Homework 5
CS2233

1. Weak Induction in Steps (6 points)

Let P (n) be the following statement:

$$\sum_{i=0}^{n} 2^i = 2^{(n+1)} - 1$$

The following sub-problems guide you through a proof by weak induction that P (n) holds for all $n \in \mathbb{N}$.

(1) (1 points) What is the statement P (0)? Show that P (0) is true, which completes the base case.

A:
$$P(0) = \sum_{i=0}^{0} 2^i = 2^{(0+1)} - 1$$
$$LHS = 2^0 = 1$$
$$RHS = 2^1 - 1 = 1$$
$$LHS = RHS$$

(2) (1 points) What is the inductive hypothesis?

$$P(k) = \sum_{i=0}^{k} 2^i = 2^{(k+1)} - 1$$

(3) (1 points) What do you need to prove in the inductive step?

$Assuming \quad P(k) \text{ is true}, \ prove\ P(k+1)\ is\ true$

$$P(k+1) = \sum_{i=0}^{k+1} 2^i = 2^{(k+2)} - 1$$

(4) (3 points) Complete the inductive step.

$$\sum_{i=0}^{k+1} 2^i = 2^{(k+2)} - 1$$
$$\sum_{i=0}^{k+1} 2^i = \sum_{i=0}^{k} 2^i + 2^{(k+1)}$$
$$\sum_{i=0}^{k} 2^i + 2^{(k+1)} = 2^{(k+2)} - 1$$
$$2^{(k+1)} - 1 + 2^{(k+1)} = 2^{(k+2)} - 1$$
$$2 * (2^{(k+1)}) - 1 = 2^{(k+2)} - 1$$
$$2^1 * 2^{(k+1)} - 1 = 2^{(k+2)} - 1$$
$$2^{(k+2)} - 1 = 2^{(k+2)} - 1$$

2. Weak Induction (10 points)

(1) (5 points) Using weak induction, prove that $3^n < n!$ for all integers n > 6.

$$P(n) := 3^n < n!$$
$$P(7) = 3^7 < 7!$$
$$LHS = 3^7 = 2187$$
$$RHS = 7! = 5040$$
$$LHS < RHS \text{ , base case true}$$

$$P(k) = 3^k < k!$$

$$P(k+1) = 3^{k+1} < (k+1)!$$
$$3^{k+1} = 3 * 3^k$$
$$3(3^k) < (k+1) * k!$$

$$3 < k+1 \text{ because } k > 6$$
$$\text{since } 3 < k+1 \text{ and } 3^k < k! \text{ , } 3 * 3^k < (k+1) * k!$$

because the product of the 2 lesser numbers is less than the product of the 2 greater, given all are > 0...

$$3^{k+1} < (k+1)!$$

(2) (5 points) Prove that log(n!) ≤ n log(n) for all integers n ≥ 1.

$$\textit{Base case}: \log(1!) \leq 1 * \log(1)$$
$$LHS = \log(1!) \rightarrow \log(1) = 0$$
$$RHS = \log(1) = 0$$
$$LHS \leq RHS \text{ ,base case istrue}$$
$$n \log n = \log(n^n)$$
$$P(k) = \log(k!) \leq \log(k)^k$$
$$k! \leq k^k$$
$$P(k+1) = \log((k+1)!) \leq \log(k+1)^{k+1}$$
$$(k+1)! \leq (k+1)^{k+1}$$
$$(k+1)! = (k+1) * k!$$
$$k! * (k+1) \leq (k+1)(k+1)^k \rightarrow k! \leq (k+1)^k$$

Since k! and $(k+1)^k$ both share the same number of terms, and k!'s terms are decreasing from k while $(k+1)^k$'s terms are made up of only (k+1) and do not decrease....

$$k! < (k+1)^k$$

as every term on the right hand side is greater than every term from the left, the right hand side is strictly greater than the left

Reminder 1: log(1) = 0.

Reminder 2: log(a ∗ b) = log(a) + log(b).

Reminder 3: If a ≤ b then log(a) ≤ log(b).

Note: The base of the logarithms doesn't matter for any of the above.

