Corporate Finance: IOE 452/MFG 455
University of Michigan
Winter 2026
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1/22/26

Contents

1 Plain-Vanilla Bond Pricing

We now formalize the valuation of fixed-income securities by deriving general pricing expressions for *plain-vanilla bonds*. A plain-vanilla bond is one with deterministic cash flows and no embedded options.

General Bond Pricing Framework

1. **Constant coupon, constant yield**

   For a standard coupon bond with constant coupon payments and a constant yield to maturity, the bond price is given by

   $$P_0 = \frac{F}{(1+y)^N} + \sum_{n=1}^{N} \frac{C}{(1+y)^n}. \tag{1}$$

   - The yield to maturity y is assumed to be the same for all periods and represents the investor's required rate of return.
   - Yields must be expressed in decimal form.
   - In the U.S., most corporate and Treasury bonds pay coupons semiannually; therefore, both yields and coupon rates must be adjusted accordingly.

2. **Variable coupon, constant yield**

   If coupon payments vary over time but the discount rate remains constant, the bond price becomes

   $$P_0 = \frac{F}{(1+y)^N} + \sum_{n=1}^{N} \frac{C_n}{(1+y)^n}. \tag{2}$$

3. **Variable coupon, variable yield**

   When both coupons and discount rates vary across periods, the bond price is

   $$P_0 = \frac{F}{(1+y_N)^N} + \sum_{n=1}^{N} \frac{C_n}{(1+y_n)^n}. \tag{3}$$

1.1 Zero-Coupon Bonds

- Zero-coupon bonds make no periodic interest payments.
- Investors receive a single payment equal to face value at maturity.
- These bonds can be viewed as coupon bonds with $C = 0$ or $r = 0$.
- The price of a zero-coupon bond is simply the present value of its face value:

$$P_0 = \frac{F}{(1 + y)^N}. \tag{4}$$

- Bond yields are often quoted in **basis points**, where 1 basis point = 0.01%. A 1% change corresponds to 100 basis points.

Determinants of Discount Rates

- For plain-vanilla bonds, coupon rates are set by the issuer and cash flows are known with certainty.
- The key valuation challenge is determining the appropriate discount rate.
- Discount rates reflect:
  1. Benchmark risk-free rates (e.g., U.S. Treasury yields),
  2. Compensation for credit, liquidity, and maturity risk,
  3. Market expectations of future economic conditions.
- The minimum required return is the yield on a default-free security of comparable maturity.
- For dollar-denominated bonds, U.S. Treasury yields serve as the benchmark.
- In many international markets, swap curves play a similar role.

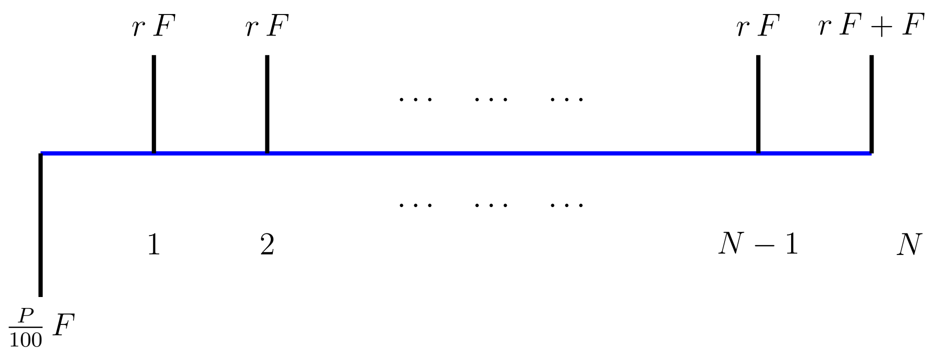
2 Explicit Bond Pricing Formulas

Bond prices depend on three fundamental parameters:

$$P = P(r, y, N),$$

where r is the coupon rate, y is the yield to maturity, and N is the number of remaining periods.

