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Hybrids Architectures to Reach Exascale

Les Architectures Hybrides pour Atteindre l'Exascale

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Thèse de doctorat d'Informatique

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Thanks everyone !

Resume

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Introduction

In the aurora 2020-2021 for USA and maybe before, like 2019 for Chinese supercomputers, the world will reach another milestone in the power of machines, the Exascale. This supercomputer will be 100 times faster than the estimated overfall operations performed by the human brain and its 10^{16} **FL**oating point **O**perations **P**er **S**econd (FLOPS) and reach a computational power of a trillion (10^{18}) FLOPS.

This odyssey started long time ago with the first vacuum tubes and the need of balistic in war. Nowdays the main aim does not changed a lot and the power of a nation is represented by its army but also the computational power of its supercomputers. Since 1962, considering the Cray CDC 6600 as the first supercomputer, the power of those machines have increase following an observation of the co-fonder of the Intel company, Alan Moore. Better known under as the "Moore's Law" it speculates in 1965 that considering the evoluation of technology the number of transistors on a dense integrated circuit will double approximately every two years. Thus the computational power, that depend intrinsectly of the number of processors on the chip, will also increase. This observation can be related to the supercomputers results through the years in the TOP500 list.

As shown on figure 1, even if estimated in early 1965, the Moore's law seems to be accurate.

Those good results are not just gave by the shrink in the semiconductor with smaller transistors. At some point in early tweenty century IBM proposed the first multi-core processor. The constructors started then to propose chips with more than one core to increase the computational power in conjonction with the shrink of semiconductors, allowing the Moore's law to thirve. This increase of the overall power of the chip comes with some complementary cost in synchronizations steps between the chips for memory access, work sharing and of course the power consumption. The general purpose CPU usually features from 2 to 16 cores on a single ship.

But since 2013-2014 a lot of companies, like Intel the Gordon Moore's company, stated that this law is over. This can be see on figure 1, in the right part of the graph, the evolution is not linear anymore and tend to dicrease. This is due to two main factors: on one hand, we slowly reach the maximal shrink size of the transistors implying hard to handle side effects and on another hand the power wall implyed by the power consumption required by so many transistors and frequency speed on the chip. And some ways were found years before to overcome this again using many-cores architectures which are now called accelerstors. Companies like Intel, NVIDIA, AMD, Altera propose their accelerator going from Xeon Phi, General Purpose Graphics Processing Unit (GPGPU) initialy used in graphics, Field Programmable Gates Array (FPGA) or even dedicated Application-Specific Integrated Circuit (ASIC). Those architecture takes the chose to implement very simple cores with low power consumption used by thousand in the same chip. This add an extrat cost in the synchronization and coordination on chip and add a need for specified memory. Those Devices, for most of them, also need to be controlled by a Host CPU adding another extra cost to shared data between the Host and Device. But used efficiently with fitting massively parallel applications, they can release a computational power way better than the classical CPU.

Even with these devices, nowaday supercomputers are facing several problems in their conception and utilization. The three mains are the power wall, the communication wall and the memory wall bounding the overall computional power of the machines. Some subproblems like

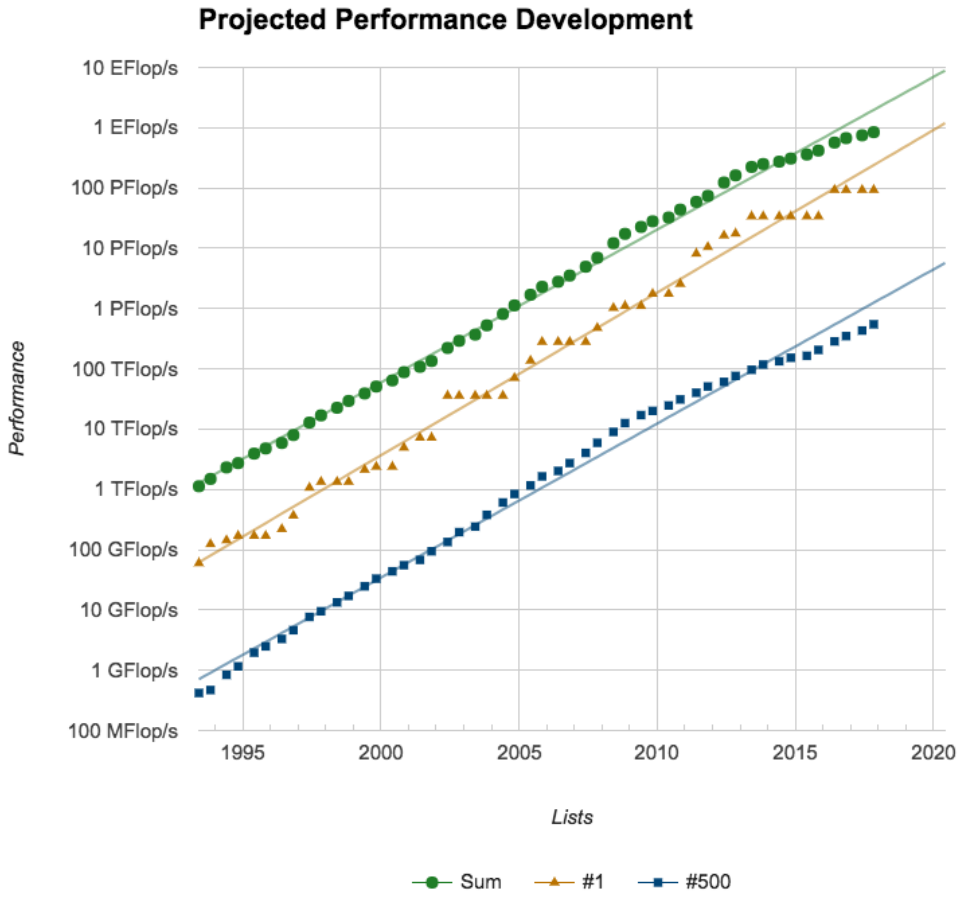

Figure 1: Computational power evolution in the TOP500 list

the the interconnect wall, resilience wall or even the complexity wall also arise and make the task even harder.

This is were this study take place, how to use those accelerators or devices in the right way and are they the way that will help us the reach exascale. We considered that the classical TOP500 ranking is not enough to target all the main wall problems of those architectures and especially accelerators. Indeed it is based on LINPACK/LAPACK with regular computation and communication that perfectly fit this architecture and does not show the reality that the domain scientists have to face in their simulation.

We propose a metric that extracts the three main issues of HPC and applyed them on accelerators architectures to figure out how to take advantage of those architectures and what are the limitations. This study is decomposed in 3 problems. The first two are targeting computation and communication wall over very irregular cases with high memory accesses, using an academic combinatorial problem and the Graph500 benchmark. The last is a computational scientific problem that cover both of the problems and appears to be hard to implement on supercomputers and even more on accelerated ones.

This thesis is composed of 3 chapters. The first will develope the state of the art in HPC from the main law to the hardware. We describe how the nowaday supercomputers are ranked and what are the main walls in their evoluiton.

In the second chapter we propose our metric to characterize supercomputers architectures. The Langford problem is described as an irregular and computationally heavy problem. This shows how the accelerators, in this case GPU, are able to support the memory and computation wall. This allowed us to beat a world record on the last instances of this academic problem. The Graph500 problem is then proposed as an irregular and communications heavy problem. We present our implementation and the logic to take advantage of the GPU computational power in an

Then, in the last chapter, we consider a problem that is heavy and irregular regarding to computation and communications. This problem is the milestone of our metric and show how nowadays supercomputers can overcome those issues. This computational science problem is based on the SPH method and we intend to provide a tool for Physisists and Astrophysists and is called FleCSPH.

The last part will conclude on this work and results and show some of the main prospects of this study and my future researches.

Chapter 1

HPC and Exascale

1.1 Introduction

The HPC world is spread over all the scientific domains nowaday and is used from usual computation to the biggest black holes merging simulations. Firstly, we present the rules, laws and organization that characterize the HPC world from the Moore's law, Amdhal with the Flynn Taxonomy and the description of the recent bottlenecks and walls.

Considering homogeneous cluster only.

1.2 Parallelism

1.2.1 Flynn taxonomy

The flynn taxonomy presents a hierarchical organization of computation machine.

In this classification [Fly72b], Michael J. Flynn present the SIMD,SISI, MISD and MIMD. Add table and present some example of machines

1.2.2 Goals

Speedup

Speedup can be separate in two parts, Latency and Throughput.

1.2.3 Bottlenecks

1.2.4 Amdahl and Gustafson

The Amdahl's[Amd67] law is use to find the theoretical speedup in latency of a program. We can separate a program in two parts, the one that can be execute in parallel and the one that is sequential. And even if we reduce the parallel part to infinite the sequential part will reach 100% of the total time.

Extracted from the Amdahl paper the law can be writen as:

$$Speedup = \frac{1}{Seq + \frac{Par}{n}} \tag{1.1}$$

Where $Seq + Par = 1$ and Seq and Par respectively the sequential and parallel ratio of a program.

On plot XXX, a representation of the perfect speedup is represented. This law does not take every cases in acount. Indeed the speedup can be much better if we grow the amount of work to be done in the same time as the number of processes that execute the program.

This is proposed by the Gustafson law presented in.

1.3 Hardware

The structure of an HPC cluster can be

1.3.1 Classical CPU

1.3.2 GPU

1.3.3 FPGA and ASICS

1.4 Clusters and Exascale

1.4.1 Benchmarking

TOP500

The TOP500 list is a list of the 500 most powerful super computers in the world. The ranking is based on the LINPACK and LAPACK suite. Initialy the LINPACK, Linear Algebra library was used but is now replaced by the new Linear Algreba Pack, LAPACK.

1.4.2 Composition and usage

1.4.3 Interconnection

1.5 Languages

1.5.1 Accelerators

1.5.2 Runtimes

1.6 Optimization

Memory locality

Vectorization

1.6.1 CPU specifications

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1.7 Conclusion

Chapter 2

Complex systems

2.1 Introduction

2.2 Combinatorial problems

2.2.1 Combinatorial search

2.2.2 Combinatorial optimization

2.3 A case study, the Langford problem

Introduction

For many years now, GPUs usage has increased in the field of High Performance Computing. The TOP500 list of the world's most powerful supercomputers contains more than about 52 systems powered by NVIDIA Kepler GPUs. In the latest list the number of hybrid machines is a fourfold increase compared with just one year before.
Since 2007, NVIDIA has offered a general GPUs programming interface: *Compute Unified Device Architecture* (CUDA). This study is based on this physical and logical architecture which requires massively parallel programming and a new vision for the implementation of resolution algorithms.

