equipped with a topology such that the multiplication and inversion maps are continuous. We equip the direct product group $G \times G$ with the product topology. Generally, any group $G$ obtains the structure of a topological group under the discrete topology.

We say that $G$ is *locally compact* if there is a compact neighbourhood of every point of $G$. $(\mathbb{R}, +)$ and $(\mathbb{C}, +)$ are some obvious examples of locally compact groups.

**Proposition 1.2.** *Every open subgroup $H$ of a topological group $G$ is also closed.*

*Proof.* If $H$ is any open subgroup of $G$, then $xH$ is also open. Thus,

$$Y := \bigcup_{x \in G \setminus H} xH$$

is also open. But $H = G \setminus Y$, and thus $H$ is closed. □

There are some important subgroups that we will consider throughout the thesis. One of these is the *maximal torus*, which is the maximal compact, connected abelian subgroup of $G$.

For some $G$ over a field $F$, the maximal torus is isomorphic to $T \cong (F^\times)^r$. The *rank* of $G$ is the dimension of the maximal torus. This is a well-defined notion as any two maximal tori are conjugate to one another [Hum98, Theorem 19.3(b)].

The *Borel subgroup* is the maximal connected, solvable subgroup of $G$.

**Example 1.3.** Consider $G = (\mathbb{Q}_p, +)$. Its discrete valuation ring $\mathbb{Z}_p$ is a subgroup of $\mathbb{Q}_p$. In terms of the $p$-adic norm,

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\},$$

and thus $\mathbb{Z}_p$ is a compact open under the metric topology. Any $x \in \mathbb{Q}_p$ is contained in some compact neighbourhood of the form $x + p^i\mathbb{Z}_p$, for some integer $i \geq 0$, and thus $\mathbb{Q}_p$ is a locally compact group.

**Example 1.4.** Let $\mathrm{Mat}_{n\times n}(\mathbb{Q}_p)$ be the set of all $n \times n$ matrices with entries in $\mathbb{Q}_p$, which we identify with $\mathbb{Q}_p^{n^2}$. Let $\mathrm{GL}_n(\mathbb{Q}_p)$ be the subset of $\mathrm{Mat}_{n\times n}(\mathbb{Q}_p)$ with non-zero determinant (that is, invertible matrices). We equip $\mathrm{GL}_n(\mathbb{Q}_p)$ with the subspace topology of $\mathbb{Q}_p^{n^2}$. Clearly, $\mathrm{GL}_n(\mathbb{Q}_p)$ is a group under matrix multiplication.

We want to show that $\mathrm{GL}_n(\mathbb{Q}_p)$ is open in $\mathrm{Mat}_{n\times n}(\mathbb{Q}_p)$. The determinant map $\det$ :

$\mathrm{Mat}_{n\times n}(\mathbb{Q}_p) \to \mathrm{Mat}_{n\times n}(\mathbb{Q}_p)$ is continuous because it is polynomial in the coordinates $(x_{ij}) \in \mathrm{Mat}_{n\times n}(\mathbb{Q}_p)$, where the $x_{ij}$'s ($1 \leq i, j \leq n$) are the entries of an $n \times n$ matrix.

The complement of $\mathrm{GL}_n(\mathbb{Q}_p)$ is the set of all matrices with determinant zero, which is closed since the pre-image of $0$ under $\det$ is closed. Therefore, $\mathrm{GL}_n(\mathbb{Q}_p)$ is open.

Let $\mathrm{SL}_n(\mathbb{Q}_p)$ be the subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$ with determinant one. Following the same argument, as above it follows that $\mathrm{SL}_n(\mathbb{Q}_p)$ is a closed subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$. One may show that $\mathrm{GL}_n(\mathbb{Z}_p)$ is the maximal compact subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$.

**Example 1.5.** Let $G = \mathrm{GL}_n(k)$, where $k$ is an algebraically closed field of characteristic zero. A standard choice for the maximal torus $T$ is the subgroup of a diagonal matrices:

$$T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}.$$

Thus, $T \cong (k^\times)^n$, and $G$ has rank $n$. The Borel subgroup is typically chosen as:

$$B = \begin{pmatrix} * & * & \cdots & * \\ & * & \ddots & \vdots \\ & & \ddots & * \\ & & & * \end{pmatrix}$$

Over an algebraically closed field, there is only one conjugacy class of Borel subgroups, and thus over $\mathrm{GL}_n(k)$ this is the typical matrix representative that is chosen.

### 1.2.1 Semisimple and Unipotent Elements

This section follows [Hum98, Chapter VI, Section 15] Let $V$ be a finite-dimensional vector space over an algebraically closed field $k$ of characteristic zero. Let $\mathrm{GL}(V)$ be the automorphism group of $V$. If one chooses a basis $\{v_1, \cdots, v_n\}$ for $V$, then $\mathrm{GL}(V) \cong \mathrm{GL}_n(k)$, the aforementioned general linear group.

**Definition 1.6.** An element $x \in \mathrm{GL}(V)$ is:

(i) *Nilpotent* if $x^n = 0$ (or equivalently, $0$ is its only eigenvalue)

(ii) *Semisimple* if it is diagonalisable over $k$

(iii) *Unipotent* if $(x-1)^n = 0$ (or equivalently, $1$ is its only eigenvalue)

**Definition 1.7.** A subgroup of $G$ is *unipotent* if all of its elements are unipotent.

There is the well-known *Jordan decomposition*:

**Theorem 1.8** (Lemma 15.1B(a), [Hum98]). *Let $x \in \mathrm{GL}(V)$. Then, there exist unique $x_s, x_u \in \mathrm{GL}(V)$ such that $x = x_u x_s = x_s x_u$, where $x_s$ and $x_u$ are semisimple and unipotent, respectively.*

**Definition 1.9.** $G$ is *reductive* if the maximal connected, unipotent normal subgroup of $G$ is trivial. Such a normal subgroup is called the *unipotent radical*.

**Example 1.10.** The *unipotent subgroup* $\mathrm{U}_n(k)$ of $\mathrm{GL}_n(k)$ is the subgroup given by

$$\begin{pmatrix} 1 & * & \cdots & * \\ & 1 & \ddots & \vdots \\ & & \ddots & * \\ & & & 1 \end{pmatrix}.$$

As the name suggests, $\mathrm{U}_n$ is a unipotent group. Define a sequence of of subgroups by:

$$\mathrm{U}_n^{(1)} = [\mathrm{U}_n, \mathrm{U}_n], \quad \mathrm{U}_n^{(n)} = [\mathrm{U}_n, \mathrm{U}_n^{(n-1)}],$$

where $[\cdot, \cdot]$ is the commutator bracket. To simplify this computation, let $n = 4$. Then,

$$U_4 = \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad U_4^{(1)} = \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathrm{U}_4^{(2)} = \begin{pmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathrm{U}_4^{(3)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and thus $\mathrm{U}_4$ is a unipotent group. Generally,

$$\mathrm{U}_n \supset \mathrm{U}_n^{(1)} \supset \mathrm{U}_n^{(2)} \supset \cdots \supset \mathrm{U}_n^{(m)} = I_n,$$

where $I_n$ is the $n \times n$ identity matrix. Recall the aforementioned Borel subgroup of $G$:

$$B = \begin{pmatrix} * & * & \cdots & * \\ & * & \ddots & \vdots \\ & & \ddots & * \\ & & & * \end{pmatrix}.$$

Taking the maximal torus $T$ of $\mathrm{GL}_n(k)$ to be the subgroup of diagonal matrices, we now have that $B = T\,\mathrm{U}_n$. Thus, $B/\,\mathrm{U}_n \cong T \cong (k^\times)^n$, and it follows then that $B$ is solvable. Further, $\mathrm{U}_n$ is the unipotent radical of $B$.

## 1.3 Root Structure of a $p$-adic Group

Let $G$ be a split, connected, reductive group over a local non-Archimidean local field $F$. Let $\mathcal{O}$ be the discrete valuation ring of $F$ with unique maximal ideal generated by the uniformiser $\pi$, and residue field $\mathcal{O}/\pi\mathcal{O} \cong \mathbb{F}_q$. Denote by $T$ the maximal torus of $G$, $N$ the unipotent radical of the Borel subgroup $B$.

### 1.3.1 The Character and Cocharacter Lattice

Define by $X^*(T) = \mathrm{Hom}(T, F^\times)$ the *character group* of $T$. If $\chi, \psi \in X^*(T)$, then $\chi + \psi \in X^*(T)$, and $-\chi$ is an inverse of $\chi$. Thus, $X^*(T)$ has the structure of an abelian group.

The *cocharacter group* is given by $X_*(T) := \mathrm{Hom}(F^\times, T)$, and is again an abelian group in a natural way.

Consider the case for which $T = F^\times$. Then, the characters of $F^\times$ are the maps $x \to x^n$ for $n \in \mathbb{Z}$. Suppose now that $G$ has rank $r$ — that is, $T \cong (F^\times)^r$. Then, for each $\nu = (n_1, \cdots, n_r) \in \mathbb{Z}^r$, define $\chi_\nu \in X^*(T)$ and $\eta_\nu \in X_*(T)$ by

$$\chi_\nu(x_1, \cdots, x_r) = x_1^{n_1} \cdots x_r^{n_r}, \quad \eta_\nu(x) = (x^{n_1}, \cdots, x^{n_r}).$$

Then, the maps $\nu \mapsto \chi_\nu$ and $\nu \mapsto \eta_\nu$ gives isomorphisms

$$X^*(T) \cong X^*(F^\times)^r \cong \mathbb{Z}^r, \quad X_*(T) \cong X_*(F^\times)^r \cong \mathbb{Z}^r,$$

respectively. It follows thus that $X_*(T)$ and $X^*(T)$ are free $\mathbb{Z}$-modules — that is, they are lattices. We will thus refer to $X^*(T)$ and $X_*(T)$ as the *character* and *cocharacter* lattices, respectively.

### 1.3.2 Root Systems

Throughout, we will assume that the reader is familiar with the notion of abstract root systems. We will state some facts about how a root system arises within $G$, but many of the details are non-trivial, and we are therefore forced to refer the reader to other sources.

First, one must construct the Lie algebra $\mathfrak{g}$ of a group $G$. We refer the reader to [CSM95, Chapter 4,5]. The adjoint representation of $G$ is $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$, and is defined by conjugation — that is, $\mathrm{Ad}(g)(X) = gXg^{-1}$. Following [CSM95, Chapter 8], $\mathfrak{g}$ decomposes as a direct sum of the subspaces:

$$\mathfrak{g} = \bigoplus_{\alpha \in X^*(T)} \mathfrak{g}_\alpha,$$

where for each $\alpha \in X^*(T)$,

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} : \mathrm{Ad}(t)X = \alpha(t)X \text{ for all } t \in T\}.$$

The non-zero $\alpha \in X^*(T)$ for which $\mathfrak{g}_\alpha \neq 0$ are the *roots* of $G$ relative to $T$. Let $\Phi$ denote the set of roots, embedded as a subset of $X^*(T) \otimes \mathbb{R}$. The *positive roots* $\Phi^+$ form a subset of $\Phi$ such that for each $\alpha \in \Phi$, one of the roots $\alpha, -\alpha$ is in $\Phi^+$. A root is *simple* if it cannot be written as the sum of two positive roots.

**Definition 1.11.** The tuple $(X^*, \Phi, X_*, \Phi^\vee)$ is the *root datum* of $G$.

Recall that $X_*(T) \cong \mathbb{Z}^r$, where $r$ is the rank of $G$. Then, since $T/T_\mathcal{O} \cong (F^\times)^r/(\mathcal{O}^\times)^r \cong \mathbb{Z}^r$, we thus obtain an isomorphism

$$X_*(T) \cong T/T_\mathcal{O}, \quad \mu \longmapsto \mu(\pi). \tag{1.1}$$

It is convenient to denote $\mu(\pi)$ by $\pi^\mu$, so that we may write elements multiplicatively. That is, for $\mu, \nu \in X_*(T)$, the element $\mu(\pi) + \nu(\pi) = (\mu+\nu)(\pi)$ corresponds to $\pi^\mu\pi^\nu = \pi^{\mu+\nu}$. Throughout this thesis, we will employ this notation.

A *root subgroup* associated to a root $\alpha$ is a one-parameter subgroup $U_\alpha \subset G$ such that $U_\alpha \cong F$. We denote by $x_\alpha : F \to U_\alpha$ the isomorphism onto the root subgroup, which we identify as an element of $U_\alpha$. We record some facts about root subgroups that we will use extensively throughout the thesis from [Bum10, Hum98]:

**Theorem 1.12.**

*(i) [Hum98, Theorem 26.3(a)] A root $\alpha \in \Phi$ uniquely defines a root subgroup $U_\alpha$.*

*(ii) [Hum98, Theorem 26.3(b)] For some $w \in W$, $\alpha \in \Phi$,*

$$wU_\alpha w^{-1} = U_{w(\alpha)}.$$

*(iii) [Bum10, Proposition 49], [Hum98, Theorem 26.3(c)] Let $\mu \in X_*(T)$, and $x_\alpha : F \xrightarrow{\simeq} U_\alpha$ be the isomorphism. Then,*

$$\pi^\mu x_\alpha(t)\pi^{-\mu} = x_\alpha(t\pi^{\langle \mu, \alpha\rangle}).$$

*(iv) [Hum98, Theorem 26.3(d)]Fix a system of positive roots $\Phi^+$, and an order on it. Then,*

$$N \cong \prod_{\alpha \in \Phi^+} U_\alpha,$$

*is the unipotent radical of $B$, normalised by $T$. Further, $TN = NT = B$.*

Throughout this thesis, we will always think of the unipotent subgroup in terms of its root subgroup decomposition. This, together with Theorem 1.12(iii), will help us with many of the computations that we will perform later.

**Example 1.13.** Let $F$ be an algebraically closed field of characteristic zero. For $G = \mathrm{SL}_n(F)$. One can denote the simple roots by $\alpha_1, \cdots, \alpha_{n-1}$, which can be represented inside $X^*(T) \cong \mathbb{Z}^n$ as $\alpha_i \mapsto e_i - e_{i+1}$, where $e_i$ is the canonical basis of $\mathbb{R}^n$.

Under this choice, the positive roots are given by $\{\alpha_{ij} = e_i - e_j : i < j\}$, where $e_i$ is the canonical basis vector of $\mathbb{R}^n$. Then, the root subgroup associated to $\alpha_{ij}$ is given by

$$U_{\alpha_{ij}} = \{I + tE_{ij} : t \in F\},$$

where $I$ is the identity matrix and $E_{ij}$ is the matrix with entry $1$ at the $(i, j)$-th position, and zero everywhere else.

Through this, the unipotent subgroup is then given by:

$$N = \begin{pmatrix} 1 & * & \cdots & * \\ & 1 & \ddots & \vdots \\ & & \ddots & * \\ & & & 1 \end{pmatrix}.$$

Different choices of simple roots give rise to differnet unipotent groups. It is clear that the matrix $TN = NT = B$, where $B$ and $T$ is the Borel subgroup and maximal torus chosen the same way as in Example 1.5, respectively. Relative to the negative roots — that is, $\{e_i - e_j : i > j\}$, we obtain the *opposite unipotent subgroup* — which we denote as $\overline{N}$. In terms of matrices, this gives us

$$\overline{N} = \begin{pmatrix} 1 & & & \\ * & 1 & & \\ \vdots & \ddots & \ddots & \\ * & \cdots & * & 1 \end{pmatrix}.$$

Then, the product $T\overline{N}$ gives us the *opposite Borel subgroup* $\overline{B}$, which is given by invertible lower-triangular matrices.

### 1.3.3 Affine Weyl Group

This subsection follows [Kir97] and [Bum10]. The *finite Weyl group* is defined in $G$ as $W := N_G(T)/T$. Let $\Phi$ be the root system of $G$, with simple roots given by $\alpha_1, \cdots, \alpha_r$, where $r$ is the rank of $G$. For each $\alpha \in \Phi$, there is a well-defined element $\alpha^\vee \in X_*(T)$ called the *coroot* of $\alpha$. There is an action of $W$ on $X_*(T)$: for $\mu \in X_*(T)$,

$$s_\alpha(\mu) = \mu - \langle \alpha, \mu \rangle \alpha^\vee, \tag{1.2}$$

where $\langle \cdot, \cdot \rangle$ is a $W$-invariant inner product. Denote by $s_{\alpha_1}, \cdots, s_{\alpha_r}$ the simple reflections corresponding to the simple roots. For brevity, we write $s_i = s_{\alpha_i}$. It is a well-known fact that $W$ is also generated by the simple roots. A presentation for $W$ is given by:

$$W := \left\langle s_1, \cdots, s_r : \begin{array}{c} s_i^2 = 1 \\ (s_i s_j)^{m_{ij}} = 1 \end{array} \right\rangle,$$

where $m_{ij}$ is the *Coxeter matrix*, with $m_{ii} = 1$, and $m_{ij} \in \{2, 3, \cdots, \infty\}$. We set $m_{ij} = \infty$ if there is no relation between $s_i$ and $s_j$ [Hum90].

**Example 1.14.** Let $G = \mathrm{SL}_{n+1}(F)$, where $F$ is a local non-Archimedean local field. Then, $W = \mathfrak{S}_n$, the symmetric group on $n$ words. It is generated by adjacent transpositions $s_i := (i\; i+1)$. As aforementioned, $G$ has rank $n$, and simple roots given by $\alpha_i := e_i - e_{i+1}$. Since $X_*(T) \cong \mathbb{Z}^n$, we write some $\mu \in X_*(T)$ as $\mu = (a_1, \cdots, a_n) \in \mathbb{Z}^n$. Then, $s_i$ acts on $\mu$ by:

$$s_i(\mu) = s_i(a_1, \cdots, a_i, a_{i+1}, \cdots, a_n) = (a_1, \cdots, a_{i+1}, a_i, \cdots, a_n).$$

The *extended affine Weyl group* $\widetilde{W}$ of $G$ is given by the quotient $N_G(T)/T_{\mathcal{O}}$. It contains the finite Weyl group $N_G(T)/T$, and a subgroup $T/T_{\mathcal{O}}$, which is isomorphic to $X_*(T)$ by (1.1).

Alternatively, one may define $\widetilde{W}$ in terms of the *affine Weyl group* $W_{\mathrm{aff}}$. This is given by the semidirect product of the coroot lattice and the finite Weyl group:

$$W_{\mathrm{aff}} := \bigoplus_{i=1}^{r} \mathbb{Z}\alpha_i \rtimes W,$$

where the $\alpha_i$'s are simple. $W$ acts on the coroot lattice in the usual way.

*Remark* 1.15. There are authors who refer to $\widetilde{W}$ as the *extended affine Weyl group* only when $G = \mathrm{GL}_n(F)$. For general $G$, they will say that $\widetilde{W}$ is the *affine Weyl group*.

Following this, one may define the extended affine Weyl group as

$$\widetilde{W} := \Omega \rtimes W_{\mathrm{aff}},$$

where $\Omega$ is the subgroup of $\widetilde{W}$ containing elements of length zero. In the case where $G$ is simply-connected, the cocharacter lattice is the same as the coroot lattice, and thus $\widetilde{W} = W_{\mathrm{aff}}$ [Bum10, Section 6].

$\widetilde{W}$ can also be written as the semi-direct product $\widetilde{W} = \Omega \rtimes W_{\text{aff}}$, where $\Omega$ acts on the affine Weyl group by Dynkin diagram automorphisms [Kir97, pg. 18-19]. That is, $\Omega$ is generated by an element $\varpi$ which permutes the simple roots by $\varpi(\alpha_i) = \alpha_{i+1}$. Thus, $\varpi$ acts on the simple reflections by:

$$\varpi s_i \varpi^{-1} = s_{i+1}.$$

The indices here are all taken modulo $r$. In a type $A$ group, one obtains the presentation for $\widetilde{W}$ [MS19, Section 1]:

$$\widetilde{W} = \left\langle \varpi, s_0, s_1, \cdots, s_r : \begin{array}{c} s_i s_j = s_j s_i \quad \text{if} \quad |i-j| \neq 1 \\ s_i^2 = 1 \quad \varpi s_i = s_{i+1}\varpi \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \end{array} \right\rangle,$$

where $s_0, \cdots, s_r$ inside $\widetilde{W}$ generate $W_{\text{aff}}$ as a subgroup. Once again, $\mathcal{H}_{\text{aff}}$ can be defined independently of the group.

### 1.3.4 Hecke Algebras

Relative to $W$, we can define the *Hecke algebra* of $W$ to be the $\mathbb{C}$-vector space $\mathcal{H}_W$ generated by elements $\{T_w : w \in W\}$ such that

$$\begin{aligned} T_{s_i}^2 &= (q-1)T_{s_i} + qT_1 \\ T_{s_i}T_{s_{i+1}}T_{s_i} &= T_{s_{i+1}}T_{s_i}T_{s_{i+1}}. \end{aligned}$$

The second relation is commonly known as *Artin's braid relation*. This is a purely algebraic construction, and can be defined independently of $G$. Similarly, the *affine Hecke algebra* $\mathcal{H}_{\text{aff}}$ is the $\mathbb{C}$-vector space generated by $\{T_w : w \in \widetilde{W}\}$ subject to the relations

$$\begin{aligned} &T_{s_i}^2 = (q-1)T_{s_i} + qT_1 \\ &T_{ww'} = T_w T_{w'} \text{ if } \ell(ww') = \ell(w) + \ell(w'). \end{aligned}$$

Here, the Artin braid relation is replaced with this length condition on elements of $\widetilde{W}$ to account for elements of length zero. While the braid relations still hold in $\mathcal{H}_{\text{aff}}$, it does not define a presentation of $\mathcal{H}_{\text{aff}}$.

