Bayesian statistics with R
5. Markov chains Monte Carlo (MCMC)

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Get posteriors with Markov chains Monte Carlo (MCMC) methods

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- $\Pr(\text{data}) = \int L(\text{data} \mid \theta)\Pr(\theta) d\theta$ is a N-dimensional integral if $\theta = \theta_1, \dots, \theta_N$
- Difficult, if not impossible to calculate!

Brute force approach via numerical integration

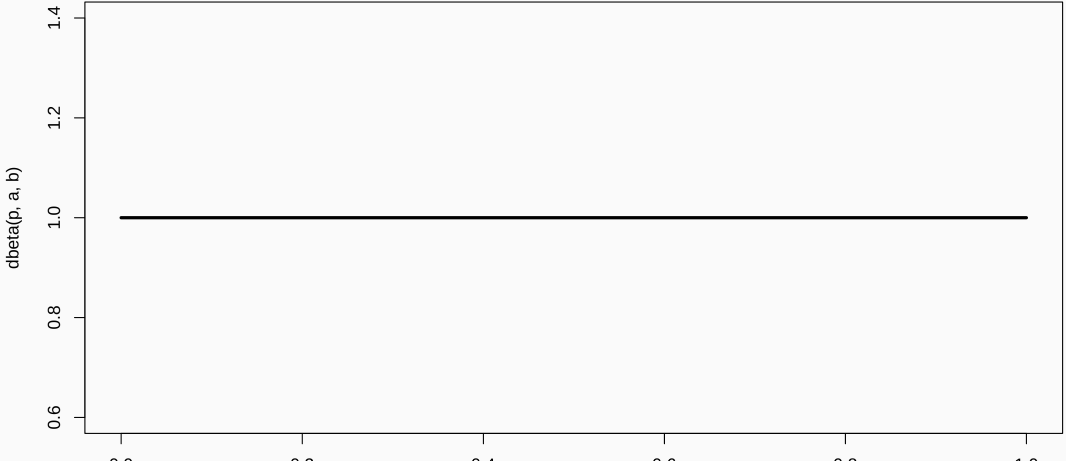
- Deer data

```r
y <- 19 # nb of success
n <- 57 # nb of attempts
```

- Likelihood Binomial$(57, \theta)$
- Prior Beta$(a = 1, b = 1)$

Beta prior

```r
a <- 1; b <- 1; p <- seq(0,1,.002)
plot(p, dbeta(p,a,b), type='l', lwd=3)
```



Apply Bayes theorem

- Likelihood times the prior: $\Pr(\text{data} \mid \theta)\ \Pr(\theta)$

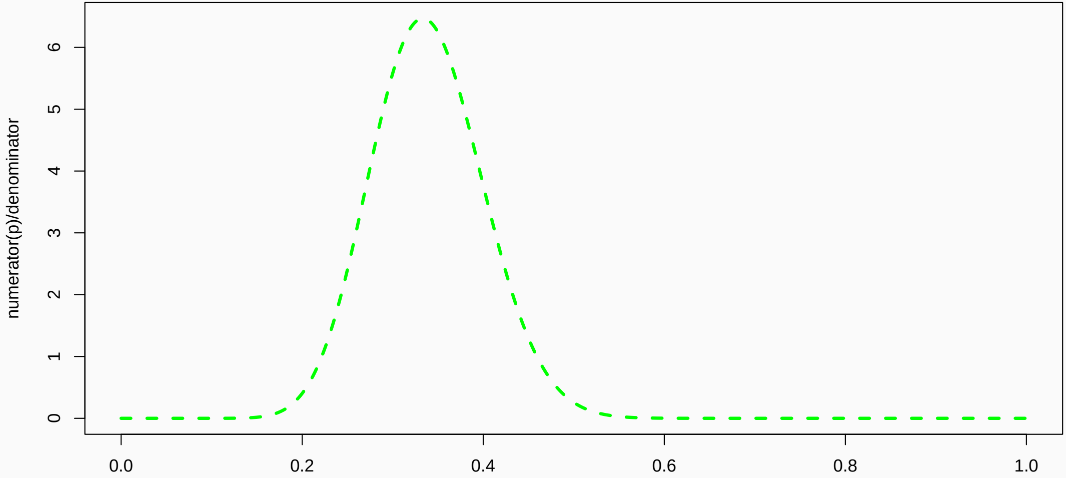
```
numerator <- function(p) dbinom(y,n,p)*dbeta(p,a,b)
```

