

ADVANCED TOPICS IN MICROECONOMETRICS: MATCHING MODELS AND THEIR APPLICATIONS

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Lecture 9. Demand models

- ▶ Beyond GEV: the pure characteristics models, the random coefficient logit model, the probit model
- ▶ Simulation methods: AR, GHK, and SARS
- ▶ The inversion theorem

- ▶ [OTME], Ch. 9.2
- ▶ McFadden (1981). “Econometric Models of Probabilistic Choice,” in C.F. Manski and D. McFadden (eds.), *Structural analysis of discrete data with econometric applications*, MIT Press.
- ▶ Berry, Levinsohn, and Pakes (1995). “Automobile Prices in Market Equilibrium,” *Econometrica*.
- ▶ Berry and Pakes (2007). “The pure characteristics demand model”. *International Economic Review*
- ▶ Train (2009). *Discrete Choice Methods with Simulation*. 2nd Edition. Cambridge University Press.
- ▶ G, Salanié (2017). “Cupids invisible hand”. Working paper.
- ▶ Bonnet, G and Shum (2017). “Yogurts choose consumers? Identification of Random Utility Models via Two-Sided Matching”. Working paper.

Section 1

THEORY

- ▶ The GEV models are convenient analytically, but not very flexible.
 - ▶ The logit model imposes zero correlation across alternatives
 - ▶ The nested logit allows for nonzero correlation, but in a very rigid way (needs to define nests).
- ▶ A good example is the probit model, where ε is a Gaussian vector. For this model, there is no close-form solution neither for G nor for G^* .
- ▶ More recently, a number of modern models don't have closed-form either. These models require simulation methods in order to approximate them by discrete models.

- ▶ The pure characteristics model (Berry and Pakes, 2007) can be motivated as follows. Assume y stands for the number of bedrooms. The logit model would assume that the random utility associated with a 2-BR is uncorrelated with a 3-BR, which is not realistic.
- ▶ Let ζ_y is the typical size of a bedroom of size y , one may introduce ϵ as the valuation of size; in which case the utility shock associated with y should be $\varepsilon_y = \epsilon \zeta_y$. More generally, the characteristics ζ_y is a d -dimensional (deterministic) vector, and $\epsilon \sim \mathbf{P}_\epsilon$ is a (random) vector of the same size standing for the valuations of the respective dimensions, so that

$$\varepsilon_y = \epsilon^\top \zeta_y.$$

- ▶ For example, if each alternative y stands for a model of car, the first component of ζ_y may be the price of car y ; the other components may be other characteristics such as number of seats, fuel efficiency, size, etc. In that case, for a given dimension $y \in \mathcal{Y}_0$, ϵ_y is the (random) valuation of this dimension by the consumer with taste vector ϵ .

- ▶ Assume without loss of generality that $\varepsilon_y = 0$, that is $\xi_0 = 0$ as we can always reduce the setting to this case by replacing ξ_y by $\xi_y - \xi_0$.
- ▶ Letting Z be the $|\mathcal{Y}| \times d$ matrix of (y, k) -term ξ_y^k , this rewrites as

$$\varepsilon = Z\epsilon.$$

- ▶ Hence, we have

$$G(U) = \mathbb{E} [\max \{U + Z\epsilon, 0\}].$$

and

$$\sigma_y(U) = \Pr \left(U_y - U_z \geq (Z\epsilon)_y - (Z\epsilon)_z \quad \forall z \in \mathcal{Y}_0 \setminus \{y\} \right).$$

- When $d = 1$ (scalar characteristics), one has
 $\sigma_y(U) = \Pr(U_y - U_z \geq (\xi_y - \xi_z)\epsilon \ \forall z \in \mathcal{Y}_0 \setminus \{y\})$, and thus

$$\sigma_y(U) = \Pr\left(\max_{z:\xi_y > \xi_z} \left\{ \frac{U_y - U_z}{\xi_y - \xi_z} \right\} \leq \epsilon \leq \min_{z:\xi_y < \xi_z} \left\{ \frac{U_y - U_z}{\xi_y - \xi_z} \right\}\right)$$

with the understanding that $\max_{z \in \emptyset} f_z = -\infty$ and $\min_{z \in \emptyset} f_z = +\infty$.