### 1.3.5 Some Explicit Examples

Many of the computations in this section employ [MS19, Lemma 2.1], which gives us very explicit ways of choosing representatives for the affine Weyl group within $\mathrm{SL}_n(F)$ or $\mathrm{GL}_n(F)$.

In this section, we perform some explicit calculations to illustrate how the objects we constructed above arise in certain groups. In particular, we begin by considering the simply connected case with $\mathrm{SL}_2$, where $\widetilde{W} = W_{\text{aff}}$ since the coweight and coroot lattice are now the same.

Given our $\mathrm{SL}_2$ root datum, it is then quite easy to generalise everything to $\mathrm{SL}_n$ (or any arbitrary $G$, as we shall see) for some suitable embedding of $\mathrm{SL}_2$ into $\mathrm{SL}_n$.

We close off with a discussion of the non-simply connected case with $\mathrm{GL}_2$ and $\mathrm{GL}_n$, where we will see how the extended affine Weyl group arises within the group.

**The $\mathrm{SL}_2(F)$ Case**

To begin, let us consider $G = \mathrm{SL}_2(F)$. Its maximal torus is given by $T = \mathrm{diag}(x, x^{-1})$, for $x \in F^\times$. Thus, $T \cong F^\times$, and $\mathrm{SL}_2(F)$ has rank one. The finite Weyl group is given by $W = \mathfrak{S}_2$. Since it is simply-connected, $\widetilde{W}_{\text{aff}} = W_{\text{aff}}$. The *affine symmetric group* is

$$\widehat{\mathfrak{S}_2} = \langle s_0, s : s_0^2 = s^2 = 1 \rangle,$$

and is isomorphic to $W_{\text{aff}}$. From above, we also have that $W_{\text{aff}} = \mathbb{Z}\alpha^\vee \oplus \mathfrak{S}_2$, where $\alpha^\vee$ is the coroot of the unique simple root $\alpha$. In particular, $\alpha \in \mathbb{R}^2$ as $\alpha = e_1 - e_2 = (1, -1)$.

We will now see how to embed $\widehat{\mathfrak{S}_2}$ inside $G$, and how we can view it as elements of $N_G(T)/T_{\mathcal{O}}$. It is customary to choose a matrix representative for the $\mathfrak{S}_2$ generator $s$ by

$$s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Using the isomorphism (1.1), we identify $\alpha^\vee$ with $\pi^{\alpha^\vee}$. One may pick the following representative for $\pi^{\alpha^\vee}$:

$$\pi^{\alpha^\vee} = \begin{pmatrix} \pi & \\ & \pi^{-1} \end{pmatrix}.$$

There is an isomorphism

$$\begin{aligned} \widehat{\mathfrak{S}_2} &\xrightarrow{\simeq} \mathbb{Z}\pi^{\alpha^\vee} \rtimes \mathfrak{S}_2, \\ s &\longmapsto s \\ s_0 &\longmapsto s\pi^{\alpha^\vee}. \end{aligned}$$

Thus, $s_0$ has representative:

$$s_0 = \begin{pmatrix} & \pi^{-1} \\ \pi & \end{pmatrix}.$$

It is easy to show that $s_0^2 = s^2 = 1$.

**The $\mathrm{SL}_n(F)$ Case**

Let us now consider $G = \mathrm{SL}_n(F)$. This is still simply-connected, with rank $n-1$. Its simple roots are given by $\alpha_i := e_i - e_{i+1}$, for $1 \le i \le n$. For each $i = 0, 1, \cdots, n-1$, there is an $\mathrm{SL}_2$ subgroup embedding [Mut18, (2.10)]:

$$\varphi_i : \mathrm{SL}_2 \longrightarrow G.$$

We define the simple reflections by:

$$s_i = \varphi_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, (1 \le i \le n-1), \quad s_0 = \varphi_i \begin{pmatrix} & \pi^{-1} \\ \pi & \end{pmatrix}, \quad \pi^{\alpha_i^\vee} = \varphi_i \begin{pmatrix} \pi & \\ & \pi^{-1} \end{pmatrix}$$

Explicitly, we lift Weyl group elements of $\mathrm{SL}_2(F)$ into $G$ in the following way:

$$s_i \longmapsto \begin{pmatrix} I_{i-1} & & & \\ & 0 & 1 & \\ & -1 & 0 & \\ & & & I_{n-i-1} \end{pmatrix}.$$

For the sake of decluttering notation, we will now write $s_i$ $(0 \le i \le n-1)$ for the simple reflections of the affine Weyl group of $\mathrm{SL}_n$. We realise the generator of the coroot lattice $\pi^{\alpha_i^\vee}$ by

$$\pi^{\alpha_i^\vee} = \begin{pmatrix} I_{i-1} & & & \\ & \pi & 0 & \\ & 0 & \pi^{-1} & \\ & & & I_{n-i-1} \end{pmatrix}.$$

The above isomorphism generalises in the following way [MS19, Equation 2.2]:

$$\begin{aligned} \widehat{\mathfrak{S}_n} &\xrightarrow{\simeq} \bigoplus_{i=1}^{n-1} \mathbb{Z}\alpha_i^\vee \rtimes \mathfrak{S}_n \\ s_i &\longmapsto s_i \quad (1 \le i \le n-1) \\ s_0 &\longmapsto s_{n-1} \cdots s_2 s_1 s_2 s_{n-1} \pi^{\alpha_1^\vee - \alpha_n^\vee}. \end{aligned}$$

Performing the computation, we find that $s_0$ has a matrix representative given by

$$s_0 = \begin{pmatrix} & & \pi^{-1} \\ & I_{n-2} & \\ (-1)^n\pi & & \end{pmatrix}.$$

It follows by a direct computation that our matrices $s_0, \cdots, s_{n-1}$, satisfy the relations

$$\begin{aligned} s_i^2 &= 1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, \\ s_i s_j &= s_j s_i \quad \text{if} \quad j \not\equiv i \pm 1 \pmod{n-1}. \end{aligned}$$

**The $\mathrm{GL}_2(F)$ Case**

Let us consider $G = \mathrm{GL}_2(F)$. Its maximal torus is given by

$$T = \begin{pmatrix} * & \\ & * \end{pmatrix} \cong (F^\times)^2,$$

and thus $G$ has rank 2. As before, it has two simple roots given by $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$. However, since $G$ is no longer simply connected, we must now contend with the extended affine Weyl group. Viewing $\widetilde{W}$ as the semidirect product $\Omega \rtimes \widehat{\mathfrak{S}_2}$, one obtains the following presentation:

$$\widetilde{W} = \left\langle \varpi, s_0, s_1 : \begin{aligned} s_0^2 = s_1^2 &= 1 \\ \varpi s_0 \varpi^{-1} &= s_1 \\ \varpi s_1 \varpi^{-1} &= s_0 \end{aligned} \right\rangle.$$

However, using our $p$-adic group structure one may realise it as the semi-direct product $\widetilde{W} = X_*(T) \rtimes \mathfrak{S}_2$. The co-weight lattice $X_*(T)$ is generated by $\pi^{\alpha_1^\vee}$, and $\pi^{\alpha_2^\vee}$. There is an isomorphism:

$$\Omega \rtimes \widehat{\mathfrak{S}_2} \cong X_*(T) \rtimes \mathfrak{S}_2. \tag{1.3}$$

We shall describe this explicitly by assigning matrix representatives to each element. In $\mathrm{GL}_2$, it is customary to choose $s$ as

$$s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The representatives for the generators of $X_*(T)$ are chosen as

$$\pi^{\alpha_1^\vee} = \begin{pmatrix} \pi & \\ & 1 \end{pmatrix}, \quad \pi^{\alpha_2^\vee} = \begin{pmatrix} 1 & \\ & \pi \end{pmatrix}.$$

Observe that

$$\begin{pmatrix} \pi & \\ & \pi^{-1} \end{pmatrix} = \pi^{\alpha_1^\vee - \alpha_2^\vee}.$$

Using our choice of matrix representatives, it is easy to check that

$$\begin{aligned} s^2 = (s\pi^{\alpha_1^\vee - \alpha_2^\vee})^2 &= 1, \\ (s\pi^{\alpha_1^\vee})s(s\pi^{\alpha_1^\vee})^{-1} &= s\pi^{\alpha_1^\vee - \alpha_2^\vee}. \\ (s\pi^{\alpha_1^\vee})s\pi^{\alpha_1^\vee - \alpha_2^\vee}(s\pi^{\alpha_1^\vee})^{-1} &= s. \end{aligned}$$

Thus, the isomorphism is defined explicitly by

$$\begin{aligned} \langle \varpi, s_0, s_1 \rangle &\xrightarrow{\simeq} \langle \pi^{\alpha_1^\vee}, \pi^{\alpha_2^\vee}, s \rangle \\ s_1 &\longmapsto s \\ s_0 &\longmapsto s\pi^{\alpha_1^\vee - \alpha_2^\vee} \\ \varpi &\longmapsto s\pi^{\alpha_1^\vee}. \end{aligned}$$

The generator $\varpi$ therefore has a matrix representative given by

$$\varpi = \begin{pmatrix} & 1 \\ \pi & \end{pmatrix}.$$

**The $\mathrm{GL}_n(F)$ Case**

For the general case in which $G = \mathrm{GL}_n(F)$, the above isomorphism is given by [MS19, Equation 2.2]:

$$\begin{aligned} \Omega \rtimes \widehat{\mathfrak{S}_n} &\xrightarrow{\simeq} \bigoplus_{i=1}^{n} \mathbb{Z}\pi^{\alpha_i^\vee} \rtimes \mathfrak{S}_n \\ s_i &\longmapsto s_i \quad (1 \leq i \leq n) \\ s_0 &\longmapsto s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1}\pi^{\alpha_1^\vee - \alpha_n^\vee} \\ \varpi &\longmapsto s_{n-1} \cdots s_1 \pi^{\alpha_1^\vee}. \end{aligned}$$

Using our map $\varphi_i$, we may also lift elements of $\Omega \rtimes \widehat{\mathfrak{S}_2}$ into $\mathrm{GL}_n$. We choose our representative for $\pi^{\alpha_i^\vee}$ as a diagonal matrix with entry $\pi$ at the $(i,i)$ position, and 1's everywhere else, following the convention of [MS19, §2.1]. Consequently, we obtain (for $1 \leq i \leq n$):

$$s_i = \begin{pmatrix} I_i & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I_{n-i} \end{pmatrix}, \quad s_0 = \begin{pmatrix} & & \pi^{-1} \\ & I_{n-1} & \\ (-1)^{n+1}\pi & & \end{pmatrix}.$$

From the isomorphism, we see that a suitable choice of matrix representative for $\varpi$ is the matrix with $1$'s on the superdiagonal, $\pi$ is on the lower-leftmost entry, and $0$'s everywhere else. That is,

$$\varpi = \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ \pi & & & \end{pmatrix}.$$

## 1.4 Haar Measures

Since the Hecke algebra is a convolution algebra, we wish to impose some notion of integration on a topological group — specifically, a locally compact group. To this end, we introduce the notion of a measure, following [Rud86]. When we begin discussing Haar measures, we will be using [BH06, Chapter 1.3] and [Fol95, Chapter 2.2].

Throughout, we will fix a topological group $G$. However, we will first ignore its group structure and look at it as a set. The *$\sigma$-algebra* of $G$ is the collection of subsets of $G$ (including $G$ itself) that are closed under set-complements, and closed under countable unions [Rud86, Definition 1.2]. The $\sigma$-algebra generated by the open sets of $G$ is called the *Borel $\sigma$-algebra*.

We call elements of a $\sigma$-algebra *measurable sets* [Rud86, Definition 1.3]. Let $\mathfrak{M}$ be a $\sigma$-algebra. The function $\mu : \mathfrak{M} \to [0, \infty]$ is a *measure* if $\mu(\emptyset) = 0$, and for disjoint sets $M_1, M_2, \cdots \in \mathfrak{M}$, the following holds:

$$\mu\left(\bigsqcup_{i=1}^{\infty} M_i\right) = \sum_{i=1}^{\infty} \mu(M_i).$$

Consequently, for any $N_1, N_2, \cdots \in \mathfrak{M}$ that are not necessarily disjoint,

$$\mu\left(\bigcup_{i=1}^{\infty} N_i\right) \leq \sum_{i=1}^{\infty} \mu(N_i).$$

Going forward, we now require the group structure of $G$.

**Definition 1.16.** A *left* (respectively, *right*) *Haar measure* $\mu$ on $G$ is a non-zero measure that is:

(i) (Left (resp. right) translation invariance) For any $g \in G$ and some measurable set $S \in \mathfrak{M}$, $\mu(gS) = \mu(S)$ (respectively, $\mu(Sg) = \mu(S)$).

(ii) For any compact $K \subseteq G$, $\mu(K)$ is finite.

Indeed, such a measure always exists on a locally compact group $G$:

**Theorem 1.17** (Fol95, Theorem 2.10)**.** *Every locally compact group $G$ possesses a left (respectively, right) Haar measure $\mu$.*

Further, Haar measures are unique up to scalar multiples:

**Theorem 1.18** (Fol95, Theorem 2.20 or BH06, Proposition 3.1)**.** *Let $\mu$ and $\lambda$ be left (respectively, right) Haar measures on $G$. Then, there exists some constant $c > 0$ such that $\mu = c\lambda$.*

**Definition 1.19.** A group $G$ is *unimodular* if any left Haar measure is also a right Haar measure.

**Definition 1.20.** Given a left Haar measure $\mu$ on $G$, the *modular character of $G$* is a homomorphism $\delta_G : G \to \mathbb{R}_{\geq 0}$ defined by

$$\delta_G(x) := \frac{\mu(Sx)}{\mu(S)},$$

where $S$ is any Borel set with finite non-zero measure.

Indeed, the value of $\delta_G(x)$ is independent of our choice of Borel set. For any Borel set $S$, and $x \in G$, the measure $\mu(Sx)$ is left-invariant. Thus, by uniqueness (Theorem 1.18), there exists some constant $c > 0$ (independent of $S$) for which $\mu(Sx) = c\mu(S)$.

Further, it follows as an immediate consequence of our definition that a group $G$ is unimodular if and only if its modular character is trivial. It thus follows that all abelian groups are unimodular.

In fact, all reductive groups are unimodular as well:

**Proposition 1.21** (Fol95, Proposition 2.29)**.** *If $G/[G, G]$ is compact, then $G$ is unimodular.*

**Corollary 1.22.** *Any reductive group $G$ is unimodular.*

**Example 1.23** (Haar Measure on $F$)**.** Let $F$ be a finite-degree extension of $\mathbb{Q}_p$. Denote by $\mathcal{O}$ its discrete valuation ring, and $\pi$ the uniformiser. The residue field is finite with cardinality $q$ — that is, $\mathcal{O}/\pi\mathcal{O} \cong \mathbb{F}_q$.
Since $F^\times$ is locally compact and abelian, it admits a unimodular Haar measure, which we denote as $\mu$. We normalise the Haar measure such that $\mu(\mathcal{O}) = 1$. We further normalise the absolute value so that $|\pi| = q^{-1}$. Since $\mathcal{O} = \bigsqcup_{x \in \mathcal{O}/\pi\mathcal{O}} (x + \pi\mathcal{O})$,

$$1 = \mu(\mathcal{O}) = \sum_{x \in \mathcal{O}/\pi\mathcal{O}} \mu(x + \pi\mathcal{O}) = q\mu(\pi\mathcal{O}),$$

and thus $\mu(\pi\mathcal{O}) = q^{-1} = |\pi|$. Further, since $\pi\mathcal{O}$ is the disjoint union of $x\pi + \pi^2\mathcal{O}$, we employ the same technique to show that $\mu(\pi^2\mathcal{O}) = q^{-2} = |\pi^2|$. Proceeding inductively, we obtain

$$\mu(\pi^j\mathcal{O}) = q^{-j} = |\pi^j|. \tag{1.4}$$

Further, observe that $\mathcal{O}^\times = \mathcal{O}\setminus\pi\mathcal{O}$, and so $\mu(\mathcal{O}^\times) = \mu(\mathcal{O}) - \mu(\pi\mathcal{O}) = 1 - q^{-1}$. The modular character $\delta_F : F^\times \to \mathbb{R}_{\geq 0}$ is given by

$$\delta_F(x) = \frac{\mu(x^{-1}A)}{\mu(A)},$$

where $A$ is some Borel set with non-zero finite measure, and $x \in F^\times$. Since $\mu(\mathcal{O}) \neq 0$, set $A = \mathcal{O}$. Then, $x$ must be of the form $x = \pi^{-j}$ for some $j \geq 0$. It follows then that the modular character is nothing but the absolute value we imposed on $F$ — that is, $\delta_F = |\cdot|$. Throughout this thesis, when we speak of a measure on $F$ we are referring to the one defined above.

**Example 1.24.** Let $G$ be a split, connected, reductive group over a non-Archimedean local field $F$ with discrete valuation ring $\mathcal{O}$ and unique maximal ideal generated by a uniformiser $\pi$.
Given Example 1.23, we can impose Haar measures on some important subgroups of $G$. If $G$ has rank $r$, then the maximal torus $T$ is isomorphic to $(F^\times)^r$. Thus, we may identify the measure on $T$ as a product measure of the measure we imposed on $F^\times$.
Recall the root subgroup decomposition of the unipotent subgroup $N$ (Theorem 1.12(i)):

$$N \cong \prod_{\alpha\in\Phi^+} U_\alpha,$$

where each root subgroup $U_\alpha$ is isomorphic to $F^\times$. Thus, we may identify the measure on $N$ with the measure on $(F^\times)^{|\Phi^+|}$.

**Example 1.25** (Haar Measure on $\mathrm{GL}_2(k)$)**.** Let $k$ be an algebraically closed field of characteristic zero. We will define a Haar measure on $\mathrm{GL}_2(k)$, which is identified as an open subset of $k^4$ by identifying $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ with $(x_{11}, x_{12}, x_{21}, x_{22})$. Then, for any Borel subset $H$ of $\mathrm{GL}_2(k)$ we define

$$\mu(H) = \int_H \frac{1}{(x_{11}x_{22} - x_{12}x_{22})^2}\,\mathrm{d}x_{11}\,\mathrm{d}x_{12}\,\mathrm{d}x_{21}\,\mathrm{d}x_{22} = \int_H \frac{1}{(\det x)^2}\,\mathrm{d}x.$$

Indeed, this measure we have imposed is unimodular. Let $T : k^4 \to k^4$ be given by

$Tu = gu$. Then,

$$\begin{aligned}\mu(gH) &= \int_{gH} \frac{1}{(\det x)^2}\,\mathrm{d}x \overset{x\mapsto gu}{=} \int_H \frac{1}{(\det(gu))^2}\,\mathrm{Jac}(T)\,\mathrm{d}u \\ &= \int_H \frac{1}{(\det g)^2(\det u)^2}(\det g)^2\,\mathrm{d}u = \int_H \frac{1}{(\det u)^2}\,\mathrm{d}u = \mu(H).\end{aligned}$$

where $\mathrm{Jac}(T)$ is the usual Jacobian on $k^4$.

# Chapter 2

# The Iwahori-Hecke Algebra

Throughout, we let $G$ be a split, connected, reductive group over a local non-Archimidean local field $F$, with discrete valuation ring $\mathcal{O}$, and uniformiser $\pi$. Let $\mathcal{O}/\pi\mathcal{O}$ be the residue field with cardinality $q$ — that is, $\mathcal{O}/\pi\mathcal{O} \cong \mathbb{F}_q$. We let $T$ be its maximal torus, and $N$ the unipotent radical of the Borel subgroup $B$. Let $K := G(\mathcal{O})$ be the maximal compact subgroup of $G$.

Let $\widetilde{W} = X_*(T) \rtimes W$ be the extended affine Weyl group. $X_*(T)$ is the cocharacter lattice, and $W$ is the finite Weyl group, generated by simple reflections $\langle s_1, \cdots, s_r \rangle$, where $r$ is the rank of $G$. The root system $\Phi$ is embedded in the space $X^*(T) \otimes \mathbb{R}$, which is equipped with a $W$-invariant inner product $\langle \cdot, \cdot \rangle$.

Within a $p$-adic group, a subgroup that will be studied extensively is the following:

**Definition 2.1.** Let $\varphi : K \to G(\mathbb{F}_q)$. The *Iwahori subgroup* is given by

$$I := \varphi^{-1}(B(\mathbb{F}_q)).$$

**Example 2.2** (Rank $n$ Iwahori Subgroup)**.** We determine the Iwahori subgroup $I$ for $G = \mathrm{GL}_n(F)$, and let $B$ be the Borel subgroup (chosen as the subgroup of upper-triangular matrices). Let $\varphi : K \to G(\mathbb{F}_q)$ denote the canonical map. Explicitly, for some matrix $(g_{ij})_{1 \leq i,j \leq n} \in K$, the map $\varphi$ sends each entry $g_{ij} \in \mathcal{O}$ to its image in $\mathcal{O}/\pi\mathcal{O} \cong \mathbb{F}_q$.