- Averaged likelihood: $\Pr(\text{data}) = \int L(\theta \mid \text{data})\ \Pr(\theta) d\theta$

```
denominator <- integrate(numerator,0,1)$value
```

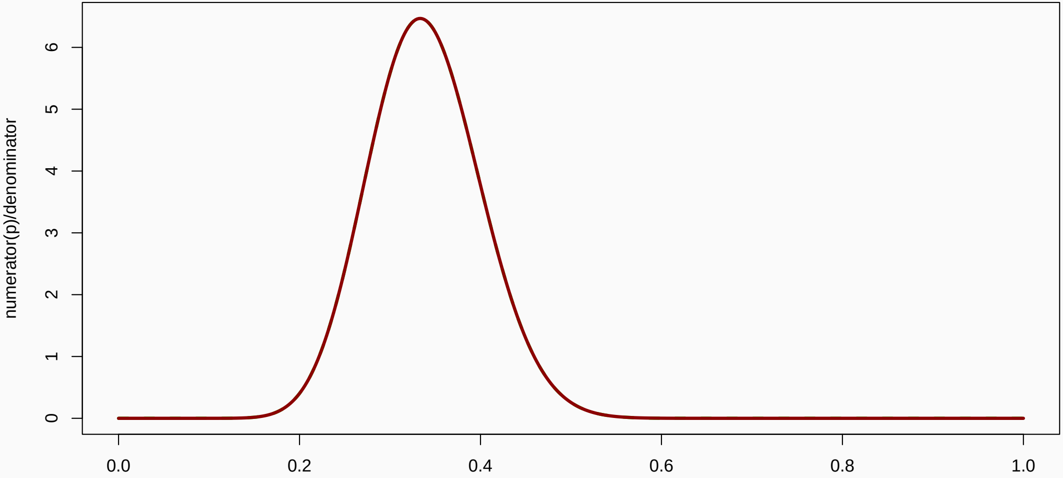
Posterior inference via numerical integration

```
plot(p, numerator(p)/denominator,type="l", lwd=3, col="green", lty=2)
```



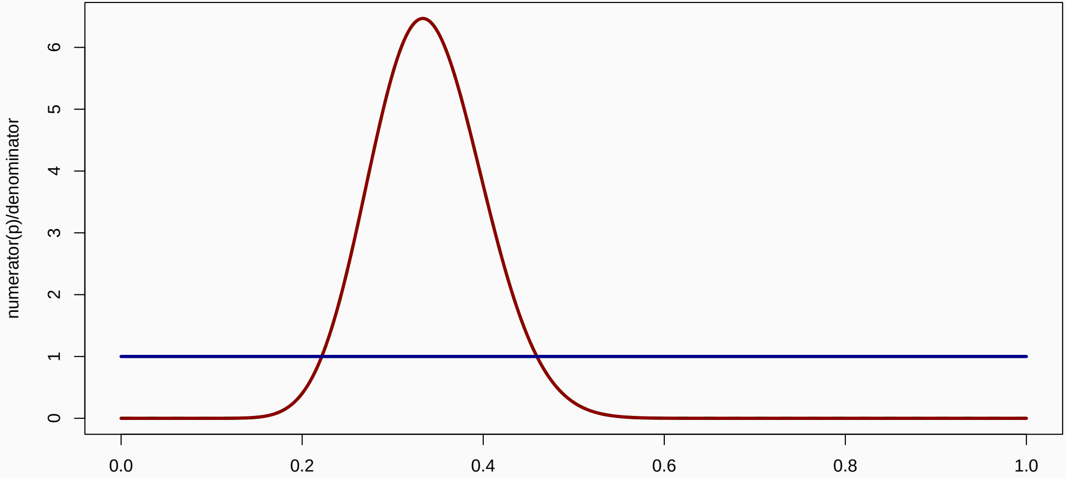
Superimpose explicit posterior distribution (Beta formula)

```r
lines(p, dbeta(p,y+a,n-y+b), col='darkred', lwd=3)
```



And the prior

```
lines(p, dbeta(p,a,b), col='darkblue', lwd=3)
```



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- Do we really wish to calculate a 3D integral?