The Langford pairing problem is a very irregular combinatorial problem and thus is a bad candidate for GPU computation which requires vectorized and regularized tasks. Hopefully there are many ways to regularize the computation in order to take advantage of the multiGPU cluster architectures.

This paper is structured as follows: we first present the background with the Langford problem and multiGPU cluster. The next section describes our method concerning the Miller algorithm on such architectures. Then we expose our multiGPU solution to solve the Langford problem based on the Godfrey algorithm. Finally we expose some concluding remarks and perspectives.

2.3.1 Background

Langford problem

C. Dudley Langford gave his name to a classic permutation problem [Gar56, Sim83]. While observing his son manipulating blocks of different colors, he noticed that it was possible to arrange three pairs of different colored blocks (yellow, red, blue) in such a way that only one block separates the red pair - noted as pair 1 - , two blocks separate the blue pair - noted as pair 2 - and finally three blocks separate the yellow one - noted as pair 3 - , see Fig. 2.1.

This problem has been generalized to any number n of colors and any number s of blocks having the same color. $L(s, n)$ consists in searching for the number of solutions to the Langford

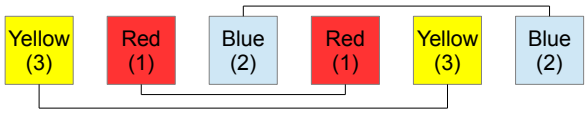

Figure 2.1: L(2,3) arrangement

Instance	Solutions	Method	Computation time
L(2,3)	1	Miller algorithm	-
L(2,4)	1		-
...	...		...
L(2,16)	326,721,800		120 hours
L(2,19)	256,814,891,280		2.5 years (1999) DEC Alpha
L(2,20)	2,636,337,861,200	Godfrey algorithm	1 week
L(2,23)	3,799,455,942,515,488		4 days with CONFIIT
L(2,24)	46,845,158,056,515,936		3 months with CONFIIT
L(2,27)	111,683,611,098,764,903,232		-
L(2,28)	1,607,383,260,609,382,393,152		-

Table 2.1: Solutions and time with differents methods

problem, up to a symmetry. In November 1967, Martin Gardner presented $L(2,4)$ (two cubes and four colors) as being part of a collection of small mathematical games and he stated that $L(2,n)$ has solutions for all n such that $n = 4k$ or $n = 4k-1$ ($k \in \mathbb{N} \setminus \{0\}$). The central resolution method consists in placing the pairs of cubes, one after the other, on the free places and backtracking if no place is available (see Fig. 2.3 for detailed algorithm).

The Langford problem has been approached in different ways: discrete mathematics results, specific algorithms, specific encoding, constraint satisfaction problem (CSP), inclusion-exclusion ... [Mil99, Wal01, Smi00, Lar09]. In 2004, the last solved instance, $L(2,24)$, was computed by our team using a specific algorithm. (see Table 2.1); $L(2,27)$ and $L(2,28)$ have just been computed but no details were given.

The main efficient known algorithms are the following: the Miller backtrack method, the Godfrey algebraic method and the Larsen inclusion-exclusion method. The Miller one is based on backtracking and can be modeled as a CSP; it allowed us to move the limit of explicits solutions building up to $L(2,21)$ but combinatorial explosion did not allow us to go further. Then, we use the Godfrey method to achieve $L(2,24)$ more quickly and then recompute $L(2,27)$ and $L(2,28)$, presently known as the last instances. The Larsen method is based on inclusion-exclusion [Lar09]; although this method is effective, practically the Godfrey one is better. The lastest known work on the Langford Problem is a GPU implementation proposed in [ABL15] in 2015. Unfortunately this study does not provide any performance considerations but just gives the number of solution of $L(2,27)$ and $L(2,28)$.

2.3.2 Miller algorithm

In this part we present our multiGPU cluster implementation of the Miller's algorithm. First, we introduce the backtrack method. Then we present our implementation in order to fit the GPUs architecture. The last section presents our results.

Backtrack resolution

As presented above the Langford problem is known to be a highly irregular combinatorial problem. We first present here the general tree representation and the ways we regularize the computation for GPUs. Then we show how to parallelize the resolution over a multiGPU cluster.

Langford's problem tree representation In [HKS02], we propose to formalize the Langford problem as a CSP (*Constraint Satisfaction Problem*), first introduced by Montanari in [Mon74], and show that an efficient parallel resolution is possible. CSP formalized problems can be transformed into tree evaluations. In order to solve $L(2,n)$, we consider the following tree of height n: see example of $L(2,3)$ in Fig. 2.2.

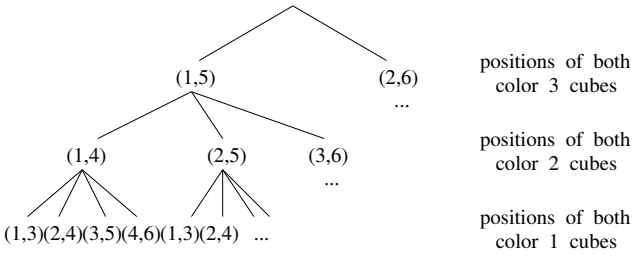

Figure 2.2: Search tree for $L(2,3)$

- Every level of the tree corresponds to a color.
- Each node of the tree corresponds to the placement of a pair of cubes without worrying about the other colors. Color p is represented at depth $n-p+1$, where the first node corresponds to the first possible placement (positions 1 and $p+2$) and i^{th} node corresponds to the placement of the first cube of color p in position i, $i \in [1,\ 2n-1-p]$.
- Solutions are leaves generated without any placement conflict.

There are many ways to browse the tree and find the solutions: *backtracking*, *forward-checking*, *backjumping*, etc [Pro93]. We limit our study to the naive *backtrack* resolution and choose to evaluate the variables and their values in a static order; in a depth-first manner, the solutions are built incrementally and if a partial assignment can be aborted, the branch is cut. A solution is found each time a leaf is reached.

The recommendation for performance on GPU accelerators is to use non test-based programs. Due to its irregularity, the basic *backtracking* algorithm, presented on Fig. 2.3, is not supposed to suit the GPU architecture. Thus a vectorized version is given when evaluating the assignments at the leaves' level, with one of the two following ways: assignments can be prepared on each tree node or totally set on final leaves before testing the satisfiability of the built solution (Fig. 2.4).

```
while not done do
 test pair            <- test
 if successful then
   if max depth then
     count solution
     higher pair
   else
     lower pair       <- remove
 else
   higher pair        <- add
```

Figure 2.3: Backtrack algorithm

```
for pair 1 positions
  assignment                    <- add
  for pair 2 positions
    assignment                  <- add
    for ...
      for pair n positions
        assignment              <- add
        if final test ok then
          count solution
```

Figure 2.4: Regularized algorithm

Data representation In order to count every Langford problem solution, we first identify all possible combinations for one color without worrying about the other ones. Each possible combination is coded within an interger, one bit to 1 corresponding to a cube presence, a 0 to its absence. This is what we called a *mask*. This way Fig. 2.5 presents the possible combinations to place the one, two and three weight cubes for the $L(2,3)$ Langford instance.

Furthermore the masks can be used to evaluate the partial placements of a chosen set of colors: all the 1s correspond to occupied positions; the assignment is consistent *iff* there are as many 1s as the number of cubes set for the assignment.

With the aim to find solutions, we just have to go all over the tree and *sum* one combination of each of the colors: a solution is found *iff* all the bits of the sum are set to 1.

Each route on the tree can be evaluated individually and independently; then it can be evaluated as a thread on the GPU. This way the problem is massively parallel and can be, indeed, computed on GPU. Fig. 2.6 represents the tree masks' representation.

	pair 1	pair 2	pair 3
1	0 0 0 1 0 1	0 0 1 0 0 1	0 1 0 0 0 1
2	0 0 1 0 1 0	0 1 0 0 1 0	1 0 0 0 1 0
3	0 1 0 1 0 0	1 0 0 1 0 0	
4	1 0 1 0 0 0		

Figure 2.5: Bitwise representation of pairs positions in $L(2,3)$

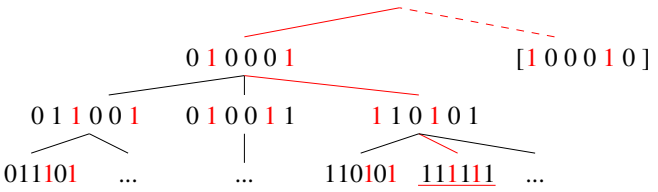

Figure 2.6: Bitwise representation of the Langford $L(2,3)$ placement tree

Specific operations and algorithms Three main operations are required in order to perform the tree search. The first one, used for both backtrack and regularized methods, aims to add a pair to a given assignment. The second one, allowing to check if a pair can be added to a given partial assignment, is only necessary for the original backtrack scheme. The last one is used for testing if a global assignment is an available solution: it is involved in the regularized version of the Miller algorithm.

Add a pair - Top of Fig. 2.7 presents the way to add a pair to a given assignment. With a *binary or*, the new mask contains the combination of the original mask and of the added pair. This operation can be performed even if the position is not available for the pair (however the resulting mask is inconsistent).

Test a pair position - On the bottom part of the same figure, we test the positioning of a pair on a given mask. For this, it is necessary to perform a *binary and* between the mask and the pair.

$= 0$: *success*, the pair can be placed here

$\neq 0$: *error*, try another position

Final validity test - The last operation is for *a posteriori* checking. For example the mask 101111, corresponding to a leaf of the tree, is inconsistent and should not be counted among the solutions. The final placement mask corresponds to a solution *iff* all the places are occupied, which can be tested as $\neg mask = 0$.

Using this data representation, we implemented both *backtrack* and *regularized* versions of the Miller algorithm, as presented in Fig. 2.3 and 2.4.
The next section presents the way we hybridize these two schemes in order to get an efficient parallel implementation of the Miller algorithm.