- Therefore, letting \mathbf{F}_ϵ be the c.d.f. associated with the distribution of ϵ , one has a closed-form expression for σ_y :

$$\sigma_y(U) = \mathbf{F}_\epsilon\left(\left[\max_{z:\xi_y > \xi_z} \left\{ \frac{U_y - U_z}{\xi_y - \xi_z} \right\}, \min_{z:\xi_y < \xi_z} \left\{ \frac{U_y - U_z}{\xi_y - \xi_z} \right\}\right]\right)$$

- ▶ When \mathbf{P}_ϵ is the $\mathcal{N}(0, S)$ distribution, then the pure characteristics model is called a Probit model; in this case,

$$\varepsilon \sim \mathcal{N}(0, \Sigma) \text{ where } \Sigma = ZSZ^\top.$$

- ▶ Note the distribution ε will not have full support unless $d \geq |\mathcal{Y}|$ and Z is of full rank.
- ▶ Computing σ in the Probit model thus implies computing the mass assigned by the Gaussian distribution to rectangles of the type

$$[l_y, u_y].$$

When Σ is diagonal (random utility terms are i.i.d. across alternatives), this is numerically easy. However, this is computationally difficult in general (more on this later).

- The random coefficient logit model (Berry, Levinsohn and Pakes, 1995) may be viewed as an interpolant between the random characteristics model and the logit model. In this case,

$$\varepsilon = (1 - \lambda) Z\epsilon + \lambda\eta$$

where $\epsilon \sim \mathbf{P}_\epsilon$, η is an EV1 distribution independent from the previous term, and λ is a interpolation parameter ($\lambda = 1$ is the logit model, and $\lambda = 0$ is the pure characteristics model).

- In this case, one may compute the Emax operator as

$$\begin{aligned} G(U) &= \mathbb{E} \left[\max_{y \in \mathcal{Y}_0} \left\{ U_y + (1 - \lambda) (Z\epsilon)_y + \lambda\eta_y \right\} \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[\max_{y \in \mathcal{Y}_0} \left\{ U_y + (1 - \lambda) (Z\epsilon)_y + \lambda\eta_y \right\} \mid \epsilon \right] \right] \\ &= \mathbb{E} \left[\lambda \log \sum_{y \in \mathcal{Y}_0} \exp \left(\frac{U_y + (1 - \lambda) (Z\epsilon)_y}{\lambda} \right) \right] \end{aligned}$$

► Recall

$$G(U) = \mathbb{E} \left[\lambda \log \sum_{y \in \mathcal{Y}_0} \exp \left(\frac{U_y + (1 - \lambda)(Z\epsilon)_y}{\lambda} \right) \right].$$

- The demand map in the random coefficients logit model is obtained by derivation of the expression of the E_{\max} , i.e.

$$\sigma_y(U) = \mathbb{E} \left[\frac{\exp \left(\frac{U_y + (1 - \lambda)(Z\epsilon)_y}{\lambda} \right)}{\sum_{y' \in \mathcal{Y}_0} \exp \left(\frac{U_{y'} + (1 - \lambda)(Z\epsilon)_{y'}}{\lambda} \right)} \right].$$

- ▶ In a number of cases, one cannot compute the choice probabilities $\sigma(U)$ using a closed-form expression. In this case, we need to resort to simulation to compute G , G^* , σ and σ^{-1} .
- ▶ The idea is that:
 - ▶ one is able to compute G and G^* for discrete distributions (more on this later)
 - ▶ the sampled versions of G , G^* , σ and σ^{-1} converge to the populations objects when the sample size is large.

- One simulates N points $\varepsilon^i \sim P$. The Emax operator associated with the empirical sample distribution P_N is

$$G_N = N^{-1} \sum_{i=1}^N \max_{y \in \mathcal{Y}} \{U_y + \varepsilon_y^i\}$$

and the demand map is given by

$$\sigma_{N,y}(U) = N^{-1} \sum_{i=1}^N 1 \left\{ U_y + \varepsilon_y^i \geq U_z + \varepsilon_z^i \quad \forall z \in \mathcal{Y}_0 \right\}$$

- In the literature, σ_N is called the *accept-reject simulator*.

