$$p(x_d \mid x_1, \dots, x_{d-1})$$ → density function (x_d)

$$= \frac{p(x_1, \dots, x_{d-1}, x_d)}{p(x_1, \dots, x_{d-1})}$$

$$= \frac{\prod_{i=1}^{d} f_i(x_i, x_{pa_i})}{p(x_1, \dots, x_{d-1})}$$

$$= \frac{f_d(x_d, x_{pa_d}) \prod_{i=1}^{d-1} f_i(x_i, x_{pa_i})}{p(x_1, \dots, x_{d-1})}$$

do not depend on x_d

$$\propto f_d(x_d, x_{pa_d})$$

→ density function (in x_d)

$$= f_d(x_d, x_{pa_d})$$

$$\Rightarrow p(x_d \mid x_1, \dots, x_{d-1}) = f(x_d, x_{pa_d})$$

$$= p(x_d \mid x_{pa_d})$$

Moreover :

$$p(x_1, \dots, x_{d-1}) = \prod_{i=1}^{d-1} f_i(x_i, x_{pa_i})$$

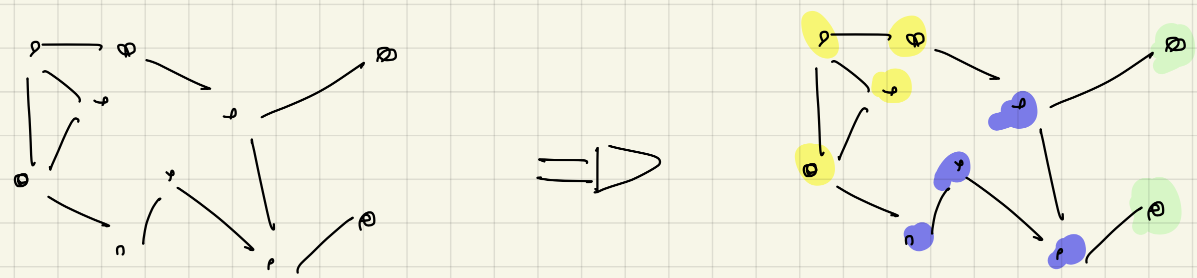
$\Big\downarrow$ by induction

$\dots$

Stochastic Block Model: (SBM)

Unsupervised learning. Clustering.

Dataset: a network $\Leftrightarrow$ a graph



(1) $Z_i \overset{iid}{\sim} \mathcal{M}(1,\pi), \forall i \in \{1, \dots, n\}$

number of nodes

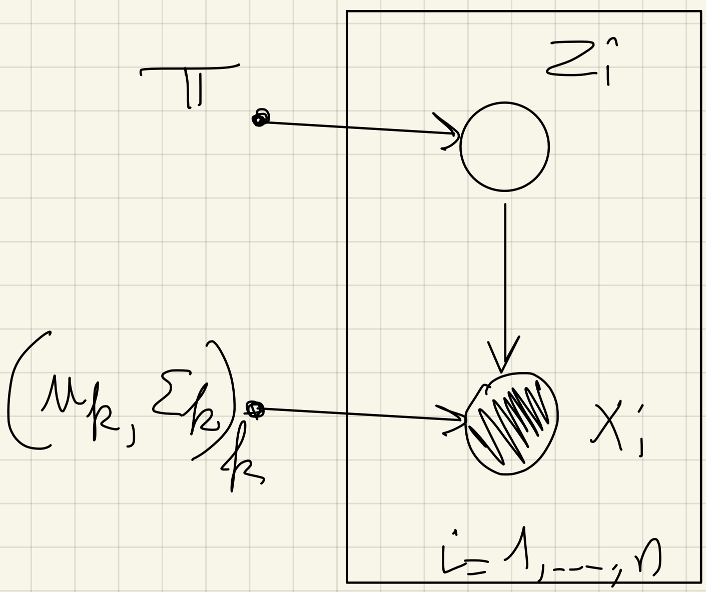
(2) $X_{ij} \mid Z_{ik}=1, Z_{jl}=1 \sim \mathcal{B}(\mu_{kl})$