Hybrid parallel implementation This part presents our methodology to implement Miller's method on a multiGPU cluster.

Tasks generation - In order to parallelize the resolution we have to generate tasks. Considering the tree representation, we construct tasks by fixing the different values of a first set of variables [pairs] up to a given level. Choosing the development level allows to generate as many

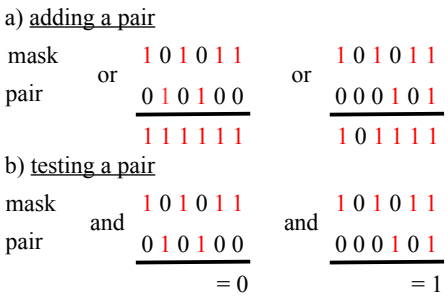

Figure 2.7: Testing and adding position

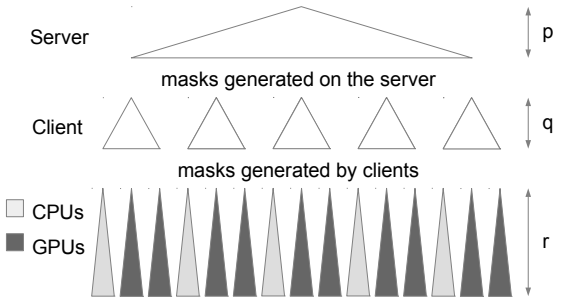

Figure 2.8: Server client distribution

tasks as necessary. This leads to a *Finite number of Irregular and Independent Tasks* (*FIIT* applications [Kra99]).

Cluster parallelization - The generated tasks are independent and we spread them in a client-server manner: a server generates them and makes them available for clients. As we consider the cluster as a set of CPU-GPU(s) machines, the clients are these machines. At the machines level, the role of the CPU is, first, to generate work for the GPU(s): it has to generate sub-tasks, by continuing the tree development as if it were a second-level server, and the GPU(s) can be considered as second-level client(s).

The sub-tasks generation, at the CPU level, can be made in parallel by the CPU cores. Depending on the GPUs number and their computation power the sub-tasks generation rhythm may be adapted, to maintain a regular workload both for the CPU cores and GPU threads: some CPU cores, not involved in the sub-tasks generation, could be made available for sub-tasks computing.

This leads to the 3-level parallelism scheme presented in Fig. 2.8, where p, q and r respectively correspond to: (p) the server-level tasks generation depth, (q) the client-level sub-tasks generation one, (r) the remaining depth in the tree evaluation, *i.e.* the number of remaining variables to be set before reaching the leaves.

***Backtrack* and *regularized* methods hybridization -** The Backtrack version of the Miller algorithm suits CPU execution and allows to cut branches during the tree evaluation, reducing the search space and limiting the combinatorial explosion effects. A regularized version had to be developed, since GPUs execution requires synchronous execution of the threads, with as few branching divergence as possible; however this method imposes to browse the entire search space and is too time-consuming.

We propose to hybridize the two methods in order to take advantage of both of them for the multiGPU parallel execution: for tasks and sub-tasks generated at sever and client levels, the tree development by the CPU cores is made using the backtrack method, cutting branches as soon as possible [and generating only possible tasks]; when computing the sub-tasks generated at client-level, the CPU cores involved in the sub-tasks resolution use the backtrack method and the GPU threads the regularized one.

Experiments tuning

In order to take advantage of all the computing power of the GPU we have to refine the way we use them: this section presents the experimental study required to choose optimal settings. This tuning allowed us to prove our proposal on significant instances of the Langford problem.

Registers, blocks and grid In order to use all GPUs capabilities, the first way was to fill the blocks and grid. To maximize occupancy (ratio between active warps and the total number of warps) NVIDIA suggests to use 1024 threads per block to improve GPU performances and proposes a CUDA occupancy calculator[1]. But, confirmed by the Volkov's results[Vol10], we

[1] `http://developer.download.nvidia.com/compute/cuda/CUDA_Occupancy_calculator.xls`

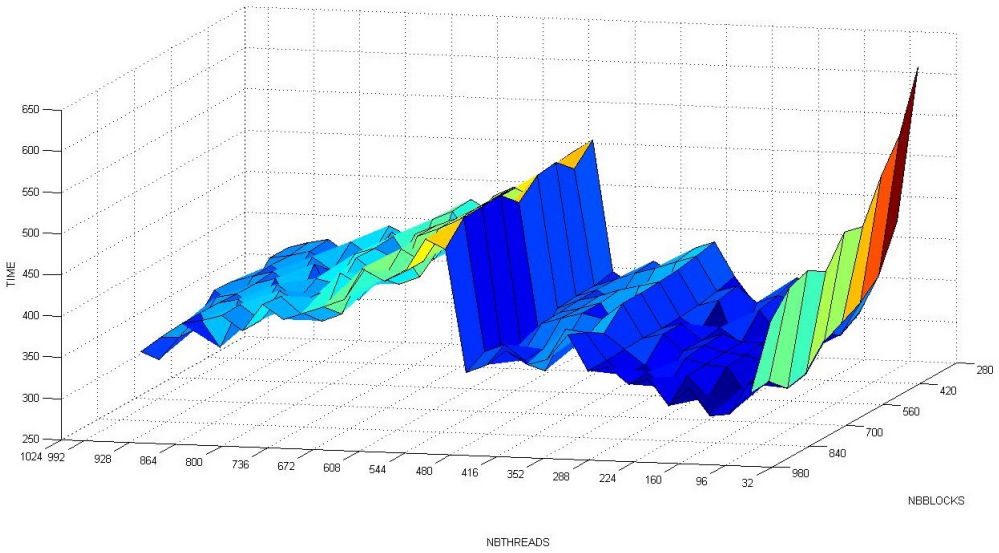

Figure 2.9: Time depending on grid and block size on $n = 15$

experimented that better performances may be obtained using lower occupancy. Indeed, another critical criterion is the inner GPU registers occupation. The optimal number of registers (57 registers) is obtained by setting 9 pairs placed on the client for $L(2, 15)$, thus 6 pairs are remaining for GPU computation.

In order to tune the blocks and grid sizes, we performed tests on the ROMEO architecture. Fig. 2.9 represents the time in relation with the number of blocks per grid and the number of threads per block. The most relevant result, observed as a local minimum on the 3D surface, is obtained near 64 or 96 threads per block; for the grid size, the limitation is relative to the GPU global memory size. It can be noted that we do not need shared memory because their are no data exchanges between threads. This allows us to use the total available memory for the L1 cache for each thread.

Streams A client has to prepare work for GPU. There are four main steps: generate the tasks, load them into the device memory, process the task on the GPU and then get the results.

CPU-GPU memory transfers cause huge time penalties (about 400 cycles latency for transfers between CPU memory and GPU *device memory*). At first, we had no overlapping between memory transfer and kernel computation because the tasks generation on CPU was too long compared to the kernel computation. To reduce the tasks generation time we used OpenMP in order to use the eight available CPU cores. Thus CPU computation was totally hidden by memory transfers and GPU kernel computation. We tried using up to 7 streams; as shown by Fig. 2.10, using only two simultaneous streams did not improve efficiency because the four steps did not overlap completely; the best performances were obtained with three streams; the slow increase in the next values is caused by synchronization overhead and CUDA streams management.

Setting up the server, client and GPU depths We now have to set the depths of each actor, server (p), client (q) and GPU (r) (see Fig. 2.8).

First we set the $r = 5$ for large instances because of the GPU limitation in terms of registers by threads, exacerbated by the use of numerous $64bits$ integers. For $r \geq 6$, we get too many registers (64) and for $r \leq 4$ the GPU computation is too fast compared to the memory load overhead.

Clients are the buffers between the server and the GPUs: $q = n - p - r$. So we have conducted tests by varying the server depth, p. The best result is obtained for $p = 3$ and

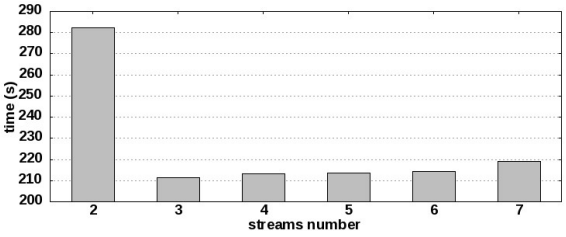

Figure 2.10: Computing time depending on streams number

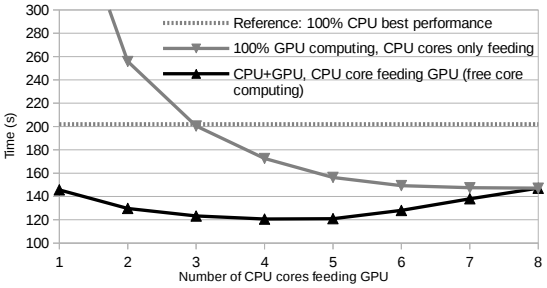

Figure 2.11: CPU cores optimal distribution for GPU feeding

performance decreases quickly for higher values. This can be explained since more levels on the server generates smaller tasks; thus GPU use is not long enough to overlap memory exchanges.

CPU: Feed the GPUs and compute The first work of CPU cores is to prepare tasks for GPU so that we can generate overlapping between memory load and kernel computation. In this configuration using eight cores to generate GPU tasks under-uses CPU computation power. It is the reason why we propose to use some of the CPU cores to take part of the sub-problems treatment. Fig. 2.11 represents computation time in relation with different task distributions between CPU and GPU. We experimentally demonstrated that only 4 or 5 CPU cores are enough to feed GPU, the other ones can be used to perform backtrack resolution in competition with GPUs.

Results

Regularized method results We now can show the results obtained for our massively parallel scheme using the previous optimizations, comparing the computation times of successive instances of the Langford problem. These tests were performed on 20 nodes of the ROMEO supercomputer, hence 40 CPU/GPU machines.