Recall that $\mathcal{O} = \varprojlim_{n \geq 1} \mathcal{O}/\pi^n\mathcal{O}$. Thus, we may view elements of $\mathcal{O}$ as coordinates $x := (a_1, a_2, a_3, \cdots)$ such that $a_i \in \mathcal{O}/\pi^i\mathcal{O}$ for each $i$, and the bonding maps are given by $f_{ij} : \mathcal{O}/\pi^j\mathcal{O} \to \mathcal{O}/\pi^i\mathcal{O}$. $(a_1, a_2, a_3, \cdots) = (0, 0, 0, \cdots)$ if and only if $x \in \pi\mathcal{O}$. Thus, the opposite portion of $I$ consists of elements in $\pi\mathcal{O}$. The diagonal entries of $B_{\mathbb{F}_q}$ must be non-zero for it to be invertible — that is, the image of the diagonal entries under $\varphi^{-1}$ are elements of $\mathcal{O} \setminus \pi\mathcal{O} = \mathcal{O}^\times$. The upper-triangular elements can be anything in $\mathcal{O}$. Thus,

$$I = \begin{pmatrix} \mathcal{O}^\times & \mathcal{O} & \cdots & \mathcal{O} \\ \pi\mathcal{O} & \mathcal{O}^\times & \ddots & \mathcal{O} \\ \vdots & \ddots & \ddots & \vdots \\ \pi\mathcal{O} & \cdots & \pi\mathcal{O} & \mathcal{O}^\times \end{pmatrix}.$$

Observe that $I$ contains $K(1) = \{g \in K : g \equiv 1 \pmod{\pi\mathcal{O}}\}$. $K(1)$ is open, and $I$ contains it as a subgroup, and thus $I$ is open. Further, $I$ is compact (see below). We can decompose the Iwahori subgroup in the following way:

**Theorem 2.3** (Iwahori Factorisation [Bum10], Proposition 50)**.** *We denote by $\overline{N}_\pi$ the $\pi\mathcal{O}$-points of $\overline{N}$. Then,*

$$I = (I \cap N)(I \cap T)(I \cap \overline{N}) = N_{\mathcal{O}} T_{\mathcal{O}} \overline{N}_\pi.$$

We say that $\mu \in X_*(T)$ is *dominant* $\langle \mu, \alpha \rangle \geq 0$ for any $\alpha \in \Phi^+$. $\mu$ is *antidominant* if $-\mu$ is dominant. If $\mu$ is dominant, then the following holds:

$$\pi^\mu (I \cap N) \pi^{-\mu} \subset I \cap N. \tag{2.1}$$

Indeed, for some $n_{\mathcal{O}} \in N_{\mathcal{O}}$ and $t \in \mathcal{O}$,

$$\pi^\mu n_{\mathcal{O}} \pi^{-\mu} = \pi^\mu \left( \prod_{\alpha \in \Phi^+} x_\alpha(t) \right) \pi^{-\mu} = \prod_{\alpha \in \Phi^+} x_\alpha(t \pi^{\langle \mu, \alpha \rangle}),$$

which is in $I \cap N = N_{\mathcal{O}}$ if and only if $\langle \mu, \alpha \rangle \geq 0$ for all $\alpha \in \Phi^+$ (otherwise, it is in $I \cap \overline{N}$).

With the above in mind, we may now define the main object of interest:

**Definition 2.4.** The *Iwahori-Hecke algebra* $\mathcal{H}$ is the space of compactly supported, $I$-bi-invariant functions on $G$ — that is, $\mathcal{H} := C_c(I \backslash G / I)$. It has an algebra structure given by the convolution product:

$$(\phi * \psi)(x) = \int_G \phi(xg^{-1}) \psi(g) \, \mathrm{d}g,$$

where the Haar measure is normalised to give $I$ measure one.

Our goal is to study the structure of $\mathcal{H}$. That is, we wish to give a basis for $\mathcal{H}$. This is obtained in the next section as we discuss the *Iwahori-Bruhat decomposition*. And given such a set of generators of $\mathcal{H}$, we may produce a presentation of $\mathcal{H}$ known as the *Iwahori-Matsumoto presentation*.

## 2.1 The Iwahori Decomposition and its Consequences

In this section, we present many group decomposition facts that will be used extensively throughout the thesis. The proofs of some of these results are quite complicated, and we are thus forced to refer the reader to other sources.

We begin with a historical interlude. In the study of Lie groups, an important result that arose was the *Bruhat decomposition*. Let $B$ be a Borel subgroup of $G$, $T$ the maximal torus, and $W = N_G(T)/T$ the finite Weyl group, and $S$ the set of simple reflections generating $W$. Then,

$$G = \bigsqcup_{w \in W} BwB.$$

This result originated in Ehresmann's work in [Ehr34], but was later generalised by Bruhat in [Bru54]. This decomposition follows from the fact that the triple $(B, N_G(T), S)$ satisfies the axioms of a *Tits system*, which is a triple $(I, \mathcal{N}, S)$ satisfying [CKM98]:

**(TS1)** $T_0 := I \cap \mathcal{N}$ is a normal subgroup of $\mathcal{N}$.

**(TS2)** There exists a Coxeter group $W$ generated by a set of simple reflections $S$, a group $\Omega$, and an isomorphism $\mathcal{N}/T_0 \cong W \rtimes \Omega$.

**(TS3)** The following hold for elements of $S$:

(a) For $w \in W \rtimes \Omega$, and $s \in S$, $wIs \subset IwsI \sqcup IwI$.

(b) For all $s \in S$, $sIs^{-1} \neq I$

**(TS4)** For $\varpi \in \Omega$, $\varpi S \varpi^{-1} = S$, and $\varpi I \varpi^{-1} = I$

**(TS5)** $G$ is generated by $I$ and $\mathcal{N}$.

*Remark* 2.5. Other sources in the literature refer to such a system as a $(I, \mathcal{N})$ *pair*.

This decomposition was then generalised to any connected, reductive algebraic group by Chevalley in [Che55].

In [IM65], Iwahori and Matsumoto proved the following:

**Theorem 2.6** (Iwahori-Bruhat Decomposition, [IM65], Theorem 2.24)**.**

$$G = \bigsqcup_{w \in \widetilde{W}} IwI. \tag{2.2}$$

Here, $I$ is the Iwahori subgroup, $\mathcal{N}$ is the normaliser of the torus $N_G(T)$. $T_0 = I \cap N_G(T) = T_{\mathcal{O}}$, and thus $\widetilde{W} = N_G(T)/T_{\mathcal{O}}$. Here, $\Omega$ is the subgroup of Dynkin diagram automorphisms, and $W$ is the affine Weyl group. $S$ is the set of simple reflections of $W_{\text{aff}}$, given by $S = \{s_0, \cdots, s_r\}$, where $s_0$ is the simple root obtained after extending $W$ by the coroot lattice [Kir97, Theorem 3.1.1].

We refer the reader to [IM65] for the details of this proof. A proof of axiom **(TS3)**(a) can be found in [Bum10, §14, pg. 76-82] in the case for which $\widetilde{W}_{\text{aff}} = W_{\text{aff}}$.

Given this Iwahori-Bruhat decomposition, we obtain a basis of indicator functions for $\mathcal{H}$. But first, we prove the lemma:

**Lemma 2.7.** *The Iwahori subgroup $I$ is compact.*

*Proof.* Using the Iwahori decomposition for $G$, we can also find an Iwahori decomposition for our maximal compact $K$. Taking the Bruhat decomposition over the residue field and pulling back by the map $\varphi : K \to G(\mathbb{F}_q)$, we see that

$$K = \bigsqcup_{w \in W} IwI.$$

This implies that $K \setminus I = \coprod_{w \in W \setminus \{1_W\}} IwI$ is open, and $I$ is a closed subset of $K$. As a consequence, $I$ is compact.

□

### 2.1.1 The Iwahori-Matsumoto Presentation

The Iwahori-Bruhat decomposition allows us to produce a basis for $\mathcal{H}$:

**Proposition 2.8.** *The elements $T_w = \mathbb{1}_{IwI}$, for $w \in \widetilde{W}$, form a $\mathbb{C}$-basis of $\mathcal{H}$.*

*Proof.* To begin, we verify that $T_w$ has compact support for each $w \in \widetilde{W}$. We view $T_w$ as functions $T_w : G \to \mathbb{C}$, and thus $\operatorname{supp} T_w = IwI$. $I$ is compact by Lemma 2.7, and thus the support of the identity element is compact. Multiplication induces a map

$$I \times \{w\} \times I \longrightarrow G.$$

Since $\{w\}$ is trivially compact, it follows by Tychonoff's theorem that $\operatorname{supp} T_w$ is compact for each $w \in \widetilde{W}$.

Consider some $f \in \mathcal{H}$ of the form $f = \sum\limits_{w \in \widetilde{W}} c_w \mathbb{1}_{IwI}$. We wish to show that this is a finite linear combination — that is, all but finitely many $c_w = 0$. This is equivalent to showing that

$$\operatorname{supp} f = \bigcup_{\substack{w \in \widetilde{W} \\ c_w \neq 0}} IwI$$

is not compact. But by the Iwahori-Bruhat decomposition, the double cosets $IwI$ are disjoint, and thus there exists no finite subcover. As such, if $f$ is not a finite-linear combination, then $f \notin \mathcal{H}$. □

A presentation for $\mathcal{H}$ in terms of this $\mathbb{C}$-basis is the so-called *Iwahori-Matsumoto presentation* of $\mathcal{H}$, first established in [IM65]. It is as follows:

**Theorem 2.9** (Iwahori-Matsumoto Presentation). *Let $s_i$ $(0 \leq i \leq r)$, where $r$ is the rank of $G$, and let $w, w' \in \widetilde{W}$. Then,*

*(i)* $T_{s_i}^2 = (q-1)T_{s_i} + qT_1$,

*(ii)* $T_w T_{w'} = T_{ww'}$ *if* $\ell(ww') = \ell(w) + \ell(w')$.

This result was first proved in [IM65, Theorem 3.3, Theorem 3.5]. The entirety of [IM65, Section 3] is dedicated to proving this result. Theorem 2.9(i) is proved in [Bum10, Theorem 21, Corollary 1], and Theorem 2.9(ii) is proved in [Bum10, Proposition 55], both for the simply-connected case.

The presentation in Iwahori and Matsumoto's paper takes a different form to how we have presented it here, but it is equivalent. More precisely, [IM65, Theorem 3.3] states that $\mathcal{H}$ is generated by the elements $T_\varpi$ $(\varpi \in \Omega)$, $T_{s_0}, \cdots, T_{s_r}$ subject to the condition that for some $\varpi \in \Omega$ and $w = s_{i_1} \cdots s_{i_k}$ a reduced decomposition:

$$T_{\varpi w} = T_\varpi T_{s_{i_1}} \cdots T_{s_{i_k}}.$$

[IM65, Theorem 3.5] gives explicit relations for the generators $T_{s_0}, \cdots, T_{s_r}$ depending on the angles between the simple roots in the affine Dynkin diagram (see [IM65, pg. 18-19]).

In the presentation we recorded, the length function accounts for length-zero elements, and the Iwahori-Matsumoto presentation seen in [Bum10, Section 15] takes this form.

## 2.2 The Bruhat Order and Double Coset Intersections

The *Bruhat order* is a partial order that can be imposed on any Coxeter group. However, this notion can be generalised to $\widetilde{W}$ in the non-simply-connected case, as we shall soon see.

We define it first for the finite Weyl group generated by simple reflections $\{s_1, \cdots, s_r\}$, following [Bum10, Section 17].

**Definition 2.10.** Let $u, v \in W$ and $v = s_{i_1} \cdots s_{i_k}$ be a reduced decomposition. Then $u \leq v$ if and only if $u$ is a subword of $v$ — that is, $u = s_{j_1} \cdots s_{j_k}$ for some subsequence $\{j_1, \cdots, j_k\} \subset \{i_1, \cdots, i_\ell\}$. if and only if there is a subsequence $\{j_1, \cdots j_\ell\}$ of $\{i_1, \cdots, i_k\}$ such that

$$u = s_{i_1} \cdots \widehat{s_{j_1}} \cdots \widehat{s_{j_\ell}} \cdots s_{i_k},$$

where elements with the hats above it denote its omission.

[Bum10, Lemma 22, 23] shows that the Bruhat order does not depend on our choice of reduced decomposition. In the simply-connected case, one may additionally define a Bruhat order for affine Weyl groups in exactly the same way.

For a Bruhat order that generalises to non-Coxeter groups, we use [BKP14, Definition B.2]:

**Definition 2.11.** Let $\alpha$ be a root, and $s_\alpha$ the simple reflection corresponding to $\alpha$. For some $w, ws_\alpha \in \widetilde{W}$, $w \leq ws_\alpha$ if $w(\alpha)$ is positive and $w \geq w(s_\alpha)$ otherwise.

Due to the presence of length zero elements in $\widetilde{W}$ (in the non-simply-connected case) the length of an element $w \in \widetilde{W}$ cannot be defined as the length of its reduced decomposition. However, [Kir97, Lemma 3.2] records a result on how one can compute the length function:

**Lemma 2.12.**

*(i)* $\ell(\varpi w) = \ell(w)$ *for* $\varpi \in \Omega$

*(ii)* $\ell(ws_i) = \begin{cases} \ell(w) + 1 \text{ if } w(\alpha_i) \in \Phi^+, \\ \ell(w) - 1 \text{ if } w(\alpha_i) \in \Phi^-. \end{cases}$

*(iii) For* $w \in W$, *and* $\mu \in X_*(T)$,

$$\ell(w(\mu)) = \sum_{\alpha \in \Phi^+} |\langle \mu, \alpha \rangle + \chi(w(\alpha))|,$$

*where* $\chi(\alpha) = 0$ *if* $\alpha \in \Phi^+$, *and* $\chi(\alpha) = 1$ *if* $\alpha \in \Phi^-$.

We will need the following result:

**Lemma 2.13** (Deo77, Theorem 1.1, Property $Z(s, w_1, w_2)$)**.** *Let* $(W, S)$ *be a Coxeter group. Then, for* $w_1, w_2 \in W$ *and* $s \in S$ *such that* $\ell(w_1) \leq \ell(w_1 s)$ *and* $\ell(w_2) \leq \ell(w_2 s)$, *one has* $w_1 \leq w_2 \iff w_2 \leq w_1 s \iff w_2 s \leq w_1 s$.

**Proposition 2.14** (HKP10, Claim 1.6.1)**.** *Let $w, x \in \widetilde{W}$. Then, if $IwI \cap NxI \neq \emptyset$, then $x \leq w$ in the Bruhat order.*

*Proof.* Suppose that $nx \in IwI$, and choose $\mu$ so dominant that $\pi^\mu n \pi^{-\mu} \in I$. Then,

$$(\pi^\mu n \pi^{-\mu})\pi^\mu x \in \pi^\mu IwI = I\pi^\mu xI,$$

and so $I\pi^\mu xI \subset I\pi^\mu IwI$. We now claim that

$$I\pi^\mu IwI \subseteq \bigsqcup_{y \leq w} I\pi^\mu yI, \tag{2.3}$$

which will thus imply that $x \leq \widetilde{w}$ in the Bruhat order.

Viewing $\widetilde{W}$ as the semidirect product $\Omega \rtimes W_{\text{aff}}$, a general element of the extended affine Weyl group has the form $\widetilde{w} = \varpi w$, for $\varpi \in \Omega$ and $w \in W_{\text{aff}}$. We first prove (2.3) for the affine Weyl group. If $w = 1$, then

$$\pi^\mu I \overset{\textbf{(TS4)}}{=} \pi^\mu \varpi I \overset{\textbf{(TS3)(a)}}{\subseteq} I\pi^\mu I.$$

Let $w = s$. Then,

$$\pi^\mu I \varpi s \overset{\textbf{(TS4)}}{=} \pi^\mu \varpi I s \overset{\textbf{(TS3)(a)}}{\subseteq} I\pi^\mu \varpi sI \sqcup I\pi^\mu \varpi I = I\pi^\mu \widetilde{w} I \sqcup I\pi^\mu I.$$

Suppose now that $I\pi^\mu I\widetilde{w}I \subseteq \bigsqcup_{y \leq \widetilde{w}} I\pi^\mu yI$, where $w_{k-1} = s_{i_1} \cdots s_{i_{k-1}}$ is a reduced decomposition in $W_{\text{aff}}$. Let $w_k = w_{k-1}s_{i_k}$. By Lemma 2.13,

$$\begin{aligned}
\pi^\mu I w_{k-1} s_{i_k} &\subseteq \left( \bigsqcup_{y \leq w_{k-1}} I\pi^\mu yI \right) s_{i_k} \\
&\subseteq \bigsqcup_{y \leq w_{k-1}} (I\pi^\mu yI \sqcup I\pi^\mu y s_{i_k} I) \\
&= \bigsqcup_{y \leq w_k} I\pi^\mu yI.
\end{aligned}$$

and therefore the result is true for $W_{\text{aff}}$. However, one can extend this result to $\widetilde{W}$ by replacing $w$ with $\varpi w$, for $\varpi \in \Omega$ and $w \in W_{\text{aff}}$ and using **(TS4)**. The results of Lemma 2.13 can be extended to the non-Coxeter case by Lemma 2.12(i). □

# Chapter 3

# Principal Series Representations

Following [HKP10, Section 1.4-1.6], we study the *universal principal series* module of $\mathcal{H}$, given by $\mathcal{M} := C_c(T_{\mathcal{O}}N\backslash G/I)$. We call $\mathcal{M}$ as such since it arises as the $I$-fixed vectors of the universal principal series of $G$ (see Chapter 3.2). Much of our exposition about principal series representations of $G$ come from [BH06].

We begin this chapter by giving an introduction to the idea of principal series representations. Then, we study the universal principal series module $\mathcal{M}$ of $\mathcal{H}$ directly, and derive some important properties of it — namely, the fact that it is a $(\mathcal{R}, \mathcal{H})$-bimodule.

Then, in the following section we will explicitly construct the universal principal series representations of $G$, from which we recover $\mathcal{M}$ by taking its $I$-fixed vectors.

In the following section, we will then turn out attention back to studying $\mathcal{M}$, and prove that it is a free, rank one $\mathcal{H}$-module.

We then conclude the chapter by applying everything that we have proved about $\mathcal{M}$ to embed $\mathcal{R}$ into $\mathcal{H}$, and produce a way to write it out in terms of the generators of $\mathcal{H}$. The key result is the isomorphism $\mathcal{R} \otimes_{\mathbb{C}} \mathcal{H}_{\text{fin}} \cong \mathcal{H}$, which is the content of [HKP10, Lemma 1.7.1]. We provide details of the proof in Theorem 3.16.

## 3.1 Principal Series Representations

**Definition 3.1.** A $G$-representation $(\rho, V)$ is *smooth* if for every $v \in V$, there is a compact open subgroup $K$ of $G$ (depending on $V$) such that $\rho(K)v = v$.

We define by $V^K$ the space of vectors in $V$ that are fixed by $\rho(K)$. That is,

$$V^K := \{v \in V : \rho(K)v = v\}.$$

We call these the *space of $K$-fixed vectors* of $V$.

**Definition 3.2.** A smooth $G$-representation is *admissible* if $V^K$ is finite-dimensional for each compact open subgroup $K$ of $G$.

Given two $G$-representations $(\rho_1, V_1)$ and $(\rho_2, V_2)$, the set $\mathrm{Hom}_G(\rho_1, \rho_2)$ is the space of intertwiners $f : V_1 \to V_2$ — that is, functions $f$ that commutes with the $G$-action:

$$f \circ \rho_1(g) = \rho_2(g) \circ f, \quad g \in G. \tag{3.1}$$

A morphism of $G$-modules is a map $f : V_1 \to V_2$ such that $f$ is a group homomorphism and $f(gv) = gf(v)$ for $g \in G$, $v \in V$. Two $G$-representations are *equivalent* if $f : V_1 \to V_2$ is a $G$-module isomorphism, and (3.1) holds. We denote the category of smooth representations by $\mathbf{Rep}(G)$.

A *multiplicative character* $\chi$ of $G$ is a group homomorphism $G \to \mathbb{C}^\times$. Since $\mathrm{Aut}_{\mathbb{C}}(\mathbb{C}) \cong \mathbb{C}^\times$, we may view any multiplicative character of $G$ as a one-dimensional representation.

*Remark* 3.3. Note that the character does not necessarily have to map into $\mathbb{C}^\times$. For instance, this thesis will consider characters mapping into $\mathcal{R}^\times$.

Given some smooth, admissible representation $(\sigma, V)$ of $G$, and an Iwahori subgroup $I$ of $G$, we can give $V^I$ the structure of an $\mathcal{H}$-module. For some $\phi \in \mathcal{H}$, $v \in V^I$, define the action:

$$\phi \cdot v = \sigma(\phi)v := \int_G \phi(g)\sigma(g)v \, \mathrm{d}g.$$

The functor $V \mapsto V^I$ defines a functor from $\mathbf{Rep}(G)$ to the category of finite-dimensional $\mathcal{H}$-modules. [Bor76, Theorem 4.10] showed that there is an exact functor from smooth, admissible $G$-modules to finite-dimensional $\mathcal{H}$-modules taking $V \mapsto V^I$. That is, we do not lose any information about the representation of $G$ by taking its $I$-fixed vectors.

### 3.1.1 Principal Series Representations

We are following [BH06, Chapter 1.2] for our definitions of various induced representations.

Let $H$ be a subgroup of $G$, and $(\sigma, W)$ a representation of $H$. Then, the $G$-representation *smoothly induced* from $H$ is:

$$\mathrm{Ind}_H^G \sigma := \{f : G \to W : f(hg) = \sigma(h)f(g)\}.$$

The map $\sigma \mapsto \operatorname{Ind}_H^G \sigma$ defines an additive and exact functor $\mathbf{Rep}(H) \to \mathbf{Rep}(G)$ [BH06, Proposition 2.4]. Here, additive here means that for any two representations $\rho, \pi$: we have that $\text{n-Ind}_H^G(\rho \oplus \pi) = \text{n-Ind}_H^G \rho \oplus \text{n-Ind}_H^G \pi$, and exact means that short exact sequences are preserved.

