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Master in Mathematics

On the real realization of the motivic spectrum ko

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Introduction

As introduced by Voevodsky in his 1998 ICM address, motivic homotopy theory is a homotopy theory for schemes that uses the affine line as a replacement for the unit interval. One can define the ∞-category of motivic spaces, which contains a representative for every smooth scheme of finite type over a fixed base, but also for every space in the usual sense. There, the affine line becomes contractible, namely equivalent to the point. In analogy with the construction of the stable ∞-category of topological spectra, one can also define the stable ∞-category of motivic spectra.

When working over $\mathbb{R}$, one connection between them is given by a functor from the ∞-category of motivic spectra to that of topological spectra, called *real Betti realization*. The goal of this Master's thesis is to identify the real Betti realization of a particular motivic spectrum called *the very effective cover of the Hermitian K-theory spectrum* and denoted by ko. We shall spend the rest of this introduction explaining what this means.

Consider a scheme X over a field k embedding into $\mathbb{R}$. We may then consider the topological space formed by its real points, with the Euclidean topology. To give two of the most basic examples, the real points of the affine line $\mathbb{A}^1_{\mathbb{R}}$ are the real line $\mathbb{R}$ itself, and the real points of the group scheme $\mathbb{G}_m = \mathsf{Spec}(\mathbb{R}[t, t^{-1}])$ are given by $\mathbb{R} \setminus \{0\}$, which is homotopy equivalent to the 0-sphere $\mathbb{S}^0$. The operation of taking the real points can be seen as a functor from the category of smooth schemes of finite type over our field k to the category of topological spaces. This functor can be extended to a functor from the ∞-category of motivic spaces to that of spaces, and even to a functor from the ∞-category of motivic spectra to that of classical topological spectra. This extension is called the real Betti realization functor, and it is often denoted by $r_{\mathbb{R}}$. This functor can be useful, for example, to make conjectures about certain phenomena in the motivic setting (apply the real realization functor, see what happens in topology, and deduce a guess about what the correct statement in motivic homotopy theory should be), or (dis)proving some claims in this setting (for instance, if a map between motivic spectra has a non-zero real realization, then it must itself be non-zero). Similar constructions may be performed replacing the field of real numbers with that of complex numbers, to obtain the complex Betti realization functor. However, we will not deal with this functor here, and "realization" will always mean "real realization" for us (except in Subsection 4.3.2, where we compute two examples of complex realizations).

Computations with the real realization functor can a priori be very complicated. Indeed, the extension of the real points functor to the ∞-category of motivic spectra involves the use of several universal properties, and thus the resulting functor is not really explicit. However, the good properties of the functor $r_{\mathbb{R}}$ save the day: for instance, it preserves colimits, finite products, and it is symmetric monoidal with respect to the smash product of motivic spectra (which can be defined and makes the ∞-category of motivic spectra into a symmetric monoidal ∞-category), respectively topological spectra. These properties make the computations with $r_{\mathbb{R}}$ a lot easier, although they remain far from trivial in general.

Motivic homotopy theory was partly introduced to study cohomology theories for schemes, for example algebraic K-theory and its variants. Roughly speaking, algebraic K-theory is a cohomology theory for a certain class of schemes, which cares about the collection of algebraic vector bundles over a given scheme, endowed with the operation of direct sum. One can also define Hermitian K-theory, which is the same but for algebraic vector bundles endowed with a chosen symmetric form instead. The ∞-category of motivic spectra is particularly well-suited to study these two examples, because they are representable in this ∞-category; we denote by KGL and KO respectively the motivic spectra representing them.

The real realizations of KGL and KO have already been studied and computed in the literature (see for instance [BH20, Lemma 3.9]). In this Master's thesis, we will compute the real realization of a closely related motivic spectrum, namely ko. The latter is a certain type of "connective cover" for the Hermitian K-theory spectrum KO, namely its *very effective cover*. In fact, KO has a certain periodicity under shifts, in particular it should not be considered "connective" in any sense that we will give to this word for motivic spectra. We will have to define what we mean by connectivity for motivic spectra. Indeed, the motivic analogs to homotopy groups are the bigraded homotopy sheaves (the bigrading being with respect to the two different kinds of circles that we have in the motivic ∞-category: the scheme $\mathbb{G}_m$, but also the topological circle $\mathcal{S}^1$), and thus it is a priori unclear how one should define this notion.

There is an extra layer of structure attached to the spectra KGL and KO. It comes from the fact that the collection of algebraic vector bundles over a given scheme is endowed with a second operation besides the direct sum, that is, the tensor product. The latter induces commutative multiplicative structures on the corresponding cohomology theories, and thus on KGL and KO. This makes KGL, KO, and actually also kgl and ko (their respective very effective covers), into $\mathcal{E}_\infty$-algebras in the ∞-category of motivic spectra. Roughly speaking, this means that they are endowed with the structure of "commutative rings up to coherent homotopy", where all the ring axioms are satisfied up to *specified* homotopies, which we may call "higher coherencies". There is also a corresponding notion for associative but non necessarily commutative multiplicative structures, namely that of $\mathcal{E}_1$-algebras (and a whole family of notions encoding "more and more commutativity": $\mathcal{E}_n$-algebras for $n \in \mathbb{N} \cup \{\infty\}$). One may show that symmetric monoidal functors preserve $\mathcal{E}_n$-algebra structures, so the real realizations of KO and ko for instance inherit $\mathcal{E}_\infty$-algebra structures in the ∞-category of spectra, also called $\mathcal{E}_\infty$-ring structures. The framework of ∞-categories is particularly well-suited to talk about such objects, and will also be convenient for the construction of the real realization functor and the description of various notions related to symmetric monoidal structures, so we adopt this point of view.

Therefore, it makes sense to ask not only what the real realization of ko (or KO, kgl, etc) is as a spectrum, but also as an $\mathcal{E}_n$-ring for different values of $n \in \mathbb{N} \cup \{\infty\}$. This adds much difficulty to the question. Note that a spectrum may admit several non-equivalent $\mathcal{E}_n$-ring structures, and these are a priori not detected on the homotopy groups (i.e., given two $\mathcal{E}_n$-ring structures, it does not suffice to show that the graded ring structures induced on the homotopy groups agree to show that the $\mathcal{E}_n$-ring structures agree in the first place).

We will only be able to identify $r_\mathbb{R}\mathsf{ko}$ as an $\mathcal{E}_1$-ring, essentially because we have a good description of certain free $\mathcal{E}_1$-algebras, which does not generalizes to $\mathcal{E}_n$-algebras for larger n. The precise reasons will become clearer in Section 5.

Without further detours, let us state our main result:

Theorem A (see Theorem 5.7)**.** *There is a Cartesian square of $\mathcal{E}_1$-rings (and thus also of spectra)*

$$\begin{tikzcd} r_\mathbb{R}\mathsf{ko} \arrow[r, "x \mapsto \beta_4/2"] \arrow[d, "x \mapsto t^4"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{ko}^{\mathsf{top}}[1/2] \arrow[d, "\mathsf{ch}"', "\beta_4/2 \mapsto t^4"] \\ \mathsf{H}\mathbb{Z}_{(2)}[t^4] \arrow[r, "t^4 \mapsto t^4"'] & \mathsf{H}\mathbb{Q}[t^4]. \end{tikzcd}$$

In particular, $r_\mathbb{R}\mathsf{ko}$ is equivalent as an $\mathcal{E}_1$-ring to $\mathsf{L}(\mathbb{R})_{\geq 0}$, the connective cover of the L-theory spectrum of $\mathbb{R}$, and there is an isomorphism of graded rings $\pi_*(r_\mathbb{R}\mathsf{ko}) \cong \mathbb{Z}[x]$ where x has degree 4.

We will define all the objects and notations appearing in the diagram precisely in Subsection 5.2; but let us already mention that the assignments of elements labeling the arrows describe the action of the maps in the square on the fourth homotopy groups. Let us also mention that $\mathsf{H}\mathbb{Q}[t^4]$ is the free $\mathcal{E}_1$-$\mathsf{H}\mathbb{Q}$-algebra on one generator t^4 in degree 4. It is an particular an $\mathsf{H}\mathbb{Q}$-module, with homotopy ring a polynomial ring over $\mathbb{Q}$ with a single generator in degree 4. The definition of $\mathsf{H}\mathbb{Z}_{(2)}[t^4]$ is similar.

The Cartesian square above involves different kinds of localizations of $r_\mathbb{R}\mathsf{ko}$: localizing away from 2 (i.e. inverting 2), localization at (2) (inverting every prime except 2) and rationalization (inverting all primes). As such, it may be called a fracture square, more precisely a 2-local fracture square in our case. We will have to identify each of these $\mathcal{E}_1$-rings separately, and also carry out a careful study of the maps appearing in the square. We will give more details about the strategy of the proof as we go, for example in the introductory paragraphs to Section 5 and its various subsections.

The identification with the connective L-theory spectrum of $\mathbb{R}$ comes from the fact that the latter admits the same 2-local fracture square; this square for $\mathsf{L}(\mathbb{R})_{\geq 0}$ appears in [HLN21].

From the main question of this Master's thesis has naturally arisen the need to determine the real realization as an $\mathcal{E}_\infty$-ring of the special linear cobordism spectrum MSL, which is a priori unrelated to our main goal. This implies further to study the relation between the motivic Thom spectrum functor and the classical Thom spectrum functor in topology, because MSL can be defined as a motivic Thom spectrum. Indeed, we had to face the problem of producing an $\mathcal{E}_1$-map from $\mathsf{H}\mathbb{Z}_{(2)}$ to $r_\mathbb{R}\mathsf{ko}_{(2)}$ in order to identify the latter as a free $\mathcal{E}_1$-algebra over the former. To do so, we take inspiration from the computation of the similar fracture square for the L-theory spectrum of $\mathbb{R}$ in [HLN21]. There, the strategy is to consider an $\mathcal{E}_\infty$-map from the oriented cobordism spectrum MSO, which is a topological Thom spectrum, to $\mathsf{L}(\mathbb{R})$, and expressing $\mathsf{H}\mathbb{Z}_{(2)}$ also as a Thom spectrum to obtain an $\mathcal{E}_2$-map from $\mathsf{H}\mathbb{Z}_{(2)}$ to $\mathsf{MSO}_{(2)}$. To reproduce this argument in the motivic setting, one can consider an $\mathcal{E}_\infty$-map $\mathsf{MSL} \to \mathsf{ko}$ and then apply the functor $r_\mathbb{R}$, because the real realization of MSL is MSO at the level of spectra, as was proven in [BH20, Cor. 4.7]. The hope is then that they actually agree as $\mathcal{E}_\infty$-rings, and this is what we will show.

Here are some results we will prove, besides the main theorem, which also might be of interest:

Proposition B (see Proposition 4.21). *As $\mathcal{E}_1$-rings, the real realization $r_\mathbb{R}(\mathsf{H}\mathbb{Z} \to \mathsf{H}\mathbb{Z}/2)$ of the cofiber of the multiplication by 2 map is equivalent to the map of free $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}/2$-algebras $\mathsf{H}\mathbb{Z}/2[t^2] \to \mathsf{H}\mathbb{Z}/2[t]$ (see Definition 1.37).*

Proposition C (see Proposition 4.26). *The real realization $r_\mathbb{R}(\mathsf{kgl} \to \mathsf{H}\mathbb{Z})$ of the cofiber of the map $T \wedge \mathsf{kgl} \to \mathsf{kgl}$ induced by the periodicity generator β_{KGL} for KGL is equivalent as a map of $\mathcal{E}_1$-rings to the map of free $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}/2$-algebras $\mathsf{H}\mathbb{Z}/2[t^4] \to \mathsf{H}\mathbb{Z}/2[t^2]$, sending t^4 to $(t^2)^2$ (see Definition 1.37).*

Theorem D (see Theorem 5.6 and Proposition 4.28). *There is an $\mathcal{E}_\infty$-map of spectra:*

$$\gamma : \mathsf{L}(\mathbb{R}) \longrightarrow r_\mathbb{R}\mathsf{KW}$$

inducing an equivalence of $\mathcal{E}_\infty$-rings $\mathsf{L}(\mathbb{R})[1/2] \simeq r_\mathbb{R}\mathsf{KW}[1/2]$. In particular, there are equivalences of $\mathcal{E}_\infty$-rings $\mathsf{KO}^{\mathsf{top}}[1/2] \simeq r_\mathbb{R}\mathsf{KW}[1/2]$ and $\mathsf{KO}^{\mathsf{top}}[1/2] \simeq r_\mathbb{R}\mathsf{KO}$.

Theorem E (see Theorem 6.29). *The real realization of the motivic multiplicative Thom spectrum functor $M : \mathcal{P}_\Sigma(\mathsf{Sm}_\mathbb{R})_{/\mathsf{SH}^\simeq} \to \mathsf{SH}$ constructed in [BH21, §16.3] is equivalent as a symmetric monoidal functor to the classical multiplicative Thom spectrum functor $M_{\mathsf{top}} : \mathsf{Spc}_{/\mathsf{Sp}^\simeq} \to \mathsf{Sp}$ constructed in [ABG18, Thm 1.6]. Here, for an ∞-category $\mathcal{C}$, $\mathcal{C}^\simeq$ denotes the maximal ∞-groupoid contained in $\mathcal{C}$, viewed as a space. More precisely, there is a commutative diagram of symmetric monoidal functors*

$$\begin{tikzcd}
\mathcal{P}_\Sigma(\mathsf{Sm}_\mathbb{R})_{/\mathsf{SH}^\simeq} \arrow[r, "r_\mathbb{R}\circ\alpha_\sharp"] \arrow[d, "M"'] & \mathsf{Spc}_{/\mathsf{Sp}^\simeq} \arrow[d, "M_{\mathsf{top}}"] \\
\mathsf{SH}(\mathbb{R}) \arrow[r, "r_\mathbb{R}"'] & \mathsf{Sp}
\end{tikzcd}$$

where $\alpha : \mathsf{SH}^\simeq(\bullet) \to \mathsf{Sp}(\mathsf{Spc}_{/r_\mathbb{R}(\bullet)})^\simeq$ is the morphism of presheaves given on $X \in \mathsf{Sm}_\mathbb{R}$ by a real realization functor $\mathsf{SH}(X)^\simeq \to \mathsf{Sp}(\mathsf{Spc}_{/r_\mathbb{R}(X)})$, and the functor $\alpha_\sharp$ sends an arrow $\mathcal{F} \to \mathsf{SH}^\simeq$ to its post-composition with α. The real realization of the presheaf $\mathsf{Sp}(\mathsf{Spc}_{/r_\mathbb{R}(\bullet)})^\simeq$ is $\mathsf{Sp}^\simeq$.

As a corollary, we have:

Theorem F (see Theorem 6.32). *There are equivalences of $\mathcal{E}_\infty$-rings*

$$r_\mathbb{R}\mathsf{MGL} \simeq \mathsf{MO}, \qquad r_\mathbb{R}\mathsf{MSL} \simeq \mathsf{MSO}, \qquad r_\mathbb{R}\mathsf{MSp} \simeq \mathsf{MU}.$$

Organization of this Master's thesis

Sections 1 to 4.2 contain recollections about the notions and techniques we will need in the proof of the main result.

Section 1 recalls the concepts of (symmetric) monoidal ∞-categories and their (commutative) monoid objects, of $\mathcal{E}_n$-algebras as a particular case of algebras over an ∞-operad, and some results about free $\mathcal{E}_1$-algebras, especially in the ∞-category of HA-modules spectra, where A is a discrete ring and HA is its Eilenberg-Mac Lane spectrum.

Section 2 contains the construction of the ∞-category of motivic spaces, and that of motivic spectra with its symmetric monoidal structure. It also contains some background material and results about notions of connectivity for motivic spectra: the effective and very effective filtrations on the ∞-category of motivic spectra, and the corresponding notions of effective and very effective slices of a given motivic spectrum.

Section 3 is a recollection of real and complex topological K-theory, represented by the topological spectra KU and $\mathsf{KO}^{\mathsf{top}}$, and also algebraic and Hermitian K-theory (for schemes), represented by the motivic spectra KGL and KO. We define them precisely and establish some parallels between the topological and motivic sides of the story, and also deal with the question of $\mathcal{E}_\infty$-structures on these topological or motivic spectra. Finally, we briefly recall the definition of L-theory.

Section 4 is about the construction of the real Betti realization functor and the computation of our first examples. As described in this introduction, we will extend the functor taking the real points of a scheme to a functor from motivic spectra to topological spectra using the construction of the ∞-category of motivic spectra in Section 2. We will see several useful tools for computations with the real realization functor, coming from its very good properties, and then apply them to compute the real realizations of KGL, kgl, KO, $\mathsf{H}\mathbb{Z}$, $\mathsf{H}\mathbb{Z}/2$, and $\widetilde{\mathsf{H}\mathbb{Z}}$ (a variant of $\mathsf{H}\mathbb{Z}$).

Section 5 contains the proof of our main result, appearing as Theorem A above. This will require us to use more or less directly every example computed in the previous section, and one result from Section 6. It also involves classical tools from homotopy theory, such as spectral sequences with a multiplicative structure, with respect to which the differentials satisfy the Leibniz rule.

Section 6 studies the relation between the motivic Thom spectrum functor and the topological one, together with their structures of symmetric monoidal functors. As mentioned earlier in this introduction, the need to study them arises from the desire, in the previous subsection, to produce an $\mathcal{E}_2$-map from $\mathsf{H}\mathbb{Z}_{(2)}$ to $r_{\mathbb{R}}\mathsf{ko}_{(2)}$, using the Thom spectrum MSO as an intermediate step. The results proven in this section make the proof of our main result complete, while possibly being of independent interest.

Finally, the Appendix is divided in three parts, containing respectively some technical results about localization of topological spectra and fracture squares (Subsection A.1), properties of the Day convolution symmetric monoidal structure on ∞-categories of presheaves (Subsection A.2), and slice ∞-categories and symmetric monoidal structures on them (Subsection A.3).

Relation to other works

- Sections 1 to 4 do not bring any new contribution; they only consist in recollections about material already present in the literature. As often as possible, we try to provide a reference for where the material in question can be found. Although they are mostly well-known results, certain examples of computations of the real realizations of some motivic spectra at the end of Section 4 do not seem to appear in the literature under this form or with these proofs; in particular Propositions 4.26 and 4.27.
- The complex realizations of KO and ko already appear in the literature; these motivic spectra realize to the expected $\mathsf{KO}^{\mathsf{top}}$ and $\mathsf{ko}^{\mathsf{top}}$ respectively, by [ARØ20, Lemma 2.13].
- The real realization of KO has been computed in [BH20, Lemma 3.9], at the level of spectra: it is equivalent to $\mathsf{KO}^{\mathsf{top}}[1/2]$. Our work includes upgrading this equivalence to one of $\mathcal{E}_\infty$-rings. This is done by improving the result in [Rön18, Thm 4.4] identifying $r_{\mathbb{R}}(\mathsf{KO}[\eta^{-1}, 1/2])$ with $\mathsf{KO}^{\mathsf{top}}[1/2]$ as spectra, to an identification of $\mathcal{E}_\infty$-rings.
- Section 5 contains our first main contribution, that is, the proof of Theorem A. This is, as far as we know, a new result.
- Section 6 combines material from the literature with new contributions. Essentially, it consists in reproducing the constructions existing in the literature, especially in [BH21], with slight differences so that they are adapted to our context. Our second main contribution is the identification of the real realization of the multiplicative motivic Thom spectrum functor with the classical multiplicative Thom spectrum functor in topology (in a sense to be made precise). As a corollary, we identify the

real realizations of MGL, MSL, and MSp with MO, MSO and MU respectively, as $\mathcal{E}_\infty$-rings. Again, the latter result was already obtained at the level of spectra in [BH20, Cor. 4.7].

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Conventions

- The reader may find any unexplained notation in the Index of notation at the end.
- For us, $\mathbb{N} = \{0, 1, 2, \dots\}$.
- For p a prime, *localization at* (p) will refer to inverting every prime different from p, whereas *localization away from* p refers to inverting p. In the first case, we use parentheses because this corresponds in algebra to a localization with respect to the prime ideal (p).
- We freely use the language of ∞-categories and stable ∞-categories following [Lur09] and [Lur17]. We also assume that the reader is familiar with triangulated categories and t-structures (see for example [Lur17, Section 1.2]).
- When no confusion seems likely, for a smooth scheme X or a space S, we will often also denote by X and S the corresponding motivic spaces (omitting the embedding into presheaves and motivic localization) or motivic spectra (omitting the infinite suspension, in the case of a pointed object).
- We use $\cong$ to denote an isomorphism of groups, rings, etc. (or in 1-categorical context in general), and $\simeq$ to denote an equivalence in an ∞-category.
- A (topological) spectrum is called connective if its homotopy groups in negative degrees are zero. It is called n-connective if it has no non-trivial homotopy groups in degree strictly less than n.

1 Symmetric monoidal ∞-categories and $\mathcal{E}_n$-rings

As mentioned in the introduction, many of the spectra we will encounter, especially those representing different variants of K-theory, are endowed with a multiplicative structure "up to homotopy". This makes sense because the category of spectra they live in also has a multiplicative structure "up to homotopy", given by the smash product. The main goal of this section is to make such statements precise. The content of this section is purely expository; we recall the notions of monoidal and symmetric monoidal ∞-categories, of (commutative) algebras in such categories, and more generally of $\mathcal{E}_n$-algebras in symmetric monoidal ∞-categories.

In this setting, the multiplicative structure "up to homotopy" we just mentioned is actually a structure of symmetric monoidal ∞-category; and the spectra endowed with a multiplicative structure "up to homotopy" are homotopy ring objects in this symmetric monoidal ∞-category:

Definition 1.1. A homotopy ring object E in a symmetric monoidal ∞-category $\mathcal{C}$ is a commutative algebra in the symmetric monoidal 1-category $h\mathcal{C}$, namely equipped with maps $\eta : \mathbb{1} \to E$ and $\mu : E \otimes E \to E$ satisfying (in $h\mathcal{C}$) the usual axioms of a commutative monoid.

The picture will be similar in the motivic case: the ∞-category of motivic spectra has a symmetric monoidal structure given by the smash product, and the algebraic and hermitian K-theory spectra are homotopy ring objects.

The axioms in the definition of a homotopy ring object, when viewed in $\mathcal{C}$, are satisfied "up to non-specified homotopies" only. But, as usual, we also want to remember the higher coherencies. This corresponds to the stronger notion of a (commutative) algebra in an ∞-category (rather that in its homotopy category, viewed as a 1-category). Moreover, in this setting one can also define the family of notions of $\mathcal{E}_n$-algebras, for $1 \leq n \leq \infty$, interpolating between (non-commutative) algebras (corresponding to $n = 1$) and commutative algebras (corresponding to $n = \infty$) in a given ∞-category. When $\mathcal{C} = \mathsf{Sp}$ or $\mathcal{C} = \mathsf{SH}(S)$ is the ∞-category of spectra or motivic spectra, $\mathcal{E}_n$-algebras are also called $\mathcal{E}_n$-rings or *highly structured (motivic) ring spectra*. As it turns out, the various K-theories we will consider, either in the topological or motivic setting, can all be endowed with such structures.

The outline of this section is as follows: we will begin by recalling a different point of view on (symmetric) monoidal 1-categories and their algebras, which will allow us to generalize the definition to the case of ∞-categories. We will then describe yet another point of view on symmetric monoidal structures, that is Lurie's construction of $\mathsf{Pr}^{\mathsf{L},\otimes}$, a symmetric monoidal ∞-category of presentable ∞-categories and left adjoint functors.

Then, we will turn to the more general case of $\mathcal{E}_n$-algebras (with algebras and commutative algebras in a symmetric monoidal ∞-category corresponding to the cases $n = 1$ and $n = \infty$ respectively). Their description for the case $1 < n < \infty$ is a bit more involved and requires us to recall the definition of ∞-operads and algebras over the latter. This is a generalization to the ∞-categorical setting of the 1-categorical notion of colored operads. We will also see how this theory encompasses both the concept of symmetric monoidal ∞-categories and that of their commutative algebras, and define ∞-operads $\mathcal{E}_n$ for all $1 \leq n \leq \infty$, and their algebras.

Finally, we will restrict our attention to the case of the ∞-category of (topological) spectra and study the construction of free $\mathcal{E}_n$-algebras in this case. This will be very useful to us in Sections 4 and 5 to identity certain $\mathcal{E}_1$-rings, because it is "easy" to produce a map out of a free object, and then, in some situations, we may show that this map is an equivalence by studying the homomorphisms it induces on the homotopy groups.

1.1 $\mathcal{E}_1$- and $\mathcal{E}_\infty$-algebras

We begin with a discussion of $\mathcal{E}_n$-algebras in the simpler cases $n = 1$ and $n = \infty$, namely the notions of an algebra and a commutative algebra in a (symmetric) monoidal ∞-category. For this subsection, our main references are [Gro20] and [Lur24, Tag 01J2].

1.1.1 The perspective of Segal and Grothendieck on (symmetric) monoidal 1-categories and their (commutative) algebras

In the case of 1-categories, the usual axioms of a monoidal structure include coherency conditions: associativity, left- and right-unitality, and the pentagon identity. Describing these coherencies in the ∞-categorical setting would require a lot of data; therefore, to define monoidal ∞-categories, one first reformulates the definition of a monoidal 1-category in a way that neatly encodes this data. This is why we begin with discussing the case of 1-categories in this subsection, and we will generalize to the ∞-categorical setting only in the next subsection.

Definition 1.2. Let C and D be 1-categories. Given a functor $p : C \to D$, a morphism $f : c \to c'$ in C is called *p-coCartesian* if for any $g : c \to c''$ and $h : p(c') \to p(c'')$ such that $h \circ p(f) = p(g)$, there exists a unique $k : c' \to c''$ with $k \circ f = g$ and $p(k) = h$

$$\begin{tikzcd}
c \arrow[r, "f"] \arrow[drr, "\forall g"] \arrow[dd, dotted, mapsto] & c' \arrow[dr, dashed, "\exists ! k"] \arrow[dd, dotted, mapsto] & \\
& & c'' \arrow[dd, dotted, mapsto] \\
p(c) \arrow[r, "p(f)"] \arrow[drr, "p(g)"'] & p(c') \arrow[dr, "\forall h"] & \\
& & p(c'').
\end{tikzcd}$$

This condition is equivalent to requiring the diagram

$$\begin{tikzcd}
C(c', c'') \arrow[r, "- \circ f"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & C(c, c'') \arrow[d] \\
D(p(c'), p(c'')) \arrow[r, "- \circ p(f)"'] & D(p(c), p(c''))
\end{tikzcd}$$

to be a pullback, or equivalently, to requiring the functor $C_{f/} \to C_{c/} \times_{D_{p(c)/}} D_{p(f)/}$ to be an isomorphism.

Using this notion, one can define a property of functors, which correspond exactly to what we need for the fibers of this functor to behave in a covariant way over the target of this functor:

Definition 1.3. A functor $p : C \to D$ is called a *coCartesian fibration* (or *Grothendieck opfibration*) if for every morphism $f : d \to d'$ in D and $c \in C$ with $p(c) = d$, there exists a p-coCartesian lift of f with source c.

One can also check that such lifts, if they exists, are essentially unique. Then, if we choose a family of such lifts (for every morphism in the target and fixed domain in the source), we obtain a covariant behaviour of the fibers as follows. Let $f : d \to d'$ be a morphism in D. We want to build a functor $C_d \to C_{d'}$. So let $c \in C_d$, by hypothesis we have chosen a p-coCartesian lift for f with source c, say $f' : c \to c'$, with necessarily $c' \in C_{d'}$. Set c' to be the image of c. Moreover, given any arrow $b \to c$ in C_d, if $b \to b'$ is the lift we have chosen for f with source b, then the diagram

$$\begin{tikzcd}
b \arrow[r] \arrow[dr] \arrow[dd, dotted, mapsto] & b' \arrow[ddr, dashed, "\exists ! k"] \arrow[d, dotted, mapsto] & \\
& c \arrow[dr] \arrow[d, dotted, mapsto] & \\
d \arrow[r, "f"] \arrow[dr, equal] & d' \arrow[ddr, equal] & c' \arrow[dd, dotted, mapsto] \\
& d \arrow[dr, "f"] & \\
& & d'
\end{tikzcd}$$

provides $k : b' \to c'$ in $C_{d'}$. One also checks that this construction preserves composition of morphisms in C_d.

However this construction is not *strictly* functorial in $f : d \to d'$, but by essential uniqueness of coCartesian lifts, it is functorial up to natural isomorphisms. The datum of these natural isomorphisms hides a lot of structure, and this is exactly what we want to use to reformulate the definition of a monoidal 1-category.

This non strict functoriality provides, for any coCartesian fibration $C \to D$, a "pseudo-functor" $D \to \mathsf{Cat}$ sending an object d to the fiber C_d, for any Grothendieck opfibration $C \to D$. Actually, this construction has a converse, called the *Grothendieck construction*: given such a pseudo-functor $F : D \to \mathsf{Cat}$, define a 1-category C with objects the pairs (d, x) with $d \in D$ and $x \in F(d)$, and morphisms between pairs (d, x) and (d', x') the sets of pairs of a morphism $f : d \to d'$ in D and a morphism $F(f)(x) \to x'$ in $F(d')$. The projection functor to D then defines a coCartesian fibration $C \to D$.

Proposition 1.4 (The Grothendieck–Segal perspective on monoidal 1-categories;[Gro20, Ex. 4.7])**.** *To every monoidal 1-category $(C, \otimes)$ corresponds a unique coCartesian fibration $C^{\otimes} \to \Delta^{\mathsf{op}}$ such that the* Segal condition *is satisfied, i.e. $C^{\otimes}_{[n]} \cong (C^{\otimes}_{[1]})^n$ via the maps induced on the fibers by the edges $\iota_i : [n] \to [1]$, opposite to the maps $[1] \to [n]$ in Δ sending 0 to $i-1$ and 1 to i, for $1 \leq i \leq n$. The converse is also true: there is a one-to-one correspondence between monoidal 1-categories and coCartesian fibrations with target Δ^{op} satisfying the Segal condition.*

Sketch of proof. Given $(C, \otimes)$ a monoidal 1-category, define $C^{\otimes}$ as the 1-category with:

- Objects: finite (possibly empty) sequences $(C_1, \dots, C_n)$ of objects in C.
- Morphisms: a morphism between two objects $(C_1, \dots, C_n) \to (C'_1, \dots, C'_m)$ is the data of a morphism $(\alpha : [m] \to [n])^{\mathsf{op}}$ in Δ^{op} and morphisms $f_i : C_{\alpha(i-1)+1} \otimes \cdots \otimes C_{\alpha(i)} \to C'_i$ in C for all $1 \leq i \leq m$. The morphism α can be thought of as encoding the domains of the f_i's.

The corresponding functor $C^{\otimes} \to \Delta^{\mathsf{op}}$ sends a sequence $(C_1, \dots, C_n)$ to $[n] \in \Delta^{\mathsf{op}}$. This suggest that, in the other direction, given a coCartesian fibration $C^{\otimes} \to \Delta^{\mathsf{op}}$, the underlying 1-category of the monoidal 1-category it defines should be the fiber $C^{\otimes}_{[1]}$. The Segal condition implies that $C^{\otimes}_{[0]} \cong *$. The image of this unique object in $C^{\otimes}_{[1]}$ under the maps induced on the fibers by the unique $[0] \to [1]$ in Δ^{op} give the monoidal unit in C. To define the tensor product of two one-object sequences $(C), (C') \in C^{\otimes}_{[1]}$, note that under the equivalence in the Segal condition, the pair $((C), (C'))$ defines some object $D \in C^{\otimes}_{[2]}$. Set $(C) \otimes (C') \in C^{\otimes}_{[1]}$ to be the image of D by the map induced on the fiber by $([1] \to [2])^{\mathsf{op}} \in \Delta^{\mathsf{op}}$, sending 0 to 0 and 1 to 2.

One then has to check that everything is well-defined and provides the desired one-to-one correspondence. To give an example, we prove that the tensor product we defined is associative, and omit the rest of the easy but tedious verifications. Our example will ultimately follow from the simplicial identity $d^2 d^1 = d^1 d^1 : [1] \to [3]$ in Δ. We denote by $[-]$ the (inverses of the) isomorphisms in the Segal condition. For example, in this notation, $C \otimes C' = (d_1)_*[C, C']$ (with $(-)_*$ denoting the map induced on the fibers). To prove associativity, given $C, C', C'' \in (C^{\otimes})_{[1]}$, a natural object to consider is $[C, C', C''] \in (C^{\otimes})_{[3]}$. Then one computes:

$$\begin{aligned}
(C \otimes C') \otimes C'' &= (d_1)_*[(d_1)_*[C, C'], C''] \\
&= ((d_1)_* \circ (d_1)_*)[C, C', C''] && (\star) \\
&= ((d_1)_* \circ (d_2)_*)[C, C', C''] && (\text{since } d_1 d_1 = d_1 d_2 \text{ in } \Delta^{\mathsf{op}}) \\
&= (d_1)_*[C, (d_1)_*[C', C'']] && (\star\star) \\
&= C \otimes (C' \otimes C'')
\end{aligned}$$

The equation $(\star)$ holds because $[(d_1)_*[C, C'], C''] = (d_1)_*[C, C', C'']$. Indeed, because of the Segal condition, it suffices to show that $(\iota_1)_*(d_1)_*[C, C', C''] = (d_1)_*[C, C']$ and $(\iota_2)_*(d_1)_*[C, C', C''] = C''$. We only show the first one, the second one being similar. We have $(\iota_1)_* \circ (d_1)_* = ((d^1 \circ (\iota_1)^{\mathsf{op}})^{\mathsf{op}})_*$ but the composition $d^1 \circ (\iota_1)^{\mathsf{op}} : [1] \to [3]$ maps 0 to 0 and 1 to 2, so it is equal to $d^3 \circ d^1$; we therefore get $(\iota_1)_*(d_1)_*[C, C', C''] = (d_1)_*(d_3)_*[C, C', C'']$. Now $(d_3)_*[C, C', C''] = [C, C']$ because the right hand side is by definition unique with the property that $(\iota_1)_*[C, C'] = C$ and $(\iota_2)_*[C, C'] = C'$. The latter holds since, similarly as above, we have

$$(\iota_1)_*(d_3)_*[C, C', C''] = (\iota_1)_*[C, C', C''] = C \text{ and } (\iota_2)_*(d_3)_*[C, C', C''] = (\iota_2)_*[C, C', C''] = C'.$$

Similarly, $[C, (d_1)_*[C', C'']] = (d_2)_*[C, C', C'']$ and so $(\star\star)$ holds. $\square$

Proposition 1.5 ([Gro20, Prop. 4.21])**.** *Under the correspondence of Proposition 1.4, algebras in $(C, \otimes)$ (also called monoid objects if the monoidal structure on C is the Cartesian one) are in one-to-one correspondence with sections of the coCartesian fibration $p : C^{\otimes} \to \Delta^{\mathsf{op}}$ carrying* convex morphisms *(i.e. injective, with image consisting in consecutive integers) to p-coCartesian edges.*

Sketch of proof. An algebra A in $(C, \otimes)$ with multiplication $\mu : A \otimes A \to A$ and unit $\eta : \mathbb{1} \to A$ (with the usual axioms), determines such a section by sending $[n]$ to the sequence with n-terms $(A, A, \ldots, A)$ in $C^{\otimes}_{[n]}$. Then, the opposite of a convex morphism $\alpha : [n] \to [m]$ in Δ is mapped to the morphism $(A_1, \ldots, A_m) \to (A_1, \ldots, A_n)$, with $A_i = A$ for all i, where all f_i's are identities $A \to A$. This is p-coCartesian because given any morphism $(\beta^{\mathsf{op}}, f_i^{\beta}) : (A_1, \ldots, A_m) \to (C_1, \ldots C_k)$ and $\gamma : [k] \to [n]$ such that $\alpha \circ \gamma = \beta$, the morphism $(A_1, \ldots, A_n) \to (C_1, \ldots, C_k)$ given by γ^{op} and f_i^{γ} is given by f_i^{β} (the same number of copies of A appear, by construction). The following diagram illustrates the situation:

$$\begin{tikzcd}
(A_1, \ldots, A_m) \arrow[r] \arrow[drr] \arrow[dd, dotted, mapsto] & (A_1, \ldots, A_n) \arrow[dr] \arrow[dd, dotted, mapsto] & \\
& & (C_1, \ldots, C_k) \arrow[dd, dotted, mapsto] \\
{[m]} \arrow[r, "\alpha^{\mathsf{op}}"] \arrow[drr, "\beta^{\mathsf{op}}"'] & {[n]} \arrow[dr, "\gamma^{\mathsf{op}}"] & \\
& & {[k].}
\end{tikzcd}$$

In the other direction, such a section $s : \Delta^{\mathsf{op}} \to C^{\otimes}$ provides an object $A := s([1]) \in (C^{\otimes})_{[1]} \cong C$, the unit map is given by $s(([1] \to [0])^{\mathsf{op}})$ (since $\mathbb{1} \in C$ corresponds to the unique object of $(C^{\otimes})_{[0]}$). For the multiplication, $s(d^1 : [2] \to [1])$ provides a map $s([2]) \to A$. Choosing a p-coCartesian lift f of d^1 with source $s([2])$, by definition of the Grothendieck construction in Proposition 1.4, the target of f is the tensor product of $(\iota_1)_*(s([2]))$ and $(\iota_2)_*(s([2]))$. Using the coCartesian property, $s([2]) \to A$ factors as a map $(\iota_1)_*(s([2])) \otimes (\iota_2)_*(s([2])) \to A$. If $(\iota_i)_*(s([2])) = A$ for $i = 1, 2$, we will have found the desired multiplication map. A coCartesian lift for $\iota_i : [2] \to [1]$ with source $s([2])$ is given by $s(\iota_i)$, since ι_i is a convex morphism and s carries them to coCartesian arrows. Since such lifts are essentially unique, we get that $(\iota_i)_*(s([2])) \cong s([1]) = A$.

We once more leave the verifications of the axioms and well-definedness of these constructions to the reader. □

We also have to describe what lax monoidal and monoidal functors become under this perspective.

Proposition 1.6 ([Gro20, Def. 4.23])**.** *In the situation of Proposition 1.4, given $(D, \boxtimes)$ another monoidal 1-category with associated coCartesian fibration $q : D^{\boxtimes} \to \Delta^{\mathsf{op}}$, there is a one-to-one correspondence between lax monoidal functors, respectively monoidal functors $(C, \otimes) \to (D, \boxtimes)$ and functors $C^{\otimes} \to D^{\boxtimes}$ over Δ^{op}, sending p-coCartesian lifts of convex morphisms, respetively all p-coCartesian arrows, to q-coCartesian arrows.*

In particular, an algebra in $(C, \otimes)$ is a lax monoidal functor $\Delta^{\mathsf{op}} \to C^{\otimes}$, where $\Delta^{\mathsf{op}} \to \Delta^{\mathsf{op}}$ is the identity fibration.

Before using this to define similar concepts in the world of ∞-categories, we briefly mention the corresponding results in the symmetric monoidal case. To encode the symmetry condition, the 1-category Δ^{op} is not sufficient. Indeed we would in particular need a map $[2] \to [2]$ corresponding to the swap morphism, and this map wouldn't be monotone then.

Definition 1.7. Let Fin_* be a skeleton of the 1-category of finite pointed sets. Its objects are the sets $\langle n \rangle := \{*, 1, \ldots, n\}$ for all $n \in \mathbb{N}$, and the morphisms are all pointed maps.

Proposition 1.8 (The Grothendieck–Segal perspective on symmetric monoidal 1-categories; [Gro20, Prop. 4.26])**.**

(i) To every symmetric monoidal 1-category $(C, \otimes)$ is associated a unique coCartesian fibration $C^{\otimes} \to \mathsf{Fin}_*$ such that the Segal condition *is satisfied, i.e. $C^{\otimes}_{[n]} \cong (C^{\otimes}_{[1]})^n$ via the maps induced on the fibers by the following maps in Fin_*: for all $1 \leq i \leq n$, $\rho_i : \langle n \rangle \to \langle 1 \rangle$ sending i to 1 and the rest to $*$. The 1-category $C^{\otimes}$ has objects finite sequences of objects in C, and a*

map $(C_1, \ldots, C_n) \to (C'_1, \ldots, C'_m)$ consists in the data of $\alpha : \langle n \rangle \to \langle m \rangle$ in Fin_* and maps $f_i : \bigotimes_{j \in \alpha^{-1}(i)} C_j \to C'_i$ for all $1 \leq i \leq m$.

The converse is also true: there is a one-to-one correspondence between symmetric monoidal 1-categories and coCartesian fibrations with target Fin_* satisfying the Segal condition.

(ii) Commutative algebras in $(C, \otimes)$ (also called commutative monoid objects if the symmetric monoidal structure on C is the Cartesian one) are in one-to-one correspondence with sections of the coCartesian fibration associated with $(C, \otimes)$ that send inert morphisms (maps such that the preimage of every element different from the basepoint is a singleton) to p-coCartesian arrows.

(iii) Lax symmetric monoidal functors, respectively symmetric monoidal functors $(C, \otimes) \to (D, \boxtimes)$, where $(C, \otimes)$ and $(D, \boxtimes)$ correspond to coCartesian fibrations $p : C^{\otimes} \to \mathsf{Fin}_*$ and $q : D^{\boxtimes} \to \mathsf{Fin}_*$, are in one-to-one correspondence with functors $C^{\otimes} \to D^{\boxtimes}$ over Fin_* sending p-coCartesian lifts of inert arrows, respectively all p-coCartesian arrows, to q-coCartesian arrows.

The forgetful functor from symmetric monoidal to monoidal 1-categories becomes in this setting pullback along the functor $\Delta^{\mathsf{op}} \to \mathsf{Fin}_*$, sending $[n]$ to $\langle n \rangle$, and a morphism $(\alpha : [m] \to [n])^{\mathsf{op}}$ to the pointed map $\langle n \rangle \to \langle m \rangle$ sending an integer j to i if $\alpha(i-1)+1 \leq j \leq \alpha(i)$ and $*$ otherwise. Note that a morphism in Δ^{op} is convex if and only if its image is inert.

1.1.2 The ∞-categorical analogs

Now that we have a good picture of the basic definitions and results in the 1-categorical setting, we are ready to generalize them to the ∞-categorical setting.

Definition 1.9. Let $p : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. Then an edge $f : c \to c'$ in $\mathcal{C}$ is called *p-coCartesian* if the natural map $\mathcal{C}_{f/} \to \mathcal{C}_{c/} \times_{\mathcal{D}_{p(c)/}} \mathcal{D}_{p(f)/}$ is an acyclic Kan fibration, or equivalently if every lifting problem of the form

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "\sigma_0"] \arrow[d] & \mathcal{C} \arrow[d, "p"] \\ \Delta^n \arrow[r, "\sigma"'] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

with $n \geq 2$ and $\Delta^1 \hookrightarrow \Lambda^n_0 \xrightarrow{\sigma_0} \mathcal{C}$ (corresponding to the inclusion of $\{0,1\}$) being equal to f admits a solution.

The functor p is called a *coCartesian fibration* if p is an inner fibration and for every edge $g : d \to d'$ in $\mathcal{D}$ and $c \in \mathcal{C}_d$, there exists a p-coCartesian lift of g with source c.

Definition 1.10.

- A *monoidal ∞-category* is a coCartesian fibration $p : \mathcal{C}^{\otimes} \to \Delta^{\mathsf{op}}$ (for some ∞-category $\mathcal{C}^{\otimes}$) such that the Segal maps (defined as in Proposition 1.4) are equivalences. Here the 1-category Δ^{op} is implicitly viewed as its nerve. We say that p induces a *monoidal structure* on $\mathcal{C} := (\mathcal{C}^{\otimes})_{[1]}$.
- A *lax monoidal functor*, respectively *monoidal functor* between the monoidal ∞-categories given bu coCartesian fibrations $p : \mathcal{C}^{\otimes} \to \Delta^{\mathsf{op}}$ and $q : \mathcal{D}^{\boxtimes} \to \Delta^{\mathsf{op}}$ is a functor $\mathcal{C}^{\otimes} \to \mathcal{D}^{\boxtimes}$ over Δ^{op} sending p-coCartesian lifts of convex edges, respectively all p-coCartesian edges, to q-coCartesian edges.
- An *(associative) algebra*, or *$\mathcal{E}_1$-algebra* in $\mathcal{C}^{\otimes}$ is a section of p sending convex arrows to p-coCartesian ones, equivalently it is a lax monoidal functor $\Delta^{\mathsf{op}} \to \mathcal{C}^{\otimes}$. If the monoidal structure on $\mathcal{C}$ is the Cartesian one (see Proposition 1.14), we may also talk about a *monoid* in $\mathcal{C}^{\otimes}$.
- The definitions in the symmetric case (and for *commutative algebras*, also called *$\mathcal{E}_\infty$-algebras*) are the same, replacing Δ^{op} by Fin_*, and convex by inert. Again, the 1-category Fin_* is implicitly viewed as its nerve.

The forgetful functor from symmetric monoidal to monoidal ∞-categories is obtained again in the same way, by pullback along $\Delta^{\mathsf{op}} \to \mathsf{Fin}_*$.

Here are some properties of these notions:

Proposition 1.11 ([Lur17, Rmk 2.1.2.20 and Def. 2.1.3.7])**.**

If $\mathcal{C}^{\otimes}$ is a (symmetric) monoidal ∞-category, then there is an induced structure of (symmetric) monoidal 1-category on $h(\mathcal{C}^{\otimes}_{[1]})$; and associative (commutative) algebras become (commutative) algebras in the 1-categorical sense; (symmetric) monoidal ∞-functors also induce (symmetric) monoidal 1-functors.

(Lax) (symmetric) monoidal functors are organized into ∞-categories ($\mathsf{Fun}^{\otimes,\mathsf{lax}}(\mathcal{C},\mathcal{D})$ and) $\mathsf{Fun}^{\otimes}(\mathcal{C},\mathcal{D})$. In particular, there are ∞-categories of associative (respectively commutative) algebras in a (symmetric) monoidal ∞-category: $\mathsf{Alg}(\mathcal{C})$ and $\mathsf{CAlg}(\mathcal{C})$ respectively.

Remark 1.12. It follows from Definition 1.10 that lax monoidal functors preserve associative algebras. Indeed, the composition of a lax monoidal functor $\Delta^{\mathsf{op}} \to \mathcal{C}^{\otimes}$ over Δ^{op} with a lax monoidal functor $\mathcal{C}^{\otimes} \to \mathcal{D}^{\boxtimes}$ is still lax monoidal, because by definition the first functor sends convex edges to p-coCartesian lifts of convex edges and then the second functor maps them to coCartesian edges again by definition. More generally, a lax monoidal functor $\mathcal{C}^{\otimes} \to \mathcal{D}^{\boxtimes}$ induces a functor $\mathsf{Alg}(\mathcal{C}^{\otimes}) \to \mathsf{Alg}(\mathcal{D}^{\boxtimes})$. The same holds in the commutative case.

Example 1.13 ([Lur17, Ex. 2.1.2.21 and 4.1.7.6]). Any 1-category with a (symmetric) monoidal structure $(C,\otimes)$, when viewed as an ∞-category $\mathsf{N}(C)$, has an induced structure of a (symmetric) monoidal ∞-category, because the nerve of the coCartesian fibration $C^{\otimes} \to \mathsf{Fin}_*$ classifying $(C,\otimes)$ as in Proposition 1.8 yields a coCartesian fibration of ∞-categories $N(C\otimes) \to \mathsf{Fin}_*$.

More generally, any (symmetric) monoidal model category $(\mathcal{M},\otimes)$ defines a (symmetric) monoidal structure on the underlying ∞-category $\mathsf{N}(\mathcal{M}^{\circ})[\mathcal{W}^{-1}]$ (where $\mathcal{M}^{\circ}$ is the subcategory of fibrant-cofibrant objects and $\mathcal{W}$ is the class of weak equivalences).

An example of symmetric monoidal structures is given by the Cartesian ones, namely when the tensor product is the usual Cartesian product:

Proposition 1.14 ([Lur17, Prop. 2.4.1.5 and Cor. 2.4.1.8])**.** *If $\mathcal{C}$ is an ∞-category admitting finite products, then there exists a coCartesian fibration $\mathcal{C}^{\times} \to \mathsf{Fin}_*$ exhibiting $\mathcal{C}$ as a symmetric monoidal ∞-category with tensor product functor given by the Cartesian product.*

Moreover, if $\mathcal{D}$ is also such an ∞-category, then a symmetric monoidal functor $\mathcal{C}^{\times} \to \mathcal{D}^{\times}$ is exactly a functor $\mathcal{C} \to \mathcal{D}$ preserving finite products.

More families of examples can be found in next subsection.

1.2 Another point of view on symmetric monoidal ∞-categories: Lurie's $\mathsf{Pr}^{\mathsf{L},\otimes}$

Heuristically, the notion of commutative algebra in a symmetric monoidal ∞-category introduced in the previous subsection encodes a multiplicative structure on such an object. But a symmetric monoidal ∞-category is itself endowed with a multiplicative structure, given by its tensor product. This is the idea behind the symmetric monoidal ∞-category $\mathsf{Pr}^{\mathsf{L},\otimes}$, built by Lurie, where Pr^{L} is a certain ∞-category whose objects are well-behaved ∞-categories (i.e. presentable ones); and its commutative algebras will exactly be the (presentably) symmetric monoidal ∞-categories. Given a presentably symmetric monoidal ∞-category $\mathcal{C}$, this point of view is in particular useful to construct symmetric monoidal structures on several ∞-categories obtained from $\mathcal{C}$, for instance the ∞-category of pointed objects $\mathcal{C}_*$ or the stabilization $\mathsf{Sp}(\mathcal{C})$.

We now make this precise and discuss the basic properties of this construction. The main reference for this subsection is [Lur17, Section 4.8].

Definition 1.15. An ∞-category $\mathcal{C}$ is called *presentable* if it admits small colimits and is generated under small colimits by a *set* of κ-compact objects, for some regular cardinal κ.

Let Pr^{L} be the ∞-category of presentable ∞-categories and colimit-preserving functors (or, equivalently (in this setting), left-adjoint functors) (see [Lur09, Def. 5.5.3.1]).

Theorem 1.16 ([Lur17, §4.8.1])**.** *The ∞-category Pr^{L} admits a closed symmetric monoidal structure, where for any $\mathcal{C},\mathcal{D} \in \mathsf{Pr}^{\mathsf{L}}$, the tensor product $\mathcal{C}\otimes\mathcal{D}$ is universal amongst objects in Pr^{L} receiving a functor from the product $\mathcal{C}\times\mathcal{D}$, which preserves colimits in both variables. Moreover, there is a natural equivalence $\mathcal{C}\otimes\mathcal{D} \simeq \mathsf{Fun}^{\mathsf{R}}(\mathcal{C}^{\mathsf{op}},\mathcal{D})$, where $\mathsf{Fun}^{\mathsf{R}}$ denotes the subcategory of right-adjoint functors.*

One says that a symmetric monoidal structure is *closed* when for every object X, the tensor functor $X \otimes -$ has a right adjoint. For a *monoidal* structure to be closed, both the functors $X \otimes -$ and

$- \otimes X$ are required to have right-adjoints.

This formalism can be used to construct a tensor product on the ∞-category of topological spectra for example, as done in [Lur17, Section 4.8.2]. The interest for us comes from the following characterization of symmetric monoidal ∞-categories (stated for example after 3.1 in [Ber20]).

Definition 1.17. A *presentably symmetric monoidal ∞-category* is a presentable ∞-category with a symmetric monoidal structure, such that the tensor product functor preserves colimits in both variables (separately).

Proposition 1.18. *The ∞-category of commutative algebras in Pr^{L} is equivalent to the ∞-category of presentably symmetric monoidal ∞-categories, with symmetric monoidal colimit-preserving functors.*

Here are some interesting features of this tensor product (necessary definitions are given after the statement):

Proposition 1.19 ([Lur17, §4.8.2] and [GGN15, Prop. 3.9])**.** *In the symmetric monoidal ∞-category $\mathsf{Pr}^{\mathsf{L},\otimes}$, and for any $\mathcal{C} \in \mathsf{Pr}^{\mathsf{L}}$, the following hold:*

(i) The ∞-category of spaces Spc is the monoidal unit.

(ii) The ∞-category of pointed objects $\mathcal{C}_*$ is equivalent to $\mathcal{C} \otimes \mathsf{Spc}_*$.

(iii) If $\mathcal{C}$ admits finite limits, then its stabilization can be described by $\mathsf{Sp}(\mathcal{C}) \simeq \mathcal{C} \otimes \mathsf{Sp}$, where Sp is the ∞-category of spectra with the usual smash product.

(iv) The ∞-category of n-truncated objects $\mathcal{C}_{\leq n}$ is equivalent to $\mathcal{C} \otimes \mathsf{Spc}_{\leq n}$.

(v) If $L : \mathsf{Pr}^{\mathsf{L}} \to \mathsf{Pr}^{\mathsf{L}}$ is a smashing localization functor, in particular if it is of the form $\mathcal{C} \mapsto \mathcal{C} \otimes \mathcal{E}$ where $\mathbb{1} \to \mathcal{E}$ is an idempotent in $\mathsf{Pr}^{\mathsf{L},\otimes}$, then for any closed presentably symmetric monoidal ∞-category $\mathcal{C}$, $L(\mathcal{C})$ has a unique closed symmetric monoidal structure making $\mathcal{C} \to L(\mathcal{C})$ into a symmetric monoidal functor. Moreover, the induced map

$$\mathsf{Fun}^{\mathsf{L},\otimes}(L(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \mathsf{Fun}^{\mathsf{L},\otimes}(\mathcal{C}, \mathcal{D})$$

where $\mathsf{Fun}^{\mathsf{L},\otimes}$ denotes colimit-preserving symmetric monoidal functors, is an equivalence when $\mathcal{D}$ is an L-local presentably symmetric monoidal ∞-category (i.e. $\mathcal{D}$ is equivalent to $L(\mathcal{D})$ in Pr^{L}).

An *idempotent* in a monoidal ∞-category $\mathcal{C}$ is an object e with a map $\mathbb{1} \to e$ such that both $e \simeq e \otimes \mathbb{1} \to e \otimes e$ and $e \simeq \mathbb{1} \otimes e \to e \otimes e$ are equivalences (here one of the conditions suffices, since the monoidal structure is symmetric). A *smashing localization* $L : \mathsf{Pr}^{\mathsf{L}} \to \mathsf{Pr}^{\mathsf{L}}$ is a localization functor (i.e. admitting a fully faithful right adjoint) which is of the form $- \otimes I$ for some object $I \in \mathsf{Pr}^{\mathsf{L}}$.

Example 1.20. Item (v) applies in particular to $L\mathcal{C} = \mathcal{C}_*$, $L\mathcal{C} = \mathcal{C}_{\leq n}$ or $L\mathcal{C} = \mathsf{Sp}(\mathcal{C})$ in view of items (ii) to (iv) in the previous Proposition, and since Spc_*, $\mathsf{Spc}_{\leq n}$ and Sp are idempotent in Pr^{L} by [Lur17, Prop. 4.8.2.11, 4.8.2.15, and 4.8.2.18].

Example 1.21. Assume $\mathcal{C}$ is an ∞-category admitting finite products, and that the induced Cartesian symmetric monoidal structure (by Proposition 1.14) is closed. Then, the symmetric monoidal structure obtained on $\mathcal{C}_*$ as in Example 1.20 is the smash product, because this is true for $\mathcal{C} = \mathsf{Spc}$, and then the explicit description of the tensor product on Pr^{L} allows us to generalize the statement. Indeed, the unique symmetric monoidal structure on Spc preserving colimits in both variables and such that the localization $(-)_+ : \mathsf{Spc} \to \mathsf{Spc}_*$ freely adding a basepoint (with right adjoint U) is symmetric monoidal, is the smash product. We quickly justify this claim. Let $\otimes$ be this tensor product, and let $X, Y \in \mathsf{Spc}_*$. We claim that $X \wedge Y \simeq X \otimes Y$. Writing $X \wedge Y = \mathsf{colim}(* \longleftarrow *_+ \wedge Y \longrightarrow U(X)_+ \wedge Y)$ (see for example [ADF17, Lemma 2.2.4]), since $\otimes$ commutes with colimits in both variables, it suffices to show the claim when X belongs to the image of $(-)_+$ (also $*$ does). But then, writing $X = x_+$ for some $x \in \mathsf{Spc}$, since Spc is generated under colimits by the point, and since $(\bullet)_+$ preserves these colimits because it is a left adjoint, proving the claim when x is the one-point space is sufficient. Then, it suffices to show that $* \otimes Y \simeq Y$, but this follows from $(-)_+$ being symmetric monoidal and thus sending the monoidal unit to the unit.

1.3 The general case: ∞-operads and $\mathcal{E}_n$-algebras

As mentioned in the introduction to this section, algebras in a symmetric monoidal ∞-category (and their commutative counterparts) studied in the previous subsections are particular cases of a family of notions encoding "more and more commutative" objects in a symmetric monoidal ∞-category, namely $\mathcal{E}_n$-algebras. They correspond to the cases $n = 1$ and $n = \infty$ respectively. We now discuss the general case, for $n \in \mathbb{N} \cup \{\infty\}$. For this subsection, we follow [Lur17, Sections 2 and 5].

1.3.1 ∞-operads and their algebras

The notion of an $\mathcal{E}_n$-space is a very classical notion in topology. In the 1-categorical setting, there exists the notion of a (topological) operad, which is a sequence of spaces with certain properties, and encodes a collection of operations together with certain relations between them. The operads $\mathcal{E}_n$ are some examples, and they encode variable notions of commutativity. Then, $\mathcal{E}_n$-spaces are *algebras* over the operad $\mathcal{E}_n$, that is, spaces with a particular structure with respect to these operads. To define $\mathcal{E}_n$-algebras in the ∞-categorical setting, we therefore first need to generalize the notion of an operad to the ∞-categorical setting. Actually, it turns out fruitful to extend the definition of a *colored* operad instead. In the 1-categorical setting, the latter simultaneously generalize symmetric monoidal 1-categories and multicategories (1-categories where the domains of morphisms may consist of several objects). The main idea of a colored operad is to encode operations that take different types of inputs, and produce different types of output, where "types" are represented by the "colors" of the operad. Here is a classical example: if R is a (discrete) ring and M a module over R, one can define a colored operad with two colors r and m, and operations in arity n (meaning with n inputs) corresponding to multilinear maps $R^{n-1} \times M \to M$. If C is a symmetric monoidal 1-category, we could consider a colored operad with one color for each object in C, and then operations in arity n for the colors $c_1, \dots, c_n, d \in C$ would correspond to morphisms $c_1 \otimes \cdots \otimes c_n \to d$ in C (here d is the color of the output of the operation).

So the notion of ∞-operad introduced by Lurie aims at generalizing to ∞-categories this notion of a colored operad; in particular it should also be a generalization of symmetric monoidal ∞-categories, and we will see that this is indeed the case.

Definition 1.22. An ∞-*operad* is a functor $p : \mathcal{O}^{\otimes} \to \mathsf{Fin}_*$ such that:

(i) Every inert morphism has a p-coCartesian lift with specified source.

(ii) For all $c, c' \in \mathcal{O}^{\otimes}$ and $f : p(c) \to p(c')$, let $\mathsf{Map}^f_{\mathcal{O}^{\otimes}}(c, c')$ be the union of connected components of $\mathsf{Map}_{\mathcal{O}^{\otimes}}(c, c')$ lying over f. Choose p-coCartesian lifts $c' \to c'_i$ for each $\rho^i : p(c') \to \langle 1 \rangle$. Then we require that $\mathsf{Map}^f_{\mathcal{O}^{\otimes}}(c, c') \to \prod_{1 \le i \le n} \mathsf{Map}^{\rho^i \circ f}_{\mathcal{O}^{\otimes}}(c, c'_i)$ is an equivalence.

(iii) Moreover, the maps ρ^i induce an equivalence $\mathcal{O}^{\otimes}_{\langle n \rangle} \simeq (\mathcal{O}^{\otimes}_{\langle 1 \rangle})^n$.

A *morphism of* ∞-*operads* from $p : \mathcal{O}^{\otimes} \to \mathsf{Fin}_*$ to $q : \mathcal{Q}^{\otimes} \to \mathsf{Fin}_*$ is a functor $\mathcal{O}^{\otimes} \to \mathcal{Q}^{\otimes}$ carrying p-coCartesian lifts of inert edges to q-coCartesian edges. Such morphisms form the ∞-category $\mathsf{Alg}_{\mathcal{O}^{\otimes}}(\mathcal{Q}^{\otimes})$ (or $\mathsf{Alg}_{\mathcal{O}}(\mathcal{Q})$) of $\mathcal{O}$-*algebras objects in* $\mathcal{Q}$.

A *coCartesian fibration of* ∞-*operads* is a coCartesian fibration $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ such that the composite $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes} \to \mathsf{Fin}_*$ exhibits $\mathcal{C}^{\otimes}$ as an ∞-operad. The latter condition is equivalent to requiring that, for every $O := [O_1, \dots, O_n] \in (\mathcal{O}^{\otimes}_{\langle 1 \rangle})^n \simeq \mathcal{O}^{\otimes}_{\langle n \rangle}$, the coCartesian lifts $O \to O_i$ of ρ^i induce an equivalence $\mathcal{C}^{\otimes}_O \xrightarrow{\simeq} \prod_{1 \le i \le n} \mathcal{C}^{\otimes}_{O_i}$.

Example 1.23. In this language, considering Fin_* as an ∞-operad (via the identity functor), it is immediate from the definitions that a symmetric monoidal ∞-category is a coCartesian fibration of ∞-operads $\mathcal{C}^{\otimes} \to \mathsf{Fin}_*$, and $\mathsf{CAlg}(\mathcal{C}^{\otimes}) = \mathsf{Alg}_{\mathsf{Fin}_*}(\mathcal{C}^{\otimes})$. In particular every symmetric monoidal ∞-category is an ∞-operad.

Moreover, with this point of view, a monoidal ∞-category is a coCartesian fibration of ∞-operads $\mathcal{C}^{\otimes} \to \mathsf{Assoc}^{\otimes}$, where $\mathsf{Assoc}^{\otimes}$ is the ∞-operad which is given by the nerve of the 1-category with the same objects as Fin_*, and a morphism $\langle m \rangle \to \langle n \rangle$ consists in the data of a similar morphism α in Fin_*, and a choice of a linear ordering on $\alpha^{-1}\{i\}$ for all $1 \le i \le n$. This is equivalent to Definition 1.10 by [Lur17, Prop. 4.1.2.10] (Δ^{op} has a map to $\mathsf{Assoc}^{\otimes}$, which is what is called an *approximation* to the ∞-operad $\mathsf{Assoc}^{\otimes}$).

1.3.2 The ∞-operads $\mathcal{E}_n$

Having defined ∞-operads, we may now restrict our attention to the examples we will be the most interested in for the rest of our discussion, namely $\mathcal{E}_n$-operads. In the classical topological setting, the operads $\mathcal{E}_n$ are also called the *little n-cubes operads*, and their definition is very geometric: they have one operation for each *rectilinear* embedding of the disjoint union of a finite number of cubes of dimension n into a cube of dimension n. Heuristically, the higher and higher commutativity they encode is visible in the fact, as the dimension n increases, there are more and more levels of "homotopies between homotopies etc" relating the different ways of permuting the little cubes embedded inside the bigger one. In fact the connectivity of the space of such embeddings increases with the dimension. In view of this description, the following definition seems reasonable:

Definition 1.24 ([Lur17, Def. 5.1.0.2])**.** Let $n \geq 1$, S be a finite set and $\mathring{I}^n := (-1,1)^n$. Then we define $\mathsf{Rect}(\mathring{I}^n \times S, \mathring{I}^n)$ the space of S-indexed tuples of rectilinear embeddings of $\mathring{I}^n$ into $\mathring{I}^n$. The latter are defined as maps $\mathring{I}^n \to \mathring{I}^n$ of the form by $(t_1, \dots, t_n) \mapsto (a_1 + t_1 b_1, \dots, a_n + t_n b_n)$ for some $a_i, b_i \in \mathbb{R}$. The space is topologized as a subset of $\mathbb{R}^{2n}$ with coordinates a_i, b_i.

Let $\mathcal{E}_n^{t,\otimes}$ be the topological category (see [Lur09, Def. 1.1.1.6]) with the same objects as Fin_*, and spaces of morphisms:

$$\mathsf{Map}_{\mathcal{E}_n^{t,\otimes}}(\langle m\rangle, \langle \ell\rangle) = \coprod_{\alpha \in \mathsf{Fin}_*(\langle m\rangle, \langle \ell\rangle)} \prod_{1 \leq j \leq \ell} \mathsf{Rect}(\mathring{I}^n \times \alpha^{-1}(\{j\}), \mathring{I}^n).$$

This could be seen as the space of "ways to embed m little n-cubes into ℓ little k-cubes".

Then there is a canonical projection functor $\mathcal{E}_n^{t,\otimes} \to \mathsf{Fin}_*$.

Let $\mathcal{E}_n^{\otimes}$ be the nerve of $\mathcal{E}_n^{t,\otimes}$ as a topological category. It therefore has a functor $\mathcal{E}_n^{\otimes} \to \mathsf{Fin}_*$, which exhibits $\mathcal{E}_n^{\otimes}$ as an ∞-operad, the *∞-operad of little n-cubes.*

For $k = \infty$, we set $\mathcal{E}_\infty^{\otimes} = \mathsf{Fin}_*$.