The previous limit with Miller's algorithm was $L(2, 19)$, obtained in 1999 after 2.5 years of sequential effort and at the same time after 2 months with a distributed approach[Mil99]. Our computation scheme allowed us to obtain it in less than 4 hours (Table 2.2), this being not only due to Moore law progress.
Note that the computation is 1.6 faster with CPU+GPU together than using 8 CPU cores. In addition, the GPUs compute 200000× more nodes of the search tree than the CPUs, with a faster time.

n	CPU (8c)	GPU (4c) + CPU (4c)
15	2.5	1.5
16	21.2	14.3
17	200.3	120.5
18	1971.0	1178.2
19	22594.2	13960.8

Table 2.2: Regularized method (seconds)

n	CPU (8c)	GPU (4c) + CPU (4c)
17	29.8	7.3
18	290.0	73.6
19	3197.5	803.5
20	–	9436.9
21	–	118512.4

Table 2.3: Backtrack (seconds)

The computation time between two different consecutive instances being multiplied by 10 approximately, this could allow us to obtain $L(2, 20)$ in a reasonable time.

Backtracking on GPUs It appears at first sight that using backtracking on GPUs without any regularization is a bad idea due to threads synchronization issues. But in order to compare

CPU and GPU computation power in the same conditions we decide to implement the original backtrack method on GPU (see Fig. 2.3) with only minor modifications. In these conditions we observe very efficient work of the NVIDIA scheduler, which perfectly handles threads desynchronization. Thus we use the same server-client distribution as in 2.3.2, each client generates masks for both CPU and GPU cores. The workload is then statically distributed on GPU and CPU cores. Executing the backtrack algorithm on a randomly chosen set of sub-problems allowed us to set the GPU/CPU distribution ratio experimentally to 80/20%,

The experiments were performed on 129 nodes of the ROMEO supercomputer, hence 258 CPU/GPU machines and one node for the server. Table 2.3 shows the results with this configuration. This method first allowed us to perform the computation of $L(2, 19)$ in less than 15 minutes, 15× faster than with the regularized method; then, we pushed the limitations of the Miller algorithm up to $L(2, 20)$ in less than 3 hours and even $L(2, 21)$ in about 33 hours[2].

This exhibits the ability of the GPU scheduler to manage highly irregular tasks. It proves that GPUs are adapted even to solve combinatorial problems, which they were not supposed to be.

2.3.3 Godfrey's algebraic method

The previous part presents the Miller algorithm for the Langford problem, this method cannot achieve bigger instances than the $L(2, 21)$.
An algebraic representation of the Langford problem has been proposed by M. Godfrey in 2002. In order to break the limitation of $L(2, 24)$ we already used this very efficient problem specific method. In this part we describe this algorithm and optimizations, and then our implementation on multiGPU clusters.

Method description

Consider $L(2, 3)$ and $X = (X_1, X_2, X_3, X_4, X_5, X_6)$. It proposes to modelize $L(2, 3)$ by $F(X, 3) = (X_1X_3 + X_2X_4 + X_3X_5 + X_4X_6) \times (X_1X_4 + X_2X_5 + X_3X_6) \times (X_1X_5 + X_2X_6)$

In this approach each term represents a position of both cubes of a given color and a solution to the problem corresponds to a term developed as $(X_1X_2X_3X_4X_5X_6)$; thus the number of solutions is equal to the coefficient of this monomial in the development. More generally, the solutions to $L(2, n)$ can be deduced from $(X_1X_2X_3X_4X_5...X_{2n})$ terms in the development of $F(X, n)$.

If $G(X, n) = X_1...X_{2n}F(X, n)$ then it has been shown that:

$$\sum_{(x_1,...,x_{2n})\in\{-1,1\}^{2n}} G(X, n)_{(x_1,...,x_{2n})} = 2^{2n+1}L(2, n)$$

So

$$\sum_{(x_1,...,x_{2n})\in\{-1,1\}^{2n}} \Big(\prod_{i=1}^{2n} x_i\Big) \prod_{i=1}^{n} \sum_{k=1}^{2n-i-1} x_k x_{k+i+1} = 2^{2n+1}L(2, n)$$

That allows to get $L(2, n)$ from polynomial evaluations. The computational complexity of $L(2, n)$ is of $O(4^n \times n^2)$ and an efficient big integer arithmetic is necessary. This principle can be optimized by taking into account the symmetries of the problem and using the Gray code: these optimizations are described below.

Optimizations

Some works focused on finding optimizations for this arithmetic method[Jai05]. Here we explain the symmetric and computation optimizations used in our algorithm.

Evaluation parity As $[F(-X, n) = F(X, n)]$, G is not affected by a global sign change. In the same way the global sign does not change if we change the sign of each pair or impair variable.

[2]Even if this instance has no interest since it is known to have no solution

Using these optimizations we can set the value of two variables and accordingly divide the computation time and result size by four.

Symmetry summing In this problem we have to count each solution up to a symmetry; thus for the first pair of cubes we can stop the computation at half of the available positions considering
$S'_1(x) = \sum_{k=1}^{n-1} x_k x_{k+2}$ instead of $S_1(x) = \sum_{k=1}^{2n-2} x_k x_{k+2}$. The result is divided by 2.

Sums order Each evaluation of $S_i(x) = \sum_{k=1}^{2n-i-1} x_k x_{k+i+1}$, before multiplying might be very important regarding to the computation time for this sum. Changing only one value of x_i at a time, we can recompute the sum using the previous one without global recomputation. Indeed, we order the evaluations of the outer sum using Gray code sequence. Then the computation time is considerably reduced.

Based on all these improvements and optimizations we can use the Godfrey method in order to solve huge instances of the Langford problem. The next section develops the main issues of our multiGPU architecture implementation.

Implementation details

In this part we present the specific adaptations required to implement the Godfrey method on a multiGPU architecture.

Optimized big integer arithmetic In each step of computation, the value of each S_i can reach $2n - i - 1$ in absolute value, and their product can reach $\frac{(2n-2)!}{(n-2)!}$. As we have to sum the S_i product on 2^{2n} values, in the worst case we have to store a value up to $2^{2n}\frac{(2n-2)!}{(n-2)!}$, which corresponds to 10^{61} for $n = 28$, with about 200 bits.

So we need few big integer arithmetic functions. After testing existing libraries like GMP for CPU or CUMP for GPU, we came to the conclusion that they implement a huge number of functionalities and are not really optimized for our specific problem implementation: product of "small" values and sum of "huge" values.

Finally, we developed a light CPU and GPU library adapted to our needs. In the sum for example, as maintaining carries has an important time penalty, we have chosen to delay the spread of carries by using buffers: carries are accumulated and spread only when useful (for example when the buffer is full). Fig. 2.12 represents this big integer handling.

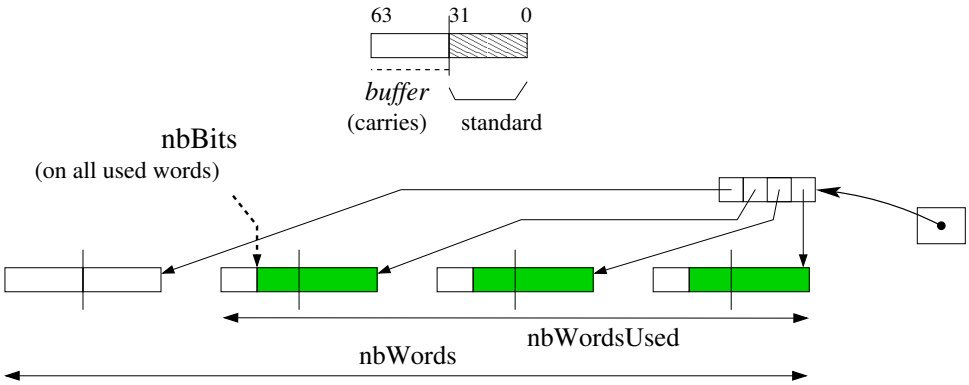

Figure 2.12: Big integer representation, 64 bits words

Gray sequence in memory The Gray sequence cannot be stored in an array because it would be too large (it would contain 2^{2n} byte values). This is the reason why only one part of the Gray code sequence is stored in memory and the missing terms are directly computed from

the known ones using arithmetic considerations. The size of the stored part of the Gray code sequence is chosen to be as large as possible to be contained in the processor's cache memory, the L1 cache for the GPUs threads: so the accesses are fastened and the computation of the Gray code is optimized. For an efficient use of the E5-2650 v2 ROMEO's CPUs, which disposes of 20 MB of level-3 cache, the CPU Gray code sequence is developed recursively up to depth 25. For the K20Xm ROMEO's GPUs, which dispose of 8 KB of constant memory, the sequence is developed up to depth 15. The rest of the memory is used for the computation itself.

Tasks generation and computation In order to perform the computation of the polynomial, two variables can be set among the $2n$ available. For the tasks generation we choose a number p of variables to generate 2^p tasks by developing the evaluation tree to depth p.

The tasks are spread over the cluster, either synchronously or asynchronously.

Synchronous computation - A first experiment was carried out with an MPI distribution of the tasks of the previous model. Each MPI process finds its tasks list based on its process id; then converting each task number into binary gives the task's initialization. The processes work independently; finally the root process ($id = 0$) gathers all the computed numbers of solutions and sums them.

Asynchronous computation - In this case the tasks can be computed independently. As with the synchronous computation, the tasks' initializations are retrieved from their number. Each machine can get a task, compute it, and then store its result; then when all the tasks have been computed, the partial sums are added together and the total result is provided.

Experimental settings

This part presents the experimental context and methodology, and the way the experiments were carried out. This study has similar goals as for the Miller's resolution experiments.

Experimental methodology We present here the way the experimental settings were chosen. Firstly we define the tasks distribution, secondly we set the number of threads per GPU block; finally, we set the CPU/GPU distribution.

Tasks distribution depth - This value being set it is important to get a high number of blocks to maintain sufficient GPU load. Thus we have to determine the best number of tasks for the distribution. As presented in part 2.3.3 the number p of bits determines 2^p tasks. On the one hand, too many tasks are a limitation for the GPU that cannot store all the tasks in its 6GB memory. On the other hand, not enough tasks means longer tasks and too few blocks to fill the GPU grid. Fig. 2.14 shows that for the $L(2, 23)$ instance the best task number is with generation depth 28.

Number of threads per block - In order to take advantage of the GPU computation power, we have to determine the threads/block distribution. Inspired by our experiments with Miller's algorithm we know that the best value may appear at lower occupancy. We perform tests on a given tasks set varying the threads/block number and grid size associated. Fig. 2.13 presents the tests performed on the $n = 20$ problem: the best distribution is around 128 threads per block.