**Definition 3.4.** The $G$-representation *compactly induced from $H$* is given by

$$\text{c-Ind}_H^G \rho = \{f : G \to V : f(hg) = \rho(h)f(g), \text{ and } f \text{ has compact support modulo } H\}.$$

**Definition 3.5.** The $G$-representation *induced normally from $H$*, which is defined as

$$\text{n-Ind}_H^G \rho = \{f : G \to V : f(hg) = \delta_G(h)^{-\frac{1}{2}}\delta_H(h)^{\frac{1}{2}}\rho(h)f(g)\},$$

where $\delta_G$ and $\delta_H$ are the modular characters of $G$ and $H$, respectively.

A *principal series representation* refers to families of induced representations of the form $\text{n-Ind}_P^G$, where $P$ is a *parabolic subgroup*, a generalisation of the Borel subgroup. In our case, we will consider induction from the Borel subgroup $B$ to $G$.

A character $\chi : F^\times \to \mathbb{C}^\times$ is *unramified* if it is trivial on $\mathcal{O}^\times$. Since $T \cong (F^\times)^r$, where $r$ is the rank of $G$, we say that a character $\chi_T : T \to \mathbb{C}^\times$ is *unramified* if it is trivial on $T_\mathcal{O}$. That is, $\mathcal{R} = C_c(T/T_\mathcal{O})$ is then the group algebra of all unramified characters.

We may identify $\chi_T$ as a $B$-representation by the decomposition $B = TN$. A *universal* (or *tautological*) character is one of the form [HKP10, Section 1.5]:

$$\chi_{\text{univ}} : T/T_\mathcal{O} \longrightarrow \mathcal{R}^\times, \quad \pi^\mu \longmapsto \pi^\mu,$$

yields the *universal* principal series representation of $G$ [Cas80, Section 2]:

$$\text{n-Ind}_B^G \chi_{\text{univ}}. \tag{3.2}$$

Note that functions in the universal principal series are $\mathcal{R}$-valued, and not $\mathbb{C}$-valued. Throughout, we will typically assume that all our principal series representations are unramified, and simply say universal principal series representation.

## 3.2 Universal Principal Series Module of $\mathcal{H}$

Before we define the universal principal series module of $\mathcal{H}$, we will need the result below. [HKP10, Section 1.1] proves this, but here, we elaborate a bit more on the details.

**Proposition 3.6.** *There is a bijection $\widetilde{W} \xrightarrow{\simeq} T_\mathcal{O}N\backslash G/I$.*

*Proof.* We take the naive map

$$w \longmapsto T_{\mathcal{O}} N \dot{w} I,$$

where $\dot{w}$ is a representative of $\widetilde{W}$ in $G$. Injectivity follows since for some $w_1, w_2 \in \widetilde{W}$, we have that if $w_1 \in T_{\mathcal{O}} N w_2 I$, then $w_1 = w_2 t_{\mathcal{O}}$ for some $t_{\mathcal{O}} \in T_{\mathcal{O}}$.

It remains to show that every $x \in T_{\mathcal{O}} N \backslash G / I$ has some $w \in \widetilde{W}$ mapping to it. That is, we want to show that some $g \in G$ lies in the double coset. By the Iwasawa decomposition, $G = BK = TNK$. In terms of elements, $g = tn'k' = \pi^{\mu} t_{\mathcal{O}} n' k'$, where $t \in T$, $t_{\mathcal{O}} \in T_{\mathcal{O}}$, $n' \in N$ and $k' \in K$. Re-writing,

$$\pi^{\mu} t_{\mathcal{O}} n' k' = \pi^{\mu} \underbrace{(t_{\mathcal{O}} n' t_{\mathcal{O}}^{-1})}_{\in N} \underbrace{t_{\mathcal{O}} k'}_{\in K},$$

and thus we may write $g = \pi^{\mu} n k$ for some $n \in N$, $k \in K$. Next, we take the Bruhat decomposition over the residue field,

$$G(\mathbb{F}_q) = \bigsqcup_{w \in W} B(\mathbb{F}_q) w B(\mathbb{F}_q) = \bigsqcup_{w \in W} N(\mathbb{F}_q) w B(\mathbb{F}_q),$$

where the second equality follows from the fact that $W$ is normalised by $T$.

Now, we pull back the element $k$ by the map $p : B(\mathcal{O}) \to B(\mathbb{F}_q)$ to obtain some $p(k) = n_1 \dot{w} b_2$, where $n_1 \in N(\mathbb{F}_q)$, $b_2 \in B(\mathbb{F}_q)$. We pull back the element $n_1$ once again to obtain some $p(n_{\mathcal{O}})$ — that is, $p(k) = p(n_{\mathcal{O}}) \dot{w} b_2$. Then, $p^{-1}(b_2) = \dot{w}^{-1} n_{\mathcal{O}}^{-1} k \in I$, and $k = n_{\mathcal{O}} \dot{w} i$. So, $k = n_{\mathcal{O}} \dot{w} i$, and $g = \pi^{\mu} n n_{\mathcal{O}} \dot{w} i$, and therefore $g$ lies in the double coset. □

The *universal principal series*[1] module is given by $\mathcal{M} := C_c(T_{\mathcal{O}} N \backslash G / I)$. Indeed, our use of the word module here is justified as $\mathcal{M}$ has the structure of a right $\mathcal{H}$-module by convolution. That is, for $h \in \mathcal{H}$, and $f \in \mathcal{M}$,

$$(f * h)(x) = \int_G f(xg^{-1}) h(g) \, \mathrm{d}g.$$

Since $I$ is open, $T_{\mathcal{O}} N w I$ is an open neighbourhood of $w \in \widetilde{W}$. By the right $I$-invariance of $\varphi \in C_c(T_{\mathcal{O}} N \backslash G / I)$, we have that $\varphi(gi) = \varphi(g)$ for each $i \in I$. So, $gi$ covers an open neighbourhood (specifically, $gI$) of $g$. As such, the function is constant on the set $gI$ containing $g$. Since our choice of $g$ is arbitrary, the function is locally constant, hence

[1] We refer to $\mathcal{M}$ as such because one obtains this module from the $I$-fixed vectors of the universal principal series of $G$ (c.f. Section 3.3 of this thesis or [HKP10, Section 1.5]). Accordingly, we emphasise that this is the universal unramified principal series module of the Iwahori-Hecke algebra, and *not* the group $G$. Further, one should note that while this is the nomenclature employed by [HKP10], but not all authors will refer to $\mathcal{M}$ in this manner.

"smooth".

The subscript "c" indicates that the functions are compactly supported on the quotient space $T_{\mathcal{O}}N\backslash G/I$ in the quotient topology. Since $I$ is open, the quotient is discrete — that is, compact support is equivalent to finite support.

[HKP10, Section 1.4] posits that the following forms a basis for $\mathcal{M}$, but no proof is provided.

**Proposition 3.7.** *For $w \in \widetilde{W}$, the $v_x := \mathbb{1}_{T_{\mathcal{O}}NwI}$ form a basis for $\mathcal{M}$.*

*Proof.* The first step is to show that $v_x$ is compact supported. We consider the basis elements $v_x$ as functions on the topological space $T_{\mathcal{O}}N\backslash G/I$. Since $I$ is open, it follows that the quotient topology is discrete. $v_x$ has single element support, it is thus compactly supported.

Since $T_{\mathcal{O}}N\backslash G/I$ is in bijection with $\widetilde{W}$ by Proposition 3.6, it follows that for any $f = \sum\limits_{x\in\widetilde{W}} c_x v_x$, we have

$$\operatorname{supp}(f) = \{x \in \widetilde{W} : c_x \neq 0\}.$$

Since each $v_x$ is compactly supported, it follows then that $\operatorname{supp} f$ is compact. Thus, it is a finite set under the discrete topology, and it would follow that thus that $f$ is a finite linear combination. $\square$

### 3.2.1 $(\mathcal{R}, \mathcal{H})$-bimodule Structure of $\mathcal{M}$

As aforementioned, we equip $\mathcal{M}$ with the structure of a right $\mathcal{H}$-module by having $\mathcal{H}$ act on the right by convolution. This action is well-defined and preserves right $I$-invariance since $(f * h)(xi) = (f * h)(x)$ for $i \in I$, $x \in G$ after applying the variable change $x \mapsto xi^{-1}$. Also, $(f * h)(tnx) = (f * h)(x)$ for $tn \in T_{\mathcal{O}}N$ by the left $T_{\mathcal{O}}N$-invariance of $f$.

There is an isomorphism $\mathbb{C}[X_*(T)] \cong C_c(T/T_{\mathcal{O}})$ given by $\pi^\mu \mapsto \chi(\pi^\mu)$. The finite Weyl group $W$ acts on $\mathcal{R}$ by conjugation — that is,

$$w \cdot \pi^\mu = w\pi^\mu w^{-1} = \pi^{w(\mu)},$$

for $w \in W$ and $\mu \in X_*(T)$. Further, we may equip $\mathcal{M}$ with the structure of a left $\mathcal{R}$-module For $\mu \in X_*(T)$ and $x \in \widetilde{W}$,

$$\pi^\mu \cdot v_x = q^{-\langle \rho, \mu \rangle} v_{\pi^\mu \cdot x}, \tag{3.3}$$

where $\rho$ is the half sum of the positive roots. For any $f \in C_c(T_{\mathcal{O}}N\backslash G/I)$, we have that $f = \sum_{x\in\widetilde{W}} c_x v_x$. Thus, for any $g \in G$,

$$(\pi^\mu \cdot f)(g) = \pi^\mu \cdot \sum_{x\in\widetilde{W}} c_x v_x(g) = q^{-\langle\rho,\mu\rangle} \sum_{x\in\widetilde{W}} c_x v_{\pi^\mu x}(g).$$

We observe that,

$$T_{\mathcal{O}}N\pi^\mu xI = (\pi^\mu T_{\mathcal{O}}\pi^{-\mu})(\pi^\mu N\pi^{-\mu})\pi^\mu xI = \pi^\mu T_{\mathcal{O}}NxI,$$

since $T$ normalises $N$. So, $\pi^{-\mu}g \in T_{\mathcal{O}}NxI$, and for any $f \in \mathcal{M}$,

$$(\pi^\mu \cdot f)(g) = q^{-\langle\rho,\mu\rangle} f(\pi^{-\mu}g). \tag{3.4}$$

**Proposition 3.8.** *The actions of $\mathcal{R}$ and $\mathcal{H}$ commute — that is,*

$$(\pi^\mu v_x) * T_y = \pi^\mu(v_x * T_y),$$

*and $\mathcal{M}$ obtains the structure of a $(\mathcal{R},\mathcal{H})$-bimodule.*

*Proof.* We verify this by a direct computation. For the left-hand side,

$$\begin{aligned}((\pi^\mu v_x) * T_y)(g) &= q^{-\langle\rho,\mu\rangle}(v_{\pi^\mu x} * T_y)(g) = q^{-\langle\rho,\mu\rangle}\int_G \mathbb{1}_{T_{\mathcal{O}}N\pi^\mu xI}(gz^{-1})\mathbb{1}_{IyI}(z)\,\mathrm{d}z \\ &= q^{-\langle\rho,\mu\rangle}\int_{IyI} v_{\pi^\mu x}(gz^{-1})\,\mathrm{d}z.\end{aligned}$$

On the right-hand side, we obtain by (3.4),

$$\begin{aligned}(\pi^\mu(v_x * T_y))(g) &= q^{-\langle\rho,\mu\rangle}(v_x * T_y)(\pi^{-\mu}g) = q^{-\langle\rho,\mu\rangle}\int_G \mathbb{1}_{T_{\mathcal{O}}NxI}(\pi^{-\mu}gz^{-1})\mathbb{1}_{IyI}(z)\,\mathrm{d}z \\ &= q^{-\langle\rho,\mu\rangle}\int_{IyI} \mathbb{1}_{T_{\mathcal{O}}NxI}(\pi^{-\mu}gz^{-1})\,\mathrm{d}z \\ &= q^{-\langle\rho,\mu\rangle}\int_{IyI} v_{\pi^\mu x}(gz^{-1})\,\mathrm{d}z,\end{aligned}$$

where the last equality is a consequence of the aforementioned equality $\pi^\mu T_{\mathcal{O}}NxI = T_{\mathcal{O}}N\pi^\mu xI$. □

[HKP10, Section 1.4] claims that the scalar $q^{-\langle\rho,\mu\rangle}$ is given by the absolute value of the determinant of the adjoint action of an element $t \in T$ on $\mathrm{Lie}(N)$, and further that $q^{-\langle\rho,\mu\rangle} = \delta_B(t)^{1/2}$, where $\delta_B$ the modular character on the Borel $B$. We give a sketch of the proof here.

**Lemma 3.9.** $q^{-\langle\rho,\mu\rangle} = \delta_B(t)^{1/2}$

*Proof (Sketch).* Let $\nu$ be a right Haar measure on $B$, and $S \subset B$ some subset of non-zero measure — then, for some $t \in T$,

$$\delta_B(t) := \frac{\nu(t^{-1}S)}{\nu(S)} = \frac{\nu(tSt^{-1})}{\nu(S)} = \frac{\nu(\mathrm{Ad}(t)S)}{\nu(S)}.$$

Since $N$ is open in $B$, it follows that $\nu(N) \neq 0$, and so

$$\delta_B(t) = \frac{\nu(\mathrm{Ad}(t)N)}{\nu(N)}.$$

Taking the Lie algebra, we write $\mathrm{Ad}(t)$ in the basis corresponding to $\mathrm{Lie}(N) = \bigoplus_{\alpha\in\Phi^+} \mathrm{Lie}(N)_\alpha$, and we find that it is diagonal with eigenvalues $\alpha(t)$. For $t = \pi^\mu$, we have

$$|\det(\mathrm{Ad}(\pi^\mu))| = \left|\det\left(\mathrm{diag}(\alpha(\pi^\mu) : \alpha \in \Phi^+)\right)\right| = \left|\prod_{\alpha\in\Phi^+} \alpha(\pi^\mu)\right|.$$

From [Spr98, pg. 46], $\alpha(\pi^\mu) = \pi^{\langle\rho,\alpha\rangle}$, and $|\pi| = q^{-1}$, by the modular character seen in Example 1.23. Thus,

$$\prod_{\alpha\in\Phi^+} |\pi|^{\langle\rho,\alpha\rangle} = q^{-\sum_{\alpha\in\Phi^+}\langle\alpha,\mu\rangle} = q^{-\langle 2\rho,\mu\rangle}.$$

□

## 3.3 Universal Principal Series Module of $G$

This section follows [HKP10, Section 1.5]. As before, we seek to provide more details about their construction. The main result is the following:

**Theorem 3.10.**

$$C_c(T_\mathcal{O}N\backslash G) \cong \text{n-Ind}_B^G\left(\text{c-Ind}_{T_\mathcal{O}N}^B \,\mathrm{triv}\right).^{2} \tag{3.5}$$

*Proof.* Since $G$ is unimodular, we have that $\delta_G(x) = 1$ for any $x \in G$. From Definition 3.4, an element of $\text{c-Ind}_{T_\mathcal{O}N}^{TN}$ is a compactly-supported, $\mathbb{C}$-valued function on $TN = B$ that is left $T_\mathcal{O}N$-invariant. Consequently,

$$\text{c-Ind}_{T_\mathcal{O}N}^{TN} \cong C_c(T_\mathcal{O}N\backslash(TN)) \cong C_c(T_\mathcal{O}\backslash T) \cong = C_c(T/T_\mathcal{O})\mathcal{R},$$

Define by

$$\chi_{\mathrm{univ}} : T/T_\mathcal{O} \longrightarrow \mathcal{R}^\times, \quad \pi^\mu \longmapsto \pi^\mu$$

the *universal character*[3]. This defines an action of $T$ on $\text{c-Ind}_{T_\mathcal{O}N}^{TN}(\mathrm{triv}) = \mathcal{R}$ by $\chi_{\mathrm{univ}}^{-1}$ — that

[2] The right-hand side of this isomorphism is the universal principal series of $G$

[3] [HKP10] refers to this as the *tautological character*.

is, for $f \in C_c(T/T_{\mathcal{O}})$, and $\pi^\mu \in T$, we set

$$\pi^\mu \cdot f(x) = f(\pi^{-\mu} x).$$

We view this representation $\chi_{\text{univ}}$ as a $B$-module by the decomposition $B = TN$.

Now, we normally induce from $B$ to $G$ using $\chi_{\text{univ}}$ — that is,

$$\text{n-Ind}_B^G(\chi_{\text{univ}}^{-1}) = \{\varphi : G \to \mathbb{C} : \varphi(tng) = \delta_B(t)^{-1/2} \chi_{\text{univ}}^{-1}(t) \varphi(g)\}.$$

We describe the isomorphism (3.5) explicitly. For $\phi \in C_c(T_{\mathcal{O}} N \backslash G)$, the corresponding $\varphi \in \text{n-Ind}_B^G(\chi_{\text{univ}}^{-1})$ is defined by

$$\varphi(g) = \sum_{t \in T/T_{\mathcal{O}}} \delta_B(t)^{-1/2} \phi(tg) \cdot t, \quad g \in G$$

Then, $\text{n-Ind}_B^G(\chi_{\text{univ}}^{-1})$ obtains a $\mathcal{R}$-module structure by the action $(r\varphi)(g) = r(\varphi(g))$. □

**Proposition 3.11.** *(3.5) induces a $(\mathcal{R}, \mathcal{H})$-bimodule isomorphism*

$$\mathcal{M} \cong \text{n-Ind}_B^G(\chi_{\text{univ}}^{-1})^I, \tag{3.6}$$

*where $\text{n-Ind}_B^G(\chi_{\text{univ}}^{-1})^I$ denotes the Iwahori-fixed vectors in $\text{Ind}_B^G \mathcal{R}$.*

*Proof (Sketch).* We argue as follows: $G$ acts on $C_c(T_{\mathcal{O}} N \backslash G)$ by right translations. The resulting representation — call it $\rho$ — then gives a right $\mathcal{H}$-action on the $I$-invariants by

$$\varphi \cdot \rho(h) = \int_H h(g^{-1})(\rho(g) \cdot \varphi) \, \mathrm{d}g = \int_G h(g^{-1}) \varphi(- \cdot g) \, \mathrm{d}g,$$

which is precisely the aforementioned convolution action.

From the isomorphism (3.5), it thus follows then that it (3.6) is an $\mathcal{H}$-module isomorphism under the induced $\mathcal{H}$ action. Since the induced action agrees with the action on $\mathcal{M}$ that we have already defined, the $\mathcal{H}$ portion of the claim is clear. One check that this is a $\mathcal{R}$-module directly. □

## 3.4 Freeness of $\mathcal{M}$ over $\mathcal{H}$

The result is from [HKP10, Lemma 1.6.1], who employed the following result by [CKM98], who derived it from Bernstein's presentation. However, we will follow [HKP10] in that we study $\mathcal{M}$ first, and then use its properties to produce Bernstein's presentation.

**Theorem 3.12** (HKP10, Lemma 1.6.1)**.** *The map $h \mapsto v_1 h$ is an isomorphism of right $\mathcal{H}$-modules from $\mathcal{H} \cong \mathcal{M}$. That is, $\mathcal{M}$ is free of rank $1$ over $\mathcal{H}$ with canonical generator $v_1$. In particular, we have a canonical isomorphism $\mathcal{H} \cong \mathrm{End}_{\mathcal{H}}(\mathcal{M})$ given by $h' \mapsto (v_1 h \mapsto v_1 h h')$.*

*Proof.* One has,

$$T_w \longmapsto v_1 * T_w = \sum_{x \in \widetilde{W}} c_{w,x} v_x.$$

From Proposition 2.14, a choice of an Iwahori subgroup fixes a Bruhat order on $\widetilde{W}$. Then, the desired result follows by proving the following:

$$c_{w,w} \neq 0 \text{ for all } w \in \widetilde{W}$$
$$c_{w,x} = 0 \text{ for all } w \leq x$$

The coefficients are precisely determined by $c_{w,x} = (v_1 * T_w)(x)$, and is non-zero if and only if $(v_1 * T_w)(x) \neq 0$. Computing directly,

$$\begin{aligned}\int_G v_1(xg^{-1}) T_w(g)\,\mathrm{d}g = \int_G \mathbb{1}_{T_{\mathcal{O}}NI}(xg^{-1}) \mathbb{1}_{IwI}(g)\,\mathrm{d}g &= \int_{IwI} \mathbb{1}_{T_{\mathcal{O}}NI}(xg^{-1})\,\mathrm{d}g \\ &= \mathrm{vol}\left(\{g \in IwI : xg^{-1} \in T_{\mathcal{O}}NI\right) \\ &= \mathrm{vol}((IwI) \cap (x^{-1}(T_{\mathcal{O}}NI))^{-1}) \\ &= \mathrm{vol}(IwI \cap (x^{-1}T_{\mathcal{O}}NI)^{-1})\end{aligned}$$

This is non-zero at $x$ if and only if

$$IwI \cap (x^{-1}T_{\mathcal{O}}NI)^{-1} \neq \emptyset.$$

Proposition 2.14 implies that if the integral is non-zero, then $x \leq w$ in the Bruhat order, and thus this matrix is upper-triangular. For $x = \widetilde{w}$,

$$\mathrm{vol}((I\widetilde{w}I) \cap (\widetilde{w}^{-1}T_{\mathcal{O}}NI)^{-1}) = \mathrm{vol}(\widetilde{w}I\widetilde{w}^{-1}I \cap T_{\mathcal{O}}NI) \geq \mathrm{vol}(I \cap I) = \mathrm{vol}(I) = 1,$$

and thus the diagonal entries are non-zero. □

Using Theorem 3.12, we are now able to produce some identities for how $\mathcal{H}$ acts on $\mathcal{M}$, which are given by [HKP10, (1.6.1), (1.6.2), (1.6.3)]. Before proving this, we prove the following identities that were recorded in [HKP10, Section 1.6]:

**Lemma 3.13.** *For some $w \in W$, and dominant $\mu \in X_*(T)$,*

*(i)* $T_{\mathcal{O}}NI \cdot IwI = T_{\mathcal{O}}NwI.$

*(ii)* $T_{\mathcal{O}}NI \cdot I\pi^{\mu}I = T_{\mathcal{O}}N\pi^{\mu}I.$

*(iii)* $T_{\mathcal{O}}NI \cap \pi^{\mu}I\pi^{-\mu}I = I.$

*Proof.*

(i) We write $\overline{N}_\pi$ for the $\pi\mathcal{O}$-points of $\overline{N}$. By the Iwahori factorisation,

$$T_{\mathcal{O}}NI \cdot IwI = T_{\mathcal{O}}NIwI = T_{\mathcal{O}}NN_{\mathcal{O}}T_{\mathcal{O}}(\overline{N}_\pi)wI = T_{\mathcal{O}}Nw(w^{-1}\overline{N}_\pi w)I.$$

Then,

$$w^{-1}\overline{N}_\pi w = \prod_{\alpha\in\Phi^-} U_{w^{-1}(\alpha)}(\pi\mathcal{O}) \subseteq I,$$

and the result follows.