Here are several important properties of these operads:

Proposition 1.25 ([Lur17, Cor. 5.1.1.5 and 5.1.1.7])**.**

(i) *There is a morphism of ∞-operads $\mathcal{E}_n^{\otimes} \to \mathcal{E}_{n+1}^{\otimes}$ for all $n \geq 1$. In particular, if $m \in \mathbb{N}$, every $\mathcal{E}_m$-algebra in $\mathcal{C}^{\otimes}$ has an $\mathcal{E}_\ell$-algebra structure for all $1 \leq \ell \leq m$.*

(ii) *The notation $\mathcal{E}_\infty^{\otimes} = \mathsf{Fin}_*$ is justified by the fact that the colimit of $\mathcal{E}_1^{\otimes} \to \mathcal{E}_2^{\otimes} \to \mathcal{E}_3^{\otimes} \to \cdots$ is equivalent to Fin_* as an ∞-operad. In particular, item (i) is also valid for $m = \infty$.*

(iii) *If the symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ has in fact an underlying ∞-category $\mathcal{C}$ which is an n-category for some finite $n \in \mathbb{N}$, then $\mathsf{CAlg}(\mathcal{C}^{\otimes}) = \mathsf{Alg}_{\mathcal{E}_\infty^{\otimes}}(\mathcal{C}^{\otimes}) \simeq \mathsf{Alg}_{\mathcal{E}_k^{\otimes}}(\mathcal{C}^{\otimes})$ for all $k > n$ via the map $\mathcal{E}_k^{\otimes} \to \mathsf{Fin}_*$.*

Corresponding to the classical statement that $\mathcal{A}_\infty$-algebras (which encode a strong notion of associativity) are $\mathcal{E}_1$-algebras, we have:

Lemma 1.26 ([Lur17, Ex. 5.1.0.7])**.** *There is an equivalence of ∞-operads $\mathcal{E}_1^{\otimes} \simeq \mathsf{Assoc}^{\otimes}$.*

By definition, we then have that $\mathcal{E}_1$-algebras in a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ form the ∞-category $\mathsf{Alg}_{\mathcal{E}_1^{\otimes}}(\mathcal{C}^{\otimes}) \simeq \mathsf{Alg}_{\mathsf{Assoc}^{\otimes}}(\mathcal{C}^{\otimes}) \simeq \mathsf{Alg}(\mathcal{C}^{\otimes})$ as defined in Subsection 1.1 (Proposition 1.11).

Remark 1.27. As in Remark 1.12, lax symmetric monoidal functors induce functors on the corresponding ∞-categories of $\mathcal{E}_k$-algebras (by composition).

1.4 $\mathcal{E}_n$-ring spectra

One of the settings in which the $\mathcal{E}_n$-algebras introduced in the previous subsection are studied the most is in the symmetric monoidal ∞-category of topological spectra. In this case, one talks about *highly structured ring spectra*, or just $\mathcal{E}_n$-rings. In this subsection, we study some of their properties.

Definition 1.28. Let $\mathsf{Sp}^{\otimes} \to \mathsf{Fin}_*$ be the coCartesian fibration corresponding to the symmetric monoidal structure on Sp given by the smash product. For $1 \leq k \leq \infty$, the *∞-category of $\mathcal{E}_k$-ring spectra* is the ∞-category $\mathsf{Alg}_{\mathcal{E}_k^{\otimes}}(\mathsf{Sp}^{\otimes})$ of $\mathcal{E}_k^{\otimes}$-algebras in $\mathsf{Sp}^{\otimes}$.

In particular, by Proposition 1.11, any such spectrum is a homotopy ring spectrum in the sense described in the introduction to this section.

Lemma 1.29. *For any $\mathcal{E}_k$-ring spectrum E, the graded $\mathbb{Z}$-module $\pi_*(E)$ is naturally endowed with the structure of an (associative, unital) graded ring, and of a graded-commutative ring when $k \geq 2$. Morphisms of $\mathcal{E}_k$-ring spectra induce ring homomorphisms in homotopy.*

Proof. The functor π_* from the ∞-category of spectra to that of graded $\mathbb{Z}$-modules factors through $h\mathsf{Sp}$. We have seen that the symmetric monoidal structure on the ∞-category Sp induces a symmetric monoidal structure in the classical sense on its homotopy category. We therefore only have to show that π_* is lax symmetric monoidal in a 1-categorical sense on $h\mathsf{Sp}$. We have a map $\mathbb{Z}[0] \to \pi_*(\mathbb{S})$ since $\pi_0(\mathbb{S}) \cong \mathbb{Z}$, and for $E, F \in h\mathsf{Sp}$, we have a map

$$\pi_k(E) \times \pi_\ell(F) = [\Sigma^\infty \mathbb{S}^k, E] \times [\Sigma^\infty \mathbb{S}^\ell, F] \to [\Sigma^\infty \mathbb{S}^k \wedge \Sigma^\infty \mathbb{S}^\ell, E \wedge F] \cong [\Sigma^\infty \mathbb{S}^{k+\ell}, E \wedge F] \cong \pi_{k+\ell}(E \wedge F)$$

natural in E and F. One checks that it is bilinear, and thus it factors through $\pi_k(E) \otimes \pi_\ell(F)$ as desired. Therefore, in virtue of Remark 1.27, it induces a functor between $\mathcal{E}_k$-ring spectra and $\mathcal{E}_k$-algebras in the 1-category of graded $\mathbb{Z}$-modules. But the latter is exactly the 1-category of associative unital graded rings for $k = 1$ and graded-commutative rings for $k \geq 2$ (because we are considering a 1-category, see Proposition 1.25(*iii*)). ☐

Lemma 1.30. *For any $\mathcal{E}_k$-ring spectrum E, its connective cover $E_{\geq 0}$ is naturally endowed with the structure of an $\mathcal{E}_k$-ring spectrum (and the same for $\mathcal{E}_k$-morphisms).*

Proof. Since lax symmetric monoidal functors preserves $\mathcal{E}_n$-algebras (Remark 1.27), we only have to show that the connective cover functor $\tau_{\geq 0}$ is lax symmetric monoidal. This follows from the "doctrinal adjunction" principle, which states that the right adjoint to a symmetric monoidal functor is canonically lax symmetric monoidal (follows from [Lur17, Cor. 7.3.2.7]). Indeed, here the adjoint is the inclusion functor, which is symmetric monoidal: it exhibits the ∞-category of connective spectra as a symmetric monoidal subcategory of Sp, because it is a full subcategory stable under equivalences and closed under the tensor product, by [Lur17, Rmk 2.2.1.2]. ☐

1.5 Free $\mathcal{E}_n$-rings

As mentioned in the introduction to this section, we will later have to consider free $\mathcal{E}_n$-algebras. More precisely, we will need the construction not of free $\mathcal{E}_n$-algebras in the ∞-category of spectra, but in the ∞-category of HA-*modules* (where A is a (discrete) ring and HA is its Eilenberg-Mac Lane spectrum). These are called free $\mathcal{E}_n$-HA-algebras. In this subsection, we will explain what this means, but first of all we have to recall some prerequisities about free $\mathcal{E}_n$-algebras in a general symmetric monoidal ∞-category $\mathcal{C}$.

Following [Lur17, §3.1.3], consider the following general situation: let $p : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a coCartesian fibration of ∞-operads, and $\mathcal{A}^\otimes \to \mathcal{B}^\otimes \to \mathcal{O}^\otimes$ be morphisms of ∞-operads. Composition with $\mathcal{A}^\otimes \to \mathcal{B}^\otimes$ induces a functor

$$\theta : \mathsf{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{C}) \to \mathsf{Alg}_{\mathcal{A}/\mathcal{O}}(\mathcal{C})$$

where $\mathsf{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{C})$ is the ∞-category of maps of ∞-operads $\mathcal{B}^\otimes \to \mathcal{C}^\otimes$ over $\mathcal{O}^\otimes$ (viewed as a subcategory of the ∞-category of functors $\mathcal{B}^\otimes \to \mathcal{C}^\otimes$ over $\mathcal{O}^\otimes$).

When θ admits a left adjoint, the latter can be seen as the "free algebra" construction previously advertised. Here are sufficient conditions for its existence.

Proposition 1.31 ([Lur17, Cor. 3.1.3.4])**.** *In the situation depicted above, if all of the following conditions are satisfied, then θ admits a left adjoint:*

- *there exists an uncountable regular cardinal κ such that $\mathcal{A}^\otimes$ and $\mathcal{B}^\otimes$ are essentially κ-small,*
- *for all $X \in \mathcal{O}$, the fiber $\mathcal{C}_X$ has all κ-small colimits,*
- *for all $X_1, \ldots, X_n, Y \in \mathcal{O}$ with maps $X_i \to Y$, the induced map $\prod_{1 \leq i \leq n} \mathcal{C}_{X_i} \to \mathcal{C}_Y$ preserves κ-small colimits in each variable separately.*

The case of interest for us is the following:

- $p : \mathcal{C}^\otimes = \mathsf{Sp}^\otimes \longrightarrow \mathcal{O}^\otimes = \mathsf{Fin}_*$ endowing Sp with its usual symmetric monoidal structure.
- $\mathcal{A}^\otimes = \mathsf{Triv}^\otimes$ the trivial ∞-operad, given by the nerve of the wide subcategory of Fin_* spanned by the inert morphisms.
- For a fixed $1 \leq k \leq \infty$, we set $\mathcal{B}^\otimes = \mathcal{E}_k^\otimes$ and $\mathsf{Triv}^\otimes \to \mathcal{E}_k^\otimes$ sending $\langle 1 \rangle \in \mathsf{Triv}$ to the unique object $\langle 1 \rangle \in (\mathcal{E}_k^\otimes)_{\langle 1 \rangle}$ (this data suffices to determine a map from the trivial operad, see the first line of the proof of [Lur17, Prop. 2.2.6.4]). The map $\mathcal{B}^\otimes = \mathcal{E}_k^\otimes \longrightarrow \mathcal{O}^\otimes = \mathsf{Fin}_*$ is the one exhibiting $\mathcal{E}_k^\otimes$ as an ∞-operad.

We want to apply Proposition 1.31 to this case. Since Sp has all colimits, and it is presentably symmetric monoidal, so that in particular the tensor product preserves colimits in each variable, we may choose κ as the first uncountable regular cardinal (also note that $\mathsf{Triv}^\otimes$ and $\mathcal{E}_k^\otimes$ have countably many objects and mapping spaces of cardinality at most $|\mathbb{R}|$). We obtain:

Proposition 1.32. *For all $1 \leq k \leq \infty$, the forgetful functor $\theta : \mathsf{Alg}_{\mathcal{E}_k}(\mathsf{Sp}) \to \mathsf{Sp}$ has a left adjoint*

$$F_{\mathcal{E}_k} : \mathsf{Sp} \to \mathsf{Alg}_{\mathcal{E}_k}(\mathsf{Sp})$$

called the free $\mathcal{E}_k$-ring spectrum functor.

Remark 1.33. In the setting of Proposition 1.31, when $\mathcal{A}^\otimes = \mathsf{Triv}^\otimes$ and $\mathcal{B}^\otimes = \mathcal{O}^\otimes$, a more explicit description of the left adjoint of θ is available (see [Lur17, Prop. 3.1.3.13]). In the case $\mathcal{O}^\otimes = \mathsf{Assoc}^\otimes$, one obtains Proposition 1.34 just below.

Proposition 1.34 ([Lur17, Cor. 4.1.1.18])**.** *Let $\mathcal{C}$ be a monoidal ∞-category with countable colimits, such that the tensor product functor preserves them in each variable separately. Then the forgetful functor $\theta : \mathsf{Alg}(\mathcal{C}) \to \mathcal{C}$ has a left adjoint $F_{\mathcal{E}_1}$. For any $X \in \mathcal{C}$, we have an equivalence in $\mathcal{C}$*

$$\theta(F_{\mathcal{E}_1}(X)) \simeq \coprod_{n \geq 0} X^{\otimes n}.$$

This applies in particular to the free $\mathcal{E}_1$-ring spectrum functor $F_{\mathcal{E}_1} : \mathsf{Sp} \to \mathsf{Alg}(\mathsf{Sp})$.

In Sections 4 and 5, we will encounter $\mathcal{E}_\infty$-rings whose homotopy rings are polynomial in one variable. We will identifying them as free $\mathcal{E}_1$-algebras, because as we saw above, we have a rather easy description of the objects underlying free $\mathcal{E}_1$-algebras, and thus of their homotopy groups. The $\mathcal{E}_n$ case for $n > 1$ turns out to be more complicated (for $n = \infty$, it involves for instance (homotopy) fixed points with respect to the permutation action of the symmetric groups on the tensor products instead of only the tensor products themselves). We therefore want to build $\mathcal{E}_1$-rings with polynomial homotopy; such that specifying an image in homotopy of the polynomial generator suffices in some sense to obtain an $\mathcal{E}_1$-map out of them (see Proposition 1.38 for the details).

More precisely, let A be a (discrete) commutative ring. We will actually consider free $\mathcal{E}_1$-algebras in the ∞-category of HA-modules. Let us briefly mention some facts we will need about modules over an $\mathcal{E}_\infty$-ring spectrum.

- For any $R \in \mathsf{CAlg}(\mathsf{Sp})$, by [Lur17, Prop. 7.1.2.7] there exists a presentably symmetric monoidal ∞-category $\mathsf{Mod}_R(\mathsf{Sp})$, whose symmetric monoidal structure is induced by that of Sp. This construction is even functorial in $R \in \mathsf{CAlg}(\mathsf{Sp})$. In particular, it has all colimits and the tensor product functor preserves them in each variable separately. An object in this ∞-category is an R-module in the usual sense, i.e. it can informally be described as a spectrum E with a unit map $R \to E$ and a multiplication map $R \wedge E \to E$, satisfying the usual axioms of a module over a ring, with all higher coherencies.
- There is a forgetful functor $U_R : \mathsf{Mod}_R(\mathsf{Sp}) \to \mathsf{Sp}$ with admits a left adjoint F_R, the *free R-module functor.* The underlying spectrum of $F_R(E)$ is $R \wedge E$ (tensor product in Sp).

Definition 1.35. Let $R \in \mathsf{CAlg}(\mathsf{Sp})$. The *$\infty$-category of $\mathcal{E}_k$-R-algebras in* Sp is $\mathsf{Alg}_{\mathcal{E}_k}(\mathsf{Mod}_R(\mathsf{Sp}))$.

We now build a free $\mathcal{E}_1$-R-algebra functor. By Proposition 1.34 and the properties we just listed, the forgetful functor $\mathsf{Alg}_{\mathcal{E}_1}(\mathsf{Mod}_R(\mathsf{Sp})) \to \mathsf{Mod}_R(\mathsf{Sp})$ admits a left adjoint.

Proposition 1.36. *The forgetful functor from $\mathcal{E}_1$-R-algebras to spectra admits a left adjoint $F_{\mathcal{E}_1,R}$, the* free $\mathcal{E}_1$-R-algebra functor*, which is obtained as a composite*

$$\mathsf{Sp} \underset{U_R}{\overset{F_R}{\rightleftarrows}} \mathsf{Mod}_R(\mathsf{Sp}) \underset{U_{\mathcal{E}_1}}{\overset{F_{\mathcal{E}_1}}{\rightleftarrows}} \mathsf{Alg}_{\mathcal{E}_1}(\mathsf{Mod}_R(\mathsf{Sp}))$$

given on the underlying spectra by $E \mapsto \bigvee_{n\geq 0} E^{\wedge n} \wedge R$.

The explicit expression on the underlying spectra comes from Proposition 1.34 and the facts listed above (note however that the coproduct and tensor products appearing in the explicit expression for $F_{\mathcal{E}_1}$ must be taken in the ∞-category of R-modules).

Definition 1.37. Let $n \geq 0$, and let A be a (discrete) commutative ring. Then $\mathsf{HA} \in \mathsf{CAlg}(\mathsf{Sp})$ and thus we may define $(\mathsf{HA})[t^n] := F_{\mathcal{E}_1,\mathsf{HA}}(\Sigma^n\mathbb{S})$ the *free $\mathcal{E}_1$-HA-algebra over one generator in degree n.* We abuse notation and also write it as $\mathsf{HA}[t^n]$, which will never mean $\mathsf{H}(\mathsf{A}[t^n])$ (Eilenberg-Mac Lane spectrum of a polynomial ring over A) for us.

Proposition 1.38. *The free $\mathcal{E}_1$-HA-algebra $\mathsf{HA}[t^n]$ has the following properties:*

(i) The underlying spectrum is given by $\bigvee_{j\geq 0} \Sigma^{nj}\mathsf{HA}$.

(ii) The homotopy groups of $\mathsf{HA}[t^n]$ form a polynomial ring on one generator in degree n, i.e. we have $\pi_*(\mathsf{HA}[t^n]) \cong A[t^n]$.

(iii) Assume E is an $\mathcal{E}_2$-ring spectrum with an $\mathcal{E}_2$-map from HA. Then E can be viewed as an $\mathcal{E}_1$-HA-algebra in a canonical way, and any element $\alpha \in \pi_n(E)$ determines a map $\mathsf{HA}[t^n] \to E$ in $\mathsf{Alg}_{\mathcal{E}_1}(\mathsf{Mod}_R(\mathsf{Sp}))$, sending t^n to α in homotopy.

Proof. Item (i) is given by Proposition 1.36. In particular, the computation of the homotopy groups follows. To finish the proof of item (ii), we must consider the ring structure. We have to show that multiplication by a generator of the n-th homotopy group induces isomorphisms in all degrees, i.e. that the composition

$$\Sigma^n\mathbb{S} \wedge \mathsf{HA}[t^n] \xrightarrow{t^n\wedge \mathsf{id}} \mathsf{HA}[t^n] \wedge \mathsf{HA}[t^n] \longrightarrow \mathsf{HA}[t^n]$$

induces isomorphisms in homotopy in non-negative degrees. This map is given by the identification

$$\Sigma^n\mathbb{S} \wedge \bigvee_{j\geq 0} \Sigma^{nj}\mathsf{HA} \simeq \bigvee_{j\geq 0} (\Sigma^n\mathbb{S} \wedge \Sigma^{nj}\mathsf{HA}) \longrightarrow \bigvee_{j\geq 0} \Sigma^{nj}\mathsf{HA}$$

mapping $\Sigma^n\mathbb{S} \wedge \Sigma^{nj}\mathsf{HA}$ to $\Sigma^{n(j+1)}\mathsf{HA}$ identically, so our claim follows.

For item (iii), the existence of a forgetful functor $\mathsf{Alg}_{\mathcal{E}_2}(\mathsf{Sp})_{\mathsf{HA}/} \to \mathsf{Alg}_{\mathcal{E}_1}(\mathsf{Mod}_{\mathsf{HA}}(\mathsf{Sp}))$ is given by [ACB19, Rmk 3.7] (which ultimately follows from [Lur17, Cor. 7.3.2.7]). Once we admit this fact, item (iii) is rephrasing the fact that $F_{\mathcal{E}_1,\mathsf{HA}}$ is left adjoint to the forgetful functor; using that $\pi_0(\mathsf{map}_{\mathsf{Sp}}(\Sigma^n\mathbb{S}, X)) \cong \pi_n(X)$. □

Remark 1.39. Item (iii) in Proposition 1.38 above may seem a bit artificial. The point is that the universal property one could hope for would be that an $\mathcal{E}_1$-ring map $\mathsf{HA} \to E$ and the choice of an element in $\pi_n(E)$ determine an $\mathcal{E}_1$-map $\mathsf{HA}[t^n] \to E$. But then, we would have needed a free object over $\Sigma^n\mathbb{S} \in \mathsf{Sp}$ in the ∞-category $\mathsf{Alg}_{\mathcal{E}_1}(\mathsf{Sp})_{\mathsf{HA}/}$. However, it turns out that this ∞-category is *not* equivalent to $\mathsf{Alg}_{\mathcal{E}_1}(\mathsf{Mod}_{\mathsf{HA}}(\mathsf{Sp}))$ (see [Lur17, Warning 7.1.3.9] and [ACB19, Remark 3.7]). More precisely, an $\mathcal{E}_1$-ring under HA is a priori not an $\mathcal{E}_1$-algebra in HA-modules. There is also a forgetful functor in the other direction. Actually, there is a forgetful functor $\mathsf{Alg}_{\mathcal{E}_1}(\mathsf{Sp})_{\mathsf{HA}/} \to \mathsf{Mod}_{\mathsf{HA}}(\mathsf{Sp})$, but the $\mathcal{E}_1$-structure we had at the beginning is not compatible with the HA-module structure we obtained. A higher commutativity (for instance E belonging to $\mathsf{Alg}_{\mathcal{E}_2}(\mathsf{Sp})_{\mathsf{HA}/}$ as we saw above) ensures compatibility of the $\mathcal{E}_1$-structure and the module structure. The lack of compatibility of the multiplicative structure is made apparent in the form that the free functor $\mathsf{Sp} \to \mathsf{Alg}_{\mathcal{E}_1}(\mathsf{Sp})_{R/}$ for $R \in \mathsf{CAlg}(\mathsf{Sp})$ takes. For example, in the case of animated rings instead of spectra, by [Sch19, Appendix to Lecture XII, p 93], it is given on the underlying objects by

$$E \longmapsto R \amalg (R \wedge E \wedge R) \amalg (R \wedge E \wedge R \wedge E \wedge R) \amalg \dots.$$

Then, the free $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}$-algebra on $\mathbb{S}$ is $\bigvee_{n\geq 0} \mathsf{H}\mathbb{Z}$ as a spectrum, and lies in $\mathsf{Sp}^{\heartsuit}$, whereas the free $\mathcal{E}_1$-ring under $\mathsf{H}\mathbb{Z}$ on $\mathbb{S}$ is given by $\bigvee_{n\geq 0} \mathsf{H}\mathbb{Z}^{\wedge n}$, which is not co-connective (and thus does not lie in $\mathsf{Sp}^{\heartsuit}$).

2 Stable and unstable motivic homotopy theory

The next section of the expository part of our discussion concerns the basic set-up of unstable and stable motivic homotopy theory. The picture is somewhat similar to the topological case: the goal is to define the ∞-categories of motivic spaces and of motivic spectra we will be working with, playing the roles of the ∞-categories of spaces and topological spectra respectively. We will also introduce some notions and elements of the basic toolbox to study them, for example the Nisnevich topology, homotopy sheaves of motivic spaces or spectra (playing the role of the homotopy groups in topology), the homotopy t-structure on the ∞-category of motivic spectra, effective and very effective motivic spectra (these are certain notions of connectivity), the slice filtration (which is to some extend a certain kind of Postnikov tower), etc.

We choose to work in the ∞-categorical setting, although the first construction of a category of motivic spaces by Morel and Voevodsky in [MV99] was performed in the setting of model categories. Another good reference in the latter setting is [Mor99].

From now on, let k be a fixed base field. We will also often consider a fixed quasi-separated Noetherian finite dimensional base scheme S (for even more general base schemes, some differences appear in the properties of the Nisnevich topos, for example it may not suffice anymore to consider only Nisnevich covers coming from elementary distinguished Nisnevich squares).

2.1 The ∞-category of motivic spaces

Motivic homotopy theory begins with the ∞-category of motivic spaces. It plays the role of the usual ∞-category of spaces in topology. To better understand the analogy between the two, let us first describe a somewhat unusual original construction of the ∞-category of spaces. Consider the 1-category of smooth manifolds SmMfd. To make it cocomplete, we embed it into the ∞-category $\mathcal{P}(\mathsf{SmMfd})$ of presheaves of spaces on SmMfd. Then, to incorporate the topology of smooth manifolds in the picture, and a notion of "homotopy invariance" represented by the contractibility of the real line $\mathbb{R}$, one considers the subcategory of sheaves with respect to the Euclidean topology on SmMfd, that are (Bousfield-)local with respect to all projections $X \times \mathbb{R} \to X$ for $X \in \mathsf{SmMfd}$. Then, this subcategory turns out to be equivalent to the usual homotopy ∞-category of spaces (see for instance [Dug01, Thm 8.3]). The ∞-category of motivic spaces is built by reproducing the same construction in the world of algebraic geometry instead. The role of SmMfd is played by the 1-category of smooth schemes of finite type over a fixed (nice enough) base scheme, the role of $\mathbb{R}$ is played by the affine line $\mathbb{A}^1$ over this base, and the Euclidean topology is replaced with the Nisnevich topology.

2.1.1 Presheaves of spaces

We introduce the 1-category of smooth schemes the construction described above begins with, and embed it into its ∞-category of presheaves.

Definition 2.1. Let S be a finite dimensional quasi-separated Noetherian scheme (e.g. $S \in \mathsf{Sm}_k$). Let Sm_S be the *1-category of smooth schemes of finite type* over S (for a more general base scheme S, one would ask for smooth schemes of finite presentation instead). Let $\mathcal{P}(\mathsf{Sm}_S) = \mathsf{Fun}(\mathsf{Sm}_S^{\mathsf{op}}, \mathsf{Spc})$ be the *∞-category of presheaves of spaces on* Sm_S.

Remark 2.2. Note that Sm_S is essentially small. Indeed, let $\{U_i\}_{i \in I}$ be a cover by affines of S, where I is a finite set (since S is quasi-compact). Then, every finite type smooth scheme X over S is given over $U_i = \mathsf{Spec}(A_i)$ for each $i \in I$ by the gluing of finitely many affine open sets $\mathsf{Spec}(A_i[x_1, \ldots, x_n]/(f_1, ..., f_r)) \to \mathsf{Spec}(A_i)$. There is only a set of possibilities for the choices of the number n of variables, the number r of polynomials, and these polynomials. Therefore, there is a set of choices of finitely many such affine open, and for each of them, a set of choices for the gluing maps. Similarly, there is only a set of maps between two fixed objects.

In the following, for any $X \in \mathsf{Sm}_S$, we denote again by X the image $y(X) \in \mathcal{P}(\mathsf{Sm}_S)$ of X by the Yoneda embedding y.

2.1.2 The Nisnevich topology

In the construction described in the introduction to this subsection, we have considered sheaves with respect to a certain topology on the 1-∞-category of smooth manifolds. In the algebraic setting, we need a Grothendieck topology on the 1-∞-category Sm_S. The first candidate would perhaps be the Zariski topology. However, it is famously too coarse for many purposes. Morel and Voevodsky chose to use in their construction of a motivic category the *Nisnevich topology* instead. The latter is finer than the Zariski topology, but coarser than the étale topology; they refer to it in [MV99] as seemingly having "the good properties of both while avoiding the bad ones". A downside of working with a finer topology is that the sheaf condition is a priori harder to check; however in the case of the Nisnevich topology (under our assumptions on S), we will see just below that there is a rather simple criterion for this.

Definition 2.3 (Nisnevich topology)**.** The *Nisnevich topology on* Sm_S is defined as the Grothendieck topology generated by *Nisnevich coverings*: these are the finite families of étale morphisms that are jointly surjective on k'-points for all fields k' (every k'-point in the target has a lift by one of the morphisms, such that this morphism induces an isomorphism at residue fields).

For a quasi-separated Noetherian finite dimensional base scheme S, this topology is determined by smaller data:

Definition 2.4. An *elementary distinguished Nisnevich square*, or just *Nisnevich square*, is a Cartesian square in Sm_S

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d, "p"] \\ U \arrow[r, hook, "\iota"] & X \end{tikzcd}$$

where ι is an open immersion and p is an étale morphism inducing an isomorphism $p^{-1}(X\backslash U) \to X\backslash U$ (with respect to the reduced scheme structures). We may denote such a square by $\{U \hookrightarrow X, V \to X\}$.

Theorem 2.5.

(i) *The family of all Nisnevich coverings $\{U \hookrightarrow X, V \to X\}$ corresponding to Nisnevich squares generates the Nisnevich topology.*

(ii) *The Nisnevich topology is subcanonical, i.e. every representable presheaf $y(X)$ for $X \in \mathsf{Sm}_S$ is a sheaf with respect to the Nisnevich topology.*

(iii) *For every $\mathcal{F} \in \mathcal{P}(\mathsf{Sm}_S)$, the following three conditions are equivalent:*

- Nisnevich excision: *For every Nisnevich square $\{U \hookrightarrow X, V \to X\}$, the natural map*

$$\mathcal{F}(X) \longrightarrow \mathcal{F}(U) \times_{\mathcal{F}(U \times_X V)} \mathcal{F}(V)$$

  is an equivalence and $\mathcal{F}(\emptyset) \simeq *$.

- Nisnevich descent: *For every Nisnevich cover $\mathcal{U}$ of a scheme X, the natural map*

$$\mathsf{lim}_{\Delta}\, \mathsf{map}_{\mathcal{P}(\mathsf{Sm}_S)}(\check{C}(\mathcal{U}), \mathcal{F}) \longrightarrow \mathcal{F}(X)$$

  is an equivalence, where map *denotes the mapping space.*

- Sheaf condition: *The presheaf $\mathcal{F}$ is a sheaf in the Nisnevich topology.*

Proof.

(i) This is [MV99, Prop. 1.4].

(ii) This follows from [Sta25, Tag 03NV], using that the étale topology is finer than the Nisnevich topology (and that representable objects are 0-truncated presheaves).

(iii) This is shown in [Lur18, Thm 3.7.5.1].

□

2.1.3 Motivic spaces

As in the topological case, the next step of the construction is to consider the subcategory of sheaves satisfying a certain notion of homotopy invariance; in the algebraic setting, this consists in defining motivic spaces as those presheaves on Sm_S satisfying $\mathbb{A}^1$-*invariance* and *Nisnevich descent* (or equivalently, $\mathbb{A}^1$-invariant Nisnevich sheaves). More precisely:

Definition 2.6. Let $\mathcal{W}_{\mathsf{mot}}$ be the collection of maps in $\mathcal{P}(\mathsf{Sm}_S)$ containing:

(*a*) The (image by the Yoneda embedding of the) projections $X \times \mathbb{A}^1 \to X$ for any $X \in \mathsf{Sm}_S$.

(*b*) The comparison maps $y(U) \amalg_{y(U \times_X V)} y(V) \to y(X)$ for any Nisnevich square $\{U \hookrightarrow X, V \to X\}$.

(*c*) The unique map $\emptyset \to y(\emptyset)$.

Let $\mathsf{L}_{\mathsf{mot}} : \mathcal{P}(\mathsf{Sm}_S) \to \mathcal{P}(\mathsf{Sm}_S)$ be the associated localization functor (in the sense of [Lur09, §5.5.4]), called *motivic localization functor*. Then, the ∞-*category of motivic spaces over* S is the full subcategory $\mathsf{Spc}(S) \subseteq \mathcal{P}(\mathsf{Sm}_S)$ spanned by the local objects.

Such localizations can be defined for any presentable ∞-category and (small) set of morphisms in it. Recall that Sm_S is essentially small by Remark 2.2, and then $\mathcal{W}_{\mathsf{mot}}$ is also essentially small for the same reasons. Therefore, $\mathcal{P}(\mathsf{Sm}_S)$ is presentable as a presheaf ∞-category on an (essentially) small ∞-category by [Lur09, Thm 5.5.1.1]. We summarize the situation in the following proposition:

Proposition 2.7. *There is an adjunction*

$$\iota : \mathsf{Spc}(S) \underset{\longleftarrow}{\overset{\perp}{\hookrightarrow}} \mathcal{P}(\mathsf{Sm}_S) : \mathsf{L}_{\mathsf{mot}}$$

which exhibits $\mathsf{Spc}(S)$ *as an accessible localization of* $\mathcal{P}(\mathsf{Sm}_S)$*. In particular,* $\mathsf{Spc}(S)$ *is presentable.*

Remark 2.8. In Definition 2.6, families of maps (*b*) and (*c*) correspond to the conditions in Theorem 2.5(*iii*). In particular, we could equivalently replace the families of maps (*b*) and (*c*) with the family (b') of maps $\check{C}(\mathcal{U}) \to X$ for $\mathcal{U}$ a Nisnevich cover, where $\check{C}(\mathcal{U})$ denotes the Čech complex. More precisely, the local objects with respect to families $(a) \cup (b) \cup (c)$ and $(a) \cup (b')$ are the same. In other terms, these families of maps have the same strong saturation and thus the localization functors they induce are the same.

One can define the smash product of two motivic spaces as usual: for $X, Y \in \mathsf{Spc}(S)_*$, let $X \wedge Y$ be the cofiber of $X \vee Y \to X \times Y$, where $\vee$ denotes the coproduct.

Proposition 2.9. *The smash product defines a closed symmetric monoidal structure on* $\mathsf{Spc}(S)_*$.

Proof. This can be showed using the formalism of Subsection 1.2 (the symmetric monoidal ∞-category $\mathsf{Pr}^{\mathsf{L},\otimes}$). Indeed, in virtue of Example 1.21, it suffices to show that $\mathsf{Spc}(S)$ has finite products and that the induced Cartesian symmetric monoidal structure of Proposition 1.14 is closed.

The existence of finite products is well-known, and can be checked in various ways. Note first that $\mathsf{Spc}(S)$ is presentable (Proposition 2.7), and thus has all limits by [Lur09, Cor. 5.5.2.4]. Then, for instance, it suffices to check that $\mathsf{L}_{\mathsf{mot}}$ preserves finite products. This holds because $\mathsf{L}_{\mathsf{mot}}$ is a countable transfinite composition of the localization functors $\mathsf{L}_{\mathbb{A}^1}$ (with respect to the family of maps (*a*) in Definition 2.6) and $\mathsf{L}_{\mathsf{Nis}}$ (with respect to the maps $(b) \cup (c)$), which both preserve finite products, by [AE17, Thm 4.27] for example. For the first one, it can be shown via its explicit description as the singular functor $\mathcal{F} \mapsto (X \mapsto |\mathcal{F}(X \times \mathbb{A}^\bullet)|)$ (see [AE17, §4.3]), and for the second one, it basically follows from the fact that it is a sheafification functor, and hence described by filtered colimits, which commute with finite products. An alternative way to show this is to view $\mathsf{Spc}(S)$ as consisting of the objects of $\mathcal{P}(\mathsf{Sm}_S)$ that are $\mathbb{A}^1$-invariant and satisfy Nisnevich excision, and these two conditions are stable under finite products.

We now show that the functor $- \times \mathcal{G}$ for $\mathcal{G} \in \mathsf{Spc}(S)$ is a left adjoint, by proving that it preserves colimits (this suffices by the adjoint functor theorem). Let $\mathsf{colim}_i \mathcal{F}_i$ be a colimit in $\mathsf{Spc}(S)$. Then, using that both $\mathsf{L}_{\mathsf{mot}}$ and ι preserve finite products, $\mathsf{L}_{\mathsf{mot}}$ preserves colimits, and $\mathsf{L}_{\mathsf{mot}} \circ \iota \simeq \mathsf{id}$, we have

$$\begin{aligned}
(\mathsf{colim}_i \mathcal{F}_i) \times \mathcal{G} &\simeq (\mathsf{colim}_i \mathsf{L}_{\mathsf{mot}} \iota \mathcal{F}_i) \times \mathcal{G} \\
&\simeq \mathsf{L}_{\mathsf{mot}} (\mathsf{colim}_i \iota \mathcal{F}_i) \times \mathsf{L}_{\mathsf{mot}} \iota \mathcal{G} \\
&\simeq \mathsf{L}_{\mathsf{mot}} ((\mathsf{colim}_i \iota \mathcal{F}_i) \times \iota \mathcal{G}) \\
&\simeq \mathsf{L}_{\mathsf{mot}} (\mathsf{colim}_i (\iota \mathcal{F}_i \times \iota \mathcal{G})) \qquad (\star) \\
&\simeq \mathsf{colim}_i \mathsf{L}_{\mathsf{mot}} (\iota(\mathcal{F}_i \times \mathcal{G})) \\
&\simeq \mathsf{colim}_i (\mathcal{F}_i \times \mathcal{G})
\end{aligned}$$

where $(\star)$ follows from the Cartesian monoidal structure on presheaves being closed (since the one on Spc is also closed). Thus the Cartesian symmetric monoidal structure on $\mathsf{Spc}(S)$ is closed. $\square$

Example 2.10. Applying the motivic localization functor, any presheaf of spaces on Sm_S defines a motivic space, for example any $X \in \mathsf{Sm}_S$, viewed as the representable presheaf (of discrete spaces) $y(X)$, but also any $E \in \mathsf{Spc}$, viewed as a constant presheaf. For example, we will often consider $\mathbb{A}^1_k = \mathsf{Spec}(k[t])$, $\mathbb{G}_m = \mathsf{Spec}(k[t, t^{-1}])$, and $\mathbb{P}^1_k$ which we also denote by T (pointed respectively at 1, 1 and ∞ when seen as objects in $\mathsf{Spc}(S)_*$). The topological spheres $\mathbb{S}^n$ also give rise to motivic spaces which we denote by $\mathcal{S}^n$ for all $n \geq 0$. In this notation, we have $\mathbb{G}_m \wedge \mathcal{S}^1 \simeq T$ in $\mathsf{Spc}(S)_*$. Moreover, suspension in $\mathsf{Spc}(S)_*$ is given by the smash product with $\mathcal{S}^1$.

2.2 The stable ∞-category of motivic spectra

Now that our algebraic analog of the ∞-category Spc of spaces has been defined, we want to define its stable companion. In topology, this is done by taking the *stabilization* $\mathsf{Sp}(\mathsf{Spc}) =: \mathsf{Sp}$, which is a construction that exists for any ∞-category with finite limits, by [Lur17, §1.4.2]. Its objects can be described as sequences $(E_0, E_1, \dots)$ of pointed spaces, with structure maps $\Sigma(E_i) \simeq \mathbb{S}^1 \wedge E_i \to E_{i+1}$ for all $i \geq 0$, such that their adjoint $E_i \to \Omega E_{i+1}$ is an equivalence (without this condition, one talks about *prespectra*). This description of the stabilization holds more generally for ∞-categories with finite limits and finite colimits.

Since $\mathsf{Spc}(S)$ admits finite limits and finite colimits (as it is presentable), the same applies: one can consider the stabilization $\mathsf{Sp}(\mathsf{Spc}(S))$. This gives rise to the *stable ∞-category of $\mathcal{S}^1$-motivic spectra*, which we denote by $\mathsf{SH}_{\mathcal{S}^1}(S)$. Its objects can again be described as sequences $(E_0, E_1, \dots)$ of pointed motivic spaces, with structure maps $\Sigma_{\mathcal{S}^1}(E_i) \simeq \mathcal{S}^1 \wedge E_i \to E_{i+1}$ for all $i \geq 0$, such that their adjoint $E_i \to \Omega_{\mathcal{S}^1} E_{i+1}$ is an equivalence.

The two constructions above corresponds to the universal way of inverting $\mathbb{S}^1$, respectively $\mathcal{S}^1$, with respect to the symmetric monoidal structure on Spc_*, respectively $\mathsf{Spc}(S)_*$, given by the smash product. Indeed, in the motivic case (and the same holds in the topological case), we have an adjunction

$$\Sigma^\infty_{\mathcal{S}^1} : \mathsf{Spc}(S)_* \underset{\longrightarrow}{\overset{\longleftarrow}{\dashv}} \mathsf{SH}_{\mathcal{S}^1}(S) \ : \Omega^\infty_{\mathcal{S}^1}$$

and the symmetric monoidal structure on $\mathsf{Spc}(S)_*$ extends to $\mathsf{SH}^{\mathcal{S}^1}(S)$, making $\Sigma^\infty_{\mathcal{S}^1}$ into a symmetric monoidal functor. Then $\mathcal{S}^1$ becomes invertible in the sense that the functor $\Sigma_{\mathcal{S}^1}(-) = \Sigma^\infty_{\mathcal{S}^1}(\mathcal{S}^1) \wedge -$ is a self-equivalence of $\mathsf{SH}_{\mathcal{S}^1}(S)$. The way in which it is universal with this property will be detailed just below.

However, it turns out that there is another kind of stabilization of $\mathsf{Spc}(S)_*$ that makes much sense. Indeed, in motivic spaces, there is a second kind of circle: the group scheme $\mathbb{G}_m$, which we would also like to invert. One way to invert both $\mathcal{S}^1$ and $\mathbb{G}_m$ is to invert their smash product $\mathbb{P}^1$. The abstract set-up to perform this inversion is as follows:

Theorem 2.11 ([Rob13, Def. 4.8 and Prop. 4.10])**.** *Let $\mathcal{C} \in \mathsf{CAlg}(\mathsf{Pr}^{\mathsf{L}})$ and $X \in \mathcal{C}$. Then, the ∞-category of X-stable objects under $\mathcal{C}$ in $\mathsf{CAlg}(\mathsf{Pr}^{\mathsf{L}})$, namely those objects for which $- \otimes X$ is an equivalence, admits an initial object $\Sigma^\infty_X : \mathcal{C} \to \mathcal{C}[X^{-1}]$.*

Example 2.12. If $\mathcal{C} = \mathsf{Spc}_*$, then $\mathcal{C}[(\mathbb{S}^1)^{-1}] \simeq \mathsf{Sp}$ as symmetric monoidal ∞-categories. Similarly, we have $\mathsf{Spc}(S)_*[(\mathcal{S}^1)^{-1}] \simeq \mathsf{SH}_{\mathcal{S}^1}(S)$.

We can relate this abstract definition of a stabilization with the usual spectra objects:

Theorem 2.13 ([Rob13, Cor. 4.24])**.** *In the setting of Theorem 2.11, if the symmetric monoidal structure on $\mathcal{C}$ is closed and if for some $n \geq 3$ the cyclic permutation on $X^{\otimes n}$ is homotopic to the identity, then the presentable ∞-category underlying $\mathcal{C}[X^{-1}]$ is equivalent to the ∞-category of sequences $(E_n)_{n \in \mathbb{N}}$ in $\mathcal{C}$, with the data of equivalences $E_n \xrightarrow{\simeq} \Omega_X(E_{n+1})$ for all $n \geq 0$, where Ω_X is the right adjoint to $- \otimes X$.*

In our case:

Lemma 2.14 ([Voe98, Lemma 4.4])**.** *Let $T = (\mathbb{P}^1, \infty) \in \mathsf{Spc}(S)_*$. The cyclic permutation on $T \wedge T \wedge T$ is homotopic to the identity.*

Proof. A proof is given in [Hoy17, Lemma 5.21] in a more general context. Here we apply this result to the trivial group and $\mathcal{E}$ the structure sheaf on S, then the representation sphere $S^{\mathcal{E}}$ is nothing but $\mathbb{A}^1/(\mathbb{A}^1 \setminus 0) \simeq \mathbb{P}^1$. The result then tells us that the transposition on $T \wedge T$ is homotopic to $-\mathsf{id} \wedge \mathsf{id} = -\mathsf{id}$. Since a 3-cycle is a composition of two transpositions, this shows our claim. □

We can therefore apply Theorem 2.13 to $\mathcal{C} = \mathsf{Spc}(S)_*$ and $X = T$:

Definition 2.15. The *∞-category of motivic $\mathbb{P}^1$-spectra*, or simply *motivic spectra*, is defined as the ∞-category $\mathsf{SH}(S) := \mathsf{Spc}(S)_*[T^{-1}]$. More explicitly, a (motivic) $\mathbb{P}^1$-spectrum (over S) is a sequence $(E_0, E_1, \dots)$ in $\mathsf{Spc}(S)_*$ with structure maps $\Sigma_T(E_i) := T \wedge E_i \to E_{i+1}$, such that their adjoint $E_i \to \Omega_T(E_{i+1})$ is an equivalence for all $i \geq 0$.

In particular, $\mathsf{SH}(S)$ is presentably symmetric monoidal, and there is a colimit-preserving symmetric monoidal functor $\Sigma^\infty_T : \mathsf{Spc}(S)_* \to \mathsf{SH}(S)$, and the adjoint functor theorem [Lur09, Cor. 5.5.2.9] implies that it has a right adjoint Ω^∞_T. Since we will work essentially with $\mathbb{P}^1$-spectra, we may also use the notation $\Sigma^\infty := \Sigma^\infty_T$ if the context is clear.

Lemma 2.16. *The ∞-categories $\mathsf{SH}_{\mathcal{S}^1}(S)$ and $\mathsf{SH}(S)$ are stable.*

Proof. For $\mathsf{SH}_{\mathcal{S}^1}(S)$ it is clear since this ∞-category is defined as $\mathsf{Sp}(\mathsf{Spc}(S)_*)$.

By [Lur17, Cor. 1.4.2.27], it suffices to show that the ∞-categorical suspension functor Σ is a self-equivalence of $\mathsf{SH}(S)$. We claim that $\Sigma E \simeq E \wedge (\Sigma^\infty_T \mathcal{S}^1)$ for all $E \in \mathsf{SH}(S)$; the claim follows because this functor has quasi-inverse $E \mapsto E \wedge (\Sigma^\infty_T \mathbb{G}_m) \wedge (\Sigma^\infty_T T)^{-1}$ (recall that $\Sigma^\infty_T T$ is invertible in $\mathsf{SH}(k)$ by definition; and that Σ^∞_T is symmetric monoidal). To prove the claim, we compute:

$$E \wedge \Sigma^\infty_T(\mathcal{S}^1) = \mathsf{colim}(E \wedge \Sigma^\infty_T(*) \longleftarrow E \wedge \Sigma^\infty_T(\mathcal{S}^0) \longrightarrow E \wedge \Sigma^\infty_T(*))$$

because both $E \wedge -$ and Σ^∞_T preserve colimits. The latter fact also implies $\Sigma^\infty_T(*) = 0$, and since Σ^∞_T is symmetric monoidal, we have $E \wedge \Sigma^\infty_T(\mathcal{S}^0) \simeq E \wedge \mathbb{1} \simeq E$. Therefore this colimit is exactly ΣE. □

The stable homotopy groups in stable homotopy theory have a bigraded analog in stable motivic homotopy theory:

Definition 2.17. Let $E \in \mathsf{SH}(S)$. The *bigraded homotopy sheaves* of E, denoted by $\underline{\pi}_i(E)_j$ with $i, j \in \mathbb{Z}$ (or also $\underline{\pi}_{k,\ell}(E) := \underline{\pi}_{k+\ell}(E)_{-\ell}$), are the Nisnevich sheaves associated with the presheaves

$$X \in \mathsf{Sm}_S \longmapsto [\Sigma^\infty(X_+) \wedge \mathcal{S}^i, \mathbb{G}_m^{\wedge j} \wedge E]$$

(where brackets denote the maps in the homotopy category, i.e. $\pi_0(\mathsf{map}(-,-))$).

We also denote $\mathcal{S}^{i,j} = \mathbb{G}_m^{\wedge j} \wedge (\mathcal{S}^1)^{\wedge i-j}$ for the *motivic (i,j)-sphere* and $\Sigma^{i,j}(-) = \Sigma^\infty \mathcal{S}^{i,j} \wedge -$ for the corresponding suspension functor.

These homotopy sheaves allow us to define a t-structure on $\mathsf{SH}(k)$.

Definition 2.18 ([Mor99, Thm. 5.2.3])**.** The *homotopy t-structure* on $\mathsf{SH}(k)$ is defined by:

$$\mathsf{SH}(k)_{\geq 0} = \{E \in \mathsf{SH}(k) \mid \underline{\pi}_i(E)_* = 0 \ \forall i < 0\}$$
$$\mathsf{SH}(k)_{\leq 0} = \{E \in \mathsf{SH}(k) \mid \underline{\pi}_i(E)_* = 0 \ \forall i > 0\}.$$

The heart of this t-structure admits an equivalent description:

Theorem 2.19 ([Mor99, Thm 5.2.6])**.** *The heart of the homotopy t-structure $\mathsf{SH}(k)^\heartsuit$ is equivalent to the 1-category of* homotopy modules. *The latter is defined as the 1-category of sequences $(M_n)_{n\in\mathbb{Z}}$ of Nisnevich sheaves of abelian groups on Sm_k such that:*

- *For all $n \in \mathbb{Z}$, M_n is strictly $\mathbb{A}^1$-invariant, i.e. for all $X \in \mathsf{Sm}_k$, the projection induces an isomorphism $H^p_{\mathsf{Nis}}(X, M_n) \to H^p_{\mathsf{Nis}}(X \times \mathbb{A}^1, M_n)$ on sheaf cohomology for all $p \geq 0$.*
- *For all $n \in \mathbb{Z}$, there is a specified isomorphism $M_n \cong (M_{n+1})_{-1}$, where for a sheaf $\mathcal{F}$, $\mathcal{F}_{-1}$ is its* contraction, *defined as the sheaf $X \in \mathsf{Sm}_k \mapsto \ker(\mathsf{ev}_1 : \mathcal{F}(X \times \mathbb{G}_m) \to \mathcal{F}(X))$, where ev_1 is induced by $X \simeq X \times \{1\} \hookrightarrow X \times \mathbb{G}_m$.*
- *Morphisms between two sequences $(M_n)_{n\in\mathbb{Z}}$ and $(M'_n)_{n\in\mathbb{Z}}$ are sequences of morphisms $M_n \to M'_n$ respecting the specified isomorphisms $M_n \cong (M_{n+1})_{-1}$ and $M'_n \cong (M'_{n+1})_{-1}$.*

For every $E \in \mathsf{SH}(k)$ and $i \in \mathbb{Z}$, the collection of homotopy sheaves $(\underline{\pi}_i(X)_j)_{j\in\mathbb{Z}}$ defines a homotopy module, and the equivalence between $\mathsf{SH}(k)^{\heartsuit}$ and the abelian 1-category of homotopy modules maps a spectrum E to the sequence $\underline{\pi}_0(X)_* = (\underline{\pi}_0(E)_n)_{n\in\mathbb{Z}}$.

Remark 2.20. The symmetric monoidal structure on $\mathsf{SH}(k)$ is compatible with the homotopy t-structure in the sense of [AN21, Def. A.10]. In particular, $\mathsf{SH}(k)_{\geq 0}$ is a symmetric monoidal subcategory of $\mathsf{SH}(k)$, and there is a unique symmetric monoidal structure on the heart $\mathsf{SH}(k)^{\heartsuit}$ such that the truncation $\mathsf{SH}(k)_{\geq 0} \to \mathsf{SH}(k)^{\heartsuit}$ is symmetric monoidal; see [AN21, Lemma A.12].

2.3 Slice filtrations and (very) effective covers

In stable homotopy theory, connective spectra play a crucial role. But since homotopy sheaves in $\mathsf{SH}(k)$ are bigraded, it is not obvious what the right notion of connectivity for motivic spectra should be. In this subsection we explore two variants and their properties. Firstly, Voevodsky defined in [Voe02] the subcategory of effective spectra, which is the zeroth level of a filtration on $\mathsf{SH}(k)$ called the *effective slice filtration.* As we will see from its definition, effectiveness provides a certain notion of connectivity with respect to $\mathbb{G}_m$, but not with respect to $\mathcal{S}^1$. This is apparent for instance in the fact that the real realization functor advertised in the introduction (and constructed in Section 4.1) can send effective spectra to spectra with non-trivial homotopy groups in arbitrarily small degrees. Indeed, if E is effective, then also $\Sigma^{-n,0}E$ is effective for all $n \geq 0$, and as we will see real realization maps $\Sigma^{-n,0}E$ to $\Sigma^{-n}(r_{\mathbb{R}}E)$. There are some other difficulties with the effective slice filtration. For example, it does not converge a priori. This means in particular, that a map between two motivic spectra which induces equivalences on all effective slices (i.e. the successive subquotients with respect to the slice filtration) is not necessarily an equivalence. For a brief presentation of the concept of effectiveness and the problems related to the effective slice filtration, we refer to the introduction of the article [Bac17].

These difficulties led Spitzweck and Østvær to introduce in [SØ12] the *very effective* slice filtration. The latter incorporates a notion of connectivity with respect to $\mathcal{S}^1$. It solves the problems mentioned above; in particular it converges and real realization maps very effective spectra to connective ones (see Lemma 4.8).

Let us now give the precise definitions (as stated in [BH21, Section 13]):

Definition 2.21. The ∞*-category of effective motivic spectra* $\mathsf{SH}(k)^{\mathsf{eff}}$ is the subcategory of $\mathsf{SH}(k)$ generated under colimits by all $\Sigma^\infty(X_+) \wedge \mathcal{S}^n$ for $X \in \mathsf{Sm}_k$ and $n \in \mathbb{Z}$.

The ∞*-category of very effective motivic spectra* $\mathsf{SH}(k)^{\mathsf{veff}}$ is the subcategory of $\mathsf{SH}(k)$ generated under colimits by all $\Sigma^\infty(X_+) \wedge \mathcal{S}^n$ for $X \in \mathsf{Sm}_k$ and $n \geq 0$.

The *effective homotopy t-structure* on $\mathsf{SH}(k)^{\mathsf{eff}}$ is defined by the data:

$$\mathsf{SH}(k)^{\mathsf{eff}}_{\geq 0} = \{E \in \mathsf{SH}(k)^{\mathsf{eff}} \mid \underline{\pi}_i(E)_0 = 0 \ \forall i < 0\}$$
$$\mathsf{SH}(k)^{\mathsf{eff}}_{\leq 0} = \{E \in \mathsf{SH}(k)^{\mathsf{eff}} \mid \underline{\pi}_i(E)_0 = 0 \ \forall i > 0\}.$$

For $n \in \mathbb{Z}$, let $\tau_{\geq_e n}$, respectively $\tau_{\leq_e n}$ be the truncation functors with respect to this t-structure.

Note that considering $n = 0$ in the definition of very effective spectra is enough because smash product with $\mathcal{S}^1$ corresponds to ∞-categorical suspension (as we saw in the proof of Lemma 2.16) which is a colimit.

Proposition 2.22.

(i) The ∞-category of effective spectra is stable, and the above data defines indeed a non-degenerate t-structure.

(ii) We have $\mathsf{SH}(k)^{\mathsf{eff}}_{\geq 0} = \mathsf{SH}(k)^{\mathsf{veff}}$.

(iii) The symmetric monoidal structure on $\mathsf{SH}(k)$ restricts to symmetric monoidal structures on both $\mathsf{SH}(k)^{\mathsf{eff}}$ and $\mathsf{SH}(k)^{\mathsf{veff}}$, such that the inclusion functors are symmetric monoidal.

Proof. To show that the ∞-category of effective spectra is stable, it suffices by [Lur17, Lemma 1.1.3.3] to show that it is stable under cofibers and translations in the stable ∞-category $\mathsf{SH}(k)$. By construction, $\mathsf{SH}(k)^{\mathsf{eff}}$ is stable under colimits, in particular it is stable under cofibers and suspensions (since it contains $*$). It is also stable under desuspension, since the latter preserves the collection of generators in the definition of $\mathsf{SH}(k)^{\mathsf{eff}}$, and desuspension is given by the functor $-\wedge \mathcal{S}^{\wedge -1}$, which commutes with colimits because $\mathsf{SH}(k)$ is presentably symmetric monoidal. The remainder of items (i) and (ii) appears in [Bac17, Prop. 4].

To prove item (iii), note that since these two subcategories are full, by [Lur17, Prop. 2.2.1.1 and Rmk 2.2.1.2] we only have to show that they are stable under the smash product. For all $m, n \in \mathbb{Z}$ and $X, Y \in \mathsf{Sm}_k$, we have

$$(\Sigma^\infty(X_+) \wedge \mathcal{S}^n) \wedge (\Sigma^\infty(Y_+) \wedge \mathcal{S}^m) \simeq \Sigma^\infty(X_+ \wedge Y_+) \wedge \mathcal{S}^{m+n} \simeq \Sigma^\infty((X \times Y)_+) \wedge \mathcal{S}^{m+n},$$

which is indeed effective, respectively very effective when $n, m \in \mathbb{N}$, by definition. Since smash product preserves colimits in both variables, and objects of the form we just considered generate $\mathsf{SH}(k)^{\mathsf{eff}}$, respectively $\mathsf{SH}(k)^{\mathsf{veff}}$ under colimits, this suffices to conclude. □

We define decreasing filtrations of $\mathsf{SH}(k)^{\mathsf{eff}}$ and $\mathsf{SH}(k)^{\mathsf{veff}}$ by shifting:

Definition 2.23. For $n \geq 0$, let $\mathsf{SH}(k)^{\mathsf{eff}}(n) := T^{\wedge n} \wedge \mathsf{SH}(k)^{\mathsf{eff}}$ be the ∞-category of *n-effective spectra* and $\mathsf{SH}(k)^{\mathsf{veff}}(n) := T^{\wedge n} \wedge \mathsf{SH}(k)^{\mathsf{veff}}$ be the ∞-category of *very n-effective spectra.*

Lemma 2.24. *The inclusion functors $\iota_n : \mathsf{SH}(k)^{\mathsf{eff}}(n) \hookrightarrow \mathsf{SH}(k)$ and $\tilde{\iota}_n : \mathsf{SH}(k)^{\mathsf{veff}}(n) \hookrightarrow \mathsf{SH}(k)$ admit right adjoints r_n and $\tilde{r}_n$ respectively, for all $n \geq 0$.*

Proof. This follows from the adjoint functor theorem [Lur09, Cor. 5.5.2.9]) once we show that ι_n and $\tilde{\iota}_n$ preserve colimits, and all ∞-categories involved are presentable. The first fact holds because these two ∞-categories are full subcategories of $\mathsf{SH}(k)$, closed under colimits in $\mathsf{SH}(k)$, so that they admit all colimits, and the latter are preserved by the respective inclusion functors.

For the second fact, note that $\mathsf{SH}(k)$ is presentable by construction (Definition 2.15 and Theorem 2.13). We do the case of effective spectra, and that of very effective ones is the same. Since the smash product preserves colimits in each variable separately, $\mathsf{SH}(k)^{\mathsf{eff}}(n)$ is generated under colimits by all $\Sigma^{2n+m,n}(X_+) = \mathcal{S}^{2n+m,n} \wedge \Sigma^\infty(X_+)$, for $m \in \mathbb{Z}$ and $X \in \mathsf{Sm}_k$. This is indeed an (essentially) small set of objects, and they are compact: for $\Sigma^\infty(X_+)$, it appears as [Hoy17, Prop. 6.4.(3)], and then for every filtered colimit $\mathsf{colim}_{i\in I}\, F(i)$ in $\mathsf{SH}(k)^{\mathsf{eff}}$, we have (using the effectiveness of $\mathcal{S}^{2n+m,n}$)

$$\begin{aligned}
\mathsf{SH}(k)^{\mathsf{eff}}(\mathcal{S}^{2n+m,n} \wedge \Sigma^\infty(X_+), \mathsf{colim}_{i\in I}\, F(i)) &\simeq \mathsf{SH}(k)^{\mathsf{eff}}(\Sigma^\infty(X_+), \mathcal{S}^{-2n-m,-n} \wedge \mathsf{colim}_{i\in I}\, F(i)) \\
&\simeq \mathsf{SH}(k)^{\mathsf{eff}}(\Sigma^\infty(X_+), \mathsf{colim}_{i\in I}\, (\mathcal{S}^{-2n-m,-n} \wedge F(i))) \\
&\simeq \mathsf{colim}_{i\in I}\, \mathsf{SH}(k)^{\mathsf{eff}}(\Sigma^\infty(X_+), \mathcal{S}^{-2n-m,-n} \wedge F(i)) \\
&\qquad\qquad \text{(by compactness of } \Sigma^\infty(X_+)) \\
&\simeq \mathsf{colim}_{i\in I}\, \mathsf{SH}(k)^{\mathsf{eff}}(\mathcal{S}^{2n+m,n} \wedge \Sigma^\infty(X_+), F(i))
\end{aligned}$$

whence the compactness of $\mathcal{S}^{2n+m,n} \wedge \Sigma^\infty(X_+)$. □

Definition 2.25. We call the compositions $\iota_n \circ r_n =: f_n$ and $\tilde{\iota}_n \circ \tilde{r}_n =: \tilde{f}_n$ the *n-effective cover* and *very n-effective cover* functors respectively. For all $E \in \mathsf{SH}(k)$, we obtain by adjunction natural maps $f_{n+1}E \to f_nE$ and $\tilde{f}_{n+1}E \to \tilde{f}_nE$ for all $n \geq 0$. Their cofibers $s_nE := \mathsf{cof}(f_{n+1}E \to f_nE)$ and $\tilde{s}_nE := \tilde{f}_{n+1} \to \tilde{f}_nE$ are called the *n-effective slice*, respectively *very n-effective slice* of E. By *(very) effective cover*, we mean the 0-th (very) effective cover.

Proposition 2.26. *The functor $\iota_0 : \mathsf{SH}(k)^{\mathsf{eff}} \to \mathsf{SH}(k)$ is right t-exact. Its right adjoint, the functor $r_0 : \mathsf{SH}(k) \to \mathsf{SH}(k)^{\mathsf{eff}}$ is t-exact and lax symmetric monoidal. It restricts to a lax symmetric monoidal functor $r_0^\heartsuit : \mathsf{SH}(k)^\heartsuit \to \mathsf{SH}(k)^{\mathsf{eff},\heartsuit}$.*

Proof. The claims about t-exactness are proven in [Bac17, Prop. 4.(3)]. As a right adjoint to the inclusion, which is symmetric monoidal (Proposition 2.22(iii)), r_0 is also lax symmetric monoidal by the doctrinal adjunction principle (follows from of [Lur17, Cor. 7.3.2.7]).

As a t-exact functor, r_0 carries the non-negative part to the non-negative part of the t-structures, and the heart to the heart. Its (co)restriction to the symmetric monoidal subcategories $\mathsf{SH}(k)_{\geq 0}$ and $(\mathsf{SH}(k)^{\mathsf{eff}})_{\geq_e 0}$ is then also lax symmetric monoidal. And its (co)restriction to the hearts $r_0^\heartsuit$ is lax symmetric monoidal because it can be written as the composition

$$\mathsf{SH}(k)^\heartsuit \hookrightarrow \mathsf{SH}(k)_{\geq 0} \xrightarrow{r_0} (\mathsf{SH}(k)^{\mathsf{eff}})_{\geq_e 0} \xrightarrow{\pi_0} \mathsf{SH}(k)^{\mathsf{eff},\heartsuit}.$$

Here, π_0 is symmetric monoidal by definition of the tensor product on the heart (using [AN21, Appendix A]). The first inclusion is lax symmetric monoidal as a right adjoint to the truncation which is symmetric monoidal by Remark 2.20. □

The (very) n-effective covers and slices behave well with respect to $\mathbb{P}^1$-suspension:

Proposition 2.27 ([Bac17, Lemma 8])**.** *For all $E \in \mathsf{SH}(k)$, we have, for all $n \geq 0$, that*

$$\begin{aligned} f_n E \wedge T &\simeq f_{n+1}(E \wedge T), \\ s_n E \wedge T &\simeq s_{n+1}(E \wedge T), \\ \tilde{f}_n E \wedge T &\simeq \tilde{f}_{n+1}(E \wedge T), \\ \tilde{s}_n E \wedge T &\simeq \tilde{s}_{n+1}(E \wedge T). \end{aligned}$$

Remark 2.28. More precisely, the first equivalence above is the dotted map in the diagram below, obtained by adjunction because $f_n E \wedge T \in \mathsf{SH}(k)^{\mathsf{eff}}(n) \wedge T = \mathsf{SH}(k)^{\mathsf{eff}}(n+1)$ (and similarly for very effective covers):

$$\begin{tikzcd} & f_{n+1}(E \wedge T) \arrow[d] \\ f_n E \wedge T \arrow[ur, dashed] \arrow[r] & E \wedge T \end{tikzcd}$$

where $f_{n+1}(E \wedge T) \to E \wedge T$ and $f_n E \to E$ are the natural maps.

The three other equivalences are obtained in similar ways.

Example 2.29. If a $\mathbb{P}^1$-spectrum $E \in \mathsf{SH}(k)$ satisfies an "m-periodicity" condition $E \wedge T^{\wedge m} \simeq E$, then the (very) effective covers and slices are also periodic, up to a shift by T. Indeed, we then have

$$f_{n+m}(E) \simeq f_{n+m}(E \wedge T^{\wedge m}) \simeq f_n E \wedge T^{\wedge m}$$

and similarly for the slices. This appears for example in the very effective slices of KO (see Theorem 5.2), or in those of KGL (KO and KGL are periodic motivic spectra defined in Subsection 3.2): $s_n(\mathsf{KGL}) \simeq s_0(\mathsf{KGL}) \wedge T^{\wedge n}$ for all $n \geq 0$ (and the zeroth effective and very effective slices are actually both $\mathsf{H}\mathbb{Z}$, see for example [ARØ20, Prop. 2.7]).

One relation between effective and very effective slices is given by the following decompositions of the very effective slices:

Proposition 2.30 (Decomposition of very effective slices, [Bac17, Lemma 11])**.** *For every $E \in \mathsf{SH}(k)$ and $n \geq 0$, there are natural cofiber sequences*

$$\Sigma^n_T s_0((\Sigma_T^{-n} X)_{\geq 1}) \longrightarrow \tilde{s}_n E \longrightarrow \Sigma^n_T f_0(\underline{\pi}_0(\Sigma_T^{-n} X)_*)$$

$$\Sigma^n_T f_1(\underline{\pi}_0(\Sigma_T^{-n} X)_*) \longrightarrow \tilde{s}_n E \longrightarrow \Sigma^n_T s_0((\Sigma_T^{-n} X)_{\geq 0})$$

where the homotopy module $\underline{\pi}_i(E)_*$ is viewed as living in $\mathsf{SH}(k)^{\heartsuit} \subseteq \mathsf{SH}(k)$.

We will only use this Proposition in the case $n = 0$, however it will be useful for us to provide another interpretation of this decomposition. Actually, for each of the cofiber sequences in the statement, its two ends correspond to the slices of E (more precisely, of $\tilde{f}_0 E$) with respect to a finer filtration on $\mathsf{SH}(k)^{\mathsf{veff}}$. We will only use the first cofiber sequence, and thus we only introduce the filtration corresponding to it; the case of the second decomposition is completely analogous.

Definition 2.31. Let $\mathcal{C}_{2n} = \Sigma^{2n,n}\mathsf{SH}(k)^{\mathsf{veff}}$ and $\mathcal{C}_{2n+1} = \Sigma^{2n+1,n}\mathsf{SH}(k)^{\mathsf{veff}}$ for all $n \in \mathbb{N}$ (the full subcategories generated by these objects).

Then for all $n \in N$, we define r'_n as the right adjoint of the inclusion $\iota'_n : \mathcal{C}_n \hookrightarrow \mathsf{SH}(k)$, and $f'_n = \iota'_n \circ r'_n$. In particular, note that $f'_{2n} = \tilde{f}_n$ for all $n \in \mathbb{N}$.

For all $X \in \mathsf{SH}(k)$, let $s'_n(X)$ be the cofiber of the natural map $f'_{n+1}(X) \to f'_n(X)$.

Remark 2.32. The existence of the right adjoint follows from the adjoint functor theorem [Lur09, Cor. 5.5.2.9]), as in the definition of the effective and very effective covers (Definition 2.25).