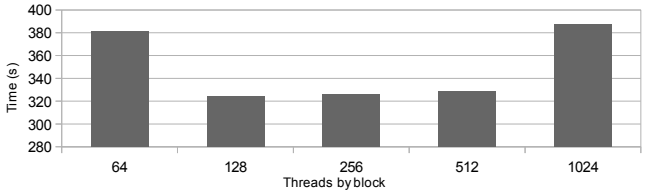

Figure 2.13: $L(2, 20)$, number of threads per block

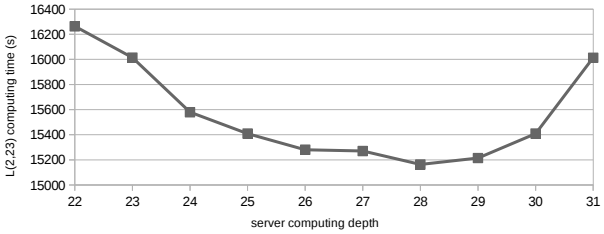

Figure 2.14: Influence on server generation depth

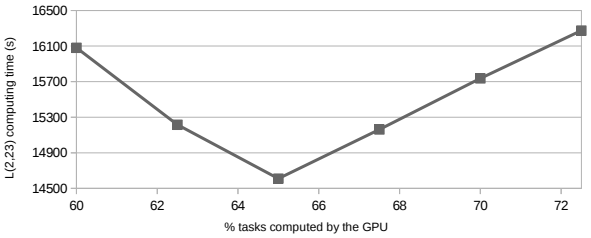

Figure 2.15: Influence of tasks repartition

CPU vs GPU distribution - The GPU and CPU computation algorithm will approximately be the same. In order to take advantage of all the computational power of both components we have to balance tasks between CPU and GPU. We performed tests by changing the CPU/GPU distribution based on simulations on a chosen set of tasks. Fig. 2.15 shows that the best distribution is obtained when the GPU handles 65% of the tasks. This optimal load repartition directly results from the intrinsics computational power of each component; this repartition should be adapted if using a more powerful GPU like Tesla K40 or K80.

Computing context As presented in part **??**, we used the ROMEO supercomputer to perform our tests and computations. On this supercomputer SLURM[JG03] is used as a reservation and job queue manager. This software allows two reservation modes: a static one-job limited reservation or the opportunity to dynamically submit several jobs in a Best-Effort manner.

Static distribution - In this case we used the synchronous distribution presented in 2.3.3. We submited a reservation with the number of MPI processes and the number of cores per process. This method is useful to get the results quickly if we can get at once a large amount of computation resources. It was used to perform the computation of small problems, and even for $L(2, 23)$ and $L(2, 24)$.
As an issue, it has to be noted that it is difficult to quickly obtain a very large reservation on such a shared cluster, since many projects are currently running.

Best effort - SLURM allows to submit tasks in the specific Best-Effort queue, which does not count in the user *fair-share*. In this queue, if a node is free and nobody is using it, the reservation is set for a job in the best effort queue for a minimum time reservation. If another user asks for a reservation and requests this node, the best effort job is killed (with, for example, a SIGTERM signal). This method, based on asynchronous computation, enables a maximal use of the computational resources without blocking for a long time the entire cluster.

For $L(2, 27)$ and even more for $L(2, 28)$ the total time required is too important to use the whole machine off a challenge period, thus we chose to compute in a Best-Effort manner. In order to fit with this submission method we chose a reasonable time-per-task, sufficient to optimize the treatments with low loading overhead, but not too long so that killed tasks are not too penalizing for the global computation time. We empirically chose to run 15-20 minute tasks and thus we considered $p = 15$ for $n = 27$ and $p = 17$ for $n = 28$.

The best effort based algorithm is presented on Fig. 2.16. The task handler maintains a maximum of 256 tasks in the queue; in addition the entire process is designed to be fault-tolerant since killed tasks have to be launched again. When finished, the tasks generate an ouput containing the number of solutions and computation time, that is stored as a file or database entry. At the end the outputs of the different tasks are merged and the global result can be provided.

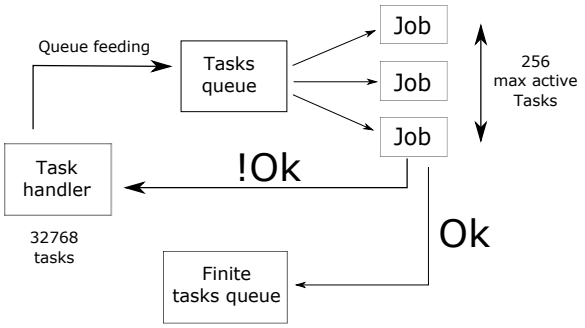

Figure 2.16: Best-effort distribution

Results

After these optimizations and implementation tuning steps, we conducted tests on the ROMEO supercomputer using best-effort queue to solve $L(2, 27)$ and $L(2, 28)$. We started the experiment after an update of the supercomputer, that implied a cluster shutdown. Then the machine was restarted and was about 50% idle for the duration of our challenge. The computation lasted less than 2 days for $L(2, 27)$ and 23 days for $L(2, 28)$. The following describes performances considerations.

Computing effort - For $L(2, 27)$, the effective computation time of the 32,768 tasks was about 30 million seconds (345.4 days), and 165,000" elapsed time (1.9 days); the average time of the tasks was 911", with a standard deviation of 20%. For the $L(2, 28)$ 131,072 tasks the total computation time was about 1365 days (117 million seconds), as 23 day elapsed time; the tasks lasted 1321" on average with a 12% standard deviation.

Best-effort overhead - With $L(2, 27)$ we used a specific database to maintain information concerning the tasks: 617 tasks were aborted [by regular user jobs] before finishing (1.9%), with an average computing time of 766" (43% of the maximum requested time for a task). This consumed 472873", which overhead represents 1.6% of the effective computing effort.

Cluster occupancy - Fig. 2.17 presents the tasks resolution over the two computation days for $L(2, 27)$. The experiment elapse time was 164700" (1.9 days). Compared to the effective computation time, we used an average of 181.2 machines (CPU-GPU couples): this represents 69.7% of the entire cluster.

Fig. 2.18 presents the tasks resolution flow during the 23 days computation for $L(2, 28)$. We used about 99 machines, which represents 38% of the 230 available nodes.

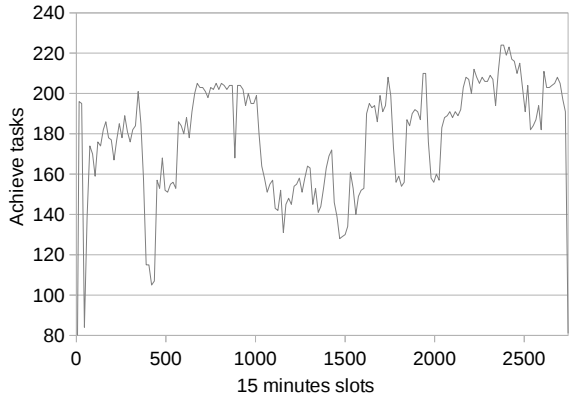

Figure 2.17: $L(2, 27)$ tasks grouped by 15" slots

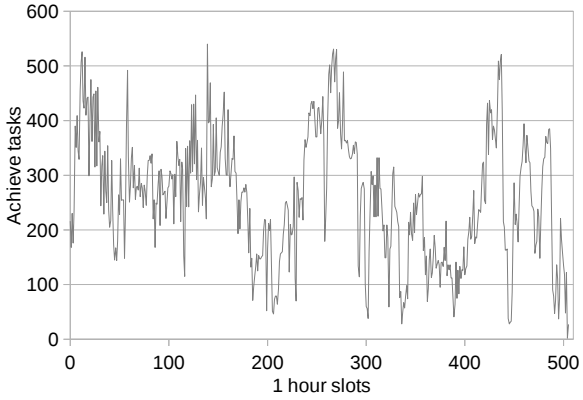

Figure 2.18: $L(2, 28)$ tasks grouped by 1 hour slots

For $L(2, 27)$, these results confirm that the computation took great advantage of the low

occupancy of the cluster during the experiment. This allowed us to obtain a weak best-effort overhead, and an important cluster occupancy. Unfortunately for $L(2, 28)$ on such a long period we got a lower part of the supercomputer dedicated to our computational project. Thus we are confident in good perspectives for the $L(2, 31)$ instance if computed on an even larger cluster or several distributed clusters.

2.3.4 Conclusion

This paper presents two methods to solve the Langford pairing problem on multiGPU clusters. In its first part the Miller's algorithm is presented. Then to break the problem limitations we show optimizations and implementation of Godfrey's algorithm.

CSP resolution method - As any combinatorial problem can be represented as a CSP, the Miller algorithm can be seen as general resolution scheme based on the backtrack tree browsing. A three-level tasks generation allows to fit the multiGPU architecture. MPI or Best-Effort are used to spread tasks over the cluster, OpenMP for the CPU cores distribution and then CUDA to take advantage of the GPU computation power. We were able to compute $L(2, 20)$ with this regularized method and to get an even better time with the basic backtrack. This proves the proposed approach and also exhibits that the GPU scheduler is very efficient at managing highly divergent threads.

MultiGPU clusters and best-effort - In addition and with the aim to beat the Langford limit we present a new implementation of the Godfrey method using GPUs as accelerators. In order to use the supercomputer ROMEO, which is shared by a large scientific community, we have implemented a distribution that does not affect the machine load, using a best-effort queue. The computation is fault-tolerant and totally asynchronous.

Langford problem results - This study enabled us to compute $L(2, 27)$ and $(L2, 28)$ in respectively less than 2 days and 23 days on the University of Reims ROMEO supercomputer. The total number of solutions is:

L(2,27) = 111,683,611,098,764,903,232
L(2,28) = 1,607,383,260,609,382,393,152

Perspectives - This study shows the benefit of using GPUs as accelerators for combinatorial problems. In Miller's algorithm they handle 80% of the computation effort and 65% in Godfrey's. As a near-term prospect, we want to scale and show that it is possible to use the order of 1000 or more GPUs for pure combinatorial problems.
The next step of this work is to generalize the method to optimization problems. This adds an order of complexity since shared information has to be maintained over a multiGPU cluster.