Adjacency matrix $X = \begin{array}{cc} & \overset{n}{\overbrace{\qquad\quad j \quad}} \\ i & \begin{bmatrix} - & - & - & 1 \\ & & 0 & \\ & & & \\ & & & \end{bmatrix} \end{array} \Big\} n$

Reminder in GMM:

① $Z_i \overset{iid}{\sim} \mathcal{M}(1, \pi), \ \forall i \in \{1, \dots, n\}$

② $X_i \mid Z_{ik} = 1 \sim \mathcal{N}(\mu_k, \Sigma_k)$



$$p(z_1, z_2, \dots, z_n \mid x_1, \dots, x_n, \pi, \theta) = \prod_{i=1}^{n} p(z_i \mid x_i, \pi, \theta)$$

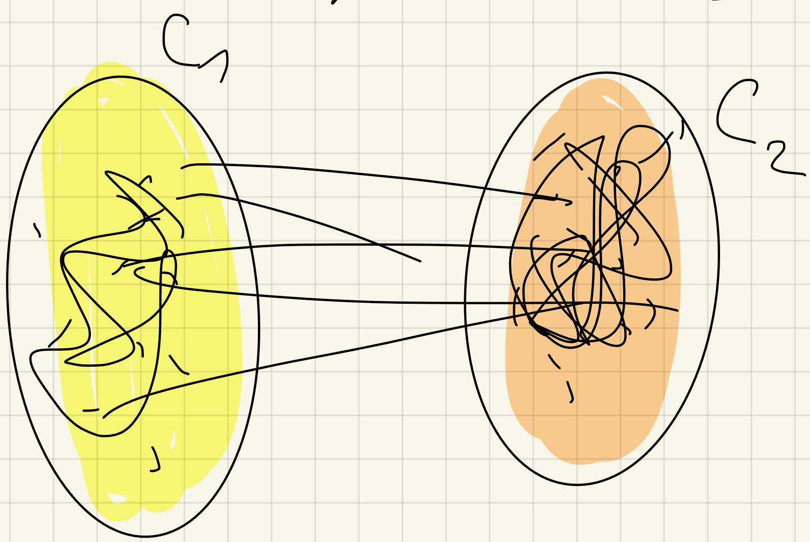
$$= \prod_{i=1}^{n} \mathcal{M}(z_i ; 1, \tau_i)$$

$\Rightarrow$ analytical expression $\Rightarrow$ EM in SBM:

$$\pi = \begin{pmatrix} \pi_1 \\ \vdots \\ \pi_K \end{pmatrix} \qquad \mu = \begin{array}{cc} & l \\ k & \begin{bmatrix} \cdots & \mu_{kl} & \\ & & \\ & & \end{bmatrix} \end{array}$$

2 examples: $K = 2$ (undirected)

$$\pi = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \qquad \mu = \begin{bmatrix} 0,9 & 0,2 \\ 0,2 & 0,95 \end{bmatrix}$$