Proposition 2.33. *For any $E \in \mathsf{SH}(k)$, the cofiber sequence $s'_1 E \to \tilde{s}_0 E \to s'_0 E$ (see the proof for its construction) is equivalent to the following cofiber sequence*

$$s_0(E_{\geq 1}) \longrightarrow \tilde{s}_0(E) \longrightarrow f_0(\underline{\pi}_0(E)_*).$$

In particular, we have $s'_{2n}(E) = \Sigma^n_T f_0(\underline{\pi}_0(\Sigma_T^{-n} X))$ and $s'_{2n+1}(E) = \Sigma^n_T s_0((\Sigma_T^{-n} X)_{\geq 1})$ for all $n \in \mathbb{N}$.

Proof. The last assertion follows from the fact that $f'_n(E) \wedge T \simeq f'_{n+2}(E \wedge T)$ for all $n \geq 0$ and $E \in \mathsf{SH}(k)$, similarly to in Proposition 2.27.

To prove the main claim, first note that the cofiber sequence $s'_1 E \to \tilde{s}_0 E \to s'_0 E$ is obtained from the octahedral axiom: in the diagram below, the vertical triangles are cofiber sequences, and we obtain the desired cofiber sequence as the dotted last row

$$\begin{tikzcd}
f'_2 E \arrow[r, equal] \arrow[d] & f'_2 E = \tilde{f}_1 E \arrow[r] \arrow[d] & f'_1 E \arrow[d] \\
f'_1 E \arrow[r] \arrow[d] & f'_0 E = \tilde{f}_0 E \arrow[r, equal] \arrow[d] & f'_0 E \arrow[d] \\
s'_1 E \arrow[r, dashed] & \tilde{s}_0 E \arrow[r, dashed] & s'_0 E.
\end{tikzcd}$$

By [Bac17, Lemma 10], we have $f'_0 \simeq \tilde{f}_0 \simeq f_0 \circ \tau_{\geq 0}$ and $f'_2 \simeq \tilde{f}_1 \simeq f_1 \circ \tau_{\geq 1}$. We will prove that $f'_1 = f_0 \circ \tau_{\geq 1}$. Then, the above diagram becomes

$$\begin{tikzcd}
f_1(E_{\geq 1}) \arrow[r, equal] \arrow[d] & f_1(E_{\geq 1}) \arrow[r] \arrow[d] & f_0(E_{\geq 1}) \arrow[d] \\
f_0(E_{\geq 1}) \arrow[r] \arrow[d] & f_0(E_{\geq 0}) \arrow[r, equal] \arrow[d] & f_0(E_{\geq 0}) \arrow[d] \\
s_0(E_{\geq 0}) \arrow[r, dashed] & \tilde{s}_0 E \arrow[r, dashed] & f_0(\underline{\pi}_0(E)_*)
\end{tikzcd}$$

by definition of s_0 and since f_0 preserves cofiber sequences.

To show our claim, we have to prove that $r_0((-)_{\geq 1}) = r_0 \circ \tau_{\geq 1}$ is the right adjoint to the inclusion $\mathcal{C}_1 = \Sigma^{1,0}\mathsf{SH}(k)^{\mathsf{veff}} \hookrightarrow \mathsf{SH}(k)$ (where r_0 is the functor from Lemma 2.24). We can factor this inclusion as

$$\begin{tikzcd}
\Sigma^{1,0}\mathsf{SH}(k)^{\mathsf{veff}} \arrow[r, hook, shift right, "\Sigma^{1,0}j"'] \arrow[r, phantom, "\top"] & \Sigma^{1,0}\mathsf{SH}(k)^{\mathsf{eff}} = \mathsf{SH}(k)^{\mathsf{eff}} \arrow[l, shift right, "\Sigma^{1,0}r"'] \arrow[r, hook, shift right, "\iota_0"'] \arrow[r, phantom, "\top"] & \mathsf{SH}(k) \arrow[l, shift right, "r_0"']
\end{tikzcd}$$

where j is the inclusion of very effective spectra into effective ones, i.e. the inclusion of the non-negative part of the effective homotopy t-structure (Proposition 2.22(ii)). Its right adjoint is therefore the truncation functor $r := \tau_{\geq_e 0}$. They remain an adjoint pair after suspending. The functors r_0 and ι_0 form an adjoint pair by definition (Lemma 2.24). So we only have to prove that $\Sigma^{1,0} r \circ r_0 = r_0 \circ \tau_{\geq 1}$. We compute:

$$\begin{aligned}
(\Sigma^{1,0} r) \circ r_0 &\simeq \Sigma^{1,0} \circ r \circ \Sigma^{-1,0} \circ r_0 \\
&\simeq \Sigma^1 \circ \tau_{\geq_e 0} \circ \Sigma^{-1} \circ r_0 \\
&\simeq \Sigma^1 \circ \Sigma^{-1} \circ \tau_{\geq_e 1} \circ r_0 && \text{(by the axioms of a t-structure)} \\
&\simeq \tau_{\geq_e 1} \circ r_0 \\
&\simeq r_0 \circ \tau_{\geq 1}
\end{aligned}$$

where the last line follows by t-exactness of r_0 (Proposition 2.26). Indeed, it is true in general for t-structures that a t-exact functor $f : \mathcal{C} \to \mathcal{D}$ commutes with truncations functors $\tau_{\geq m}$ for all $m \in \mathbb{Z}$. Indeed, for $m = 0$ it is the definition of being t-exact, and then for m arbitrary, we have:

$$\begin{aligned}
\tau_{\geq k} \circ f &= \Sigma^k \circ \tau_{\geq 0} \circ \Sigma^{-k} \circ f && \text{(by the axioms of a t-structure)} \\
&= \Sigma^k \circ \tau_{\geq 0} \circ f \circ \Sigma^{-k} && \text{(by exactness of } f\text{)} \\
&= \Sigma^k \circ f \circ \tau_{\geq 0} \circ \Sigma^{-k} && \text{(by the case } m = 0\text{)} \\
&= f \circ \Sigma^k \circ \tau_{\geq 0} \circ \Sigma^{-k} && \text{(by exactness of } f\text{)} \\
&= f \circ \tau_{\geq k}. && \text{(by the axioms of a t-structure)}
\end{aligned}$$

This finishes the proof. □

Example 2.34. Using the computations in the proof of [Bac17, Thm 16], one can show that the slices of KO with respect to the filtration from Definition 2.31 are given by $s'_n\mathsf{KO} = \Sigma^{n,\lfloor n/2 \rfloor} \wedge s'^{0}_{n \bmod 8}$ where

$$s'^{0}_0 = \widetilde{\mathsf{H}\mathbb{Z}}, \quad s'^{0}_1 = \mathsf{H}\mathbb{Z}/2, \quad s'^{0}_2 = \mathsf{H}\mathbb{Z}/2, \quad s'^{0}_3 = 0, \quad s'^{0}_4 = \mathsf{H}\mathbb{Z}, \quad s'^{0}_5 = 0, \quad s'^{0}_6 = 0, \quad s'^{0}_7 = 0,$$

see Subsections 4.3.1 and 4.3.3 for the definitions of these motivic spectra.

3 Different types of K-theory and L-theory

Our next section of prerequisites concerns different variants of K-theory. The final goal is to be able to define the motivic spectrum ko (and its $\mathcal{E}_\infty$-structure). The latter is the very effective cover of the motivic spectrum KO, which represents *hermitian K-theory* in the stable motivic homotopy ∞-category. The spectrum KO has an $\mathcal{E}_\infty$-structure, and the $\mathcal{E}_\infty$-structure we are considering on its effective cover ko is inherited from it. The construction of the ∞-categories of motivic spaces and motivic spectra in the previous section, together with the discussion of $\mathcal{E}_n$-rings in Section 1, will now allow us to define such objects and their multiplicative structure precisely.

This section contains no new result; we only recall the definitions of some classical objects we will be working with in the remaining sections. However, it is useful to first recall the definition of K-theory in the topological context. K-theory, and its variants, are certain cohomology theories for spaces, and are representable in the stable ∞-category of topological spectra by spectra admitting certain multiplicative structures. We first introduce complex and real K-theory (of spaces), and give some properties of the topological spectra representing them. This example constitutes a model for all the next ones: we will see that the constructions of all the other variants are following somewhat similar steps. We then introduce their algebraic analogs, namely algebraic and hermitian K-theory, and again study the question of representability. In particular, we will see that they are represented in the ∞-category of motivic spectra by $\mathcal{E}_\infty$-rings denoted by KGL and KO respectively. Finally, we recall the definition of L-theory; this will be particularly relevant to us in view of Theorem A advertised in the introduction and proven in Section 5: we will identify the real realization of ko with the connective part of the L-theory spectrum of the real numbers. L-theory will also feature in Section 4, when we compute the real realization of KO.

3.1 Complex and real topological K-theory

In this subsection, we recall the topological side of the story of K-theory. In a nutshell, complex topological K-theory is a generalized cohomology theory (for spaces) which classifies complex vector bundles. The collection of such objects on a fixed space is endowed with the direct sum operation, which is associative and unital. We therefore get monoids, but to obtain a cohomology theory, we need groups. There exists an operation called *group completion* which allows us to build groups out of these monoids. The different variants of K-theory we will consider follow the same idea, except that they deal with other types of vector bundles, or schemes instead of spaces.

Definition 3.1. An object $X \in \mathsf{CMon}(\mathsf{Spc}) = \mathsf{CAlg}(\mathsf{Spc}^\times)$ is called *group-like* if the monoid structure induced on $\pi_0(X)$ is actually a group structure. We denote the ∞-category of such objects by $\mathsf{CMon}(\mathsf{Spc})^{\mathsf{gp}}$. Consider the inclusion $\mathsf{CMon}(\mathsf{Spc})^{\mathsf{gp}} \hookrightarrow \mathsf{CMon}(\mathsf{Spc})$; this admits a left adjoint $(-)^{\mathsf{gp}}$, called the *group completion* functor. An explicit model for the group completion functor is given by $X \mapsto \Omega BX$, where B is the classifying space functor (which can be explicitly described by bar constructions) by [May74, Thm 1.6]).

Definition 3.2. The *complex, respectively real, topological K-theory space of* $X \in \mathsf{Spc}$ is the space $(\mathsf{Vect}_{\mathbb{C}}(X)^{\simeq})^{\mathsf{gp}}$, respectively $(\mathsf{Vect}_{\mathbb{R}}(X)^{\simeq})^{\mathsf{gp}}$, where $\mathsf{Vect}_{\mathbb{C}}(X)$ is the 1-category of complex vector bundles on X, viewed as an essentially small ∞-category (respectively, real vector bundles for $\mathbb{C}$ replaced with $\mathbb{R}$), and $\mathsf{Vect}_{\mathbb{C}}(X)^{\simeq}$ is the maximal ∞-groupoid it contains, viewed as a space. Here, the functor $(-)^{\simeq}$ is right adjoint to the forgetful functor from essentially small ∞-groupoids to Cat_∞. Then $\mathsf{Vect}_{\mathbb{C}}(X)^{\simeq}$ is viewed as an object in $\mathsf{CMon}(\mathsf{Spc})$ via the direct sum of vector bundles.

The *complex topological K-groups* of X are $\mathsf{KU}_i(X) := \pi_i(B(\mathsf{Vect}_{\mathbb{C}}(X)^{\simeq})^{\mathsf{gp}})$ (respectively $\mathsf{KO}_i^{\mathsf{top}}(X)$ and $\mathbb{R}$ for real topological K-groups).

We chose the notation $\mathsf{KO}^{\mathsf{top}}$ to avoid confusion with the motivic spectrum KO which will feature a lot in the next sections.

The following results are classical:

Theorem 3.3 ([BR05] for *(ii)*, [Ati66] for *(iii)* and *(iv)*, also see [Str92] for *(iv)*)**.**

(i) There exist spectra KU, $\mathsf{KO}^{\mathsf{top}} \in \mathsf{Sp}$ *representing complex, respectively real, topological K-theory.*

(ii) The spectra KU *and* $\mathsf{KO}^{\mathsf{top}}$ *admit* $\mathcal{E}_\infty$*-ring spectra structures, where multiplication is induced by the tensor product of vector bundles.*

(iii) The spectrum KU *is 2-periodic, i.e. there is a map* $\beta_{\mathsf{KU}} : \mathbb{S}^2 \to \Omega^\infty \mathsf{KU}$ *such that multiplication by* β_{KU}

$$\Sigma^\infty \mathbb{S}^2 \wedge \mathsf{KU} \xrightarrow{\beta_{\mathsf{KU}} \wedge \mathrm{id}} \mathsf{KU} \wedge \mathsf{KU} \xrightarrow{\mu} \mathsf{KU}$$

is an equivalence. Here μ *is the multiplication map for the* $\mathcal{E}_\infty$*-structure on* KU*.*

Similarly, $\mathsf{KO}^{\mathsf{top}}$ *is 8-periodic, with a map* $\beta_{\mathsf{KO}^{\mathsf{top}}} : \mathbb{S}^8 \to \Omega^\infty \mathsf{KO}^{\mathsf{top}}$ *inducing an equivalence* $\Sigma^\infty \mathbb{S}^8 \wedge \mathsf{KO}^{\mathsf{top}} \to \mathsf{KO}^{\mathsf{top}}$*.*

(iv) The spectrum KU *may be written explicitly as the sequence* $(\mathbb{Z} \times BU, U, \mathbb{Z} \times BU, \dots)$*, where* U *is the infinite unitary group, and multiplication by* β_{KU} *induces a map* $\Sigma^2(\mathbb{Z} \times BU) \to \mathbb{Z} \times BU$*, whose adjoint* $\mathbb{Z} \times BU \to \Omega^2(\mathbb{Z} \times BU)$ *is an equivalence. The spectrum* $\mathsf{KO}^{\mathsf{top}}$ *may be written explicitly as* $(\mathbb{Z} \times BO, O, O/U, U/Sp, \mathbb{Z} \times BSp, Sp, Sp/U, U/O, \dots)$*, where* O *is the infinite orthogonal group, and multiplication by* $\beta_{\mathsf{KO}^{\mathsf{top}}}$ *induces a map* $\Sigma^8(\mathbb{Z} \times BO) \to \mathbb{Z} \times BO$ *whose adjoint* $\mathbb{Z} \times BO \to \Omega^8(\mathbb{Z} \times BO)$ *is an equivalence.*

Items (*iii*) and (*iv*) are often referred to as *Bott periodicity*, and the elements β, either viewed as maps or as classes in homotopy, are called *Bott elements*.

There are interesting relations between KU and $\mathsf{KO}^{\mathsf{top}}$. We only cite them informally to give some intuition about the constructions we will do in the motivic setting in Subsection 3.2.

Theorem 3.4 ([Bou90, §1.1] for (*i*) and [Ati66] for (*ii*))**.**

- *There is a cofiber sequence, called the* Wood cofiber sequence

$$\Sigma \mathsf{KO}^{\mathsf{top}} \xrightarrow{\eta} \mathsf{KO}^{\mathsf{top}} \xrightarrow{c} \mathsf{KU}$$

  where $\eta \in \pi_1(\mathbb{S})$ *is the Hopf map and* c *is induced by complexification of real vector bundles.*

- *Complex conjugation on complex vector bundles induces an action of* C_2 *on* KU *(compatible with the* $\mathcal{E}_\infty$*-ring structure) whose (homotopy) fixed points recover the real topological* K*-theory spectrum:* $(\mathsf{KU})^{hC_2} \simeq \mathsf{KO}^{\mathsf{top}}$*.*

Remark 3.5. Different variants of C_2-equivariant spectra KU appear in the literature; the one we considered is also called Atiyah's Real K-theory. The terminology might be misleading, because we saw that KU represents *complex* topological K-theory, however here the word "Real" does not refer to the real numbers directly, but rather to the notion of "real structures" (anti-linear homeomorphisms on *complex* vector bundles).

3.2 Algebraic and hermitian K-theory

Algebraic K-theory can be seen as an analog of complex topological K-theory for schemes, indeed the construction is the same, replacing spaces by affine schemes (in Sm_k, say) and complex vector bundles by algebraic vector bundles. We will see that, as in the topological case, algebraic K-theory is representable by a motivic $\mathcal{E}_\infty$-ring KGL in $\mathsf{SH}(k)$. Then another justification of the analogy between topological K-theory and algebraic K-theory will be given in Section 4, by the computation that the complex realization of KGL is nothing but the spectrum KU which represents complex K-theory of spaces (Lemma 4.23). In this perspective, hermitian K-theory, which is defined in the same way as algebraic K-theory, replacing algebraic vector bundles by algebraic bundles with a symmetric form, is an analog of real topological K-theory: indeed, the complex realization of the motivic spectrum KO representing it is the topological spectrum $\mathsf{KO}^{\mathsf{top}}$ representing real K-theory. In this subsection, we make all these definitions precise, and give analogs in the motivic setting to several topological results we have seen in the previous subsection about real and complex K-theory.

The definitions given in this subsection for Hermitian K-theory and the Grothendieck-Witt construction are valid under the assumption that the field k is *not* of characteristic 2. This assumption causes no trouble for us later, since to talk about real realization we will later assume that k has a fixed embedding into $\mathbb{R}$, and thus has characteristic 0.

Definition 3.6. The *algebraic* K*-theory space of an affine scheme* $X \in \mathsf{Sm}_k$ is $(\mathsf{Vect}(X)^{\simeq})^{\mathsf{gp}} \in \mathsf{Spc}_*$, where $\mathsf{Vect}(X)$ is the 1-category of algebraic vector bundles (in particular, of finite rank) over X, viewed as an essentially small ∞-category, and $\mathsf{Vect}(X)^{\simeq}$ is the associated ∞-groupoid, viewed as a space. The latter is viewed as an object in $\mathsf{CMon}(\mathsf{Spc})$ via the direct sum of vector bundles.

The K*-groups of* X are $K_i(X) := \pi_i(B(\mathsf{Vect}(X)^{\simeq})^{\mathsf{gp}})$ for all $i \geq 0$.

The motivic ∞-category was partly introduced with the goal in mind to obtain a homotopy theory of schemes where algebraic K-theory would be representable. This indeed is the case, both stably and unstably:

Theorem 3.7 ([Qui75], [TT90], [MV99])**.**

(i) *There exists a motivic space $K \in \mathsf{Spc}(k)_*$ such that for all affine schemes $X \in \mathsf{Sm}_k$, the sections $K(X)$ are the K-theory space from Definition 3.6. More explicitly,*

$$K \simeq \mathsf{L_{mot}}(\mathbb{Z} \times BGL) \simeq \mathsf{L_{mot}}(B(\mathsf{Vect}(-)^{\simeq})^{\mathsf{gp}}$$

(see [AE17, §6.2] for more details). The tensor product of vector bundles induces a multiplication map $K \wedge K \to K$.

(ii) *There is a map $\beta_{\mathsf{KGL}} : \mathbb{P}^1 \to K$ in $\mathsf{Spc}(k)$ inducing by multiplication a map $\mathbb{P}^1 \wedge K \to K$ whose adjoint $K \to \Omega_T K$ is an equivalence. In particular, $\mathsf{KGL} := (K, K, \dots)$ defines a motivic spectrum. The map β_{KGL} classifies the virtual bundle $\mathcal{O}(1) - \mathcal{O}$ on $\mathbb{P}^1$, where $\mathcal{O}$ and $\mathcal{O}(1)$ are respectively the trivial and tautological bundles.*

(iii) *The spectrum KGL represents algebraic K-theory, i.e. for all $p, q \in \mathbb{Z}$ with $p - 2q \geq 0$ and $X \in \mathsf{Sm}_k$ affine, we have*

$$[\Sigma^{p,q}\Sigma^{\infty}_{+}X, \mathsf{KGL}] \cong K_{p-2q}(X).$$

Remark 3.8. Several observations follow from this Theorem. In item (i), the fact that $K \in \mathsf{Spc}(k)_*$ implies in particular that algebraic K-theory is $\mathbb{A}^1$-invariant and has Nisnevich descent on affine schemes. From item (ii) above, it follows that KGL is $(2,1)$-periodic, in the sense that β_{KGL} induces an equivalence $\Sigma^{2,1}\mathsf{KGL} \simeq \mathsf{KGL}$. In particular, KGL should not be a connective motivic spectrum in any reasonable sense. We will therefore consider later its very effective cover kgl (which turns out to agree with its effective cover). Item (ii) is an analog to the Bott periodicity theorem in the classical case. We may therefore call β_{KGL} the *Bott element* for KGL. As mentioned in the introduction to this subsection, the spectrum KGL can be viewed as a motivic analog to complex K-theory KU.

We now refine the construction of KGL to show it acquires the structure of a motivic $\mathcal{E}_\infty$-ring spectrum, and then we will directly construct the motivic spectrum KO mentioned above as an $\mathcal{E}_\infty$-ring spectrum applying the same method. We follow [BH20, §3.2] very closely.

Definition 3.9. Let $\mathsf{CMon}(\mathsf{Gpd}) := \mathsf{CAlg}(\mathsf{Gpd}^{\times})$ be the ∞-category of commutative algebra objects of the Cartesian symmetric monoidal ∞-category of ∞-groupoids. Let $\mathsf{CMon}(\mathsf{Gpd})^{\mathsf{gp}}$ be the subcategory consisting of objects whose underlying commutative monoid in $h\mathsf{Gpd}$ is a commutative group object.

The *group completion functor* or *K-theory* functor is defined as the composition

$$(-)^{\mathsf{gp}} : \mathsf{CMon}(\mathsf{Gpd}) \longrightarrow \mathsf{CMon}(\mathsf{Gpd})^{\mathsf{gp}} \longrightarrow \mathsf{Spc}_*$$

where the first functor is left adjoint to the inclusion and the second functor is the forgetful one (under the equivalence $\mathsf{Gpd} \simeq \mathsf{Spc}$).

Proposition 3.10. *The ∞-category $\mathsf{CMon}(\mathsf{Gpd})$ admits a non-Cartesian symmetric monoidal structure $\mathsf{CMon}(\mathsf{Gpd})^{\otimes}$, making the free functors $\mathsf{Gpd} \to \mathsf{CMon}(\mathsf{Gpd})$, respectively $\mathsf{Gpd}_* \to \mathsf{CMon}(\mathsf{Gpd})$ symmetric monoidal.*

With respect to this structure and the smash product symmetric monoidal structure on Spc_*, the K-theory functor is well-defined and lax symmetric monoidal. In particular, it induces a functor

$$(-)^{\mathsf{gp}} : \mathsf{CAlg}(\mathsf{CMon}(\mathsf{Gpd})^{\otimes}) \longrightarrow \mathsf{CAlg}(\mathsf{Spc}_*^{\otimes}).$$

Proof. The first part of the statement follows from [GGN15, Thm 5.1].

In particular, the first functor in the composition $(-)^{\mathsf{gp}} : \mathsf{CMon}(\mathsf{Gpd}) \longrightarrow \mathsf{CMon}(\mathsf{Gpd})^{\mathsf{gp}} \longrightarrow \mathsf{Spc}_*$ defining $(-)^{\mathsf{gp}}$ is well-defined and symmetric monoidal. The second functor in this composition is lax symmetric monoidal because it is right adjoint to the free functor, which is symmetric monoidal again by [GGN15, Thm 5.1]. Then, there is an induced functor on the ∞-categories of commutative algebra objects on both sides by Remark 1.12. This proves the second part of the statement. □

Definition 3.11. The *algebraic K-theory* motivic space K is defined as the motivic localization of the presheaf

$$\mathsf{Sm}_k^{\mathsf{op}} \xrightarrow{\mathsf{Vect}(-)^{\simeq}} \mathsf{CAlg}(\mathsf{CMon}(\mathsf{Gpd})^{\otimes}) \xrightarrow{(-)^{\mathsf{gp}}} \mathsf{CAlg}(\mathsf{Spc}_*^{\otimes})$$

where for $X \in \mathsf{Sm}_k$, the ∞-groupoid $\mathsf{Vect}(X)^{\simeq}$ acquires the structure of a commutative monoid object with respect to both cartesian product and tensor product via the direct sum, respectively the tensor product of algebraic vector bundles. Then, taking the nerve allows us to view it as an object in $\mathsf{CAlg}(\mathsf{CMon}(\mathsf{Gpd})^{\otimes})$ (the nerve functor preserves finite products). Functoriality of $\mathsf{Vect}(-)$ is provided by pullback.

Remark 3.12. There is a canonical equivalence $\mathsf{Fun}(\mathsf{Sm}_k^{\mathsf{op}}, \mathsf{CAlg}(\mathsf{Spc}_*)) \simeq \mathsf{CAlg}(\mathcal{P}(\mathsf{Sm}_k)_*)$ by [Lur12, Ex. 1.1.3], and similarly for sheaves instead of presheaves. Note that the motivic localization functor $\mathsf{L}_{\mathsf{mot}} : \mathcal{P}(\mathsf{Sm}_k)_* \to \mathsf{Spc}(k)_*$ is symmetric monoidal. Indeed, in the non-pointed case, it preserves finite products and therefore is symmetric monoidal for the Cartesian structures. This suffices, by our construction of the tensor product on $\mathsf{Spc}(k)_*$. Thus we may view the presheaf in the above definition as an object in $\mathsf{CAlg}(\mathcal{P}(\mathsf{Sm}_k)_*)$, and K as an object in $\mathsf{CAlg}(\mathsf{Spc}(k)_*)$. In particular, K can be seen as an $\mathbb{A}^1$-invariant Nisnevich sheaf with values in $\mathsf{CAlg}(\mathsf{Spc}_*^{\otimes})$.

Definition 3.13. The *Hermitian K-theory motivic space GW*, or *Grothendieck-Witt K-theory space* is defined as the motivic localization of the presheaf

$$\mathsf{Sm}_k^{\mathsf{op}} \xrightarrow{\mathsf{Bil}(-)^{\simeq}} \mathsf{CAlg}(\mathsf{CMon}(\mathsf{Gpd})^{\otimes}) \xrightarrow{(-)^{\mathsf{gp}}} \mathsf{CAlg}(\mathsf{Spc}_*^{\otimes})$$

where for $X \in \mathsf{Sm}_k$, $\mathsf{Bil}(X)$ denotes the 1-category of symmetric vector bundles over X (vector bundles with a symmetric bilinear form). The ∞-groupoid $\mathsf{Bil}(X)^{\simeq}$ is made into an object in $\mathsf{CAlg}(\mathsf{CMon}(\mathsf{Gpd})^{\otimes})$ by the direct sum and tensor product of algebraic vector bundles and their symmetric structures, similarly to the case of $\mathsf{Vect}(X)^{\simeq}$.

By definition, the forgetful map $\mathsf{Bil}(-) \to \mathsf{Vect}(-)$, which preserves direct sums and tensor products of vector bundles, can be upgraded to a transformation of functors valued in $\mathsf{CAlg}(\mathsf{CMon}(\mathsf{Gpd})^{\otimes})$, and thus induces an $\mathcal{E}_\infty$-morphism $GW \to K$ in $\mathsf{Spc}(k)_*$.

We can now define the corresponding motivic spectra:

Definition 3.14. We define the *algebraic K-theory spectrum* KGL and *Hermitian K-theory spectrum* KO respectively as:

$$\mathsf{KGL} := (\Sigma^\infty K)[\beta_{\mathsf{KGL}}^{-1}]$$
$$\mathsf{KO} := (\Sigma^\infty GW)[\beta_{\mathsf{KO}}^{-1}].$$

This definition of KGL agrees with the one in Theorem 3.7.

Here the notation $X[f^{-1}]$ for X an object in a symmetric monoidal ∞-category $\mathcal{C}$ and f a morphism in $\mathcal{C}$ refers to *periodization* in the sense of [Hoy20, Section 3]. It is a localization with respect to all maps $\mathsf{id}_Y \otimes f$ for $Y \in \mathcal{C}$.

The map β_{KGL} is defined as in Theorem 3.7(*ii*). To define β_{KO}, let $\mathbb{HP}^1 = HGr(2, H_-^{\oplus 2})$ be the subscheme of the Grassmanian $Gr(2,4)$ of 2-dimensional subspaces in the trivial bundle of rank 4, consisting of subspaces such that the restriction of the standard alternating form $H_-^{\oplus 2}$ is non-degenerate on these subspaces. Here H_- is represented by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Let $H_- \to \mathbb{HP}^1$ denote the trivial bundle of rank 2 with the standard alternating form, and let $U_{2,4}$ be the tautological bundle (informally the fiber over each point, represented by a 2-dimensional subspace of $H_-^{\oplus 2}$, consists of the subspace itself). Then $\beta_{\mathsf{KO}} \in \pi_{8,4}(GW)(\mathbb{R}) \cong [T^{\wedge 4}, GW]$ is the class represented by the virtual vector bundle $(U_{2,4} - H_-)^{\otimes 2}$ on $(\mathbb{HP}^1)^{\wedge 2} \simeq T^{\wedge 4}$.

Remark 3.15. The motivic spectrum KO is also often denoted by KQ in the literature.

As in Subsection 1.4, we may consider $\mathcal{E}_n$-algebra objects in the symmetric monoidal ∞-category of motivic spectra. We will call them motivic $\mathcal{E}_n$-ring spectra, or just (motivic) $\mathcal{E}_n$-rings if the context is clear enough.

Since Σ^∞ is symmetric monoidal, we get $\mathcal{E}_\infty$-structures on $\Sigma^\infty K$ and $\Sigma^\infty GW$. The periodizations KO and KGL then inherit $\mathcal{E}_\infty$-structures by the results in [Hoy20, Section 3]. There is also an $\mathcal{E}_\infty$-map $c : \mathsf{KO} \to \mathsf{KGL}$, which we call the *forgetful map*, built as follows. Consider the composite $\Sigma^\infty GW \to \Sigma^\infty K \to (\Sigma^\infty K)[\beta_{\mathsf{KGL}}^{-1}] = \mathsf{KGL}$, where the first map is induced by the $\mathcal{E}_\infty$-map $GW \to K$ discussed above, and the second map is localization. Then, this composite factors through $\mathsf{KO} = (\Sigma^\infty GW)[\beta_{\mathsf{KO}}^{-1}]$ because the image of β_{KO} under the induced map $[T^{\wedge 4}, GW] \to [T^{\wedge 4}, K]$ is β^4_{KGL} by [BH20, §3.2.5] and [RØ16, Prop. 3.3], and thus is already invertible on KGL.

Let us now give some interpretation about the cohomology theory represented by KO. Classically, the (zeroth) Grothendieck–Witt group of an affine scheme $X \in \mathsf{Sm}_k$ is defined in the same way as K-theory, but considering vector bundles endowed with symmetric bilinear forms instead. Namely, we set $GW_0(X) := \pi_0(B(\mathsf{Bil}(X)^{\simeq})^{\mathsf{gp}})$, which is the set of connected components of the Grothendieck-Witt space $GW(X) = B(\mathsf{Bil}(X)^{\simeq})^{\mathsf{gp}}$. Writing KO as a sequence of pointed motivic spaces, this corresponds to $\mathsf{KO}_0(X)$ (still for X affine). The other levels of KO are *shifted Grothendieck–Witt spaces.* To define them, we briefly mention another way to view the construction of the Grothendieck–Witt space.

To any *k-differential-graded 1-category with weak equivalences and duality* $(\mathcal{D}, \mathcal{W}, *, \nu)$, one can associate a space $GW(\mathcal{D}, \mathcal{W}, (-)^*, \nu)$ (for the construction of this space, see [Sch10, Def. 3]). A duality on such a 1-category is a functor $(-)^* : \mathcal{D}^{\mathsf{op}} \to \mathcal{D}$ together with a natural transformation $\nu : \mathsf{id} \to (-)^* \circ ((-)^*)^{\mathsf{op}}$ such that $\mathsf{id}_{d^*} = (\nu_d)^* \circ \nu_{d^*}$ for all $d \in \mathcal{D}$. Then $GW(X) := GW^{[0]}(X)$ is given by $GW(\mathsf{Ch}^{\mathsf{b}}(\mathsf{Vect}(X)), \mathsf{quasi\text{-}iso}, (-)^*, \nu)$ where Ch^{b} denotes the 1-category of bounded chain complexes, $*$ is the usual duality for chain complexes, and ν is the usual double-dual transformation. We can use this to define spaces

$$GW^{[n]}(X) = GW(\mathsf{Ch}^{\mathsf{b}}(\mathsf{Vect}(X)), \mathsf{quasi\text{-}iso}, (-)^{*,(n)}, \nu^{(n)})$$

where $(E_\bullet)^{*,(n)} := (E_\bullet)^*[n]$ and $\nu^{(n)} = (-1)^{n(n-1)/2}\nu$ for all $n \geq 0$, which actually constitute the sections on X of the levels of KO.

Then, one proves the following periodicity statement:

Proposition 3.16 ([Kum20, Lemma 2.5.1])**.** *There are equivalences of k-differential graded 1-categories with weak equivalences and duality:*

$$\begin{aligned}(\mathsf{Ch}^{\mathsf{b}}(\mathsf{Vect}(X)), \mathsf{quasi\text{-}iso}, (-)^{*,(n+4m)}, \nu^{(n+4m)}) &\xrightarrow{\simeq} (\mathsf{Ch}^{\mathsf{b}}(\mathsf{Vect}(X)), \mathsf{quasi\text{-}iso}, (-)^{*,(n)}, \nu^{(n)}) \\ E_\bullet &\longmapsto E_\bullet[2m]\end{aligned}$$

$$\begin{aligned}(\mathsf{Ch}^{\mathsf{b}}(\mathsf{Vect}(X)), \mathsf{quasi\text{-}iso}, (-)^{*,(n+4m+2)}, \nu^{(n+4m+2)}) &\xrightarrow{\simeq} (\mathsf{Ch}^{\mathsf{b}}(\mathsf{Vect}(X)), \mathsf{quasi\text{-}iso}, (-)^{*,(n)}, -\nu^{(n)}) \\ E_\bullet &\longmapsto E_\bullet[2m+1].\end{aligned}$$

Moreover, $GW^{[2]}(X) = B(\mathsf{Alt}(X)^{\simeq})^{\mathsf{gp}}$ for all affine schemes $X \in \mathsf{Sm}_k$, where $\mathsf{Alt}(X)$ is the 1-category of symplectic algebraic vector bundles over X, i.e. bundles with an alternating bilinear form.

The following statement summarizes the different properties we have seen and is an analog of Theorem 3.7 for the motivic spectrum underlying the $\mathcal{E}_\infty$-ring KO. It also provides a motivic version of Theorem 3.4(i):

Theorem 3.17 ([Kum20, §4.3] for (i) and (ii); [ST15] for (iii); [RØ16, Thm 3.4] for (iv))**.**

- *(i) The spectrum KO represents Hermitian K-theory: it is given as a sequence of motivic spaces by $(GW^{[n]})_{n\in\mathbb{N}}$, with $GW^{[0]}(X) = B(\mathsf{Bil}(X)^{\simeq})^{\mathsf{gp}}$ and $GW^{[2]}(X) = B(\mathsf{Alt}(X)^{\simeq})^{\mathsf{gp}}$ if $X \in \mathsf{Sm}_k$ is affine.*
- *(ii) The spectrum KO is $(8,4)$-periodic. More precisely, the virtual vector bundle $(U_{2,4} - H_-)^{\otimes 2}$ on $(\mathbb{HP}^1)^{\wedge 2} \simeq T^{\wedge 4}$ defines a map $\beta_{\mathsf{KO}} : T^{\wedge 4} \to \mathsf{KO}$, inducing by multiplication an equivalence $\Sigma^{8,4}\mathsf{KO} \xrightarrow{\simeq} \mathsf{KO}$.*
- *(iii) As a prespectrum KO is given by $(\mathbb{Z} \times B_{\mathsf{et}}O, Sp/GL, \mathbb{Z} \times BSp, O/GL, ...)$, where $B_{\mathsf{et}}(-)$ denotes the classifying space construction as an étale sheaf, and O an Sp are the infinite orthogonal and symplectic groups respectively.*
- *(iv) There is a Wood cofiber sequence*

$$\Sigma^{1,1}\mathsf{KO} \xrightarrow{\eta} \mathsf{KO} \xrightarrow{c} \mathsf{KGL}$$

where $\eta : \mathbb{G}_m \to \mathbb{1}$ is the motivic Hopf map and c is the forgetful map discussed above.

As we mentioned earlier, periodicity implies in particular that the spectra in question should not be considered connective. We will therefore consider their very effective covers.

Lemma 3.18. *Let $1 \leq n \leq \infty$. The (very) effective cover of a motivic $\mathcal{E}_n$-ring spectrum is itself naturally a motivic $\mathcal{E}_n$-ring spectrum. In particular, the very effective covers $\mathsf{kgl} := \tilde{f}_0(\mathsf{KGL})$ and $\mathsf{ko} := \tilde{f}_0(\mathsf{KO})$ have induced structures of $\mathcal{E}_\infty$-ring spectra.*

Proof. This is the same proof as Lemma 1.30: lax symmetric monoidal functors preserves $\mathcal{E}_n$-algebra objects (Remark 1.27). The functors f_0 and $\tilde{f}_0$ are lax symmetric monoidal functors in virtue of the doctrinal adjunction principle (follows from [Lur17, Cor. 7.3.2.7]): they are right adjoints to the symmetric monoidal functors ι_0 and $\tilde{\iota}_0$ respectively (Proposition 2.22(*iii*)). $\square$

In their study of the homotopy sheaves of the motivic sphere spectrum, Morel and Hopkins defined Milnor–Witt K-theory (see [Mor99, §6.3]). We will also encounter two variants, the Milnor K-theory and Witt K-theory.

Definition 3.19. Let L be a field. The *Milnor-Witt K-theory* of L is a $\mathbb{Z}$-graded ring $\underline{K}_*^{MW}(L)$ defined as the quotient of the free non-commutative ring $\mathbb{Z}\langle\{[a] \mid a \in L \setminus \{0\}\} \cup \{\eta\}\rangle$, where η has degree -1 and all $[a]$ have degree 1, by the relations:

$$\begin{aligned}
&[ab] = [a] + [b] + \eta[a][b] \ \forall a, b \in L \setminus \{0\} \\
&[a][1-a] = 0 \ \forall a \in L \setminus \{0, 1\} \\
&[a]\eta = \eta[a] \ \forall a \in L \setminus \{0\} \\
&2\eta + \eta^2[-1] = 0.
\end{aligned}$$

Let $h := 2 + \eta[-1]$. The *Milnor K-theory* of L is then defined as $\underline{K}_*^M(L) := \underline{K}_*^{MW}(L)/\eta$, and the *Witt theory* of L is the quotient $\underline{K}_*^W(L) := \underline{K}_*^{MW}(L)/h$.

Theorem 3.20 ([Mor99, after Thm 6.4.7])**.** *Milnor-Witt K-theory is the zeroth homotopy module of the sphere spectrum: as homotopy modules, we have that*

$$\underline{\pi}_0(\mathcal{S})_* \simeq \underline{K}_*^{MW}.$$

Under this identification, the element in the homotopy of the sphere spectrum corresponding to $\eta \in \underline{K}_{-1}^{MW}(K)$ is the *motivic Hopf map*, given by the quotient map $\mathcal{S}^{3,2} \simeq \mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1 \simeq \mathcal{S}^{2,1}$.

3.3 L-theory

To finish off this section, we have to recall a construction closely related to that of Hermitian K-theory, namely L-theory. As mentioned in the introduction to this section, we will later compare the real realization of KO and its very effective cover ko with some L-theory spectra.

Corresponding to the construction of the Grothendieck-Witt space of a k-differential graded 1-category with weak equivalences and duality in previous subsection, for $X \in \mathsf{Sm}_k$ is defined in [Sch17, Def. 5.4] a *Grothendieck-Witt spectrum $GW_\bullet(X)$* whose infinite loop space is precisely the aforementioned Grothendieck-Witt space. We have

$$GW_\bullet(X) := GW_\bullet(\mathsf{Ch^b}(\mathsf{Vect}(X)), \mathsf{quasi\text{-}iso}, *, \nu) \in \mathsf{Sp}.$$

In [Sch17, Def. 5.7] are also defined shifted Grothendieck–Witt spectra $GW_\bullet^{[n]}(X)$ in the same way as for shifted Grothendieck–Witt spaces.

They come with a multiplicative structure: a "cup product" (see [Sch17, §5.4]) which in our case reads as maps

$$GW_\bullet^{[n]}(k) \wedge GW_\bullet^{[m]}(X) \longrightarrow GW_\bullet^{[n+m]}(X)$$

for all $X \in \mathsf{Sm}_k$, and $m, n \in \mathbb{Z}$.

Then, one defines:

Definition 3.21 ([Sch17, Def. 7.1])**.** The *L-theory spectrum* of $X \in \mathsf{Sm}_k$ is defined as the η-localization of the Grothendieck-Witt spectrum $GW_\bullet(X)$. More precisely,

$$\mathsf{L}(X) := \mathsf{colim}\ \Big(GW_\bullet(X) \xrightarrow{\eta} \mathcal{S}^1 \wedge GW_\bullet^{[-1]}(X) \xrightarrow{\eta} \mathcal{S}^2 \wedge GW_\bullet^{[-2]}(X) \dots\Big)$$

where η is smash product with the map

$$\mathcal{S}^0 \xrightarrow{\langle 1\rangle} GW_\bullet^{[0]}(k) \xrightarrow{-\delta} \mathcal{S}^1 \wedge GW_\bullet^{[-1]}(k)$$

(using the cup product $GW_\bullet^{[-1]}(k) \wedge GW_\bullet(X) \to GW_\bullet^{[-1]}(X)$), where $\langle 1\rangle$ represents the element of $\pi_0(GW_\bullet^{[0]}(k))$ corresponding to the symmetric bundle of dimension 1 given by product in k, and δ is a certain connecting homomorphism in a cofiber sequence for the GW-construction of a cone, see [Sch17, beginning of §6, p 56].

Proposition 3.22. *The L-theory spectrum* $\mathsf{L}(X)$ *of* $X \in \mathsf{Sm}_k$ *is an* $\mathcal{E}_\infty$*-ring.*

Proof. By the previous definition, $\mathsf{L}(X)$ is the η-periodization of $GW_\bullet(X)$, so it inherits an $\mathcal{E}_\infty$-ring structure by [Hoy20, Section 3]. □

4 Real (and complex) Betti realization

In this section, we finally construct the real and complex realization functors advertised in the introduction and mentioned many times in the previous sections, and then we compute our first examples. Subsections 4.1 and 4.2 consist again only of recollections about material taken from the literature or well-known statements. The results of the computations of Subsection 4.3 are mostly well-known, although some of them might not appear in the literature under this form or with these proofs, for example Propositions 4.26 and 4.27. The results of Subsection 4.3.4 seem to appear in the literature only at the level of spectra and not $\mathcal{E}_\infty$-rings.

In Subsection 4.1, we will construct the real realization functor $r_{\mathbb{R}} : \mathsf{SH}(\mathbb{R}) \to \mathsf{Sp}$ (and then the construction of the complex realization functor is entirely similar, replacing $\mathbb{R}$ by $\mathbb{C}$ whenever appropriate). More details about the general strategy we follow are given in the first paragraph of the subsection in question. Subsection 4.2 constitutes a first toolbox for computations with the real realization functor. We will see how the functor $r_{\mathbb{R}}$ relates to the operation of applying it levelwise to the motivic spaces constituting a motivic spectrum, and recall how real realization can be viewed as a Bousfield localization with respect to the inclusion $\rho : \mathcal{S}^0 \to \mathbb{G}_m$. We will review other general properties of the functor $r_{\mathbb{R}}$, in particular the fact that it maps very effective spectra to connective ones, and that it preserves certain localizations of spectra (at a prime, away from a prime, and rationalization). Finally, Subsection 4.3 contains a collection of examples, where we apply the techniques of our toolbox to determine the real realizations of several classical examples of motivic spectra: KGL and kgl, for which we also determine the complex realizations, and KO, $\mathsf{ko}[1/2]$, but also $\mathsf{H}\mathbb{Z}$, $\mathsf{H}\mathbb{Z}/2$, and $\widetilde{\mathsf{H}\mathbb{Z}}$ (we will first recall their definitions). Almost all of the tools of the previous subsection come in handy in these computations, and we also make a heavy use of the long exact sequence in homotopy both in the motivic and topological settings. To identify a certain connecting homomorphism in one of these sequences, we will also have to deal with Steenrod operations in the motivic Steenrod algebra and the motivic Cartan relations (in the proof of Proposition 4.26). We will also use the free $\mathcal{E}_1$-HA-algebras constructed in Subsection 1.5.

4.1 Construction of the real realization functor

We now construct the real Betti realization functor $r_{\mathbb{R}} : \mathsf{SH}(\mathbb{R}) \to \mathsf{Sp}$ (more generally, $r_{\mathbb{R}} : \mathsf{SH}(k) \to \mathsf{Sp}$ for any field k with a fixed embedding $\alpha : k \hookrightarrow \mathbb{R}$). Recall from the introduction that this functor was supposed to extend the functor $\mathsf{Sm}_k \to \mathsf{Spc}$, which associates to a scheme X the space of its real points $X(\mathbb{R})$ with the Euclidean topology. In view of the construction of the ∞-category of motivic spectra in Section 2, starting from Sm_k to go to $\mathsf{Spc}(k)$ and then $\mathsf{SH}(k)$, we will follow the same steps to progressively extend the functor of real points $(-)(\mathbb{R}) : \mathsf{Sm}_k \to \mathsf{Spc}$. Indeed, it is in general hard to write down explicitly functors between ∞-categories, but the various universal properties of the ∞-categories appearing in the construction of $\mathsf{SH}(k)$ will help us to extend our functor step-by-step. We will first define the functor of real points $(-)(\mathbb{R})$ precisely, and then proceed to the construction properly speaking. Here is an outline of the strategy:

1. Since Sm_k was first embedded into its presheaf ∞-category, we use the universal property of the latter (free cocompletion) to extend $(-)(\mathbb{R})$ to a colimit-preserving functor $r_{\mathbb{R}} : \mathcal{P}(\mathsf{Sm}_k) \to \mathsf{Spc}$.
2. Then, we considered a Bousfield localization of $\mathcal{P}(\mathsf{Sm}_k)$, imposing $\mathbb{A}^1$-invariance and Nisnevich descent. Again, using the universal property of such a localization, we get a functor $r_{\mathbb{R}} : \mathsf{Spc}(k) \to \mathsf{Spc}$ (and its pointed analog $r_{\mathbb{R}} : \mathsf{Spc}(k)_* \to \mathsf{Spc}_*$).
3. The ∞-category of motivic spectra was obtained by inverting $\mathbb{P}^1$ with respect to the symmetric monoidal structure on $\mathsf{Spc}(k)_*$, and it was universal with this property, in a suitable sense. We will see that the composition $r_{\mathbb{R}} : \mathsf{Spc}(k)_* \to \mathsf{Spc}_* \xrightarrow{\Sigma^\infty} \mathsf{Sp}$ is a symmetric monoidal functor inverting $\mathbb{P}^1$ (whose image in Spc_* is $\mathbb{S}^1$) and thus it factors through $\mathsf{SH}(k)$. Therefore we get $r_{\mathbb{R}} : \mathsf{SH}(k) \to \mathsf{Sp}$.

The numbering of the steps is repeated below when we actually perform the construction.

Remark 4.1. In all this subsection, the field of real number can be replaced with the field of complex numbers, to build the complex realization functor $r_{\mathbb{C}} : \mathsf{SH}(k) \to \mathsf{Sp}$, for any field k with a specified embedding into $\mathbb{C}$, extending the functor of complex points $(-)_{\mathbb{C}} : \mathsf{Sm}_k \to \mathsf{Spc}$. Note that $\mathbb{R}$ may be viewed as the C_2-fixed points of the conjugation action on $\mathbb{C}$. Adopting an equivariant perspective allows one to compute the real realization from the complex one; or more precisely as the geometric C_2-fixed points of some genuine equivariant spectrum.

Definition 4.2. Let $X \in \mathsf{Sm}_k$. The *real points of* X form a space, denoted by $X(\mathbb{R})$, with underlying set $\mathsf{Sm}_k(\mathsf{Spec}(\mathbb{R}), X) \simeq \mathsf{Sm}_{\mathbb{R}}(\mathsf{Spec}(\mathbb{R}), X \times_k \mathsf{Spec}(\mathbb{R}))$, and topology induced as follows: since X is of finite type over k, we may pick an open cover of X by affine subsets of the form $U = \mathsf{Spec}(k[X_1, \dots, X_n]/(f_1, \dots, f_r)) \to \mathsf{Spec}(k)$. Then, the topology on

$$\mathsf{Sm}_k(\mathsf{Spec}(\mathbb{R}), U) \simeq \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid f_i(x_1, \dots, x_n) = 0 \ \forall i \leq r\}$$

is induced by the Euclidean one. This construction extends to a functor $(-)(\mathbb{R}) : \mathsf{Sm}_k \to \mathsf{Spc}$.

For well-definedness, one has to show that this does not depend on the choice of the cover, nor of the presentations of the rings involved. This is elementary, but we provide a proof for completeness.

Assume there is an isomorphism of k-algebras

$$\psi : k[x_1, \dots, x_n]/(f_1, \dots, f_r) \xrightarrow{\sim} k[y_1, \dots, y_m]/(g_1, \dots, g_s).$$

Then, the map $\{\bar{x} \in \mathbb{R}^n \mid f_i(\bar{x}) = 0 \, \forall i \leq r\} \to \{\bar{y} \in \mathbb{R}^m \mid g_j(\bar{y}) = 0 \, \forall j \leq s\}$ induced on the real points is given by $\bar{x} \mapsto (Q_j(\bar{x}))_j$ where Q_j is any lift in $k[x_1, \dots, x_n]$ of $\psi^{-1}(y_j)$. Similarly ψ^{-1} induces a map in the other direction, such that $\bar{y} \mapsto (P_i(\bar{y}))_i$ where P_i is a lift in $k[y_1, \dots, y_m]$ for $\psi(x_i)$. Both map are polynomial, whence continuous, and they are inverse to one another because ψ is an isomorphism. In particular, the topology we defined does not depend on the choice of the presentations of the k-algebras involved.

Given two covers $\{U_i\}$, $\{U'_j\}$ of X by affines as in the definition, we may find a cover refining both, consisting only of affine open subsets that are distinguished in both the U_i's and the U'_j's. Then, using the previous bullet point, we only have to show that if $\{U_i\}$ is a cover of X as in the definition, then the topology induced by any refinement of this cover by distinguished open subsets is the same. Assume $U_i = \mathsf{Spec}(k[x_1, \dots, x_{n_i}]/(f_1^{(i)}, \dots, f_{r_i}^{(i)}))$ and the refinement is $U_i = \bigcup_{f \in A_i} (U_i)_f$ where A_i is a set of polynomials in $\mathsf{Spec}(k[x_1, \dots, x_{n_i}])$. The topology on the real points lying in U_i is that of $\{\bar{x} \in \mathbb{R}^{n_i} \mid f_j^{(i)}(\bar{x}) = 0 \ \forall j \leq r_i\}$. For $f \in A_i$, we choose the presentation

$$(U_i)_f = \mathsf{Spec}(k[x_1, \dots, x_{n_i}, x_{n_i+1}]/(f_1^{(i)}, \dots, f_{r_i}^{(i)}, x_{n_i+1}f - 1 = 0)),$$

and then the corresponding topology is that of $\{\bar{x} \in \mathbb{R}^{n_i+1} \mid f_j^{(i)}(\bar{x}) = 0 \ \forall j \leq r_i, \ x_{n_i+1}f(\bar{x}) = 1\}$. However, the inclusion $(U_i)_f \to U_i$ induces on the real points the map $(x_1, \dots, x_{n_i+1}) \mapsto (x_1, \dots, x_{n_i})$, which is seen to be a homeomorphism onto the open subset $\{\bar{x} \in \mathbb{R}^{n_i} \mid f_j^{(i)}(\bar{x}) = 0 \, \forall j \leq r_i, f(\bar{x}) \neq 0\}$ of the real points of U_i (the inverse is given by $\bar{x} \mapsto (\bar{x}, 1/f(\bar{x}))$). These observations imply our claim.

Remark 4.3. The bijection $\mathsf{Sm}_k(\mathsf{Spec}(\mathbb{R}), X) \simeq \mathsf{Sm}_{\mathbb{R}}(\mathsf{Spec}(\mathbb{R}), X \times_k \mathsf{Spec}(\mathbb{R}))$ implies that we could only consider the case $k = \mathbb{R}$: otherwise, it suffices to first use the base change functor $\mathsf{Sm}_k \to \mathsf{Sm}_{\mathbb{R}}$ induced by $\alpha : k \hookrightarrow \mathbb{R}$.

We now apply the strategy in four steps exposed above to construct the real Betti realization functor:

1. We start with the functor $(-)(\mathbb{R})$ from Definition 4.2. The slogan: "every presheaf is a colimit of representable objects" gives rise to the following formal statement:

   Theorem 4.4 ([Lur09, Thm 5.1.5.6]). *Let $\mathcal{C}$ be an essentially small ∞-category and $\mathcal{D}$ be an ∞-category admitting colimits. The Yoneda embedding induces an equivalence of ∞-categories*

   $$\mathsf{Fun}^{\mathsf{L}}(\mathcal{P}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \mathsf{Fun}(\mathcal{C}, \mathcal{D})$$

   where $\mathsf{Fun}^{\mathsf{L}}$ denotes the subcategory of colimit preserving functors. The inverse is given by left Kan extension along the fully faithful Yoneda embedding $\mathcal{C} \to \mathcal{P}(\mathcal{C})$. In other terms, the ∞-category of presheaves of spaces is freely generated under colimits by $\mathcal{C}$.

   Applying this proposition to $\mathcal{C} = \mathsf{Sm}_k$ and $\mathcal{D} = \mathsf{Spc}$ yields a functor $r_{\mathbb{R}} : \mathcal{P}(\mathsf{Sm}_k) \to \mathsf{Spc}$ extending $(-)(\mathbb{R})$. Recall that we have seen that Sm_k was essentially small in Remark 2.2.

2. The fact that $\mathsf{Spc}(k)$ is an accessible localization of the presheaf ∞-category (Proposition 2.7) also translates to the following statement about functor ∞-categories:

Proposition 4.5 ([Lur09, Prop. 5.5.4.20]). *Let $\mathcal{D}$ be an ∞-category with colimits. The localization functor $\mathsf{L}_{\mathsf{mot}}$ induces a fully faithful embedding*

$$\mathsf{Fun}^{\mathsf{L}}(\mathsf{Spc}(k), \mathcal{D}) \hookrightarrow \mathsf{Fun}^{\mathsf{L}}(\mathcal{P}(\mathsf{Sm}_k), \mathcal{D})$$

with essential image the functors carrying the maps in $\mathcal{W}_{\mathsf{mot}}$ (Definition 2.6) to equivalences.

We can apply [Lur09, Prop. 5.5.4.20] to our particular case, because we already saw that $\mathcal{P}(\mathsf{Sm}_k)$ was a presentable ∞-category (after Definition 2.6).

To apply Proposition 4.5 to $r_{\mathbb{R}} : \mathcal{P}(\mathsf{Sm}_k) \to \mathsf{Spc}$, we want to show that it inverts $\mathcal{W}_{\mathsf{mot}}$. It suffices to check this on a generating set of $\mathcal{W}_{\mathsf{mot}}$, which is given by the families of maps (a) and (b') from Remark 2.8). Since these maps are between representable objects or colimits thereof, and since $r_{\mathbb{R}} : \mathcal{P}(\mathsf{Sm}_k) \to \mathsf{Spc}$ was extended from Sm_k, we only have to check that $(-)(\mathbb{R})$ already carries the families of maps (before applying the Yoneda embedding) to equivalences.

(a) The functor $\mathsf{Sm}_k(\mathsf{Spec}(\mathbb{R}), -)$ preserves limits, and therefore $(-)(\mathbb{R})$ preserves (fiber) products. This is compatible with the topology we defined. Thus $X \times \mathbb{A}^1 \to X$ is mapped to the projection $X(\mathbb{R}) \times \mathsf{Spec}(k[t])(\mathbb{R}) \to X(\mathbb{R})$ which is a (homotopy) equivalence since $\mathsf{Spec}(k[t])(\mathbb{R}) = \mathbb{R}$.

(b') If $\mathcal{U}$ is a Nisnevich cover of $X \in \mathsf{Sm}_k$, we claim that $\mathsf{colim}\,(\check{C}(\mathcal{U})(\mathbb{R})) \to X(\mathbb{R})$ is an equivalence. Since the functor $(-)(\mathbb{R})$ commutes with (fiber) products, it carries this map to $\mathsf{colim}\,\check{C}(r_{\mathbb{R}}(\mathcal{U})) \to r_{\mathbb{R}}(X)$ (where $r_{\mathbb{R}}(\mathcal{U})$ is obtained by applying $r_{\mathbb{R}}$ to each map in the cover $\mathcal{U}$). Now, the proof of [DI04, Prop. 4.10], tells us that this map is an equivalence in Spc, provided that $r_{\mathbb{R}}(\mathcal{U})$ is locally split over X. Note first that $r_{\mathbb{R}}(\mathcal{U})$ is jointly surjective. Indeed, by definition, a Nisnevich cover is surjective on k'-points for any field k', in particular for $k' = \mathbb{R}$. Now, since Nisnevich covers are in particular étale covers, it suffices to show that the image under the real points functor of an étale map is locally split on its image. By [Sta25, Tag 02GH], an étale map $V \to X$ of smooth schemes of finite type over k is locally of the form

$$p : \mathsf{Spec}(A[y]_h/(g)) \cong \mathsf{Spec}(A[y,z]/(g, zh-1)) \to \mathsf{Spec}(A)$$

with $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$ is a finite type k-algebra, and $g, h \in A[y]$, with g monic and $g'(y) = \frac{\partial}{\partial y} g(x_1, \ldots, x_n, y)$ a unit in $A[y,z]/(g, zh-1)$. Taking real points yields the map

$$\begin{aligned} \{(\bar{x}, y, z) \in \mathbb{R}^{n+2} \mid f_i(\bar{x}) = 0\ \forall i \leq r, g(\bar{x}, y) = 0, zh(\bar{x}, y) = 1\} &\longrightarrow \{\bar{x} \in \mathbb{R}^n \mid f_i(\bar{x}) = 0\ \forall i \leq r\} \\ (\bar{x}, y, z) &\longmapsto \bar{x} \end{aligned}$$

where we also write g and h for lifts of g and h to $k[x_1, \ldots, x_n, y]$, viewed in $\mathbb{R}[x_1, \ldots, x_n, y]$. For a point $\bar{x}_0$ in the image of $r_{\mathbb{R}}(p)$, pick a preimage $(\bar{x}_0, y_0, z_0)$ lifting it. Consider the smooth function $f : \mathbb{R}^{n+2} \to \mathbb{R}^2$, $(\bar{x}, y, z) \mapsto (g(\bar{x}, y), z \cdot h(\bar{x}, y) - 1)$. Then f vanishes at $(\bar{x}_0, y_0, z_0)$ and its Jacobian with respect to the last two variables is given at this point by $\frac{\partial}{\partial y} g(\bar{x}_0, y_0) \cdot h(\bar{x}_0, y_0)$. Now, the hypotheses on g and $(\bar{x}_0, y_0, z_0)$ precisely imply that this Jacobian does not vanish. By the implicit function theorem, there exists $U \subseteq \mathbb{R}^n$ open, and $k : U \to \mathbb{R}^2$, such that $k(\bar{x}_0) = (y_0, z_0)$, and $f(\bar{x}, k(\bar{x})) = 0$ for all $\bar{x} \in U$. Restricting to $U \cap \{\bar{x} \in \mathbb{R}^n \mid f_i(\bar{x}) = 0\ \forall i \leq r\}$, the assignment $\bar{x} \mapsto (\bar{x}, k(\bar{x}))$ provides the section we need.

We thus obtain a colimit preserving functor $r_{\mathbb{R}} : \mathsf{Spc}(k) \to \mathsf{Spc}$, or $r_{\mathbb{R}} : \mathsf{Spc}(k)_* \to \mathsf{Spc}_*$ via the same construction.

3. Post-composition with the infinite suspension functor $\Sigma^\infty : \mathsf{Spc}_* \to \mathsf{Sp}$, which is a left-adjoint, yields a colimit preserving functor $\Sigma^\infty \circ r_{\mathbb{R}} : \mathsf{Spc}(k)_* \to \mathsf{Sp}$. By construction, $\mathsf{SH}(k)$ is the universal symmetric monoidal ∞-category inverting $\mathbb{P}^1$ in $\mathsf{Spc}(k)_*$. Indeed, it was constructed in Definition 2.15 as the initial object of $\mathsf{CAlg}(\mathsf{Pr}^{\mathsf{L}})$ under $\mathsf{Spc}(k)_*^{\otimes}$ (with the smash product) with $- \otimes \mathbb{P}^1$ an equivalence. This tells us exactly that:

Proposition 4.6. *Precomposition with $\Sigma^\infty : \mathsf{Spc}(S)_* \to \mathsf{SH}(S)$ induces an embedding*

$$\mathsf{Fun}^{\mathsf{L},\otimes}(\mathsf{SH}(S), \mathsf{Sp}) \longrightarrow \mathsf{Fun}^{\mathsf{L},\otimes}(\mathsf{Spc}(S)_*, \mathsf{Sp})$$

with image the functors sending $\mathbb{P}^1$ to an invertible object (and natural transformations whose $\mathbb{P}^1$-component is an equivalence).

We therefore have prove that $\Sigma^\infty \circ r_{\mathbb{R}} : \mathsf{Spc}(k)_* \to \mathsf{Sp}$ is equipped with symmetric monoidal structure and inverts $\mathbb{P}^1$. To prove the first claim, since Σ^∞ is symmetric monoidal, it suffices to check the claim for $r_{\mathbb{R}}$ itself. Recall that the symmetric monoidal structure on $\mathsf{Spc}(k)_*$ was obtained from the Cartesian structure on $\mathsf{Spc}(k)$ by tensoring with Spc_* in $\mathsf{Pr}^{\mathsf{L},\otimes}$ (Proposition 2.9). Thus we show that $r_{\mathbb{R}} : \mathsf{Spc}(k) \to \mathsf{Spc}$ preserves finite products, then $r_{\mathbb{R}}$ is symmetric monoidal with respect to the Cartesian structures by Proposition 1.14.

We claim that it is enough to prove that $r_{\mathbb{R}} : \mathcal{P}(\mathsf{Sm}_k) \to \mathsf{Spc}$ preserves finite products. We temporarily denote this functor by $r_{\mathbb{R},\mathcal{P}(\mathsf{Sm}_k)}$ to distinguish it from $r_{\mathbb{R}} : \mathsf{Spc}(k) \to \mathsf{Spc}$. Indeed, if this holds, then for any $\mathcal{F}$ and $\mathcal{G} \in \mathsf{Spc}(k)$, we have

$$\begin{aligned}
r_{\mathbb{R}}(\mathcal{F} \times \mathcal{G}) &\simeq r_{\mathbb{R}}(\mathsf{L}_{\mathsf{mot}}\iota\mathcal{F} \times \mathsf{L}_{\mathsf{mot}}\iota\mathcal{G}) \\
&\simeq r_{\mathbb{R}}(\mathsf{L}_{\mathsf{mot}}(\iota\mathcal{F} \times \iota\mathcal{G})) && (\mathsf{L}_{\mathsf{mot}} \text{ preserves products (proof of Proposition 2.9)}) \\
&\simeq r_{\mathbb{R},\mathcal{P}(\mathsf{Sm}_k)}(\iota\mathcal{F} \times \iota\mathcal{G}) \\
&\simeq r_{\mathbb{R},\mathcal{P}(\mathsf{Sm}_k)}(\iota\mathcal{F}) \times r_{\mathbb{R},\mathcal{P}(\mathsf{Sm}_k)}(\iota\mathcal{G}) && \text{(by assumption)} \\
&\simeq r_{\mathbb{R}}(\mathsf{L}_{\mathsf{mot}}\iota\mathcal{F}) \times r_{\mathbb{R}}(\mathsf{L}_{\mathsf{mot}}\iota\mathcal{G}) \\
&\simeq r_{\mathbb{R}}(\mathcal{F}) \times r_{\mathbb{R}}(\mathcal{G}).
\end{aligned}$$

Thus we now show that $r_{\mathbb{R}} : \mathcal{P}(\mathsf{Sm}_k) \to \mathsf{Spc}$ preserves finite products. We already saw that this holds for representable objects. Let $E = \mathsf{colim}_{i\in I}\, y(X_i)$ and $F = \mathsf{colim}_{j\in J}\, y(Y_j)$ be presheaves, written as colimits of representable objects. Then, using universality of colimits in $\mathcal{P}(\mathsf{Sm}_k)$ and the claim for representable objects, we can compute that

$$\begin{aligned}
r_{\mathbb{R}}(E \times F) &\simeq r_{\mathbb{R}}(\mathsf{colim}_{i\in I}\,\mathsf{colim}_{j\in J}\, y(X_i \times Y_j)) \\
&\simeq \mathsf{colim}_{i\in I}\,\mathsf{colim}_{j\in J}\, r_{\mathbb{R}}(X_i \times Y_j) && (\text{since } r_{\mathbb{R}} \text{ preserves colimits}) \\
&\simeq \mathsf{colim}_{i\in I}\,\mathsf{colim}_{j\in J}\, r_{\mathbb{R}}(X_i) \times r_{\mathbb{R}}(Y_j) \\
&\simeq \mathsf{colim}_{i\in I}\, r_{\mathbb{R}}(X_i) \times \mathsf{colim}_{j\in J}\, r_{\mathbb{R}}(Y_j) \\
&\simeq r_{\mathbb{R}}(E) \times r_{\mathbb{R}}(F)
\end{aligned}$$

as desired.

Finally, to show the second claim about $\Sigma^\infty r_{\mathbb{R}}(\mathbb{P}^1)$ being invertible, we can use the previous claim to compute

$$\begin{aligned}
r_{\mathbb{R}}(T) &\simeq r_{\mathbb{R}}(\mathcal{S}^1 \wedge \mathbb{G}_m) \simeq r_{\mathbb{R}}(\mathcal{S}^1) \wedge r_{\mathbb{R}}(\mathbb{G}_m) \\
&\simeq r_{\mathbb{R}}(\Sigma(* \amalg *)) \wedge r_{\mathbb{R}}(\mathsf{Spec}(k[t,t^{-1}])) \\
&\simeq \Sigma(* \amalg *) \wedge (\mathbb{R} \setminus 0) && (\text{since } r_{\mathbb{R}}(*) \simeq r_{\mathbb{R}}(\mathsf{Spec}(\mathbb{R})) \simeq *) \\
&\simeq \mathbb{S}^1 \wedge \mathbb{S}^0 \simeq \mathbb{S}^1
\end{aligned}$$

whose infinite suspension spectrum is by definition invertible in Sp.