(ii) Using the Iwahori factorisation again,

$$\begin{aligned} T_{\mathcal{O}}NI \cdot I\pi^\mu I &= T_{\mathcal{O}}NN_{\mathcal{O}}T_{\mathcal{O}}(I\cap\overline{N})\pi^\mu I \\ &= T_{\mathcal{O}}N(I\cap\overline{N})\pi^\mu \\ &= T_{\mathcal{O}}N\pi^\mu(\pi^{-\mu}(I\cap\overline{N})\pi^\mu)I. \end{aligned}$$

From (2.1), the dominance of $\mu$ implies that $\pi^{-\mu}(I\cap\overline{N})\pi^\mu \subset I\cap\overline{N} \subset I$, and the desired result follows.

(iii) The $\supseteq$ direction is obvious. Some element $x$ in this intersection has the form

$$x = t_{\mathcal{O}}n_{\mathcal{O}}i_1 = \pi^\mu i_2\pi^{-\mu},$$

where $i_2 = t'_{\mathcal{O}}n_{\mathcal{O}}i_3$, and $i_3 \in I\cap\overline{N}$. Thus,

$$\pi^\mu i_2\pi^{-\mu} = \underbrace{(\pi^\mu t'_{\mathcal{O}}\pi^{-\mu})}_{\in T_{\mathcal{O}}\subset I}\underbrace{(\pi^\mu n_{\mathcal{O}}\pi^{-\mu})}_{\in N_{\mathcal{O}}\subset I}(\pi^\mu i_3\pi^{-\mu}).$$

Since $i_3 \in I\cap\overline{N}$, it follows that $\pi^\mu i_3\pi^{-\mu}$ is in the opposite unipotent. But $\pi^\mu i_3\pi^{-\mu}i_1^{-1} = t_{\mathcal{O}}n_{\mathcal{O}}$, where $t_{\mathcal{O}}n_{\mathcal{O}}$ is in the unipotent. Thus, $t_{\mathcal{O}}n_{\mathcal{O}} = 1$, and we have

$$\pi^\mu i_3\pi^{-\mu} = i_1.$$

Together,

$$x = \underbrace{(\pi^\mu t'_{\mathcal{O}}\pi^{-\mu})}_{\in T_{\mathcal{O}}\subset I}\underbrace{(\pi^\mu n_{\mathcal{O}}\pi^{-\mu})}_{\in N_{\mathcal{O}}\subset I}\underbrace{(\pi^\mu i_3\pi^{-\mu})}_{\in I\cap\overline{N}\subset I},$$

and thus $x \in I$.

□

**Theorem 3.14** (HKP10, (1.6.1), (1.6.2), (1.6.3)). *The following identities hold:*

*(i)* $v_1T_w = v_w$ *for every* $w \in W$,

*(ii)* $v_{\pi^\mu} T_w = v_{\pi^\mu w}$ *for each* $w \in W$ *and* $\mu \in X_*(T)$,

*(iii)* $v_1 T_{\pi^\mu} = v_{\pi^\mu}$ *for dominant* $\mu \in X_*(T)$.

*Proof.*

(i) Computing, we have

$$\begin{aligned} v_1 * T_w(x) &= \mathrm{vol}(IwI \cap (x^{-1}T_\mathcal{O}NI)^{-1}) \\ &= \mathrm{vol}(IwI \cap INT_\mathcal{O}x) \\ &= \mathrm{vol}(IwIx^{-1} \cap INT_\mathcal{O}) \\ &= \mathrm{vol}(xIw^{-1}I \cap T_\mathcal{O}NI). \end{aligned}$$

First, we wish to show that if $x \notin T_\mathcal{O}NwI$, then $xIw^{-1}I \cap T_\mathcal{O}NI = \emptyset$. If this set is non-empty, then there is some $z \in xIw^{-1}I \cap T_\mathcal{O}NI$ such that $x(i_1w^{-1}i_2) = z \in T_\mathcal{O}NI$. That is, $x \in T_\mathcal{O}NI \cdot IwI$. By Lemma 3.13(i), we thus have that $x \in T_\mathcal{O}NwI$. That is, $v_1T_w$ is supported on $T_\mathcal{O}NwI$, and thus $v_1T_w = \lambda v_w$ for some $\lambda \in \mathbb{C}$.

It remains to show that $\lambda = 1$ — that is, $\mathrm{vol}(wIw^{-1}I \cap T_\mathcal{O}NI) = 1$. This follows from showing that $wIw^{-1}I \cap T_\mathcal{O}NI = I$. The $\supseteq$ direction is obvious.

Note that $wIw^{-1}I \subseteq K$, and that $T_\mathcal{O}NI \cap K = I$. Since $T_\mathcal{O}, N \subset I$, we have that

$$t_\mathcal{O}ni = k \in K \implies n = t_\mathcal{O}^{-1}ki^{-1},$$

which is in $I$ if and only if $k \in I$. Thus, $T_\mathcal{O}NI \subseteq I$, as we required.

(ii) Applying (3.3) and computing directly,

$$v_{\pi^\mu}T_w = q^{\langle\rho,\mu\rangle}(\pi^\mu v_1)T_w = q^{\langle\rho,\mu\rangle}\pi^\mu(v_1T_w) = q^{\langle\rho,\mu\rangle}\pi^\mu \cdot v_w = v_{\pi^\mu w}$$

(iii) Computing directly,

$$\begin{aligned} (v_1 * T_{\pi^\mu})(x) &= \int_G \mathbb{1}_{T_\mathcal{O}NI}(xg^{-1})\mathbb{1}_{I\pi^\mu I}(g)\,\mathrm{d}g = \mathrm{vol}(INT_\mathcal{O}x \cap I\pi^\mu I) \\ &= \mathrm{vol}(xI\pi^{-\mu}I \cap T_\mathcal{O}NI), \end{aligned}$$

where the last equality follows by taking inverses. Indeed, if $x = \mu$, then $(v_1 * T_{\pi^\mu})(\pi^\mu) = 1$ by Lemma 3.13(iii). Employing the same argument as in (i), if $y \in xI\pi^{-\mu}I \cap T_\mathcal{O}N$, Then, $x \in T_\mathcal{O}NI \cdot I\pi^\mu I$, and by Lemma 3.13(ii), we have that $x \in T_\mathcal{O}N\pi^\mu I$, and the result follows.

□

## 3.5 The Bernstein Basis

Recall the Iwahori-Matsumoto presentation:

$$T_{s_i}^2 = (q-1)T_{s_i} + qT_1,$$
$$T_w T_{w'} = T_{ww'} \text{ if } \ell(ww') = \ell(w) + \ell(w')$$

If one lets $q \to 1$, then $\mathcal{H}$ becomes $\mathbb{C}[\widetilde{W}]$. As such, one may think of the Iwahori-Hecke algebra as a $q$-deformation of the affine Weyl group $\widetilde{W}$. One may write the affine Weyl group as the semi-direct product

$$\widetilde{W} = X_*(T) \rtimes W.$$

As such, it thus follows that one should be able to obtain deformations of $X_*(T)$ and $W$ as well. We already have a deformation of $\mathbb{C}[W]$ given by the *finite Hecke algebra* $\mathcal{H}_{\text{fin}}$. It now remains to produce some deformation of $\mathbb{C}[X^*(T)] = \mathcal{R}$.

We recall that $\mathcal{R}$ acts on $C_c(T_{\mathcal{O}}\backslash G/I)$ by (3.4), and that the actions of $\mathcal{H}$ and $\mathcal{R}$ commute — that is, $\mathcal{M}$ is a $(\mathcal{R}, \mathcal{H})$-bimodule. Thus, we may identify an element $r \in \mathcal{R}$ with an endomorphism $m \mapsto rm$, where $m \in \mathcal{M}$. That is, we may identify $r$ precisely with some element in $\mathrm{End}_{\mathcal{H}}(\mathcal{M})$.

By the isomorphism $\mathcal{H} \cong \mathrm{End}_{\mathcal{H}}(\mathcal{M})$ in Theorem 3.12, there exists an element of $\mathcal{H}$ corresponding to the same endomorphism as the one defined by $r$. Let $\lambda \in X_*(T)$ be dominant. Then, we have an embedding

$$\mathcal{R} \longhookrightarrow \mathcal{H}, \quad \pi^\lambda \longmapsto \Theta_\lambda$$

defined by the action $\pi^\lambda \cdot v_1 = v_1 \cdot \Theta_\lambda$. Let us assume that $\lambda \in X^*(T)$ is dominant. Then, we have that $\pi^\lambda v_1 = q^{-\langle -\rho, \lambda\rangle} v_{\pi^\lambda} = v_1 \cdot \Theta_\lambda$. Then, applying Theorem 3.14(iii), we obtain $q^{-\langle \rho, \lambda\rangle} v_1 T_{\pi^\lambda} = v_1 \cdot \Theta_\lambda$. So,

$$\Theta_\lambda = q^{-\langle \rho, \lambda\rangle} T_{\pi^\lambda},$$

for dominant $\lambda \in X_*(T)$.

**Theorem 3.15** (HKP10, Remark 1.7.2)**.** *For any $\lambda \in X_*(T)$ such that $\lambda = \lambda_1 - \lambda_2$ with $\lambda_1, \lambda_2$ dominant,*

$$\Theta_\lambda = q^{\langle \rho, -\lambda_1 + \lambda_2\rangle} T_{\pi^{\lambda_1}} (T_{\pi^{\lambda_2}})^{-1}$$

*Proof.* We have $\pi^\lambda = \pi^{-\lambda_2 + \lambda_1} = \pi^{-\lambda_2} \cdot \pi^{\lambda_1}$. Using Theorem 3.14(iii), we compute directly to obtain

$$\begin{aligned}\pi^{\lambda} \cdot v_1 &= \pi^{-\lambda_2} \cdot \pi^{\lambda_1} \cdot v_1 \\ &= \pi^{-\lambda_2} q^{-\langle \rho, \lambda_1 \rangle} v_{\pi_1^{\lambda}} \\ &= q^{-\langle \rho, \lambda_1 \rangle} \pi^{-\lambda_2} v_1 T_{\pi^{\lambda_1}} \\ &= q^{-\langle \rho, \lambda_1 \rangle} (\pi^{\lambda_2})^{-1} v_1 T_{\pi^{\lambda_1}}.\end{aligned}$$

Since $\pi^{-\lambda_2} = (\pi^{\lambda_2})^{-1}$, we have that $\Theta_{-\lambda_2} = (\Theta_{\lambda_2})^{-1}$. Substituting this into the above gives us

$$\pi^{\lambda} \cdot v_1 = q^{\langle \lambda_2 - \lambda_1, \rho \rangle} T_{\pi^{\lambda_1}} (T_{\pi^{\lambda_2}})^{-1},$$

as required. □

One readily shows by a direct computation that for $\lambda, \mu \in X_*(T)$,

$$\Theta_{\lambda} \Theta_{\mu} = \Theta_{\lambda + \mu}.$$

This then gives us the desired $q$-deformation of $\mathcal{R}$. The next result produces a basis of $\mathcal{H}$ in terms of the $\Theta_{\lambda}$'s and $T_w$'s.

**Theorem 3.16** (HKP10, Lemma 1.7.1). *Multiplication in $\mathcal{H}$ induces a vector space isomorphism*

$$\mathcal{R} \otimes_{\mathbb{C}} \mathcal{H}_{\text{fin}} \cong \mathcal{H}, \quad \pi^{\mu} \otimes_{\mathbb{C}} h \longmapsto \Theta_{\mu} h$$

*Proof.* We compose the above map with the isomorphism $h \mapsto v_1 h$ seen in Theorem 3.12 to obtain

$$\begin{array}{ccccc} \mathcal{R} \otimes_{\mathbb{C}} \mathcal{H}_{\text{fin}} & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{M} \\ \\ \pi^{\mu} \otimes_{\mathbb{C}} h & \longmapsto & \Theta_{\mu} h & \longmapsto & v_1(\Theta_{\mu} h) \end{array}$$

Using Theorem 3.14(ii), we have that for some $w \in W$,

$$v_1(\Theta_{\mu} h) = (v_1 \Theta_{\mu}) h = (\pi^{\mu} v_1) T_w = q^{-\langle \rho, \mu \rangle} v_{\pi^{\mu}} T_w = q^{-\langle \rho, \mu \rangle} v_{\pi^{\mu} w}$$

Thus, the composition maps $\pi^{\mu} \otimes_{\mathbb{C}} T_w \mapsto q^{-\langle \lambda, \rho \rangle} v_{\pi^{\mu} w}$, and is thus a vector space isomorphism. By Theorem 3.12, $\mathcal{H} \cong \mathcal{M}$ as right $\mathcal{H}$-modules. Therefore, it is necessarily the case that $\mathcal{R} \otimes_{\mathbb{C}} \mathcal{H}_{\text{fin}} \cong \mathcal{H}$. □

# Chapter 4

# The Bernstein Presentation

Having now comprehensively established the structure of $\mathcal{M}$, the proof of the Bernstein presentation begins in earnest. In this section, we establish some properties of the intertwiner $\mathcal{I}_w$ in the sense of [HKP10], from which we eventually produce Bernstein's relation, thus giving us our desired presentation of $\mathcal{H}$.

## 4.1 Intertwiners

Following [HKP10, Section 1.10-1.11], [BKP14, Chapter 6, pg. 29-34], we define an intertwiner $\mathcal{I}_w$ by

$$\mathcal{I}_w(\varphi)(g) = \int_{N_w} \varphi(\dot{w}^{-1}ng)\,\mathrm{d}n,$$

where $N_w = N \cap w\overline{N}w^{-1}$, and the Haar measure is normalised to give $N_w \cap K$ measure $1$.

This is an intertwining operator of some suitable completion of $\mathcal{M}$. In particular, the aforementioned authors fix some subset $J$ of positive coroots, and complete $\mathcal{M}$ in such a way that that one can take infinite sums of elements of the form $\pi^{\alpha^\vee}$ for $\alpha^\vee \in J$. The details of the construction are involved, and will not be used throughout this thesis. Thus, we refer to reader to the above sources.

Instead, we will prove some properties of the intertwiner $\mathcal{I}_w$. These results can be seen in [HKP10, Lemma 1.11.1]. As usual, we will outline the proof of this result in more detail.

**Lemma 4.1** (HKP10, Lemma 1.11.1)**.**

*(i)* $\pi^{w(\mu)} \circ \mathcal{I}_w = \mathcal{I}_w \circ \pi^\mu$ *for all* $\mu \in X_*(T)$,

*(ii)* $\mathcal{I}_{w_1 w_2} = \mathcal{I}_{w_1} \circ \mathcal{I}_{w_2}$ *if* $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$,

*(iii)* $\mathcal{I}_w$ *is a right* $\mathcal{H}$*-module homomorphism.*

*Proof.*

(i) It is sufficient to check this for any any basis element $v_x$, as we can extend linearly. On the left-hand side, we have

$$\pi^{w(\mu)}(\mathcal{I}_w(v_x))(g) = q^{-\langle\rho,w(\mu)\rangle}\mathcal{I}_w(v_x)(\pi^{-w(\mu)}g) = q^{-\langle\rho,w(\mu)\rangle}\int_{N_w} v_x(\dot{w}^{-1}n\pi^{-w(\mu)}g)\,\mathrm{d}n.$$

Now, for the right-hand side,

$$\mathcal{I}_w(\pi^{\mu}(v_x))(g) = \int_{N_w} \pi^{\mu}(v_x)(\dot{w}^{-1}ng)\,\mathrm{d}n = q^{-\langle\rho,\mu\rangle}\int_{N_w} v_x(\pi^{-\mu}\dot{w}^{-1}ng)\,\mathrm{d}n.$$

Observe that

$$\begin{aligned}\pi^{-\mu}w^{-1}ng &= w^{-1}(w\pi^{-\mu}w^{-1})ng = w^{-1}\pi^{-w(\mu)}ng\\ &= w^{-1}(\pi^{-w(\mu)}n\pi^{w(\mu)})\pi^{-w(\mu)}g\end{aligned}$$

So,

$$\mathcal{I}_w(\pi^{\mu}(v_x))(g) = q^{-\langle\rho,\mu\rangle}\int_{N_w} v_x(\dot{w}^{-1}(\pi^{-w(\mu)}n\pi^{w(\mu)})g)\,\mathrm{d}n. \tag{4.1}$$

By [Bum10, Lemma 25], we have that

$$N \cap w\overline{N}w^{-1} = \prod_{\beta\in\Phi^+\cap w(\Phi^-)} U_\beta, \tag{4.2}$$

where the product can be taken in any fixed order. $\Phi^+ \cap w(\Phi^-)$ is the inversion set of $w$, written $\mathrm{Inv}(w^{-1})$. Let $x_\beta : F \to U_\beta$ be the isomorphism onto the root subgroup. Then, we consider the variable change $n \mapsto \pi^{w(\mu)}n\pi^{-w(\mu)}$. To obtain the required result, we wish to show that this variable change has Jacobian $q^{-\langle\rho,w(\mu)-\mu\rangle}$.

From (4.2), our element $n$ has the form $n = \prod_{\alpha\in\mathrm{Inv}(w^{-1})} x_\alpha(t)$. Then,

$$\pi^{w(\mu)}n\pi^{-w(\mu)} = \prod_{\alpha\in\mathrm{Inv}(w^{-1})} \pi^{w(\mu)}x_\alpha(t)\pi^{-w(\mu)} = \prod_{\alpha\in\mathrm{Inv}(w^{-1})} x_\alpha(t\pi^{\langle\alpha,w(\mu)\rangle}).$$

Since $N \cong F^{|\,\mathrm{Inv}(w^{-1})|}$ by its root subgroup decomposition, we may view the Haar measure on $N$ as a product measure of the measure we imposed on $F$ in Example 1.23. Thus, this variable change gives us a Jacobian of the form

$$\prod_{\alpha\in\mathrm{Inv}(w^{-1})} q^{-\langle\alpha,w(\mu)\rangle} = q^{-\sum_{\alpha\in\mathrm{Inv}(w^{-1})}\langle\alpha,w(\mu)\rangle}. \tag{4.3}$$

Observe that

$$\Phi^+ \setminus w(\Phi^+) = \mathrm{Inv}(w^{-1}), \quad \Phi^- \setminus w(\Phi^-) = -\mathrm{Inv}(w^{-1}).$$

Thus,

$$\rho - w(\rho) = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha - \frac{1}{2}\sum_{\alpha\in w(\Phi^+)}\alpha = 1\sum_{\Phi^+\cap w(\Phi^-)}\alpha.$$

Using the $W$-invariance of the inner product $\langle\cdot,\cdot\rangle$,

$$\langle\rho, \mu - w(\mu)\rangle = \langle w(\rho) - \rho, w(\mu)\rangle = -\sum_{\alpha\in\mathrm{Inv}(w^{-1})}\langle\alpha, w(\mu)\rangle.$$

Substituting this into (4.3), we are done.