2.4 Complex systems as a benchmark

2.4.1 Introduction

The most commonly used search algorithms for graphs are Breadth First Search (BFS) and Depth First Search (DFS). Many graph analysis methods, such as the finding of shortest path for unweighted graphs and centrality, are based on BFS.

As it is a standard approach method in graph theory, its implementation and optimization require extensive work. This algorithm can be seen as frontier expansion and exploration. At each step the frontier is expanded with the unvisited neighbors. The sequential and basic algorithm is well known and is presented on Algorithm 1.

This algorithm is very famous thanks to its use in many applications but also thanks to the world supercomputer ranking called Graph500[3]. This benchmark is designed to measure the performance on very irregular problems like BFS on a large scale randomized generated graph. The first Graph500 list was released in November 2010. The last list, issued in November 2015, is

[3] `http://www.graph500.org`

Algorithm 1 Sequential BFS

```
1: function COMPUTE_BFS(G = (V, E): graph representation, v_s: source vertex, In: current
   level input, Out: current level output, Vis: already visited vertices)
2:     In ← {v_s};
3:     Vis ← {v_s};
4:     P(v) ←⊥ ∀v ∈ V;
5:     while In ≠ ∅ do
6:         Out ← ∅
7:         for u ∈ In do
8:             for v|(u, v) ∈ E do
9:                 if v ∉ Vis then
10:                    Out ← Out ∪ {v};
11:                    Vis ← Vis ∪ {v};
12:                    P(v) ← u;
13:                end if
14:            end for
15:        end for
16:        In ← Out
17:    end while
18: end function
```

composed of 201 machines ranked using a specific metric: Traversed Edges Per Second, denoted as TEPS. The aim is to perform a succession of 64 BFS on a large scale graph in the fastest possible way. Then the ratio of edges traversed per the time of computation is used to rank the machines.

This benchmark is more representative of communication and memory accesses than computation itself. Other benchmarks can be used to rank computational power such as LINPACK for the TOP500 list. Indeed the best supercomputers (K-Computer, Sequoia, Mira, ...) on the ladder have a very specific communication topology and sufficient memory, and are large enough to quickly visit all the nodes of the graph.

In this study we focus on GPU optimization. There are many CPU algorithms available, which are listed on the Graph500 website. In order to rank the ROMEO supercomputer we had to create a dedicated version of the Graph500 benchmark in order to fit the supercomputer architecture. As this supercomputer is accelerated by GPUs, three successive approaches had to be applied: first create an optimized CPU algorithm; second provide a GPU specific version and third take advantage of both CPU and GPU computation power.

This paper is organized as follows. The first section performs a survey of graph representation and analysis; it also describes some specific implementations. The second section describes the Graph500 protocol and focuses on the Kronecker graph generation method and the BFS validation. The third section presents the chosen methods to implement graph representation and work distribution over the supercomputer nodes. It particularly focuses on the interest of a hybrid CSR and CSC representation. We concude by examining the results for different graph scales and load distributions.

2.4.2 Related work

The most efficient algorithm to compute BFS traversal is used and detailed in [CPW+12]. It uses a 2D partition of the graph which will be detailed later. This algorithm is used on the BlueGene/P and BlueGene/Q architectures but can be easily adapted to any parallel cluster.

We use another key study in order to build our Graph500 CPU/GPU implementation. This paper [MGG15] proposes various effective methods on GPU for BFS. Merrill & al. explaine and teste a few efficient methods to optimize memory access and work sharing between threads

on a large set of graphs. It focuses on Kronecker graphs in particular. First they propose several methods for neighbor-gathering with a serial code versus a warp-based and a CTA-based approach. They also use hybridization of these methods to reach the performance level. In a second part they describe the way to perform label-lookup, to check if a vertex is already visited or not. They propose to use a bitmap representation of the graph with texture memory on the GPU for fast random accesses. In the last phase, they propose methods to suppress duplicated vertices generated during the neighbor exploration phase. Then based on these operations they propose *expand-contract*, *contract-expand*, *two-phase* and finally *hybrid* algorithms to adapt the method with all the studied graph classes. The last part they propose a multi-GPU implementation. They use a 1D partition of the graph and each GPU works on its subset of vertices and edges.

In [FDB+14], a first work is proposed to implement a multi-GPU cluster version of the Graph500 benchmark. The scheme used in their approach is quite similar to the one in our study but with a more powerful communication network, namely FDR InfiniBand.

In our work we focus on the GPUDirect usage on the ROMEO supercomputer.

2.4.3 Environment

As previously mentioned, a CPU implementation is available on the official Graph500 website. A large range of software technology is covered with MPI, OpenMP, etc. All these versions use the same generator and the same validation pattern which is described in this part below.

The Graph500 benchmark is based on the following stages:

- *Graph generation.* The first step is to generate the Kronecker graph and mix the edges and vertices. The graph size is chosen by the user (represented as a based-2 number of vertices). The $EDGEFACTOR$, average ratio of edges by vertex, is always 16. Self-loop and multiple edges are possible with Kronecker graphs. Then 64 vertices for the BFS are randomly chosen. The only rule is that a chosen vertex must have at least one link with another vertex in the graph. *This stage is not timed*;

- *Structure generation.* The specific code part begins here. Based on the edge list and its structure the user is free to distribute the graph over the machines. In a following section we describe our choices for the graph representation. *This stage is timed*;

- *BFS iterations.* This is the key part of the ranking. Based on the graph representation, the user implements a specific optimized BFS. Starting with a root vertex the aim is to build the correct BFS tree (up to a race condition at every level), storing the result in a predecessor list for each vertex;

- *BFS verification.* The user-computed BFS is validated. The number of traversed edges is determined during this stage.

The process is fairly simple and sources can be found at `http://www.graph500.org`. The real problem is to find an optimized way to use parallelism at several levels: node distribution, CPU and GPU distribution and then massive parallelism on accelerators.

Generator

The *Kronecker graphs*, based on Kronecker products, represent a specific graph class imposed by the Graph500 benchmark. These graphs represent realistic networks and are very useful in our case due to their irregular aspect [LCK+10]. The main generation method uses the Kronecker matrix product. Based on an initiator adjacency matrix K_1, we can generate a Kronecker graph of order $K_1^{[k]}$ by multiplying K_1 by itself k times. The Graph500 generator uses Stochastic Kronecker graphs, avoiding large scale matrix multiplying, to generate an edge list which is utterly mixed (vertex number and edge position) to avoid locality.

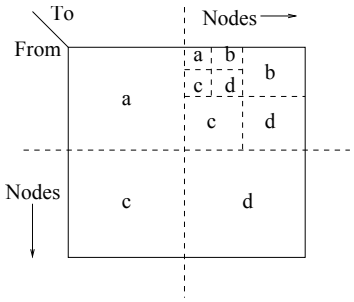

Figure 2.19: Kronecker generation scheme based on edge probability

As presented on Fig 2.19, the generation is based on edge presence probability on a part of the adjacency matrix. For the Graph500 the probabilities are $a = 0.57$, $b = c = 0.19$ and $d = 0.05$. The generator handle can be stored in a file or directly split in the RAM memory of each process. The first option is not very efficient and imposes a lot of I/O for the generation and verification stage but can be very useful for large scale problems. The second option is faster but uses a part of the RAM thus less ressources are available for the current BFS execution.

Validation

The validation stage is completed after the end of each BFS. The aim is to check if the tree is valid and if the edges are in the original graph. This is why we must keep a copy of the original graph in memory, file or RAM. This validation is based on the following stages, presented on the official Graph500 website. First, the BFS tree is a tree and does not contain cycles. Second, each tree edge connects vertices whose BFS levels differ by exactly one. Third, every edge in the input list has vertices with levels that differ by at most one or that both are not in the BFS tree. Finally, he BFS tree spans an entire connected component's vertices, and a node and its parent are joined by an edge of the original graph.

In order to meet the Graph500 requirements we use the proposed verification function provided in the official code.

2.4.4 BFS traversal

In this section we present the actual algorithm we used to perform the BFS on a multi-GPU cluster. In a first part we introduce the data structure; then we present the algorithm and the optimizations used.

Data structure

We performed tests of several data structures. In a first work we tried to work with bitmap. Indeed the regularity of computation can fit very well with the GPU architecture. But this representation imposes a significant limitation on the graph size. This representation is used on the BlueGene/Q architecture. Indeed they have some specific hardware bit-wise operations implemented in their processors and have a large amount of memory, allowing them to perform very large scale graph analysis.

In a second time we used common graph representations, Compressed Sparse Row (CSR) and Compressed Sparse Column (CSC) representation, which fit very well with sparse graphs such as the Graph500 ones. The following example illustrates the CSR representation:

$$M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$R = \{0, 2, 4, 7, 8\}$$

$$C = \{1, 2, 0, 2, 0, 1, 3, 2\}$$

The M adjacency matrix represents the graph. R vector contains the cumulative number of neighbors for each vertex, of size $(\#vertices + 1)$. C, of size $(\#edges)$, is, for each index of R, the edges of a vertex. This representation is very compact and very efficient to work with sparse graphs.

General algorithm

When looking at the latest Graph500 list we see that the best machines are the BlueGene ones. We count about 26 BlueGene/Q and BlueGene/P machines in the first 50 machines. This is due to a quite specific version of the BFS algorithm proposed in [CPW+12]. It proposes a very specific 2D distribution for parallelism and massive use of the 5D torus interconnect.

In the BFS algorithm, like other graph algorithms, parallelism can take several shapes. We can split the vertices into partitions using 1D partition. Each thread/machine can then work on a subset of vertices. The main issue with this method is that the partitions are not equal since the number of edges per vertex can be very different; moreover in graphs like Kronecker ones where some vertices have a very high degree compared to other ones. Thus we are confronted with a major load balancing problem.