Bayesian computation

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THE JOURNAL OF CHEMICAL PHYSICS VOLUME 21, NUMBER 6 JUNE, 1953

Equation of State Calculations by Fast Computing Machines

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(Received March 6, 1953)

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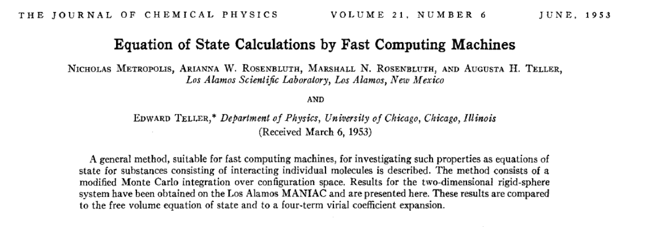

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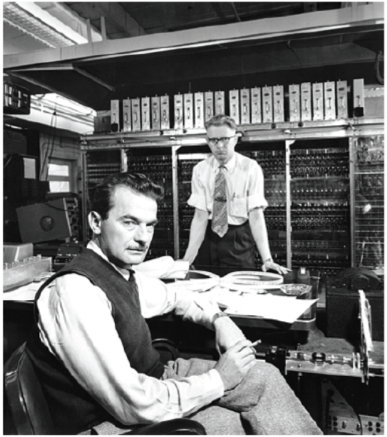
- Avoid explicit calculation of integrals in Bayes formula.
- Instead, approximate posterior to arbitrary degree of precision by drawing large sample.
- Markov chain Monte Carlo = MCMC; boost to Bayesian statistics!

MANIAC:
Mathematical Analyzer, Numerical Integrator, and Computer



MANIAC:
1000 pounds
5 kilobytes of memory
70k multiplications/sec

Your laptop:
4–7 pounds
2–8 million kilobytes
Billions of multiplications/sec

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- Several ways of constructing these chains: e.g., Metropolis-Hastings, Gibbs sampler, Metropolis-within-Gibbs.
- How to implement them in practice?!

The Metropolis algorithm

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- We illustrate sampling from the posterior distribution of winter survival.
- We write functions in `R` for the likelihood, the prior and the posterior.

```r
# deer data, 19 "success" out of 57 "attempts"
survived <- 19
released <- 57

# log-likelihood function
loglikelihood <- function(x, p){
  dbinom(x = x, size = released, prob = p, log = TRUE)
}

# prior density
logprior <- function(p){
  dunif(x = p, min = 0, max = 1, log = TRUE)
}
```

```r
# posterior density function (log scale)
posterior <- function(x, p){
  loglikelihood(x, p) + logprior(p) # - log(Pr(data))
}
```

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4. We spin a continuous spinner that lands anywhere from 0 to 1 — call the random spin X. If X is smaller than R, we move to the candidate location, otherwise we remain at the current location.

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5. We repeat 2-4 a number of times called **steps** (many steps).

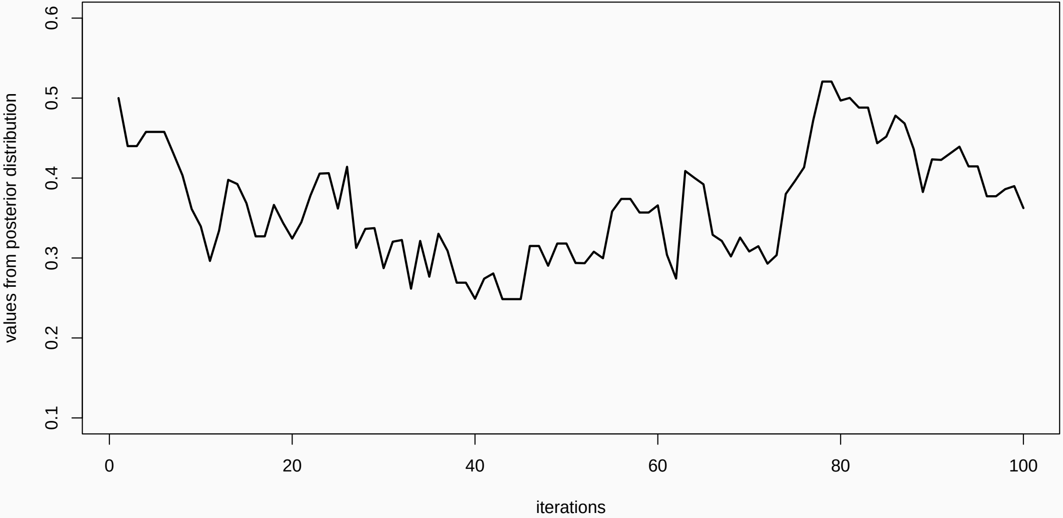
```r
# propose candidate value
move <- function(x, away = .2){
  logitx <- log(x / (1 - x))
  logit_candidate <- logitx + rnorm(1, 0, away)
  candidate <- plogis(logit_candidate)
  return(candidate)
}

# set up the scene
steps <- 100
theta.post <- rep(NA, steps)
set.seed(1234)
```