(ii) By a direct computation,

$$\begin{aligned}
\mathcal{I}_{w_1}(\mathcal{I}_{w_2}(\varphi))(g) &= \int_{N_{w_1}} \mathcal{I}_{w_2}(\varphi)(\dot{w}_1{}^{-1}n_1g)\,\mathrm{d}n_1 = \int_{N_{w_1}}\int_{N_{w_2}} \varphi(\dot{w}_2{}^{-1}n_2\dot{w}_1{}^{-1}ng)\,\mathrm{d}n_2\,\mathrm{d}n_1 \\
&= \int_{N\cap w_1\overline{N}w_1^{-1}}\int_{N\cap w_2\overline{N}w_2^{-1}} \varphi(\dot{w}_2{}^{-1}n_2\dot{w}_1{}^{-1}n_1g)\,\mathrm{d}n_2\,\mathrm{d}n_1 \\
&= \int_{N\cap w_1\overline{N}w_1^{-1}}\int_{N\cap w_2\overline{N}w_2^{-1}} \varphi(\dot{w}_2^{-1}\dot{w}_1{}^{-1}\dot{w}_1n'\dot{w}_1{}^{-1}ng)\,\mathrm{d}n_2\,\mathrm{d}n_1 \\
&= \int_{N\cap w_1\overline{N}w_1^{-1}}\int_{N\cap w_2\overline{N}w_2^{-1}} \varphi((\dot{w}_1\dot{w}_2)^{-1}(\dot{w}_1n_2\dot{w}_1{}^{-1})n_1g)\,\mathrm{d}n_2\,\mathrm{d}n_1.
\end{aligned}$$

Observe that $w_1n_2w_1^{-1} \in w_1(N\cap w_2\overline{N}w_2^{-1})w_1^{-1}$, and so we may re-write the above equation to

$$\int_{N\cap w_1\overline{N}w_1^{-1}}\int_{w_1(N\cap w_2\overline{N}w_2^{-1})w_1^{-1}} \varphi((\dot{w}_1\dot{w}_2)^{-1}n_2n_1g)\,\mathrm{d}n.$$

If $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$, then multiplication induces a bijection

$$w_1(N\cap w_2\overline{N}w_2^{-1})w_1^{-1} \times N\cap w_1\overline{N}w_1^{-1} \longrightarrow N\cap(w_1w_2)\overline{N}(w_1w_2)^{-1}.$$

Then, by Fubini's theorem,

$$\begin{aligned}
&\int_{N\cap w_1\overline{N}w_1^{-1}}\int_{w_1(N\cap w_2\overline{N}w_2^{-1})w_1^{-1}} \varphi((\dot{w}_1\dot{w}_2)^{-1}n_2n_1g)\,\mathrm{d}n \\
&= \int_{(N\cap w_1\overline{N}w_1^{-1})\times w_1(N\cap w_2\overline{N}w_2^{-1})w_1^{-1}} \varphi((\dot{w}_1\dot{w}_2)^{-1}n_2n_1g)\,\mathrm{d}n_2\,\mathrm{d}n_1 \\
&= \int_{N\cap(w_1w_2)\overline{N}(w_1w_2)^{-1}} \varphi((\dot{w}_1\dot{w}_2)^{-1}ng)\,\mathrm{d}n \\
&= \mathcal{I}_{w_1w_2}(\varphi)(g),
\end{aligned}$$

as desired.

(iii) Let $w, w' \in W$. Then, by Fubini:

$$\begin{aligned}\mathcal{I}_w(\varphi * T_{w'})(x) &= \int_{N_w} (\varphi * T_{w'}(\dot{w}^{-1}nx)\,\mathrm{d}n = \int_{N_w}\int_G \varphi(\dot{w}^{-1}nxg^{-1})T_{w'}(g)\,\mathrm{d}g\,\mathrm{d}n \\ &= \int_G T_{w'}(g)\int_{N_w} \varphi(\dot{w}^{-1}nxg^{-1})\,\mathrm{d}n\,\mathrm{d}g \\ &= (\mathcal{I}_w(\varphi) * T_{w'})(x).\end{aligned}$$

□

### 4.1.1 Intertwiners in the Rank 1 Case

This section follows [HKP10, Section 1.12], giving more detail about the computations that were performed. We impose a Haar measure on $F$ as in Example 1.23. Let $G = \mathrm{SL}_2(F)$. Then, $G$ is semisimple and rank 1. Let $\alpha$ be the unique positive root of $G$, and we choose the coroot corresponding to $\alpha$ as $\alpha^\vee = \frac{2\alpha}{\langle\alpha,\alpha\rangle}$, and $s_\alpha$ the simple reflection.

Here, $s_\alpha \overline{N} s_\alpha^{-1} = N$, so $N_w = N$. Following [HKP10], consider $\mathcal{I}_{s_\alpha}(v_1)$, and let $J(j,w)$ ($j \in \mathbb{Z}, w \in W$) for the value of $\mathcal{I}_{s_\alpha}(v_1)$ at the element $\pi^{j\alpha^\vee} w \in \widetilde{W} = X_*(T) \rtimes W$ — that is,

$$J(j,w) = \mathcal{I}_{s_\alpha}(v_1)(\pi^{j\alpha^\vee}) = \int_N \mathbb{1}_{T_\mathcal{O}NI}(\dot{s_\alpha n \pi^{j\alpha^\vee}} w)\,\mathrm{d}n.$$

Since $\mathcal{I}_{s_\alpha}(v_1) \in C_c(T_\mathcal{O}N\backslash G/I)$, it is expressible as a linear combination in the basis:

$$\begin{aligned}\mathcal{I}_{s_\alpha}(v_1) &= \sum_{x\in\widetilde{W}} c_x v_x \overset{x\mapsto \pi^{j\alpha^\vee}w}{=} \sum_{\pi^{j\alpha^\vee}w\in\widetilde{W}} c_{\pi^{j\alpha^\vee}w} v_{\pi^{j\alpha^\vee}w} = \sum_{\substack{w\in W\\ j\alpha^\vee\in X_*(T)}} c_{\pi^{j\alpha^\vee}w} v_{\pi^{j\alpha^\vee}w} \\ &= \sum_{\substack{w\in W\\ j\in\mathbb{Z}}} J(j,w) v_{\pi^{j\alpha^\vee}w}.\end{aligned}$$

If $w = 1$, then

$$J(j,1) = \int_N \mathbb{1}_{T_\mathcal{O}NI}(s_\alpha^{-1} n \pi^{j\alpha^\vee})\,\mathrm{d}n.$$

So,

$$\pi^{j\alpha^\vee} \in n s_\alpha T_\mathcal{O} N I = n T_\mathcal{O} N s_\alpha I = T_\mathcal{O} N s_\alpha I.$$

Consequently, $J(j,1) = 0$ if $j < 0$. In the case for which $w = s_\alpha$, we obtain $\pi^{j\alpha^\vee} s_\alpha \in T_\mathcal{O} N s_\alpha I$, from which we observe that $J(j, s_\alpha) = 0$ for $j < 0$. Thus, we will now assume that $j \geq 0$.

Throughout, we will have

$$s_\alpha = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad \pi^{j\alpha^\vee} = \begin{pmatrix} \pi^j & 0 \\ 0 & \pi^{-j} \end{pmatrix},$$

for some $x \in F$.

Consider the case for which $j = 0$. Observe that $s_\alpha n w \in T_\mathcal{O} N K$ if and only if $n \in N_\mathcal{O}$. For $n \in N_\mathcal{O}$, the element $s_\alpha n w$ additionally belongs in $K$. In rank 1, the Iwahori subgroup is

$$\begin{pmatrix} \mathcal{O}^\times & \mathcal{O} \\ \pi\mathcal{O} & \mathcal{O}^\times \end{pmatrix}.$$

Thus, $s_\alpha n w \in K$ belongs in $T_\mathcal{O} N I$ if and only if its lower left entry is contained in the prime ideal $(\pi)$ of $\mathcal{O}^\times$. So, for $w = 1$,

$$s_\alpha n = \begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix},$$

and so $J(0, 1) = 0$ since $1 \notin (\pi)$. For $w = s_\alpha$, we have that

$$s_\alpha n s_\alpha = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}.$$

Computing, we have that

$$J(0, s_\alpha) = \int_N \mathbb{1}_{T_\mathcal{O} N I}(s_\alpha n s_\alpha)\, \mathrm{d}n = \mathrm{vol}(T_\mathcal{O} N I \cap s_\alpha N s_\alpha).$$

Let

$$\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} i_1 & i_2 \\ i_3 & i_4 \end{pmatrix} \in T_\mathcal{O} N I, \quad \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \in s_\alpha N s_\alpha = \overline{N}.$$

Then, an element $z \in T_\mathcal{O} N I \cap \overline{N}$ has the form

$$z = \begin{pmatrix} t_1 i_1 + y i_2 & t_1 i_2 + y_4 \\ t_2 i_3 & t_2 i_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}.$$

Since $t_2 \in \mathcal{O}^\times$ and $i_3 \in \pi\mathcal{O}$, it follows that $t_2 i_3 \in \pi\mathcal{O}$. The other entries of the matrix are fixed, and therefore $\mathrm{vol}(T_\mathcal{O} N I \cap \overline{N})$ is the measure of $\pi\mathcal{O}$ — that is, $J(0, s_\alpha) = q^{-1}$.

Now, let $j > 0$. We have,

$$s_\alpha n \pi^{j\alpha^\vee} = \begin{pmatrix} 0 & -\pi^{-j} \\ \pi^j & x\pi^{-j} \end{pmatrix}.$$

Now, observe that in order for $s_\alpha n\pi^{j\alpha^\vee}w$ to be in $T_{\mathcal{O}}NK$, it is necessary that $x \in \pi^j\mathcal{O}^\times$. We therefore assume this and write $x = \pi^j u$ for some unit $u \in \mathcal{O}^\times$. Then, after some computation we may write

$$s_\alpha n\pi^{j\alpha^\vee} = \begin{pmatrix} u^{-1} & -\pi^{-j} \\ \pi^j & x\pi^{-j} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u^{-1}\pi^j & 1 \end{pmatrix},$$

where the first factor lies in $T_{\mathcal{O}}N$, and the second factor lies in $K$. It therefore follows that $s_\alpha n\pi^{j\alpha^\vee} \in T_{\mathcal{O}}NI$ if and only if the second factor lies in $I$. But $I \subset K$, and so this is always the case. Thus — by a similar argument above — $J(j,1)$ is the measure of $\pi^j\mathcal{O}^\times$, which is $q^{-j}(1-q^{-1})$.

Observe that $s_\alpha n\pi^{j\alpha^\vee}s_\alpha \in T_{\mathcal{O}}NI$ if and only if

$$\begin{pmatrix} 1 & 0 \\ u^{-1}\pi^j & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -u^{-1}\pi^j \end{pmatrix} \in I,$$

which is never the case. Therefore, $J(j, s_\alpha) = 0$. We have proved the following:

**Lemma 4.2** (HKP10, Lemma 1.12.1)**.**

$$\mathcal{I}_{s_\alpha}(v_1) = q^{-1}v_{s_\alpha} + (1-q^{-1})\sum_{j=1}^{\infty} q^{-j} v_{\pi^{j\alpha^\vee}}.$$

As a consequence of what we did above, we additionally obtain the following result:

**Lemma 4.3** (HKP10, Lemma 1.12.2)**.**

$$\mathcal{I}_{s_\alpha}(\mathbb{1}_{T_{\mathcal{O}}NK}) = q^{-1}\mathbb{1}_{T_{\mathcal{O}}Ns_\alpha K} + \sum_{j=0}^{\infty} q^{-j}(1-q^{-1})\mathbb{1}_{T_{\mathcal{O}}N\pi^{j\alpha^\vee}K} = \frac{1-q^{-1}\pi^{\alpha^\vee}}{1-\pi^{\alpha^\vee}}\mathbb{1}_{T_{\mathcal{O}}NK}.$$

*Proof.* By the Iwahori-Bruhat decomposition, $K = I \sqcup Is_\alpha I$. Therefore,

$$\mathbb{1}_{T_{\mathcal{O}}NK} = v_1 + v_{s_\alpha}.$$

Then, by Theorem 3.14(i) and Theorem 4.1(iii),

$$\mathcal{I}_{s_\alpha}(\mathbb{1}_{T_{\mathcal{O}}NK}) = \mathcal{I}_{s_\alpha}(v_1 + v_{s_\alpha}) = \mathcal{I}_{s_\alpha}(v_1) + \mathcal{I}_{s_\alpha}(v_{s_\alpha}) = \mathcal{I}_{s_\alpha}(v_1) + \mathcal{I}_{s_\alpha}(v_1)T_{s_\alpha}.$$

Applying Theorem 3.14(i), (ii),

$$\begin{aligned}
\mathcal{I}_{s_\alpha}(\mathbb{1}_{T_\mathcal{O}NK}) &= q^{-1}v_{s_\alpha} + q^{-1}v_1(T_{s_\alpha})^2 + (1-q^{-1})\sum_{j\geq 1} q^{-j}\left(v_{\pi^{j\alpha^\vee}} + v_{\pi^{j\alpha^\vee}s_\alpha}\right) \\
&= q^{-1}v_{s_\alpha} + (1-q^{-1})v_{s_\alpha} + v_1 + (1-q^{-1})\sum_{j\geq 1} q^{-j}\mathbb{1}_{T_\mathcal{O}N\pi^{j\alpha^\vee}K} \\
&= q^{-1}(v_1+v_{s_\alpha}) + (1-q^{-1})(v_1+v_{s_\alpha}) + (1-q^{-1})\sum_{j\geq 1} q^{-j}\mathbb{1}_{T_\mathcal{O}N\pi^{j\alpha^\vee}K} \\
&= q^{-1}\mathbb{1}_{T_\mathcal{O}NK} + (1-q^{-1})\sum_{j\geq 0} q^{-j}\mathbb{1}_{T_\mathcal{O}N\pi^{j\alpha^\vee}K}
\end{aligned}$$

By (3.3),

$$\begin{aligned}
\mathcal{I}_{s_\alpha}(\mathbb{1}_{T_\mathcal{O}NK}) &= q^{-1}\mathbb{1}_{T_\mathcal{O}NK} + \sum_{j\geq 0}(1-q^{-1})q^{-j}q^{\langle\rho, j\alpha^\vee\rangle}\pi^{j\alpha^\vee}\mathbb{1}_{T_\mathcal{O}NK} \\
&= \left(q^{-1} + \sum_{j\geq 0}(1-q^{-1})\pi^{j\alpha^\vee}\right)\mathbb{1}_{T_\mathcal{O}NK} \\
&= \left(q^{-1} + \frac{1-q^{-1}}{1-\pi^{\alpha^\vee}}\right)\mathbb{1}_{T_\mathcal{O}NK} \\
&= \frac{1-q^{-1}\pi^{\alpha^\vee}}{1-\pi^{\alpha^\vee}}\mathbb{1}_{T_\mathcal{O}NK}.
\end{aligned}$$

□

## 4.2 General Results on Intertwiners

**Theorem 4.4** (HKP10, Lemma 1.13.1). *Let $s_\alpha$ be the simple reflection corresponding to a simple root $\alpha$. Then,*

*(i)*

$$\mathcal{I}_{s_\alpha}(v_1) = q^{-1}v_{s_\alpha} + (1-q^{-1})\sum_{j\geq 1}\pi^{j\alpha^\vee}v_1.$$

*(ii)*

$$\mathcal{I}_{s_\alpha}(v_1+v_{s_\alpha}) = \left(\frac{1-q^{-1}\pi^{\alpha^\vee}}{1-\pi^{\alpha^\vee}}\right)(v_1+v_{s_\alpha}).$$

*(iii)*

$$\mathcal{I}_{s_\alpha}(\mathbb{1}_{T_\mathcal{O}NK}) = \left(\frac{1-q^{-1}\pi^{\alpha^\vee}}{1-\pi^{\alpha^\vee}}\right)\mathbb{1}_{T_\mathcal{O}NK}.$$

*Proof.*

(i), (ii) Reduces to rank one case.

(iii) Note that $\mathbb{1}_{T_\mathcal{O}NK} = \sum_{w\in W} v_w$ since $K = \bigsqcup_{w\in W} IwI$. The same computation as that in Lemma 4.3 yields the desired result.

□

**Corollary 4.5** (Gindikin-Karpelevich Formula)**.**

$$\mathcal{I}_w(\mathbb{1}_{T_\mathcal{O}NK}) = \left(\prod_{\alpha\in\mathrm{Inv}(w^{-1})} \frac{1-q^{-1}\pi^{\alpha^\vee}}{1-\pi^{\alpha^\vee}}\right)\mathbb{1}_{T_\mathcal{O}NK}.$$

*Proof.* Immediate by Lemma 4.1(ii) and Theorem 4.4(iii). □

## 4.3 Bernstein's Relation

Following [HKP10, Section 1.14], we cancel the denominator from the intertwiner $\mathcal{I}_w$ by defining

$$\mathcal{J}_w := \left(\prod_{\alpha\in\mathrm{Inv}(w^{-1})} (1-\pi^{\alpha^\vee})\right)\cdot\mathcal{I}_w.$$

The intertwiner $\mathcal{J}_w$ is normalised such that it restricts to a $\mathcal{H}$-endomorphism $\mathcal{M}\to\mathcal{M}$. Indeed, for $w = s_\alpha$:

$$\begin{aligned}\mathcal{J}_{s_\alpha}(v_1) &= (1-\pi^{\alpha^\vee})\left(q^{-1}v_{s_\alpha} + (1-q^{-1})\sum_{j\geq 1}\pi^{j\alpha^\vee}v_1\right)\\ &= (1-\pi^{\alpha^\vee})q^{-1}v_{s_\alpha} + (1-q^{-1})(1-\pi^{\alpha^\vee})\left(-v_1 + \frac{1}{1-\pi^{\alpha^\vee}}v_1\right)\\ &= (1-\pi^{\alpha^\vee})q^{-1}v_{s_\alpha} + (1-q^{-1})(\pi^{\alpha^\vee}v_1) \in \mathcal{M},\end{aligned}$$

and thus $\mathcal{J}_w \in \mathrm{End}_\mathcal{H}(\mathcal{M})$. So, $\mathcal{J}_w$ can be identified with an element of $\mathcal{H}$ by Theorem 3.12. Using by Theorem 4.4(i), we find that for a simple root $\alpha$, the element of $\mathcal{H}$ corresponding to $\mathcal{J}_{s_\alpha}$ is,

$$(1-q^{-1})\Theta_{\alpha^\vee} + q^{-1}(1-\Theta_{\alpha^\vee})T_{s_\alpha}. \tag{4.4}$$

It follows as a result of Lemma 4.1(i) that

$$\mathcal{J}_{s_\alpha}\circ\Theta_\mu = \Theta_{s_\alpha(\mu)}\circ\mathcal{J}_{s_\alpha}. \tag{4.5}$$

Then, (4.4), together with (4.5) for $w = s_\alpha$, gives us *Bernstein's relation*:

$$T_{s_\alpha}\Theta_\mu = \Theta_{s_\alpha(\mu)}T_{s_\alpha} + (1-q)\frac{\Theta_{s_\alpha(\mu)}-\Theta_\mu}{1-\Theta_{-\alpha^\vee}}. \tag{4.6}$$

The right-hand side, though written as a fraction, still lies in the image of $\mathcal{R} \hookrightarrow \mathcal{H}$, since

$$\frac{\Theta_{s_\alpha(\mu)} - \Theta_\mu}{1 - \Theta_{-\alpha^\vee}} = \Theta_\mu + \Theta_{\mu-\alpha^\vee} + \Theta_{\mu-2\alpha^\vee} + \cdots + \Theta_{s_\alpha(\mu)+\alpha^\vee}.$$

Employing Theorem 3.16 in conjunction with Bernstein's relation (4.6), we thus obtain:

**Corollary 4.6** (Bernstein Presentation). *The Iwahori-Hecke algebra has a basis $\{T_w\Theta_\lambda : w \in W, \lambda \in X_*(T)\}$, with the following presentation: for $\lambda, \mu \in X_*(T)$, $s_\alpha$ the simple reflection, and $w, w \in W$ such that $\ell(ww') = \ell(w) + \ell(w')$,*

$$\begin{aligned} T_{s_\alpha}^2 &= (q-1)T_{s_\alpha} + qT_1, \\ T_wT_{w'} &= T_{ww'}, \\ \Theta_\lambda\Theta_\mu &= \Theta_{\lambda+\mu}, \\ T_{s_\alpha}\Theta_\mu &= \Theta_{s_\alpha(\mu)}T_{s_\alpha} + (1-q)\frac{\Theta_{s_\alpha(\mu)} - \Theta_\mu}{1 - \Theta_{-\alpha^\vee}}. \end{aligned}$$

An advantage of the Bernstein presentation is that it allows one can construct a basis for the centre $Z(\mathcal{H})$ of $\mathcal{H}$ by summing the generators $\Theta_\lambda$ over the Weyl-orbits of the coweights $\lambda$. We will see this in the following chapter.

The centre of $\mathcal{H}$ also appears in other important results, such as the Satake isomorphism (see Chapter 6), and also allows one to compute Macdonald's formula for spherical functions [HKP10, Section 5].

# Chapter 5

# The Satake Isomorphism

The statement of the Satake isomorphism is as follows:

$$\mathcal{H}_{\mathrm{sph}} \cong \mathcal{R}^W,$$

where the left-hand side is the *spherical Hecke algebra* (to be defined in the first section). We prove this in Theorem 5.7. A lot of the techniques that we employ here are similar to the ideas employed in proving the Bernstein presentation. The first two sections of this chapter are devoted to producing the Satake isomorphism. We follow [HKP10, Section 4.1-4.3]. Throughout this chapter, we seek to be very explicit with our computations.

Chapter 6.3 and Chapter 6.4.1-6.4.2 is then devoted to highlighting the utility of the Satake isomorphism in the theory of automorphic representations, and the local Langland correspondence for $G = \mathrm{GL}_n$, respectively.

More precisely, in Chapter 6.3, we highlight how the Satake isomorphism allows one to characterise irreducible spherical representations of $G$. This has applications in representations of the adelic group $G(\mathbb{A}_{\mathbb{Q}})$, as all but finitely many of the local components of an automorphic representation are spherical, as highlighted by Flath's theorem.

Chapter 6.4 is dedicated to expositing special cases of the local Langland correspondence for $G = \mathrm{GL}_n(\mathbb{Q}_p)$. This conjecture seeks a canonical bijection between admissible, irreducible $\mathrm{GL}_n(\mathbb{Q}_p)$-modules and $F$-semisimple Weil-Deligne representations.
In particular, Chapter 6.4.1 talks about the unramified local Langland correspondences, which posits a bijection between unramified $\mathrm{GL}_n(\mathbb{Q}_p)$-modules and unramified and unramified Weil-Deligne representations. This follows by using the Satake isomorphism and arguments using highest weight theory.
Chapter 6.4.2 concerns itself with the tamely ramified local Langland correspondence,

also known as the *Deligne-Langlands correspondence*. [KL87] was the first to provide a proof, and gives a geometric construction of irreducible tamely ramified representations of $\mathcal{H}$. This is then related to tamely ramified Weil-Deligne representations to prove the conjecture.