In [CPW+12] they propose a new vision of graph traversal, here BFS, on distributed-memory machines. Instead of using standard 1D distribution their BFS is based on a 2D distribution. The adjacency matrix is split into blocks of same number of vertices. If we consider $l \times l$ blocks $A_{i,j}$ we can split the matrix as follows:

$$M = \begin{bmatrix} A_{0,0} & A_{0,1} & \cdots & A_{0,l-1} \\ A_{1,0} & A_{1,1} & \cdots & A_{1,l-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{l-1,0} & A_{l-1,1} & \cdots & A_{l-1,l-1} \end{bmatrix}$$

Each bloc $A_{x,y}$ is a subset of edges. We notice that blocks $A_{0,l-1}$ and $A_{l-1,0}$ have the same edges but in a reverse direction for undirected graphs. Based on this distribution they use *virtual processors*, which are either machines or nodes, each associated with a block. This has several advantages. First we reduce the load balancing overhead and a communication pattern can be set up. Indeed each column shares the same *in_queue* and each row will generate an *out_queue* in the same range. Thus for all the exploration stages, communications are only on line and we just need a column communication phase to exchange the queues for the next BFS iteration. Algorithm 2 presents the BlueGene/Q and BlueGene/P parallel BFS.

This algorithm is based on the exploration phase, denoted by *ExploreFrontier()*. It performs the exploration phase independently on all the machines. Then several communication phases follow. The first two phases are performed on the same processes line. The last one is performed on a processes column.

- On line 15, an exclusive scan is performed for each process on the same line, all the $A_{i,x}$ with $i \in [0, l-1]$. This operation allows us to know which vertices have been discovered in this iteration.

- On line 19, a broadcast of the current *out_queue* is sent to the processes on the same line. With this information they would be able to update the predecessor list only if they are the first parent of a vertex.

- On line 24, a global communication on each column is needed to prepare the next iteration. The aim is to replace the previous *in_queue* by the newly computed *out_queue*.

Algorithm 2 Parallel BFS on BlueGene

1: $Vis_{i,j} \leftarrow In_{i,j}$
2: $P(N_{i,j}, v) \leftarrow \perp$ **for all** $v \in R^{1D}_{i,j}$
3: **if** $v_s \in R^{1D}_{i,j}$ **then**
4: $\quad P(N_{i,j}, v_s) \leftarrow v_s$
5: **end if**
6: **while** *true* **do**
7: $\quad (Out_{i,j}, Marks_{i,j} \leftarrow$ ExploreFrontier();
8: $\quad done \leftarrow \bigwedge_{0 \le k,l \le n} (Out_{k,l} = \emptyset)$
9: $\quad$ **if** *done* **then**
10: $\quad\quad$ **exit loop**
11: $\quad$ **end if**
12: $\quad$ **if** $j = 0$ **then**
13: $\quad\quad prefix_{i,j} = \emptyset$
14: $\quad$ **else**
15: $\quad\quad$ **receive** $prefix_{i,j}$ from $N_{i,j-1}$
16: $\quad$ **end if**
17: $\quad assigned_{i,j} \leftarrow Out_{i,j} \setminus prefix_{i,j}$
18: $\quad$ **if** $j \neq n - 1$ **then**
19: $\quad\quad$ **send** $prefix_{i,j} \cup Out_{i,j}$ **to** $N_{i,j+1}$
20: $\quad$ **end if**
21: $\quad Out_{i,j} \leftarrow \bigcup_{0 \le k \le n} Out_{i,k}$
22: $\quad$ WritePred()
23: $\quad Vis_{i,j} \leftarrow Vis_{i,j} \cup Out_{i,j}$
24: $\quad In_{i,j} \leftarrow Out_{j,i}$
25: **end while**

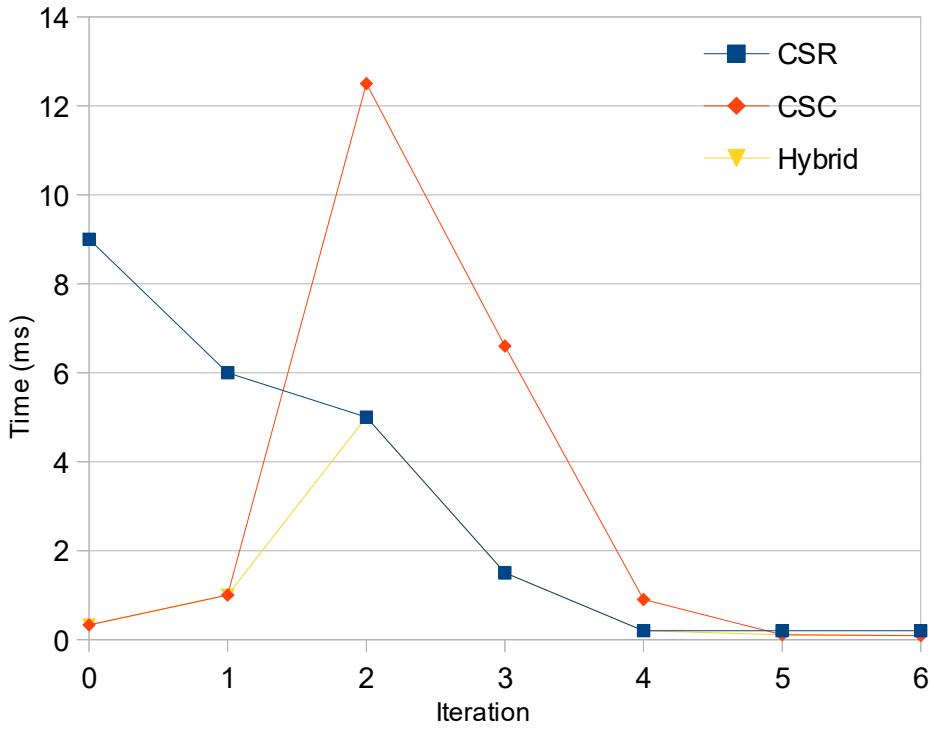

Figure 2.20: CSR and CSC approach comparison. On a 6 iterations BFS, the time with the two method is compared. The hybridization just takes the best time of each method

Two functions are not specified: *ExploreFrontier()* converts the *in_queue* into *out_queue* taking account of the previously visited vertices; *WritePred()* aims to generate the BFS tree and therefore store the predecessor list. In this algorithm the predecessor distribution is still in 1D to avoid vertex duplication. This part can be done using RDMA communication to update predecessor value or with traditional MPI all-to-all exchanges. It can be done during each iteration stage or at the end of the BFS but this requires using a part of the memory to store this data.

This algorithm, which is the basis of many implementations, is the main structure of our distribution.

Direction optimization

In order to get an optimized computation in terms of TEPS we decided to sacrifice a small part of the memory for storing both the CSC and CSR representations. Indeed during the different BFS iterations the *in_queue* size varies a lot and, taking this into account, it is wiser to perform exploration from *top-down* or *bottom-up*. So, as proposed in [BAP13], we perform a direction-optimized BFS.

In the first case, *top-down*, we start from the vertices in the *in_queue* and check all the neighbors verifying each time if this neighbor has ever been visited. Then if not, it is added to the *out_queue*. When the *in_queue* is sparse, like for the first and latest iterations, this method is very efficient. In the second case, *bottom-up*, we start the exploration by the not-yet-visited vertices and verify if there is a link between those vertices and the *in_queue* ones. If yes, the not-yet-visited vertex is added to the *out_queue*. Fig 2.20 presents the two approches, whith the time visiting all the edges, and the benefits of their hybridization.

GPU optimization

In algorithm 2, two parts are not developed. namely *ExploreFrontier()* and *WritePred()*. Indeed these phases are optimized using the GPU. Based on the Merill et al. implementation, the algorithm is optimized to use the shared memory and the texture memory of the GPU. For our version we decided to keep the bitmap implementation for the communications and the queues. So we have to fit the CSR and CSC implementations. On algorithm 3 we present the CSR algorithm; CSC is based on the same approach but starting from the *visited* queue.

In the CSR version each warp is attached to a 32 bit word of the *in_queue* bitmap. Then if this word is empty the whole warp is released; if it contains some vertices, the threads collaborate to load the entire neighbor list. Then they access the coalescent area in the main memory to load the neighbor list. A texture memory is used to accelerate the verification concerning this vertex. Indeed this memory is optimized to be randomly accessed. Then the vertex is added in the bitmap *out_queue*.

Communications

Based on the algorithm 2 communications pattern, we first used MPI with the CPU transferring the data. But the host-device transfer time between the CPU and the GPU was too time-consuming. In order to accelerate the transfers between the GPUs, we used a specific GPU MPI-aware library. This library allows direct MPI operations from the memory of one GPU to another and also implements direct GPU collective operations. GPUDirect can be used coupled with this library. In the last version we used this optimization with GDRCopy.

2.4.5 Results

Working environment

All the tests were led on the ROMEO cluster available at the University of Reims Champagne-Ardenne (France). It provides 130 nodes, each composed of 2 Ivy Bridge CPUs (8 cores), 2.6GHz and 2 Tesla K20Xm GPUs.

We used the nodes as two independent machines with one eight core CPU and one GPU attached, linked by PCIe-v3. This allows having 260 machines for computation, each containing 32GB RAM memory and linked with QDR InfiniBand (up to 10 Gbps). The nodes are connected with a FatTree topology. A K20Xm GPU has 6GB memory, 250GB/s bandwidth, 2688 CUDA cores including 896 double precision cores.

The supercomputer supports MPI with GPU Aware and GPUDirect. The developments were made using MVAPICH-GDR 2.2a with CUDA 7.5 [Nvi08] and OpenMP for the CPU implementation.

CPU and GPU comparison

On Fig 2.21 we present the single node implementation. Here we compare the best CPU implementation proposed by the Graph500 benchmark with our GPU implementation. On our cluster we worked with K20Xm GPUs. The GPU result is twice times better than the CPU one. We also carried out tests on some "general public" GPUs like GTX980 and GTX780Ti. The result is better on these GPUs because they do not implement the ECC memory and do not provide double precision CUDA cores. Indeed all the cores can be used for the Exploration phase.