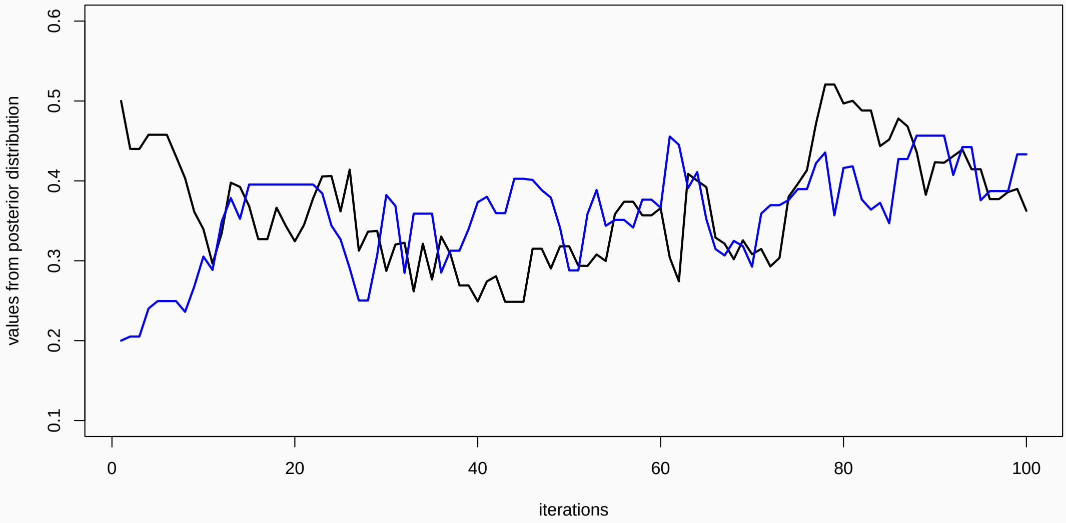
```r
# pick starting value (step 1)
inits <- 0.5
theta.post[1] <- inits
```

```r
for (t in 2:steps){ # repeat steps 2-4 (step 5)

  # propose candidate value for prob of success (step 2)
  theta_star <- move(theta.post[t-1])

  # calculate ratio R (step 3)
  pstar <- posterior(survived, p = theta_star)
  pprev <- posterior(survived, p = theta.post[t-1])
  logR <- pstar - pprev
  R <- exp(logR)

  # decide to accept candidate value or to keep current value (step 4)
  accept <- rbinom(1, 1, prob = min(R, 1))
  theta.post[t] <- ifelse(accept == 1, theta_star, theta.post[t-1])
}
```