(3) (5 points) Prove for all integers n $\geq$ 1 that if A_1, A_2, ..., A_n and B are sets, then:

$$\left(\bigcap_{i=0}^{n} A_i\right) \cup B = \bigcap_{i=0}^{n} (A_i \cup B)$$

Notation 1: $\bigcap_{i=0}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$.

Notation 2: $\bigcap_{i=0}^{n} (A_i \cup B) = (A_1 \cup B) \cap (A_2 \cup B) \cap \ldots \cap (A_n \cup B)$.

Hint 1: $\bigcap_{i=0}^{k+1} A_i = \left(\bigcap_{i=0}^{k} A_i\right) \cap A_{k+1}$ (true for all $k \geq 1$).

Hint 2: Use the fact that (X ∩ Y) ∪ Z = (X ∪ Z) ∩ (Y ∪ Z) where X, Y, and Z are sets.

$$\left(\bigcap_{i=0}^{1} A_i\right) \cup B = (A_0 \cap A_1) \cup B = (B \cup A_0) \cap (B \cup A_1)$$

$$\bigcap_{i=0}^{1} (A_i \cup B) = (A_0 \cup B) \cap (A_1 \cup B)$$

$$\text{therefore} \left(\bigcap_{i=0}^{1} A_i\right) \cup B = \bigcap_{i=0}^{1} (A_i \cup B) \text{ and the base case is true}$$

$$\text{assuming } \left(\bigcap_{i=0}^{k} A_i\right) \cup B = \bigcap_{i=0}^{k} (A_i \cup B) \text{ is true...}$$

$$\left(\bigcap_{i=0}^{k+1} A_i\right) \cup B = \bigcap_{i=0}^{k+1} (A_i \cup B)$$

$$\left(\bigcap_{i=0}^{k+1} A_i\right) \cup B = \left(A_{k+1} \cap \bigcap_{i=0}^{k} A_i\right) \cup B = (A_{k+1} \cup B) \cap \left(\bigcap_{i=0}^{k} A_i \cup B\right) = (A_{k+1} \cup B) \cap \bigcap_{i=0}^{k} (A_i \cup B) = LHS$$

$$\bigcap_{i=0}^{k+1} (A_i \cup B) = (A_{k+1} \cup B) \cap \bigcap_{i=0}^{k+1} (A_i \cup B) = RHS$$

$$LHS = RHS$$

3. Strong Induction (11 points)

(1) (6 points) Let P (n) be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The Induction and Recursion parts of this exercise outline a strong induction proof that P (n) is true for n ≥ 18.

(a) (1 points) Show that P (18), P (19), and P (20) are true, which completes the base case.

$a := 4$ *cent stamp*

$b := 7$ *cent stamp*

$P(18) = 1*(a) + 2*(b)$

$P(19) = 3*(a) + 1*(b)$

$P(20) = 5*(a) + 0*(b)$

base case is true

(b) (1 points) What is the inductive hypothesis?

$P(k) = (m*a) + (n*b)$, where m & n are integers, for $k \geq 18$

(c) (1 points) What do you need to prove in the inductive step?

$P(k+1) = P(k-3) + a$

(d) (3 points) Complete the inductive step for k ≥ 20.

Assume that for every postage of m-cents, $18 \leq m \leq k$ it can be expressed exactly using 4-cent and 7-cent stamps

Since $P(k-3)$ is true by assumption, all we need to do is add a 4-cent stamp to obtain $P(k+1)$, so $P(k+1)$ must be true

(2) (5 points) Use strong induction to show that every positive integer can be written as a sum of distinct powers of two

$(i.e., 2^0 = 1, 2^1 = 2, 2^2 = 4, 2^3 = 8, 2^4 = 16, \ldots)$.

For example: 19 = 16 + 2 + 1 = $2^4 + 2^1 + 2^0$

Hint: For the inductive step, separately consider the case where k + 1 is even and where it is odd. When it is even, note that (k + 1)/2 is an integer

Base Case : $n = 1$

$P(1) = 2^0$

$1 = 1$ *base case true*

step 1 Assume P(k) is true for all $1 \leq k \leq n$

Prove $P(k+1)$ *is true*

step 2 *case* 1: $k+1$ *is even*

divide by 2, because in doing so we subtract 1 from the power of every 2^n present in the summation

case 2 : $k+1$ *is odd*

If $k+1 = 1$ then done. Otherwise subtract 2^0 from the summation