- ▶ GHK simulator (Geweke, Hajivassiliou and Keane) improves on the AR simulator.
- ▶ Without loss of generality, we shall focus on the market share of $y = 0$, and label the elements of \mathcal{Y} as $\mathcal{Y} = \{1, \dots, M\}$ where $M = |\mathcal{Y}|$. Then if \mathbf{F}_η is the c.d.f. of the random vector η valued in $\mathbb{R}^{\mathcal{Y}}$ defined by $\eta_y := \varepsilon_y - \varepsilon_0$ for $z \in \mathcal{Y}$, then

$$\mu_0 = 1 - \mathbf{1}'_{\mathcal{Y}} \nabla G(U) = \mathbf{F}_\eta(z),$$

where $z_y = -U_y$, for all $y \in \mathcal{Y}$.

- ▶ Note that $\mathbf{F}_\eta(z) = \Pr(\eta_1 \leq z_1, \dots, \eta_M \leq z_M)$ can be expressed as

$$\mu_0 = \prod_{j=1}^M \Pr(\eta_j \leq z_j | \eta_1 \leq z_1, \dots, \eta_{j-1} \leq z_{j-1}) \quad (1)$$

with the understanding that the first term in this product (associated with $j = 1$) is simply the unconditional probability distribution $\Pr(\eta_1 \leq z_1)$.

- The fundamental assumption behind the GHK method is that the Rosenblatt quantile associated with the distribution of η is known.
- Recall from the exercise discussed in the previous lecture that the Rosenblatt quantile is the map T such that $T\#\mu = P$, and such that the Jacobian DT of T is lower triangular with nonnegative diagonal. That is,

$$\begin{cases} \eta_1 = T_1(U_1) \\ \eta_2 = T_2(U_1, U_2) \\ \dots \\ \eta_M = T_M(U_1, U_2, \dots, U_M) \end{cases},$$

where $U \sim U([0, 1]^d)$, and $T_i(u)$ depends only on u_1, \dots, u_i and is a nondecreasing function of u_i . In order to evaluate quantity $\mathbf{F}_\eta(z)$ in (1), one needs to evaluate $\pi_j = \Pr(\eta_j \leq z_j | \eta_1 \leq z_1, \dots, \eta_{j-1} \leq z_{j-1})$, that is

$$\pi_j = \mathbb{E} [1 \{ T_j(U_1, U_2, \dots, U_j) \leq z_j \} | T_1(U_1) \leq z_1, \dots, T_{j-1}(U_1, U_2, \dots, U_{j-1}) \leq z_{j-1}] \quad (2)$$

- Denote $T_i^{-1}(z; u_1, \dots, u_{i-1})$ the inverse of $u_i \mapsto T_i^{-1}(u_1, \dots, u_{i-1}, u_i)$ for fixed values of u_1, \dots, u_{i-1} .
- Then, observe that if $\tilde{U} \sim U([0, 1]^d)$, and if

$$\begin{cases} \hat{U}_1 = \tilde{U}_1 T_1^{-1}(z_1) \\ \hat{U}_2 = \tilde{U}_2 T_2^{-1}(z_2; \hat{U}_1) \\ \dots \\ \hat{U}_M = \tilde{U}_M T_M^{-1}(z_M; \hat{U}_1, \dots, \hat{U}_{M-1}) \end{cases} . \quad (3)$$

- Then the conditional expectation

$$\pi_j = \mathbb{E} [1 \{ T_j(U_1, U_2, \dots, U_j) \leq z_j \} \mid T_1(U_1) \leq z_1, T_2 \leq \text{etc.}]$$

coincides with the unconditional expectation

$$\pi_j = \mathbb{E} \left[T_j^{-1}(z_j; \hat{U}_1, \hat{U}_2, \dots, \hat{U}_{j-1}) \right] . \quad (4)$$

- ▶ The GHK simulator can be interpreted as an importance sampling simulation procedure.
- ▶ Indeed, expression

$$\pi_j = \mathbb{E} [1 \{ T_j (U_1, U_2, \dots, U_j) \leq z_j \} \mid T_1 (U_1) \leq z_1, T_2 \leq \text{etc.}]$$

is a conditional expectation; one may compute it by accept-reject but this is computationally suboptimal, as it leads us to discard a fraction of draws – which can be a significant fraction.

- ▶ In contrast, in expression

$$\pi_j = \mathbb{E} \left[T_j^{-1} (z_j; \hat{U}_1, \hat{U}_2, \dots, \hat{U}_{j-1}) \right].$$

is an unconditional expectation; one shall compute it by drawing K i.i.d. draws of $\tilde{U} \sim U([0, 1]^d)$, computing the \hat{U} 's, and averaging over all the values of $T_j^{-1} (\hat{U}_1, \hat{U}_2, \dots, \hat{U}_{j-1}, z_j)$ simulated that way. In the second method, we have not discarded any draws, which is more efficient.