## 5.1 The Spherical Hecke Algebra

We begin by recording the result from [Bru64, Théorème 12.2]:

**Theorem 5.1** (Cartan Decomposition)**.**

$$G = \bigsqcup_{\substack{\mu \in X_*(T) \\ \mu \text{ dominant}}} K\pi^\mu K.$$

See [Bum10, Proposition 35] for a proof of this for $G = \mathrm{GL}_n(F)$.

Then, define the algebra

$$\mathcal{H}_{\mathrm{sph}} := C_c(K\backslash G/K),$$

called the *spherical Hecke algebra*. We equip it with an algebra structure by convolution, with Haar measure giving $K$ measure one. The result that we eventually wish to produce is the following isomorphism

$$\mathcal{H}_{\mathrm{sph}} \cong \mathcal{R}^W,$$

where $\mathcal{R}^W$ is the set of $W$-invariant elements of $\mathcal{R}$. That is,

$$\mathcal{R}^W = \{\pi^\mu : w(\mu) = \mu \text{ for all } \mu \in X_*(T)\} = \{\pi^\mu : w\pi^\mu w^{-1} = \pi^\mu \text{ for all } \mu \in X_*(T)\}.$$

This isomorphism is the so-called *Satake isomorphism*.

By Theorem 5.1, $\mathcal{H}_{\mathrm{sph}}$ has a $\mathbb{C}$-basis given by $\{K\pi^\mu K : \mu \text{ dominant}\}$. Define by $e_K$ the idempotent element $e_K := \mathbb{1}_K / \mathrm{vol}(K)$. Indeed,

$$(e_K * e_K)(x) = \frac{1}{(\mathrm{vol}(K))^2} \int_K \mathbb{1}_K(xg^{-1})\,\mathrm{d}g = \frac{\mathrm{vol}(Kx \cap K)}{\mathrm{vol}(K)} = \begin{cases} \frac{1}{\mathrm{vol}(K)} \text{ if } x \in K \\ 0 \text{ otherwise} \end{cases} = e_K(x).$$

Recall that

$$K = \bigsqcup_{w \in W} IwI,$$

and that as a consequence $\mathbb{1}_{T_{\mathcal{O}}NK} = \sum_{w\in W} v_w$. Further, observe that

$$e_K = \frac{\mathbb{1}_K}{\mathrm{vol}(K)} \frac{\sum_{w\in W} T_w}{\sum_{w\in W} \mathrm{vol}(IwI)} = \frac{\sum_{w\in W} T_w}{\sum_{w\in W} q^{\ell(w)}}, \tag{5.1}$$

and so $e_K \in \mathcal{H}$. One may also show the idempotency of $e_K$ from the identity (5.1), using the Iwahori-Matsumoto presentation of $\mathcal{H}$.

As subrings of the Iwahori-Hecke algebra $\mathcal{H}$, we have the following equality:

**Proposition 5.2.** $\mathcal{H}_{\mathrm{sph}} = e_K \mathcal{H} e_K$.

*Proof.* Let $h_K \in \mathcal{H}_{\mathrm{sph}}$. We wish to show that $h_K = e_K h e_K$ for some $h \in \mathcal{H}$. But by the idempotency of $e_K$, we have that $e_K h_K e_K = e_K h e_K$, and thus it is sufficient to show that $h_K = e_K h e_K$. We have,

$$(e_K * \mathbb{1}_{K\pi^\mu K})(x) = \frac{1}{\mathrm{vol}(K)} \int_K \mathbb{1}_{K\pi^\mu K}(xg^{-1})\,\mathrm{d}k = \mathrm{vol}(K\pi^\mu K \cap Kx) = \begin{cases} 1 \text{ if } x \in K\pi^\mu K \\ 0 \text{ otherwise.} \end{cases}$$

We now show the $\supseteq$ direction. Let $h \in \mathcal{H}$, and $k_1, k_2 \in K$. Then,

$$(e_K * h * e_K)(k_1 x k_2) = \frac{1}{(\mathrm{vol}(K))^2} \iint_K h((k_1xk_2)g_1^{-1}g_2^{-1})\,\mathrm{d}g_1\,\mathrm{d}g_2 = \iint_K h(xg_1^{-1}g_2^{-1})\,\mathrm{d}g_1\,\mathrm{d}g_2,$$

where the last equality follows since $h$ is a linear combination of indicator functions on the double cosets $K\pi^\mu K$, for $\mu$ dominant. Thus, $h \in \mathcal{H}_{\mathrm{sph}}$, as required. □

This shows that

$$e_K * \mathbb{1}_{K\pi^\mu K} = \mathbb{1}_{K\pi^\mu K}.$$

Let

$$\mathcal{M}_{\mathrm{sph}} := C_c(T_{\mathcal{O}}N\backslash G/K),$$

which we may identify within our universal principal series module $\mathcal{M}$:

**Proposition 5.3.** *$\mathcal{M}_{\mathrm{sph}} = \mathcal{M}e_K$, and $\mathcal{M}e_K$ is a $\mathcal{H}$-submodule of $\mathcal{M}$.*

*Proof.* There is a well-defined right $\mathcal{H}$-action on $\mathcal{M}$, and since $e_K \in \mathcal{H}$ by (5.1), it follows that $\mathcal{M}e_K$ is a $\mathcal{H}$-submodule of $\mathcal{M}$. $e_K$ is also an element of $\mathcal{H}_{\mathrm{sph}}$ by construction.

For the $\subseteq$, it is sufficient to show that $m_K = m_K e_K$ for $m_K \in \mathcal{M}_{\mathrm{sph}}$ by the idempotency of $e_K$. By a direct computation,

$$(m_K * e_K)(x) = \frac{1}{\mathrm{vol}(K)} \int_K m_K(xk^{-1})\,\mathrm{d}k = \frac{1}{\mathrm{vol}(K)} \int_K m_K(x)\,\mathrm{d}k = m_K(x),$$

where the second equality follows by the right $K$-invariance of $m_K$, and the last equality follows since $\mathrm{d}k$ is normalised to give $K$ measure one.

For the $\supseteq$ side, let $v_w$ be a basis element of $\mathcal{M}$, $w \in \widetilde{W}$. Then, for some $k \in K$,

$$(v_w * e_K)(xk) = \frac{1}{\mathrm{vol}(K)} \int_K \mathbb{1}_{T_\mathcal{O}NwI}(xkg^{-1})\,\mathrm{d}g = \frac{1}{\mathrm{vol}(K)} \int_K \mathbb{1}_{T_\mathcal{O}NwI}(xg^{-1}),$$

and thus $v_w * e_K$ is right $K$-invariant, and an element of $\mathcal{M}_{\mathrm{sph}}$. □

It follows then that $\mathcal{M}_{\mathrm{sph}}$ is spanned by elements of the form $v_w * e_K$, which form a $\mathbb{C}$-basis for it.

**Proposition 5.4.** $v_w * e_K = \mathbb{1}_{T_\mathcal{O}NwK}$.

*Proof.* Computing directly,

$$(v_w * e_K)(x) = \frac{1}{\mathrm{vol}(K)} \int_K \mathbb{1}_{T_\mathcal{O}NwI}(xg^{-1})\,\mathrm{d}g = \frac{\mathrm{vol}(x^{-1}T_\mathcal{O}NwI \cap K)}{\mathrm{vol}(K)}.$$

Let $x \in T_\mathcal{O}NwK$, then $x^{-1} \in Kw^{-1}NT_\mathcal{O}$. Observe that

$$Kw^{-1}NT_\mathcal{O} \cdot T_\mathcal{O}NwI = KNwI \supseteq K,$$

and thus $\mathrm{vol}(x^{-1}T_\mathcal{O}NwI \cap K) = \mathrm{vol}(K)$.

Suppose now that $x \notin T_\mathcal{O}NwK$, but that $x^{-1}T_\mathcal{O}NwI \cap K \neq \emptyset$. Thus, there is some element $y = x^{-1}t_\mathcal{O}nwi = k$, which implies that $x \in T_\mathcal{O}NwI \cdot K = T_\mathcal{O}NwK$, a contradiction. Therefore,

$$\frac{\mathrm{vol}(x^{-1}T_\mathcal{O}NwI \cap K)}{\mathrm{vol}(K)} = \begin{cases} 1 \text{ if } x \in T_\mathcal{O}NwK \\ 0 \text{ otherwise} \end{cases} = \mathbb{1}_{T_\mathcal{O}NwK}.$$

□

There is a right $\mathcal{H}_{\mathrm{sph}}$ action on $\mathcal{M}_{\mathrm{sph}}$ given by convolution. Further, there is a left $\mathcal{R}$-action on $\mathcal{M}_{\mathrm{sph}}$ inherited from the one on $\mathcal{M}$:

$$\pi^\mu \cdot (v_w * e_K) = (\pi^\mu v_w) * e_K = v_{\pi^\mu w} * e_K = \mathbb{1}_{T_\mathcal{O}N\pi^\mu wK}.$$

One readily checks that these actions commute, and $\mathcal{M}_{\mathrm{sph}}$ is a $(\mathcal{R}, \mathcal{H}_{\mathrm{sph}})$-bimodule.

Let $\pi^\mu w \in \widetilde{W}$, where $w \in W$. Then, taking a representative of $w$ in $K$, the $\mathbb{C}$-basis of $\mathcal{M}_{\mathrm{sph}}$ reduces to $\{\mathbb{1}_{T_\mathcal{O}N\pi^\mu K} : \mu \in X_*(T)\}$. In the case where $\mu = 0$, the vector

$$v_{\mathrm{sph}} := \mathbb{1}_{T_\mathcal{O}NK},$$

is called the *spherical vector*. Indeed, this is not the first time we have encountered this, and it will also play an important role in our computations in this chapter.

**Proposition 5.5.** *$\mathcal{M}_{\mathrm{sph}}$ is free of rank one as a $\mathcal{R}$-module, with generator $v_{\mathrm{sph}} := \mathbb{1}_{T_{\mathcal{O}}NK}$.*

*Proof.* By the left $\mathcal{R}$-module structure of $\mathcal{M}_{\mathrm{sph}}$, we have $\mathbb{1}_{T_{\mathcal{O}}N\pi^\mu K} = q^{\langle\rho,\mu\rangle}\pi^\mu \cdot v_{\mathrm{sph}}$. So, there is a left $\mathcal{R}$-module isomorphism

$$\mathcal{R} \cong \mathcal{M}_{\mathrm{sph}}, \quad \pi^\mu \longmapsto \pi^\mu \cdot v_{\mathrm{sph}}.$$

Consequently, we have a ring isomorphism

$$\mathcal{R} \cong \mathrm{End}_{\mathcal{R}}(\mathcal{M}_{\mathrm{sph}}), \quad r \longmapsto (r'v_{\mathrm{sph}} \longmapsto rr'v_{\mathrm{sph}}),$$

and the result follows. □

## 5.2 The Satake Transform

We now define a homomorphism $\mathcal{H}_{\mathrm{sph}} \to \mathcal{R}$, called the *Satake transform*. By the $(\mathcal{R}, \mathcal{H}_{\mathrm{sph}})$-bimodule structure of $\mathcal{M}_{\mathrm{sph}}$, it follows that some $h \in \mathcal{H}_{\mathrm{sph}}$ is an $\mathcal{R}$-module endomorphism of $\mathcal{M}_{\mathrm{sph}}$ since

$$r \cdot (m * h) = (r \cdot m) * h,$$

for $r \in \mathcal{R}$, $m \in \mathcal{M}_{\mathrm{sph}}$. Thus, there is a composition of homomorphisms

$$\begin{tikzcd} \mathcal{H}_{\mathrm{sph}} \arrow[r] \arrow[rr, bend left, "\mathcal{S}at"] & \mathrm{End}_{\mathcal{R}}(\mathcal{M}_{\mathrm{sph}}) \arrow[r, "\simeq"] & \mathcal{R} \end{tikzcd}$$

where $\mathcal{S}at$ is the Satake transform. We denote by $h^\vee$ the image of $h$ under $\mathcal{S}at$. Then, by Proposition 5.5 and the bimodule structure on $\mathcal{M}_{\mathrm{sph}}$, $\mathcal{S}at$ is characterised by the property:

$$m * h = h^\vee \cdot m, \tag{5.2}$$

for some $m \in \mathcal{M}_{\mathrm{sph}}$.

Let $\mathcal{L} := \operatorname{Frac}\mathcal{R}$. Thus, $\mathcal{L}^W$ is a subfield of $\mathcal{L}$. Then, define:

$$\mathcal{H}_{\mathrm{gen}} := \mathcal{L}^W \otimes_{\mathcal{R}^W} \mathcal{H}, \quad \mathcal{M}_{\mathrm{gen}} := \mathcal{L} \otimes_{\mathcal{R}} \mathcal{M},$$

which allows us to define a new intertwiner $\mathcal{K}_w \in \mathrm{End}_{\mathcal{H}_{\mathrm{gen}}}(\mathcal{M}_{\mathrm{gen}})$ by:

$$\mathcal{K}_w := \left( \prod_{\alpha \in \mathrm{Inv}(w)} \frac{1 - \pi^{\alpha^\vee}}{1 - q^{-1}\pi^{\alpha^\vee}} \right) \mathcal{I}_w. \tag{5.3}$$

For simple $\alpha$, we have that $\mathcal{K}_{s_\alpha}^2 = 1$, and it follows from Lemma 4.1(ii) that

$$\mathcal{K}_{w_1 w_2} = \mathcal{K}_{w_1} \circ \mathcal{K}_{w_2},$$

for $w_1, w_2 \in W$. By our construction, it follows as a consequence of the Gindikin-Karpelevich formula (Corollary 4.5) that $\mathcal{K}_w$ fixes the spherical vector — that is, $\mathcal{K}_w(v_{\mathrm{sph}}) = v_{\mathrm{sph}}$. It is also immediate by our construction that $\mathcal{K}_w$ satisfies all the properties of Lemma 4.1.

Applying $\mathcal{K}_w$ to both sides of (5.2), we see that:

$$\begin{aligned} &(h^\vee \circ \mathcal{K}_w)(v_{\mathrm{sph}}) = (\mathcal{K}_w \circ w(h^\vee))(v_{\mathrm{sph}}) \\ \iff &h^\vee \cdot v_{\mathrm{sph}} = w(h^\vee) \cdot v_{\mathrm{sph}} \\ \iff &h^\vee = w(h^\vee), \end{aligned}$$

where the last implication follows from the freeness of $\mathcal{M}_{\mathrm{sph}}$. Thus $h^\vee \in \mathcal{R}^W$, and the Satake transform is in fact a map $\mathcal{H}_{\mathrm{sph}} \to \mathcal{R}^W$.

**Proposition 5.6** (HKP10, Section 4.2). *The coefficient of $\pi^\nu$ in $h^\vee$ is given by*

$$a_\nu = \delta_B^{-1/2}(\pi^\nu) \int_N h(n\pi^\nu)\, \mathrm{d}n.$$

*Proof.* Writing $h^\vee$ in terms of basis elements of $\mathcal{R}$, we have:

$$h^\vee = \sum_{\mu \in X_*(T)} a_\mu \pi^\mu.$$

Then, substituting into (5.2),

$$\begin{aligned} h^\vee \cdot v_{\mathrm{sph}} &= \sum_{\mu \in X_*(T)} a_\mu \pi^\mu \cdot v_{\mathrm{sph}} = \sum_{\mu \in X_*(T)} a_\mu \sum_{w \in W} \pi^\mu \cdot v_w \\ &= \sum_{\mu \in X_*(T)} q^{-\langle \rho, \mu \rangle} a_\mu \sum_{w \in W} v_{\pi^\mu w}. \end{aligned}$$

Evaluating at $\pi^\nu$, we find that

$$v_{\pi^\mu w}(\pi^\nu) = \begin{cases} 1 \text{ if } \mu = \nu,\ w = 1, \\ 0 \text{ otherwise}, \end{cases}$$

and so:

$$(h^\vee \cdot v_{\text{sph}})(\pi^\nu) = q^{-\langle \rho,\nu\rangle} a_\nu.$$

Recall that $\delta_B^{1/2}(\pi^\mu) = q^{-\langle \rho,\mu\rangle}$. Then, using (5.2), we find that

$$a_\nu = q^{\langle \rho,\nu\rangle}(v_{\text{sph}} * h)(\pi^\nu) = \delta_B^{-1/2}(\pi^\nu)\int_G v_{\text{sph}}(g)h(\pi^\nu g^{-1})\,\mathrm{d}g.$$

Then, by the Iwasawa decomposition and Fubini's theorem,

$$\begin{aligned} a_\nu &= \delta_B^{-1/2}(\pi^\nu)\int_G \mathbb{1}_{T_{\mathcal{O}}NK}(\pi^\nu g^{-1})h(g)\,\mathrm{d}g \\ &\overset{g\mapsto g\pi^\nu}{=} \delta_B^{-1/2}(\pi^\nu)\int_G \mathbb{1}_{T_{\mathcal{O}}NK}(g^{-1})h(g\pi^\nu)\,\mathrm{d}g \\ &= \delta_B^{-1/2}(\pi^\nu)\int_K\int_N\int_{T_{\mathcal{O}}} h(knt\pi^\nu)\,\mathrm{d}t\,\mathrm{d}n\,\mathrm{d}k \\ &= \delta_B^{-1/2}(\pi^\nu)\int_N h(n\pi^\nu)\,\mathrm{d}n, \end{aligned}$$

where the last equality follows by the $K$-biinvariance of $h$, and the fact that the measures $\mathrm{d}t$ and $\mathrm{d}k$ are normalised to give $T_{\mathcal{O}}$ and $K$ volume one, respectively. □

We can now prove the following [HKP10, Section 4.3]:

**Theorem 5.7** (Satake Isomorphism)**.** *The Satake transform $\mathcal{S}at : \mathcal{H}_{\text{sph}} \to \mathcal{R}^W$ is an isomorphism.*

*Proof.* Let $h_\mu := \mathbb{1}_{K\pi^\mu K}$, with $\mu$ a dominant co-weight. By the Cartan decomposition, this is a $\mathbb{C}$-basis for $\mathcal{H}_{\text{sph}}$. Recall that $\{\pi^\mu : \mu \in X_*(T)\}$ form a $\mathbb{C}$-basis for $\mathcal{R}$. Then, for $\nu$ a dominant co-weight, the elements

$$s_\nu := \sum_{\lambda\in W\nu}\pi^\lambda,$$

form a $\mathbb{C}$-basis for $\mathcal{R}^W$.

Then, following the same argument used in Proposition 5.6, the coefficient of $s_\nu$ in $h_\mu^\vee$ is given by

$$c_{\mu\nu} = \delta_B^{-1/2}(\pi^\nu)\int_N \mathbb{1}_{K\pi^\mu K}(n\pi^\nu)\,\mathrm{d}n = \delta_B^{-1/2}(\pi^\nu)\,\mathrm{vol}(K\pi^\mu K\cap N\pi^\nu).$$

The isomorphism then follows from proving the following:

$$\text{If } K\pi^\mu K\cap N\pi^\nu \neq \emptyset, \text{ then } \nu \le \mu.$$

The proof of this fact can be found in [BT72, 4.4.4]. Here, $\nu \leq \mu$ if $\mu - \nu$ is a $\mathbb{Z}_+$-linear combination of positive roots. Let $\mu = \nu$. Then, take inverses,

$$c_{\mu\mu} = \mathrm{vol}(K\pi^{\mu}K\pi^{-\mu} \cap N),$$

and observe that $1 = \mathrm{vol}(N \cap K) \leq c_{\mu\mu}$, and thus the diagonal entries are non-zero. $\square$

[HKP10, Lemma 2.3.1] states that the centre of $\mathcal{H}$ — denoted by $Z(\mathcal{H})$ — is isomorphic to $\mathcal{R}^W$. Following this, the corollary is immediate:

**Corollary 5.8.** $Z(\mathcal{H}) \cong \mathcal{H}_{\mathrm{sph}}$.

We remark that $Z(\mathcal{H}) \neq \mathcal{H}_{\mathrm{sph}}$, and are merely isomorphic. For instance, $1 \in \mathcal{H}$ is the identity of $Z(\mathcal{H})$, but the identity in $\mathcal{H}_{\mathrm{sph}}$ is the idempotent element $e_K$. So, one may find two isomorphic copies of $\mathcal{R}^W$ inside $\mathcal{H}$.

## 5.3 Spherical Principal Series Representations

For this section, we follow [Mac13].

**Definition 5.9.** A smooth representation $(\sigma, V)$ of $G$ is *unramified* (or *spherical*) if it contains a non-zero $K$-fixed vector — that is, $V^K \neq 0$.

There is a $\mathcal{H}_{\mathrm{sph}}$-action on $V^K$ which gives it the structure of a $\mathcal{H}_{\mathrm{sph}}$-module. If $V$ is a spherical irreducible $G$-module, then $V^K$ is an irreducible $\mathcal{H}_{\mathrm{sph}}$-module. Thus, there is an exact functor

$$(-)^K : \mathbf{Rep}(G) \longrightarrow \mathbf{Rep}(\mathcal{H}_{\mathrm{sph}}), \quad V \longmapsto V^K,$$

from the category of spherical $G$-modules to the category of $\mathcal{H}_{\mathrm{sph}}$-modules which maps irreducible objects to irreducible objects.