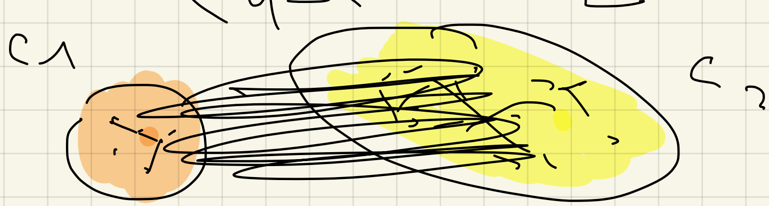


2 communities

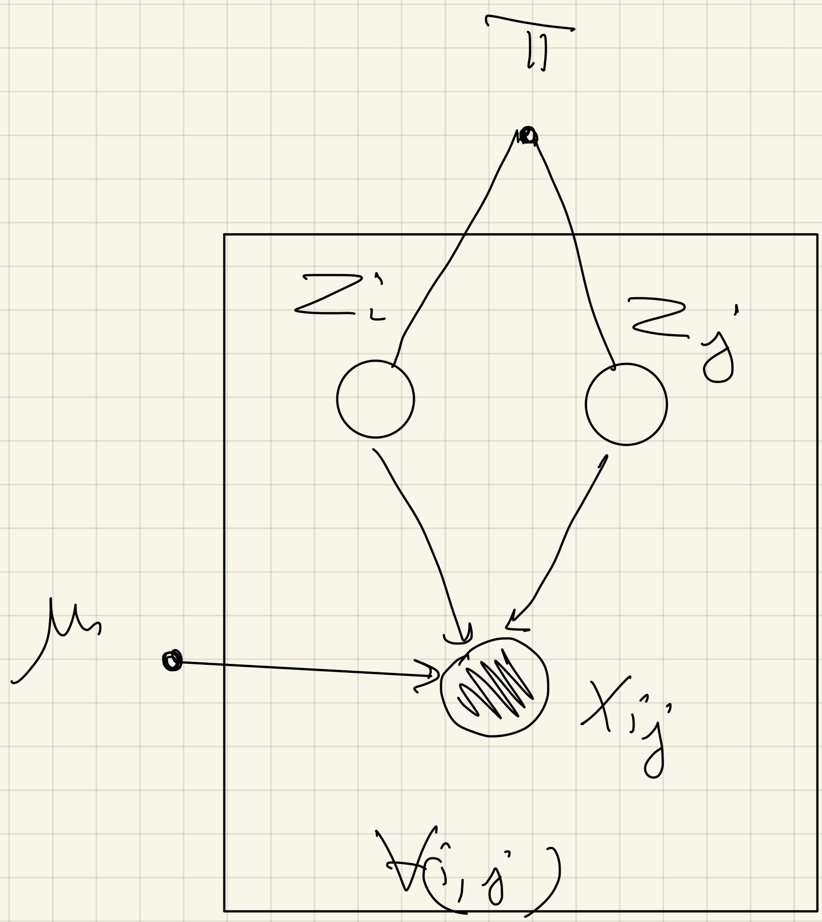
$K = 2$ (undirected)

$$\pi = \begin{pmatrix} 0,1 \\ 0,9 \end{pmatrix} \quad \mu = \begin{bmatrix} 0,2 & 0,9 \\ 0,9 & 0,2 \end{bmatrix}$$

disassortative mixing (star pattern)



Directed graphical model of SBM:



adjacency matrix

$$P(z_1, z_2, \dots, z_n \mid X, \pi, \mu)$$

$$\neq \prod_{i=1}^{n} P(z_i \mid \underline{\quad\quad})$$

↳ no analytical expression

↳ no EM

↳ cannot maximise exactly the likelihood $\Longrightarrow$ VEM

VEM: consider an ELBO:

$$\mathcal{L}\left(R((z_i)_i)\,;\,\pi,\alpha\right) = \sum_{(z_i)_i} R((z_i)_i) \log \frac{P(X,(z_i)_i \mid \pi,\mu)}{R((z_i)_i)}$$

Assumptions: ($\Rightarrow$ approximations)

we set: $R(z_i) = \mathcal{M}(z_i\,;\,1,\tau_i)$

$\Rightarrow$ the ELBO becomes a function of $\left[\pi,\alpha,(\tau_i)_i\right]$

VEM: VM: fix $(\tau_i)_i$. Maximise $\mathcal{L}$ with respect to π,α

VE: fix π,α. Maximise $\mathcal{L}$ with respect the $(\tau_i)_i$

> until convergence of $\mathcal{L}$

Variational decomposition:

$$\log p(X \mid \pi, \mu) = \mathcal{L} + KL$$