So we obtain the desired extension $r_{\mathbb{R}} : \mathsf{SH}(k) \to \mathsf{Sp}$, which we call the *real Betti realization* functor. We might also talk about real Betti realization for any of the functors in the intermediate steps above.

4.2 Tools to compute real realizations

Since the functor $r_{\mathbb{R}}$ was constructed in the previous subsection using universal properties, it is a priori not clear how to compute the real realization of a given motivic spectrum. And indeed, one could argue that there is no general strategy for doing this. But the functor $r_{\mathbb{R}}$ has very good properties, such as being symmetric monoidal and preserving colimits and finite products, and thus the situation is not hopeless. This subsection gathers some tools we shall use later to compute the realizations of a number of motivic spectra. In this context, a theorem of Bachmann (Theorem 4.10 below) will turn out to be crucial for us. Roughly speaking, it states that the real realization functor can be computed by inverting the map ρ in $\mathsf{SH}(\mathbb{R})$, where ρ is the the inclusion $\mathcal{S}^0 \to \mathbb{G}_m$. More details are given below. But let us begin with some easier results.

The following result compares the real realization of a motivic spectrum with the sequence of pointed spaces obtained by applying the real realization functor levelwise to it.

Lemma 4.7 (Levelwise computation of $r_{\mathbb{R}}$)**.** *Let $E = (E_0, E_1, \dots) \in \mathsf{SH}(k)$. Then, the real realization of E is computed as the spectrification of the prespectrum $(r_{\mathbb{R}}(E_n))_{n\in\mathbb{N}}$. More explicitely,*

$$r_{\mathbb{R}}(E)_n \simeq \mathsf{colim}_m \Omega^m r_{\mathbb{R}}(E_{n+m}).$$

In particular, $\pi_n(r_{\mathbb{R}}(E)) \cong \mathsf{colim}_m(\pi_{n+m}(r_{\mathbb{R}}(E_m))$.

Proof. First note that $(r_{\mathbb{R}}(E_n))_{n\in\mathbb{N}}$ is a prespectrum via the structure maps obtained by real realization of those of E. Indeed, a map $T \wedge E_n \to E_{n+1}$ realizes to a map $\mathbb{S}^1 \wedge r_{\mathbb{R}}(E_n) \to r_{\mathbb{R}}(E_{n+1})$. The spectrification functor for topological prespectra sends a sequence $(E_n)_{n\in\mathbb{N}}$ to $\mathsf{colim}_n\Sigma^{-n}\Sigma^{\infty}E_n$ where the maps in the colimit are induced by the structure maps $\Sigma E_n \to E_{n+1}$. Similarly, for motivic prespectra, spectrification sends a sequence $(E_n)_{n\in\mathbb{N}}$ to $\mathsf{colim}_n\Sigma_T^{-n}\Sigma_T^{\infty}E_n$ with maps induced by $\Sigma^T E_n \to E_{n+1}$.

If E as in the statement is already a spectrum, it is its own spectrification and therefore we have

$$r_{\mathbb{R}}(E) \simeq r_{\mathbb{R}}(\mathsf{colim}_n(T^{\wedge(-n)} \wedge \Sigma_T^{\infty}E_n)) \simeq \mathsf{colim}_n(\mathbb{S}^{\wedge(-n)} \wedge \Sigma_{\mathbb{S}^1}^{\infty} r_{\mathbb{R}}(E_n))$$

since $r_{\mathbb{R}}$ commutes with colimits, is symmetric monoidal, maps T to $\mathbb{S}^1$, and commutes with infinite suspension functors by construction (it is extended from $\mathsf{Spc}(k)_*$). The right-hand side is exactly the spectrification of the sequence $(r_{\mathbb{R}}(E_n))_{n\in\mathbb{N}}$ with structure maps given by the real realizations of those of E. $\square$

Also, the properties of $r_{\mathbb{R}}$ provide one reason for which very effectiveness (Section 2.3) is a reasonable notion of connectivity for motivic spectra:

Lemma 4.8. *The restriction of $r_{\mathbb{R}}$ to $\mathsf{SH}(k)^{\mathsf{veff}}(m)$ takes values in m-connective spectra.*

Proof. Since $\mathsf{SH}(k)^{\mathsf{veff}}(m) = T^{\wedge m} \wedge \mathsf{SH}(k)^{\mathsf{veff}}$ by definition and $r_{\mathbb{R}}(T^{\wedge m}) = \mathbb{S}^m$, it suffices to prove the statement for $m = 0$. By definition, $\mathsf{SH}(k)^{\mathsf{veff}}$ is generated under colimits by all objects of the form $\mathcal{S}^n \wedge \Sigma^{\infty}(X_+)$ for $n \geq 0$ and $X \in \mathsf{Sm}_k$. The latter all realize to connective spectra because $r_{\mathbb{R}}(\mathcal{S}^n \wedge \Sigma^{\infty}(X_+)) = \mathbb{S}^n \wedge \Sigma^{\infty} r_{\mathbb{R}}(X)_+$ is connective for all $n \geq 0$ and $X \in \mathsf{Sm}_k$. Then, since $r_{\mathbb{R}}$ commutes with colimits, it suffices to argue that the subcategory of connective spectra is closed under colimits. This holds because a spectrum X is connective if and only if the mapping spectrum $\mathsf{Map}(X, Y)$ is 0 for all $Y \in \mathsf{Sp}_{\leq -1}$, but this condition is closed under colimits. $\square$

Note that the corresponding statement for effective motivic spectra is clearly false; since $\mathcal{S}^1$ is invertible in $\mathsf{SH}(k)^{\mathsf{eff}}$ but $r_{\mathbb{R}}(\mathcal{S}^1) \simeq \mathbb{S}^1$ is not invertible in connective spectra.

Example 4.9. We computed in the real case $r_{\mathbb{R}}(T) \simeq \mathbb{S}^1$. In comparison, the same computation in the complex case yields $r_{\mathbb{C}}(T) \simeq r_{\mathbb{C}}(\mathcal{S}^1) \wedge r_{\mathbb{C}}(\mathbb{G}_m) \simeq \mathbb{S}^1 \wedge \mathbb{C} \setminus \{0\} \simeq \mathbb{S}^2$. This provides one interpretation of the motivic bigrading: the first index can be seen as the complex dimension and the second index as the difference between the complex and real dimensions. Indeed, $r_{\mathbb{R}}(\mathcal{S}^{p,q}) \simeq \mathbb{S}^{p-q}$, whereas $r_{\mathbb{C}}(\mathcal{S}^{p,q}) \simeq \mathbb{S}^p$.

Note that the real realization of the map $\rho : \mathcal{S}^0 \to \mathbb{G}_m$, given by the inclusion of $\mathcal{S}^0$ into $\mathbb{G}_m$ (as the points corresponding to 1 and -1), is an equivalence $\mathbb{S} \xrightarrow{\simeq} \mathbb{S}$. But much more is true: indeed, the fact that ρ becomes invertible is sufficient to describe real realization (required notation is explained after the statement).

Theorem 4.10 ([Bac18, Thm 35])**.** *Let S be a Noetherian scheme of finite dimension. There is an equivalence of ∞-categories*

$$\mathsf{SH}(S)[\rho^{-1}] \xrightarrow{\simeq} \mathsf{SH}(\mathsf{Shv}(RS)).$$

In particular, there is an equivalence $\mathsf{SH}(\mathbb{R})[\rho^{-1}] \xrightarrow{\simeq} \mathsf{Sp}$ such that the composite

$$\mathsf{SH}(\mathbb{R}) \longrightarrow \mathsf{SH}(\mathbb{R})[\rho^{-1}] \xrightarrow{\simeq} \mathsf{Sp}$$

is naturally equivalent to the real realization functor $r_{\mathbb{R}}$.

Here, RS is the *real spectrum of* S. It is some topological space whose underlying set is the set of pairs $(x, \mathfrak{o})$ with x a point in S and $\mathfrak{o}$ an ordering on the residue field of x. The target of the first equivalence is the ∞-category of spectra associated with the topos of sheaves of spaces on the topological space RS. The source of this equivalence is the ∞-category of ρ-local motivic spectra, in the sense of Bousfield localization.

Remark 4.11. Although the real and complex realization functors share many properties, it is important to note that the second part of the statement of Theorem 4.10 is of course very specific to the *real* realization functor, and is completely false in the case of complex realization.

Proposition 4.12 ([Bac18, Lemma 15])**.** *The canonical localization functor* $\mathsf{SH}(S) \to \mathsf{SH}(S)[\rho^{-1}]$ *is equivalent to the functor*

$$E \mapsto \mathsf{colim}(E \xrightarrow{\rho} E \wedge \mathbb{G}_m \xrightarrow{\rho} E \wedge \mathbb{G}_m^{\wedge 2} \to \cdots).$$

This result holds in general for localizations in symmetric monoidal ∞-categories, under some compactness assumptions, for ρ replaced by a map $\mathbb{1} \to A$ where A is invertible with respect to the monoidal structure (see [Bac17, Lemma 15]).

Lemma 4.13. *For any* $E \in \mathsf{SH}(\mathbb{R})$*, the homotopy groups of its real realization can be computed as*

$$\pi_m(r_{\mathbb{R}}(E)) = \mathsf{colim}_n(\cdots \to \underline{\pi}_m(E)_n(\mathsf{Spec}(\mathbb{R})) \xrightarrow{\rho} \underline{\pi}_m(E)_{n+1}(\mathsf{Spec}(\mathbb{R})) \to \cdots).$$

Proof. Call F the equivalence $\mathsf{SH}(\mathbb{R})[\rho^{-1}] \to \mathsf{Sp}$ above, and $L_\rho : \mathsf{SH}(\mathbb{R}) \to \mathsf{SH}(\mathbb{R})[\rho^{-1}]$ the canonical localization functor. Then, for all $m \in \mathbb{Z}$ and $E \in \mathsf{SH}(\mathbb{R})$, we have:

$$\begin{aligned}
\pi_m(r_{\mathbb{R}}E) &\cong [\mathbb{S}^m, r_{\mathbb{R}}E] \\
&\cong [r_{\mathbb{R}}(\mathcal{S}^m), r_{\mathbb{R}}E] \\
&\cong [F(L_\rho(\mathcal{S}^m)), F(L_\rho(E))] \\
&\cong [L_\rho(\mathcal{S}^m), L_\rho(E)] \\
&\cong [\mathcal{S}^m, \mathsf{colim}_n\, E \wedge \mathbb{G}_m^{\wedge n}] && \text{(since } L_\rho \text{ is a left-adjoint)} \\
&\cong \mathsf{colim}_n\, [\mathcal{S}^m, E \wedge \mathbb{G}_m^{\wedge n}] && \text{(by compactness of (the infinite suspension of) } \mathcal{S}^m\text{)} \\
&\cong \mathsf{colim}_n\, \underline{\pi}_m(E)_n(\mathsf{Spec}(\mathbb{R}))
\end{aligned}$$

where the last isomorphism follows from the fact that the sections of the homotopy sheaves and presheaves of E agree on $\mathsf{Spec}(\mathbb{R})$. This holds because sheafification preserves stalks, and sections over $\mathsf{Spec}(\mathbb{R})$ is a stalk functor of the Nisnevich site on $\mathsf{Sm}_{\mathbb{R}}$. Indeed, by [MV99, p99], every functor $\mathcal{F} \in \mathsf{Shv}_{\mathsf{Nis}}(\mathsf{Sm}_{\mathbb{R}}) \longmapsto \mathsf{colim}_{\mathsf{Spec}(\mathcal{O}^{\mathsf{hen}}_{U,u}) \to X \to \mathsf{Spec}(\mathbb{R})} \mathcal{F}(X)$ where $U \in \mathsf{Sm}_{\mathbb{R}}$ and $u \in U$ is a point in the usual sense defines a point of the Nisnevich site (this is even a conservative family of points). The colimit is indexed by the opposite of the ∞-category of factorizations of $\mathsf{Spec}(\mathcal{O}^{\mathsf{hen}}_{U,u}) \to \mathsf{Spec}(\mathbb{R})$ in $\mathbb{R}$-schemes through some $X \in \mathsf{Sm}_{\mathbb{R}}$. Here, for $U = \mathbb{R}$, this functor boils down to taking real sections since $\mathbb{R}$ is a henselian local ring, and the identity factorization is terminal in the ∞-category indexing the colimit. □

We will need later two other properties of the real realization functor:

Proposition 4.14. *The real realization functor has the following properties:*

(i) The functor $r_{\mathbb{R}}$ *commutes with localization away from* p *a prime, localization at* (p)*, and rationalization.*

(ii) For $1 \leq n \leq \infty$*, real realization induces a functor from the* ∞*-category of* $\mathcal{E}_n$*-algebras in* $\mathsf{SH}(k)$ *to that of* $\mathcal{E}_n$*-rings spectra.*

Proof. (i) These operations are described by colimits (see Appendix A.1), which we saw were preserved by the functor $r_{\mathbb{R}}$.

(ii) The claim follows from Remark 1.27: (lax) symmetric monoidal functors preserve $\mathcal{E}_n$-algebras. □

4.3 The realizations of some classical motivic spectra

We can now put into practice the methods and tools we just saw, to compute the real and complex realizations of the motivic spectra KGL and kgl, and the real realizations of some other classical examples of motivic spectra: $\mathsf{H}\mathbb{Z}$ and $\mathsf{H}\mathbb{Z}/2$, which represent motivic cohomology, and their variant $\widetilde{\mathsf{H}\mathbb{Z}}$, but also KO and $\mathsf{ko}[1/2]$, which is a first step towards our main result: the real realization of ko. All these examples have the structure of $\mathcal{E}_\infty$-rings. For many of them, we are actually able to say something about the $\mathcal{E}_\infty$- or $\mathcal{E}_1$-ring structures of their realizations as well.

4.3.1 The realizations of $\mathsf{H}\mathbb{Z}/2$ and $\mathsf{H}\mathbb{Z}$

Voevodsky's motivic cohomology spectrum, which we may denote by $\mathsf{H}\mathbb{Z}$, admits several equivalent definitions:

Definition 4.15. Over a perfect base field k, the *motivic Eilenberg Mac Lane spectrum*, denoted by $\mathsf{H}\mathbb{Z}$, is equivalently defined as:

(i) The motivic spectrum representing the motivic cohomology theory in $\mathsf{SH}(k)$, i.e. for all $X \in \mathsf{Sm}_k$ and $p, q \in \mathbb{Z}$, we have $H^{p,q}(X, \mathbb{Z}) \cong [\Sigma^\infty_+ X, \Sigma^{p,q}\mathsf{H}\mathbb{Z}]$.

(ii) The zeroth effective slice of the sphere spectrum in $\mathsf{SH}(k)$.

(iii) The effective cover of the *Milnor K-theory sheaf*: $\mathsf{H}\mathbb{Z} = f_0\underline{K}^M_*$ (see Definition 3.19).

The equivalence between (i) and (ii) is proven in [Lev08]. The equivalence between (iii) and the previous definitions is [Bac17, Lemma 12].

Similarly, we can also define:

Definition 4.16. The motivic spectrum $\mathsf{H}\mathbb{Z}/2$ is equivalently defined as:

(i) The motivic spectrum representing motivic cohomology $\mathsf{mod}\, 2$ in $\mathsf{SH}(k)$.

(ii) The cofiber of the multiplication by 2 map on $\mathsf{H}\mathbb{Z}$ (which justifies the notation).

(iii) The effective cover of the Milnor K-theory $\mathsf{mod}\, 2$ sheaf: $\mathsf{H}\mathbb{Z}/2 = f_0(\underline{K}^M_*/2)$.

Since the effective cover functor preserves cofiber sequences (and multiplication by 2), we have $f_0(\underline{K}^M_*/2) \simeq f_0(\underline{K}^M_*)/2$ where all cofibers are taken in $\mathsf{SH}(k)$. This is also true if the cofiber $\underline{K}^M_*/2$ is taken in $\mathsf{SH}(k)^\heartsuit$ by [Bac17, Lemma 12].

Remark 4.17. When computing the sections over $\mathbb{R}$ of the homotopy sheaves of $\mathsf{H}\mathbb{Z}/2$, we will see many of them are non-trivial (and thus for $\mathsf{H}\mathbb{Z}$ too); and so they are not really reminiscent of the very simple homotopy advertized by the name "Eilenberg-Mac Lane spectrum".

In both cases, Definition (iii) allows us to define an $\mathcal{E}_\infty$-structure on $\mathsf{H}\mathbb{Z}$ and $\mathsf{H}\mathbb{Z}/2$: indeed, precomposition of r_0 by the inclusion defines a lax symmetric monoidal functor $\mathsf{SH}(k)^\heartsuit \to \mathsf{SH}(k)^{\mathsf{eff}}$ (proof of Proposition 2.26) and thus carries $\mathcal{E}_\infty$-algebras to $\mathcal{E}_\infty$-algebras. In particular, we only have to show $\underline{K}^M_*, \underline{K}^M_*/2 \in \mathsf{CAlg}(\mathsf{SH}(k)^\heartsuit)$. However, the heart of a t-structure on an ∞-category is a 1-category by [Lur17, Rmk 1.2.1.12], and therefore its commutative algebras are commutative algebras in the 1-categorical sense. By Theorem 3.20, we have $\underline{K}^{MW}_* \simeq \underline{\pi}_0(\mathcal{S})_*$, which belongs to $\mathsf{SH}(k)^\heartsuit$. Since the truncation functor $\mathsf{SH}(k)_{\geq 0} \to \mathsf{SH}(k)^\heartsuit$ is symmetric monoidal by definition of the tensor product on the target, it sends the commutative algebra object $\mathcal{S}$ to a commutative algebra object $\underline{\pi}_0(\mathcal{S})_*$. Because $\underline{K}^M_*$ and $\underline{K}^M_*/2$ are quotients of $\underline{K}^{MW}_*$, they also define commutative algebras, as desired.

This argument also shows that $\mathsf{H}\mathbb{Z} \to \mathsf{H}\mathbb{Z}/2$ is a map of motivic $\mathcal{E}_\infty$-rings.

Proposition 4.18. *The motivic spectra $\mathsf{H}\mathbb{Z}$, $\mathsf{H}\mathbb{Z}/2$, and $\widetilde{\mathsf{H}\mathbb{Z}} := f_0\underline{K}^{MW}_*$ are very effective.*

Proof. By [Bac17, Lemma 6], and the remark below it, $\underline{K}^{MW}_*$, $\underline{K}^M_*$ and $\underline{K}^M_*/2$ are effective homotopy modules, i.e. they belong to the image of $\iota_0^\heartsuit : \mathsf{SH}(k)^{\mathsf{eff},\heartsuit} \to \mathsf{SH}(k)^\heartsuit$. By right t-exactness of ι_0 (Proposition 2.26), we have $\iota_0^\heartsuit = \tau_{\leq 0} \circ \iota_0$. For an effective homotopy module X, we prove that $f_0(X)$ is very effective, and this suffices to prove the proposition, by definition of $\mathsf{H}\mathbb{Z}$, $\mathsf{H}\mathbb{Z}/2$, and $\widetilde{\mathsf{H}\mathbb{Z}}$.

Let E be a preimage for X by $\iota_0^\heartsuit$. To show that f_0X is very effective, we show that it is equivalent to its very effective cover. We compute

$$\begin{aligned}
f_0X &\simeq \iota_0 r_0(\tau_{\leq 0}\iota_0 E) \\
&\simeq \iota_0\tau_{\leq_e 0}(r_0\iota_0 E) && \text{(since } r_0 \text{ is t-exact by Proposition 2.26)} \\
&\simeq \iota_0\tau_{\leq_e 0}E \\
&\simeq \iota_0 E \in \mathsf{SH}(k)_{\geq 0}. && \text{(because } E \in \mathsf{SH}(k)^{\mathsf{eff},\heartsuit} \text{ and } \iota_0 \text{ is right t-exact by Proposition 2.26)}
\end{aligned}$$

Then, we have

$$\begin{aligned}
\tilde{f}_0(f_0X) &\simeq f_0\tau_{\geq 0}(f_0X) && \text{(by [Bac17, Lemma 10], } \tilde{f}_0 = f_0 \circ \tau_{\geq 0}) \\
&\simeq f_0(f_0X) && \text{(we just computed that } f_0X \in \mathsf{SH}(k)_{\geq 0}) \\
&\simeq f_0X
\end{aligned}$$

as desired. □

The global sections of the homotopy sheaves of $\mathsf{H}\mathbb{Z}/2$ are known:

Theorem 4.19 ([Voe96])**.** *There are isomorphism of graded rings*

$$\underline{K}^M_*(k)/2 \cong H^*_{\acute{e}t}(k,\mathbb{Z}/2) \cong \underline{\pi}_0(\mathsf{H}\mathbb{Z}/2)_*(k).$$

In particular, $\underline{\pi}_0(\mathsf{H}\mathbb{Z}/2)_*(\mathbb{R}) \cong \mathbb{Z}/2[\rho]$ is a polynomial ring over a generator of degree 1, which can be identified with ρ.

Proof. Milnor's conjecture, which was proven by Voevodsky in [Voe96], states in particular that for a field k' of characteristic not 2 (in our case $k \hookrightarrow \mathbb{R}$ has characteristic 0), we have

$$\underline{K}^M_*(k')/2 \cong H^*_{\text{ét}}(k',\mathbb{Z}/2).$$

In words, the Milnor K-theory $\mathsf{mod}\,2$ of k' is equal to the étale cohomology $\mathsf{mod}\,2$ of k', or equivalently the group cohomology of the absolute Galois group of k' with coefficients in $\mathbb{Z}/2$. In our case, we learn that $\underline{K}^M_*(\mathbb{R})/2$ is the group cohomology $\mathsf{mod}\,2$ of $\mathbb{Z}/2$ (for the trivial action). By [Bro94, §III.1, Example 2], it is thus equal to $\mathbb{Z}/2$ in every degree, and as a ring it is polynomial on one generator, namely ρ.

Other consequences of Voevodsky's proof are the fact that $\underline{K}^M_n(k')/2 \cong H^{n,n}(k',\mathbb{Z}/2)$ (where the right hand-side is motivic cohomology mod 2, so it is $\underline{\pi}_{n,n}(\mathsf{H}\mathbb{Z}/2)(k)$ if $k = k'$), and that cup product with a generator of $H^{0,1}(k',\mathbb{Z}/2) \cong \mu_2(k') \cong \mathbb{Z}/2$ (here μ_2 denotes the square roots of unity), which we denote by τ, induces isomorphisms $H^{p,q}(k',\mathbb{Z}/2) \cong H^{p,q+1}(k',\mathbb{Z}/2)$ when $0 \leq p \leq q$ (see for example [RØ16, Lemma 6.1 and the end of the proof of Lemma 6.9]). In particular, if $k = k'$, we get $\underline{K}^M_n(k)/2 \cong [\mathsf{Spec}(k), \Sigma^{n,n}\mathsf{H}\mathbb{Z}/2] \cong \underline{\pi}_{-n,-n}(\mathsf{H}\mathbb{Z}/2)(k)$. □

Definition 4.20. As in the previous proof, let $\tau \in H^{0,1}(k,\mathbb{Z}/2) \cong \mu_2(k) \cong \mathbb{Z}/2$ be a generator, i.e. corresponding to $-1 \in \mu_2(k)$.

Proposition 4.21. *As $\mathcal{E}_1$-rings, the real realization $r_{\mathbb{R}}(\mathsf{H}\mathbb{Z} \to \mathsf{H}\mathbb{Z}/2)$ of the cofiber of the multiplication by 2 map is equivalent to the map of free $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}/2$-algebras $\mathsf{H}\mathbb{Z}/2[t^2] \to \mathsf{H}\mathbb{Z}/2[t]$ (see Definition 1.37).*

Proof. We proceed in three step. We first identify the homotopy groups of the real realization of $\mathsf{H}\mathbb{Z}/2$ and their structure as a graded ring. Then, we use this and a theorem of Hopkins and Mahowald to identify $r_{\mathbb{R}}(\mathsf{H}\mathbb{Z}/2)$ as a free $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}/2$-algebra. Finally, we deduce the homotopy ring and the $\mathcal{E}_1$-structure of $r_{\mathbb{R}}(\mathsf{H}\mathbb{Z})$ from the cofiber sequence $\mathsf{H}\mathbb{Z} \xrightarrow{\cdot 2} \mathsf{H}\mathbb{Z} \to \mathsf{H}\mathbb{Z}/2$.

Step 1: Computation of the homotopy ring of $r_{\mathbb{R}}(\mathsf{H}\mathbb{Z}/2)$. This is easier for $\mathsf{H}\mathbb{Z}/2$ rather than $\mathsf{H}\mathbb{Z}$ because the real sections of its bigraded motivic homotopy ring are known, as we have seen in Theorem 4.19.

In view of Lemma 4.13, we have $\pi_k(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}/2) = \mathsf{colim}_n\,\underline{\pi}_k(\mathsf{H}\mathbb{Z}/2)_n(\mathbb{R}) = \mathsf{colim}_n\,\pi_{k-n,-n}(\mathsf{H}\mathbb{Z}/2)(\mathbb{R})$, which can then for $k \geq 0$ and $n \geq k$ be rewritten as

$$\mathsf{colim}_n H^{n-k,n}(\mathbb{R},\mathbb{Z}/2) \cong \mathsf{colim}_n \tau^k H^{n-k,n-k}(\mathbb{R},\mathbb{Z}/2) \cong \mathsf{colim}_n \tau^k \underline{K}^M_{n-k}(\mathbb{R}) \cong \mathsf{colim}_n \mathbb{Z}/2\{\tau^k\rho^{n-k}\}.$$

This colimit is $\mathbb{Z}/2$ with generator $(\tau\rho^{-1})^k \in \underline{\pi}_k(\mathsf{H}\mathbb{Z}/2)_0(\mathbb{R})$. It follows that $\pi_*(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}/2) \cong \mathbb{Z}/2[t]$ is a polynomial ring on a single generator $t := r_{\mathbb{R}}(\tau\rho^{-1})$ in degree 1.

Step 2: Structure of $r_{\mathbb{R}}(\mathsf{H}\mathbb{Z}/2)$ as an $\mathcal{E}_1$-ring. We will show that there exists a map of $\mathcal{E}_2$-rings $\mathsf{H}\mathbb{Z}/2 \to r_{\mathbb{R}}(\mathsf{H}\mathbb{Z}/2)$. Here, on the left-hand side, $\mathsf{H}\mathbb{Z}/2$ denotes the usual topological Eilenberg-Mac Lane spectrum, whereas on the right-hand side it is the motivic Eilenberg-Mac Lane spectrum from Definition 4.16. Then, by Proposition 1.38(iii), this will induce a map $\mathsf{H}\mathbb{Z}/2[t] \to r_{\mathbb{R}}(\mathsf{H}\mathbb{Z}/2)$ sending the generator t we adjoint on the left hand-side to the element $t \in \pi_1(r_{\mathbb{R}}(\mathsf{H}\mathbb{Z}/2))$ we had above. Then, since Proposition 1.38(ii) there is an isomorphism of rings $\pi_*(\mathsf{H}\mathbb{Z}/2[t]) \cong \mathbb{Z}/2[t]$, and the map $\pi_*(\mathsf{H}\mathbb{Z}/2[t]) \to \pi_*(r_{\mathbb{R}}(\mathsf{H}\mathbb{Z}))$ is a ring homomorphism, we deduce that the latter is an isomorphism, so $\mathsf{H}\mathbb{Z}/2[t] \to r_{\mathbb{R}}(\mathsf{H}\mathbb{Z}/2)$ is an equivalence of $\mathcal{E}_1$-rings.

To produce an $\mathcal{E}_2$-map $\mathsf{H}\mathbb{Z}/2 \to r_{\mathbb{R}}(\mathsf{H}\mathbb{Z}/2)$, we appeal to the so-called Hopkins–Mahowald theorem:

Theorem 4.22 ([Mah79], [MNN15, Thm 4.18])**.** *The free $\mathcal{E}_2$-ring with $2 = 0$ is $\mathsf{H}\mathbb{Z}/2$. More precisely, there is a pushout diagram of $\mathcal{E}_2$-rings*

$$\begin{tikzcd}
F_{\mathcal{E}_2}(\mathbb{S}^0) \arrow[r, "2"] \arrow[d, "0"'] & \mathbb{S} \arrow[d] \\
\mathbb{S} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near start] & \mathsf{H}\mathbb{Z}/2.
\end{tikzcd}$$

where the maps 2 and 0 are obtained by the universal property of the free $\mathcal{E}_2$-ring, applied to the maps $\cdot 2 : \mathbb{S} \to \mathbb{S}$ and $0 : \mathbb{S} \to \mathbb{S}$.

The original theorem in [Mah79] expresses $\mathsf{H}\mathbb{Z}/2$ as the Thom spectrum of a map $\Omega^2\mathbb{S}^3 \to \mathsf{BGL}_1(\mathbb{S})$. The theorem as we cited it is proven under this form in [MNN15]. We will come back to the formalism of (multiplicative) Thom spectra in Section 6.

Since $2 = 0 \in \pi_*(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}/2)$ by the previous computation, we have to choose a homotopy between the multiplication by 2 map and the zero map on $r_{\mathbb{R}}(\mathsf{H}\mathbb{Z}/2)$, and then the theorem provides an $\mathcal{E}_2$-map $\mathsf{H}\mathbb{Z}/2 \to r_{\mathbb{R}}(\mathsf{H}\mathbb{Z}/2)$, and thus the $\mathcal{E}_1$-map $\mathsf{H}\mathbb{Z}/2[t] \to r_{\mathbb{R}}(\mathsf{H}\mathbb{Z}/2)$ we were looking for, concluding this part of the proof.

Step 3: Realization of $\mathsf{H}\mathbb{Z}$, and of the quotient map to $\mathsf{H}\mathbb{Z}/2$. Consider once more the cofiber sequence $\mathsf{H}\mathbb{Z} \xrightarrow{\cdot 2} \mathsf{H}\mathbb{Z} \to \mathsf{H}\mathbb{Z}/2$. We claim that $r_{\mathbb{R}}(\cdot 2) = \cdot 2 = 0$ on $r_{\mathbb{R}}\mathsf{H}\mathbb{Z}$. This holds because, by definition, we have $\mathsf{H}\mathbb{Z} = f_0\underline{K}^M_*$. However, in Milnor K-theory, we have $2\rho = 0$. Indeed, $\rho \in [\mathcal{S}^0, \mathbb{G}_m] \cong \underline{\pi}_0(()_1\mathcal{S})(k) \cong K_1^{MW}(k)$ corresponds to $[-1]$ in the Milnor-Witt K-theory of k by [Mor99, Thm 6.3.3 and 6.4.1]. But then in Milnor K-theory, since $[1] = [1^2] = [1] + [1]$, we obtain $2\rho = [-1] + [-1] = [(-1)^2] = [1] = 0$.

Thus, after inverting ρ, the multiplication by 2 map becomes the zero map. Therefore, after real realization, we obtain a long exact sequence (here $n \geq 0$)

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{n+1}(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}) \arrow[r, "\cdot 2=0"] & \pi_{n+1}(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}) \arrow[r, hook, "q_{n+1}"] & \pi_{n+1}(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}/2) \cong \mathbb{Z}/2 \arrow[dll, two heads, "\partial_n"'] \\
& \pi_n(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}) \arrow[r, "\cdot 2=0"'] & \pi_n(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}) \arrow[r, hook, "q_n"'] & \pi_n(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}/2) \cong \mathbb{Z}/2 \arrow[r] & \cdots .
\end{tikzcd}$$

Note that all spectra involved are connective by Lemma 4.8 (since $\mathsf{H}\mathbb{Z}$ and $\mathsf{H}\mathbb{Z}/2$ are very effective by Proposition 4.18). Therefore, q_0 is also surjective; it is an isomorphism. Then $\pi_0(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}) \cong \mathbb{Z}/2$ and ∂_0 is a surjection $\mathbb{Z}/2 \to \mathbb{Z}/2$, namely an isomorphism. Thus the injective map q_1 has trivial image, and so $\pi_1(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}) = 0$. We are back to the original situation: the long exact sequence repeats in exactly the same way. Therefore, $\pi_n(r_{\mathbb{R}}\mathsf{H}\mathbb{Z})$ is $\mathbb{Z}/2$ if n is even and non-negative and 0 else. Let u be a generator of $\pi_2(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}) \cong \mathbb{Z}/2$. Then, since q_2 is an isomorphism, the real realization of the quotient map sends it to the generator t^2 of $\pi_2(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}/2)$. Since this quotient map induces a ring homomorphism on homotopy, and an isomorphism in even degrees, we also learn that u^n is sent to t^{2n} by the isomorphism q_{2n}, so it generates $\pi_{2n}(r_{\mathbb{R}}\mathsf{H}\mathbb{Z})$. The identification with $\mathsf{H}\mathbb{Z}[t^2]$ as $\mathcal{E}_1$-rings follows as before from the Hopkins–Mahowald theorem. This finishes the proof. $\square$

4.3.2 The realizations of KGL and kgl

To expand our list of examples, we compute in this subsection the complex and real Betti realizations of KGL and kgl. The computation of $r_{\mathbb{R}}\mathsf{kgl}$ will actually be useful to us in Section 5 to determine $r_{\mathbb{R}}\mathsf{ko}$.

Lemma 4.23. *There is an equivalence of spectra $r_{\mathbb{C}}\mathsf{KGL} \simeq \mathsf{KU}$ where $\mathsf{KU} = (\mathbb{Z} \times BU, U, \mathbb{Z} \times BU, \dots)$ is the usual K-theory spectrum.*

Proof. We have seen that $r_{\mathbb{C}}(\mathbb{P}^1) \simeq \mathbb{S}^2$. Therefore, the complex realization functor can more naturally be seen as taking values in the ∞-category of $\mathbb{S}^2$-spectra ("double speed spectra") $\mathsf{Sp}_{\mathbb{S}^2}$. This approach is completely equivalent to the one introduced before, because there is an equivalence of categories between $\mathsf{Sp}_{\mathbb{S}^2}$ and Sp (sending a sequence $(E_0, E_1, \dots)$ to the sequence $(E_0, \Omega E_1, E_1, \Omega E_2, \dots)$). This can be proven for example using Theorems 2.11 and 2.13. The same argument as in the proof of Lemma 4.7 shows that $r_{\mathbb{C}}\mathsf{KGL}$ is the spectrification in $\mathbb{S}^2$-spectra of its levelwise complex realization. Since $\mathsf{KGL} = (K, K, \dots)$, we first compute, by construction of the realization functor

$$r_{\mathbb{C}}(K) = r_{\mathbb{C}}(\mathsf{L}_{\mathsf{mot}}(\mathbb{Z} \times BGL)) \simeq r_{\mathbb{C}}\left(\coprod_{\mathbb{Z}} BGL\right) \simeq \coprod_{\mathbb{Z}} r_{\mathbb{C}}(BGL) \underset{(\star)}{\simeq} \mathbb{Z} \times B\left(r_{\mathbb{C}}\left(\bigcup_{n\in\mathbb{N}^*} GL_n\right)\right)$$
$$\simeq \mathbb{Z} \times B\left(\bigcup_{n\in\mathbb{N}^*} GL_n(\mathbb{C})\right) \underset{(\star\star)}{\simeq} \mathbb{Z} \times B\left(\bigcup_{n\in\mathbb{N}^*} U_n\right) \simeq \mathbb{Z} \times BU$$

where $(\star\star)$ follows from the well-known fact that $GL_n(\mathbb{C})$ deformation retracts onto the unitary group $U_n(\mathbb{C})$, and these deformation retracts can be made compatible as n varies. One way to see this, is to note that one the one hand $U_n(\mathbb{C})$ is the intersection in $GL_{2n}(\mathbb{R})$ of $GL_n(\mathbb{C})$ and the orthogonal group $O_{2n}(\mathbb{R})$, and on the other hand, for $m \in \mathbb{N}$, the groups $GL_m(\mathbb{R})$ compatibly deformation retract onto the groups $O_m(\mathbb{R})$ by Gram-Schmidt orthogonalization. To justify $(\star)$, we use the following model for the classifying space (see [AE17, §5.1]): sectionwise, $B(-)$ is given by the geometric realization of the bar construction. Thus we have that

$$BG = \mathsf{colim}_{\Delta^{\mathsf{op}}}\left(G \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; G \times G \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\; G \times G \times G \cdots\right).$$

But since $r_{\mathbb{R}}$ commutes with finite products and colimits as we have seen, it therefore commutes with this model for the classifying space functor.

Then, $r_{\mathbb{C}}\mathsf{KGL}$ is the $\mathbb{S}^2$-spectrification of $(\mathbb{Z} \times BU, \mathbb{Z} \times BU, \dots)$. But this is already an $\mathbb{S}^2$-spectrum, corresponding to the $\mathbb{S}^1$-spectrum $(\mathbb{Z} \times BU, U, \mathbb{Z} \times BU, U, \dots) = \mathsf{KU}$ by the usual Bott periodicity theorem (see for example [Ati66]). $\square$

Lemma 4.24. *There is an equivalence of spectra $r_{\mathbb{R}}\mathsf{KGL} \simeq 0$.*

Proof. Let us show that $\pi_n(r_{\mathbb{R}}\mathsf{KGL}) = 0$ for all $n \in \mathbb{Z}$. We use Lemma 4.7 again (levelwise computation of the real realization). By the same computation as in Lemma 4.23, but in the real case this time, we have that

$$r_{\mathbb{R}}(K) \simeq \mathbb{Z} \times B\left(\bigcup_{n\in\mathbb{N}^*} GL_n(\mathbb{R})\right) \simeq \mathbb{Z} \times BO$$

since $GL_n(\mathbb{R})$ deformation retracts onto the orthogonal group $O_n(\mathbb{R})$ by Gram-Schmidt orthogonalization as mentioned above, and these deformation retracts are compatible as n varies. It follows that $\pi_n(r_{\mathbb{R}}\mathsf{KGL}) \cong \mathsf{colim}_m\, \pi_{n+m}(\mathbb{Z} \times BO)$. However, it is a classical result that the homotopy groups of $\mathbb{Z} \times BO$ are given by $\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0$ repeating with period 8 (see for example [nLa25c]). Thus, there is a cofinal subsequence of zeroes appearing in the colimit, and therefore this colimit must be zero. $\square$

Lemma 4.25. *There is an equivalence of spectra $r_{\mathbb{C}}\mathsf{kgl} \simeq \mathsf{KU}_{\geq 0} =: \mathsf{ku}$.*

Proof. Since kgl is by definition very effective, its real realization is connective, and thus we have the following commutative diagram

$$\begin{tikzcd} r_{\mathbb{C}}\mathsf{KGL} \arrow[r, "\simeq"] & \mathsf{KU} \\ r_{\mathbb{C}}\mathsf{kgl} \arrow[u] \arrow[r, dashed, "\gamma"] & \mathsf{KU}_{\geq 0} \arrow[u] \end{tikzcd}$$

where the top horizontal equivalence is that of Lemma 4.23, and both vertical maps are the natural maps for the very effective cover, respectively connective cover. Our goal is to show that γ is an equivalence. By [ARØ20, Prop. 2.7], there is a cofiber sequence $T \wedge \mathsf{kgl} \to \mathsf{kgl} \to \mathsf{H}\mathbb{Z}$ induced by the periodicity generator β_{KGL} for KGL (which induces $\beta_{\mathsf{kgl}} : T \to \mathsf{kgl}$) (analog to the cofiber sequence $\mathbb{S}^2 \wedge \mathsf{KU}_{\geq 0} \to \mathsf{KU}_{\geq 0} \to \mathsf{H}\mathbb{Z}$ in Sp induced by the classical Bott element). Also, the *complex* realization of the motivic Eilenberg-Mac Lane spectrum is the classical Eilenberg-Mac Lane spectrum $\mathsf{H}\mathbb{Z}$ by [Lev14, in particular Thm 5.5]. Since complex realization preserves cofiber sequences, we obtain a long exact sequence

$$\cdots \longrightarrow \pi_{n+1}(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}) \longrightarrow \pi_{n-2}(r_{\mathbb{R}}\mathsf{kgl}) \longrightarrow \pi_n(r_{\mathbb{R}}\mathsf{kgl}) \longrightarrow \pi_n(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}) \longrightarrow \cdots .$$

By connectivity of $r_{\mathbb{C}}\mathsf{kgl}$, setting $n = 0$ or $n = 1$, we obtain isomorphisms $\pi_0(r_{\mathbb{C}}\mathsf{kgl}) \cong \pi_0(r_{\mathbb{C}}\mathsf{H}\mathbb{Z}) \cong \mathbb{Z}$ and $\pi_1(r_{\mathbb{C}}\mathsf{kgl}) \cong \pi_1(r_{\mathbb{C}}\mathsf{H}\mathbb{Z}) \cong 0$. Then, since $\pi_n(r_{\mathbb{C}}\mathsf{H}\mathbb{Z}) = 0$ for all $n \geq 1$, we obtain isomorphisms $\pi_{n-2}(r_{\mathbb{C}}\mathsf{kgl}) \cong \pi_n(r_{\mathbb{C}}\mathsf{kgl})$ for all $n \geq 2$, induced by multiplication by the complex realization of β_{kgl}. This tells us that $\pi_*(r_{\mathbb{C}}\mathsf{kgl})$ is a polynomial ring on one generator β (corresponding to $r_{\mathbb{C}}(\beta_{\mathsf{kgl}})$) in degree 2. We know that $\pi_*(\mathsf{KU}_{\geq 0})$ has the same ring structure. Therefore, to show that γ is an equivalence, we only have to show that it sends β to a generator in degree 2 for $\pi_*(\mathsf{KU}_{\geq}0)$. But by construction, $r_{\mathbb{C}}(\beta_{\mathsf{kgl}})$ maps under the vertical map on the left-hand side of the diagram to $r_{\mathbb{C}}(\beta_{\mathsf{KGL}})$. Since $\beta_{\mathsf{KGL}} : T \wedge \mathsf{KGL} \to \mathsf{KGL}$ is an equivalence, multiplication by $r_{\mathbb{C}}(\beta_{\mathsf{KGL}})$ induces isomorphisms $\pi_{k-2}(r_{\mathbb{C}}\mathsf{KGL}) \cong \pi_k(r_{\mathbb{C}}\mathsf{KGL})$ for all $k \in \mathbb{Z}$. When k is even, these groups are both isomorphic to $\mathbb{Z}$. In particular, $r_{\mathbb{C}}(\beta_{\mathsf{KGL}})$ generates $\pi_2(r_{\mathbb{C}}\mathsf{KGL})$ and thus $r_{\mathbb{C}}(\beta_{\mathsf{kgl}})$ is mapped to a generator of $\pi_2(\mathsf{KU}_{\geq 0})$ by γ, as desired (the natural map for the connective cover induces an isomorphism in homotopy in non-negative degrees). □

Proposition 4.26. *The real realization $r_{\mathbb{R}}(\mathsf{kgl} \to \mathsf{H}\mathbb{Z})$ of the cofiber of the map $T \wedge \mathsf{kgl} \to \mathsf{kgl}$ induced by the periodicity generator β_{KGL} for KGL is equivalent as a map of $\mathcal{E}_1$-rings to the map of free $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}/2$-algebras $\mathsf{H}\mathbb{Z}/2[t^4] \to \mathsf{H}\mathbb{Z}/2[t^2]$, sending t^4 to $(t^2)^2$ (see Definition 1.37).*

Proof. We consider the cofiber sequence $T \wedge \mathsf{kgl} \to \mathsf{kgl} \to \mathsf{H}\mathbb{Z}$ of the statement (see [ARØ20, Prop. 2.7]). We computed in Subsection 4.3.1 that $r_{\mathbb{R}}(\mathsf{H}\mathbb{Z}) \simeq \mathsf{H}\mathbb{Z}/2[t^2]$. We claim that $r_{\mathbb{R}}(\beta_{\mathsf{kgl}}) = 0$, which we will prove at the end. Using this fact, the long exact sequence induced in homotopy after real realization splits as short exact sequences

$$0 \longrightarrow \pi_n(r_{\mathbb{R}}\mathsf{kgl}) \longrightarrow \pi_n(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}) \longrightarrow \pi_{n-2}(r_{\mathbb{R}}\mathsf{kgl}) \longrightarrow 0$$

for all $n \in \mathbb{Z}$. For n odd, since the middle term vanishes, this implies that all odd homotopy groups of $r_{\mathbb{R}}\mathsf{kgl}$ vanish as well. We know that $r_{\mathbb{R}}\mathsf{kgl}$ is connective, so for $n = 0$, we obtain an isomorphism $\pi_0(r_{\mathbb{R}}\mathsf{kgl}) \to \pi_0(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}) \cong \mathbb{Z}/2$. For $n = 2$, the map on the right of the sequence is therefore a surjection $\mathbb{Z}/2 \to \mathbb{Z}/2$, so it is an isomorphism, and $\pi_2(r_{\mathbb{R}}\mathsf{kgl}) = 0$. Then for $n = 4$ we obtain an isomorphism $\pi_n(r_{\mathbb{R}}\mathsf{kgl}) \to \pi_n(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}) \cong \mathbb{Z}/2$. Continuing this argument inductively, $\mathsf{kgl} \to \mathsf{H}\mathbb{Z}$ induces after real realization isomorphisms in homotopy in degrees divisible by 4, and the other homotopy groups of $r_{\mathbb{R}}\mathsf{kgl}$ vanish. The fact that $r_{\mathbb{R}}\mathsf{kgl} \simeq \mathsf{H}\mathbb{Z}/2[t^4]$ as $\mathcal{E}_1$-rings is then proven exactly as in Subsection 4.3.1 for $r_{\mathbb{R}}\mathsf{H}\mathbb{Z}$. In this construction, since $\mathsf{kgl} \to \mathsf{H}\mathbb{Z}$ is an $\mathcal{E}_\infty$-map, using the Hopkins–Mahowald theorem, its realization becomes a map of $\mathcal{E}_2$-rings under $\mathsf{H}\mathbb{Z}/2$. In particular, it is a map of $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}/2$-algebras, and then $\mathsf{H}\mathbb{Z}/2[t^4]$ being a free object implies that the real realization of $\mathsf{kgl} \to \mathsf{H}\mathbb{Z}$ is exactly the map of $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}/2$-algebras in the statement.

To finish the proof, we are left to show our claim that $r_{\mathbb{R}}(\beta_{\mathsf{kgl}}) = 0$. In the remainder of the proof we abbreviate $\beta := \beta_{\mathsf{kgl}}$. Note that the map $T \wedge \mathsf{kgl} \to \mathsf{kgl}$ induces after real realization multiplication by $r_{\mathbb{R}}(\beta)$ in homotopy, where $r_{\mathbb{R}}(\beta)$ is viewed as an element in $\pi_1(r_{\mathbb{R}}\mathsf{kgl})$ (because $\beta : T \to \mathsf{kgl}$ and $r_{\mathbb{R}}(T) = \mathbb{S}^1$). The goal is then to show that $r_{\mathbb{R}}(\beta) \cdot 1 = 0$ where $1 \in \pi_0(r_{\mathbb{R}}\mathsf{kgl})$, or in other terms, that the map $\pi_0(r_{\mathbb{R}}\mathsf{kgl}) \to \pi_1(r_{\mathbb{R}}\mathsf{kgl})$ induced by β is the zero map. To do so, consider the cofiber sequence appearing as the last row in the following diagram, obtained by the octahedral axiom

$$\begin{tikzcd}
\Sigma^{4,2}\mathsf{kgl} \arrow[r, equal] \arrow[d, "\Sigma^{2,1}\beta"'] & \Sigma^{4,2}\mathsf{kgl} \arrow[r, "\Sigma^{2,1}\beta"] \arrow[d, "\beta^2"'] & \Sigma^{2,1}\mathsf{kgl} \arrow[d, "\beta"'] \\
\Sigma^{2,1}\mathsf{kgl} \arrow[r, "\beta"] \arrow[d, "\Sigma^{2,1}\gamma"'] & \mathsf{kgl} \arrow[r, equal] \arrow[d] & \mathsf{kgl} \arrow[d, "\gamma"'] \\
\Sigma^{2,1}\mathsf{H}\mathbb{Z} \arrow[r, dashed] & \mathsf{kgl}/\beta^2 \arrow[r, dashed] & \mathsf{H}\mathbb{Z}.
\end{tikzcd}$$

Then we have a commutative diagram

$$\begin{tikzcd}
 & & \pi_0(r_{\mathbb{R}}\mathsf{kgl}) \arrow[r, "\beta"] \arrow[d, "\cong"] & \pi_1(r_{\mathbb{R}}\mathsf{kgl}) \arrow[d, "\cong"] & \\
\cdots \arrow[r] & \pi_2(r_{\mathbb{R}}(\mathsf{kgl}/\beta)) \arrow[r, "\partial"] & \pi_0(r_{\mathbb{R}}(\mathsf{kgl}/\beta)) \arrow[r] & \pi_1(r_{\mathbb{R}}(\mathsf{kgl}/\beta^2)) \arrow[r] & \cdots
\end{tikzcd}$$

where the bottom row is the long exact sequence induced by this cofiber sequence after realization, and the square comes from the left bottom square in the previous diagram, which commutes by the octahedral axiom.

- We claim that it suffice to show that ∂ is surjective; in other terms, that it hits $1 \in \pi_0(\mathsf{kgl}/\beta) \cong \mathbb{Z}/2$.

  Indeed, note that in the beginning of the proof, we did not use that $r_{\mathbb{R}}(\beta) = 0$ to obtain the isomorphisms $\pi_0(r_{\mathbb{R}}\mathsf{kgl}) \cong \pi_0(r_{\mathbb{R}}(\mathsf{kgl}/\beta)) \cong \pi_0(r_{\mathbb{R}}(\mathsf{H}\mathbb{Z})) \cong \mathbb{Z}/2$. So we can already use this fact, and in particular it tells us that the vertical map on the left-hand side in the diagram is an isomorphism. The vertical map on the right-hand side is an isomorphism because in the long exact sequence induced by the real realization of the cofiber sequence $\Sigma^{4,2}\mathsf{kgl} \xrightarrow{\beta^2} \mathsf{kgl} \to \mathsf{kgl}/\beta^2$, the groups $\pi_i(r_{\mathbb{R}}(\Sigma^{4,2}\mathsf{kgl})) \cong \pi_{i-2}(r_{\mathbb{R}}\mathsf{kgl})$ vanish for $i = 0, 1$ by connectivity of $r_{\mathbb{R}}\mathsf{kgl}$. Thus, by exactness, if ∂ is surjective, then β is the zero map.

- In view of the previous bullet point, it suffices to show that $\partial(t^2) = 1$ where t^2 is viewed as an element in $\pi_*(r_{\mathbb{R}}(\mathsf{kgl}/\beta)) \cong \pi_*(r_{\mathbb{R}}(\mathsf{H}\mathbb{Z})) \cong \mathbb{Z}/2[t^2]$. To do so, we reduce to a computation in the homotopy of $\mathsf{H}\mathbb{Z}/2 \in \mathsf{SH}(k)$. We have proven in Proposition 4.21 that the quotient map $\mathsf{H}\mathbb{Z} \to \mathsf{H}\mathbb{Z}/2$ induces an isomorphism in even homotopy groups after real realization. Let $d : \mathsf{H}\mathbb{Z} \to \Sigma^{3,1}\mathsf{H}\mathbb{Z}$ be the boundary map which represents ∂ before real realization. We claim firstly that the following diagram is commutative

$$\begin{tikzcd}
\mathsf{H}\mathbb{Z} \arrow[r, "d"] \arrow[d, "\mathsf{mod}\,2"'] & \Sigma^{3,1}\mathsf{H}\mathbb{Z} \arrow[d, "\mathsf{mod}\,2"] \\
\mathsf{H}\mathbb{Z}/2 \arrow[r, "\mathsf{Sq}^3"] & \Sigma^{3,1}\mathsf{H}\mathbb{Z}/2
\end{tikzcd}$$

  where Sq^3 is the third motivic Steenrod square (see [Voe03]), and secondly that $\mathsf{Sq}^3(\tau^2) = \rho^3$.

- Let us see how the claims allow us to finish the proof, that is, how they imply that $\partial(t^2) = 1$. Indeed, $t^2 \in \pi_2(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}/2) \cong \mathsf{colim}_n\ \underline{\pi}_2(\mathsf{H}\mathbb{Z}/2)_n(\mathbb{R})$ corresponds to $\rho^{-2}\tau^2 \in \underline{\pi}_2(\mathsf{H}\mathbb{Z}/2)_0(\mathbb{R})$ as we saw, or equivalently, in the colimit, to $\tau^2 \in \underline{\pi}_2(\mathsf{H}\mathbb{Z}/2)_2(\mathbb{R})$. It is therefore mapped to $\rho^3 \in \underline{\pi}_0(\mathsf{H}\mathbb{Z}/2)_3(\mathbb{R})$, corresponding in the colimit $\mathsf{colim}_n\ \underline{\pi}_0(\mathsf{H}\mathbb{Z}/2)_n(\mathbb{R}) \cong \pi_0(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}/2)$ to $1 \in \underline{\pi}_0(\mathsf{H}\mathbb{Z}/2)_0(\mathbb{R})$ and thus a generator of $\mathbb{Z}/2 \cong \pi_0(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}/2)$. This suffices because in the commutative square above, the vertical maps induce isomorphisms on the homotopy groups in even degrees after real realization.

- To prove the first claim, we use that by [RØ16, Lemma A.4], we have

$$[\mathsf{H}\mathbb{Z}, \Sigma^{3,1}\mathsf{H}\mathbb{Z}/2] = H^{0,0}(\mathsf{Spec}(\mathbb{R}), \mathbb{Z}/2)\{\mathsf{Sq}^3 \circ (\mathsf{H}\mathbb{Z} \to \mathsf{H}\mathbb{Z}/2)\} = \mathbb{Z}/2\{\mathsf{Sq}^3 \circ (\mathsf{H}\mathbb{Z} \to \mathsf{H}\mathbb{Z}/2)\}$$

  because $H^{0,0}(\mathsf{Spec}(\mathbb{R}), \mathbb{Z}/2) \cong \underline{K}_0^M/2(\mathbb{R}) \cong \mathbb{Z}/2$ as we have seen. Then, $(\mathsf{mod}\,2) \circ d$ is either $\mathsf{Sq}^3 \circ (\mathsf{mod}\,2)$ or 0. To distinguish between them, we take complex realizations. The complex realization of the sequence $\Sigma^{2,1}\mathsf{kgl} \to \mathsf{kgl} \xrightarrow{\gamma} \mathsf{H}\mathbb{Z}$ is the sequence $\Sigma^2\mathsf{ku} \to \mathsf{ku} \to \mathsf{H}\mathbb{Z}$ (see the proof of Lemma 4.25). Therefore, the complex realization of d is exactly the suspension of the map $\Sigma^{-1}\mathsf{H}\mathbb{Z} \to \Sigma^2\mathsf{H}\mathbb{Z}$ called the first k-invariant (defined in Remark 5.4) of ku. This map is the non-zero operation Q_1 (whose reduction $\mathsf{mod}\, 2$ is non-zero as well), see [Bru, Section 2] for a proof and more details.

- The last step to finish the proof is to show our second claim, namely that $\mathsf{Sq}^3(\tau^2) = \rho^3$. By [Voe03], we have the following properties of the motivic Steenrod squares: firstly, $\mathsf{Sq}^2(\tau) = 0$ since $\mathsf{Sq}^{2n}(u) = 0$ for any u of bidegree (p, q) with $n > p-q$ and $n \geq q$, and then $\mathsf{Sq}^3(\tau) = \mathsf{Sq}^1\mathsf{Sq}^2(\tau) = 0$. Secondly, $\mathsf{Sq}^1 = \beta$ and $\beta\tau = \rho$. Finally, we have a motivic Cartan relation
$$\mathsf{Sq}^{2i+1}(u \smile v) = \sum_{r=0}^{2i+1} \mathsf{Sq}^r(u) \smile \mathsf{Sq}^{2i+1-r}(v) + \rho \sum_{r=0}^{i-1} \mathsf{Sq}^{2r+1}(u) \smile \mathsf{Sq}^{2i-2r-1}(v).$$
Using these facts, we compute
$$\begin{aligned}\mathsf{Sq}^3(\tau^2) &= \mathsf{Sq}^0(\tau)\mathsf{Sq}^3(\tau) + \mathsf{Sq}^1(\tau)\mathsf{Sq}^2(\tau) + \mathsf{Sq}^2(\tau)\mathsf{Sq}^1(\tau) + \mathsf{Sq}^3(\tau)\mathsf{Sq}^0(\tau) + \rho\mathsf{Sq}^1(\tau)\mathsf{Sq}^1(\tau)\\ &= \rho(\beta\tau)^2\\ &= \rho^3,\end{aligned}$$
as desired.

□

4.3.3 The realization of $\widetilde{\mathsf{H}\mathbb{Z}}$

We can compute another example which will be useful to us in Section 5: the real realization of $\widetilde{\mathsf{H}\mathbb{Z}} := f_0\underline{K}_*^{MW}$, because it appears in a description of the zeroth very effective slice of ko (see Proposition 5.2).

Proposition 4.27. *On the homotopy rings, the real realization of the quotient map*
$$\widetilde{\mathsf{H}\mathbb{Z}} := f_0\underline{K}_*^{MW} \longrightarrow f_0(\underline{K}_*^{MW}/\eta) = f_0\underline{K}_*^{M} = \mathsf{H}\mathbb{Z}$$
identifies with the quotient map $\mathbb{Z}[t^2]/(2t^2) \to \mathbb{Z}[t^2]/2$.

Proof. The homotopy ring of $r_\mathbb{R}\mathsf{H}\mathbb{Z}$ was computed in Proposition 4.21. By [Bac17, Lemma 17], the quotient map in the statement induces isomorphisms on homotopy sheaves $\underline{\pi}_n(-)_*$ for all $n \neq 0$. In particular, for all $n \neq 0$, $\pi_n(r_\mathbb{R}\widetilde{\mathsf{H}\mathbb{Z}}) \cong \mathsf{colim}_m\underline{\pi}_n(\widetilde{\mathsf{H}\mathbb{Z}})_m(\mathbb{R}) \cong \mathsf{colim}_m\underline{\pi}_n(\mathsf{H}\mathbb{Z})_m(\mathbb{R}) \cong \pi_n(r_\mathbb{R}\mathsf{H}\mathbb{Z})$. Since $\widetilde{\mathsf{H}\mathbb{Z}}$ is very effective by Proposition 4.18, its real realization is connective and we are only left with the zeroth homotopy group to deal with. Consider the following commutative diagram
$$\begin{array}{ccc}\widetilde{\mathsf{H}\mathbb{Z}} = f_0\underline{K}_*^{MW} & \longrightarrow & \underline{K}_*^{MW}\\ \big\downarrow & & \big\downarrow\\ \mathsf{H}\mathbb{Z} = f_0\underline{K}_*^{M} & \longrightarrow & \underline{K}_*^{M}.\end{array}$$
By [Bac17, Lemma 17] again, the top horizontal map induces isomorphisms on the homotopy sheaves $\underline{\pi}_0(-)_*$, but these homotopy sheaves are then also that of the sphere spectrum (Theorem 3.20). In particular, arguing as above, $\pi_0(r_\mathbb{R}\widetilde{\mathsf{H}\mathbb{Z}}) \cong \pi_0(r_\mathbb{R}\underline{K}_*^{MW}) \cong \pi_0(r_\mathbb{R}\mathcal{S}) = \pi_0(\mathbb{S}) = \mathbb{Z}$. We therefore have in degree 0 a ring map $\mathbb{Z} \to \mathbb{Z}/2$, which must be the quotient by 2 map. This suffices to conclude our proof, because we have determined $\pi_*(r_\mathbb{R}\widetilde{\mathsf{H}\mathbb{Z}}) \to \pi_*(r_\mathbb{R}\mathsf{H}\mathbb{Z})$ in every degree, we know the ring structure on the right hand-side, and we know the map is a ring homomorphism. □

4.3.4 The realization of KO

We are finally getting closer to our main focus, namely Hermitian K-theory. We can identify the real realization of KO:

Proposition 4.28 ([BH20, Lemma 3.9])**.** *There is an equivalence of spectra* $r_\mathbb{R}\mathsf{KO} \simeq \mathsf{KO}^{\mathsf{top}}[1/2]$.

Proof. Recall the Wood cofiber sequence

$$\Sigma^{1,1}\mathsf{KO} \xrightarrow{\eta} \mathsf{KO} \longrightarrow \mathsf{KGL}$$

from Theorem 3.17(*iv*). Also note that in Milnor–Witt K-theory, we have $h\rho^2 = 0$. Indeed, by [Mor12, Cor. 3.8], Milnor–Witt K-theory is "$(1-h)$-graded commutative", so $\rho^2 = (1-h)p^2$.

Therefore, after inverting ρ, $h = 2 + \rho\eta$ becomes 0 and thus after real realization $\rho\eta = -2$, so that the real realization of η is inducing multiplication by -2 in homotopy. But also, we have seen in Lemma 4.24 that $r_{\mathbb{R}}\mathsf{KGL} \simeq 0$. This means that the cofiber of the multiplication by 2 map on $r_{\mathbb{R}}\mathsf{KO}$ is zero, in other terms $\cdot 2$ is an equivalence, and thus $r_{\mathbb{R}}\mathsf{KO} \simeq r_{\mathbb{R}}(\mathsf{KO}[1/2])$ via the canonical localization map. But now $r_{\mathbb{R}}(\mathsf{KO}[1/2])$ identifies with $\mathsf{KO}[\rho^{-1}, 1/2] \simeq \mathsf{KO}[\rho^{-1}, 1/2, \eta^{-1}]$ (in view of the relation $\rho\eta = -2$) under the equivalence $\mathsf{SH}(\mathbb{R})[\rho^{-1}] \simeq \mathsf{Sp}$ (Theorem 4.10). Let $\mathsf{KW} = \mathsf{KO}[\eta^{-1}]$. The latter is also an object of independent interest; it is a $(1,1)$- and $(4,0)$-periodic motivic spectrum which represents *Balmer–Witt K-theory* (see for example [Hor05, Appendix A]). It is often also denoted by KT.

Then $r_{\mathbb{R}}\mathsf{KO}$ identifies with $\mathsf{KO}[\eta^{-1}][1/2][\rho^{-1}]$, which in turn identifies with $r_{\mathbb{R}}(\mathsf{KW}[1/2])$. By [Rön18, Thm 4.4], the latter is equivalent as a spectrum to $\mathsf{KO}^{\mathsf{top}}[1/2]$. This can be viewed as a version of the so-called Brumfiel's theorem (see for example [KSW16, Thm 6.2]). This suffice to prove the statement, however we will upgrade this to an equivalence of $\mathcal{E}_\infty$-rings in Theorem 5.6, studying the ring spectrum $\mathsf{L}(\mathbb{R})[1/2]$ as an intermediate step. $\square$

Actually, by [ARØ20, Prop. 2.11], the Wood cofiber sequence used in the proof refines to a cofiber sequence of very effective covers

$$\Sigma^{1,1}\mathsf{ko} \xrightarrow{\eta} \mathsf{ko} \longrightarrow \mathsf{kgl}.$$

In the proof above, 2 being invertible on the realization greatly simplifies our task. But here, since $r_{\mathbb{R}}\mathsf{kgl} \not\simeq 0$ (Proposition 4.26), the same strategy does not work (and indeed, we will see that 2 is not invertible on $r_{\mathbb{R}}\mathsf{ko}$ (Proposition 5.1)). But we can localize away from 2, and then with the same proof strategy we obtain the following result:

Proposition 4.29. *There is an equivalence of spectra $r_{\mathbb{R}}(\mathsf{ko}[1/2]) \simeq \mathsf{ko}^{\mathsf{top}}[1/2]$.*

Proof. As in the proof of Proposition 4.28, it suffices to compute the realization of $\mathsf{ko}[\eta^{-1}][1/2]$. The point here is that $\mathsf{ko}[\eta^{-1}] = \mathsf{KW}_{\geq 0}$ (see [BH20, Ex. 4.2 and §1.2]), and then the claim follows as in the previous proof, where we saw $r_{\mathbb{R}}(\mathsf{KW}[1/2]) \simeq \mathsf{KO}^{\mathsf{top}}[1/2]$ as spectra. Indeed, $r_{\mathbb{R}}$ is right t-exact for the homotopy t-structure: it is exact and sends the non-negative part of the homotopy t-structure to connective spectra, as can be seen by combining Definition 2.18 and Lemma 4.13. Again, we will upgrade the equivalence of spectra in the statement to one of $\mathcal{E}_\infty$-rings in Theorem 5.6. $\square$

5 The fracture square for the real realization of ko

This section contains our first main contribution: the identification of $r_{\mathbb{R}}\mathsf{ko}$ as an $\mathcal{E}_1$-ring. Indeed, recall that, by the results of Subsection 4.1, the spectrum $r_{\mathbb{R}}\mathsf{ko}$ inherits an $\mathcal{E}_\infty$-ring structure from that of ko (and thus ultimately from that of KO). We will discuss a bit later why our strategy only allows us to identify the $\mathcal{E}_1$-ring structure and not the $\mathcal{E}_\infty$-ring one.

A first challenge in this computation, in comparison to the examples from Subsection 4.3, is that we have no useful description of the pointed motivic spaces appearing in the sequence that defines the motivic spectrum ko. Indeed, even with the knowledge of those of KO, taking the very effective cover may completely change them. A second challenge is that we have very little information about the homotopy sheaves of ko. On the other hand, we do have some grasp on ko thanks to the Wood cofiber sequence $\Sigma^{1,1}\mathsf{ko} \xrightarrow{\eta} \mathsf{ko} \xrightarrow{c} \mathsf{kgl}$ (recalled in the proof of Proposition 4.29). Moreover, the very effective slices of ko have been computed in [Bac17], and their description (see Theorem 5.2 below) involves only motivic spectra whose real realization we already determined, see Subsection 4.3. This will allow us to compute the homotopy ring of $r_{\mathbb{R}}\mathsf{ko}$ in Subsection 5.1. We prove that it is a polynomial ring over $\mathbb{Z}$ with a single generator in degree 4.