Algorithm.

For $k = 1, \dots, K$: Draw $\tilde{U}^k \sim \mathcal{U}([0, 1]^d)$. Compute $(\hat{U}_1^k, \dots, \hat{U}_M^k)$ from $(\tilde{U}_1^k, \dots, \tilde{U}_M^k)$ using transformation (3). Compute $\pi_j^K = K^{-1} \sum_{k=1}^K T_j^{-1} \left(z_j; \hat{U}_1^k, \hat{U}_2^k, \dots, \hat{U}_{j-1}^k \right)$ for $j = 1, \dots, M$. Return the GHK simulator

$$\mu_0 = \prod_{j=1}^M \pi_j^K.$$

Remark: The practical difficulty with the implementation of this algorithm is the knowledge of the Rosenblatt quantile in closed form. A leading example where this object is readily available is given by the case when \mathbf{P} is Gaussian, which is called the probit model.

- The Probit model is characterized by $\mathbf{P} = \mathcal{N}(0, \Sigma)$, with Σ is a $M \times M$ symmetric semidefinite positive matrix, hence $\text{cov}(\varepsilon_y, \varepsilon_{y'}) = \Sigma_{yy'}$. In this case, the Rosenblatt quantile is known. Let $\Sigma = LL^\top$ be the Choleski decomposition of Σ , where L is lower triangular with a positive diagonal. Then the Rosenblatt quantile T is such that

$$\begin{cases} T_1(u) = L_{11}\Phi^{-1}(u_1) \\ T_2(u) = L_{21}\Phi^{-1}(u_1) + L_{22}\Phi^{-1}(u_2) \\ \dots \\ T_M(u) = L_{M1}\Phi^{-1}(u_1) + \dots + L_{MM}\Phi^{-1}(u_M) \end{cases}$$

where Φ is the c.d.f. of the standard normal (univariate) distribution.

In this case, \hat{U} is obtained from $U \sim \mathcal{U}([0, 1]^M)$ by

$$\begin{cases} \hat{U}_1 = \tilde{U}_1\Phi\left(\frac{z_1}{L_{11}}\right) \\ \hat{U}_2 = \tilde{U}_2\Phi\left(\frac{z_2 - L_{21}\Phi^{-1}(\hat{U}_1)}{L_{22}}\right) \\ \dots \\ \hat{U}_M = \tilde{U}_M\Phi\left(\frac{z_M - L_{M1}\Phi^{-1}(\hat{U}_1) - \dots - L_{M(M-1)}\Phi^{-1}(\hat{U}_{M-1})}{L_{MM}}\right) \end{cases}$$

- ▶ McFadden's smoothed accept-reject simulator (SARS) consists in sampling $\varepsilon \sim P$: $\varepsilon^1, \dots, \varepsilon^N$, and replacing the max by the smooth-max

$$\sigma_{N,T,y}(U) = \sum_{i=1}^N \frac{1}{N} \frac{\exp((U_y + \varepsilon_y^i)/T)}{\sum_z \exp((U_z + \varepsilon_z^i)/T)}$$

- ▶ One seeks U so that the induced choice probabilities are s , that is

$$s_y = \sum_{i=1}^N \frac{1}{N} \frac{\exp((U_y + \varepsilon_y^i)/T)}{\sum_z \exp((U_z + \varepsilon_z^i)/T)}.$$

- ▶ The associated Emax operator is

$$G_{N,T}(U) = \mathbb{E}_{\mathbf{P}_N} \left[G_{\text{logit}}(U + \varepsilon^i) \right]$$

so the underlying random utility structure is a random coefficient logit.

THEOREM (G AND SALANIÉ)

Consider a solution $(u(\varepsilon), v_y)$ to the dual Monge-Kantorovich problem with cost $\Phi(\varepsilon, y) = \varepsilon_y$, that is:

$$\begin{aligned} \min_{u, v} \int u(\varepsilon) d\mathbf{P}(\varepsilon) + \sum_{y \in \mathcal{Y}_0} v_y s_y \\ \text{s.t. } u(\varepsilon) + v_y \geq \Phi(\varepsilon, y) \end{aligned} \tag{5}$$

Then:

- (i) $U = \sigma^{-1}(s)$ is given by $U_y = v_0 - v_y$.
- (ii) The value of Problem (5) is $-G^*(s)$.

