Research project (Sept. – Dec. 2020)
(Cole)

Computing optimal designs using Python and R

- D-optimal designs
- A-optimal designs
- C-optimal designs
- I-optimal designs
- other designs

Problem 1: Compute D-optimal design for quadratic regression model using Python.

- The problem is described in the next two pages.

- Write Python code to solve the problem. Detailed documentation is needed in the code.

Problem 1: Consider a linear regression model

$$y_i = \theta_0 + \theta_1 x_i + \theta_2 x_i^2 + \varepsilon_i, \qquad i = 1, 2, \cdots, n.$$

where $x_i \in [-1, 1]$,
$\varepsilon_i \sim$ i.i.d $N(0, \sigma^2)$.

- Choose $x_1, x_2, \cdots, x_n \in [-1, 1]$ such that the least squares estimator of $(\theta_0, \theta_1, \theta_2)$ is the "most efficient."

- optimal design problem

(i) Define $u_i = -1 + 2(i-1)/(N-1)$, $i = 1, 2, \cdots, N$, and N is given. For example $N = 21$.

(ii) For $i = 1, 2, \cdots, N$, let

$$A_i = \begin{pmatrix} 1 \\ u_i \\ u_i^2 \end{pmatrix} (1 \;\; u_i \;\; u_i^2).$$

Note that A_i are 3×3 matrices.

(iii) Let $W = (w_1, w_2, \cdots, w_N)$, where $w_i \geq 0$ for $i = 1, \cdots, N$, and $\sum_{i=1}^{N} w_i = 1$.

Here W is the unknown vector in the optimal design problem.

(iv) Define

$$D(W) = \sum_{i=1}^{N} w_i A_i, \quad \text{— } 3\times 3 \text{ positive definite matrix}.$$

(v) Convex optimization problem

$$\begin{cases} \min\limits_{W} -\log\left(\det\left(D(W)\right)\right) \\ \text{s.t.} \quad w_i \geq 0, \ i = 1, 2, \cdots, N, \\ \qquad \sum\limits_{i=1}^{N} w_i = 1. \end{cases}$$

(vi) Use CVx to solve the optimization problem to get W.

* Start with Python.
* The solution may look like this:

$$W = (0.3333, 0, 0, \cdots, 0, 0.3333, 0, \cdots, 0, 0.3333).$$