But this is of course not enough to identify $r_{\mathbb{R}}\mathsf{ko}$ as an $\mathcal{E}_1$-ring. Indeed, we have already seen an example of $\mathcal{E}_1$-ring with a similar homotopy ring in Subsection 1.5, namely the free $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}$-algebra $\mathsf{H}\mathbb{Z}[t^4]$. However, as we will see in Remark 5.4, they are actually not the same.

To deal with the $\mathcal{E}_1$-ring structure, we therefore adopt another strategy in Subsection 5.2. Recall the result mentioned in the proof Proposition 4.29 that the real realization of $r_{\mathbb{R}}(\mathsf{ko})[1/2]$ is $\mathsf{ko}^{\mathsf{top}}[1/2]$ *as an $\mathcal{E}_\infty$-ring.* We will finish the proof of the latter in Subsection 5.2, but in the meantime, this is still a hint in the direction of considering a 2-local fracture square. Indeed, such a square expresses $r_{\mathbb{R}}\mathsf{ko}$ in terms of various localizations: it is the pullback as $\mathcal{E}_\infty$-rings of the localization away from 2 $r_{\mathbb{R}}(\mathsf{ko})[1/2]$ and the localization at (2) $r_{\mathbb{R}}(\mathsf{ko})_{(2)}$, over the rationalization $r_{\mathbb{R}}(\mathsf{ko})_{\mathbb{Q}}$ (see Proposition A.6 in the Appendix, and Subsection 5.2 below). We already identified one term in this pullback. Our approach will be to try to compute the two other terms. However, the identification of the localization at (2) turns out to be particularly challenging. We will only be able to identify it as an $\mathcal{E}_1$-ring instead of an $\mathcal{E}_\infty$-ring, because we will use the description of free $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}_{(2)}$-algebras and their homotopy rings from Proposition 1.38. Indeed, they have polynomial homotopy rings, whereas the $\mathcal{E}_\infty$-case is more complicated. But even in this simpler case, we will still have to make $r_{\mathbb{R}}(\mathsf{ko})_{(2)}$ an $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}_{(2)}$-algebra. As we will see in Proposition 5.3, this will involve studying the real realization of the special linear cobordism spectrum MSL, and the formalism of Thom spectrum functors. We will dedicate Section 6 to this. This construction only allows us to produce a map of $\mathcal{E}_2$-rings $\mathsf{H}\mathbb{Z}_{(2)} \to r_{\mathbb{R}}(\mathsf{ko})_{(2)}$, and thus only an $\mathcal{E}_1$-map of $\mathsf{H}\mathbb{Z}_{(2)}$ modules. This is another reason for which we only deal with the $\mathcal{E}_1$-structure instead of the $\mathcal{E}_\infty$ one.

After the more explicit identification of the spectra appearing in the fracture square, and a carefully analysis of the maps in the square (which involves computations with spectral sequences and various tricks), we will be able to compare the real realization of ko and the L-theory spectrum of $\mathbb{R}$ in Subsection 5.3. More precisely, it has been shown in [HLN21] that $\mathsf{L}(\mathbb{R})$ fits into a similar fracture square; this has inspired a lot our computations. Finally, by comparing the two fracture squares, we will be able to conclude that $r_{\mathbb{R}}\mathsf{ko}$ is equivalent as an $\mathcal{E}_1$-ring to the connective cover of the L-theory spectrum of the real numbers $\mathsf{L}(\mathbb{R})_{\geq 0}$ (Definition 3.21).

5.1 The homotopy ring of $r_{\mathbb{R}}\mathsf{ko}$

The first step of our identification of $r_{\mathbb{R}}\mathsf{ko}$ is to compute its homotopy groups and their graded ring structure. We proceed in four steps (the numbering matches the one in the proof of the proposition below):

1. We first show that these homotopy groups are finitely generated in every degree, using the description of the very effective slices of ko.
2. Since these groups are abelian, we may analyze their torsion part and their free part separately. To begin with, we show that they have no 2-torsion, using the Wood cofiber sequence.
3. We then show that they have no torsion at the other primes, and that they are actually free of

rank 1 in every non-negative degree divisible by 4, and 0 else. To do so, we reduce to a statement after inverting 2, and use that we already have computed $r_{\mathbb{R}}\mathsf{ko}[1/2]$ (as a spectrum).

4. Finally, we describe the ring structure, studying again the Wood cofiber sequence.

Here is the result:

Proposition 5.1. *The homotopy ring of $r_{\mathbb{R}}\mathsf{ko}$ is given by*

$$\pi_*(r_{\mathbb{R}}\mathsf{ko}) \cong \mathbb{Z}[x]$$

where x is a generator in degree 4, mapping to a generator of $\pi_4(r_{\mathbb{R}}\mathsf{kgl}) \cong \mathbb{Z}/2$ in the Wood cofiber sequence $\Sigma^{1,1}\mathsf{ko} \to \mathsf{ko} \to \mathsf{kgl}$ (Subsection 4.3.4, before Proposition 4.29).

Proof. **Step 1:** The homotopy groups of $r_{\mathbb{R}}\mathsf{ko}$ are finitely generated abelian groups. We will use the very effective slices (Subsection 2.3) of ko. Indeed, we claim that the realizations of these slices have degreewise finitely generated homotopy groups (call this claim $(\star)$). Then, we show by induction on $m \geq -1$ the following statement: for all $n \in \mathbb{N}$, $\pi_{n+m}(r_{\mathbb{R}}(\tilde{f}_n\mathsf{KO}))$ is finitely generated. This holds for $m \leq -1$ since real realization on $\mathsf{SH}(\mathbb{R})^{\mathsf{veff}}(n)$ takes values in n-connective spectra as seen in Lemma 4.8. This implies our result because for $n = 0$, we get that for all $m \in \mathbb{N}$, $\pi_m(r_{\mathbb{R}}(\tilde{f}_0\mathsf{KO})) = \pi_m(r_{\mathbb{R}}\mathsf{ko})$ is finitely generated. Assume that the induction hypothesis holds for some fixed $m \geq -1$. Then for all $n \in \mathbb{N}$, the cofiber sequence $\tilde{f}_{n+1}\mathsf{KO} \to \tilde{f}_n\mathsf{KO} \to \tilde{s}_n\mathsf{KO}$ defining the very effective slices induces a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r, "a"] & \pi_{n+1+m}(r_{\mathbb{R}}(\tilde{f}_{n+1}\mathsf{KO})) \arrow[r] & \pi_{n+m+1}(r_{\mathbb{R}}(\tilde{f}_n\mathsf{KO})) \arrow[r] & \pi_{n+m+1}(r_{\mathbb{R}}(\tilde{s}_n\mathsf{KO})) \arrow[dll, "b"'] \\ & \pi_{n+m}(r_{\mathbb{R}}(\tilde{f}_{n+1}\mathsf{KO})) \arrow[r] & \pi_{n+m}(r_{\mathbb{R}}(\tilde{f}_n\mathsf{KO})) \arrow[r] & \cdots \end{tikzcd}$$

and therefore $\pi_{n+m+1}(r_{\mathbb{R}}(\tilde{f}_n\mathsf{KO}))$ is an extension of $\mathsf{ker}(b)$ and $\mathsf{coker}(a)$. The former is finitely generated because it is a subgroup of $\pi_{n+m+1}(r_{\mathbb{R}}(\tilde{s}_n\mathsf{KO}))$, which is finitely generated by $(\star)$. The latter is also finitely generated since it is a quotient of $\pi_{n+1+m}(r_{\mathbb{R}}(\tilde{f}_{n+1}\mathsf{KO}))$, which is finitely generated by induction hypothesis. Therefore, $\pi_{n+m+1}(r_{\mathbb{R}}(\tilde{f}_n\mathsf{KO}))$ is finitely generated, finishing our proof by induction.

Before passing to the next step of the proof, we still have to show that $(\star)$ holds. The very effective slices of ko are known:

Theorem 5.2 ([Bac17, Thm 16])**.** *The very effective slices of KO (equivalently, ko), are given by $\tilde{s}_n\mathsf{KO} \simeq \tilde{s}_n\mathsf{ko} \simeq \Sigma^{2n,n}\tilde{S}_{n \bmod 4}$ where*

$$\tilde{S}_0 = \tilde{s}_0\mathsf{KO}, \quad \tilde{S}_1 = \mathsf{H}\mathbb{Z}/2, \quad \tilde{S}_2 = \mathsf{H}\mathbb{Z}, \quad \tilde{S}_3 = 0.$$

Moreover, the cofiber sequence

$$s_0(E_{\geq 1}) \longrightarrow \tilde{s}_0(E) \longrightarrow f_0(\underline{\pi}_0(E)_*)$$

of Proposition 2.30 becomes for $E = \mathsf{KO}$

$$\Sigma^{1,0}\mathsf{H}\mathbb{Z}/2 \longrightarrow \tilde{s}_0\mathsf{KO} \longrightarrow \widetilde{\mathsf{H}\mathbb{Z}}.$$

By the computation of the real realizations of $\mathsf{H}\mathbb{Z}$, $\mathsf{H}\mathbb{Z}/2$ and $\widetilde{\mathsf{H}\mathbb{Z}}$ in Subsections 4.3.1 and 4.3.3, we know that their homotopy groups are finitely generated. Thus that of the realizations of very effective slices of ko are too (for $\tilde{s}_0$, we use the same inductive argument as we just did).

Step 2: There is no 2-torsion in $\pi_*(r_{\mathbb{R}}\mathsf{ko})$. Recall the Wood cofiber sequence $\Sigma^{1,1}\mathsf{ko} \xrightarrow{\eta} \mathsf{ko} \xrightarrow{c} \mathsf{kgl}$ seen in the proof of Proposition 4.29, where η is induced by the Hopf map and c is induced by the forgetful map from Hermitian K-theory to algebraic K-theory.

As we computed in the case of KO, the real realization of η is the multiplication by 2 map on the sphere spectrum, and by Proposition 4.24, $r_{\mathbb{R}}\mathsf{kgl}$ is the free $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}/2$-algebra $\mathsf{H}\mathbb{Z}/2[t^4]$. Assume for a contradiction that $\pi_m(r_{\mathbb{R}}\mathsf{ko})$ has 2-torsion for some $m \geq 0$ (and in particular is non-zero). Then in the long exact sequence induced by the Wood cofiber sequence

$$\cdots \longrightarrow \pi_{m+1}(r_{\mathbb{R}}\mathsf{kgl}) \xrightarrow{a} \pi_m(r_{\mathbb{R}}\mathsf{ko}) \xrightarrow{\cdot 2 = b} \pi_m(r_{\mathbb{R}}\mathsf{ko}) \xrightarrow{c} \pi_m(r_{\mathbb{R}}\mathsf{kgl}) \longrightarrow \cdots$$

the kernel of the map $\cdot 2 = b$ is non-trivial, so a is non-zero and $\pi_{m+1}(r_{\mathbb{R}}\mathsf{kgl}) \neq 0$, i.e. $k \equiv 3 \mod 4$. Then $\pi_m(r_{\mathbb{R}}\mathsf{kgl}) = 0$, meaning that multiplication by 2 is surjective on $\pi_m(r_{\mathbb{R}}\mathsf{ko})$. In Step 1, we saw that this group what finitely generated abelian. Using the classification of such groups, this is impossible for non-zero groups possessing 2-torsion.

Step 3: $\pi_k(r_{\mathbb{R}}\mathsf{ko})$ is torsion-free of rank 1 in degrees $m \in 4\mathbb{N}$, and zero otherwise. Indeed, by Proposition 4.29 and since real realization and homotopy groups commutes with localization away from 2 (Proposition 4.14), we have

$$\begin{aligned} \pi_*(r_{\mathbb{R}}\mathsf{ko})[1/2] &\cong \pi_*((r_{\mathbb{R}}\mathsf{ko})[1/2]) \cong \pi_*(r_{\mathbb{R}}(\mathsf{ko}[1/2])) \cong \pi_*(\mathsf{ko}^{\mathsf{top}}[1/2]) \\ &\cong \left(\mathbb{Z}[\alpha_1, \beta_4, \lambda_8] \big/ (\alpha_1^3, 2\alpha_1, \alpha_1\beta_4, \beta_4^2 - 4\lambda_8)\right)[1/2] \cong \mathbb{Z}[1/2][\beta_4] \end{aligned}$$

(where $|\alpha_1| = 1$, $|\beta_4| = 4$, and $|\lambda_8| = 8$) by a classical result in topology, see for instance [Str92, §1.1]. In particular, the homotopy groups of $r_{\mathbb{R}}\mathsf{ko}$ cannot have any p-torsion for $p \neq 2$ either. Otherwise, their localizations at 2 would still have torsion, but $\mathbb{Z}[1/2]$ does not. Therefore, $\pi_*(r_{\mathbb{R}}\mathsf{ko})$ is degreewise free. Comparing with the localization away from 2, we deduce that it consists in a copy of $\mathbb{Z}$ is every degree divisible by 4, and the trivial group else. Indeed, if $\mathbb{Z}^{\oplus m}[1/2] \cong \mathbb{Z}[1/2]$ for some $m \in \mathbb{N}$, then the quotients by the $\mathbb{Z}$-module map $\cdot 3$ on both sides are isomorphic as well. They are respectively $(\mathbb{Z}/3)^{\oplus m}$ and $\mathbb{Z}/3$, whence $m = 1$ for cardinality reasons.

Note however that the comparison with $r_{\mathbb{R}}\mathsf{ko}[1/2]$ does not suffice to describe the ring structure of $\pi_*(r_{\mathbb{R}}\mathsf{ko})$. Indeed, if $x \in \pi_4(r_{\mathbb{R}}\mathsf{ko}) \cong \mathbb{Z}$ denotes a generator, we would obtain a polynomial ring after localization away from 2, regardless of whether x^2 equals a generator for $\pi_8(r_{\mathbb{R}}\mathsf{ko})$, or twice a generator for example.

Step 4: We describe the ring structure. Using again the Wood cofiber sequence, for all $m \geq 0$, we have an exact sequence

$$\cdots \longrightarrow \pi_{4m}(r_{\mathbb{R}}\mathsf{ko}) \xrightarrow{c_{4m}} \pi_{4m}(r_{\mathbb{R}}\mathsf{kgl}) \cong \mathbb{Z}/2 \longrightarrow \pi_{4m-1}(r_{\mathbb{R}}\mathsf{ko}) = 0 \longrightarrow \cdots.$$

Let x be a generator of $\pi_4(r_{\mathbb{R}}\mathsf{ko}) \cong \mathbb{Z}$. Then since c_4 is surjective, $c_4(x)$ generates $\pi_4(r_{\mathbb{R}}\mathsf{kgl}) \cong \mathbb{Z}/2$. The forgetful map $c : \mathsf{ko} \to \mathsf{kgl}$ in the Wood cofiber sequence is a map of $\mathcal{E}_\infty$-rings, as we saw below Definition 3.14, and thus induces a ring homomorphism in homotopy. Therefore, for $m \geq 0$, $c_{4m}(x^m) = c_4(x)^m$ generates $\pi_{4m}(r_{\mathbb{R}}\mathsf{kgl}) \cong \mathbb{Z}/2$ as well (since $\pi_*(r_{\mathbb{R}}\mathsf{kgl})$ is a polynomial ring). This means that x^m generates $\pi_{4m}(r_{\mathbb{R}}\mathsf{ko})$. This holds because if $x^m = a \cdot g_m$ where $a \in \mathbb{Z}$ and g_m is a generator of $\pi_{4m}(r_{\mathbb{R}}\mathsf{ko})$, if a is even then $c_{4m}(x^m) = a \cdot c_{4m}(g_m) = 0 \in \mathbb{Z}/2$ which is a contradiction, and if $|a| > 1$ is odd then after localization away from 2 we do not find back the polynomial ring structure of $\pi_*(\mathsf{ko}^{\mathsf{top}})[1/2]$. Indeed, then $a \cdot g_m$ must be a generator of the degree $4m$ part of $\mathbb{Z}[1/2][\beta_4]$ as a $\mathbb{Z}[1/2]$-module, in particular multiplication by $a \in \mathbb{Z}$ is invertible on $\mathbb{Z}[1/2]$, which is impossible if $|a| > 1$ is odd. □

5.2 Structure as an $\mathcal{E}_1$-ring

Our discussion in Subsection 4.3.4 showed us the interest of considering a 2-local fracture square for $r_{\mathbb{R}}\mathsf{ko}$. Indeed, we were already able to identify $r_{\mathbb{R}}\mathsf{ko}[1/2]$ as a spectrum in Proposition 4.29. The existence of fracture square for $\mathcal{E}_n$-rings is Proposition A.6. As mentioned in the introduction to this section, we will only be able to identify the $\mathcal{E}_1$-ring structure, so we might as well apply straight away the latter result to $n = 1$. We choose $M = r_{\mathbb{R}}\mathsf{ko}$, and $p = 2$, to obtain a pullback square of $\mathcal{E}_1$-rings

$$\begin{tikzcd} r_{\mathbb{R}}\mathsf{ko} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & (r_{\mathbb{R}}\mathsf{ko})[1/2] \arrow[d] \\ (r_{\mathbb{R}}\mathsf{ko})_{(2)} \arrow[r] & (r_{\mathbb{R}}\mathsf{ko})_{\mathbb{Q}}. \end{tikzcd}$$

Therefore, we have to identify $(r_{\mathbb{R}}\mathsf{ko})_{(2)}$ (Subsection 5.2.1), $(r_{\mathbb{R}}\mathsf{ko})_{\mathbb{Q}}$ (Subsection 5.2.2), and $(r_{\mathbb{R}}\mathsf{ko})[1/2]$ (Subsection 5.2.3) as $\mathcal{E}_1$-rings, and describe the maps in the square we will obtain (Subsections 5.2.4 to 5.2.7). However, for the localization at (2), we will have to appeal to non-trivial results regarding the real realization of the special linear cobordism spectrum MSL and the formalism of Thom spectrum functors, whose discussion we postpone to Section 6. We will use Proposition 4.14(i) several times without mentioning it, which allows us to commute real realization with the localizations appearing in the fracture square.

5.2.1 Identifying the localization at (2)

Having computed $\pi_*(r_{\mathbb{R}}\mathsf{ko}) \cong \mathbb{Z}[x]$ in Subsection 5.1, we can immediately deduce from Proposition A.2 that $\pi_*((r_{\mathbb{R}}\mathsf{ko})_{(2)}) \cong \pi_*(r_{\mathbb{R}}\mathsf{ko}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(2)} \cong \mathbb{Z}_{(2)}[x]$ as rings. We want to use this to identify $(r_{\mathbb{R}}\mathsf{ko})_{(2)}$ with the free $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}_{(2)}$-algebra on one generator in degree 4. As mentioned before, to prove this we appeal to a non-trivial result from Section 6.

Proposition 5.3. *There is an equivalence of $\mathcal{E}_1$-rings*

$$\mathsf{H}\mathbb{Z}_{(2)}[t^4] \xrightarrow{\simeq} (r_{\mathbb{R}}\mathsf{ko})_{(2)}$$

sending $t^4 \in \pi_4(\mathsf{H}\mathbb{Z}_{(2)}[t^4])$ to (the image in the localization of) the generator $x \in \pi_4(r_{\mathbb{R}}\mathsf{ko})$.

Proof. By [HJNY22, Thm 10.1], there is an $\mathcal{E}_\infty$-ring morphism $\mathsf{MSL} \to \mathsf{ko}$. The motivic spectrum MSL is the *special linear cobordism spectrum*, and classifies "special linear orientations" of a motivic spectrum. Now, it was proven in [BH20, Cor. 4.7], that the real realization of MSL is equivalent as a topological spectrum to the *oriented Thom spectrum* MSO (which represents the cohomology theory of oriented cobordism). By Theorem 6.32, they are equivalent as $\mathcal{E}_\infty$-rings. By [HLN21, Cor. 3.7], MSO admits after localization at (2) a map of $\mathcal{E}_2$-rings $\mathsf{H}\mathbb{Z}_{(2)} \to \mathsf{MSO}_{(2)}$. This map is obtained using the formalism of Thom spectra. We will study the latter in Section 6, in order to prove Theorem 6.32.

Therefore, we obtain maps of $\mathcal{E}_2$-rings $\mathsf{H}\mathbb{Z}_{(2)} \to r_{\mathbb{R}}(\mathsf{MSL})_{(2)}$ and $\mathsf{H}\mathbb{Z}_{(2)} \to (r_{\mathbb{R}}\mathsf{ko})_{(2)}$ by composition.

Then, by Proposition 1.38(*iii*), to obtain a map of $\mathcal{E}_1$-rings $\mathsf{H}\mathbb{Z}_{(2)}[t] \to (r_{\mathbb{R}}\mathsf{ko})_{(2)}$, it suffices to specify an element of $\pi_4((r_{\mathbb{R}}\mathsf{ko})_{(2)})$. We choose the image in the localization of $x \in \pi_4(r_{\mathbb{R}}\mathsf{ko})$). By Proposition 1.38(*ii*), and since by Proposition A.2, $\pi_*((r_{\mathbb{R}}\mathsf{ko})_{(2)}) \cong \pi_*(r_{\mathbb{R}}\mathsf{ko}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(2)} \cong \mathbb{Z}_{(2)}[x]$, this map induces isomorphisms on the homotopy groups (note that $\mathbb{Z}_{(2)}$ is a module over itself in a unique way), and we have obtained the desired equivalence. □

Remark 5.4. The localization at (2) is crucial here, because MSO is not an $\mathsf{H}\mathbb{Z}$-module, since it has non-trivial k-invariants. The k-invariants of $E \in \mathsf{Sp}$ are the horizontal compositions in the diagram

$$\begin{tikzcd}
\Sigma^{n+2}\mathsf{H}(\pi_{n+2}E) \arrow[r] & E_{\leq n+2} \arrow[d] & \\
\Sigma^{n+1}\mathsf{H}(\pi_{n+1}E) \arrow[r] & E_{\leq n+1} \arrow[r] \arrow[d] & \Sigma^{n+3}\mathsf{H}(\pi_{n+2}E) \\
& E_{\leq n} \arrow[r] & \Sigma^{n+2}\mathsf{H}(\pi_{n+1}E)
\end{tikzcd}$$

where the right-down-right zig-zags (one showed in blue, the other one in green) are distinguished triangles, or maps defined in the same way using the Whitehead tower instead of the Postnikov tower as the middle column.

The strategy above cannot be used to identify $r_{\mathbb{R}}\mathsf{ko}$ as an $\mathcal{E}_1$-ring with $\mathsf{H}\mathbb{Z}[t^4]$ directly. Actually, we can even prove that this is wrong, since otherwise we would have $\mathsf{ko}^{\mathsf{top}}[1/2] \simeq r_{\mathbb{R}}\mathsf{ko}[1/2] \simeq \mathsf{H}\mathbb{Z}[1/2][t^4]$ as $\mathcal{E}_1$-rings. We would then have $\mathsf{ku}[1/2] \wedge \mathsf{H}\mathbb{Z} \simeq \mathsf{ko}^{\mathsf{top}}[1/2] \wedge \mathsf{H}\mathbb{Z} \oplus \Sigma^2\mathsf{ko}^{\mathsf{top}}[1/2] \wedge \mathsf{H}\mathbb{Z}$ (using the proof of [LNS23, Thm 9.3]), and thus

$$\mathsf{ku}[1/2] \wedge \mathsf{H}\mathbb{Z} \simeq \mathsf{H}\mathbb{Z}[1/2][t^4] \wedge \mathsf{H}\mathbb{Z} \oplus \Sigma^2\mathsf{H}\mathbb{Z}[1/2][t^4] \wedge \mathsf{H}\mathbb{Z}.$$

Reducing $\mathsf{mod}\,3$, this yields $\mathsf{ku}[1/2] \wedge \mathsf{H}\mathbb{Z}/3 \simeq \mathsf{H}\mathbb{Z}/3[t^4] \wedge \mathsf{H}\mathbb{Z} \oplus \Sigma^2\mathsf{H}\mathbb{Z}/3[t^4] \wedge \mathsf{H}\mathbb{Z}$. However, using formal group laws, one shows that the left-hand side is trivial, while the right-hand side is not, see for example [GS].

5.2.2 Identifying the rationalization

Using our knowledge of the localization at (2) of $r_{\mathbb{R}}\mathsf{ko}$, we can identify its rationalization.

Lemma 5.5. *There is an equivalence of $\mathcal{E}_1$-rings*

$$\mathsf{H}\mathbb{Q}[t^4] \xrightarrow{\simeq} (r_{\mathbb{R}}\mathsf{ko})_{\mathbb{Q}}$$

sending $t^4 \in \pi_4(\mathsf{H}\mathbb{Q}[t^4])$ to (the image in the localization of) the generator $x \in \pi_4(r_{\mathbb{R}}\mathsf{ko})$.

Proof. The rationalization of $r_{\mathbb{R}}\mathsf{ko}$ is the localization away from 2 of its localization at (2), namely $\mathsf{H}\mathbb{Z}_{(2)}[t^4]$, and we claim the result of this is the free $\mathcal{E}_1$-$\mathsf{H}\mathbb{Q}$-algebra $\mathsf{H}\mathbb{Q}[t^4]$. Indeed, the unique ring map $\mathbb{Z}_{(2)} \to \mathbb{Q}$ (which induces an $\mathcal{E}_\infty$-map $\mathsf{H}\mathbb{Z}_{(2)} \to \mathsf{H}\mathbb{Q}$ and thus a map of $\mathsf{H}\mathbb{Z}_{(2)}$-modules $\mathsf{H}\mathbb{Z}_{(2)} \to \mathsf{H}\mathbb{Q}[t^4]$) and the element $t^4 \in \pi_4(\mathsf{H}\mathbb{Q}[t^4])$ induce a map of $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}_{(2)}$-algebras $\mathsf{H}\mathbb{Z}_{(2)}[t^4] \to \mathsf{H}\mathbb{Q}[t^4]$. By our description of the homotopy rings of free $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}_{(2)}$-algebras (Proposition 1.38(*ii*)), we see that after localization away from 2, this map induces isomorphisms in homotopy. This proves our claim. $\square$

5.2.3 Identifying the localization away from 2

Recall that we have already proven that there is an equivalence of spectra $r_{\mathbb{R}}\mathsf{ko}[1/2] \simeq \mathsf{ko}^{\mathsf{top}}[1/2]$ in Proposition 4.29. We have to reprove this result in a more careful way, keeping track of the equivalences involved, in order to later be able to identify precisely the maps appearing in the fracture square, and upgrade this identification as one of $\mathcal{E}_\infty$-rings (and thus, in particular, $\mathcal{E}_1$-rings), as already claimed in Proposition 4.29. In the proof of the latter, what was preventing us from having an equivalence of $\mathcal{E}_\infty$-rings was the identification of $\mathsf{KO}^{\mathsf{top}}[1/2]$ with $r_{\mathbb{R}}(\mathsf{KW}[1/2])$ as spectra only. We now improve this result:

Theorem 5.6. *There is an $\mathcal{E}_\infty$-map*

$$\gamma : \mathsf{L}(\mathbb{R}) \longrightarrow r_{\mathbb{R}}\mathsf{KW}$$

inducing an equivalence of $\mathcal{E}_\infty$-rings $\mathsf{L}(\mathbb{R})[1/2] \simeq r_{\mathbb{R}}\mathsf{KW}[1/2]$. In particular, there are equivalences of $\mathcal{E}_\infty$-rings

$$\mathsf{KO}^{\mathsf{top}}[1/2] \simeq r_{\mathbb{R}}\mathsf{KW}[1/2] \simeq r_{\mathbb{R}}\mathsf{KO}[1/2],$$
$$\mathsf{ko}^{\mathsf{top}}[1/2] \simeq r_{\mathbb{R}}\mathsf{kw}[1/2] \simeq r_{\mathbb{R}}\mathsf{ko}[1/2].$$

Proof. The map from the statement is obtained as the following composition of maps of $\mathcal{E}_\infty$-rings (where each step is explained below)

$$\begin{tikzcd}
& \mathsf{L}(\mathbb{R}) \arrow[r, "\simeq"] & \mathrm{colim}\left(GW^{[0]}(\mathbb{R}) \xrightarrow{\eta} \Sigma GW^{[-1]}(\mathbb{R}) \xrightarrow{\eta} \cdots\right) \arrow[d, "\gamma_1", "\simeq"'] \\
& \mathbb{L}(\mathbb{R}) \arrow[r, "\simeq"] & \mathrm{colim}\left(\mathbb{GW}^{[0]}(\mathbb{R}) \xrightarrow{\eta} \Sigma\mathbb{GW}^{[-1]}(\mathbb{R}) \xrightarrow{\eta} \cdots\right) \arrow[d, "\gamma_2", "\simeq"'] \\
& & \mathrm{colim}\left(\mathsf{L}_{\mathbb{A}^1}\mathbb{GW}^{[0]}(\mathbb{R}) \xrightarrow{\eta} \mathsf{L}_{\mathbb{A}^1}\Sigma\mathbb{GW}^{[-1]}(\mathbb{R}) \xrightarrow{\eta} \cdots\right) \arrow[d, "\gamma_3", "\simeq"'] \\
& \Gamma(\mathbb{R}, \mathsf{KW}) \arrow[r, "\simeq", "\gamma_4"'] \arrow[d, "\gamma_5"] & \mathrm{colim}\left(\Gamma(\mathbb{R}, \mathsf{KO}) \xrightarrow{\eta} \Gamma(\mathbb{R}, \mathbb{G}_m^{\wedge -1} \wedge \mathsf{KO}) \xrightarrow{\eta} \cdots\right) \\
r_{\mathbb{R}}\mathsf{KW} \arrow[r, "\simeq", "\gamma_6"'] & \mathrm{colim}\left(\Gamma(\mathbb{R}, \mathsf{KW}) \xrightarrow{\rho} \Gamma(\mathbb{R}, \mathsf{KW} \wedge \mathbb{G}_m) \xrightarrow{\rho} \cdots\right). &
\end{tikzcd}$$

1. Recall the Grothendieck-Witt spectrum functor GW introduced in Subsection 3.3 to define L-theory. One can define similarly a *stabilized* Grothendieck-Witt spectrum functor $\mathbb{GW}$, which yields *stabilized* L-theory $\mathbb{L}(-)$ (see [Sch17, 8.12] and [KSW16, Section 6]). The two top-most horizontal equivalences are then definitions. The map γ_1 is induced by the natural maps between the functors GW and $\mathbb{GW}$. The latter are maps of $\mathcal{E}_\infty$-ring spectra on the non-shifted parts, and maps of modules over this $\mathcal{E}_\infty$-ring for the shifted parts, so they induce a map of $\mathcal{E}_\infty$-rings between the L-theory and stabilized L-theory spectra. Since $\mathsf{Spec}(\mathbb{R})$ is regular, this map is actually an equivalence by [KSW16, beginning of p7]. In general, it is only an equivalence after inverting 2, by [Sch17, Lemma 8.16].

2. The map γ_2 is the levelwise application of the $\mathbb{A}^1$-localization functor; indeed, the functors $\mathbb{G}\mathsf{W}^{[n]}$ can be viewed as presheaves of spectra on Sm_k. Moreover, $\mathsf{L}_{\mathbb{A}^1}$ is a left adjoint and thus commutes with suspensions (since $\mathsf{L}_{\mathbb{A}^1}(*) = *$) and taking stalks (and thus with $(-)(\mathbb{R})$). This map is actually an equivalence because $\mathsf{L}_{\mathbb{A}^1}$ commutes with the outer colimit as well, but $\mathbb{L}(\mathbb{R}) \to \mathsf{L}_{\mathbb{A}^1}\mathbb{L}(\mathbb{R})$ is an equivalence. Indeed, the homotopy groups of the stabilized L-theory spectrum are Karoubi's stabilized Witt groups (see [Sch17, after Def. 8.12]), which are already $\mathbb{A}^1$-invariant for affine schemes by [Kar06, Thm 2.1(a)]. Our claim then follows using the explicit formula for the $\mathbb{A}^1$-localization $\mathsf{L}_{\mathbb{A}^1}(E) = |E(- \times \Delta^\bullet_{\mathbb{A}^1})|$ (as shown in [MV99, p87] or [AE17, §4.3]). Alternatively, we could use that $\mathbb{L}(-)$ is actually $\mathbb{A}^1$-invariant whenever the base scheme is regular, by [KSW21, Section 2].

3. Then γ_3 is an equivalence provided by [HJNY22, Prop. 7.6], which states that there are equivalences $\Gamma(S, \Sigma^n_T KO) \simeq \mathsf{L}_{\mathbb{A}^1}\mathbb{G}\mathsf{W}^{[n]}(S)$ for any qcqs scheme S with 2 invertible. Here $\Gamma(S,-)$ is the section functor, if we view a motivic spectrum as a presheaf of topological spectra (see for example [CHR24]). In particular, $\Gamma(S, \Sigma^\infty_T E) = \Sigma^\infty(E(S))$ if $E \in \mathsf{Spc}(k)_*$ and $S \in \mathsf{Sm}_k$. Moreover, the restriction of this functor to ρ-local motivic spectra is nothing but the equivalence $\mathsf{SH}(\mathbb{R})[\rho^{-1}] \simeq \mathsf{Sp}$ in Theorem 4.10.

   We have to check that $\Sigma^n\Gamma(\mathbb{R}, \Sigma_T^{-n} KO) \simeq \Gamma(\mathbb{R}, \mathbb{G}_m^{\wedge -n} KO)$. Since $\Gamma(\mathbb{R}, -)$ is a stalk functor (proof of Lemma 4.13) it commutes with colimits and thus with suspension. We can then conclude using that $\Sigma^n_{\mathcal{S}^1}\Sigma^{-n}_T \simeq \mathbb{G}_m^{\wedge -n} \wedge -$.

4. Another consequence of the fact that $\Gamma(\mathbb{R}, -)$ commutes with colimits is that we can pull it out from the sequential colimit under consideration, and thus we obtain $\Gamma(\mathbb{R}, \mathsf{KW})$. The transition maps still correspond to the motivic η, because both are given by the image of $\eta \in \underline{\pi}_{-1,-1}(\mathcal{S})(\mathbb{R})$ (and we have maps of commutative ring spectra, which must preserve the unit).

5. The map γ_5 is the natural map to the colimit.

6. The map γ_6 is then an equivalence, because once more the functor $\Gamma(\mathbb{R}, -)$ commutes with the sequential colimit we are considering, and by Theorem 4.10, real realization can be written as the composition $\mathsf{SH}(\mathbb{R}) \to \mathsf{SH}(\mathbb{R})[\rho^{-1}] \simeq \mathsf{Sp}$, where the first map is localization at ρ, i.e. exactly the colimit we are considering, and the equivalence is given by $\Gamma(\mathbb{R}, -)$. This also proves that γ_5 becomes an equivalence after inverting 2, because ρ is already invertible on $\mathsf{KW}[1/2] \simeq \mathsf{KO}[\eta^{-1}, 1/2]$ as we saw earlier (proof of Proposition 4.28).

As for the last assertion, by [LNS23, Ex. 9.2], there is an $\mathcal{E}_\infty$-map $\tau_\mathbb{R} : \mathsf{ko}^{\mathsf{top}} \to L(\mathbb{R})_{\geq 0}$, inducing an equivalence after inverting 2. Moreover, on the 8th homotopy groups, this maps sends λ_8 (a generator for $\pi_8(\mathsf{ko}) \cong \mathbb{Z}$) to $16b^2$ (where b is a generator of $\pi_4(\mathsf{L}(\mathbb{R})) \cong \mathbb{Z}$). Therefore, there is an equivalence

$$\mathsf{ko}^{\mathsf{top}}[1/2][\lambda_8^{-1}] \simeq \mathsf{L}(\mathbb{R})_{\geq 0}[1/2][(16b^2)^{-1}] \simeq \mathsf{L}(\mathbb{R})_{\geq 0}[1/2][b^{-1}]$$

but the left-hand side is $\mathsf{KO}^{\mathsf{top}}[1/2]$ since $\mathsf{ko}^{\mathsf{top}}[\lambda_8^{-1}] \simeq \mathsf{KO}^{\mathsf{top}}$ by periodicity (see for example [LN18, Cor. 5.1]), and similarly the right-hand side is $\mathsf{L}(\mathbb{R})[1/2]$ because $\mathsf{L}(\mathbb{R})_{\geq 0}[b^{-1}] \simeq \mathsf{L}(\mathbb{R})$ (same reference). Moreover, the equivalence $r_\mathbb{R}\mathsf{KW}[1/2] \simeq r_\mathbb{R}\mathsf{KO}[1/2]$ and the passage to the connective and very effective covers is proven exactly as in Propositions 4.29 and 4.28. □

5.2.4 Main result: identifying the maps and assembling the square

We can now state our main result:

Theorem 5.7 (The real realization of ko). *As an $\mathcal{E}_1$-ring, the real realization of* ko *is uniquely characterized by the following Cartesian square of $\mathcal{E}_1$-rings*

$$\begin{tikzcd} r_\mathbb{R}\mathsf{ko} \arrow[r, "x\mapsto \beta_4/2"] \arrow[d, "x\mapsto t^4"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{ko}^{\mathsf{top}}[1/2] \arrow[d, "\mathsf{ch}"', "\beta_4/2\mapsto t"] \\ \mathsf{H}\mathbb{Z}_{(2)}[t^4] \arrow[r, "t^4\mapsto t^4"'] & \mathsf{H}\mathbb{Q}[t^4] \end{tikzcd}$$

where the vertical map on the right hand side is the Chern character *defined below (Definition 5.8). The assignments labeling the arrows describe the maps induced on the fourth homotopy groups.*

Definition 5.8. Let $\mathsf{H}\mathbb{Q}[t^4]$ and $\mathsf{H}\mathbb{Q}[u^2]$ be the free $\mathcal{E}_1$-$\mathsf{H}\mathbb{Q}$-algebras on one generator in degrees 4 and 2 respectively. We call *Chern character* any of the following maps:

- the map $\mathsf{KU}^{\mathsf{top}}_{\geq 0} \to \mathsf{H}\mathbb{Q}[u^2]$ given by the (connective cover of the) classical Chern character as defined in [BH58, Ex. 9.1]. On the homotopy rings, it induces the inclusion $\mathbb{Z}[\tilde{u}^2] \to \mathbb{Q}[U^2]$ where $\tilde{u}^2 \in \pi_2(\mathbb{Z} \times BU) \cong \mathbb{Z}$ is a generator.
- the map $\mathsf{KO}^{\mathsf{top}}_{\geq 0} = \mathsf{ko}^{\mathsf{top}} \to \mathsf{H}\mathbb{Q}[t^4]$ obtained by pre-composition of the previous map with the connective cover the the complexification map $\mathsf{KO}^{\mathsf{top}} \to \mathsf{KU}^{\mathsf{top}}$, and corestricting along the inclusion $\mathsf{H}\mathbb{Q}[t^4] \hookrightarrow \mathsf{H}\mathbb{Q}[u^2]$ given by $t^4 \mapsto (u^2)^2$. (The complexification map, as the name indicates, is induced by complexification of vector bundles. On the homotopy rings, it induces the map
$$\mathbb{Z}[\alpha_1, \beta_4, \lambda_8]\big/(\alpha_1^3, 2\alpha_1, \alpha_1\beta_4, \beta_4^2 - 4\lambda_8) \to \mathbb{Z}[\tilde{u}^2]$$
sending α_1 to 0, β_4 to $2(\tilde{u}^2)^2$ (and thus λ_8 to $(\tilde{u}^2)^4$) (see for example [Rog08, §5.3]).
- the maps induced by the two previous ones after localization away from 2 of the source.

Remark 5.9. The Chern character $\mathsf{ch} : \mathsf{ko}^{\mathsf{top}} \to \mathsf{H}\mathbb{Q}[t^4]$ factors through the rationalization, and the map $\mathsf{ko}^{\mathsf{top}}_{\mathbb{Q}} \xrightarrow{\mathsf{ch}_{\mathbb{Q}}} \mathsf{H}\mathbb{Q}[t^4]$ obtained in this way is an equivalence. Indeed, $\pi_*(\mathsf{ko}^{\mathsf{top}}_{\mathbb{Q}}) \cong \pi_*(\mathsf{ko}^{\mathsf{top}}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}[\beta_4]$ (by Proposition A.2) and $\pi_*(\mathsf{H}\mathbb{Q}[t^4]) \cong \mathbb{Q}[t^4]$ (by Proposition 1.38(ii)), with $\mathsf{ch}_{\mathbb{Q}}$ sending β_4 to $2t^4$: it induces isomorphisms in homotopy.

Proof of Theorem 5.7. In our description of the spectra appearing in the square, we have implicitly fixed ourselves equivalences f and g such that the diagram

$$\begin{tikzcd}
r_{\mathbb{R}}\mathsf{ko} \arrow[r] \arrow[d] & (r_{\mathbb{R}}\mathsf{ko})[1/2] \arrow[r, "\simeq", "g"'] \arrow[d] & \mathsf{ko}^{\mathsf{top}}[1/2] \arrow[d] \arrow[dr, "\mathsf{ch}"] & \\
(r_{\mathbb{R}}\mathsf{ko})_{(2)} \arrow[r] \arrow[d, "f"', "\simeq"] & (r_{\mathbb{R}}\mathsf{ko})_{\mathbb{Q}} \arrow[r, "\simeq", "g_{\mathbb{Q}}"'] & (\mathsf{ko}^{\mathsf{top}}[1/2])_{\mathbb{Q}} \arrow[r, "\simeq", "\mathsf{ch}_{\mathbb{Q}}"'] & \mathsf{H}\mathbb{Q}[t^4] \\
\mathsf{H}\mathbb{Z}_{(2)}[t^4] \arrow[urrr, bend right, "k"] & & &
\end{tikzcd}$$

commutes, where $\mathsf{ch}_{\mathbb{Q}}$ is the map from Remark 5.9. The outer square is exactly the pushout square of the statement, and the top left square is that of Proposition A.6. We therefore have to determine what is the map k, i.e. where it sends t^4.

- Let us determine a precise construction for g that will allow us to keep track of the maps involved. Consider the following diagram of $\mathcal{E}_\infty$-rings

$$\begin{tikzcd}
& \mathsf{ko}^{\mathsf{top}} \arrow[r, "\tau_{\mathbb{R}}"] \arrow[d, equal] & \mathsf{L}(\mathbb{R}) \arrow[r, "\gamma"] & r_{\mathbb{R}}\mathsf{KW} & r_{\mathbb{R}}\mathsf{KO} \arrow[l] \\
& \mathsf{ko}^{\mathsf{top}} \arrow[r] \arrow[d] & \mathsf{L}(\mathbb{R})_{\geq 0} \arrow[r] \arrow[u] \arrow[d] & r_{\mathbb{R}}(\mathsf{KW})_{\geq 0} = r_{\mathbb{R}}\mathsf{kw} \arrow[u] \arrow[d, "\simeq"] & r_{\mathbb{R}}\mathsf{ko} \arrow[l] \arrow[u] \arrow[d] \\
\mathsf{ko}^{\mathsf{top}}[1/2] & \mathsf{ko}^{\mathsf{top}}[1/2] \arrow[l, "\simeq", "\psi^2"'] \arrow[r, "\simeq"] & \mathsf{L}(\mathbb{R})_{\geq 0}[1/2] \arrow[r, "\simeq"] & r_{\mathbb{R}}(\mathsf{kw}[1/2]) & r_{\mathbb{R}}(\mathsf{ko}[1/2]) \arrow[l, "\simeq"'] \arrow[llll, bend left, "g"]
\end{tikzcd} \tag{1}$$

constructed in the following manner:

  - The top row is constructed first:
    * The first map $\tau_{\mathbb{R}} : \mathsf{ko}^{\mathsf{top}} \to \mathsf{L}(\mathbb{R})$ is taken from [LNS23, Ex. 9.2].
    * The second map γ is the one from Proposition 5.6 above.
    * Finally, the map $r_{\mathbb{R}}\mathsf{KO} \to r_{\mathbb{R}}\mathsf{KW}$ is the real realization of the localization at η.

– The two left-most maps of the second row are the maps induced on the connective covers by that of the first row; the last map in the row is obtained as the real realization of the map $\mathsf{ko} \to \mathsf{kw}$ induced on the very effective covers.

– Finally, in the third row, ψ^2 is the second Adams operation (by [Kar78, Section IV, 7.13, 7.19 and 7.25], this is an $\mathcal{E}_\infty$-map and it induces multiplication by 4 on the fourth homotopy group), and the other maps are obtained by localization away from 2 of the second row. The three right-most maps are equivalences respectively by [LNS23, Ex. 9.2] (or [LN18, Cor. 5.4]), by Theorem 5.6, and by the proof of Proposition 4.29.

- Recall from Subsection 3.2 the element $\beta_{\mathsf{ko}} \in \pi_{8,4}(\mathsf{ko})$ induced by the Bott element β_{KO} exhibiting KO as an $(8,4)$-periodic spectrum. We want to study $g_*(r_{\mathbb{R}}(\beta_{\mathsf{ko}}))$ instead of $g_*(x)$ because $r_{\mathbb{R}}(\beta_{\mathsf{ko}})$ comes from the element $r_{\mathbb{R}}(\beta_{\mathsf{KO}}) \in \pi_4(r_{\mathbb{R}}\mathsf{KO})$ and thus we can study instead the first row in the diagram. We claim that, on the fourth homotopy groups, this row looks as follows

$$\mathbb{Z}\{\beta_4\} \xrightarrow{\beta_4 \mapsto 8b} W_4(\mathbb{R}) \cong \mathbb{Z}\{b\} \xrightarrow{b \mapsto b} \mathbb{Z}[1/2]\{b\} \xleftarrow{16b \leftarrow\!\shortmid r_{\mathbb{R}}(\beta_{\mathsf{ko}})} \pi_4(r_{\mathbb{R}}\mathsf{ko})$$

where b is a generator in $\pi_4(\Gamma(\mathbb{R}, \mathsf{KW})) = \pi_4(\mathbb{L}(\mathbb{R})) \cong \mathbb{Z}$ (see Theorem 5.6). Indeed:

– The fact that the first map induces multiplication by 8 on the fourth homotopy group is [LNS23, Ex. 9.2]. We outline the proof provided there in Proposition 5.10 below.

– For the second map γ, having a closer look at its construction in Theorem 5.6, we see that it is the composition of the two maps $\mathsf{L}(\mathbb{R}) \to \Gamma(\mathbb{R}, \mathsf{KW})$ and then the natural map

$$\Gamma(\mathbb{R}, \mathsf{KW}) \to \mathsf{colim}(\Gamma(\mathbb{R}, \mathsf{KW}) \xrightarrow{\rho} \Gamma(\mathbb{R}, \mathsf{KW} \wedge \mathbb{G}_m) \xrightarrow{\rho} \cdots) \simeq r_{\mathbb{R}}\mathsf{KW}.$$

The first map is an equivalence even before inverting 2 as we saw, and the fourth homotopy groups of its domain and codomain are both $\mathbb{Z}$ (since L-theory of $\mathbb{R}$ is known, see [HLN21, Prop. 4.1]). Using that η is an equivalence on KW, and the fact that $\rho\eta = -2$ after real realization, the colimit rewrites as $\mathsf{colim}(\Gamma(\mathbb{R}, \mathsf{KW}) \xrightarrow{\cdot 2} \Gamma(\mathbb{R}, \mathsf{KW}) \xrightarrow{\cdot 2} \cdots)$ (see diagram 2 below), which gives the isomorphism $\mathbb{Z}[1/2] \cong \pi_4(r_{\mathbb{R}}\mathsf{KW})$, where the unit element of the ring on the left corresponds to the image of the generator b: we thus denote this ring as $\mathbb{Z}[1/2]\{b\}$.

– Finally, we determine the image of $r_{\mathbb{R}}\beta_{\mathsf{KO}}$ in $\pi_4(r_{\mathbb{R}}\mathsf{KW})$. Since the map $r_{\mathbb{R}}\mathsf{KO} \to r_{\mathbb{R}}\mathsf{KW}$ was induced by localization at η, we have a comparison diagram of the colimits very similar to the one in the proof of Proposition 5.6

$$\begin{tikzcd}
\pi_4(r_{\mathbb{R}}\mathsf{KO}) \cong \mathsf{colim}\Big(\cdots & {[\mathcal{S}^4, \mathbb{G}_m^{\wedge -4} \wedge \mathsf{KO}]} \arrow[r, dashed, "\rho"] \arrow[d] & \cdots \arrow[r, dashed, "\rho"] & {[\mathcal{S}^4, \mathsf{KO}]} \arrow[r, dashed, "\rho"] \arrow[d] & {[\mathcal{S}^4, \mathbb{G}_m^{\wedge 1} \wedge \mathsf{KO}]} \arrow[d] & \cdots \Big) \\
\pi_4(r_{\mathbb{R}}\mathsf{KW}) \cong \mathsf{colim}\Big(\cdots & {[\mathcal{S}^4, \mathbb{G}_m^{\wedge -4} \wedge \mathsf{KW}]} \arrow[r, dashed, "\rho"] & \cdots \arrow[r, dashed, "\rho"] & {[\mathcal{S}^4, \mathsf{KW}]} \arrow[r, dashed, "\rho"] \arrow[dr, "\cdot(-2)"] & {[\mathcal{S}^4, \mathbb{G}_m^{\wedge 1} \wedge \mathsf{KW}]} \arrow[d, "\eta"] & \cdots \Big) \\
 & {[\mathcal{S}^4, \mathsf{KW}]} \arrow[u, "\eta^4"] \arrow[urr, "\cdot(-2)^4"] & & & {[\mathcal{S}^4, \mathsf{KW}]} &
\end{tikzcd} \tag{2}$$

where the vertical maps are induced by suspensions of the localizations at η. Now, β_{KO} lives in $[\mathcal{S}^4, \mathbb{G}_m^{\wedge -4} \wedge \mathsf{KO}]$. The composite of the η-localization map and of $(\eta^4)^{-1}$ sends it to a generator in $[\mathcal{S}^4, \mathsf{KW}] \cong \pi_4(\Gamma(\mathbb{R}, \mathsf{KW})) \cong \mathbb{Z}$ by [BH20, §6.3.2, before Lemma 6.9]. Therefore, in the colimit $\mathbb{Z}\{b\}[1/2]$ of the gray diagram, it is mapped to $2^4 b = 16b$.

- Since ψ^2 induces multiplication by 4 on the fourth homotopy groups, the image of (the localization away from 2 of) $r_{\mathbb{R}}(\beta_{\mathsf{ko}}) \in \pi_4(r_{\mathbb{R}}\mathsf{ko}[1/2])$ by g is equal to (the localization of) $8\beta_4 \in \pi_4(\mathsf{ko}^{\mathsf{top}}[1/2])$. That is, $g(x) = \beta_4/2$, because by Proposition 5.11 below, we have $r_{\mathbb{R}}(\beta_{\mathsf{ko}}) = 16x$. Therefore, in diagram 1 at the beginning of the proof, we must have $k(t^4) = t^4$, and k is the map of $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}_{(2)}$-algebras $\mathsf{H}\mathbb{Z}_{(2)}[t^4] \to \mathsf{H}\mathbb{Q}[t^4]$ sending t^4 to t^4 in the fourth homotopy groups, as desired.

□

5.2.5 Results used in the proof of the main result, part I

In the above proof, we used some claims without proving them, it is now time to justify these:

Proposition 5.10 ([LNS23, Thm A and Ex. 9.2]). *Consider the map $\tau_{\mathbb{R}} : \mathsf{ko}^{\mathsf{top}} \to \mathsf{L}(\mathbb{R})$ obtained as the $\mathbb{R}$-component of the unique lax symmetric monoidal transformation from connective K-theory to L-theory of real C^*-algebras $\tau : K(-)_{\geq 0} \to L(-)$. Then $\tau_{\mathbb{R}}$ induces multiplication by 8 on the fourth homotopy groups (which are both isomorphic to $\mathbb{Z}$)*

Proof. We give a rough outline of the argument in [LNS23, Ex. 9.2], relying on the results of [LN18]. The claim is proven by comparison with the K- and L- theories of *complex* C^*-algebras. By [LNS23, Lemma 9.1], there is a commutative diagram

$$\begin{tikzcd} K(-)_{\geq 0} \arrow[r, "\tau"] \arrow[d] & L(-) \arrow[d] \\ K((-)_{\mathbb{C}})_{\geq 0} \arrow[r, "\tau'_{(-)_{\mathbb{C}}}"] & L((-)_{\mathbb{C}}) \end{tikzcd}$$

where $(-)_{\mathbb{C}}$ denotes complexification and τ' is similarly a unique lax symmetric monoidal transformation in the complex case, studied in [LN18]. Taking the $\mathbb{R}$-component, the diagram becomes

$$\begin{tikzcd} \mathsf{ko}^{\mathsf{top}} \arrow[r, "\tau_{\mathbb{R}}"] \arrow[d] & L(\mathbb{R}) \arrow[d] \\ \mathsf{ku} \arrow[r, "\tau'_{\mathbb{C}}"] & L(\mathbb{C}). \end{tikzcd}$$

The vertical map on the left is the complexification map already encountered in Definition 5.8, and induces multiplication by 2 on the fourth homotopy groups, both equal to $\mathbb{Z}$. On the homotopy of the connective parts, the vertical map on the right is the inclusion of a polynomial ring on a generator in degree 4 in a polynomial ring on a generator in degree 2: $\mathbb{Z}[b] \subseteq \mathbb{Z}[a]$ where $b \mapsto a^2$. Finally, the lower horizontal map is multiplication by 4 on the fourth homotopy groups by [LN18, Lemma 4.9]. $\square$

Proposition 5.11. *In the notation of Proposition 5.1, we have $r_{\mathbb{R}}(\beta_{\mathsf{ko}}) = 16x \in \pi_4(r_{\mathbb{R}}\mathsf{ko})$.*

Proof. The proof is divided in various results, proven in the remainder of this subsection. Here we show how they fit together and try to give some intuition for the proof.

- We want to understand the action of $r_{\mathbb{R}}\beta_{\mathsf{ko}}$ on the homotopy groups of $r_{\mathbb{R}}\mathsf{ko}$, by considering a cofiber sequence where this map appears together with only a small number of easily computable groups. We will thus consider slices. The following appears as Lemma 5.13 below:

  Let $a \in \mathbb{Z}$ with $r_{\mathbb{R}}(\beta_{\mathsf{ko}}) = ax$ (recall that x generates $\pi_4(r_{\mathbb{R}}\mathsf{ko})$). Then $a = |\pi_4(r_{\mathbb{R}}C)|$, where C is the cofiber of the natural map $\tilde{f}_4\mathsf{ko} \to \mathsf{ko}$.

  This result holds because, as we will see in its proof, the natural map $\tilde{f}_4\mathsf{ko} \to \mathsf{ko}$ corresponds to β_{ko} under the identification $\tilde{f}_4\mathsf{ko} \simeq \tilde{f}_4\mathsf{KO} \simeq \tilde{f}_4(\Sigma^{8,4}\mathsf{KO}) \simeq T^{\wedge 4} \wedge \tilde{f}_0\mathsf{KO} \simeq T^{\wedge 4} \wedge \mathsf{ko}$, where the first equivalence is induced by β_{KO} (periodicity of KO) and the second identification is Proposition 2.27.
- Thus, we have to compute $|\pi_4(r_{\mathbb{R}}C)|$. A powerful tool in computing homotopy groups is spectral sequences. In general, under sufficiently nice assumptions, such spectral sequences exist and converge for any filtered object in a stable ∞-category. We will consider the very effective filtration on $C = \mathsf{ko}/\tilde{f}_4\mathsf{ko}$, and after applying real realization, this will give us a filtration on $r_{\mathbb{R}}C$. The E^1-page of such spectral involves the homotopy groups of the successive cofibers of the maps in the filtered object. In our case, these will be the homotopy groups of the real realizations of the very effective slices of C, which are the first four very effective slices of ko. Recall that the latter were described in Theorem 5.2.
- However, the problem is that the only grasp we have on the 0-th slice is given by a decomposition as a cofiber sequence. We need some additional technology to compute the homotopy groups of $r_{\mathbb{R}}(\tilde{s}_0\mathsf{ko})$ (the long exact sequence associated with the cofiber sequence will not suffice, because the connecting homomorphisms are unknown). We will compute these homotopy groups using another spectral sequence, for the filtration given this time by the effective homotopy t-structure (Definition 2.21). We do this in Lemma 5.14, and then are able to compute $|\pi_4(r_{\mathbb{R}}C)| = 16$ using the first spectral sequence we mentioned in Lemma 5.15.

This finishes the proof. $\square$

5.2.6 Recollections about spectral sequences and their multiplicative structure

Before proving Lemmas 5.14 and 5.15 used in the proof of Proposition 5.11, we have to recall some prerequisites about spectral sequences for cofiltered objects in stable ∞-categories.

Theorem 5.12. *Let $\mathcal{C}$ be a stable ∞-category and let $\pi : \mathcal{C} \to \mathcal{A}$ be a homological functor (i.e. with an induced long exact sequence in $\mathcal{A}$ for every cofiber sequence in $\mathcal{C}$) where $\mathcal{A}$ is an abelian 1-category admitting sequential limits. Let $\pi_n = \pi \circ \Sigma^{-n}$ for all $n \in \mathbb{Z}$.*

Consider a cofiltered object $X := \lim_n(\cdots \to X_{-1} \to X_0 \to X_1 \to \cdots)$, such that $\mathsf{colim}_n X_n$ exists, and $\forall n \in \mathbb{Z}$, assume that $\pi_n(X_r) = 0$ for $r \ll 0$, and $\mathsf{colim}_{m\geq 0}\,\mathsf{cof}(X_{-n} \to X_{m-n})$ exists and is preserved by π.

Under these conditions, there is a spectral sequence with $E^1_{p,q} = \pi_p(\mathsf{cof}(X_{-q-1} \to X_{-q}))$ converging strongly to $\pi_p(\mathsf{colim}_n X_n)$: for all $p, q \in \mathbb{Z}$, we have:

- $E^\infty_{p,q} = \mathsf{coker}(\pi_p(X_{-q}) \to \pi_p(X_{-q+1}))$
- $\pi_p(\mathsf{colim}_n X_n) = \mathsf{colim}_q\, \mathsf{coker}(\pi_p(X_{-q}) \to \pi_p(\mathsf{colim}_n X_n))$.

We call the cofibers $\mathsf{cof}(X_{-q-1} \to X_{-q})$ the subquotients *associated to the cofiltered object $(X_n)_{n\in\mathbb{Z}}$.*

This result appears in dual form in [nLa25d, Prop. 2.24], and is proven in a more restricted form (but sufficient for our purposes) in [Lur17, Prop. 1.2.2.14].

As for the grading, our convention is that the differential on the E^r page has degree $(-1, r)$.

Moreover, if the cofiltered object we started with admits some kind of multiplicative structure, the spectral sequence also inherits a multiplicative structure, which respect to which the differentials satisfy the Leibniz rule. This can help to determine certain differentials in such spectral sequences. More precisely, choosing $\mathcal{C} = \mathsf{Sp}$, $\mathcal{A} = \mathsf{Ab}$ and $\pi = \pi_0$ in Theorem 5.12, any pairing on the cofiltered object $(X_n)_{n\in\mathbb{Z}}$ induces a pairing on the subquotients and therefore a multiplicative structure on the associated spectral sequence. We will briefly explain how this works, but for more precise statements, see [Dug03] (especially section 6 and Theorem 6.2).

A pairing on $(X_n)_{n\in\mathbb{Z}}$ is the data of "a coherent graded multiplication" taking the form of compatible maps $X_n \wedge X_m \to X_{n+m}$ for all $n, m \in \mathbb{Z}$. It then descends to a pairing on the subquotients, which itself descends to a bigraded pairing on the spectral sequence, taking the form of maps $E^r_{p,q} \otimes E^r_{s,t} \to E^r_{p+s,q+t}$ for all $r \geq 1$ and $p, q, s, t \in \mathbb{Z}$; and the differentials d^r satisfy the (graded) Leibniz rule with respect to this pairing. More precisely, the pairing on the subquotients arises as the maps $\gamma_{n,m}$ induced between the cofibers of α and the zero map in the bottom face of the following diagram (where $S_n := \mathsf{cof}(X_{n-1} \to X_n)$ for all $n \in \mathbb{Z}$):

$$
\begin{tikzcd}
X_{m-1}\wedge X_{n-1} \arrow[rr] \arrow[dr] \arrow[dd] & & X_{m+n-2} \arrow[dr] \arrow[dd, dashed] & & \\
& X_m \wedge X_{n-1} \arrow[rr] \arrow[dd] & & X_{m+n-1} \arrow[dd] & \\
X_{m-1}\wedge X_n \arrow[rr, dashed] \arrow[dr] \arrow[dd, "\alpha'"] & & X_{m+n-1} \arrow[dr, dashed] \arrow[dd, dashed] & & \\
& X_m \wedge X_n \arrow[rr] \arrow[dd] & & X_{m+n} \arrow[dd] & \\
X_{m-1}\wedge S_n \arrow[rr, dashed, "\lambda_{m-1,n}"] \arrow[dr, "\alpha"] & & S_{m+n-1} \arrow[dr, dashed, "0"] & & \\
& X_m \wedge S_n \arrow[rr, "\lambda_{m,n}"] \arrow[dr] & & S_{m+n} \arrow[dr] & \\
& & S_m \wedge S_n \arrow[rr, "\gamma_{m,n}"] & & S_{m+n}
\end{tikzcd}
\qquad (3)
$$

All vertical composites are cofiber sequences; the maps in blue in the bottom face are the maps induced on the cofibers by the two top layers of horizontal maps.

5.2.7 Results used in the proof, part II

We can now use multiplicative spectral sequences to prove the last results we needed in the proof of the main theorem.

Lemma 5.13. *Let $a \in \mathbb{Z}$ with $r_{\mathbb{R}}(\beta_{\mathsf{ko}}) = ax$ (recall that x generates $\pi_4(r_{\mathbb{R}}\mathsf{ko})$). Then $a = |\pi_4(r_{\mathbb{R}}C)|$, where C is the cofiber of the natural map $\tilde{f}_4\mathsf{ko} \to \mathsf{ko}$.*

Proof. The cofiber sequence $\tilde{f}_4\mathsf{ko} \to \mathsf{ko} \to C$ induces a long exact sequence

$$\cdots \longrightarrow \pi_4(r_{\mathbb{R}}\tilde{f}_4\mathsf{ko}) \longrightarrow \pi_4(r_{\mathbb{R}}\mathsf{ko}) \longrightarrow \pi_4(r_{\mathbb{R}}C) \longrightarrow \pi_3(r_{\mathbb{R}}\tilde{f}_4\mathsf{ko}) \longrightarrow \cdots .$$

By Proposition 2.27, we have $\tilde{f}_4\mathsf{ko} \simeq \Sigma^{8,4}\mathsf{ko}$ and therefore the first map in the sequence is a map $\mathbb{Z} \to \mathbb{Z}$, given by multiplication by some integer a', while the last group in the sequence is zero. Therefore, we have $\pi_4(r_{\mathbb{R}}C) \cong \mathbb{Z}/a'$, and we only have to show that $a = a'$, i.e. $r_{\mathbb{R}}(\beta_{\mathsf{ko}}) = a'x$.

Consider the commutative diagram

$$\begin{tikzcd}
\tilde{f}_4\mathsf{ko} \arrow[r, "\simeq"] \arrow[ddr, bend right] & \tilde{f}_4\mathsf{KO} \arrow[d] & \tilde{f}_4(\Sigma^{8,4}\mathsf{KO}) \arrow[l, "\tilde{f}_4(\beta_{\mathsf{KO}})"', "\simeq"] \arrow[d] \\
 & \mathsf{KO} & \Sigma^{8,4}\mathsf{KO} \arrow[l, "\simeq", "\beta_{\mathsf{KO}}"'] \\
 & \mathsf{ko} \arrow[u] & \Sigma^{8,4}\mathsf{ko}. \arrow[l, "\beta_{\mathsf{ko}}"] \arrow[u] \arrow[uu, dashed, bend right, "\simeq"']
\end{tikzcd}$$

The dotted arrow exists since $\Sigma^{8,4}\mathsf{ko} \in \mathsf{SH}(\mathbb{R})^{\mathsf{veff}}(4)$, and it is exactly the equivalence from Proposition 2.27 between $\tilde{f}_n(T^{\wedge m} \wedge -)$ and $T^{\wedge m} \wedge \tilde{f}_{n-m}(-)$ by construction. We therefore get that $\tilde{f}_4\mathsf{ko} \to \mathsf{ko}$ is, after real realization, exactly given on the fourth homotopy group by the real realization of β_{ko}, that is, multiplication by a. $\square$

Lemma 5.14. *The homotopy groups of $r_{\mathbb{R}}(\tilde{s}_0\mathsf{ko})$ are given for $k \geq 0$ by*

$$\pi_k(r_{\mathbb{R}}\tilde{s}_0\mathsf{ko}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}/2 & \text{if } k \equiv 2 \bmod 4 \text{ or } k \equiv 3 \bmod 4 \\ 0 & \text{if } k \equiv 1 \bmod 4 \end{cases}$$

and in positive degrees divisible by 4, $\pi_k(r_{\mathbb{R}}\tilde{s}_0\mathsf{ko})$ is an extension of two copies of $\mathbb{Z}/2$, in particular it has cardinality 4.

Proof. We apply Theorem 5.12 to $\mathcal{C} = \mathsf{Sp}$, $\mathcal{A} = \mathsf{Ab}$ the 1-category of abelian groups, $\pi = \pi_0$ is the usual stable homotopy group functor (then the notation π_n is coherent with the usual stable higher homotopy groups), and $X_n = r_{\mathbb{R}}((\tilde{s}_0\mathsf{ko})_{\geq_e -n})$ if $n \leq 0$ and $\tilde{s}_0\mathsf{ko}$ else. The assumptions of the theorem hold, by the same argument as in the proof of Lemma 5.15.

Step 1: In the E^1-page, only the zeroth and first line have non-trivial terms, i.e. only the zeroth and first subquotients are non-trivial.