The Satake isomorphism implies that $\mathcal{H}_{\mathrm{sph}}$ is abelian, and thus by Schur's lemma $\dim V^K = 1$ for irreducible $V^K$. That is, $\mathcal{H}_{\mathrm{sph}}$ acts by a character.

Through the Satake isomorphism, we view this as a character on $\mathcal{R}^W$. Since $\mathcal{R}$ is the Iwahori-Hecke algebra of $T$, given by $C_c(T/T_{\mathcal{O}})$, we may identify a character on $\mathcal{R}^W$ with a $W$-invariant, unramified character.

We can do this the other way — that is, we wish to take an unramified character $\chi$ of $T$, and then produce some unramified representation of $G$. Let $\chi$ be an unramified character of $T$, which we view as a character of $B$ through the decomposition $B = TN$. Then, we normally induce from $B$ to $G = BK$:

$$\text{n-Ind}_B^{BK}(\chi) = \{f : G \to \mathbb{C} : f(bk) = \delta_B^{1/2}(b)\chi(b)f(b)\}. \tag{5.4}$$

Such a representation (5.4) is called a *spherical principal series representation*.

Since $\chi$ is trivial on $T_{\mathcal{O}} = T \cap K$, it follows that $\chi$ is also trivial on $B \cap K$. As such,

$$\dim \text{n-Ind}_B^{BK}(\chi)^K = 1, \tag{5.5}$$

and this is thus an irreducible $\mathcal{H}_{\text{sph}}$-module. The $K$-fixed vectors of $\text{n-Ind}_B^G \chi^K$, as a $\mathcal{H}_{\text{sph}}$-module, is the unique character $\chi : \mathcal{R}^W \to \mathbb{C}$. It follows by the Satake isomorphism that this character $\chi$ is $W$-invariant.

It is known that $\text{n-Ind}_B^G \chi$ has precisely one irreducible spherical subquotient [Mac13]. Using the Satake isomorphism, one may then prove the following bijection:

$$\left\{\begin{matrix}\text{Irreducible, unramified}\\ \text{representations of } G\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Unramified characters}\\ \text{of } T \text{ up to the } W\text{-action}\end{matrix}\right\}, \tag{5.6}$$

where the right-hand side can be thought of irreducible $\mathcal{H}_{\text{sph}}$-modules.

The importance of this bijection lies in the theory of *automorphic representations*. [Bum09, Chapter 3] gives a treatment of this subject. To begin, we first define the *ring of adèles of* $\mathbb{Q}$ $\mathbb{A}_{\mathbb{Q}}$. This is the restricted product of all completions of $\mathbb{Q}$ with respect to its rings of integers. That is,

$$\mathbb{A}_{\mathbb{Q}} := \left\{(x_v) \in \mathbb{R} \times \prod_{v \text{ prime}} \mathbb{Q}_v : x_v \in \mathbb{Z}_v \text{ for all but finitely many places } v\right\}.$$

One typically denotes the restricted product in the following way:

$$\mathbb{A}_{\mathbb{Q}} = \mathbb{R} \times \prod_{v \text{ prime}}' {}'\mathbb{Q}_v.$$

One then shows that

$$G(\mathbb{A}_{\mathbb{Q}}) = \prod_{\mathbb{Z}_v}' G(\mathbb{Q}_v).$$

And using a result of [Fla79], there is a way of characteristing most admissible representations of $G(\mathbb{A}_{\mathbb{Q}})$ using its spherical representations:

**Theorem 5.10** ([Fla79]). *Any irreducible admissible representation $\rho$ of $G(\mathbb{A}_{\mathbb{Q}})$ is uniquely a* *restricted tensor product of irreducible smooth representations $\rho_v$ of $G(\mathbb{Q}_v)$, with $\rho_v$ spherical for all but finitely many $v$.*

## 5.4 Local Langlands Correspondence for $\mathrm{GL}_n(\mathbb{C})$

For the remainder of this thesis, we specialise $G$ to be $\mathrm{GL}_n(\mathbb{Q}_p)$. Then, the *local Langlands correspondence* for $\mathrm{GL}_n$ posits the following canonical bijection:

$$\left\{\begin{matrix}\text{Admissible, irreducible}\\ \text{complex representations}\\ \text{of } \mathrm{GL}_n(\mathbb{Q}_p)\end{matrix}\right\}_{/\cong} \longleftrightarrow \left\{\begin{matrix}\text{Finite-dimensional}\\ F\text{-semisimple}\\ \text{Weil-Deligne representations}\end{matrix}\right\}_{/\cong} \tag{5.7}$$

**Definition 5.11.** Consider the short-exact sequence:

$$0 \longrightarrow I_{\mathbb{Q}_p} \longrightarrow \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \xrightarrow{p} \mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \cong \widehat{\mathbb{Z}} \longrightarrow 0$$

where $I_{\mathbb{Q}_p} := \ker p$ is the *inertia subgroup*. Then, the *Weil-Deligne group* $W_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ is defined by

$$W_{\mathbb{Q}_p} := p^{-1}(\mathbb{Z}).$$

**Definition 5.12.** An $n$-dimensional *Weil-Deligne representation* is a triple $(\rho, V, N)$ where $V$ is an $n$-dimensional complex vector space, and

$$\rho : W_{\mathbb{Q}_p} \longrightarrow \mathrm{GL}(V)$$

is a continuous representation, and $N \in \mathrm{End}(V)$ is a nilpotent element such that

$$\rho(x)N\rho(x)^{-1} = |x|N,$$

for all $x \in W_{\mathbb{Q}_p}$. We say that a Weil-Deligne representation is *$F$-semisimple* if $\rho$ is semisimple.

The general proof of (5.7) is involved (see [Wed00] for an exposition), and we will not seek to exposit this. We refer the interested reader to [BH06], who gives an extensive treatment on the techniques used to prove this conjecture in the $G = \mathrm{GL}_2(F)$ case. For us, we restrict our attention to unramified and tamely ramified representations in the following subsections, and we will attempt to showcase the utility of the Satake isomorphism in proving these results.

Throughout, we will follow a set of lecture notes written by Romanov and Williamson [RW21]. In particular, the chapters that we employ are [RW21, Chapter 9.4, Chapter 10.3, Chapter 11.2, Chapter 11.3].

### 5.4.1 Unramified Local Langlands Correspondence

For this section, we follow [RW21, Chapter 9.4, Chapter 11.2].

**Definition 5.13.** A *Weil-Deligne* representation is *unramified* if $N = 0$ and $\rho$ factors through $\mathbb{Z}$ — that is,

$$W_{\mathbb{Q}_p} \twoheadrightarrow \mathbb{Z} \hookrightarrow \mathrm{GL}(V)$$

The *unramified Local Langlands conjecture* for $\mathrm{GL}_n$ is then given by:

$$\left\{\begin{matrix}\text{Unramified, admissible}\\ \text{complex representations}\\ \text{of } \mathrm{GL}_n(\mathbb{Q}_p)\end{matrix}\right\}_{/\cong} \longleftrightarrow \left\{\begin{matrix}\text{Finite-dimensional}\\ \text{unramified}\\ \text{Weil-Deligne representations}\end{matrix}\right\}_{/\cong} \tag{5.8}$$

Let ${}^LG$ be the group with root datum dual to that of $G$, called the *Langlands dual group*. By our construction, one obtains an isomorphism

$$X_*(T) \cong X^*({}^LT).$$

In the case where $G = \mathrm{GL}_n$, ${}^L\mathrm{GL}_n = \mathrm{GL}_n$. However, we will not omit the superscripted $L$ to emphasise that we are working with the Langlands dual group.

Recall that the irreducible representations of ${}^L\,\mathrm{GL}_n(\mathbb{Q}_p)$ are in bijection with dominant weights. And since dominant weights are in bijection with elements of $\mathbb{C}[X^*({}^LT)]^W$, the Satake isomorphism thus implies that

$$\mathcal{H}_{\mathrm{sph}} \cong R({}^L\mathrm{GL}_n(\mathbb{Q}_p)),$$

where $R(-)$ is the representation ring of ${}^L\,\mathrm{GL}_n(\mathbb{Q}_p)$. As such, irreducible representations of $\mathcal{H}_{\mathrm{sph}}$ can be identified with characters of the representation ring

$$\chi : R({}^L\,\mathrm{GL}_n(\mathbb{Q}_p)) \longrightarrow \mathbb{C}.$$

By the Chevalley restriction theorem, we have that $R({}^L\,\mathrm{GL}_n(\mathbb{Q}_p)) \cong R({}^LT)^W$, and thus characters of the representation ring are in bijection with conjugacy classes of semisimple elements on ${}^L\,\mathrm{GL}_n(\mathbb{Q}_p)$. The above argument using highest weight theory can be summarised using the commutative diagram:

$$\begin{tikzcd} R({}^L\,\mathrm{GL}_n(\mathbb{Q}_p)) \arrow[r, hook] \arrow[dr, "\simeq"'] & R({}^LT) = \mathbb{C}[X^*({}^LT)] \\ & \mathbb{C}[X^*({}^LT)]^W \arrow[u, hook] \end{tikzcd}$$

This conjugacy class is then in bijection with unramified Weil-Deligne representations by definition. In short, the argument seen in [RW21, Chapter 9.4] for the unramified local

Langlands correspondence can be summarised as follows:

$$
\begin{aligned}
\left\{ \begin{matrix} \text{Unramified} \\ \text{representations} \\ \text{of } \mathrm{GL}_n(\mathbb{Q}_p) \end{matrix} \right\}_{/\cong} &\xleftrightarrow{\text{Chapter 6.3}} \left\{ \text{Irreducible } \mathcal{H}_{\mathrm{sph}}\text{-modules} \right\}_{/\cong} \\
&\xleftrightarrow{\text{Commutativity of } \mathcal{H}_{\mathrm{sph}}} \left\{ \begin{matrix} \text{Characters of } \mathcal{H}_{\mathrm{sph}} \\ \text{up to the } W\text{-action} \end{matrix} \right\} \\
&\xleftrightarrow{\text{Satake Isomorphism}} \left\{ \begin{matrix} \text{Characters of} \\ R({}^L\mathrm{GL}_n(\mathbb{Q}_p)) \end{matrix} \right\} \\
&\xleftrightarrow{\text{Chevalley Restriction}} \left\{ \begin{matrix} \text{Semisimple conjugacy} \\ \text{classes of } {}^L\mathrm{GL}_n(\mathbb{Q}_p) \end{matrix} \right\} \\
&\xleftrightarrow{\text{Definition 5.13}} \left\{ \begin{matrix} \text{Unramified} \\ \text{Weil-Deligne} \\ \text{Representations} \end{matrix} \right\}_{/\cong}
\end{aligned}
$$

The above diagram is essentially borrowed from [RW21, Chapter 9.4, pg. 73]. We have re-labelled some of the arrows so that it makes sense within the context of this thesis.

### 5.4.2 Deligne-Langlands Correspondence

For this section, we follow [RW21, Chapter 10.3, Chapter 11.3]. As before, we also let $G = \mathrm{GL}_n(\mathbb{Q}_p)$. We say that a representation of $G$ is *tamely ramified* if it admits an Iwahori-fixed vector. In line with our notation, we let $I$ denote the Iwahori subgroup of $G$.

On the other hand, we say that a Weil-Deligne representation $(\rho, V, N)$ is *tamely ramified* if $\rho$ acts trivially on the inertia subgroup $I_{\mathbb{Q}_p}$. Then, $F$-semisimple Weil-Deligne representations are equivalent to conjugacy classes of pairs $(s, N)$, where $s$ and $N$ are semisimple and nilpotent elements respectively such that $sNs^{-1} = qN$. The Deligne-Langlands correspondence posits the following bijection:

$$
\left\{ \begin{matrix} \text{Tamely ramified} \\ G\text{-representations} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Tamely ramified} \\ \text{Weil-Deligne} \\ \text{representations} \end{matrix} \right\} \tag{5.9}
$$

This was originally proved in [KL87, Theorem 7.12]. Further, [Bor76] showed that there is an equivalence of categories:

$$
\left\{ \begin{matrix} \text{Admissible} \\ \text{Representations of } G \\ \text{with } I\text{-fixed vectors} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Finite-dimensional} \\ \text{Representations of } \mathcal{H} \end{matrix} \right\}.
$$

Accordingly, we will re-write the bijection:

$$\{\text{Irreducible finite-dimensional } \mathcal{H}\text{-modules}\} \longleftrightarrow \left\{\begin{smallmatrix}\text{Tamely ramified}\\ \text{Weil-Deligne}\\ \text{representations}\end{smallmatrix}\right\} \tag{5.10}$$

Let $\mathrm{Vect}^{\mathbb{C}}_G(X)/\cong$ be the isomorphism classes of $G$-equivariant vector bundles over $X$. The Whitney sum $\oplus$ (see [Hat17, Chapter 1.1]) of vector bundles give this the structure of a semigroup. Then, the *$G$-equivariant $K$-group of $X$* is:

$$K^G(X) := \mathrm{Groth}(\mathrm{Vect}^{\mathbb{C}}_G(X)/\cong, \oplus).$$

Now, let $\mathcal{N}$ and $\mathcal{B}$ be the variety of nilpotent elements and Borel subgroups of $G$, respectively. Then, the *Steinberg variety* (introduced first in [Ste76]) of $G$ is:

$$\mathrm{St} := \{(x, B, B') \in \mathcal{N} \times \mathcal{B} \times \mathcal{B} : x \in \mathrm{Lie}(B) \cap \mathrm{Lie}(B')\}.$$

Having now established the above notation, we record a non-trivial result from [KL87]:

**Theorem 5.14** (Theorem 3.5, KL87)**.**

$$\mathcal{H} \cong K^{{}^LG\times\mathbb{C}^\times}(\mathrm{St}).$$

Now, let $\pi : T^*\mathcal{B} \to \mathcal{N}$ be the Springer resolution (see [CG09]), and let $\mathcal{B}_x := \pi^{-1}(x)$ the Springer fibre of an element $x \in \mathcal{N}$. Let

$$Z_{{}^LG\times\mathbb{C}^\times}(x) := \{(g, z) \in {}^LG \times \mathbb{C}^\times : z \cdot gxg^{-1} = x\}.$$

Then, Theorem 5.14 implies that there is an action of $\mathcal{H}$ on $K^{Z_{{}^LG\times\mathbb{C}^\times}(x)}(\mathcal{B}_x)$. [KL87] used this fact to show that the $K$-theory of all Springer fibres provides all irreducible $\mathcal{H}$-modules. Then, to prove the Deligne-Langlands correspondence, they then related the $K$-theory of Spring fibres to tamely ramified Weil-Deligne representations. All of these (extremely non-trivial) facts, together, then leads to a proof of the Deligne-Langlands correspondence (5.10).

# Bibliography


[BH06] C.J. Bushnell and G. Henniart. *The Local Langlands Conjecture for $GL(2)$*. A Series of Comprehensive Studies in Mathematics vol. 335, Springer-Verlag, New York-Berlin, 2006.

[BKP14] A. Braverman, D. Kazhdan, and M. Patnaik. Iwahori-Hecke Algebras for $p$-adic Loop Groups. *arXiv:1403.0602*, 2014.

[Bor76] A. Borel. Admissible Representations of a Semi-Simple Group over a Local Field with Vectors Fixed under an Iwahori Subgroup. *Inventiones Mathematicae, Springer-Verlag*, 35(pg. 233-259), 1976.

[Bru54] F. Bruhat. Representations induites des groupes de lie semisimples complexes. *Comptes Rendues Acad. Sci. Paris*, 238(pg. 437-439), 1954.

[Bru64] F. Bruhat. Sur une classe de sous-groupes de Chevalley sur un corps p-adique. *Publications mathématiques de l'I.H.É.S.*, 23(pg. 45-74), 1964.

[BT72] F. Bruhat and J. Tits. Groupes reductifs sur un corps local. i. *Publications mathématiques de l'I.H.É.S.*, 41(pg. 5-251), 1972.

[Bum09] D. Bump. *Automorphic Forms and Representations*. Cambridge Studies in Advanced Mathematics (55), 2009.

[Bum10] D. Bump. Hecke Algebras. 2010.

[Cas80] W. Casselman. The Unramified Principal Series of $p$-adic Groups I. The Spherical Function. *Composito Mathematica*, 40(3), 1980.

[CG09] N. Chriss and V. Ginzburg. *Representation Theory and Complex Geometry*. Springer Science and Business Media, 2009.

[Che55] C. Chevalley. Sur certains groupes simples. *Tôhoku Math.*, 7(pg. 14-66), 1955.

[CKM98] N. Chriss and K. Khuri-Makdisi. On the Iwahori-Hecke algebra of a $p$-adic group. *International Math. Res. Notices*, (pg. 85-100), 1998.

[CSM95] R. Carter, G. Segal, and I. MacDonald. *Lectures on Lie Groups and Lie Algebras*. London Mathematical Society, Student Texts(32), 1995.

[Deo77] V.V. Deodhar. Some Characterizations of Bruhat Ordering on a Coxeter Group and Determination of the Relative Möbius Function. *Inventiones Mathematicae*, 39(pg. 187-198), 1977.

[Ehr34] C. Ehresmann. Sur la topologie de certains espaces homogenes. *Annals of Mathematics*, 35(2), 1934.

[Fla79] D. Flath. Decomposition of representations into tensor products. *Automorphic forms, representations and L-functions, Proc. Sympos. Pure Math.*, XXXIII(pg. 179-183), 1979.

[Fol95] G.B. Folland. *A Course in Abstract Harmonic Analysis*. CRC Press, 1995.

[Gin96] V. Ginzburg. Perverse sheaves on a loop group and Langlands duality. *Preprint*, 1996.

[Hat17] A. Hatcher. *Vector Bundles and $K$-Theory*. Allen Hatcher, 2017.

[HKP10] T.J. Haines, R.E Kottwitz, and A. Prasad. Iwahori-Hecke Algebras. *Journal of the Ramanujuan Mathematical Society*, 25(pg. 113-145), 2010.

[Hum90] J.E. Humphreys. *Reflection Groups and Coxeter Groups*. Cambridge University Press, 1990.

[Hum98] J. Humphreys. *Linear Algebraic Groups*. Graduate Texts in Mathematics, vol. 21, Springer-Verlag, New York-Berlin, 1998.

[IM65] N. Iwahori and H. Matsumoto. On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups. *Publications mathématiques de l'I.H.É.S.*, 25(pg. 5-48), 1965.

[Iwa86] K. Iwasawa. *Local Class Field Theory*. Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1986.

[Jon87] V. Jones. Hecke algebra representations of braid groups and link polynomials. *Annals of Mathematics*, 126(pg. 335-388), 1987.

[Kir97] A.A. Kirillov. Lectures on Affine Hecke Algebras and MacDonald's Conjectures. *Bulletin of the American Mathematical Society*, 34(3), 1997.

[KL79] D. Kazhdan and G. Lusztig. Representations of Coxeter groups and Hecke algebras. *Inventiones, Mathematicae, Springer-Verlag*, 53(2), 1979.

[KL87] D. Kazhdan and G. Lusztig. Proof of the Deligne-Langlands conjecture for Hecke algebras. *Invent. Math.*, 87(1, pg. 153-215), 1987.

[Lus83] G. Lusztig. Singularities, character formulas, and a $q$-analog of weight multiplicities. *Analyse et topologie sur les espaces singuliers, II, III*, 101(pg. 208-229), 1983.

[Mac13] Macky. Lecture 4: Spherical representations and Satake isomorphism. Number Theory Learning Seminar 2013-2014, Stanford University, from https://virtualmath1.stanford.edu/~conrad/JLseminar/Notes/, 2013.

[MS19] V. Miemetz and C. Stroppel. Affine Quiver Schur Slgebras and $p$-adic $\mathrm{GL}_n$. *Selecta Mathematica*, 2019.

[Mut18] D. Muthiah. On Iwahori-Hecke Algebras for $p$-adic Loop Groups: Double Coset Basis and Bruhat Order. *American Journal of Mathematics*, 140(1, pg. 221-244), 2018.

[MV00] I. Mirkovic and K. Vilonen. Perverse sheaves on affine Grassmanians and Langlands duality. *Math. Res. Lett.*, 7(no. 1, pg. 13-24), 2000.

[MV07] I. Mirković and K. Vilonen. Geometric Langlands duality and representations of algebraic groups over commutative rings. *Annals of Mathematics*, 166(45-143), 2007.

[Pra05] A. Prasad. On Bernstein's Presentation of Iwahori-Hecke algebras and representations of split reductive groups over non-archimedean local fields. *Bulletin of Kerala Mathematics Association (Special Issue on Harmonic Analysis and Quantum Groups*, 2005.

[Rud86] W. Rudin. *Real and Complex Analysis*. McGraw-Hill Book Company, 1986.

[RW21] A. Romanov and G. Williamson. Langlands correspondence and Bezrukavnikov's equivalence. *arXiv:2103.02329*, 2021.

[Sat63] I. Satake. Theory of spherical functions on reductive algebraic groups over $p$-adic fields. *Publications mathématiques de l'I.H.É.S.,*, 18(pg. 5-69), 1963.

[Spr98] T.A. Springer. *Linear Algebraic Groups*. Modern Birkhäuser Classics, Second Edition, 1998.

[Ste76] R. Steinberg. On the desingularisation of the unipotent variety. *Invent. Math.*, 36(pg. 209-224), 1976.

[Wedoo] T. Wedhorn. The local Langlands correspondnce for $\mathrm{GL}(n)$ over $p$-adic fields. *Mathematisches Institut der Universität zu Köln*, Lecture given at the School on Automorphic forms on $\mathrm{GL}(n)$(31 July to 11 August), 2000.