Starting at the value 0.5 and running the algorithm for 100 iterations.

```r
head(theta.post)
#> [1] 0.5000000 0.4399381 0.4399381 0.4577124 0.4577124 0.4577124
tail(theta.post)
#> [1] 0.4145878 0.3772087 0.3772087 0.3860516 0.3898536 0.3624450
```


values from posterior distribution
0.1
0.2
0.3
0.4
0.5
0.6
0
20
40
60
80
100
iterations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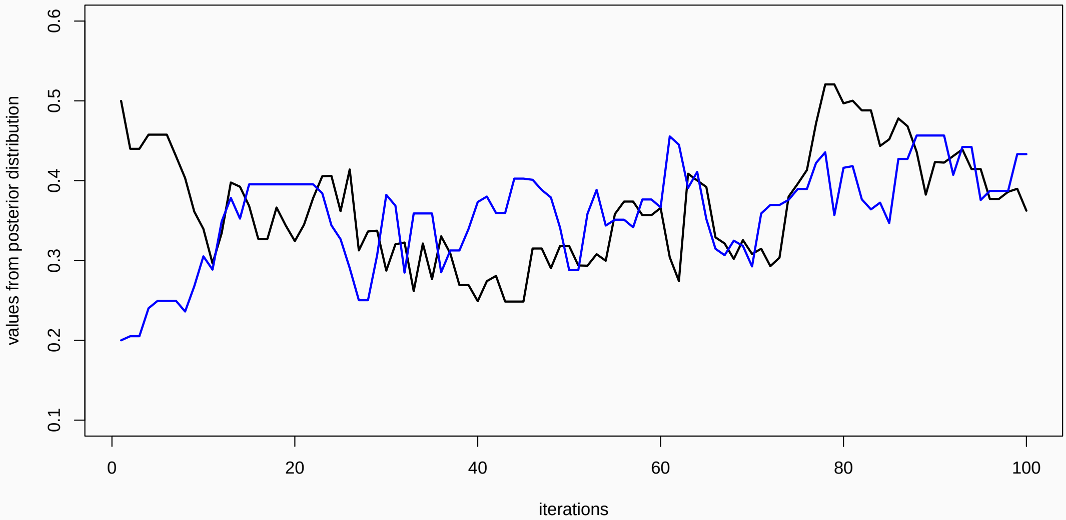
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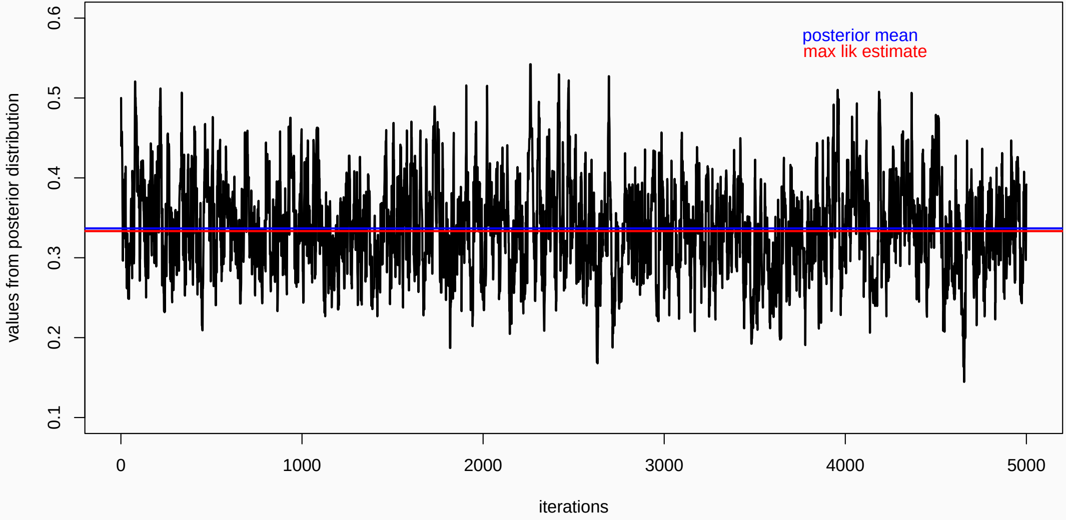
posterior mean
max lik estimate
values from posterior distribution
iterations
0.1
0.2
0.3
0.4
0.5
0.6
0
1000
2000
3000
4000
5000

Animating the Metropolis algorithm - 1D example

https://gist.github.com/oliviergimenez/5ee33af9c8d947b72a39ed1764040bf3

Animating the Metropolis algorithm - 2D example

https://mbjoseph.github.io/posts/2018-12-25-animating-the-metropolis-algorithm/

The Markov-chain Monte Carlo Interactive Gallery

https://chi-feng.github.io/mcmc-demo/