Indeed, we have by definition $\tilde{s}_0\mathsf{ko} \in \mathsf{SH}(\mathbb{R})^{\mathsf{veff}} = \mathsf{SH}(\mathbb{R})^{\mathsf{eff}}_{\geq 0}$ (so that $(\tilde{s}_0\mathsf{ko})_{\geq_e 0} = \tilde{s}_0\mathsf{ko}$). By Claim (1) in the proof of [Bac17, Lemma 11], we have $(\tilde{s}_0\mathsf{ko})_{\geq_e 1} \simeq s_0(\mathsf{ko}_{\geq 1})$, and the latter is the first term in the cofiber sequence of Theorem 5.2, namely $\Sigma^{1,0}\mathsf{H}\mathbb{Z}/2$. The natural map $(\tilde{s}_0)_{\geq_e 1} \to \tilde{s}_0$ is exactly the map $\Sigma^{1,0}\mathsf{H}\mathbb{Z}/2 \to \tilde{s}_0$ appearing in this cofiber sequence.

In particular, the 0-th subquotient is nothing but the third term of this cofiber sequence, $\widetilde{\mathsf{H}\mathbb{Z}}$. In view of Proposition 2.33, it therefore makes sense to denote $s_1'\mathsf{ko} = (\tilde{s}_0\mathsf{ko})_{\geq_e 1} \simeq \Sigma^{1,0}\mathsf{H}\mathbb{Z}/2$ and $s_0'\mathsf{ko} = (\tilde{s}_0\mathsf{ko})/((\tilde{s}_0\mathsf{ko})_{\geq_e 1}) \simeq \widetilde{\mathsf{H}\mathbb{Z}}$. The first subquotient is actually $s_1'\mathsf{ko}$ since $(\tilde{s}_0\mathsf{ko})_{\geq_e k} \simeq 0$ for all $k \geq 2$, because in the cofiber sequence

$$(\Sigma^{1,0}\mathsf{H}\mathbb{Z}/2)_{\geq_e k} \to (\tilde{s}_0\mathsf{ko})_{\geq_e k} \to (\widetilde{\mathsf{H}\mathbb{Z}})_{\geq_e k},$$

the outer parts are zero. Indeed, we have $\mathsf{H}\mathbb{Z}/2, \widetilde{\mathsf{H}\mathbb{Z}} \in \mathsf{SH}(\mathbb{R})^{\mathsf{eff},\heartsuit}$ (Proposition 4.18), meaning in particular that $(\widetilde{\mathsf{H}\mathbb{Z}})_{\geq_e k} \simeq 0$ for all $k \geq 1$ and $(\Sigma^{1,0}\mathsf{H}\mathbb{Z}/2)_{\geq_e k} \simeq \Sigma^{1,0}(\mathsf{H}\mathbb{Z}/2_{\geq_e k-1}) \simeq 0$ for all $k \geq 2$.

Step 2: We write down the E^1-page.

By Subsections 4.3.3 and 4.3.1, we know the homotopy groups of the real realizations of the slices $s'_0\mathsf{ko}$ and $s'_1\mathsf{ko}$. We therefore obtain the following E^1-page for our spectral sequence computing the homotopy groups of $r_{\mathbb{R}}\tilde{s}_0\mathsf{ko}$:

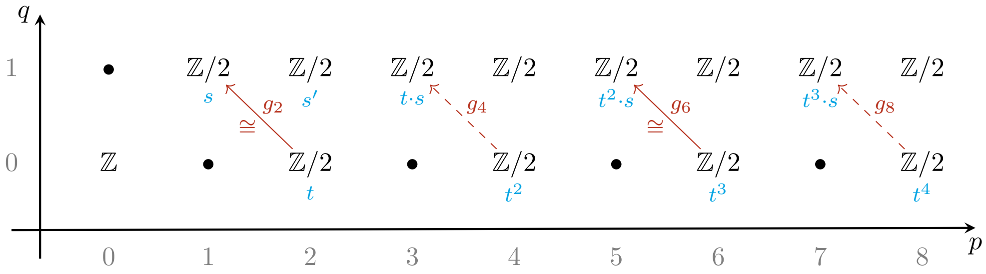

Figure 1: E^1-page for the spectral sequence for $r_{\mathbb{R}}\tilde{s}_0\mathsf{ko}$ with respect to the effective homotopy t-structure.

The bullets "•" represent trivial groups. In red are some d^1 differentials, we will explain how to determine them just below. The elements written in blue under some groups in the E^1-page are our notation for generators for these groups.

Step 3: We identify the differential g_2.

To begin with, note that we must have $\pi_1(r_{\mathbb{R}}\tilde{s}_0\mathsf{ko}) \cong 0$. Indeed, this group is the $E^1_{1,0}$-term in the spectral sequence given by the theorem for $X'_n = r_{\mathbb{R}}(\tilde{f}_{-n}\mathsf{ko})$ or $r_{\mathbb{R}}\mathsf{ko}$ if $n > 0$ (we will see in the proof of Lemma 5.15 why the assumptions are the theorem are verified in this case). We obtain a spectral sequence concentrated in the first quadrant with $E^1_{p,q} = \pi_p(\tilde{s}_q\mathsf{ko})$ for all $p, q \in \mathbb{N}$, converging strongly to $\pi_p(r_{\mathbb{R}}\mathsf{ko})$. In particular, the $E^1_{1,0}$-term will never receive a non-trivial differential. Moreover, since the first column of this same spectral contains only zeroes except in position $(0,0)$ (we have $\pi_0(r_{\mathbb{R}}\tilde{s}_i\mathsf{ko}) \cong 0$ for all $i \geq 1$), it will never be the source of a non-trivial differential either, and will survive to the E^∞-page. But since $\pi_1(r_{\mathbb{R}}\mathsf{ko}) \cong 0$ as we computed before, nothing in the first column can survive. This means that $\pi_1(r_{\mathbb{R}}\tilde{s}_0\mathsf{ko}) \cong 0$. In particular, in the spectral sequence for $r_{\mathbb{R}}(\tilde{s}_0\mathsf{ko})$ and the effective homotopy t-structure, $E^1_{1,1} \cong \mathbb{Z}/2$ must eventually die. Since it cannot be the source of any non-trivial differential, the only possibility is that it receives a non-trivial differential g_2 from $E^1_{2,0} \cong \mathbb{Z}/2$, which must then be an isomorphism.

Step 4: We use the multiplicative structure that this spectral sequence admits to compute the next differentials.

Consider the discussion in Subsection 5.2.6 about the multiplicative structure on spectral sequences. Let $E = \tilde{s}_0\mathsf{ko}$. In our case, for $X_n := r_{\mathbb{R}}(E_{\geq_e -n})$ if $n \leq 0$ and $r_{\mathbb{R}}E$ else, the pairings $X_{-n} \wedge X_{-m} \to X_{-n-m}$ are the real realizations of the maps

$$E_{\geq_e n} \wedge E_{\geq_e m} \to E \wedge E \to E$$

given by the natural maps followed by the multiplication on $E = \tilde{s}_0\mathsf{ko}$ (the latter is itself induced by the pairing in the tower for the very effective slice filtration on ko, as we will see below), which lifts to $E_{\geq_e n+m}$. Indeed, we have

$$\mathsf{SH}(\mathbb{R})^{\mathsf{eff}}_{\geq n} \wedge \mathsf{SH}(\mathbb{R})^{\mathsf{eff}}_{\geq m} = \Sigma^{n,0}\mathsf{SH}(\mathbb{R})^{\mathsf{eff}}_{\geq 0} \wedge \Sigma^{m,0}\mathsf{SH}(\mathbb{R})^{\mathsf{eff}}_{\geq 0} \subseteq \Sigma^{m+n,0}\mathsf{SH}(\mathbb{R})^{\mathsf{eff}}_{\geq 0} = \mathsf{SH}(\mathbb{R})^{\mathsf{eff}}_{\geq m+n}$$

because the effective homotopy t-structure is compatible with the symmetric monoidal structure (in the sense of [AN21, Def. A.10]). The multiplication on $E = \tilde{s}_0\mathsf{ko}$ is induced by the towers for the very effective slice filtration on ko.

In particular, we have pairings

$$\widetilde{\mathsf{H}\mathbb{Z}} \wedge \widetilde{\mathsf{H}\mathbb{Z}} \simeq s'_0\mathsf{ko} \wedge s'_0\mathsf{ko} \to s'_0\mathsf{ko} = \widetilde{\mathsf{H}\mathbb{Z}}$$

and

$$\widetilde{\mathsf{H}\mathbb{Z}} \wedge \Sigma^{1,0}\mathsf{H}\mathbb{Z}/2 \simeq s'_0\mathsf{ko} \wedge s'_1\mathsf{ko} \to s'_1\mathsf{ko} = \Sigma^{1,0}\mathsf{H}\mathbb{Z}/2.$$

We aim to show that these are exactly the pairings induced by the ring structure on $\widetilde{\mathsf{H}\mathbb{Z}}$, respectively the composition $\widetilde{\mathsf{H}\mathbb{Z}} \wedge \mathsf{H}\mathbb{Z}/2 \to \mathsf{H}\mathbb{Z} \wedge \mathsf{H}\mathbb{Z}/2 \to \mathsf{H}\mathbb{Z}/2$ induced by the natural map from Milnor–Witt K-theory to Milnor K-theory and the $\mathsf{H}\mathbb{Z}$-algebra structure on $\mathsf{H}\mathbb{Z}/2$. To do so, we prove that such maps are uniquely determined by their precomposition with the unit $\mathbb{1} \to \widetilde{\mathsf{H}\mathbb{Z}}$; and we show that both for the pairing coming from the filtration and for our candidate pairings, the map obtained in this way is the unitor $\mathbb{1} \wedge X \to X$, for $X = \widetilde{\mathsf{H}\mathbb{Z}}$ or $\Sigma^{1,0}\mathsf{H}\mathbb{Z}/2$ respectively.

Indeed, for any $E \in \mathsf{SH}(\mathbb{R})^{\mathsf{eff},\heartsuit}$, using that $\widetilde{\mathsf{H}\mathbb{Z}} = \mathbb{1}_{\leq_e 0}$ (by [Bac17, Lemma 12]), we have

$$\begin{aligned}\mathsf{SH}(\mathbb{R})(\widetilde{\mathsf{H}\mathbb{Z}} \wedge E, E) &\simeq \mathsf{SH}(\mathbb{R})((\widetilde{\mathsf{H}\mathbb{Z}} \wedge E)_{\leq_e 0}, E) \simeq \mathsf{SH}(\mathbb{R})((\mathbb{1}_{\leq_e 0} \wedge E_{\leq_e 0})_{\leq_e 0}, E)\\ &\simeq \mathsf{SH}(\mathbb{R})((\mathbb{1} \wedge E)_{\leq_e 0}, E) \simeq \mathsf{SH}(\mathbb{R})(\mathbb{1} \wedge E, E)\end{aligned}$$

since, in general, for $F, G \in \mathsf{SH}(\mathbb{R})^{\mathsf{eff}}_{\geq 0}$, we have $(F \wedge G)_{\leq_e 0} \simeq (F_{\leq_e 0} \wedge G_{\leq_e 0})_{\leq_e 0}$ via the natural maps $F \to F_{\leq_e 0}$ and $G \to G_{\leq_e 0}$. Indeed, almost by definition of a t-structure, we have in $\mathsf{SH}(\mathbb{R})^{\mathsf{eff}}_{\geq 0}$ a cofiber sequence $F_{\geq_e 1} \wedge G \to F \wedge G \to F_{\leq_e 0} \wedge G$. The associated long exact sequence for homotopy objects then gives an equivalence $(F \wedge G)_{\leq_e 0} \xrightarrow{\simeq} (F_{\leq_e 0} \wedge G)_{\leq_e 0}$ (since we have $F_{\geq_e 1} \wedge G \in \mathsf{SH}(\mathbb{R})^{\mathsf{eff}}_{\geq_e 1} \wedge \mathsf{SH}(\mathbb{R})^{\mathsf{eff}}_{\geq_e 0} \subseteq \mathsf{SH}(\mathbb{R})^{\mathsf{eff}}_{\geq_e 1}$). Re-applying the same argument exchanging the roles of F and G, we obtain our claim.

Moreover, the equivalence of mapping spaces obtained above is induced by precomposition with $\mathbb{1} \to \mathbb{1}_{\leq_e 0} \simeq \widetilde{\mathsf{H}\mathbb{Z}}$ (tensored with E). Indeed, there is a commutative diagram

$$\begin{tikzcd}
\mathbb{1} \wedge E \arrow[r, "\tau_1"] \arrow[d, "\tau_2"'] & \mathbb{1}_{\leq_e 0} \wedge E_{\leq_e 0} \simeq \widetilde{\mathsf{H}\mathbb{Z}} \wedge E \arrow[r, "\vartheta"] \arrow[d] & E_{\leq_e 0} \simeq E \\
(\mathbb{1} \wedge E)_{\leq_e 0} \arrow[r, "\tau_3"] & (\mathbb{1}_{\leq_e 0} \wedge E_{\leq_e 0})_{\leq_e 0}. \arrow[ur, dashed, "\vartheta'"'] &
\end{tikzcd}$$

where all maps in the square are induced by the natural maps in the truncation.

The equivalence above sends a map ϑ to the composite $\vartheta' \circ \tau_3 \circ \tau_2 \simeq \vartheta \circ \tau_1$, as desired.

We now apply this result to $E = \widetilde{\mathsf{H}\mathbb{Z}}$, or $E = \mathsf{H}\mathbb{Z}/2$ (recall that $\widetilde{\mathsf{H}\mathbb{Z}}, \mathsf{H}\mathbb{Z}/2 \in \mathsf{SH}(\mathbb{R})^{\mathsf{eff},\heartsuit}$ by Proposition 4.18), since $\mathsf{SH}(\mathbb{R})(\widetilde{\mathsf{H}\mathbb{Z}} \wedge \Sigma^{1,0}\mathsf{H}\mathbb{Z}/2, \Sigma^{1,0}\mathsf{H}\mathbb{Z}/2) \simeq \mathsf{SH}(\mathbb{R})(\widetilde{\mathsf{H}\mathbb{Z}} \wedge \mathsf{H}\mathbb{Z}/2, \mathsf{H}\mathbb{Z}/2)$, we only have to show that the pairings induced on the subquotients of the tower correspond to the unitor after precomposition by the natural map $\mathbb{1} \to \mathbb{1}_{\leq_e 0} \simeq \widetilde{\mathsf{H}\mathbb{Z}}$. This can be proven by chasing through the construction of the pairing induced on the subquotients; which is ultimately obtained from the multiplication on ko, which of course is unital. More precisely, the unit $\mathbb{1} \to \mathbb{1}_{\leq_e 0} = \widetilde{\mathsf{H}\mathbb{Z}}$ is exactly the one induced by

$$\mathbb{1} \longrightarrow \mathsf{ko} \longrightarrow \tilde{s}_0\mathsf{ko} = f_0(\tilde{s}_0\mathsf{ko}) \longrightarrow f_0((\tilde{s}_0\mathsf{ko})_{\leq 0}) = f_0(\underline{\pi}_0(\tilde{s}_0\mathsf{ko})) = s'_0\mathsf{ko}.$$

To check that $\mathbb{1} \wedge s'_0\mathsf{ko} \to s'_0\mathsf{ko} \wedge s'_0\mathsf{ko} \to s'_0\mathsf{ko}$ is the unitor, for example, using diagram 3, we want to show that it holds true for $\mathbb{1} \wedge (\tilde{s}_0\mathsf{ko})_{\geq_e i} \to (\tilde{s}_0\mathsf{ko})_{\geq_e i}$ for $i = 0, 1$. Since these pairings are induced by one we had on $\tilde{s}_0$, which is induced from that on the very effective filtration, we want to show it for the maps $\mathbb{1} \wedge \tilde{f}_i\mathsf{ko} \to \tilde{f}_i\mathsf{ko}$ for $i = 0, 1$, and these where ultimately induced by $\mathbb{1} \wedge \mathsf{ko} \to \mathsf{ko} \wedge \mathsf{ko} \to \mathsf{ko}$, whence the claim follows.

What we just showed proves that the pairing $S_0 \wedge S_0 \to S_0$ in our spectral sequence is induced by the ring structure of $\pi_*(r_\mathbb{R}\widetilde{\mathsf{H}\mathbb{Z}}) = \mathbb{Z}[t^2]/(2t^2)$ (Proposition 4.27), with $\widetilde{\mathsf{H}\mathbb{Z}} \to \mathsf{H}\mathbb{Z}/2$ killing 2. Furthermore, the pairing $S_0 \wedge S_1 \to S_1$ with $S_1 = \Sigma^{1,0}\mathsf{H}\mathbb{Z}/2$ is given after desuspension by $\mathbb{Z}[t^2]/(2t^2) \otimes \mathbb{Z}/2[t] \to \mathbb{Z}/2[t^2] \otimes \mathbb{Z}/2[t] \to \mathbb{Z}/2[t]$ where the first map is the quotient and the second one is multiplication (viewing $\mathbb{Z}/2[t^2] \subseteq \mathbb{Z}/2[t]$ by Proposition 4.21). This explains why the generators are the ones displayed in the spectral sequence in Figure 1. Since g_2 is an isomorphism, we have $d^1(t) = s$. It follows that g_{2k} is zero for k even and an isomorphism for k odd. Indeed by the Leibniz rule $g_{2k}(t^k) = d^1(t^k) = kt^{k-1}d^1(t) \in \mathbb{Z}/2$ is 0 if k is even and equals $t^{k-1} \cdot s$ otherwise (i.e. a generator in the target).

This proves the statement of the lemma. □

Lemma 5.15. *With the same notation as in Lemma 5.13, we have $|\pi_4(r_\mathbb{R}C)| = 16$.*

Proof. We apply Theorem 5.12 again to $\mathcal{C} = \mathsf{Sp}$, $\mathcal{A} = \mathsf{Ab}$ the 1-category of abelian groups, $\pi = \pi_0$, and $X_n = r_{\mathbb{R}}(\tilde{f}_{-n}\mathsf{ko})$ or $r_{\mathbb{R}}\mathsf{ko}$ if $n > 0$. The theorem applies because all colimits exists in Sp; we have $\pi_n(X_r) = \pi_n(r_{\mathbb{R}}(\tilde{f}_{-r}\mathsf{ko})) = 0$ for $r < -n$ (then $\tilde{f}_{-r}\mathsf{ko} \in \mathsf{SH}(\mathbb{R})^{\mathsf{veff}}(-r)$, and real realization maps this subcategory to $(-r)$-connective spectra (Lemma 4.8); and π_0 preserves sequential colimits. Moreover, $\mathsf{colim}_n X_n = \mathsf{colim}(\cdots \to r_{\mathbb{R}}(\tilde{f}_2\mathsf{ko}) \to r_{\mathbb{R}}(\tilde{f}_1\mathsf{ko}) \to r_{\mathbb{R}}\mathsf{ko} \to r_{\mathbb{R}}\mathsf{ko} \to \cdots) = r_{\mathbb{R}}\mathsf{ko}$. We obtain a spectral sequence concentrated in the first quadrant with $E^1_{p,q} = \pi_p(\tilde{s}_q\mathsf{ko})$ for all $p, q \in \mathbb{N}$, converging strongly to $\pi_p(r_{\mathbb{R}}\mathsf{ko})$. We know the latter groups, and we will use this to compute several differentials in the E_1-page. Then, since the same spectral sequence for ko replaced with C is given by the truncation above the third line (i.e. we only keep indices $0 \le p \le 4$ and $p \in \mathbb{N}$), this will give us information about the homotopy of $r_{\mathbb{R}}C$.

It follows from the computation of the very effective slices of ko (Theorem 5.2) and the homotopy groups of their real realizations (Propositions 4.21 and 4.27, and Lemma 5.14) that the E^1-page of our spectral sequence for $r_{\mathbb{R}}\mathsf{ko}$ with respect to the very effective slice filtration looks as follows:

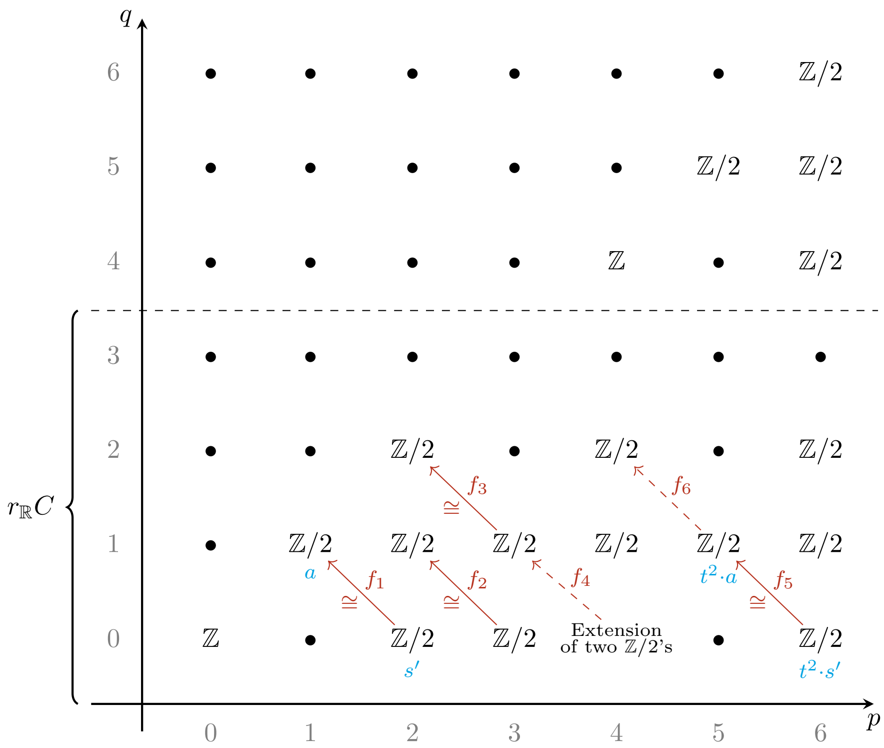

Figure 2: E^1-page for the spectral sequence for $r_{\mathbb{R}}\mathsf{ko}$ with respect to the very effective slice filtration.

The notation is as in Figure 1. We now explain how to determine the differentials.

Since $r_{\mathbb{R}}\mathsf{ko}$ has non trivial homotopy only in degrees divisible by 4, in columns whose index is not divisible by 4, nothing must survive to the E^{∞}-page.

- No non-zero differential with source $E^1_{1,1} = \mathbb{Z}/2$ can exist. So it must receive a non-trivial differential on some page; the only possibility is that $d^1 : E^1_{2,0} \to E^1_{1,1}$, namely f_1, is an isomorphism.
- The same argument shows that f_2 is an isomorphism.
- The $E^1_{2,2}$-term could a priori also receive a non-trivial differential d^2 from $E^2_{3,0}$ on the E^2-page; but since f_2 is an isomorphism, $E^2_{3,0} = 0$. Thus f_3 must be an isomorphism.
- Since $d^1 \circ d^1 = 0$, we have $f_3 \circ f_4 = 0$, so by the previous bullet point $f_4 = 0$. We conclude that the $E^1_{4,0}$-term survives to the E_{∞}-page, and the $E^1_{4,1}$-term too since it cannot neither be the source nor the target of a non-trivial differential.

Unfortunately, these arguments do not allow us to determine f_5 and f_6. The multiplicative structure on this spectral sequence provides the missing information.

The multiplicative structure comes this time from the pairing $\tilde{f}_n\mathsf{KO}\wedge\tilde{f}_m\mathsf{KO}\to\tilde{f}_{n+m}\mathsf{KO}$ obtained as follows: for any $n,m\geq 0$, the composition $\tilde{f}_n\mathsf{KO}\wedge\tilde{f}_m\mathsf{KO}\to\mathsf{KO}\wedge\mathsf{KO}\to\mathsf{KO}$ given by the natural maps followed by multiplication lifts to $\tilde{f}_{n+m}\mathsf{ko}$ because

$$\begin{aligned}\mathsf{SH}(\mathbb{R})^{\mathsf{veff}}(n)\wedge\mathsf{SH}(\mathbb{R})^{\mathsf{veff}}(m)&=(T^{\wedge n}\wedge\mathsf{SH}(\mathbb{R})^{\mathsf{veff}})\wedge(T^{\wedge m}\wedge\mathsf{SH}(\mathbb{R})^{\mathsf{veff}})\\&\subseteq(T^{\wedge(n+m)}\wedge\mathsf{SH}(\mathbb{R})^{\mathsf{veff}})=\mathsf{SH}(\mathbb{R})^{\mathsf{veff}}(n+m).\end{aligned}$$

The compatibility between these pairings come from the ($\mathcal{E}_\infty$-)ring axioms for KO. Then the pairing descend to the slices as in diagram 3.

We claim that the pairing $\tilde{s}_0\mathsf{ko}\wedge\tilde{s}_1\mathsf{ko}\to\tilde{s}_1\mathsf{ko}$ factors through $(\tilde{s}_0\mathsf{ko})/(s_1'\mathsf{ko})\ \wedge\ \tilde{s}_1\mathsf{ko}\simeq s_0'\mathsf{ko}\wedge\tilde{s}_1\mathsf{ko}$. Indeed, the latter is the cofiber of the map $s_1'\mathsf{ko}\wedge\tilde{s}_1\mathsf{ko}\to\tilde{s}_0\mathsf{ko}\wedge\tilde{s}_1\mathsf{ko}$, whose post composition with the pairing is zero. Indeed, $\Sigma_T^{-1}(s_1'\mathsf{ko}\wedge\tilde{s}_1\mathsf{ko}\to\tilde{s}_1\mathsf{ko})$ is a map $\Sigma^{1,0}\mathsf{H}\mathbb{Z}/2\wedge\mathsf{H}\mathbb{Z}/2\to\mathsf{H}\mathbb{Z}/2$, so it must be zero by the axioms of a t-structure (the source belongs to $\mathsf{SH}(\mathbb{R})^{\mathsf{eff}}_{\geq 1}$ whereas the right hand side belongs to $\mathsf{SH}(\mathbb{R})^{\mathsf{eff}}_{\leq 0}$ (Proposition 4.18). Moreover, the map $\tilde{s}_0\mathsf{ko}\wedge s_1'\mathsf{ko}\to\tilde{s}_0\mathsf{ko}\wedge\tilde{s}_0\mathsf{ko}\to\tilde{s}_0\mathsf{ko}$ induced by the natural map and the pairing lifts to $s_1'\mathsf{ko}$ on the target, and then by the same argument as above this lift factors through $s_0'\mathsf{ko}\wedge s_1'\mathsf{ko}$.

Then, since $\tilde{s}_1\mathsf{ko}\simeq\Sigma^{2,1}\mathsf{H}\mathbb{Z}/2$, we can use the exact same strategy as we did for $s_1'\mathsf{ko}\simeq\Sigma^{1,0}\mathsf{H}\mathbb{Z}/2$ to show that $s_0'\mathsf{ko}\wedge\tilde{s}_1\mathsf{ko}\to\tilde{s}_1\mathsf{ko}$ is the $(2,1)$ suspension of the pairing $\widetilde{\mathsf{H}\mathbb{Z}}\wedge\mathsf{H}\mathbb{Z}/2\to\mathsf{H}\mathbb{Z}/2$ previously considered. In particular, we deduce the following:

- Let us also denote by s' a generator of $\pi_2(r_\mathbb{R}\tilde{s}_0\mathsf{ko})\cong\mathbb{Z}/2$ (it comes from the generator s' of $\pi_2(r_\mathbb{R}s_1'\mathsf{ko})$ in Figure 1). Then, a preimage in $\pi_4(r_\mathbb{R}\tilde{s}_0\mathsf{ko})$ for $t^2\in\pi_4(r_\mathbb{R}s_0'\mathsf{ko})$ acts upon s' by sending it to the image of $t^2\cdot s'\in\pi_6(r_\mathbb{R}s_1'\mathsf{ko})$ in $\pi_6(r_\mathbb{R}\tilde{s}_0\mathsf{ko})\cong\mathbb{Z}/2$, which is a generator, as we can see from the spectral sequence for $r_\mathbb{R}\tilde{s}_0\mathsf{ko}$ (Figure 1).
- Let a be a generator of $\pi_1(r_\mathbb{R}\tilde{s}_1\mathsf{ko})\cong\mathbb{Z}/2$. Then a preimage in $\pi_4(r_\mathbb{R}\tilde{s}_0\mathsf{ko})$ for $t^2\in\pi_4(r_\mathbb{R}s_0'\mathsf{ko})$ acts upon a by sending it to a generator in $\pi_5(r_\mathbb{R}\tilde{s}_1\mathsf{ko})\cong\mathbb{Z}/2$ (this is the action of $r_\mathbb{R}\widetilde{\mathsf{H}\mathbb{Z}}$ on $r_\mathbb{R}\Sigma^{2,1}\mathsf{H}\mathbb{Z}/2$ as we have seen it before, for the spectral sequence in Figure 1).
- Therefore, since by the Leibniz rule for d^1, we have

  $$f_5(t^2\cdot s')=d^1(t^2\cdot s')=t^2\cdot d^1(s')+2s'd^1(t)=t^2\cdot a\in E^1_{5,1}\cong\mathbb{Z}/2$$

  (recall that f_1 is an isomorphism), f_5 maps a generator to a generator and is an isomorphism. In particular, since $d^1\circ d^1=0$, we also get $f_6=0$, and the $E^1_(4,2)$-term survives to the E_∞-page.

Now the spectral sequence for $r_\mathbb{R}C=r_\mathbb{R}(\mathsf{ko}/\tilde{f}_4\mathsf{ko})$ is the truncation of that for $r_\mathbb{R}\mathsf{ko}$ above the third horizontal line (indeed, $\tilde{s}_iC=\tilde{s}_i\mathsf{ko}$ for $0\leq i\leq 3$, and 0 otherwise). In particular, the differentials must be the same and in the fourth column, all terms on the E^1-page survive to the $\mathcal{E}_\infty$-page. Therefore, $\pi_4(r_\mathbb{R}C)$ is an extension of four copies of $\mathbb{Z}/2$; it has cardinality 16 (we can even identify it as $\mathbb{Z}/16\mathbb{Z}$ because in the proof of Lemma 5.13 that it was cyclic). This concludes the proof. $\square$

5.3 Comparison with the L-theory of $\mathbb{R}$

As mentioned in the introduction to this section, our interest in a 2-local fracture square for $r_\mathbb{R}\mathsf{ko}$ is also driven by the hope of comparing it to the 2-local fracture square for the L-theory spectrum of $\mathbb{R}$ appearing in [HLN21] (with a typo[1]), which reads as follows:

Theorem 5.16 ([HLN21, Rmk (3) on p3]). *There is a Cartesian square of $\mathcal{E}_1$-rings*

$$\begin{tikzcd}\mathsf{L}(\mathbb{R}) \arrow[r, "b\mapsto\beta_4/2"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathsf{KO}^{\mathsf{top}}[1/2] \arrow[d, "\mathsf{ch}"] \\ \mathsf{H}\mathbb{Z}_{(2)}[t^4,(t^4)^{-1}] \arrow[r, "t^4\mapsto t^4"'] & \mathsf{H}\mathbb{Q}[t^4,(t^4)^{-1}]\end{tikzcd}$$

where the bottom left and right corner are respectively the free $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}_{(2)}$-algebra on one invertible generator in degree 4, and the free $\mathcal{E}_1$-$\mathsf{H}\mathbb{Q}$-algebra on one invertible generator in degree 4. The map

[1] Private communication with Markus Land

ch *is the Chern character from Definition 5.8. The element $b \in \pi_4(\mathsf{L}(\mathbb{R})) \cong \mathbb{Z}$ is a generator. In particular, taking connective covers, we obtain a Cartesian square of $\mathcal{E}_1$-rings*

$$\begin{tikzcd} \mathsf{L}(\mathbb{R})_{\geq 0} \arrow[r, "b\mapsto\beta_4/2"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{ko}^{\mathsf{top}}[1/2] \arrow[d, "\mathsf{ch}"] \\ \mathsf{H}\mathbb{Z}_{(2)}[t^4] \arrow[r, "t^4\mapsto t^4"'] & \mathsf{H}\mathbb{Q}[t^4]. \end{tikzcd}$$

Proof. We only prove the connective version, since this is the one we want to compare $r_{\mathbb{R}}\mathsf{ko}$ to. Proposition A.6 shows the existence of a 2-local fracture square. We now identify more explicitly the corners of the Cartesian square and the maps appearing in it.

- To identify the localization away from 2, we consider the same map $\mathsf{ko}^{\mathsf{top}} \to \mathsf{L}(\mathbb{R})_{\geq 0}$ as in the proof of Theorem 5.7, from [LNS23]; recall that this map is an equivalence after localizing at 2, and induces multiplication by 8 on the fourth homotopy groups before localization. We post-compose it with the automorphism of $\mathsf{ko}^{\mathsf{top}}[1/2]$ given by the (connective cover of the) second Adams operation ψ^2 as in the proof of Theorem 5.7, and obtain $\mathsf{L}(\mathbb{R})_{\geq 0} \to \mathsf{ko}^{\mathsf{top}}[1/2]$ sending b to $\beta_4/2$ in the fourth homotopy groups.

- To identify the localization at (2), we proceed similarly as in the proof of Theorem 5.7: there is a map of $\mathcal{E}_\infty$-rings $\mathsf{MSO} \to \mathsf{L}(\mathbb{R})$, and by [HLN21, Cor. 3.7], $\mathsf{MSO}_{(2)}$ receive a map of $\mathcal{E}_2$-rings from $\mathsf{H}\mathbb{Z}_{(2)}$. Therefore $\mathsf{L}(\mathbb{R})_{(2)}$ receives a $\mathcal{E}_2$-ring map from $\mathsf{H}\mathbb{Z}_{(2)}$ (see). Then it also receives a map of $\mathcal{E}_1$-rings from the free $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}_{(2)}$-algebra $\mathsf{H}\mathbb{Z}_{(2)}[t^4]$, sending t^4 to b in π_4. Since the domain is connective, this map lifts to $\mathsf{L}(\mathbb{R})_{\geq 0}$. The ring structure on the homotopy groups on both sides implies that this map is an equivalence ($\pi_*(\mathsf{L}(\mathbb{R})) \cong \mathbb{Z}[b]$ with $|b| = 4$ by [HLN21, Prop. 4.1], and $\pi_*(\mathsf{H}\mathbb{Z}_{(2)}[t^4])$ is the same ring, generated by t^4, by Proposition 1.38(ii)).

- We have seen that $\mathsf{ko}^{\mathsf{top}}[1/2] \to \mathsf{ko}^{\mathsf{top}}[1/2]_{\mathbb{Q}} \xrightarrow{\simeq} \mathsf{H}\mathbb{Q}[t^4]$ via the Chern character in Remark 5.9.

- Finally, we have to determine what is the map $\mathsf{H}\mathbb{Z}_{(2)}[t^4] \to \mathsf{H}\mathbb{Q}[t^4]$ in the pullback square, corresponding to our choices of identifications. Studying the actions of these maps on the fourth homotopy groups, we see that the composite $\mathsf{L}(\mathbb{R})_{\geq 0} \to \mathsf{ko}^{\mathsf{top}}[1/2] \to \mathsf{H}\mathbb{Q}[t^4]$ maps b to t^4. Since the map $\mathsf{L}(\mathbb{R})_{\geq 0} \to \mathsf{H}\mathbb{Z}_{(2)}[t^4]$ we have chosen sends b to t^4, the bottom horizontal map must send t^4 to t^4.

This finishes the proof. □

Comparing with Theorem 5.7, we deduce the following:

Theorem 5.17. *The real realization of ko is equivalent as an $\mathcal{E}_1$-ring to the connective cover of the L-theory spectrum of the real numbers*

$$r_{\mathbb{R}}\mathsf{ko} \simeq \mathsf{L}(\mathbb{R})_{\geq 0}.$$

Remark 5.18. For comparison, recall the results from Proposition 4.28 and Theorem 5.6 stating that $r_{\mathbb{R}}\mathsf{KO} \simeq \mathsf{L}(\mathbb{R})[1/2] \simeq \mathsf{KO}^{\mathsf{top}}[1/2]$ as $\mathcal{E}_\infty$-rings.

6 Motivic and topological multiplicative Thom spectra

In this Section, our goal is to to show that $r_{\mathbb{R}}\mathsf{MSL} \simeq \mathsf{MSO}$ as $\mathcal{E}_\infty$-rings. We need this result to produce an $\mathcal{E}_2$-map $\mathsf{H}\mathbb{Z}_{(2)} \to r_{\mathbb{R}}(\mathsf{MSL})_{(2)}$ (and thus $\mathsf{H}\mathbb{Z}_{(2)} \to r_{\mathbb{R}}(\mathsf{ko})_{(2)}$) in the proof of Proposition 5.3. This is actually a stronger result than we would need, but it seems to be of independent interest.

The $\mathcal{E}_\infty$-structures on MSL and MSO are both obtained by expressing them as *Thom spectra*; indeed a symmetric monoidal Thom spectrum functor can be constructed both in the topological and motivic setting. What we will actually show is that the real realization of the motivic Thom spectrum functor (also called motivic colimit functor in a more general setting) and the topological Thom spectrum functor agree as symmetric monoidal functors, in a sense yet to be made precise.

We will first recall how the topological and motivic multiplicative Thom spectrum functors are constructed (in Subsections 6.1 and 6.2 respectively). Our main references are [BH21] for motivic multiplicative Thom spectra, [BEH22] for the motivic colimit functors, and [ACB19] and [ABG18] for topological multiplicative Thom spectra. We will then compare these formalisms in Subsection 6.3, via real realization. Many subtleties must be taken into account here. For example, we have to show that our diagrams commute in the ∞-category of symmetric monoidal ∞-categories and symmetric monoidal functors, and not only at the level of functors between mere ∞-categories. And this, to ensure that the functors they induce between the categories of $\mathcal{E}_\infty$-algebras on both sides agree. Our arguments will show that $r_{\mathbb{R}}\mathsf{MSL} \simeq \mathsf{MSO}$ "as multiplicative Thom spectra" (in particular, as $\mathcal{E}_\infty$-rings), but the general set-up in which we work allows us to consider other classical examples. For instance, it follows using the same methods that $r_{\mathbb{R}}\mathsf{MGL} \simeq \mathsf{MO}$ and $r_{\mathbb{R}}\mathsf{MSp} \simeq \mathsf{MU}$ "as multiplicative Thom spectra". This extends [BH20, Cor. 4.7], where these equivalences are proven at the level of spectra.

6.1 Topological multiplicative Thom spectra

Definition 6.1. Let $\mathsf{Sp}^{\simeq}$ be the maximal ∞-groupoid in the ∞-category Sp. The (topological) *Thom spectrum functor*

$$M_{\mathsf{top}} : \mathsf{Spc}_{/\mathsf{Sp}^{\simeq}} \longrightarrow \mathsf{Sp}$$

is the colimit preserving functor sending an arrow $\mathsf{Spc} \longrightarrow \mathsf{Sp}^{\simeq}$ to the colimit of the diagram

$$\mathsf{Spc} \longrightarrow \mathsf{Sp}^{\simeq} \hookrightarrow \mathsf{Sp}.$$

More precisely, under the equivalence $\mathsf{Spc}_{/\mathsf{Sp}^{\simeq}} \simeq \mathcal{P}(\mathsf{Sp}^{\simeq})$ from Lemma A.15 (at the level of ∞-categories), the functor M_{top} is the left Kan extension of the embedding $\mathsf{Sp}^{\simeq} \hookrightarrow \mathsf{Sp}$.

Remark 6.2. In the above definition, we would like to look at $\mathsf{Sp}^{\simeq}$ as a space. This raises set-theoretic issues. Indeed, the ∞-category Sp is not small, so $\mathsf{Sp}^{\simeq}$ is a large ∞-groupoid. The same problem will arise with the ∞-groupoid $\mathsf{SH}(S)^{\simeq}$ in Subsection 6.2. There are various ways of dealing with such problems in the context of ∞-categories, as explained in [Lur09, §1.2.15]. The approach Lurie chooses is to work with *Grothendieck universes* associated with certain cardinal numbers. In this way, one can distinguish (depending on the cardinal chosen) between "small" and "large" objects.

In our case, we could replace $\mathsf{Sp}^{\simeq}$ with the essentially small ∞-groupoid $\mathsf{Pic}(\mathsf{Sp})$ of invertible objects in Sp. This is in fact the more classical definition of the Thom spectrum functor, and the examples we are interested in (see Example 6.19 and proof of Proposition 5.3) are anyway obtained by applying the Thom spectrum functors to maps $X \to \mathsf{Sp}^{\simeq}$ which factor through $\mathsf{Pic}(\mathsf{Sp})$. The same remark applies to $\mathsf{SH}(S)^{\simeq}$.

We will however continue to work with $\mathsf{Sp}^{\simeq}$ and $\mathsf{SH}(S)^{\simeq}$, because we want to study the construction of the motivic Thom spectrum functor in [BH21, §16.3], which uses $\mathsf{SH}^{\simeq}$, and then compare it to the topological Thom spectrum functor. We will see in Subsection 6.3.3 that this comparison is most naturally expressed using $\mathsf{Sp}^{\simeq}$ if we use $\mathsf{SH}^{\simeq}$ to being with.

We would like to upgrade M_{top} to a symmetric monoidal functor, so that it induces in particular a functor $\mathsf{CAlg}(\mathsf{Spc}_{/\mathsf{Sp}^{\simeq}}) \to \mathsf{CAlg}(\mathsf{Sp})$, which is used to define the $\mathcal{E}_\infty$-structure on MSO (respectively MO and MU). To begin with, we have to define a symmetric monoidal structure on the source of M_{top}. Proposition A.14 provides us with such a structure, if we choose a symmetric monoidal structure on Spc and an $\mathcal{E}_\infty$-algebra structure on $\mathsf{Sp}^{\simeq} \in \mathsf{Spc}$. We endow Spc with its Cartesian structure, and then $\mathsf{Sp}^{\simeq}$ inherits an $\mathcal{E}_\infty$-structure from that of $\mathsf{Sp} \in \mathsf{CAlg}(\mathsf{Cat}_\infty^{\times})$. Indeed, the functor

$(-)^{\simeq} : \mathsf{Cat}_\infty \to \mathsf{Gpd} \simeq \mathsf{Spc}$ is symmetric monoidal with respect to the Cartesian structure, because it preserves products (as a right adjoint to the forgetful functor), and so it preserves $\mathcal{E}_\infty$-algebras.

Theorem 6.3 ([ABG18, Thm 1.6])**.** *The functor M_{top} admits the structure of a symmetric monoidal functor with respect to the symmetric monoidal structure on the slice ∞-category described in Proposition A.14 (equivalent by Lemma A.15 to the Day convolution structure) and the usual smash product of spectra. It is left Kan extended as a symmetric monoidal functor (in the sense of Definition A.10) from the inclusion $\mathsf{Sp}^{\simeq} \hookrightarrow \mathsf{Sp}$. It thus induces a functor $\mathsf{CAlg}(\mathsf{Spc}_{/\mathsf{Sp}^{\simeq}}) \to \mathsf{CAlg}(\mathsf{Sp})$ which we may call the* (topological) multiplicative Thom spectrum functor*.*

The symmetric monoidal left Kan extension described in the theorem agrees with the construction in [ABG18, Thm 1.6] because, by Proposition A.11, it is determined as a symmetric monoidal functor by its restriction to $*_{/\mathsf{Sp}^{\simeq}} \simeq \mathsf{Sp}^{\simeq}$.

Example 6.4 (see for instance [Bea23]). The spectrum MO is obtained as $M_{\mathsf{top}}(BO \to \mathsf{Sp}^{\simeq})$ where the map $BO \to \mathsf{Sp}^{\simeq}$ is the *j-homomorphism.* To define the latter, following [HY20, p. 2], one first considers the symmetric monoidal functor $\coprod_{n\in\mathbb{N}} BO_n \longrightarrow \mathsf{Sp}$. Here the left hand side is viewed as the maximal ∞-groupoid in the 1-category of real vector spaces viewed as an ∞-category (or in other terms real vector bundles on the point), and a vector bundle is sent to the infinite suspension of its Thom space (in this case, the one-point compactification of our vector space). This map takes values in the subgroupoid of invertible spectra (also denoted $\mathsf{Pic}(\mathsf{Sp})$), and thus factors through the group completion $\mathbb{Z} \times BO$ of $\coprod_{n\in\mathbb{N}} BO_n$. Restricting to $\{0\} \times BO$ (corresponding to rank 0 virtual vector bundles on the point), we obtain the j-homomorphism $BO \to \mathsf{Pic} \hookrightarrow \mathsf{Sp}^{\simeq}$.

Similarly, MSO and MU are obtained as $M_{\mathsf{top}}(BSO \to BO \to \mathsf{Sp}^{\simeq})$ and $M_{\mathsf{top}}(BU \to BO \to \mathsf{Sp}^{\simeq})$ respectively.

By Proposition A.16, any map of $\mathcal{E}_\infty$-spaces $X \to \mathsf{Sp}^{\simeq}$ defines a commutative algebra in $\mathsf{Spc}_{/\mathsf{Sp}^{\simeq}}$. We thus have $\mathcal{E}_\infty$-algebra structures on $BO \to \mathsf{Sp}^{\simeq}$, $BSO \to \mathsf{Sp}^{\simeq}$, and $BU \to \mathsf{Sp}^{\simeq}$ in the slice ∞-category (since the j-homomorphism is an $\mathcal{E}_\infty$-map by its construction as a symmetric monoidal functor). The Thom spectrum functor being symmetric monoidal, we obtain $\mathcal{E}_\infty$-structures on MO, MSO, and MU respectively.

As a consequence of Theorem 6.3 and Proposition A.16 (which describes $\mathcal{E}_n$-algebras in the slice ∞-category), for any $n \in \mathbb{N} \cup \{\infty\}$ and $X \in \mathsf{Alg}_{\mathcal{E}_n}(\mathsf{Spc})$, there are induced functors

$$M_{\mathsf{top}} : \mathsf{map}_{\mathsf{Alg}_{\mathcal{E}_n}(\mathsf{Spc})}(X, \mathsf{Sp}^{\simeq}) \hookrightarrow \mathsf{Alg}_{\mathcal{E}_n}(\mathsf{Spc}_{/\mathsf{Sp}^{\simeq}}) \longrightarrow \mathsf{Alg}_{\mathcal{E}_n}(\mathsf{Sp}).$$

Under the equivalence $\mathsf{map}_{\mathsf{Alg}_{\mathcal{E}_n}(\mathsf{Spc})}(X, \mathsf{Sp}^{\simeq}) \simeq \mathsf{map}_{\mathsf{Alg}_{\mathcal{E}_n}(\mathsf{Cat}_\infty)}(X, \mathsf{Sp})$, we obtain that M_{top} endows the colimit of an $\mathcal{E}_n$-monoidal functor $X \to \mathsf{Sp}$ with the structure of an $\mathcal{E}_n$-algebra in Sp. The following proposition is a result generalizing this fact.

Proposition 6.5 ([ACB19, Thm 2.8 and Cor. 2.11])**.** *Let $\mathcal{O}$ be an ∞-operad, and let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{O}$-monoidal ∞-categories, with $\mathcal{D}$ cocomplete. There is a canonical structure of $\mathcal{O}$-algebra on the colimit of any lax $\mathcal{O}$-monoidal functor $\mathcal{C} \to \mathcal{D}$. More precisely, if $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ exhibits $\mathcal{C}$ as an $\mathcal{O}$-monoidal ∞-category, then the colimit functor defines a left adjoint to precomposition by p*

$$\mathsf{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D}) = \mathsf{Fun}^{\mathsf{lax}}_{\mathcal{O}}(\mathcal{C}, \mathcal{D}) \underset{-\circ p}{\overset{\mathsf{colim}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{Alg}_{\mathcal{O}}(\mathcal{D}).$$

Remark 6.6. In the setting of Proposition A.9, if $F : \mathcal{C} \to \mathcal{D}$ is a lax symmetric monoidal functor, let $G := \mathsf{LKE}^{\otimes}(F)$. Then G is a lax symmetric monoidal functor $\mathcal{P}(\mathcal{C}) \to \mathcal{D}$. If $\mathcal{F} \in \mathsf{Alg}_{\mathcal{E}_n}(\mathcal{P}(\mathcal{C}))$, then the $\mathcal{E}_n$-structure induced on $G(\mathcal{F}) \simeq \mathsf{colim}_{c\in\mathcal{C}_{/\mathcal{F}}} F(c)$ is exactly the $\mathcal{E}_n$-structure from Proposition 6.5 on the colimit of the lax symmetric monoidal functor $\mathcal{C}_{/\mathcal{F}} \to \mathcal{C} \xrightarrow{F} \mathcal{D}$.

6.2 Motivic multiplicative Thom spectra

In this subsection, we define the motivic analog M to M_{top}. More precisely, for any base scheme S (say, Noetherian of finite dimension, or belonging to Sm_k), there is a motivic multiplicative Thom spectrum funtor $M_S : \mathcal{P}_\Sigma(\mathsf{Sm}_S)_{\mathsf{SH}^{\simeq}} \to \mathsf{SH}(S)$, where $\mathcal{P}_\Sigma(\mathsf{Sm}_S)$ is the ∞-category of spherical presheaves (Definition 6.11), and $\mathsf{SH}^{\simeq}$ is the presheaf sending $X \in \mathsf{Sm}_S$ to the ∞-groupoid $\mathsf{SH}(X)^{\simeq}$. We will compare in Subsection 6.3 the functors $M_{\mathbb{R}}$ and M_{top} under real realization. The motivic Thom spectrum functor allows us in particular to define the motivic $\mathcal{E}_\infty$-rings MSL, MGL and MSp.

The construction of this functor relies heavily on the study of the functoriality in S of these construction, in particular of the assignment $S \mapsto \mathsf{SH}(S)$, and so this is what we will start with. Then, we will be able to define the motivic Thom spectrum functor following [BH21, Section 16 and Appendix C].

6.2.1 Functoriality of $\mathsf{SH}(-)$

Ayoub has constructed in his thesis [Ayo07] a full six-functors formalism in the stable motivic set-up. We will only need some of these functors and properties here. Our starting point will be to consider SH as a spherical presheaf of symmetric monoidal ∞-categories, i.e. a functor $\mathsf{Sm}_S^{\mathsf{op}} \to \mathsf{CAlg}(\mathsf{Cat}_\infty^\times)$ preserving finite products (having in mind the case $S = \mathsf{Spec}(\mathbb{R})$ to which we later want to specialize).

As we now explain, this data is equivalent to that of a functor $\mathsf{Span}(\mathsf{Sm}_S, \mathsf{all}, \mathsf{fold}) \to \mathsf{Cat}_\infty$ which preserves products. The source ∞-category is defined as follows:

Definition 6.7. Let $\mathcal{C}$ be an ∞-category, and let left and right be two classes of edges in $\mathcal{C}$, both containing equivalences and closed by pullback along one another. Then, there is an ∞-category $\mathsf{Span}(\mathcal{C}, \mathsf{left}, \mathsf{right})$ with vertices the objects of $\mathcal{C}$, and for any $X, Z \in \mathcal{C}$, the edges from X to Z are given by all spans $X \longleftarrow Y \longrightarrow Z$ where $Y \in \mathcal{C}$, and $(Y \to X) \in \mathsf{left}$, $(Y \to Z) \in \mathsf{right}$. The composition of two spans $V \leftarrow W \to X$ and $X \leftarrow Y \to Z$ is the span $V \leftarrow W \times_X Y \to Z$.

The full definition as an ∞-category is given in [Bar17, Section 5].

In our case, $\mathsf{left} = \mathsf{all}$ denotes the class of all edges in Sm_S, and $\mathsf{right} = \mathsf{fold}$ denotes the class of finite coproducts of finite fold maps, i.e. maps of the form $\coprod_{i \le n} S_i^{\amalg m_i} \to \coprod_{i \le n} S_i$ (where n and m_i are non-negative integers), whose restriction to $S_i^{\amalg m_i}$ is the fold map $S_i^{\amalg m_i} \to S_i$ for all $i \le n$. In this notation, we have:

Proposition 6.8 ([BH21, Prop. C.1])**.** *Let $\mathcal{D}$ be an ∞-category with finite products. Consider the functor*

$$\theta : \mathsf{Fun}(\mathsf{Span}(\mathsf{Fin}, \mathsf{all}, \mathsf{all}), \mathcal{D}) \longrightarrow \mathsf{Fun}(\mathsf{Fin}_*, \mathcal{D})$$

induced by restriction along

$$\begin{aligned} \mathsf{Fin}_* &\longrightarrow \mathsf{Span}(\mathsf{Fin}, \mathsf{all}, \mathsf{all}) \\ \langle n \rangle &\longmapsto \langle n \rangle^\circ \\ (f : \langle n \rangle \to \langle m \rangle) &\longmapsto (\langle n \rangle^\circ \hookleftarrow f^{-1}(\langle m \rangle^\circ) \xrightarrow{f} \langle m \rangle^\circ) \end{aligned}$$

where $\langle n \rangle^\circ$ denotes the finite set $\{1, \dots, n\}$ viewed as a subset of $\langle n \rangle = \{*, 1, \dots, n\}$.

Then, θ restricts to an equivalence of ∞-categories

$$\mathsf{Fun}^\times(\mathsf{Span}(\mathsf{Fin}, \mathsf{all}, \mathsf{all}), \mathcal{D}) \xrightarrow{\simeq} \mathsf{CAlg}(\mathcal{D}^\times).$$

This statement generalizes to presheaves with values in $\mathsf{CAlg}(\mathcal{D}^\times)$:

Proposition 6.9 ([BH21, Prop. C.5])**.** *For an extensive ∞-category $\mathcal{C}$ (see [BH21, Def. 2.3]) (in particular, for Sm_S), consider the functor*

$$\begin{aligned} \mathcal{C}^{\mathsf{op}} \times \mathsf{Span}(\mathsf{Fin}, \mathsf{all}, \mathsf{all}) &\longrightarrow \mathsf{Span}(\mathcal{C}, \mathsf{all}, \mathsf{fold}) \\ (c, \langle n \rangle^\circ) &\longmapsto c^{\amalg n} \\ (c' \xrightarrow{f^{\mathsf{op}}} c, \langle n \rangle^\circ \xleftarrow{a} \langle m \rangle^\circ \xrightarrow{b} \langle k \rangle^\circ) &\longmapsto (c'^{\amalg n} \leftarrow c^{\amalg m} \to c^{\amalg k}) \end{aligned}$$

where the restriction of $c^{\amalg m} \to c^{\amalg k}$ to the i-th component is the inclusion as the $b(i)$-th component, and the restriction of $c^{\amalg m} \to (c')^{\amalg}$ to the i-th component is given by f followed by the inclusion as the $a(i)$-th component.

Then, for $\mathcal{D}$ an ∞-category with finite products, restriction along this functor induces

$$\Theta : \mathsf{Fun}(\mathsf{Span}(\mathcal{C}, \mathsf{all}, \mathsf{fold}), \mathcal{D}) \to \mathsf{Fun}(\mathcal{C}^{\mathsf{op}} \times \mathsf{Span}(\mathsf{Fin}, \mathsf{all}, \mathsf{all}), \mathcal{D})$$

which restricts to an equivalence

$$\mathsf{Fun}^\times(\mathsf{Span}(\mathcal{C}, \mathsf{all}, \mathsf{fold}), \mathcal{D}) \xrightarrow{\simeq} \mathsf{Fun}^\times(\mathcal{C}^{\mathsf{op}}, \mathsf{CAlg}(\mathcal{D}^\times)).$$

In our case, we have to study the functoriality of the assignment $X \in \mathsf{Sm}_S \longmapsto \mathsf{SH}(X) \in \mathsf{Cat}_\infty$. The latter has very good functoriality properties, both covariantly and contravariantly:

Theorem 6.10 ([Ayo07, §1.4.1])**.** *Let $f : X \to Y$ be a morphism in Sm_S.*

(i) There are equivalences $\mathsf{SH}(A \amalg B) \simeq \mathsf{SH}(A) \times \mathsf{SH}(B)$ for all $A, B \in \mathsf{Sm}_S$. Under these identifications, a map $(\nabla : Y \to Z) \in \mathsf{fold}$ induces $\nabla_\otimes : \mathsf{SH}(Y) \to \mathsf{SH}(X)$ corresponding to the smash product.

(ii) There are induced pullback-pushforward adjunctions

$$\begin{tikzcd} \mathcal{P}(\mathsf{Sm}_X) \arrow[r, "\mathsf{L}_{\mathrm{mot}}"] \arrow[d, shift right, "f_*"'] & \mathsf{Spc}(X) \arrow[r, "\Sigma^\infty_+"] \arrow[d, shift right, "f_*"'] & \mathsf{SH}(X) \arrow[d, shift right, "f_*"'] \\ \mathcal{P}(\mathsf{Sm}_Y) \arrow[r, "\mathsf{L}_{\mathrm{mot}}"] \arrow[u, shift right, "f^*"', "\dashv"] & \mathsf{Spc}(Y) \arrow[r, "\Sigma^\infty_+"] \arrow[u, shift right, "f^*"', "\dashv"] & \mathsf{SH}(Y) \arrow[u, shift right, "f^*"', "\dashv"] \end{tikzcd}$$

where the squares involving f^* commute. The squares formed by f_* and the right adjoints $\iota \vdash \mathsf{L}_{\mathrm{mot}}$, $\Omega^\infty \vdash \Sigma^\infty_+$ commute.

(iii) All pushforward functors f^* in (ii) are symmetric monoidal, with respect to the Cartesian structure on the ∞-categories of presheaves and motivic spaces, and the smash product on the stable motivic ∞-categories.

(iv) If f is moreover smooth, there are induced adjunctions

$$\begin{tikzcd} \mathcal{P}(\mathsf{Sm}_X) \arrow[r, "\mathsf{L}_{\mathrm{mot}}"] \arrow[d, shift right, "f_\sharp"'] & \mathsf{Spc}(X) \arrow[r, "\Sigma^\infty_+"] \arrow[d, shift right, "f_\sharp"'] & \mathsf{SH}(X) \arrow[d, shift right, "f_\sharp"'] \\ \mathcal{P}(\mathsf{Sm}_Y) \arrow[r, "\mathsf{L}_{\mathrm{mot}}"] \arrow[u, shift right, "f^*"', "\dashv"] & \mathsf{Spc}(Y) \arrow[r, "\Sigma^\infty_+"] \arrow[u, shift right, "f^*"', "\dashv"] & \mathsf{SH}(Y) \arrow[u, shift right, "f^*"', "\dashv"] \end{tikzcd}$$

where the squares involving either f^* or $f_\sharp$ commute.

(v) In the situation of (iv), the projection formula holds for all three functors $f_\sharp$. This means that for every A in the source of $f_\sharp$ and B in its target, we have

$$f_\sharp(A \otimes f^*B) \simeq f_\sharp(A) \otimes B.$$

(vi) The constructions of (i) to (iv) are functorial in f.

(vii) For every commutative square in Sm_S

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

with f and f' smooth, there is an exchange transformation

$$\mathsf{Ex}^*_\sharp : f'_\sharp(g')^* \Longrightarrow g^* f_\sharp$$

(defined in the proof below) of functors $\mathsf{SH}(X) \to \mathsf{SH}(Y')$. It is an equivalence if the square is Cartesian in Sm_S.

(viii) For every diagram in Sm_S of the form

$$\begin{tikzcd} W \arrow[d, "g"'] & Y' \arrow[l] \arrow[r, "\nabla'"] \arrow[d, "u'"] & Z' \arrow[d, "u"] \\ X & Y \arrow[l, "f"'] \arrow[r, "\nabla"] & Z \end{tikzcd}$$

with u and u' smooth and $\nabla, \nabla' \in \mathsf{fold}$, there is a distributivity transformation

$$\mathsf{Dis}_{\sharp\otimes} : u_\sharp \nabla'_\otimes (f')^* \Longrightarrow \nabla_\otimes (\pi_Y)_\sharp \pi_W^*$$

(defined in the proof below) of functors $\mathsf{SH}(W) \to \mathsf{SH}(Z)$. It is an equivalence when the square on the right hand side is a pullback and $Z' = R_{Y/Z}(W \times_X Y)$ is the Weil restriction (see [BH21, §2.3], and the proof below for an easier description in our case).

Proof. We prove parts of this very long statement and give references for the rest.

(i) We have $\mathsf{Sm}_{A\amalg B} \simeq \mathsf{Sm}_A \times \mathsf{Sm}_B$, where a scheme X with structure map $X \xrightarrow{f} A \amalg B$ is sent to $(X \times_{A\amalg B} A \to A, X \times_{A\amalg B} B \to B)$. Rewriting $X \to A \amalg B$ as $f^{-1}(A) \amalg f^{-1}(B) \to A \amalg B$, both components are Zariski open subschemes in X. In particular, every Nisnevich sheaf on $\mathsf{Sm}_{A\amalg B}$ can be reconstructed from its restrictions to Sm_A and Sm_B by taking products. Whence $\mathsf{Spc}(A \amalg B)_* \simeq \mathsf{Spc}(A)_* \times \mathsf{Spc}(B)_*$, and thus the same is true for the stable motivic ∞-categories (see [BH21, Lemma 9.6]. Informally, inverting $\mathbb{P}^1_{A\amalg B}$ on the left hand side corresponds to inverting $(\mathbb{P}^1_A, \mathbb{P}^1_B)$ on the right, or equivalently inverting $(\mathbb{P}^1_A, \mathbb{1})$ and then $(\mathbb{1}, \mathbb{P}^1_B)$, since $(\mathbb{P}^1_A, \mathbb{P}^1_B)$ is their tensor product).

In particular, if $Y = \coprod_{i \le n} X_i^{\amalg m_i} \to \coprod_{i\le n} X_i = X$ is a map in fold, then the smash product induces a map $\mathsf{SH}(Y) \simeq \prod_{i \le n} \mathsf{SH}(X_i)^{m_i} \to \prod_{i\le n} \mathsf{SH}(X_i) \simeq \mathsf{SH}(X)$ given by the product of the maps $\mathsf{SH}(X_i)^{m_i} \to \mathsf{SH}(X_i)$ each given by the m_i-fold tensor product.

(ii) At the level of presheaves f^* is the left Kan extension of the pullback functor, which sends a scheme $A \in \mathsf{Sm}_Y$ to $A \times_Y X \in \mathsf{Sm}_X$. By the universal property of the presheaf ∞-category this functor preserves colimits and thus admits a right adjoint which we denote by f_*. The functor f_* is precomposition of the presheaves with the pullback map $\mathsf{Sm}_Y^{\mathsf{op}} \to \mathsf{Sm}_X^{\mathsf{op}}$. We then extend these constructions by a procedure similar to the construction of $r_{\mathbb{R}}$ is Section 4.1. Indeed, since pullback sends projections of the form $A \times_Y \mathbb{A}^1_Y \to A$ to projections of the same form with respect to $\mathbb{A}^1_X$, and Nisnevich squares are preserved under base change, the composite $\mathsf{L}_{\mathsf{mot}} f^* : \mathcal{P}(\mathsf{Sm}_Y) \to \mathsf{Spc}(X)$ factors through $\mathsf{Spc}(Y)$ as a colimit-preserving functor $\mathsf{Spc}(Y) \to \mathsf{Spc}(X)$ which we again denote by f^*. In particular, the square in statement (ii) with the two functors f^* and the horizontal maps $\mathsf{L}_{\mathsf{mot}}$ commutes. As a colimit preserving functor, $f^* : \mathsf{Spc}(Y) \to \mathsf{Spc}(X)$ admits a right adjoint f_*. Moreover, note that f^* preserves finite products in $\mathsf{Spc}(X)$. Indeed, $\mathsf{L}_{\mathsf{mot}} f^* \iota = f^* \mathsf{L}_{\mathsf{mot}} \iota = f^*$, and all three functors on the left preserve finite products; for f^* on presheaves it already holds at the level of schemes, which generate presheaves under colimits, and in our case these colimits commute with finite products. Therefore, f^* extends to a symmetric monoidal functor $f^* : \mathsf{Spc}(Y)_* \to \mathsf{Spc}(X)_* \xrightarrow{\Sigma^\infty} \mathsf{SH}(X)$ sending $\mathbb{P}^1_Y = T_Y$ to $\Sigma^\infty(T_Y \times_Y X) \simeq \Sigma^\infty T_X$, the latter is invertible in $\mathsf{SH}(X)$ by definition (some compatibility with respect to basepoints has to be checked). We deduce that f^* extends to a symmetric monoidal colimit preserving functor $\mathsf{SH}(Y) \to \mathsf{SH}(X)$ (Proposition 4.6), making the square on the right-hand side of the diagram in statement (ii) commute, and admitting a right adjoint f_*.

(iii) Proven in (ii).

(iv) If f is smooth, we obtain a functor $\mathsf{Sm}_X \to \mathsf{Sm}_Y$ given by post-composition of the structure maps with f. It is left adjoint to f^*, since by definition a morphism of schemes $B \to A \times_Y X$ over X is exactly a map $B \to A$ making the relevant square over Y commute, i.e. a morphism in $\mathsf{Sm}_Y(B \to X \to Y, A \to Y)$. It also has a left Kan extension to the relevant ∞-categories of presheaves. Now, $\mathsf{L}_{\mathsf{mot}} \circ f_\sharp : \mathsf{Sm}_X \to \mathsf{Spc}(Y)$ inverts motivic equivalences. Indeed, for $A \in \mathsf{Sm}_X$, we have $\mathbb{A}^1_X \times_X A = \mathsf{Spec}(\mathbb{Z}[t]) \times_{\mathsf{Spec}(\mathbb{Z})} (X \times_X A) = \mathsf{Spec}(\mathbb{Z}[t]) \times_{\mathsf{Spec}(\mathbb{Z})} (Y \times_Y A) = \mathbb{A}^1_Y \times_Y A$ and the projection map is sent to the projection map. Also, changing the structure map doesn't affect Nisnevich squares. Thus we obtain a functor $f_\sharp : \mathsf{Spc}(X) \to \mathsf{Spc}(Y)$ making the square on the left-hand side in statement (iv) commute. It is still left adjoint to f^* because for all $\mathcal{F} \in \mathsf{Spc}(X)$ and $\mathcal{G} \in \mathsf{Spc}(Y)$, we have:

$$\begin{aligned}
\mathsf{Spc}(X)(\mathcal{F}, f^*\mathcal{G}) &\simeq \mathcal{P}(X)(\iota\mathcal{F}, \iota f^*\mathcal{G}) && (\star)\\
&\simeq \mathcal{P}(X)(\iota\mathcal{F}, \iota \mathsf{L}_{\mathsf{mot}} f^* \iota\mathcal{G}) \\
&\simeq \mathcal{P}(X)(\iota\mathcal{F}, f^*\iota\mathcal{G}) && (f^*\iota\mathcal{G} \text{ is already local})\\
&\simeq \mathcal{P}(Y)(f_\sharp\iota\mathcal{F}, \iota\mathcal{G}) && (\text{by adjointness at the level of presheaves})\\
&\simeq \mathsf{Spc}(Y)(\mathsf{L}_{\mathsf{mot}} f_\sharp \iota\mathcal{F}, \mathcal{G}) \\
&\simeq \mathsf{Spc}(Y)(f_\sharp \mathsf{L}_{\mathsf{mot}} \iota\mathcal{F}, \mathcal{G}) \\
&\simeq \mathsf{Spc}(Y)(f_\sharp\mathcal{F}, \mathcal{G}),
\end{aligned}$$

where for $(\star)$ we used that ι is fully faithful, as it is right adjoint to the localization functor $\mathsf{L}_{\mathsf{mot}}$. For the extension of $f_\sharp$ to the stable ∞-categories, making the square on the right-hand side commute, see [Hoy17, Lemma 6.2 and below].