Strong and weak scaling

On Fig 2.23 and Fig 2.22 we see the result of strong and weak scaling. In the strong scaling we used a $SCALE$ of 21 for different numbers of GPUs. The application scales up to 16 GPUs but then the data exchanges are too penalizing; performance for 64 GPUs is lower. Indeed

Algorithm 3 Exploration kernel based on CSR

Constants:
$NWARP$: number of WARPS per block

Variables:
pos_word: position of the word in in_queue
$word$: value of the word in in_queue
$lane\_id$: thread ID in the WARP
$warp\_id$: WARP number if this block
$comm[NWARP][3]$: shared memory array
$shared\_vertex[NWARP]$: vertex in shared memory

Begin
if $word = 0$ **then**
    free this WARP
end if
if $word \ \&1 \ << lane\_id$ **then**
    $id\_sommet \leftarrow= pos\_word * 32 + lane\_id$
    $range[0] \leftarrow C[id\_sommet]$
    $range[1] \leftarrow C[id\_sommet + 1]$
    $range[2] \leftarrow range[1] - range[0]$
end if
while $\_any(range[2])$ **do**
    if $range[2]$ **then**
        $comm[warp\_id][0] \leftarrow lane\_id$
    end if
    if $comm[warp\_id][0] \leftarrow lane\_id$ **then**
        $comm[warp\_id][0] \leftarrow range[0]$
        $comm[warp\_id][0] \leftarrow range[1]$
        $range[2] \leftarrow 0$
        $share\_vertex[warp\_id] = id\_sommet$
    end if
    $r\_gather \leftarrow comm[warp\_id][0] + lane\_id$
    $r\_gather\_end \leftarrow comm[warp\_id][2]$
    while $r\_gather < r\_gather\_end$ **do**
        $voisin \leftarrow R[r\_gather]$
        if $not \in tex\_visited$ **then**
            Adding in $tex\_visited$
            AtomicOr(out_queue,$voisin$)
        end if
        $r\_gather \leftarrow r\_gather + 32$
    end while
end while

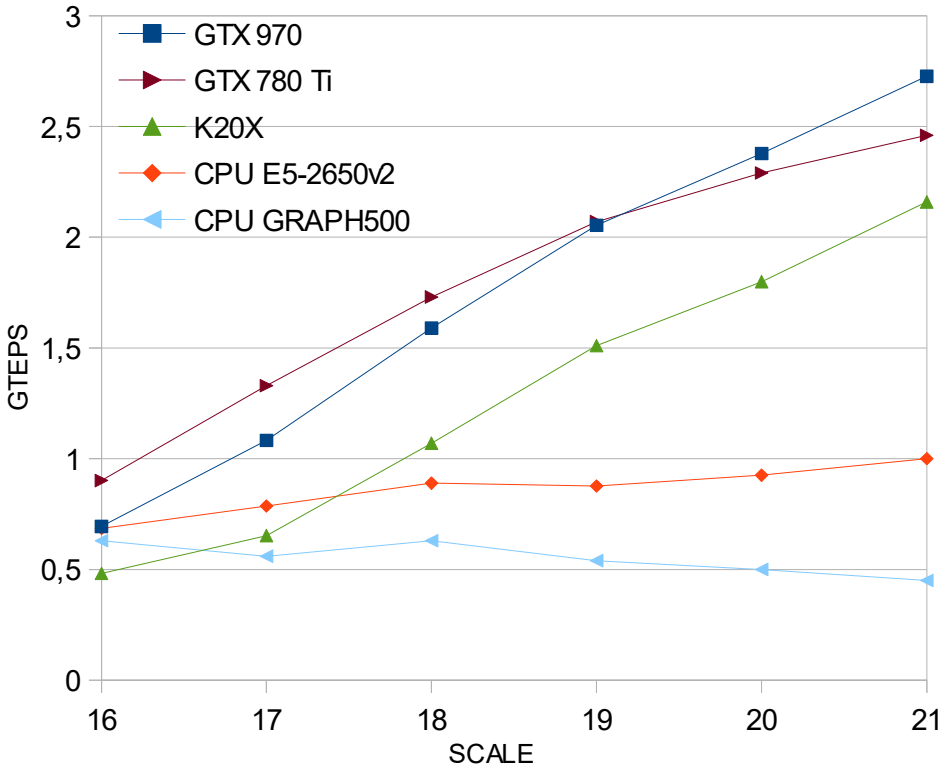

Figure 2.21: Single CPU and accelerators comparison. CPU Graph500 represent the best implementation proposed by the Graph500 website.

as the problem scale does not change, the computational part is reduced compared to the communication one. Using 16 GPUs we were able to perform up to 4.80 GTEPS.

For the weak scaling, the *SCALE* evolves with the number of GPUs. So the computation part grows and the limitation of communications is reduced. On Fig 2.22, the problem SCALE is presented on each point. With our method we were able to reach up to 12 GTEPS using this scaling.

Communications and GPUDirect

Each node of the ROMEO supercomputer is composed of two CPU sockets and two GPUs, named GPU 0 and GPU 1. Yet the node just has one HCA (Host Channel Adapters), linked with CPU 0 and GPU 0. In order to use this link GPU 1 has to pass through a Quick Path Interconnect link (QPI) between the two CPU sockets. This link considerably reduces the bandwidth available for node-to-node communication. Another problem is that the two GPUs have to share the same HCA for their communication.

On Fig 2.24, the tests are based on the GPU-only implementation. First we worked with the two GPUs of the nodes. We were able to perform up to to *SCALE* 29 with 12 GTEPS. The GPUDirect implementation does not allow the communication with a QPI link. So in order to compare the results, we used only the GPU 0 of each node of the supercomputer. Based on our algorithm implementation we need to use a number 2^{2n} of GPUs. Then the tests on Fig 2.24 are for 256 GPUs (with GPU 0 and GPU 1) and with 64 GPUs (using just GPU 0 only). Thus we were able to reach a better value of GTEPS. As the major limitation is the communications stage, using only GPU 0 allowed us to obtain about 13.70 GTEPS on the ROMEO supercomputer.

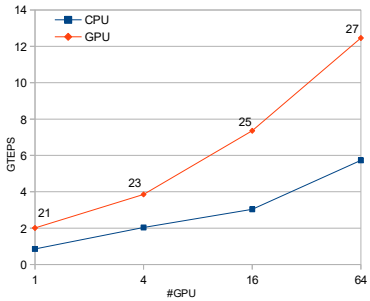

Figure 2.22: CPU *vs* GPU weak scaling. The number of CPUs is the same as the number of GPUs.

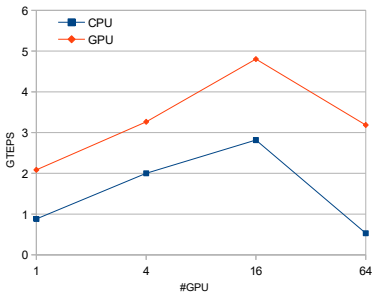

Figure 2.23: CPU *vs* GPU strong scaling. The *SCALE* is showed on the GPU line. The number of CPUs is the same as the number of GPUs.

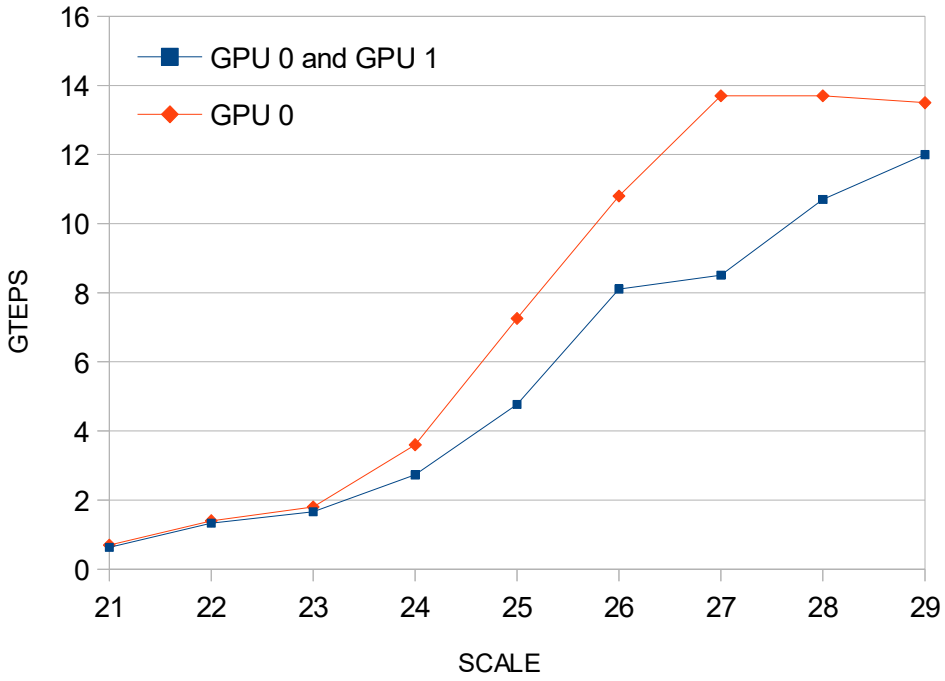

Figure 2.24: Full node GPUs *vs* GPU 0. The GPU 0 implementation not use the QPI link of the two CPU socket.

2.4.6 Conclusions

In this study we present an optimized implementation of the Graph500 benchmark for the ROMEO multi-GPU cluster. It is based on the BlueGene/Q algorithm and GPU optimization for BFS traversal by Merrill et al. This work highlights different key points. First, we have chosen a hybrid memory representation of graphs using both CSR and CSC. Although this representation requires more memoriy, it significantly reduces the computation workload and allows us to achieve outstanding performance. Second, the inter-node and intra-node communication is a critical bottleneck. Each compute node has two GPUs, however only one shares the same PCIe bridge with the Infiniband HCA that allows to take advantage of the GPUDirect technology. Third, due to the low compute power needed for BFS traversal, we get better performance by fully loading GPUs. Otherwise communication time cannot be overlapped with computation time. Thus to achieve the best performance we had to use only half of each node. Finally, using all these optimizations, we achieved satisfactory results. Indeed, by using GPUDirect on 64 GPUs, we are able to achieve 13,70 GTEPS. In this configuration CPUs are only used to

synchronize GPUs kernels. All the communications are directly GPU to GPU using a CUDA-aware MPI library and GPUDirect.

These results will be published in the next Graph500 list. With a total of 13.70 GTEPS the ROMEO supercomputer could be ranked at the 91th position.

Today we can identify some interesting perspectives to carry on the study. Communication cost is the major limitation and a better control of load distribution is needed between communication and computation in order to obtain even better performance. Part of the solution might come from new technologies developed by Nvidia, such as the new PASCAL architecture or NVlink buses.

2.5 Conclusion

Chapter 3

Application

Conclusion

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Annexes

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