Class notes for day 1 (raw and unpolished)

Driver i chooses z

$$\max_z \{u_{iz}(p_z)\}$$

$$s_z(p) = \sum_i 1\left\{z \in \arg\max_{z'} \{u_{iz'}(p_{z'})\}\right\}$$

Claim: this is approximated by

$$s_z^T(p) = \sum_i \frac{\exp(u_{iz}(p_z)/T)}{\sum_{z'} \exp(u_{iz'}(p_{z'})/T)}$$

Logit framework for driver i, $\max_z \{u_{iz}(p_z) + T\varepsilon_{iz}\}$, where $(\varepsilon_{iz})_z \sim iid$ Gumbel

$\Pr(i \text{ chooses } z) = \frac{\exp(u_{iz}(p_z)/T)}{\sum_{z'} \exp(u_{iz'}(p_{z'})/T)}$

0.1 Log-sum-exp trick

For all a, b and c

$$\begin{aligned}
T\log\left(e^{a/T} + e^{b/T}\right) &= T\log\left(e^{c/T}e^{(a-c)/T} + e^{c/T}e^{(b-c)/T}\right) \\
&= T\log\left(e^{c/T}\left(e^{(a-c)/T} + e^{(b-c)/T}\right)\right) \\
&= c + T\log\left(e^{(a-c)/T} + e^{(b-c)/T}\right)
\end{aligned}$$

Take $c = \max(a, b)$, then we get

$T\log\left(e^{a/T} + e^{b/T}\right) = \max(a, b) + T\log\left(e^{(\min(0,b-a))/T} + e^{\min(0,a-b)/T}\right)$

We have

$$\frac{e^{a/T}}{e^{a/T} + e^{b/T}} = \frac{e^{(a-c)/T}}{e^{(a-c)/T} + e^{(b-c)/T}} = \frac{e^{(a-\max(a,b))/T}}{e^{(a-\max(a,b))/T} + e^{(b-\max(a,b))/T}}$$

1 Coordinate update

Solve equation in p'_z

$$e_z(p'_z; p_{-z}) = q_z$$

2 Other methods

Solve $e(p) = 0$

$e_z(p) > 0$ then z is oversupplied

Tatonnement: $p_z(t+1) = p_z(t) - \epsilon e(p(t))$

General equilibrium: Tatonnement converges for GS

Newton-Smale

$p_z(t+1) = p_z(t) - \epsilon (Dp(t))^{-1} e(p(t))$

3 Gross substitutes: definition

Assume $e_z(p) =$ excess supply for z is:

- increasing in p_z
- is decreasing in $p_{z'}$ $z' \neq z$
- is continous

It's the case here

$e_z(p) = \sum_i 1\{z \in \arg\max_{z'} u_{iz'}(p_{z'})\} - \sum_j 1\{z \in \arg\min c_{jz'}(p_{z'})\}$

$e_z(p) = \sum_i \frac{\exp u_{iz}(p_z)}{\sum_z \exp(u_{iz'}(p_{z'}))} - xxx$

$$\frac{\partial e_z}{\partial p_{z'}} = -\sum_i \frac{\exp u_{iz}(p_z) \exp u_{iz'}(p_{z'}) u'_{iz'}(p_{z'})}{\left(\sum_z \exp(u_{iz'}(p_{z'}))\right)^2} \leq 0$$

4 Convergence of Jacobi

$e(p) = 0$

Assume we have p^0 such that $e_z(p^0) \leq 0$ for all z

$e_z(p_z^1, p_{-z}^0) = 0$

$e_z(p_z^1, p_{-z}^0) = 0 \geq e_z(p_z^0, p_{-z}^0)$

then because $e_z(., p_{-z}^0)$ is increasing, I have

$p_z^1 \geq p_z^0$

Let's $e_z(p^1) \leq 0$. Why?

$e_z(p^1) = e_z(p_z^1, p_{-z}^1) \leq e_z(p_z^1, p_{-z}^0) = 0$

Get a sequence $(p_z^t)_z$ such that $p_z^t \leq p_z^{t+1}$.

Assume this sequence is bounded. Then p^t converges, call p^* its limit.

Show $e(p^*) = 0$.

We have by definition

$$e_z(p_z^{t+1}, p_z^t) = 0$$

thus by continuity

$$e_z(p_z^*, p_z^*) = 0$$

thus

$$e_z(p^*) = 0.$$

5 Link with fixed point theory

$e_z(cu_z(p), p_{-z}) = 0$ in other words, $cu_z(p) = p'_z$ such that $e_z(p'_z, p_{-z}) = 0$.

The algorithm we just saw (Jacobi) consists of

$$p^{t+1} = cu\left(p^t\right)$$

Let's study the coordinate update function $cu(p)$.

1. $cu_z(p)$ does not depend on p_z.
2. $cu_z(p)$ is monotone in $p_{z'}$ for $z' \neq z$. Indeed,

$$e_z(cu_z(p), p_{-z}) = 0$$

derive wrt $p_{z'}$ for $z' \neq z$, one has

$$\frac{\partial e_z}{\partial p_z}(cu_z(p), p_{-z}) \frac{\partial cu_z}{\partial p_{z'}}(p) + \frac{\partial e_z}{\partial p_{z'}}(cu_z(p), p_{-z}) = 0$$

$$\frac{\partial cu_z}{\partial p_{z'}}(p) = -\frac{\frac{\partial e_z}{\partial p_{z'}}(cu_z(p), p_{-z})}{\frac{\partial e_z}{\partial p_z}(cu_z(p), p_{-z})} \geq 0$$

This means that cu is an order preserving map. Thus if
$p^0 \leq p^*$
then
$cu\left(p^0\right) \leq cu(p^*)$
$p^1 \leq p^*$

6 SOR methods

$p^{t+1} = \theta p^t + (1-\theta)\, cu(p)$