(v) See [Hoy17, Lemma 6.2 and below]. The idea is to reduce to the case of motivic spaces since every $A \in \mathsf{SH}(X)$ is a colimit of objects of the form $\Sigma^n_T \Sigma^\infty_+ U$, with U varying in Sm_X.

(vi) See [Ayo07, §1.4.1].

(vii) The exchange transformation is defined as the composition

$$\mathsf{Ex}^*_\sharp : f'_\sharp (g')^* \xRightarrow{\eta_f} f'_\sharp (g')^* f^* f_\sharp \simeq f'_\sharp f'^* g^* f_\sharp \xRightarrow{\varepsilon_{f'}} g^* f_\sharp$$

where η_f and $\varepsilon_{f'}$ are the (co)units for the adjunctions $f_\sharp \dashv f^*$ and $f'_\sharp \dashv f'^*$ respectively, and the equivalence in the middle witnesses the commutativity of the square in statement (*vii*).

It is an equivalence for pullback squares by [Ayo07, §1.4.1] or [Hoy17, Cor. 6.12].

(viii) The distributivity transformation is defined as the composition

$$\mathsf{Dis}_{\sharp\otimes} : u_\sharp \nabla'_\otimes (f')^* \xRightarrow{\mathsf{Ex}_{\sharp,\otimes}} \nabla_\otimes u'_\sharp (f')^* \simeq \nabla_\otimes (\pi_Y)_\sharp (f' \times u')_\sharp (f' \times u')^* \pi^*_W \xRightarrow{\varepsilon_{f' \times u'}} \nabla_\otimes (\pi_Y)_\sharp \pi^*_W$$

where the equivalence in the middle uses $u' \simeq \pi_Y \circ (f' \times u')$ and $f' \simeq \pi_W \circ (f' \times u')$, and

$$\mathsf{Ex}_{\sharp,\otimes} : u_\sharp \nabla'_\otimes \xRightarrow{\eta_{u'}} u_\sharp \nabla'_\otimes (u')^* u'_\sharp \simeq u_\sharp u^* \nabla_\otimes u'_\sharp \xRightarrow{\varepsilon_u} \nabla_\otimes u'_\sharp$$

with the equivalence in the middle given by the symmetric monoidality of u^* (since the $\nabla_\otimes$ and $\nabla'_\otimes$ are given by the smash product).

It is an equivalence when the conditions of statement (*viii*) are satisfied by [BH21, Prop. 5.10]. Note that we don't need the quasi-projectiveness assumptions on the map h in this proposition, because this is only used to ensure the existence of the Weil restriction, however the latter always exists for maps in fold. Indeed, the Weil restriction $R_{Y/X}$ is defined by the property that

$$\mathsf{Sm}_X(U, R_{Y/X} V) \simeq \mathsf{Sm}_Y(U \times XY, V).$$

In the case of a map $\left(Y = \coprod_{i \leq n} X_i^{\amalg m_i} \to \coprod_{i \leq n} X_i = X\right) \in \mathsf{fold}$, we have

$$R_{Y/X}(V) = \coprod_{i \leq n} \left(V_{X_{i,1}} \times_{X_i} \cdots \times_{X_i} V_{X_{i,m_i}}\right) \longrightarrow \coprod_{i \leq n} X_i,$$

where $X_{i,j}$ is the j-th component in $X_i^{\amalg m_i}$ and $V_{X_{i,j}}$ is the component of V living over $X_{i,j}$, for all $i \leq n$ and $1 \leq j \leq m_i$.

□

6.2.2 Construction of the motivic Thom spectrum functor

We will now use the results of the previous subsection to construct functors $\mathcal{P}_\Sigma(\mathsf{Sm}_S)_{/\mathsf{SH}^\simeq} \to \mathsf{SH}(S)$ for all $S \in \mathsf{Sm}_\mathbb{R}$.

Definition 6.11. Let $\mathcal{C}$ be an ∞-category with finite coproducts. Then $\mathcal{P}_\Sigma(\mathcal{C})$ is the ∞-category of functors $\mathcal{C}^{\mathsf{op}} \to \mathsf{Spc}$ preserving finite products (i.e. sending finite coproducts in $\mathcal{C}$ to products in Spc). It is sometimes called the *∞-category of spherical presheaves (of spaces) on $\mathcal{C}$*, or the *non-abelian derived category of $\mathcal{C}$*. This construction extends to a functor $\mathcal{P}_\Sigma(\bullet) : \mathsf{Cat}_\infty \to \mathsf{Cat}^{\mathsf{sift}}_\infty$, left adjoint to the inclusion of ∞-categories admitting all sifted colimits (and functors preserving them) into all ∞-categories. In other terms $\mathcal{P}_\Sigma(\mathcal{C})$ is the free sifted-cocomplete ∞-category generated by $\mathcal{C}$, and every spherical presheaf is a sifted colimit of representable presheaves (which are in particular spherical).

Actually, the construction of the functor $\mathcal{P}_\Sigma(\mathsf{Sm}_S)_{/\mathsf{SH}^\simeq} \to \mathsf{SH}(S)$ is itself functorial in S, in the sense that we will view both the source and target as presheaves of symmetric monoidal ∞-categories on $\mathsf{Sm}_{S'}$ for some base scheme S', and construct a natural symmetric monoidal transformation between these presheaves. Even more generally, in order to later compare the motivic Thom spectrum functor with the topological one, we will perform such a construction for SH replaced with any presheaf of symmetric monoidal ∞-categories which satisfies a number of good properties, inspired by (*i*), (*iv*), (*vii*) and (*viii*) in Theorem 6.10. This formalism is that of motivic colimit functors (see [BEH22]), here taking into account the symmetric monoidal structure.

Theorem 6.12 ([BH21, §16.3])**.** *Denote* $\mathsf{Span} := \mathsf{Span}(\mathsf{Sm}_S, \mathsf{all}, \mathsf{fold})$*. Let* $\mathcal{F} : \mathsf{Span} \longrightarrow \mathsf{Cat}_\infty$ *be a functor preserving finite products, such that:*

(i) for any $f : Y \to X$ *in* Sm_S smooth*,* $f^* := \mathcal{F}(X \leftarrow Y = Y) : \mathcal{F}(X) \to \mathcal{F}(Y)$ *admits a left adjoint denoted by* $f_\sharp$*.*

(ii) for any Cartesian square in Sm_S

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

with f *and* f' *smooth morphisms, the exchange transformation*

$$\mathsf{Ex}^*_\sharp : f'_\sharp (g')^* \Longrightarrow g^* f_\sharp$$

of functors $\mathcal{F}(X) \to \mathcal{F}(Y')$ *(defined as in Theorem 6.10) is an equivalence.*

(iii) Given a fold map $\nabla : Y \to Z$*, let* $\nabla_\otimes := \mathcal{F}(Y = Y \longrightarrow Z) : \mathcal{F}(Y) \to \mathcal{F}(Z)$*. Then* ∇ *encodes the tensor product on the symmetric monoidal* ∞*-category* $\mathcal{F}(Z)$ *(Proposition 6.9). For every diagram in* Sm_S *of the form*

$$\begin{tikzcd} W \arrow[d, "g"'] & R_{Y/X}(W \times_X Y) \times_Z Y \arrow[l] \arrow[r, "\nabla'"] \arrow[d, "u'"] \arrow[dr, phantom, "\lrcorner", very near start] & R_{Y/X}(W \times_X Y) \arrow[d, "u"] \\ X & Y \arrow[l, "f"'] \arrow[r, "\nabla"] & Z \end{tikzcd}$$

with u *and* u' *smooth morphisms and* $\nabla, \nabla' \in \mathsf{fold}$*, the distributivity transformation*

$$\mathsf{Dis}_{\sharp\otimes} : u_\sharp \nabla'_\otimes (f')^* \Longrightarrow \nabla_\otimes (\pi_Y)_\sharp \pi_W^*$$

(defined as in Theorem 6.10) of functors $\mathcal{F}(W) \to \mathcal{F}(Z)$ *is an equivalence.*

Then, there exists a natural transformation $\widetilde{M}_{\mathcal{F}} : (\mathsf{Sm}_\bullet)_{/\mathcal{F}} \to \mathcal{F}$ *of functors* $\mathsf{Span} \to \mathsf{CAlg}(\mathsf{Cat}_\infty^\times)$ *(i.e. of presheaves of symmetric monoidal* ∞*-categories). Moreover, if* $\mathcal{F}$ *lifts to the* ∞*-category* $\mathsf{Cat}_\infty^{\mathsf{sift}}$ *of small sifted-cocomplete* ∞*-categories and functors preserving sifted colimits, then we obtain a natural transformation* $\widetilde{M}_{\mathcal{F}} : \mathcal{P}_\Sigma(\mathsf{Sm}_\bullet)_{/\mathcal{F}^\simeq} \to \mathcal{F}$ *of functors* $\mathsf{Span} \to \mathsf{CAlg}(\mathsf{Cat}_\infty^{\mathsf{sift}})$ *(i.e. of presheaves of sifted cocomplete symmetric monoidal* ∞*-categories).*

Remark 6.13. In the notation $\mathcal{P}_\Sigma(\mathsf{Sm}_\bullet)_{/\mathcal{F}^\simeq}$, $\mathcal{F} : \mathsf{Span} \to \mathsf{Cat}_\infty$ is viewed as a functor associating to $X \in \mathsf{Sm}_S$ a presheaf of spaces on Sm_X, given by

$$\mathsf{Sm}_X^{\mathsf{op}} \hookrightarrow \mathsf{Sm}_S^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathsf{Cat}_\infty \xrightarrow{(-)^\simeq} \mathsf{Spc}.$$

Thus $\mathcal{P}_\Sigma(\mathsf{Sm}_\bullet)_{/\mathcal{F}^\simeq}$ associates to X the actual slice ∞-category $\mathcal{P}_\Sigma(\mathsf{Sm}_X)_{/\mathcal{F}^\simeq}$, viewing $\mathcal{F}^\simeq \in \mathcal{P}_\Sigma(\mathsf{Sm}_X)$. Note however that it is a priori unclear whether the symmetric monoidal structure on the slice ∞-category induced by this construction agrees with the one in Proposition A.14. We will come back to this question in Subsection 6.3.3.

On the other hand, any presheaf of spaces can also be viewed as a presheaf of ∞-categories via the inclusion $\iota : \mathsf{Spc} \hookrightarrow \mathsf{Cat}_\infty$. Then the notation $\mathcal{P}_\Sigma(\mathsf{Sm}_X)_{/\mathcal{F}}$ also makes sense. We point out that the latter ∞-category is actually equivalent to $\mathcal{P}_\Sigma(\mathsf{Sm}_{\mathbb{R}})_{/\mathcal{F}^\simeq}$ (see [BH21, below Definition 16.1]) because any map from a presheaf of spaces to $\mathcal{F}$ lifts to $\mathcal{F}^\simeq$ (by adjointness of $\iota \dashv (-)^\simeq$). Then, this allows us to make sense of the notation $(\mathsf{Sm}_X)_{/\mathcal{F}}$ as the subcategory of $\mathcal{P}_\Sigma(\mathsf{Sm}_X)_{/\mathcal{F}}$ containing representable presheaves. This agrees with Definition A.12.

Proof of Theorem 6.12. We repeat the construction in [BH21, §16.3]. We will use the theory of straightening and unstraightening, for which we refer to [Lur09, §3.2].

Let $F : \mathcal{E} \to \mathsf{Span}^{\mathsf{op}}$ be the Cartesian fibration classified by the functor $\mathcal{F}$. Let $\mathsf{Fun}_{\mathsf{sm}}(\Delta^1, \mathsf{Span})$ be the full subcategory of $\mathsf{Fun}(\Delta^1, \mathsf{Span})$ consisting of functors whose image is a span of the form $X \xleftarrow{f} Y = Y$ where f is a smooth morphism of schemes in Sm_S. The composition

$$\mathsf{Fun}_{\mathsf{sm}}(\Delta^1, \mathsf{Span}) \times \Delta^1 \xrightarrow{\mathrm{ev}} \mathsf{Span} \xrightarrow{\mathcal{F}} \mathsf{Cat}_\infty$$

where the first functor is evaluation, can be understood as a natural transformation between the functors $\mathcal{F} \circ s$ and $\mathcal{F} \circ t$, where s (respectively t) is the source (respectively target) functor

$$\mathsf{Fun}_{\mathsf{sm}}(\Delta^1, \mathsf{Span}) \longrightarrow \mathsf{Fun}_{\mathsf{sm}}(\Delta^0, \mathsf{Span}) \simeq \mathsf{Span}$$

induced by the inclusion $\{0\} \to \Delta^1$ (respectively $\{1\} \to \Delta^1$). Since precomposition of functors corresponds to pullbacks of Cartesian fibrations by [Lur09, Def. 3.3.2.2], the functors $\mathcal{F} \circ s$ and $\mathcal{F} \circ t$ classify respectively the Cartesian fibrations s^*F and t^*F such that

$$\begin{tikzcd} s^*\mathcal{E} \arrow[r] \arrow[d, "s^*F"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "F"] \\ \mathsf{Fun}_{\mathsf{sm}}(\Delta^1, \mathsf{Span})^{\mathsf{op}} \arrow[r, "s^{\mathsf{op}}"] & \mathsf{Span}^{\mathsf{op}} \end{tikzcd} \qquad \begin{tikzcd} t^*\mathcal{E} \arrow[r] \arrow[d, "t^*F"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "F"] \\ \mathsf{Fun}_{\mathsf{sm}}(\Delta^1, \mathsf{Span})^{\mathsf{op}} \arrow[r, "t^{\mathsf{op}}"] & \mathsf{Span}^{\mathsf{op}} \end{tikzcd}$$

and the natural transformation between $\mathcal{F} \circ s$ and $\mathcal{F} \circ t$ gives a morphism of Cartesian fibrations $\phi : s^*\mathcal{E} \to t^*\mathcal{E}$ (i.e. a functor over $\mathsf{Fun}_{\mathsf{sm}}(\Delta^1, \mathsf{Span})^{\mathsf{op}}$ which preserves Cartesian edges). Consider the following commutative diagram

$$\begin{tikzcd} t^*\mathcal{E} \arrow[drr, "t^*F"'] & & s^*\mathcal{E} \arrow[ll, "\phi"'] \arrow[r, "\chi"] \arrow[d, "s^*F"'] & \mathcal{E} \arrow[d, "F"] \\ & & \mathsf{Fun}_{\mathsf{sm}}(\Delta^1, \mathsf{Span})^{\mathsf{op}} \arrow[r, "s^{\mathsf{op}}"] & \mathsf{Span}^{\mathsf{op}}. \end{tikzcd}$$

We now pause to give an outline of the proof that will follow. The goal is to show that the composition $t^*\mathcal{E} \to \mathsf{Fun}_{\mathsf{sm}}(\Delta^1, \mathsf{Span})^{\mathsf{op}} \to \mathsf{Span}^{\mathsf{op}}$ is a Cartesian fibration which is classified by a presheaf of symmetric monoidal ∞-categories $(\mathsf{Sm}_\bullet)_{/\!\!/\mathcal{F}}$, of which $(\mathsf{Sm}_\bullet)_{/\mathcal{F}}$ is a subfunctor; and that we can construct a map $t^*\mathcal{E} \to \mathcal{E}$ endowing the natural transformation $\widetilde{M_{\mathcal{F}}}$ we are looking for. To do so, the steps are the following:

1. Show that s^{op} is a Cartesian fibration, so that the composition $s^{\mathsf{op}} \circ t^*F : t^*\mathcal{E} \to \mathsf{Span}^{\mathsf{op}}$ is a Cartesian fibration (note that t^*F is a Cartesian fibration because it is a pullback of F).
2. Define a *relative left adjoint* ψ to ϕ, which will in particular be a map $t^*\mathcal{E} \to s^*\mathcal{E}$.
3. Show that the composition $\chi \circ \psi$ is a morphism of Cartesian fibrations over $\mathsf{Span}^{\mathsf{op}}$, and thus corresponds to a natural transformation between the functors classifying these Cartesian fibrations.
4. Show that $s^{\mathsf{op}} \circ t^*F$ is indeed classified by some presheaf $(\mathsf{Sm}_\bullet)_{/\!\!/\mathcal{F}}$, which admits $(\mathsf{Sm}_\bullet)_{/\mathcal{F}}$ as a subfunctor, and deduce the first part of the statement of the theorem.
5. Consider the case where $\mathcal{F}$ lifts to $\mathsf{Cat}_\infty^{\mathsf{sift}}$.

Let us apply this strategy.

Step 1: s^{op} is a Cartesian fibration. Equivalently, we have to show that s is a coCartesian fibration. Firstly, it is an inner fibration because the source functor $\mathsf{Fun}(\Delta^1, \mathsf{Span}) \to \mathsf{Span}$ is an inner fibration by [Lur09, Cor. 2.4.7.11], and the restriction of an inner fibration to a full subcategory is still an inner fibration by [Lur24, Tag 01CU]. Secondly, a coCartesian edge over $X \xleftarrow{f} Y \xrightarrow{\nabla} Z$ in Span with source $X \xleftarrow{j} W = W$ is given by the edge in $\mathsf{Fun}_{\mathsf{sm}}(\Delta^1, \mathsf{Span})$ (i.e. natural transformation)

$$\begin{tikzcd} W \arrow[d, equal] & R_{Y/Z}(W \times_X Y) \times_Z Y \arrow[l] \arrow[r] \arrow[d, equal] \arrow[dl, phantom, "\ulcorner"] & R_{Y/Z}(W \times_X Y) \arrow[d, equal] \\ W \arrow[d, "j"'] & R_{Y/Z}(W \times_X Y) \times_Z Y \arrow[l, "f'"'] \arrow[r, "\nabla'"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & R_{Y/Z}(W \times_X Y) \arrow[d] \\ X & Y \arrow[l, "f"] \arrow[r, "\nabla"] & Z \end{tikzcd}$$

which we call e (for a grid like this to define a natural transformation between the vertical spans on the left and right hand sides, we need the top left and bottom right squares to be pullbacks, which is the case here). Indeed, given another edge e' with the same source

$$\begin{tikzcd}
W & B \arrow[l] \arrow[r] & D \\
W \arrow[u, equal] \arrow[d, "j"'] & B \arrow[l, "g'"'] \arrow[r, "\widetilde{\nabla}'"] \arrow[u, equal] \arrow[d] \arrow[ul, phantom, "\ulcorner"] \arrow[dr, phantom, "\lrcorner"] & D \arrow[u, equal] \arrow[d] \\
X & A \arrow[l, "g"] \arrow[r, "\widetilde{\nabla}"] & C
\end{tikzcd}$$

and a commutative diagram in Span

$$\begin{tikzcd}
X & Y \arrow[l, "f"'] \arrow[r, "\nabla"] & Z \\
& A \arrow[ul, "g"] \arrow[dr, "\widetilde{\nabla}"'] & E \arrow[u, "h"'] \arrow[d, "\overline{\nabla}"] \\
& & C
\end{tikzcd}$$

(in particular $A \simeq Y \times_Z E$), the following edge e'' in $\mathsf{Fun}_{\mathsf{sm}}(\Delta^1, \mathsf{Span})$ lifts $Z \leftarrow E \to C$ and provides a commutative diagram $e'' \circ e \simeq e'$

$$\begin{tikzcd}
R_{Y/Z}(W \times_X Y) & D \times_C E \arrow[l] \arrow[r] & D \\
R_{Y/Z}(W \times_X Y) \arrow[u, equal] \arrow[d] & D \times_C E \arrow[l, "h'"'] \arrow[r] \arrow[u, equal] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & D \arrow[u, equal] \arrow[d] \\
Z & E \arrow[l, "h"] \arrow[r, "\overline{\nabla}"] & C.
\end{tikzcd}$$

To construct h', note that by definition of the Weil restriction, there are equivalences

$$\begin{aligned}
\mathsf{Sch}_Z(D \times_C E, R_{Y/Z}(W \times_X Y)) &\simeq \mathsf{Sch}_Y((D \times_C E) \times_Z Y, W \times_X Y) \\
&\simeq \mathsf{Sch}_Y(D \times_C A, W \times_X Y) \\
&\simeq \mathsf{Sch}_Y(B, W \times_X Y).
\end{aligned}$$

Then h' is defined as the preimage under these equivalences of the map in $\mathsf{Sch}_Y(B, W \times_X Y)$ induced by $g' : B \to W$.

Note that we had to choose the fiber product $D \times_C E$ for our lift in order to obtain a morphism of spans. Up to equivalence this lift is unique.

Step 2: ϕ has a relative left adjoint. This means (see [Lur17, Def. 7.3.2.2]) that there exists a functor ψ in the other direction, and a transformation $\varepsilon : \psi\phi \to \mathsf{id}$ exhibiting ψ as a left adjoint to ϕ, and moreover ψ commutes with the structure maps, in the sense that $s^*F \circ \psi \simeq t^*F \circ \phi \circ \psi \overset{\varepsilon}{\Rightarrow} t^*F$ is an equivalence. By the criterion from [Lur17, Prop. 7.3.2.11], since ϕ is a morphism of Cartesian fibrations, it suffices to show that ϕ has fiberwise a left-adjoint. Over $(X \overset{f}{\leftarrow} Y = Y) \in \mathsf{Fun}_{\mathsf{sm}}(\Delta^1, \mathsf{Span})$ (so $f : Y \to X$ is smooth), the functor ϕ is by definition the corresponding component of the natural transformation encoding it. Therefore, it is given by $f^* : \mathcal{F}(X) \to \mathcal{F}(Y)$, which admits a left adjoint $f_\sharp$ by Assumption (i). We thus obtain a relative left adjoint $\psi : t^*\mathcal{E} \to s^*\mathcal{E}$.

Step 3: $\chi \circ \psi : t^*\mathcal{E} \to s^*\mathcal{E} \to \mathcal{E}$ is a morphism of Cartesian fibrations over $\mathsf{Span}^{\mathsf{op}}$. Compatibility with the structure maps down to $\mathsf{Span}^{\mathsf{op}}$ holds by construction. So we have to show that $\chi \circ \psi$ preserves Cartesian edges. We first need a lemma about composition of Cartesian fibrations:

Lemma 6.14. *Let $p : \mathcal{E} \to \mathcal{C}$ and $q : \mathcal{C} \to \mathcal{D}$ be Cartesian fibrations. Then an edge e in $\mathcal{E}$ is Cartesian with respect to the composite Cartesian fibration $q \circ p$ if and only if e is p-Cartesian and $p(e)$ is q-Cartesian.*

Proof. The "if" direction follows from checking the universal property of Cartesian edges; this is also how one proves that the composition of Cartesian fibrations is Cartesian (see [Lur09, Prop. 2.4.2.3]).

Since q is a Cartesian fibration, $q(p(e))$ has a q-Cartesian lift e' with the same target as $p(e)$. And since p is a Cartesian fibration, e' has itself a p-Cartesian lift e'' with the same target as e. Then, e'' is a $(q \circ p)$-Cartesian lift of $q(p(e))$, as is e by assumption. Therefore e and e'' are equivalent. In particular, e is p-Cartesian, and $p(e)$ is equivalent to $p(e'') = e'$, so it is q-Cartesian. □

Coming back to our case, let e be a $(s^{\mathsf{op}} \circ t^*F)$-Cartesian edge. Then e is t^*F Cartesian, and $(t^*F)(e)$ is s^{op}-Cartesian. It follows that $(t^*F)(e)$ is the opposite of an edge in $\mathsf{Fun}_{\mathsf{sm}}(\Delta^1, \mathsf{Span})$ of the form

$$\begin{tikzcd}
W \arrow[equal, d] & Y' \arrow[l] \arrow[r] \arrow[equal, d] & Z' \arrow[equal, d] \\
W \arrow[d, "g"'] & Y' \arrow[l, "f'"'] \arrow[r, "\nabla'"] \arrow[d, "u'"] \arrow[dr, phantom, "\lrcorner", very near start] & Z' \arrow[d, "u"] \\
X & Y \arrow[l, "f"] \arrow[r, "\nabla"'] & Z
\end{tikzcd}$$

where $Y' := R_{Y/Z}(W \times_X Y) \times_Z Y$ and $Z' = R_{Y/Z}(W \times_X Y)$. Here g, u, and u' are smooth. Since e is now a (t^*F)-Cartesian lift of this edge, it is of the form $\alpha : (Z \leftarrow Z' = Z', H) \to (X \leftarrow W = W, E)$ with $H \in \mathcal{F}(Z')$, $E \in \mathcal{F}(W)$, and α consists in the data of $(t^*F)(e)$ (the diagram above) together with an equivalence $H \xrightarrow{\simeq} \nabla'_{\otimes} f'^* E$.

Then $\chi\psi(e)$ is an edge $(Z, u_\sharp(H)) \to (X, g_\sharp(E))$ given by the morphisms $(X \leftarrow Y \to Z)^{\mathsf{op}}$ and $u_\sharp H \to \nabla_{\otimes} f^* g_\sharp E$ is given by the composite

$$u_\sharp H \longrightarrow u_\sharp \nabla'_{\otimes} f'^* E \xrightarrow{\mathsf{Dis}_{\sharp,\otimes}} \nabla_{\otimes} (\pi_Y)_\sharp \pi_W^* E \xrightarrow{\mathsf{Ex}_\sharp^*} \nabla_{\otimes} f^* g_\sharp E,$$

where $\pi_Y : Y \times_X W \to Y$ is the projection and similarly for π_W.

In order for $\chi\psi$ to be F-Cartesian, this map must be an equivalence. But we saw above that the first map was an equivalence, and the two next ones are equivalences by assumptions (iii) and (ii) in the statement respectively.

Step 4: We deduce the first part of the statement. The morphism of Cartesian fibrations $\chi \circ \psi$ corresponds to a natural transformation $\widetilde{M}_{\mathcal{F}}$ of functors $\mathsf{Span} \to \mathsf{Cat}_\infty^\infty$ between the functors classified by $s^{\mathsf{op}} \circ t^*F$ and F respectively. This first functor associates to $X \in \mathsf{Sm}_S$ the fiber of $s^{\mathsf{op}} \circ t^*F$ over X. The fiber of s^{op} over X can be identified with Sm_X; and over an object Y in Sm_X, the fiber of t^*F is by definition $\mathcal{F}(Y)$. Therefore, the functor classified by $s^{\mathsf{op}} \circ t^*F$ may be denoted by $(\mathsf{Sm}_\bullet)_{/\!/\mathcal{F}}$. Indeed, it associates to $X \in \mathsf{Sm}_S$ the ∞-category $(\mathsf{Sm}_X)_{/\!/\mathcal{F}}$ of pairs (U, E) where $U \in \mathsf{Sm}_X$ and $E \in \mathcal{F}(U)$, and a morphism $(U, E) \to (U', E')$ is the data of $f : U \to U'$ in Sm_X and a map $E \to \mathcal{F}(f^{\mathsf{op}})(E')$ in $\mathcal{F}(U)$, which is *not* required to be an equivalence. These ∞-categories are spliced into a presheaf as follows: a morphism $X \xleftarrow{f} Y \xrightarrow{\nabla} Z$ is sent to the functor $(\mathsf{Sm}_X)_{/\!/\mathcal{F}} \to (\mathsf{Sm}_Z)_{/\!/\mathcal{F}}$ sending a pair (W, E) with $W \in \mathsf{Sm}_X$ and $E \in \mathcal{F}(X')$ to the pair consisting of $R_{Y/Z}(X' \times_X Y) \in \mathsf{Sm}_Z$ and $\nabla'_{\otimes} f'^* E \in \mathcal{F}(R_{Y/Z}(X' \times_X Y))$, re-using the notation for a Cartesian lift in Step 1.

The slice ∞-category $(\mathsf{Sm}_X)_{/\mathcal{F}}$ as considered in Remark 6.13 is viewed as the wide subcategory of $(\mathsf{Sm}_X)_{/\!/\mathcal{F}}$ where the maps $E \to \mathcal{F}(f^{\mathsf{op}})(E')$ in $\mathcal{F}(U)$ in the description above are required to be equivalences. As such, $(\mathsf{Sm}_\bullet)_{/\mathcal{F}}$ forms a sub-presheaf of $(\mathsf{Sm}_\bullet)_{/\!/\mathcal{F}}$, and we may restrict the natural transformation $\widetilde{M}_{\mathcal{F}}$ to it, giving the first part of the statement.

Step 5: We prove the second part of the statement. If $\mathcal{F}$ lifts to $\mathsf{Cat}_\infty^{\mathsf{sift}}$, consider the following diagram

$$\begin{tikzcd}
\mathsf{Span} \arrow[rr, "(\mathsf{Sm}_\bullet)_{/\mathcal{F}}"] \arrow[rr, bend right=20] \arrow[rr, phantom, shift right=2ex, "\Downarrow \widetilde{M}_{\mathcal{F}}"] \arrow[dr, "\mathcal{F}"'] & & \mathsf{Cat}_\infty \arrow[dr, "\mathcal{P}_\Sigma(-)"] & \\
& \mathsf{Cat}_\infty^{\mathsf{sift}} \arrow[ur] \arrow[rr, bend left=20] \arrow[rr, equal] \arrow[rr, phantom, shift left=2ex, "\Downarrow \varepsilon"] & & \mathsf{Cat}_\infty^{\mathsf{sift}}
\end{tikzcd}$$

where the curved arrows are the composites and ε is the counit of the adjunction $\mathcal{P}_\Sigma(-) \dashv \iota$ (since spherical presheaves define the free sifted cocomplete ∞-category, ι is the inclusion here). The composition of the two natural transformations provides a natural transformation $\mathcal{P}_\Sigma\left((\mathsf{Sm}_\bullet)_{/\mathcal{F}}\right) \to \mathcal{F}$ of functors $\mathsf{Span} \to \mathsf{Cat}_\infty^{\mathsf{sift}}$, where each component is a left Kan extension. Now, the left hand side has a subfunctor $\mathcal{P}_\Sigma(\mathsf{Sm}_\bullet)_{/\mathcal{F}}$, with an induced symmetric monoidal structure, and so the transformation restricts to a morphism of presheaves of symmetric monoidal ∞-categories $\widetilde{M_\mathcal{F}} : \mathcal{P}_\Sigma(\mathsf{Sm}_\bullet)_{/\mathcal{F}} \to \mathcal{F}$, as desired. ☐

Definition 6.15. For $\mathcal{F}$ a functor as in Theorem 6.12, we define $M_\mathcal{F}$ as the S-component of the transformation $\widetilde{M_\mathcal{F}}$, and called it the *motivic colimit functor associated with $\mathcal{F}$*.

Proposition 6.16. *Let $\mathcal{F}$ be as in Theorem 6.12. Then the associated motivic colimit functor $M_\mathcal{F}$ is a symmetric monoidal, colimit preserving functor sending the arrow $u : y(X) \to \mathcal{F}^\simeq$ for $X \in \mathsf{Sm}_S$ with structure map $f : X \to S$ to $f_\sharp(E)$, where $E = u(X)(\mathsf{id}_X) \in \mathcal{F}^\simeq(X)$ is classified by u. More generally, the image of an arrow $\gamma : \mathcal{G} \to \mathcal{F}^\simeq$ in $\mathcal{P}(\mathsf{Sm}_S)_{/\mathcal{F}^\simeq}$ is given by*

$$M_\mathcal{F} \simeq \mathsf{colim}_{(x,X)\in(\mathsf{Sm}_S)_{/\!\!/\mathcal{G}}}(p_X)_\sharp\gamma(x).$$

Proof. The last assertion is [BH21, Rmk 16.5] or [BEH22, Rmk 2.6], and is deduced from the expression of an element of the slice as a colimit of representable objects, and the fact that $M_\mathcal{F}$ preserves colimits (or viewing $M_\mathcal{F}$ as a left Kan extension). ☐

We now apply Definition 6.15 to the functor SH:

Proposition 6.17. *The functor $\mathsf{SH} : \mathsf{Sm}_\mathbb{R}^{\mathsf{op}} \to \mathsf{CAlg}(\mathsf{Cat}_\infty^\times)$ satisfies all hypotheses of Theorem 6.12 and lifts to $\mathsf{Cat}_\infty^{\mathsf{sift}}$. In particular, it defines a functor*

$$M := M_{\mathsf{SH}} : \mathcal{P}_\Sigma(\mathsf{Sm}_\mathbb{R})_{/\mathsf{SH}^\simeq} \longrightarrow \mathsf{SH}(\mathbb{R})$$

which we call the motivic (multiplicative) Thom spectrum functor*.*

Proof. By Proposition 6.9, as a spherical presheaf of symmetric monoidal ∞-categories (by Theorem 6.10, parts (i), (iii) and (vi)), SH defines a functor $\mathsf{Span} \to \mathsf{Cat}_\infty$ which preserves finite products.

We first show that SH lifts to $\mathsf{Cat}_\infty^{\mathsf{sift}}$. Note first that for all $X \in \mathsf{Sm}_\mathbb{R}$, $\mathsf{SH}(X)$ is cocomplete, in particular it admits sifted colimits. Moreover, if $(X \xleftarrow{f} Y \xrightarrow{\nabla} Z) \in \mathsf{Span}(X, Z)$, then the induced functor $\nabla_\otimes \circ f^* : \mathsf{SH}(X) \to \mathsf{SH}(Z)$ preserves sifted colimits. Indeed, f^* preserves all colimits as a left adjoint, and ∇ is induced by the tensor product functor, which preserves sifted colimits since it preserves colimits in both variables separately. More precisely, if $\nabla : Z \amalg Z \to \mathbb{Z}$ is the simplest fold map, then $\nabla_\otimes : \mathsf{SH}(Z) \times \mathsf{SH}(Z) \to \mathsf{SH}(Z)$ is the tensor product. In this situation, for any functor $\mathcal{G} : \mathcal{D} \to \mathsf{SH}(Z) \times \mathsf{SH}(Z)$ with source a sifted category, we have:

$$\begin{aligned}
\nabla_\otimes(\mathsf{colim}_\mathcal{D}\ \mathcal{G}) &\simeq \nabla_\otimes(\mathsf{colim}_\mathcal{D}\ (\pi_1\mathcal{G}), \mathsf{colim}_\mathcal{D}\ (\pi_2\mathcal{G})) \\
&\simeq (\mathsf{colim}_\mathcal{D}\ (\pi_1\mathcal{G})) \otimes (\mathsf{colim}_\mathcal{D}\ (\pi_2\mathcal{G})) \\
&\simeq \mathsf{colim}_{\mathcal{D}\times\mathcal{D}}\ (\pi_1\mathcal{G}\pi_1 \otimes \pi_2\mathcal{G}\pi_2) \quad (\otimes \text{ preserves colimits in both variables separately}) \\
&\simeq \mathsf{colim}_\mathcal{D}\ (\pi_1\mathcal{G} \otimes \pi_2\mathcal{G}) \qquad (\text{by definition of a sifted category}) \\
&\simeq \mathsf{colim}_\mathcal{D}\ p_\otimes\mathcal{G}
\end{aligned}$$

where π_1 and π_2 are the canonical projections for the products $\mathsf{SH}(Z) \times \mathsf{SH}(Z)$ and $\mathcal{D} \times \mathcal{D}$. The same holds for more complicated fold maps.

The fact that assumptions (i) to (iii) in Theorem 6.12 hold follows from Theorem 6.10. ☐

Remark 6.18. By Proposition 6.16, we get the the following explicit expression for the motivic Thom spectrum functor:

$$M(\mathcal{G} \to \mathsf{SH}^\simeq) \simeq \mathsf{colim}_{(x,X)\in(\mathsf{Sm}_\mathbb{R})_{/\!\!/\mathcal{G}}}(p_X)_\sharp\gamma(x),$$

where p_X is the structure map $X \to \mathsf{Spec}(\mathbb{R})$.

Example 6.19 ([BH21, Ex. 16.22], [BH20, below Lemma 4.6]). The motivic $\mathcal{E}_\infty$-ring spectrum MGL is the image under the motivic multiplicative Thom spectrum functor of the arrow $j : K^\circ \longrightarrow \mathsf{SH}^\simeq$ where j is the motivic j-homomorphism, and K° is the rank 0 summand of algebraic K-theory (defined precisely later in this example). As we will now see, j is constructed as a morphism of presheaves of symmetric monoidal ∞-categories, and thus it acquires a structure of commutative algebra object in the source ∞-category of the functor M, by Proposition 6.20. The construction, as described in

[BH21, §16.2], is the following. For any scheme $X \in \mathsf{Sm}_{\mathbb{R}}$, there is a symmetric monoidal ∞-category $\mathsf{Vect}(X)$ of algebraic vector bundles on X, where the tensor product is the direct sum of vector bundles. (Contravariant) functoriality in X is given by the pullback of vector bundles. There is a symmetric monoidal functor $\mathsf{Vect}(X) \to \mathsf{Pic}(X)$ sending a vector bundle ξ to $\Sigma^\infty(\xi/(\xi \setminus \{0\})$, where $\mathsf{Pic}(X)$ is the subgroupoid spanned by the invertible objects in $\mathsf{SH}(X)^{\simeq}$. This extends to a morphism of presheaves of $\mathcal{E}_\infty$-spaces $\mathsf{Vect}^{\simeq} \to \mathsf{Pic}$. It factors through the group completion $(\mathsf{Vect}(X)^{\simeq})^{\mathsf{gp}}$ because Pic is already group-like by construction. Thus, we obtain a morphism $K \to \mathsf{Pic} \to \mathsf{SH}^{\simeq}$ of spherical presheaves of group-like $\mathcal{E}_\infty$-spaces. In parallel, the rank map $\mathsf{Vect}(X)^{\simeq} \to \mathbb{N}$ is a morphism of monoids, and taking group completion yields a morphism $K \to \mathbb{Z}$ of spherical presheaves of group-like $\mathcal{E}_\infty$-spaces, whose fiber is the rank zero part K°. Restricting our morphism $K \to \mathsf{SH}^{\simeq}$ to K°, we obtain the motivic j-homomorphism $j : K^\circ \longrightarrow \mathsf{SH}^{\simeq}$.

Similarly, the motivic spectra MSL and MSp are respectively given by $M(KSL^\circ \to K^\circ \xrightarrow{j} \mathsf{SH}^{\simeq})$ and $M(KSp^\circ \to K^\circ \xrightarrow{j} \mathsf{SH}^{\simeq})$, where KSL° and KSp° are defined similarly as K°, but with respect to even-dimensional oriented bundles, respectively even-dimensional symplectic bundles. The even-dimension requirement is to ensure the symmetry of the direct sum of *oriented* vector bundles. It holds that $K^\circ \simeq \mathsf{L}_{\mathsf{mot}} BGL \simeq \mathsf{L}_{\mathsf{mot}} \mathsf{colim}_{n\in\mathbb{N}} BGL_n$, and similarly $KSL^\circ \simeq \mathsf{L}_{\mathsf{mot}} BSL$, and $KSp^\circ \simeq \mathsf{L}_{\mathsf{mot}} BSp$ (see [BH20, after Lemma 4.6] and [BH21, before Thm 16.13]).

Proposition 6.20. *There is a functor*

$$\mathsf{Fun}^{\times}(\mathsf{Span}, \mathsf{Spc})_{/\mathsf{SH}} \longrightarrow \mathsf{CAlg}\left((\mathcal{P}_\Sigma(\mathsf{Sm}_{\mathbb{R}})_{/\mathsf{SH}}\right),$$

where $\mathsf{Fun}^{\times}$ *denotes the ∞-category of functors preserving finite products (and compatible natural transformations).*

In particular, a morphism of spherical presheaves of symmetric monoidal ∞-categories $A \to \mathsf{SH}^{\simeq}$ defines a commutative algebra object in the slice $\mathcal{P}_\Sigma(\mathsf{Sm}_{\mathbb{R}})_{/\mathsf{SH}^{\simeq}}$ with the symmetric monoidal structure from the last assertion in Theorem 6.12.

Proof. This follows from the last displayed composition of functors in [BH21, proof of Prop. 16.17 and Rmk 16.18]. $\square$

Remark 6.21. This is a partial analog to Proposition A.16. Indeed, we know by Proposition A.16 that a morphism of commutative algebras in $\mathcal{P}_\Sigma(\mathsf{Sm}_{\mathbb{R}})$ with target $\mathsf{SH}^{\simeq}$ yields a commutative algebra in the slice $\mathcal{P}_\Sigma(\mathsf{Sm}_{\mathbb{R}})^{\otimes}_{/\mathsf{SH}^{\simeq}}$ with respect to the symmetric monoidal structure from Proposition A.14. We don't know if the latter agrees with the symmetric monoidal structure on $\mathcal{P}_\Sigma(\mathsf{Sm}_{\mathbb{R}})_{/\mathsf{SH}^{\simeq}}$ arising from the construction in Theorem 6.12, but Proposition 6.20 tells us that, at least, we may construct commutative algebras with respect to this alternative structure in the same way as for the former structure.

6.3 Comparison of topological and motivic multiplicative Thom spectra

In this subsection we aim to show that the motivic multiplicative Thom spectrum functor corresponds under real realization to the topological multiplicative Thom spectrum, in the appropriate sense.

In order to study real realization of motivic Thom spectra, we consider another motivic colimit functor which already incorporates real realization in its definition (Subsection 6.3.2). Since we want to compare the latter with the motivic Thom spectrum functor, we first study in Subsection 6.3.1 naturality in $\mathcal{F}$ of the construction from Definition 6.15. We then proceed in Subsection 6.3.3 to the comparison of this other motivic colimit functor with the topological Thom spectrum functor, and we finally deduce the missing elements to prove Theorem 5.7 in Subsection 6.3.4.

6.3.1 Naturality of the motivic colimit functor in the presheaf $\mathcal{F}$

We begin by studying the naturality of the construction of $\widetilde{M_{\mathcal{F}}}$ in Theorem 6.12 in the functor $\mathcal{F}$.

Proposition 6.22. *Let $\mathcal{F}, \mathcal{G} : \mathsf{Span} \to \mathsf{Cat}_\infty$ be functors satisfying the assumptions of Theorem 6.12. Assume $\tau : \mathcal{F} \to \mathcal{G}$ is a natural transformation such that, in the notation of Assumption (i) in Theorem 6.12, for any smooth morphism $f : Y \to X$ in $\mathsf{Sm}_{\mathbb{R}}$, the transformation τ preserves the adjunctions $f_\sharp \dashv f^*$ corresponding to $\mathcal{F}$ and $\mathcal{G}$.*

Then, there is a natural transformation $\tau_\sharp : (\mathsf{Sm}_\bullet)_{/\mathcal{F}} \to (\mathsf{Sm}_\bullet)_{/\mathcal{G}}$, given by postcomposition with τ, such that the following square is a commutative diagram of transformations of functors $\mathsf{Span} \to \mathsf{Cat}_\infty$ preserving finite products.

$$\begin{tikzcd} (\mathsf{Sm}_\bullet)_{/\mathcal{F}} \arrow[r, "\tau_\sharp"] \arrow[d, "\widetilde{M}_{\mathcal{F}}"'] & (\mathsf{Sm}_\bullet)_{/\mathcal{G}} \arrow[d, "M_{\mathcal{G}}"] \\ \mathcal{F} \arrow[r, "\tau"'] & \mathcal{G}. \end{tikzcd} \tag{4}$$

Moreover, if $\mathcal{F}$ and $\mathcal{G}$ lift to $\mathsf{Cat}_\infty^{\mathsf{sift}}$, and τ descends to a transformation of functors $\mathsf{Span} \to \mathsf{Cat}_\infty^{\mathsf{sift}}$, then the same holds for $\mathcal{P}_\Sigma(\mathsf{Sm}_\bullet)_{/\mathcal{F}^\simeq}$ and $\mathcal{P}_\Sigma(\mathsf{Sm}_\bullet)_{/\mathcal{G}^\simeq}$ instead of $(\mathsf{Sm}_\bullet)_{/\mathcal{F}}$ and $(\mathsf{Sm}_\bullet)_{/\mathcal{G}}$.

Proof. Firstly, to construct $\tau_\sharp$, we have to produce a morphism between the Cartesian fibrations classified by $(\mathsf{Sm}_\bullet)_{/\mathcal{F}}$ and $(\mathsf{Sm}_\bullet)_{/\mathcal{G}}$. To do so, we chase through the construction in the proof of Theorem 6.12, and re-use its notation.

The natural transformation τ induces a morphism between the Cartesian fibrations $F : \mathcal{E} \to \mathsf{Span}^{\mathsf{op}}$ and $G : \mathcal{H} \to \mathsf{Span}^{\mathsf{op}}$ classified by $\mathcal{F}$ and $\mathcal{G}$ respectively. It therefore pulls back to morphisms of Cartesian fibrations $s^*\tau : s^*\mathcal{E} \to s^*\mathcal{H}$ and $t^*\tau : t^*\mathcal{E} \to t^*\mathcal{H}$ over $\mathsf{Fun}_{\mathsf{sm}}(\Delta^1, \mathsf{Span})^{\mathsf{op}}$. In particular, the latter defines a morphism of Cartesian fibrations over $\mathsf{Span}^{\mathsf{op}}$ between $s^{\mathsf{op}} \circ t^*F$ and $s^{\mathsf{op}} \circ t^*G$. Indeed, assume e is an $(s^{\mathsf{op}} \circ t^*F)$-Cartesian edge in $t^*\mathcal{E}$. Then, by Lemma 6.14, e is t^*F-Cartesian and therefore $(t^*\tau)(e)$ is (t^*G)-Cartesian. Now $(t^*G)((t^*\tau)(e)) = (t^*F)(e)$ is s^{op}-Cartesian (again by Lemma 6.14) and so $t^*\tau(e)$ is $(s^{\mathsf{op}} \circ t^*G)$-Cartesian (reverse direction in Lemma 6.14).

Secondly, we show the commutativity of the square 4. In the above we have obtained a diagram of morphisms of Cartesian fibrations over $\mathsf{Span}^{\mathsf{op}}$

$$\begin{tikzcd} t^*\mathcal{E} \arrow[rr, "\psi_{\mathcal{F}}"] \arrow[dr, "t^*\tau"'] \arrow[ddrrrrr, dashed, "t^*F"] & & s^*\mathcal{E} \arrow[rr, "\chi_{\mathcal{F}}"] \arrow[dr, "s^*\tau"] & & \mathcal{E} \arrow[dr, "\tau"] \arrow[ddr, dashed, "F"'] & \\ & t^*\mathcal{H} \arrow[rr, "\psi_{\mathcal{G}}"] \arrow[drrrr, "t^*G"'] & & s^*\mathcal{H} \arrow[rr, "\chi_{\mathcal{G}}"] & & \mathcal{H} \arrow[d, "G"] \\ & & & & & \mathsf{Span}^{\mathsf{op}} \end{tikzcd}$$

(in particular all the triangles down to $\mathsf{Span}^{\mathsf{op}}$ commute) and what we have to show now is that the top rectangle commutes. The square on its right-hand side commutes by construction of $s^*\tau$. All is left to show is commutativity of the square to its left-hand side. Consider an edge in $t^*\mathcal{E}$, it takes the form of a commutative diagram with four squares as in Step 3 of the proof of Theorem 6.12, together with objects $H \in \mathcal{F}(Z')$ and $E \in \mathcal{F}(W)$, with a morphism $\kappa : H \to \nabla'_\otimes f'^*E$ in $\mathcal{F}(Z')$. The difference compared to the aforementionned Step 3 is that Y' and Z' are not required to be Weil restrictions, and κ must not be an equivalence. Then the top-right composite $s^*\tau \circ \psi_{\mathcal{F}}$ maps this edge to the edge consisting in the data of the same diagram and the morphism in $\mathcal{G}(Z)$

$$\tau_Z\left(u_\sharp H \xrightarrow{u_\sharp \kappa} u_\sharp \nabla'_\otimes f'^* E \xrightarrow{\mathsf{Dis}_{\sharp,\otimes}} \nabla_\otimes (\pi_Y)_\sharp \pi_W^* E \xrightarrow{\mathsf{Ex}_\sharp^*} \nabla_\otimes f^* g_\sharp E\right)$$

where the $(-)_\sharp$, $(-)_\otimes$, and $(-)*$ refer to the functoriality of $\mathcal{F}$, whereas the left-bottom composite maps it to

$$u_\sharp \tau_{Z'}(H) \xrightarrow{\tau_{Z'}(\kappa)} u_\sharp \tau_W(\nabla'_\otimes f'^*(E)) \simeq u_\sharp \nabla'_\otimes f'^* \tau_W(E) \xrightarrow{\mathsf{Dis}_{\sharp,\otimes}} \nabla_\otimes (\pi_Y)_\sharp \pi_W^* \tau_W(E) \xrightarrow{\mathsf{Ex}_\sharp^*} \nabla_\otimes f^* g_\sharp \tau_W(E),$$

where the $(-)_\sharp$, $(-)_\otimes$, and $(-)*$ now refer to the functoriality of $\mathcal{G}$.

By assumption, τ is compatible with the exchange transformation $\mathsf{Ex}_\sharp^*$ and with $u_\sharp$. For compatibility with the distributivity transformation, note that it is by definition the composition of the exchange transformation $\mathsf{Ex}_{\sharp,\otimes}$ with the counit of the adjunction $(-)_\sharp \dashv (-)^*$. The transformation τ is compatible with them by assumption, since compatibility of τ with $(-)_\otimes$ is given by it being a morphism of presheaves of symmetric monoidal ∞-categories (naturality for forward morphisms in Span, i.e. spans consisting of one identity and one fold map). We conclude that the diagram commutes.

Finally, by construction of $t^*\tau$ as a pullback, the induced transformations $(\mathsf{Sm}_\bullet)_{/\mathcal{F}} \to (\mathsf{Sm}_\bullet)_{/\mathcal{G}}$ is indeed given over $X \in \mathsf{Sm}_\mathbb{R}$ by the functor $(\mathsf{Sm}_X)_{/\mathcal{F}} \to (\mathsf{Sm}_X)_{/\mathcal{G}}$ induces by post-composition by τ (in the sense that a pair $(Y \in \mathsf{Sm}_X, E \in \mathcal{F}(Y))$ is sent to $(Y, \tau_Y(E) \in \mathcal{G}(Y))$). This justifies the notation $\tau_\sharp$ in the commutative square of the statement.

The argument when everything lifts to $\mathsf{Cat}_\infty^{\mathsf{sift}}$ is the same as in the proof of Theorem 6.12. In this case, the transformation $\tau_\sharp$ between the slices is really given by post-composition with τ. $\square$

6.3.2 A motivic colimit functor $M_\mathcal{R}$ related to real realization

We now will construct a functor $\mathcal{R} : \mathsf{Span} \to \mathsf{Cat}_\infty$ which satisfies the assumption of Theorem 6.12, and provides us with a motivic colimit functor related to real realization. This functor will be our intermediate step between the motivic and topological Thom spectrum functors, because the naturality result of the previous subsection allows us to compare the motivic Thom spectrum functor with this new motivic colimit functor, and we will be able to compare the latter to the topological Thom spectrum functor in the next subsection.

Definition 6.23. Let $\mathcal{R} : \mathsf{Span} = \mathsf{Span}(\mathsf{Sm}_\mathbb{R}, \mathsf{all}, \mathsf{fold}) \longrightarrow \mathsf{Cat}_\infty$ be the spherical presheaf of symmetric monoidal ∞-categories defined by $X \in \mathsf{Sm}_\mathbb{R} \mapsto \mathsf{Sp}(\mathsf{Spc}_{/r_\mathbb{R}(X)})$, where functoriality is given by pullback (the necessary properties are showed in Lemma 6.24 below). The symmetric monoidal structure on $\mathsf{Sp}(\mathsf{Spc}_{/r_\mathbb{R}(X)})$ is induced as in Example 1.20 by the Cartesian structure on $\mathsf{Spc}_{/r_\mathbb{R}(X)}$ (note that the slice ∞-category admits finite limits by [nLa25a, Prop. 4.6].

To apply Definition 6.15 to $\mathcal{F} = \mathcal{R}$, we show

Lemma 6.24. *The functor $\mathcal{R}$ from Definition 6.23 satisfies the assumptions of Theorem 6.12 and lifts to $\mathsf{Cat}_\infty^{\mathsf{sift}}$. It also satisfies a projection formula as in Theorem 6.10(v). In particular, we obtain a symmetric monoidal motivic colimit functor*

$$M_\mathcal{R} : \mathcal{P}_\Sigma(\mathsf{Sm}_\mathbb{R})_{/\mathcal{R}^\simeq} \longrightarrow \mathsf{Sp}\left(\mathsf{Spc}_{/r_\mathbb{R}(\mathsf{Spec}(\mathbb{R}))}\right) \simeq \mathsf{Sp}.$$

Proof. **Step 1:** we prove that $\mathcal{R}$ is a spherical presheaf of symmetric monoidal ∞-categories and satisfies Assumption (i). For any morphism $f : X \to Y$ in $\mathsf{Sm}_\mathbb{R}$, we have a diagram

$$\begin{tikzcd}
\mathsf{Spc}_{/r_\mathbb{R}(X)} \arrow[rr, shift left=4, "f_\sharp"] \arrow[rr, phantom, shift left=2, "\perp"] \arrow[rr, phantom, shift right=2, "\perp"] \arrow[rr, shift right=4, "f_*"'] \arrow[d, "\Sigma^\infty_+"'] & & \mathsf{Spc}_{/r_\mathbb{R}(Y)} \arrow[ll, "f^*"'] \arrow[d, "\Sigma^\infty_+"] \\
\mathsf{Sp}\left(\mathsf{Spc}_{/r_\mathbb{R}(X)}\right) \arrow[rr, shift left=2, "f_\sharp"] \arrow[rr, phantom, "\perp"] & & \mathsf{Sp}\left(\mathsf{Spc}_{/r_\mathbb{R}(Y)}\right) . \arrow[ll, shift left=2, "f^*"]
\end{tikzcd}$$

The adjunction $f_\sharp \dashv f^*$ is built as in the case of schemes, where f^* is the pullback functor and $f_\sharp$ is induced by post-composition of structure maps. In particular, f^* preserves products because it is a right adjoint, so it is symmetric monoidal with respect to the Cartesian symmetric monoidal structure on both ∞-categories. The right adjoint f_* to f^* exists because pullbacks commute with colimits in spaces (universality of colimits). Since both $f_\sharp$ and f^* are left-adjoints, they preserve colimits and thus descend to an adjunction on the stabilization of both ∞-categories. The induced functor f^* becomes symmetric monoidal for the smash product. Moreover, we have, for all $X, Y \in \mathsf{Sm}_\mathbb{R}$

$$\begin{aligned}
\mathcal{R}(X \amalg Y) &\simeq \mathsf{Sp}\left(\mathsf{Spc}_{/r_\mathbb{R}(X \amalg Y)}\right) \\
&\simeq \mathsf{Sp}\left(\mathsf{Spc}_{/r_\mathbb{R}(X)} \times \mathsf{Spc}_{/r_\mathbb{R}(Y)}\right) \\
&\simeq \mathsf{Sp}\left(\mathsf{Spc}_{/r_\mathbb{R}(X)}\right) \times \mathsf{Sp}\left(\mathsf{Spc}_{/r_\mathbb{R}(Y)}\right) \\
&\simeq \mathcal{R}(X) \times \mathcal{R}(Y).
\end{aligned}$$

This proves assumption (i) and establishes $\mathcal{R}$ as a presheaf of symmetric monoidal ∞-categories.

Step 2: We prove Assumption (*ii*). Consider a square as in the statement of Assumption (*ii*). We first prove a projection formula, more precisely, that for all $S \in \mathcal{R}(X)$ and $T \in \mathcal{R}(Y)$, we have $f_\sharp(S \wedge f^*T) \simeq f_\sharp(S) \wedge T$. Indeed, since both sides preserve colimits in both S and T separately, it suffices to show that the formula holds for the infinite suspension spectra of some $S \in \mathsf{Spc}_{/r_\mathbb{R}(X)}$ and $T \in \mathsf{Spc}_{/r_\mathbb{R}(Y)}$. We have

$$\begin{aligned} f_\sharp(\Sigma^\infty_+ S \wedge f^*\Sigma^\infty_+ T) &\simeq f_\sharp(\Sigma^\infty_+ S \wedge \Sigma^\infty_+ f^*T) \\ &\simeq f_\sharp(\Sigma^\infty_+(S \times_{r_\mathbb{R}(X)} f^*T)) \\ &\simeq \Sigma^\infty_+ f_\sharp(S \times_{r_\mathbb{R}(X)} f^*T)) \\ &\simeq \Sigma^\infty_+(f_\sharp(S) \times_{r_\mathbb{R}(Y)} T) \qquad (\star) \\ &\simeq f_\sharp(\Sigma^\infty_+(S)) \wedge \Sigma^\infty_+ T \end{aligned}$$

where $(\star)$ follows from the pasting law for pullbacks (dual of [Lur09, Lemma 4.4.2.1]).

Now, to prove that the exchange transformation is an equivalence in this pullback square, it suffices similarly to show it before stabilization. Then, for all $S \in \mathsf{Spc}_{/r_\mathbb{R}(X)}$ with structure map $x : S \to r_\mathbb{R}(X)$, we have

$$\begin{aligned} g^* f_\sharp(S \to r_\mathbb{R}(X)) &\simeq g^*(S \to r_\mathbb{R}(X) \to r_\mathbb{R}(Y)) \\ &\simeq (S \times_{r_\mathbb{R}(Y)} r_\mathbb{R}(Y') \to r_\mathbb{R}(Y')) \\ &\simeq f'_\sharp(S \times_{r_\mathbb{R}(X)} r_\mathbb{R}(X') \to r_\mathbb{R}(X')) \qquad (\star) \\ &\simeq f'_\sharp g'^*(S \to r_\mathbb{R}(X)) \end{aligned}$$

where $(\star)$ follows from the following diagram and the pasting law for pullbacks (recall that real realization preserves pullbacks)

$$\begin{tikzcd} S \times_{r_\mathbb{R}(Y)} r_\mathbb{R}(Y') \arrow[r] \arrow[d] & S \arrow[d, "x"] \\ r_\mathbb{R}(X') \arrow[r, "g'"'] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & r_\mathbb{R}(X) \arrow[d, "f"] \\ r_\mathbb{R}(Y') \arrow[r, "g"] & r_\mathbb{R}(Y). \end{tikzcd}$$

Thus the exchange transformation is an equivalence.

Step 3: We prove Assumption (*iii*). Consider a diagram as in the statement of Assumption (*iii*). To lighten notation we consider the case where $Y = Z^{\amalg n}$, the case of direct sums of fold maps being similar. Write $U := W \times_X Y$, and viewing it as a scheme over Y we can write $U = \coprod_{i \le n} U_i$ where U_i lives over the i-th copy Z_i of Z in Y. Let $e : Y' \to U$ be the natural map; it is given over Z_i by the projection on the i-th component $\prod_{j \le n} U_j \to U_i$. Then, for any A over $r_\mathbb{R}(W)$, let $B = (\pi_W)^*(A)$, which lives over $r_\mathbb{R}(U)$. Write B_i the component of B over $r_\mathbb{R}(U_i)$.

Now $e^*(B)$, as a space over $r_\mathbb{R}(Y') = r_\mathbb{R}(R_{Y/Z} U \times_Z Y) = \coprod_{i \le n} r_\mathbb{R}(U_1) \times_{r_\mathbb{R}(Z)} \cdots \times_{r_\mathbb{R}(Z)} r_\mathbb{R}(U_n)$, is given by

$$\coprod_{i \le n} r_\mathbb{R}(U_1) \times_{r_\mathbb{R}(Z)} \cdots \times_{r_\mathbb{R}(Z)} B_i \times_{r_\mathbb{R}(Z)} \cdots \times_{r_\mathbb{R}(Z)} r_\mathbb{R}(U_n).$$

In the remained of this proof step, unless specified otherwise, all products are fiber products over $r_\mathbb{R}(Z)$. Then $u_\sharp \nabla'_\otimes f'^*(A) = u_\sharp \nabla'_\otimes e^*(B)$ is given by the fiber product over $(r_\mathbb{R}(U_1) \times \cdots \times r_\mathbb{R}(U_n))$ of the factors $r_\mathbb{R}(U_1) \times \cdots \times B_i \times \cdots \times r_\mathbb{R}(U_n)$ for $1 \le i \le n$, that is

$$u_\sharp \nabla'_\otimes e^*(T) = B_1 \times \cdots \times B_n \longrightarrow r_\mathbb{R}(Z),$$

which coincides with $\nabla_\otimes (\pi_Y)_\sharp (\pi_W)^*(A) = \nabla_\otimes (\pi_Y)_\sharp(B)$. Now $\nabla_\otimes u'_\sharp f'^*(A) = \nabla_\otimes u'_\sharp e^*(B)$ is given by the fiber product over $r_\mathbb{R}(Z)$

$$\prod_{i \le n} r_\mathbb{R}(U_1) \times \cdots \times B_i \times \cdots \times r_\mathbb{R}(U_n) \longrightarrow r_\mathbb{R}(Z).$$

The distributivity transformation is now the map

$$B_1 \times \cdots \times B_n \longrightarrow \prod_{i \le n} r_\mathbb{R}(U_1) \times \cdots \times B_i \times \cdots \times r_\mathbb{R}(U_n) \longrightarrow B_1 \times \cdots \times B_n,$$

which is seen to be the identity by chasing through the construction. Thus the distributivity transformation is an equivalence in the desired case.

Step 4: We show that $\mathcal{R}$ lifts to $\mathsf{Cat}_\infty^{\mathsf{sift}}$. Note first that for all $X \in \mathsf{Sm}_\mathbb{R}$, the slice $\mathsf{Spc}_{/r_\mathbb{R}(X)}$ is presentable because Spc is a presentable ∞-category by [nLa25b, Prop. 4.15]. Then, using [Lur17, Prop. 1.4.4.4], the ∞-category $\mathsf{Sp}(\mathsf{Spc}_{/r_\mathbb{R}(X)})$ is cocomplete. Moreover, for any map $f \in \mathsf{Sm}_\mathbb{R}$, the functor f^* preserves colimits; and for any fold map ∇, the functor $\nabla_\otimes$ preserves sifted colimits since it is induced by the tensor product functor on the ∞-categories $\mathcal{R}(X)$, which are presentably symmetric monoidal by construction and hence the tensor product functor preserves colimits in each variable separately. Since the operations f^* and $\nabla_\otimes$ describe the functoriality of $\mathcal{R} : \mathsf{Span} \to \mathsf{Cat}_\infty$, the latter lifts to $\mathsf{Cat}_\infty^{\mathsf{sift}}$, as desired. $\square$

We want to apply Proposition 6.22 to a transformation $\alpha : \mathsf{SH} \to \mathcal{R}$ of functors $\mathsf{Span} \to \mathsf{Cat}_\infty^{\mathsf{sift}}$, which we now define. Such a natural transformation is a morphism of spherical presheaves of sifted-cocomplete symmetric monoidal ∞-categories. For every $S \in \mathsf{Sm}_\mathbb{R}$, by the results in Subsection 4 we used to construct the real realization functor, to produce a colimit-preserving symmetric monoidal functor $\mathsf{SH}(S) \to \mathsf{Sp}(\mathsf{Spc}_{/r_\mathbb{R}(S)})$, it suffices to produce a functor $\mathsf{Spc}(S) \to \mathsf{Spc}_{/r_\mathbb{R}(S)}$ preserving colimits and finite products, such that the induced functor on pointed objects maps $\mathbb{P}^1_S$ to an object whose infinite suspension is invertible with respect to the tensor product. We use real realization for this functor: send $X \in \mathsf{Sm}_S$ to the real realization of its structure map $r_\mathbb{R}(X) \to r_\mathbb{R}(S)$. The left Kan extension $\mathcal{P}(\mathsf{Sm}_S) \to \mathsf{Spc}_{/r_\mathbb{R}(S)}$ then factors through $\mathsf{Spc}(S)$ because real realization does. Moreover, $\mathbb{P}^1_S = \mathsf{colim}(S \leftarrow \mathsf{Spec}(\mathbb{Z}[t,t^{-1}]) \times S \to S)$ maps to $\mathsf{colim}(r_\mathbb{R}(S) \leftarrow r_\mathbb{R}(S) \amalg r_\mathbb{R}(S) \to r_\mathbb{R}(S))$ which is the suspension of $*_+$ in the slice $\mathsf{Spc}_{/r_\mathbb{R}(S)}$ and thus its infinite suspension spectrum is by definition invertible in $\mathsf{Sp}(\mathsf{Spc}_{/r_\mathbb{R}(S)})$. By construction, $\alpha_\mathbb{R} \simeq r_\mathbb{R}$ under the identification $\mathcal{R}(\mathbb{R}) = \mathsf{Sp}\left(\mathsf{Spc}_{/r_\mathbb{R}(\mathbb{R})}\right) \simeq \mathsf{Sp}$ (the latter is an equivalence of symmetric monoidal ∞-categories because the symmetric monoidal structure on $\mathsf{Sp}\left(\mathsf{Spc}_{/r_\mathbb{R}(\mathbb{R})}\right)$ is induced in Pr^L by tensoring with Sp the Cartesian symmetric monoidal ∞-category $\mathsf{Spc}_{/r_\mathbb{R}(\mathbb{R})} = \mathsf{Spc}_{/*}$.