PROOF.

$\sigma^{-1}(s) = \arg \max_{U: U_0=0} \{ \sum_{y \in \mathcal{Y}} s_y U_y - G(U) \}$, thus, letting $v = -U$, v is the solution of

$$\min_{v: v_0=0} \left\{ \sum_{y \in \mathcal{Y}_0} s_y v_y + G(-v) \right\}$$

which is exactly problem (5). □

- ▶ It follows from the inversion theorem that the problem of demand inversion in the pure characteristics model is a semi-discrete transport problem, a point made in Bonnet, G and Shum (2017).
- ▶ Indeed, the correspondence is:
 - ▶ an alternative y is a fountain
 - ▶ the characteristics of an alternative is a fountain location
 - ▶ the systematic utility associated with alternative y is minus the price of fountain y
 - ▶ the market share of alternative y coincides with the capacity of fountain y
 - ▶ the random vector ϵ is the location of an inhabitant

- Cf. Bonnet, G. and Shum (2017). Let $u_i = T \log \sum_z \exp((U_z + \varepsilon_z^i)/T)$. One has

$$\begin{cases} s_y = \sum_{i=1}^N \frac{1}{N} \exp((U_y - u_i + \varepsilon_y^i)/T) \\ \frac{1}{N} = \sum_y \frac{1}{N} \exp((U_y - u_i + \varepsilon_y^i)/T) \end{cases}.$$

- As a result, (u_i, U_y) are the solution of the regularized OT problem

$$\min_{u, U} \sum_{i=1}^N \frac{1}{N} u_i - \sum s_y U_y + \sum_{i,y} \frac{1}{N} \exp((U_y - u_i + \varepsilon_y^i)/T).$$

- Consider the IPFP algorithm for solving the latter problem:

$$\begin{cases} \exp(u_i^{k+1}/T) = \sum_z \exp((U_z^k + \varepsilon_z^i)/T) \\ \exp(U_y^{k+1}/T) = \frac{Ns_y}{\sum_{i=1}^N \exp((-u_i^{k+1} + \varepsilon_y^i)/T)} \end{cases}$$

- This rewrites as

$$\exp U_y^{k+1}/T = \frac{Ns_y}{\sum_{i=1}^N \frac{\exp(\varepsilon_y^i/T)}{\sum_z \exp((U_z^k + \varepsilon_z^i)/T)}}, \text{ i.e.}$$

$$U_y^{k+1} = T \log s_y - T \log \sum_{i=1}^N \frac{1}{N} \frac{\exp(\varepsilon_y^i/T)}{\sum_z \exp((U_z^k + \varepsilon_z^i)/T)}$$

which is exactly the contraction mapping algorithm of Berry, Levinsohn and Pakes (1995, appendix 1).

Section 2

CODING

- ▶ We shall code the AR simulator for the probit model and then invert it using the inversion theorem.

- ▶ Take a vector of systematic utilities:

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U_y = c(1.6, 3.2, 1.1,0)
```

- ▶ Simulate the market shares using the AR simulator:

```
epsilon_iy = matrix(rnorm(nbDraws*nbY),ncol=nbY) %*%  
SqrtCovar  
u_iy = t(t(epsilon_iy)+U_y)  
ui = apply(X = u_iy, MARGIN = 1, FUN = max)  
s_y = apply(X = u_iy - ui, MARGIN = 2,FUN = function(v)  
(length(which(v==0)))) / nbDraws
```

- To invert the market share, simply run the optimal assignment problem:
A1 =
kronecker(matrix(1,1,nbY),sparseMatrix(1:nbDraws,1:nbDraws))
A2 =
kronecker(sparseMatrix(1:nbY,1:nbY),matrix(1,1,nbDraws))
A = rbind2(A1,A2)
result = gurobi (
list(A=A,obj=c(epsilon_iy),modelsense="max",
rhs=c(rep(1/nbDraws,nbDraws),s_y) ,sense="="),
params=list(OutputFlag=0))
Uhat_y = - result\$pi[(1+nbDraws):(nbY+nbDraws)] +
result\$pi[(nbY+nbDraws)]