To show naturality, we have to check that for any $f : X \to Y$ in Sm_S, the diagram

$$\begin{tikzcd} \mathsf{SH}(Y) \arrow[r, "\alpha_Y"] \arrow[d, "f^*"] & \mathsf{Sp}\left(\mathsf{Spc}_{/r_\mathbb{R}(Y)}\right) \arrow[d, "f^*"] \\ \mathsf{SH}(X) \arrow[r, "\alpha_X"] & \mathsf{Sp}\left(\mathsf{Spc}_{/r_\mathbb{R}(X)}\right) \end{tikzcd}$$

is commutative.

Since all functors in this diagram are symmetric monoidal and colimit-preserving, we check the claim for the restriction to Sm_Y (i.e. on smooth schmes and maps between them). By definition, for any $Z \in \mathsf{Sm}_Y$ with structure map $z : Z \to Y$, we have

$$\alpha_X \circ f^*(\Sigma^\infty_+ Z) \simeq \alpha_X(\Sigma^\infty_+(Z \times_Y X)) \simeq \Sigma^\infty_+(r_\mathbb{R}(Z \times_Y X) \xrightarrow{r_\mathbb{R} z'} r_\mathbb{R}(X))$$

where z' is the pullback of z along f. Also

$$f^* \circ \alpha_Y(\Sigma^\infty_+ Z) \simeq f^*(\Sigma^\infty_+(r_\mathbb{R}(Z) \xrightarrow{r_\mathbb{R} z} r_\mathbb{R}(Y))) \simeq \Sigma^\infty_+(r_\mathbb{R}(Z) \times_{r_\mathbb{R}(Y)} r_\mathbb{R}(X) \to r_\mathbb{R}(X))$$

which is equivalent to $\Sigma^\infty_+(r_\mathbb{R}(Z \times_Y X) \xrightarrow{r_\mathbb{R} z'} r_\mathbb{R}(X)$ since real realization preserves pullbacks.

As promised, we apply Proposition 6.22 to the transformation α:

Proposition 6.25. *There is a commutative diagram of symmetric monoidal ∞-categories and symmetric monoidal functors*

$$\begin{tikzcd} \mathcal{P}(\mathsf{Sm}_\mathbb{R})_{/\mathsf{SH}^\simeq} \arrow[r, "\alpha_\sharp"] \arrow[d, "M"'] & \mathcal{P}(\mathsf{Sm}_\mathbb{R})_{/\mathcal{R}^\simeq} \arrow[d, "M_\mathcal{R}"] \\ \mathsf{SH}(\mathbb{R}) \arrow[r, "\alpha_\mathbb{R}"] & \mathsf{Sp}\left(\mathsf{Spc}_{/r_\mathbb{R}(\mathbb{R})}\right) \simeq \mathsf{Sp}. \end{tikzcd}$$

Proof. We have to check that α satisfies the assumption of Proposition 6.22, i.e. that α preserves the $(-)_\sharp \dashv (-)^*$ adjunctions. The compatibility with f^* is the commutativity of real Betti realization with pullbacks. To show that $f_\sharp \alpha_X \simeq \alpha_Y f_\sharp : \mathsf{SH}(X) \to \mathcal{R}(Y)$, note that their restrictions to representable presheaves agree because real realization commutes with the post-composition of structure maps. To extend to all spectra, since $\mathsf{SH}(X)$ is generated under colimits by terms of the form $\mathcal{S}^{a,b} \wedge \Sigma^\infty_+ Z$ for all $a, b \in \mathbb{Z}$ and $Z \in \mathsf{Sm}_X$. Then, since all the functors involved preserve colimits, we only have to show that they agree when restricted to such objects. This last fact follows from the projection formula for both SH (Theorem 6.10) and for $\mathcal{R}$ (Proposition 6.24). Indeed, we have:

$$\begin{aligned}
\alpha_Y f_\sharp(\mathcal{S}^{a,b}_X \wedge \Sigma^\infty_+ Z) &\simeq \alpha_Y f_\sharp(f^*(\mathcal{S}^{a,b}_Y) \wedge \Sigma^\infty_+ Z) \\
&\simeq \alpha_Y(\mathcal{S}^{a,b}_Y \wedge f_\sharp(\Sigma^\infty_+ Z)) \\
&\simeq \mathbb{S}^{a-b}_Y \wedge \Sigma^\infty_+ \alpha_Y(f_\sharp(Z)) \\
&\simeq \mathbb{S}^{a-b}_Y \wedge f_\sharp(\Sigma^\infty_+ \alpha_X(Z)) \\
&\simeq f_\sharp(f^*(\mathbb{S}^{a-b}_Y) \wedge \Sigma^\infty_+ \alpha_X(Z)) \\
&\simeq f_\sharp(\mathbb{S}^{a-b}_X \wedge \Sigma^\infty_+ \alpha_X(Z)) \\
&\simeq f_\sharp(\alpha_X(\mathcal{S}^{a-b} \wedge \Sigma^\infty_+ Z)).
\end{aligned}$$

The fact that α is compatible with the unit and counit of the transformation follows again from the fact that real realization preserves pullbacks. □

6.3.3 Relating the motivic colimit functor $M_{\mathcal{R}}$ to the topological Thom spectrum functor

As mentioned at the beginning of the previous subsection, the goal is now to compare $M_{\mathcal{R}}$ with the topological Thom spectrum functor. We saw while constructing $r_{\mathbb{R}}$ that the real realization functor $\mathcal{P}(\mathsf{Sm}_{\mathbb{R}}) \to \mathsf{Spc}$ factored through the ∞-category of $\mathbb{A}^1$-invariant Nisnevich sheaves $\mathsf{Spc}(\mathbb{R})$. Actually, a bit more is true:

Lemma 6.26. *The real realization functor* $\mathsf{Spc}(\mathbb{R}) \to \mathsf{Spc}$ *factors through the* ∞*-category of* $\mathbb{A}^1$*-invariant real-étale sheaves.*

Proof. The real realization of the Čech complex of a real étale cover $U \to X$, i.e. an étale map inducing a surjection $RU \to RX$ is equivalent to $r_{\mathbb{R}}(X)$ (recall the real spectrum $R(-)$, introduced below Theorem 4.10). In step 3 of the construction of the real realization functor in Subsection 4.1, when we saw that $r_{\mathbb{R}}$ had Nisnevich descent, we saw that real realization of étale maps were locally split on their image, and that Čech complexes of locally split maps were equivalent to the codomain. Here we have surjectivity by assumption, so the real realization of the cover is locally split, and thus induces the desired equivalence. □

The above result implies that $\mathcal{R}$ is an $\mathbb{A}^1$-invariant real étale sheaf (and $\mathcal{R}^{\simeq}$ too, since being a sheaf is a limit condition and $(-)^{\simeq}$ is a right adjoint). Therefore, the functor $M_{\mathcal{R}} : \mathcal{P}(\mathsf{Sm}_{\mathbb{R}})_{/\mathcal{R}^{\simeq}} \to \mathsf{Sp}$ factors through the ∞-category $L_{\mathbb{A}^1,\mathsf{ret}}(\mathcal{P}(\mathsf{Sm}_{\mathbb{R}}))_{/\mathcal{R}^{\simeq}}$ of $\mathbb{A}^1$-invariant real étale sheaves over $\mathcal{R}^{\simeq}$, by [BEH22, Prop. 2.11] (the proposition deals with the interaction between localizing a slice ∞-category and slicing a localization of an ∞-category).

This allows a very interesting identification with the topological case:

Theorem 6.27 ([ABEH25, Thm 1.1])**.** *The (unstable) real realization functor* $r_{\mathbb{R}} : \mathcal{P}(\mathsf{Sm}_{\mathbb{R}}) \to \mathsf{Spc}$ *factors through the* ∞*-category* $L_{\mathbb{A}^1,\mathsf{ret}}(\mathcal{P}(\mathsf{Sm}_{\mathbb{R}}))$ *of* $\mathbb{A}^1$*-invariant real étale sheaves, and the induced functor* $L_{\mathbb{A}^1,\mathsf{ret}}(\mathcal{P}(\mathsf{Sm}_{\mathbb{R}})) \to \mathsf{Spc}$*, given by taking sections on* $\mathsf{Spec}(\mathbb{R})$*, is an equivalence*

$$\begin{tikzcd}
\mathcal{P}(\mathsf{Sm}_{\mathbb{R}}) \arrow[r] \arrow[d, "L_{\mathbb{A}^1,\mathsf{ret}}"'] & \mathsf{Spc} \\
L_{\mathbb{A}^1,\mathsf{ret}}(\mathcal{P}(\mathsf{Sm}_{\mathbb{R}})) \arrow[ur, dashed, "\simeq"']
\end{tikzcd}$$

whose inverse is given by left Kan extension.

Since the global sections of $\mathcal{R}^{\simeq}$ are the space $\mathsf{Sp}^{\simeq}$, the functor $M_{\mathcal{R}}$ induces a functor $\mathsf{Spc}_{/\mathsf{Sp}^{\simeq}} \to \mathsf{Sp}$ which we denote by M'_{top}. We summarize the situation as the following commutative diagram of symmetric monoidal ∞-categories and symmetric monoidal functors

$$\begin{tikzcd}
\mathcal{P}(\mathsf{Sm}_{\mathbb{R}})_{/\mathsf{SH}^{\simeq}} \arrow[r, "\alpha_\sharp"] \arrow[d, "M"'] & \mathcal{P}(\mathsf{Sm}_{\mathbb{R}})_{/\mathcal{R}^{\simeq}} \arrow[r, "L_{\mathbb{A}^1,\mathsf{ret}}"] \arrow[d, "M_{\mathcal{R}}"] \arrow[rr, bend left, "r_{\mathbb{R}}"] & L_{\mathbb{A}^1,\mathsf{ret}}(\mathcal{P}(\mathsf{Sm}_{\mathbb{R}}))_{/\mathcal{R}^{\simeq}} \arrow[r, "\simeq"] \arrow[dl, dashed] & \mathsf{Spc}_{/\mathsf{Sp}^{\simeq}} \arrow[dll, dashed, "M'_{\mathsf{top}}"] \\
\mathsf{SH}(\mathbb{R}) \arrow[r, "r_{\mathbb{R}}"] & \mathcal{R}(\mathbb{R}) \simeq \mathsf{Sp}. & &
\end{tikzcd} \tag{5}$$

Here the symmetric monoidal structure on $\mathsf{Spc}_{/\mathsf{Sp}^{\simeq}}$ is induced by that of $L_{\mathbb{A}^1,\mathsf{ret}}(\mathcal{P}(\mathsf{Sm}_{\mathbb{R}}))_{/\mathcal{R}^{\simeq}}$ viewed as a symmetric monoidal subcategory of $\mathcal{P}_{\Sigma}(\mathsf{Sm}_{\mathbb{R}})_{/\mathcal{R}^{\simeq}}$ (since its a localization of the latter, as we saw below the proof of Lemma 6.26). We denote this symmetric monoidal structure by $\mathsf{Spc}^{\otimes'}_{/\mathsf{Sp}^{\simeq}}$ to distinguish it from the one Lurie constructs (Proposition A.14), and which we denote by $\mathsf{Spc}^{\otimes}_{/\mathsf{Sp}^{\simeq}}$.

The following result justifies the notation:

Lemma 6.28. *The functors M'_{top} and M_{top} agree on objects.*

Proof. By Proposition 6.16, for all $\mathcal{F} \in \mathcal{P}_{\Sigma}(\mathsf{Sm}_{\mathbb{R}})$ and $(\gamma : \mathcal{F} \to \mathcal{R}) \in \mathcal{P}_{\Sigma}(\mathsf{Sm}_{\mathbb{R}})_{/\mathcal{R}^{\simeq}}$, we have

$$M_{\mathcal{R}}(\gamma : \mathcal{F} \to \mathcal{R}) \simeq \mathsf{colim}_{(x,X) \in (\mathsf{Sm}_{\mathbb{R}})_{/\!\!/\mathcal{F}}} (p_X)_\sharp \gamma(x).$$

Now if $\Gamma : \chi \to \mathsf{Sp}^{\simeq}$ is an object in $\mathsf{Spc}_{/\mathsf{Sp}^{\simeq}}$, let $\gamma : \mathcal{F} \to \mathcal{R}^{\simeq}$ be the corresponding $\mathbb{A}^1$-invariant real-étale sheaf over $\mathcal{R}^{\simeq}$, i.e. the left Kan extension of Γ by Theorem 6.27. Then

$$\begin{aligned}
M'_{\mathsf{top}}(\Gamma : \chi \to \mathsf{Sp}^{\simeq}) &\simeq M'_{\mathsf{top}}(r_{\mathbb{R}}(\gamma : \mathcal{F} \to \mathcal{R}^{\simeq})) \\
&\simeq M_{\mathcal{R}}(\gamma : \mathcal{F} \to \mathcal{R}^{\simeq}) \\
&\simeq \mathsf{colim}_{(X,x) \in (\mathsf{Sm}_{\mathbb{R}})_{/\!\!/\mathcal{F}}} (p_X)_\sharp \gamma(x) \\
&\simeq \mathsf{colim}_{x \in \mathcal{F}(\mathbb{R})} \gamma(x) \qquad (\star) \\
&\simeq \mathsf{colim}_{x \in \chi} \Gamma(x) \\
&\simeq M_{\mathsf{top}}(\Gamma : \chi \to \mathsf{Sp}^{\simeq}).
\end{aligned}$$

Here, $(\star)$ follows from the fact that $\mathcal{F}$ is by definition the left Kan extension of its restriction to the point $\{\mathsf{Spec}(\mathbb{R})\}$. In particular, the functor $(\mathsf{Sm}_{\mathbb{R}})_{/\!\!/\mathcal{F}} \to \mathsf{Sp}$ whose colimit we are considering is the left Kan extension of the functor $*_{/\!\!/\mathcal{F}} \simeq \mathcal{F}(\mathbb{R}) \to \mathsf{Sp}$, given by the $\mathbb{R}$-component of γ followed by the embedding $\mathsf{Sp}^{\simeq} \hookrightarrow \mathsf{Sp}$. But the colimit of a left Kan extended functor is the colimit of the original functor, whence $(\star)$ follows. $\square$

Inspired by the proof of the previous lemma, we may show a stronger result, namely that the real realization of the motivic multiplicative Thom spectrum functor agrees with the topological multiplicative Thom spectrum functor:

Theorem 6.29. *The symmetric monoidal structures $\mathsf{Spc}^{\otimes'}_{/\mathsf{Sp}^{\simeq}}$ (obtained from the construction of Theorem 6.12, i.e. from [BH21, §16.3], and diagram 5) and $\mathsf{Spc}^{\otimes}_{/\mathsf{Sp}^{\simeq}}$ (described in Proposition A.14) agree. Moreover, the functors M'_{top} and M_{top} agree as symmetric monoidal functors.*

Proof. One the one hand, by Proposition A.15, the symmetric monoidal structure $\mathsf{Spc}^{\otimes}_{/\mathsf{Sp}^{\simeq}}$ from Proposition A.14 is equivalently the Day convolution structure on $\mathcal{P}(*_{/\mathsf{Sp}^{\simeq}}) \simeq \mathcal{P}(\mathsf{Sp}^{\simeq})$, where $\mathsf{Sp}^{\simeq}$ is viewed as a symmetric monoidal subcategory of Sp with the usual smash product.

On the other hand, the underlying ∞-category of $\mathsf{Spc}^{\otimes'}_{/\mathsf{Sp}^{\simeq}}$ is the same as that of $\mathsf{Spc}^{\otimes}_{/\mathsf{Sp}^{\simeq}}$, namely $\mathcal{P}(\mathsf{Sp}^{\simeq})$. Then, by Proposition A.8, to show that the symmetric monoidal structure is equivalent to the Day convolution one, it suffices to show that:

1. the tensor product $\otimes'$ preserves colimits in both variables,
2. and the Yoneda embedding $\mathsf{Sp}^{\simeq} \to \mathcal{P}(\mathsf{Sp}^{\simeq})$ can be made into a symmetric monoidal functor with respect to the smash product on the left hand side and the structure $\mathsf{Spc}^{\otimes'}_{/\mathsf{Sp}^{\simeq}}$ on the right hand side.

By construction, the coCartesian fibration $\mathsf{Spc}^{\otimes'}_{/\mathsf{Sp}^{\simeq}} \to \mathsf{Fin}_*$ representing the symmetric monoidal structure is classified by the composition

$$\mathsf{Fin}_* \xrightarrow{H} \mathsf{Cat}_\infty \xrightarrow{\mathcal{P}(-)} \mathsf{Pr}^{\mathsf{L}} \xrightarrow{U} \mathsf{Cat}_\infty$$

where H classifies the coCartesian fibration $\mathsf{Sp}^{\simeq,\otimes'} \longrightarrow \mathsf{Fin}_*$ encoding some symmetric monoidal structure on $\mathsf{Sp}^{\simeq}$, coming from diagram 5, which we don't know yet; and U is the forgetful functor. In particular, the tensor product functor, which is the image of the map of finite sets $\langle 2\rangle \to \langle 1\rangle$ sending 1 and 2 to 1, preserves colimits in both variables. This proves (1).

Let η be the unit of the adjunction $\mathcal{P}(-) \dashv U$. It induces a natural transformation

$$\begin{tikzcd}
\mathsf{Fin}_* \arrow[r, "H"] & \mathsf{Cat}_\infty \arrow[rr, equal] \arrow[rr, bend right, "U\circ\mathcal{P}(-)"'] \arrow[dr, "\mathcal{P}(-)"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathsf{Cat}_\infty \\
& & \mathsf{Pr}^{\mathsf{L}} \arrow[ur, "U"'] &
\end{tikzcd}$$

between the functors $\mathsf{Fin}_* \longrightarrow \mathsf{Cat}_\infty$ classifying $\mathsf{Sp}^{\simeq,\otimes'}$ and $\mathsf{Spc}^{\otimes'}_{/\mathsf{Sp}^{\simeq}}$ respectively. This induces a symmetric monoidal functor $\mathsf{Sp}^{\simeq,\otimes'} \longrightarrow \mathsf{Spc}^{\otimes'}_{/\mathsf{Sp}^{\simeq}}$, whose underlying functor is by construction the Yoneda embedding. This implies that the restriction of M'_{top} to $*^{\otimes'}_{/\mathsf{Sp}^{\simeq}} \simeq \mathsf{Sp}^{\simeq,\otimes'}$ is a symmetric monoidal functor $\mathsf{Sp}^{\simeq,\otimes'} \longrightarrow \mathsf{Sp}^{\otimes}$, where the symmetric monoidal structure on Sp is the usual one. But we also know that the underlying functor is the inclusion. Therefore, $\mathsf{Sp}^{\simeq,\otimes'}$ is endowed with the symmetric monoidal structure induced from that of Sp. This proves (2).

Thus $\mathsf{Spc}^{\otimes'}_{/\mathsf{Sp}^{\simeq}}$ and $\mathsf{Spc}^{\otimes'}_{/\mathsf{Sp}^{\simeq}}$ are both equivalent as symmetric monoidal categories to $\mathcal{P}(\mathsf{Sp}^{\simeq})$ with the Day convolution, where $\mathsf{Sp}^{\simeq}$ is viewed as a symmetric monoidal subcategory of Sp with the usual smash product. By Proposition A.11, to show that M_{top} and M'_{top} agree as symmetric monoidal functors, it then suffices to show that their restrictions to $\mathsf{Sp}^{\simeq}$ agree as symmetric monoidal functors. In both cases, we have seen that this restriction was the embedding of the symmetric monoidal subcategory $\mathsf{Sp}^{\simeq} \hookrightarrow \mathsf{Sp}$. This concludes the proof. ☐

The statement of Theorem 6.29 gives rise to the following question.

Question 6.30. *In the situation of Theorem 6.12, do the symmetric monoidal structures $\mathcal{P}_\Sigma(\mathsf{Sm}_S)^{\otimes'}_{/\mathcal{F}}$ (obtained from the construction of Theorem 6.12, i.e. from [BH21, §16.3]) and $\mathcal{P}_\Sigma(\mathsf{Sm}_S)^{\otimes}_{/\mathcal{F}}$ (described in Proposition A.14) agree?*

We conjecture that this question can be answered by the affirmative, but we were unable to prove it. Since both symmetric monoidal structures are by construction Day convolution with respect to some symmetric monoidal structures on the slice $(\mathsf{Sm}_S)_{/\mathcal{F}}$, it would suffice to prove that the structures $\otimes$ and $\otimes'$ agree at this level.

Let us give a hint in the direction of a positive answer to Question 6.30:

Proposition 6.31. *In the situation of Theorem 6.12, the tensor product of any two objects for the symmetric monoidal structures $\mathcal{P}_\Sigma(\mathsf{Sm}_S)^{\otimes'}_{/\mathcal{F}}$ and $\mathcal{P}_\Sigma(\mathsf{Sm}_S)^{\otimes}_{/\mathcal{F}}$ agree.*

Proof. Let $g : X \to \mathcal{F}$ and $h : Y \to \mathcal{F}$ be two objects in $(\mathsf{Sm}_S)_{/\mathcal{F}}$, represented by $x \in \mathcal{F}(X)$ and $y \in \mathcal{F}(Y)$. By the informal description of the tensor product in $\mathcal{P}_\Sigma(\mathsf{Sm}_S)^{\otimes}_{\mathcal{F}}$ in Proposition A.14, we have

$$(X \to \mathcal{F}) \otimes (Y \to \mathcal{F}) = X \times Y \xrightarrow{g\times h} \mathcal{F}\times\mathcal{F} \xrightarrow{\mu} \mathcal{F}$$

where μ gives the multiplicative structure on the presheaf of symmetric monoidal ∞-categories $\mathcal{F}$.

In other terms, their tensor product is the map $X \times Y \longrightarrow \mathcal{F}$, represented by the element $\mathcal{F}(\pi_X^{\mathsf{op}})(x) \otimes_{X\times Y} \mathcal{F}(\pi_Y^{\mathsf{op}})(y) \in \mathcal{F}(X\times Y)$, where for $Z \in \mathsf{Sm}_{\mathbb{R}}$, the functor $\otimes_Z$ is the tensor product on $\mathcal{F}(Z)$, namely $\mathcal{F}(Z^{\amalg 2} = Z^{\amalg 2} \longrightarrow Z)$, viewing $\mathcal{F}$ as a functor $\mathsf{Span} \to \mathsf{Cat}_\infty$ preserving finite products.

On the other hand, recall that the presheaf of symmetric monoidal ∞-categories $\mathcal{P}_\Sigma(\mathsf{Sm}_\bullet)^{\otimes'}_{/\mathcal{F}}$ is classified by the Cartesian fibration $s^{\mathsf{op}} \circ t^*F : t^*\mathcal{E} \to \mathsf{Span}^{\mathsf{op}}$, re-using the notation of the proof of Theorem 6.12. Thus the tensor product of g and h is the source of a $(s^{\mathsf{op}} \circ t^*F)$-Cartesian lift for $(S^{\amalg 2} = S^{\amalg 2} \longrightarrow S)^{\mathsf{op}}$ with target $(X \amalg Y, (x,y) \in \mathcal{F}(X \amalg Y) \simeq \mathcal{F}(X) \times \mathcal{F}(Y)) \in t^*\mathcal{E}$. We have seen in the proof of Theorem 6.12 that an s^{op}-Cartesian lift for such an edge was given by (the opposite of)

$$\begin{tikzcd}
X \amalg Y \arrow[d] & (X \times Y)^{\amalg 2} \arrow[l] \arrow[r, "\nabla'"] \arrow[d] & X \times Y = R_{S^{\amalg 2}/S}(X \amalg Y) \arrow[d] \\
S^{\amalg 2} \arrow[r, equal] & S^{\amalg 2} \arrow[r, "\nabla"] & S
\end{tikzcd}$$

and thus the source of the $(s^{\mathsf{op}} \circ t^*F)$-Cartesian lift we are looking for is

$$(X \times Y, \mathcal{F}(X \amalg Y \longleftarrow (X \times Y)^{\amalg 2} \longrightarrow X \times Y)(x,y)),$$

which is exactly $\mathcal{F}(\pi_X^{\mathsf{op}})(x) \otimes_{X \times Y} \mathcal{F}(\pi_Y^{\mathsf{op}})(y) \in \mathcal{F}(X \times Y)$ by construction, as desired. $\square$

6.3.4 The real realizations of MGL, MSL and MSp

We now specialize to our case of interest: the motivic spectrum MSL. As in Example 6.19, similar considerations apply to MSp and MGL.

Theorem 6.32. *There are equivalences of $\mathcal{E}_\infty$-rings*

$$r_{\mathbb{R}}\mathsf{MGL} \simeq \mathsf{MO}, \qquad r_{\mathbb{R}}\mathsf{MSL} \simeq \mathsf{MSO}, \qquad r_{\mathbb{R}}\mathsf{MSp} \simeq \mathsf{MU}.$$

Proof. We saw in Example 6.19 that $r_{\mathbb{R}}\mathsf{MGL} = r_{\mathbb{R}}(M(K^\circ \xrightarrow{j} \mathsf{SH}^{\simeq}))$. Using diagram 5 and Theorem 6.29, this is equivalent as an $\mathcal{E}_\infty$-ring to $M_{\mathsf{top}}(r_{\mathbb{R}}(K^\circ \to \mathsf{SH}^{\simeq} \to \mathcal{R}^{\simeq}))$.

Recall from Lemma 6.28 and Example 6.4 that $\mathsf{MO} = M_{\mathsf{top}}(BO \xrightarrow{j} \mathsf{Sp}^{\simeq})$. So what we have to show is that $r_{\mathbb{R}}(K^\circ \to \mathsf{SH}^{\simeq} \to \mathcal{R}^{\simeq}) \simeq (BO \xrightarrow{j} \mathsf{Sp}^{\simeq})$ as commutative algebras in $\mathsf{Spc}^{\otimes}_{/\mathsf{Sp}^{\simeq}}$, or equivalently, as maps of $\mathcal{E}_\infty$-spaces with target $\mathsf{Sp}^{\simeq}$ (by Proposition A.16). Recall that $K^\circ \to \mathsf{SH}^{\simeq} \to \mathcal{R}^{\simeq}$ is a morphism of spherical presheaves of $\mathcal{E}_\infty$-spaces, viewed as a commutative algebra in $\mathcal{P}_\Sigma(\mathsf{Sm}_{\mathbb{R}})^{\otimes'}_{/\mathcal{R}^{\simeq}}$ using Proposition 6.20. This is how its real realization is viewed as a map of $\mathcal{E}_\infty$-spaces. In other terms, we have to show that the real realization of the motivic j-homomorphism, followed by $\alpha : \mathsf{SH}^{\simeq} \to \mathcal{R}^{\simeq}$, is the topological j-homomorphism as a map of $\mathcal{E}_\infty$-spaces.

This follows from the constructions of the motivic and topological j-homomorphisms as symmetric monoidal functors in Examples 6.4 and 6.19. Indeed, the steps of the construction, starting from ∞-groupoids of vector bundles and then extending to the group completion, correspond to each other under real realization. To see this, recall that $r_{\mathbb{R}}(BGL_n) \simeq BO_n$, and that $K^\circ \simeq \mathsf{L}_{\mathsf{mot}}BGL$ by example 6.19, so that $r_{\mathbb{R}}(K^\circ) \simeq BO$ by similar computations as in Lemma 4.24.

The proof follows in the exact same way for MSL and MSp, using Example 6.19 and the fact that $r_{\mathbb{R}}(KSL^\circ) = BSO$ and $r_{\mathbb{R}}(KSp^\circ) = BU$ (with the same proof as in the case of K°). $\square$

This was the missing piece to prove our main result, because we used in the proof of Proposition 5.3 that the real realization of MSL is equivalent to MSO as an $\mathcal{E}_\infty$-ring spectrum.

7 Conclusion

The goal of this Master's thesis was to compute the real realization of ko, the very effective cover of the motivic Hermitian K-theory motivic spectrum. This question can be viewed at several different levels, since this real realization $r_{\mathbb{R}}\mathsf{ko}$, which is a topological spectrum, inherits an $\mathcal{E}_\infty$-ring structure from ko. We have focused on identifying $r_{\mathbb{R}}\mathsf{ko}$ as an $\mathcal{E}_1$-ring, and found it was equivalent to the connective L-theory spectrum of the field of real numbers, $\mathsf{L}(\mathbb{R})_{\geq 0}$. The comparison between them arises through their 2-local fracture squares, which agree. In particular, both can be expressed as a pullback of $\mathsf{ko}^{\mathsf{top}}[1/2]$, the connective topological K-theory spectrum with 2 inverted, with $\mathsf{H}\mathbb{Z}_{(2)}[t^4]$ over $\mathsf{H}\mathbb{Q}[t^4]$, the two latter being free $\mathcal{E}_1$-algebras in the ∞-categories of modules over $\mathsf{H}\mathbb{Z}_{(2)}$ and $\mathsf{H}\mathbb{Q}$ respectively. In particular, we have also seen that $\pi_*(r_{\mathbb{R}}\mathsf{ko}) \cong \mathbb{Z}[x]$ is a polynomial ring with x a generator in degree 4. On our way, we computed the real realizations of several other interesting motivic spectra, such as the Eilenberg-Mac Lane spectra $\mathsf{H}\mathbb{Z}$ and $\mathsf{H}\mathbb{Z}/2$ representing motivic cohomology, the very effective cover of algebraic K-theory kgl, and also the motivic Thom spectra MGL, MSL, MSp representing different variants of algebraic cobordism. The latter realize, as $\mathcal{E}_\infty$-rings, to the classical Thom spectra MO, MSO and MU respectively, representing various types of cobordism in the topological setting. This was not the initial goal of this project, but the motivic spectrum MSL naturally entered the proof, and it led us to compare the topological and motivic multiplicative Thom spectrum functors, showing that they correspond to one another under real realization.

Naturally, one can ask about the $\mathcal{E}_\infty$-structure on $r_{\mathbb{R}}\mathsf{ko}$, rather than only the $\mathcal{E}_1$-structure. The L-theory spectrum $\mathsf{L}(\mathbb{R})$, and its connective cover, are also $\mathcal{E}_\infty$-ring spectra. The question is then open to know whether $r_{\mathbb{R}}\mathsf{ko}$ agrees with $\mathsf{L}(\mathbb{R})_{\geq 0}$ as an $\mathcal{E}_\infty$-ring as well. More generally, one may wish to generalize the 2-local fracture square we have computed to a pullback square of $\mathcal{E}_\infty$-rings. It is not directly obvious how to proceed; for instance, it is unclear whether $\mathsf{H}\mathbb{Z}_{(2)}[t^4]$ and $\mathsf{H}\mathbb{Q}[t^4]$ admit an $\mathcal{E}_\infty$-structure since they were only defined as free $\mathcal{E}_1$-objects. And even in this case, there is no reason to expect them to be free as $\mathcal{E}_\infty$-objects, so producing $\mathcal{E}_\infty$-maps out of them (with target $(r_{\mathbb{R}}\mathsf{ko})_{(2)}$ or $(r_{\mathbb{R}}\mathsf{ko})_{\mathbb{Q}}$) might be difficult.

Perhaps considering p-local fracture squares for other primes $p \neq 2$ could be interesting. However, the prime 2 is special in motivic homotopy; we have seen that inverting it allows us to add η-localizations (for E a motivic spectrum, $r_{\mathbb{R}}(E[1/2]) \simeq r_{\mathbb{R}}(E[1/2, 1/\eta])$). This has helped us in identifying $r_{\mathbb{R}}\mathsf{ko}[1/2]$. The special role played by the prime 2 is witnessed by the existence of a splitting $\mathsf{SH}(k)[1/2] \simeq \mathsf{SH}(k)[1/2]^+ \times \mathsf{SH}(k)[1/2]^-$, where η acts as zero on the *plus* part and in an invertible way on the *minus* part. Under this splitting, the plus part of a motivic spectrum $E[1/2] \in \mathsf{SH}(k)[1/2]$ is its η-completion $E[1/2]^\wedge_\eta$, whereas its minus part is its localization away from η, namely $E[1/2, 1/\eta]$.

Another direction to explore is related to the alternative perspective on real realization we briefly mentioned in Remark 4.1. That is, there exists a realization functor from motivic spectra to genuine C_2-equivariant spectra, the idea being to think of this as the complex realization, with the C_2 action given by complex conjugation. The real realization can then be recovered by computing the *geometric* fixed points. It might be interesting to use this other method to identify the real realization of ko (at the various possible levels) and see how the difficulty of the computation and notions appearing in the proof compare to our approach.

A Appendix

A.1 Localization of spectra

For our discussion in Section 5, which uses p-local fracture squares, we need to talk about localization of spectra away from a prime p, at the prime (p), and also rationalization. These examples fit into the general notion of Bousfield localization of spectra: for example, for $E \in \mathsf{Sp}$, its localization at (p), denoted by $E_{(p)}$, is classically defined as the Bousfield localization of E with respect to the Moore spectrum $S\mathbb{Z}_{(p)}$. In this case it turns out that $E_{(p)} \simeq E \wedge S\mathbb{Z}_{(p)}$, and also it can be computed as a certain sequential colimit, as we will see below. However, we will only use these properties and equivalent descriptions rather than the definition a a Bousfield localization, therefore we may *define* these localizations by these equivalent descriptions.

Definition A.1. Let $E \in \mathsf{Sp}$ be a spectrum, p be a prime, and for all $n \geq 1$ let P_n be the set of the first n prime numbers.

(i) Let $E[1/p] = \mathsf{colim}(E \xrightarrow{p} E \xrightarrow{p} \dots)$ be the *localization of E away from p* (also called *E with p inverted*).

(ii) Let $E_{(p)} = \mathsf{colim}_n(E \to E[1/p_1] \to E[1/p_1][1/p_2] \to \dots)$, where $\{p_1, p_2, \dots\}$ is the set of primes different from p, be the *localization of E at (p)*.

(iii) Let $E_{\mathbb{Q}} = \mathsf{colim}_n(E \to E[1/p_1] \to E[1/p_1][1/p_2] \to \dots)$, where $\{p_1, p_2, \dots\}$ is the set of primes, be the *rationalization of E*.

The following properties are consequences of the definitions:

Proposition A.2. *Let $E \in \mathsf{Sp}_{\geq 0}$ and p be a prime.*

(i) We have $E[1/p] \simeq E \wedge \mathbb{S}[1/p]$, and $\pi_*(E[1/p]) \cong \pi_*(E)[1/p]$. The functor $E \to E[1/p]$ is symmetric monoidal.

(ii) We have $E_{(p)} \simeq E \wedge \mathbb{S}_{(p)}$, and $\pi_*(E_{(p)}) \cong \pi_*(E)_{(p)}$. The functor $E \to E_{(p)}$ is symmetric monoidal.

(iii) We have $E_{\mathbb{Q}} \simeq (E[1/p])_{(p)} \simeq (E_{(p)})[1/p] \simeq E \wedge \mathbb{S}_{\mathbb{Q}}$, and $\pi_*(E_{\mathbb{Q}}) \cong \pi_*(E) \otimes_Z \mathbb{Q}$. The functor $E \to E_{\mathbb{Q}}$ is symmetric monoidal.

Proof. In item (i), the first claim follows from the fact that smash product of spectra commute with colimits in each variable (and the definition of the multiplication by p map). The second claim uses that homotopy groups of spectra commute with sequential colimits. The last claim follows from the fact that $\mathbb{S}[1/p]$ is idempotent in Sp. Indeed, since smash product commute with colimits, we have

$$\mathbb{S}[1/p] \wedge \mathbb{S}[1/p] \simeq \mathsf{colim}((\mathbb{S} \wedge \mathbb{S}[1/p]) \xrightarrow{p} (\mathbb{S} \wedge \mathbb{S}[1/p]) \xrightarrow{p} \dots) \simeq \mathsf{colim}(\mathbb{S}[1/p] \xrightarrow{p} \mathbb{S}[1/p] \xrightarrow{p} \dots)$$

but multiplication by p is an equivalence on $\mathbb{S}[1/p]$, because using the description as a colimit it is immediate to construct an inverse for it.

The proof of items (ii) and (iii) is similar, using the fact that colimits commute with colimits. $\square$

Remark A.3. Definition A.1 recovers the usual notions of Bousfield localization with respect to the spectra $S\mathbb{Z}[1/p]$, $S\mathbb{Z}_{(p)}$, and $S\mathbb{Q} = \mathsf{H}\mathbb{Q}$ respectively. These are in particular smashing localizations [Bou79, Prop. 2.4]. Indeed, following the discussion in [Hoy], defining a Moore spectrum for an abelian group A as some spectrum E with the property that $E \wedge \mathsf{H}\mathbb{Z} = \mathsf{H}\mathsf{A}$ (which makes sense in the context of localization of *connective* spectra), then if $A = \mathsf{colim}_{i \in I} A_i$ is described as a filtered colimit of abelian groups, we obtain that a colimit of Moore spectra for the A_i's, say $\mathsf{colim}_{i \in I} S(A_i)$, is also a Moore spectrum for A. Indeed, $(\mathsf{colim}_{i \in I} S(A_i)) \wedge \mathsf{H}\mathbb{Z} \simeq \mathsf{colim}_{i \in I}(S(A_i) \wedge \mathsf{H}\mathbb{Z}) \simeq \mathsf{colim}_{i \in I} \mathsf{HA}_i \simeq \mathsf{HA}$ since the Eilenberg-Mac Lane spectrum functor preserves filtered colimits of abelian groups. Writing $\mathbb{Z}[1/p]$, $\mathbb{Z}_{(p)}$, and $\mathbb{Q}$ under the forms of colimits similar to the ones of Definition A.1 for E replaced by $\mathbb{Z}$, and noticing that $\mathbb{S}$ is with this definition a Moore spectrum for $\mathbb{Z}$, we get that $\mathbb{S}[1/p]$, $\mathbb{S}_{(p)}$ and $\mathbb{S}_{\mathbb{Q}}$ are Moore spectra for the aforementioned groups.

Remark A.4. In virtue of Remark 1.27, since all three functors from Definition A.1 are symmetric monoidal, they induce an endofunctor on $\mathsf{Alg}_{\mathcal{E}_n}(\mathsf{Sp}_{\geq 0})$. In particular, if E is an $\mathcal{E}_n$-ring spectra, $E[1/p]$, $E_{(p)}$ and $E_{\mathbb{Q}}$ can naturally be viewed as $\mathcal{E}_n$-ring spectra.

Certain subtleties arise in localization of $\mathcal{E}_\infty$-objects in general; see for example [Hoy20, Section 3.2]. However, in the cases we are interested in, the definitions above suffice for our purposes (localization with respect to multiplication by an integer on a stable ∞-category is usually well-behaved, see the aforementioned article).

The following statement was used in the proof of Proposition A.6:

Lemma A.5. *Let $p \in \mathbb{N}$ be an integer. A map $f : E \to F$ in Sp is an equivalence if and only if the maps $f_p : E[1/p] \to F[1/p]$ (induced between the localizations) and $f/p : E/p \to F/p$ (induced between the respective cofibers of the multiplication by p maps) are equivalences.*

Proof. The "only if" direction is immediate. For the other direction, it suffices to show that if $(f_p)_* : \pi_*(E)[1/p] \to \pi_*(E)[1/p]$ and $(f/p)_* : \pi_*(E/p) \to \pi_*(F/p)$ are isomorphisms in every degree, then the same holds for $f_* : \pi_*(E) \to \pi_*(F)$. This follows by diagram chasing in the following comparison between long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{k+1}(E) \arrow[r, "q_{k+1}"] \arrow[d, "f_{k+1}"] & \pi_{k+1}(E/p) \arrow[r, "\partial_k"] \arrow[d, "(f/p)_{k+1}"] & \pi_k(E) \arrow[r, "\cdot p"] \arrow[d, "f_{k+1}"] & \pi_k(E) \arrow[r, "q_k"] \arrow[d, "f_k"] & \pi_k(E/p) \arrow[r] \arrow[d, "(f/p)_k"] & \cdots \\
\cdots \arrow[r] & \pi_{k+1}(F) \arrow[r, "q'_{k+1}"] & \pi_{k+1}(F/p) \arrow[r, "\partial'_k"] & \pi_k(F) \arrow[r, "\cdot p"] & \pi_k(F) \arrow[r, "q'_k"] & \pi_k(F/p) \arrow[r] & \cdots .
\end{tikzcd}$$

We begin by showing surjectivity of f_m for all $m \in \mathbb{Z}$. Let $b \in \pi_m(F)$. Then by surjectivity of $(f_p)_*$, $\exists a \in \pi_m(E)$, $n, j \in \mathbb{N}$, $p^n(f_*(a) - p^j b) = 0$. Call $P_{n,j}$ the property

$$\text{“ } p^n(f_*(a) - p^j b) = 0 \implies b \in \mathsf{Im}(f_*) \text{ ”}.$$

- If $n = 0$ and $j = 0$, then $b = f_*(a)$ as desired, so $P_{0,0}$ holds.
- If $n = 0$ but $j \geq 1$, we have $f_*(a) = p^j b \in \mathsf{Im}(\cdot p) = \ker(q'_m)$. Thus $(f/p)_* q_m(a) = 0$, but $(f/p)_*$ is injective, so $q_m(a) = 0$ and $\exists a' \in \pi_m(E)$, $pa' = a$. Thus $p(f_* a' - p^{j-1} b) = 0$ and we have reduced to the case $n = 1$ and $j - 1$. Therefore $P_{1,j-1} \implies P_{0,j}$.
- If $n \geq 1$, we have $p^{n-1}(f_*(a) - p^j b) \in \mathsf{ker}(\cdot p)$. Then by exactness, there exists $u \in \pi_{m+1}(F/p)$ such that $\partial'_m(u) = p^{n-1}(f_*(a) - p^j b) = f_* \partial_m(\tilde{u})$ for some $\tilde{u}$ since $(f/p)_*$ is surjective and we have $f_*(p^{n-1}a - \partial_k \tilde{u}) - p^{n+j-1} b = 0$. So we are back to the case 0 and $n + j - 1$. Therefore $P_{0,\ell} \implies P_{n,\ell+1-n}$ for all $1 \leq n \leq \ell + 1$. In particular, for $n = 1$, using the previous item, also $P_{0,\ell+1}$ holds.
- This sequence of implications on the properties P suffices to conclude ($P_{0,0}$ implies the property indexed by pairs of integers summing to 1, in particular it implies $P_{0,1}$, which then implies the property for all pairs of integers summing to 2, and so on).

We then show injectivity. If $x \in \pi_m(E)$ satisfies $f_*(x) = 0$, then by injectivity of $(f_p)_*$ we have $p^n x = 0$ for some $n \in \mathbb{N}$. If $n = 0$, $x = 0$ as desired, and otherwise we proceed by induction. Assume that $p^n y = 0$ and $f_*(y) = 0$ implies $y = 0$ for any $y \in \pi_m(E)$. Then if $p^{n+1} x = 0$, we have $p^n x \in \mathsf{ker}(\cdot p)$, so $\exists u \in \pi_{m+1}(E/p), p^n x = \partial_m(u)$. Then $f_* \partial_m u = 0 = \partial'_m((f/p)_* u)$, so $\exists v \in \pi_{m+1} F, q'_{m+1}(v) = (f/p)_* u$, and by surjectivity of f_* (as we just showed), there exists w with $v = f_*(w)$. But then $(f/p)_*(u) = q'_{m+1} f_* w = (f/p)_* q_{m+1} w$, and by injectivity $u = q_{m+1} w$. Finally, $p^n x = \partial_m q_{m+1} w = 0$, and we conclude by our inductive hypothesis. ☐

Localizations allows us to write p-local fracture squares; which means that any spectrum, or $\mathcal{E}_n$-ring, can be written as a pullback of its localizations:

Proposition A.6 (p-local fracture squares for $\mathcal{E}_n$-rings)**.** *Let $1 \leq n \leq \infty$, let $M \in \mathsf{Alg}_{\mathcal{E}_n}(\mathsf{Sp})$ be an $\mathcal{E}_n$-ring, and let $p \in \mathbb{N}$ be a prime. Then there is a Cartesian square of $\mathcal{E}_n$-rings*

$$\begin{tikzcd}
M \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & M[1/p] \arrow[d] \\
M_{(p)} \arrow[r] & M_{\mathbb{Q}}
\end{tikzcd}$$

where all maps are the canonical localization maps.

Proof. This statement in the case of spectra appears as [Bou90, Prop. 2.10]. Here is a proof in the case of $\mathcal{E}_n$-rings. Let M' be the pullback of the cospan $M_{(p)} \longrightarrow M_{\mathbb{Q}} \longleftarrow M[1/p]$ formed by the localization maps. We want to show that the natural map $M \to M'$ is an equivalence, which can be checked at the level of spectra. The forgetful functor from $\mathcal{E}_n$-rings to spectra being a right adjoint (Proposition 1.32), it preserves pullbacks. In Sp, to check that a map $f : E \to F$ is an equivalence, it suffices to check that the maps $f[1/p] : E[1/p] \to F[1/p]$ and $f/p : E/p \to F/p$ (induced between the cofibers of the multiplication by p map) are equivalences (Lemma A.5). Since these two constructions are described by colimits, they commute with pushouts, or equivalently, pullbacks in Sp. After inverting p, respectively moding out by p, we therefore obtain pullback squares

$$\begin{tikzcd} M'[1/p] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & M[1/p] \arrow[d] \\ M_{\mathbb{Q}} \arrow[r] & M_{\mathbb{Q}} \end{tikzcd} \qquad \begin{tikzcd} M'/p \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & 0 \arrow[d] \\ M/p \arrow[r] & 0. \end{tikzcd}$$

Then, $M \to M'$ induces equivalences $M[1/p] \to M'[1/p]$ and $M_{(p)} \to M'_{(p)}$, so $M \to M'$ is an equivalence, as desired. $\square$

A.2 Day convolution on ∞-categories of presheaves

The ∞-category of functors between two symmetric monoidal ∞-categories can be endowed with a symmetric monoidal structure, called Day convolution.

Theorem A.7 ([Gla16, Prop. 2.11] and [Lur17, Ex. 2.2.6.9])**.** *Let $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ be symmetric monoidal ∞-categories. Assume that $\mathcal{D}$ has all small colimits and that the tensor product on $\mathcal{D}$ preserves them in each variable separately. Then the ∞-category $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ admits a symmetric monoidal structure $\mathsf{Fun}(\mathcal{C}, \mathcal{D})^{\otimes}$, called* Day convolution. *It has in particular the property that there is an equivalence of ∞-categories*

$$\mathsf{CAlg}(\mathsf{Fun}(\mathcal{C}, \mathcal{D})^{\otimes}) \simeq \mathsf{Fun}^{\mathsf{lax}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}).$$

For any symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$, in the particular case $\mathcal{D}^{\otimes} = \mathsf{Spc}^{\times}$, Day convolution endows the category of presheaves $\mathcal{P}(\mathcal{C})$ with a symmetric monoidal structure (because the opposite of $\mathcal{C}$ has an induced symmetric monoidal structure, see [Lur17, Rmk 2.4.2.7]). This symmetric monoidal structure has the following universal property:

Proposition A.8 ([Lur17, Cor. 4.8.1.12 and Rmk 4.8.1.13])**.** *Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. Then Day convolution is the essentially unique symmetric monoidal structure on the ∞-category $\mathcal{P}(\mathcal{C})$ such that:*

1. *the tensor product preserves colimits in both variables,*
2. *and the Yoneda embedding $y : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ can be extended into a symmetric monoidal functor $y^{\otimes}$.*

More generally, Day convolution structures on ∞-categories of presheaves have the following universal property:

Proposition A.9. *Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category, and consider $\mathcal{P}(\mathcal{C})^{\otimes}$ with the Day convolution symmetric monoidal structure. Let $\mathcal{D}^{\otimes} \in \mathsf{CAlg}(\mathsf{Pr}^{\mathsf{L}})$.*

Then, the symmetric monoidal Yoneda embedding $y^{\otimes} : \mathcal{C}^{\otimes} \to \mathcal{P}(\mathcal{C})^{\otimes}$ induces an equivalence

$$\mathsf{Fun}^{\mathsf{L},\mathsf{lax}}(\mathcal{P}(\mathcal{C})^{\otimes}, \mathcal{D}^{\otimes}) \xrightarrow[-\circ y^{\otimes}]{\simeq} \mathsf{Fun}^{\mathsf{lax}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}),$$

where $\mathsf{Fun}^{\mathsf{L},\mathsf{lax}}$ denotes colimit-preserving lax symmetric monoidal functors, and $\mathsf{Fun}^{\mathsf{lax}}$ all lax symmetric monoidal functors.

Definition A.10. In the setting of Proposition A.9, let $\mathsf{LKE}^{\otimes}$ be an inverse for the equivalence $-\circ y^{\otimes}$. Given $F \in \mathsf{Fun}^{\mathsf{lax}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$, we say that the lax symmetric monoidal functor $\mathsf{LKE}^{\otimes}(F) : \mathcal{P}(\mathcal{C})^{\otimes} \to \mathcal{D}^{\otimes}$ is *left Kan extended as a lax symmetric monoidal functor* from F.

Proof of Proposition A.9. Day convolution makes $\mathsf{Fun}(\mathcal{P}(\mathcal{C}),\mathcal{D})$, respectively $\mathsf{Fun}(\mathcal{C},\mathcal{D})$, into symmetric monoidal ∞-categories $\mathsf{Fun}(\mathcal{P}(\mathcal{C}),\mathcal{D})^{\otimes}$, respectively $\mathsf{Fun}(\mathcal{C},\mathcal{D})^{\otimes}$. We first claim that the Yoneda embedding induces a symmetric monoidal equivalence between $\mathsf{Fun}(\mathcal{C},\mathcal{D})^{\otimes}$ and the symmetric monoidal subcategory $\mathsf{Fun}^{\mathsf{L}}(\mathcal{P}(\mathcal{C}),\mathcal{D})^{\otimes} \subseteq \mathsf{Fun}(\mathcal{P}(\mathcal{C}),\mathcal{D})^{\otimes}$ of colimit preserving functors. Indeed, by [Nik16, Cor. 3.8], precomposition by a lax symmetric monoidal functor induces a lax symmetric monoidal functor with respect to the Day convolution on the functor categories. In our case, this means that precomposition with the symmetric monoidal Yoneda embedding induces a lax symmetric monoidal functor $\mathsf{Fun}(\mathcal{P}(\mathcal{C}),\mathcal{D})^{\otimes} \to \mathsf{Fun}(\mathcal{C},\mathcal{D})^{\otimes}$. The usual universal property of the presheaf category (free cocompletion) implies that the underlying functor restricts to an equivalence $\mathsf{Fun}^{\mathsf{L}}(\mathcal{P}(\mathcal{C}),\mathcal{D}) \to \mathsf{Fun}(\mathcal{C},\mathcal{D})$. We thus get the desired lax symmetric monoidal (and thus symmetric monoidal) equivalence.

Therefore, precomposition by the Yoneda embedding induces an equivalence

$$\mathsf{CAlg}(\mathsf{Fun}^{\mathsf{L}}(\mathcal{P}(\mathcal{C}),\mathcal{D})^{\otimes}) \xrightarrow{\simeq} \mathsf{CAlg}(\mathsf{Fun}(\mathcal{C},\mathcal{D})^{\otimes}).$$

By Theorem A.7, the left and right hand sides are respectively equivalent to $\mathsf{Fun}^{\mathsf{L},\mathsf{lax}}(\mathcal{P}(\mathcal{C})^{\otimes},\mathcal{D}^{\otimes})$ and $\mathsf{Fun}^{\mathsf{lax}}(\mathcal{C}^{\otimes},\mathcal{D}^{\otimes})$. This concludes the proof. $\square$

Proposition A.11. *The equivalence of Proposition A.9 restricts to an equivalence:*

$$\mathsf{Fun}^{\mathsf{L},\otimes}(\mathcal{P}(\mathcal{C})^{\otimes},\mathcal{D}^{\otimes}) \xrightarrow[-\circ y^{\otimes}]{\simeq} \mathsf{Fun}^{\otimes}(\mathcal{C}^{\otimes},\mathcal{D}^{\otimes})$$

between subcategories of (strongly) symmetric monoidal functors.

Proof. Since $y^{\otimes}$ is strongly symmetric monoidal, the functor $-\circ y^{\otimes}$ preserves strong monoidality. On the other hand, if $F \in \mathsf{Fun}^{\mathsf{lax}}(\mathcal{C}^{\otimes},\mathcal{D}^{\otimes})$ is actually strongly symmetric monoidal (which, once we have specified a lax monoidal structure, is a property and does not require additional data), let $H = \mathsf{LKE}^{\otimes}(F)$. Then, we have to show that H is strongly symmetric monoidal, i.e. that for any $\mathcal{F},\mathcal{G} \in \mathcal{P}(\mathcal{C})$, the map

$$H(\mathcal{F}) \otimes_{\mathcal{D}} H(\mathcal{G}) \longrightarrow H(\mathcal{F} \otimes_{\mathcal{P}(\mathcal{C})} \mathcal{G})$$

(coming from the lax symmetric monoidal structure on H) is an equivalence. By assumption, this holds for representable presheaves $\mathcal{F}$ and $\mathcal{G}$. Moreover, the class of presheaves $\mathcal{F}$ and $\mathcal{G}$ for which this holds is closed under colimits in both $\mathcal{F}$ and $\mathcal{G}$. Indeed, H preserves colimits by construction, and tensor products in both $\mathcal{D}$ and $\mathcal{P}(\mathcal{C})$ preserve colimits in each variable (by Proposition A.8). Since every presheaf is a colimit of representable ones, this concludes the proof. $\square$

A.3 Slice ∞-categories

For $\mathcal{C}$ an ∞-category and $X \in \mathcal{C}$, Lurie defines in [Lur09, Prop. 1.2.9.2] an ∞-category $\mathcal{C}_{/X}$, called the *slice of* $\mathcal{C}$ *over* X, or an *overcategory*. This can be thought of as the category of edges in $\mathcal{C}$ with target X, and morphisms over X. It comes with a forgetful functor $\mathcal{C}_{/X} \to \mathcal{C}$, remembering the source of an edge. Slice ∞-categories are an essential ingredient of our discussion in Section 6. Here are some results about them that we have used.

Definition A.12. Let $\mathcal{C}$ be an ∞-category, $X \in \mathcal{C}$ and $\mathcal{D} \subseteq \mathcal{C}$ a subcategory. Then the ∞-category $\mathcal{D}_{/X}$ is defined as the fiber of the forgetful functor $\mathcal{C}_{/X} \to \mathcal{C}$ over $\mathcal{D}$.

Proposition A.13 (see for example [ABG18, §5.3])**.** *Let* $\mathcal{C}$ *be an* ∞*-category, and* $\mathcal{F} \in \mathcal{P}(\mathcal{C})$*. Then there is an equivalence of* ∞*-categories*

$$\mathcal{P}(\mathcal{C})_{/\mathcal{F}} \simeq \mathcal{P}(\mathcal{C}_{/\mathcal{F}})$$

where $\mathcal{C}_{/\mathcal{F}}$ *is as in Definition A.12, with* $\mathcal{C}$ *viewed as a subcategory of* $\mathcal{P}(\mathcal{C})$ *via the Yoneda embedding.*

Slices of symmetric monoidal ∞-categories over commutative algebra objects inherit a symmetric monoidal structure:

Proposition A.14 ([Lur17, Thm 2.2.2.4 and Rmk 2.2.2.5])**.** *Let* $\mathcal{D}^{\otimes}$ *be a symmetric monoidal* ∞*-category and* $X \in \mathsf{CAlg}(\mathcal{D})$*. Then the slice* $\mathcal{D}_{/X}$ *admits a symmetric monoidal structure* $\mathcal{D}^{\otimes}_{/X}$*, making in particular the projection* $\mathcal{D}^{\otimes}_{/X} \to \mathcal{D}^{\otimes}$ *symmetric monoidal. Informally, the tensor product of two objects* $c \to X$ *and* $d \to X$ *in the slice can be described by*

$$(c \to X) \otimes (d \to X) = (c \otimes d \to X \otimes X \to X)$$

where $X \otimes X \to X$ *is given by the multiplicative structure of* X*.*

Lemma A.15 ([ABG18, §6.2])**.** *In the situation of Proposition A.13, if $\mathcal{C} = *$ (so that $\mathcal{P}(\mathcal{C}) \simeq \mathsf{Spc}$) and $X \in \mathsf{CMon}(\mathsf{Spc})$, there is an equivalence of symmetric monoidal ∞-categories*

$$\mathsf{Spc}_{/X} \simeq \mathcal{P}(X),$$

where $\mathsf{Spc}_{/X}$ is endowed with the symmetric monoidal structure from Proposition A.14, and $\mathcal{P}(X)$ is the ∞-category of presheaves on X with the Day convolution symmetric monoidal structure (see Subsection A.2), viewing X as a symmetric monoidal ∞-category.

The symmetric monoidal structure on the slice ∞-category has a very convenient universal property:

Proposition A.16 ([ACB19, Lemma 2.12])**.** *In the situation of Proposition A.14, let $q : \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ be a symmetric monoidal ∞-category and $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ a lax symmetric monoidal functor. Then:*

- *Lax symmetric monoidal functors $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}_{/X}$ lifting F correspond to symmetric monoidal natural transformations $F \to X \circ q$, where X is viewed as a section of $p : \mathcal{D}^{\otimes} \to \mathsf{Fin}_*$. More precisely, there is an equivalence*

$$\mathsf{map}_{\mathsf{Fun}^{\mathsf{lax}}(\mathcal{C}^{\otimes},\mathcal{D}^{\otimes})}(F, X \circ q) \simeq \mathsf{Fun}^{\mathsf{lax}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}_{/X}) \times_{\mathsf{Fun}^{\mathsf{lax}}(\mathcal{C}^{\otimes},\mathcal{D}^{\otimes})} \{F\}.$$

- *In particular, for $\mathcal{C}^{\otimes} = \mathcal{E}_n^{\otimes}$, viewing F as an $\mathcal{E}_n$-algebra $Y \in \mathsf{Alg}_{\mathcal{E}_n}(\mathcal{D}^{\otimes})$, then $\mathcal{E}_n$-algebras in $\mathcal{D}^{\otimes}_{/X}$ lifting $Y \in \mathsf{Alg}_{\mathcal{E}_n}(\mathcal{D}^{\otimes})$ correspond to morphisms of $\mathcal{E}_n$-algebras $Y \to X$ in $\mathcal{D}^{\otimes}$. More precisely, there is an equivalence*

$$\mathsf{map}_{\mathsf{Alg}_{\mathcal{E}_n}(\mathcal{D}^{\otimes})}(Y, X) \simeq \mathsf{Alg}_{\mathcal{E}_n}(\mathcal{D}^{\otimes}_{/X}) \times_{\mathsf{Alg}_{\mathcal{E}_n}(\mathcal{D}^{\otimes})} \{Y\}.$$

Remark A.17. Both Propositions A.14 and A.16 are proven in a more general setting in the references we gave; namely for $\mathcal{O}^{\otimes}$-monoidal ∞-categories, for some ∞-operad $\mathcal{O}^{\otimes}$. We will only need the case $\mathcal{O}^{\otimes} = \mathsf{Fin}_*$.

Index of notation

$\mathbb{A}^1$	Affine line over some base scheme	
$\mathsf{Alg}(\mathcal{C}^\otimes)$, $\mathsf{CAlg}(\mathcal{C}^\otimes)$	∞-categories of $\mathcal{E}_1$-, resp. $\mathcal{E}_\infty$-algebras in a symmetric monoidal ∞-category $\mathcal{C}^\otimes$	1.11
$\mathsf{Alg}_{\mathcal{O}^\otimes}(\mathcal{C}^\otimes)$	∞-category of algebras over an ∞-operad $\mathcal{O}^\otimes$, in $\mathcal{C}^\otimes$ symmetric monoidal	1.22
$\mathcal{C}^\simeq$	Maximal ∞-groupoid in an ∞-category $\mathcal{C}$ (right adjoint to the forgetful functor)	3.2
Δ	1-category of finite non-empty linearly ordered sets and monotone functions	§1.1
η	Motivic Hopf map	3.17
$\mathcal{E}_n$	∞-operad of little n-cubes	1.24
f_n, $\tilde{f}_n$	n-effective, respectively very n-effective cover	2.25
Fin	1-category of finite sets (including $\emptyset$)	1.7
$\mathsf{Fun}^\times(\mathcal{C},\mathcal{D})$	∞-category of functors $\mathcal{C}\to\mathcal{D}$ preserving finite products	6.20
$\mathsf{Fun}^\mathsf{L}(\mathcal{C},\mathcal{D})$	∞-category of colimit-preserving functors $\mathcal{C}\to\mathcal{D}$	1.19
$\mathsf{Fun}^\otimes(\mathcal{C}^\otimes,\mathcal{D}^\otimes)$	∞-category of symmetric monoidal functors $\mathcal{C}^\otimes\to\mathcal{D}^\otimes$	1.10
$\mathsf{Fun}^\mathsf{lax}(\mathcal{C}^\otimes,\mathcal{D}^\otimes)$	∞-category of lax symmetric monoidal functors $\mathcal{C}^\otimes\to\mathcal{D}^\otimes$	1.10
$\mathbb{G}_m$	Group scheme $S\times_{\mathsf{Spec}(\mathbb{Z})}\mathsf{Spec}(\mathbb{Z}[t,t^{-1}])$ over a base scheme S	
$h\mathcal{C}$	1-category given by the homotopy category of an ∞-category $\mathcal{C}$	
$\mathsf{HA}[t^n]$	Free $\mathcal{E}_1$-HA-algebra on one generator in degree n	1.37
$\mathsf{H}\mathbb{Z}, \mathsf{H}\mathbb{Z}/2$	Motivic Eilenberg-Mac Lane spectra (represent motivic cohomology)	4.15
$\widetilde{\mathsf{H}\mathbb{Z}}$	Effective cover of the Milnor-Witt K-theory sheaf: $f_0\underline{K}_*^{MW}$	4.27
$\underline{K}_*^{MW}$, $\underline{K}_*^{M}$, $\underline{K}_*^{W}$	Milnor-Witt, Milnor, and Witt K-theory sheaves	3.19
$\mathsf{KGL}, \mathsf{kgl}$	Algebraic K-theory motivic spectrum and its very effective cover	3.14
KO, ko	Hermitian K-theory motivic spectrum and its very effective cover	3.14
$\mathsf{KO}^\mathsf{top}, \mathsf{ko}^\mathsf{top}$	Real topological K-theory spectrum and its connective cover	3.3
KU	Complex topological K-theory spectrum	3.3
KW, kw	Balmer-Witt K-theory motivic spectrum and its very effective cover	4.28
$\mathsf{L}(-)$	L-theory spectrum functor	3.21
L_mot	Motivic localization functor	2.6
$\mathsf{map}_\mathcal{C}$	Mapping space in an ∞-category $\mathcal{C}$	
$\mathsf{MGL}, \mathsf{MSL}, \mathsf{MSp}$	General linear, special linear, and symplectic cobordism motivic Thom spectra	6.19
$\mathsf{MO}, \mathsf{MSO}, \mathsf{MU}$	Orthogonal, special orthogonal, and complex cobordism Thom spectra	6.4
M_top	Topological Thom spectrum functor	6.3
$\langle n\rangle$, $\langle n\rangle^\circ$	Finite pointed set $\{*,1,\ldots,n\}$ in Fin_*, resp. finite set $\{1,\ldots,n\}$ in Fin	1.7
$\Omega^\infty, \Omega^\infty_{\mathcal{S}^1}, \Omega^\infty_T$	Infinite loop space functors for classical, resp. motivic $\mathcal{S}^1$- and $\mathbb{P}^1$-spectra	§2.2
$\underline{\pi}_*(-)_*$, $\underline{\pi}_{*,*}(-)$	Homotopy sheaves of a motivic space or spectrum	2.17
$\mathcal{P}(\mathcal{C})$	∞-category of presheaves of spaces on an ∞-category $\mathcal{C}$	2.1
$\mathcal{P}_\Sigma(\mathcal{C})$	∞-category of functors $\mathcal{C}^\mathsf{op}\to\mathsf{Spc}$ preserving finite products	6.11
Pr^L	∞-category of presentable ∞-categories and left-adjoint functors	1.15
ρ	Inclusion $\mathcal{S}^0\to\mathbb{G}_m$ on the points corresponding to 1 and -1	§4.2
$r_\mathbb{R}$, $r_\mathbb{C}$	Real and complex Betti realization functors	§4.1
Spc	∞-category of spaces	
$\mathcal{S}$	Motivic sphere spectrum	
$\mathcal{S}^n$	(Simplicial) n-sphere in the ∞-category of motivic spaces or motivic spectra	
$\mathcal{S}^{p,q}$	Motivic bigraded spheres, $\mathcal{S}^{p,q}=\mathcal{S}^{p-q}\wedge\mathbb{G}_m^{\wedge q}$	
s_n, $\tilde{s}_n$	n-effective, respectively very n-effective slice	2.25
$\mathbb{S}$	Topological sphere spectrum	
$\mathbb{S}^n$	n-sphere in the ∞-category of spaces or spectra	
$\mathsf{SH}(S)$	∞-category of motivic spectra over S	2.15
$\mathsf{SH}(S)^\mathsf{eff}$, $\mathsf{SH}(S)^\mathsf{veff}$	Subcategories of effective, respectively very effective spectra over S	2.21
$\mathsf{SH}(S)^\mathsf{eff}(n)$, $\mathsf{SH}(S)^\mathsf{veff}(n)$	Subcategories of n-effective, respectively very n-effective spectra over S	2.23
$\Sigma^\infty, \Sigma^\infty_{\mathcal{S}^1}, \Sigma^\infty_T$	Infinite suspension functors for classical, resp. motivic $\mathcal{S}^1$- and $\mathbb{P}^1$-spectra	§2.2
Sm_S	1-category of smooth schemes of finite type over the base scheme S	2.1
Sp	∞-category of (topological) spectra	
$\mathsf{Spc}(S)$	∞-category of motivic spaces over the base scheme S	2.6
T	Pointed motivic space or spectrum corresponding to $(\mathbb{P}^1,\infty)$	
$\tau_{\geq n}, \tau_{\leq n}$	Truncation functors for a t-structure	
$\mathcal{W}_\mathsf{mot}$	Generating family for the localization L_mot	2.6
$y(-)$	Yoneda embedding	
$(-)_+$	Disjoint base point functor (free functor from unpointed to pointed objects)	

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