- There are many equivalent formulation of the bond price as a function of its parameters. Consider the following particular characterization of the bond and its time diagram:



- **Bond Formula I:** The price of a bond with the above time diagram is given by

$$P = 100\,r\left(\frac{1}{(1+y)} + \cdots + \frac{1}{(1+y)^N}\right) + \frac{100}{(1+y)^N} \tag{5}$$

  where in Eq. (5), the bond's current price at time 0 is P (expressed as a percentage of F) , y is the constant yield expressed in decimal, and r is the coupon interest as a fraction of the face value F.

- **Proof–**

- Note that in this notation, the cost of the bond is $\frac{P}{100}\,F$. This is the price of the bond at time 0.
- The future value of the bond price on its last coupon, i.e., period N, is $\frac{P}{100}\,F\,(1+y)^N$. This is the future value of the cost of the bond!
- Next, let us calculate the future values of all coupon payments:
- The future value of the first coupon $r\,F$ on the maturity date is $r\,F\,(1+y)^{N-1}$.
- The future value of the second coupon $r\,F$ on the maturity date is $r\,F\,(1+y)^{N-2}$.
- $\cdots \quad \cdots \quad \cdots$

- The future value of the last coupon $r\,F$ on maturity is $r\,F$.
- The future value of the face value F, is F.
- By equating the two future values, we get

$$\begin{aligned}
\frac{P}{100}\,F\,(1+y)^N &= r\,F\left((1+y)^{N-1} + (1+y)^{N-2} + \cdots + (1+y) + 1\right) + F \\
\frac{P}{100}\,(1+y)^N &= r\left((1+y)^{N-1} + (1+y)^{N-2} + \cdots + (1+y) + 1\right) + 1 \qquad (6)
\end{aligned}$$

- Eq. (6) can be written in various ways!
- If we multiply Eq. (6) by $100\,(1+y)^{-N}$, we get the following desired form:

$$P = 100\,r\left(\frac{1}{(1+y)} + \cdots + \frac{1}{(1+y)^N}\right) + \frac{100}{(1+y)^N} \qquad (7)$$

• Another characterization of bond price that is in use is the following form. This particular representation recognizes the summation form in Eq. (6) and is given by:

• **Bond Formula II: Present-Value Representation**

$$P = 100\,r\sum_{k=1}^{N}(1+y)^{-k} + 100\,(1+y)^{-N} \qquad (8)$$

• Note that in Eq. (8) we can interpret the bond price as the present value of the expected future cash flows.

This expression emphasizes that a bond's price equals the present value of its expected future cash flows.

• **Bond Formula III: Annuity Form**

$$P = 100\left((r-y)\times\frac{1-(1+y)^{-N}}{y} + 1\right) \qquad (9)$$

• This formulation makes the relationship between coupon rate and yield explicit and is especially useful for identifying whether a bond trades at a premium, par, or discount.

- **Proof–**

Rewrite Eq. (7) in the form:

$$P = \frac{100\, r}{1+y}\left(1 + \frac{1}{1+y} + \cdots + \frac{1}{(1+y)^{N-1}}\right) + \frac{100}{(1+y)^{N}}$$

Recognize the terms inside parentheses as a geometric series and a simple manipulation.

3 Yield to Maturity and Internal Rate of Return

- Yield to maturity (**YTM**) measures the total return an investor earns if the bond is held to maturity.
- YTM is the discount rate that equates the present value of cash flows to the bond's market price.
- Mathematically, **YTM** solves for y in the following

$$P = \sum_{n=1}^{N} \frac{C_n}{(1+y)^n}$$

  That is, find y such that

$$P - \frac{C_1}{(1+y)^1} - \frac{C_2}{(1+y)^2} - \cdots - \frac{C_N}{(1+y)^N} = 0$$

- **YTM** is equivalent to the bond's internal rate of return (**IRR**).
- Computing YTM requires solving a nonlinear equation and is typically done numerically using software such as Excel.
- Note that for an N period bond, one must solve for the roots of an N^{th} degree polynomial to determine **IRR**.
- Therefore, finding **IRR** amounts to implementing a root finding procedure. To see: Let $v = \frac{1}{1+y}$. Then, **IRR** is the solution of

$$P - C_1\, v - C_1\, v^2 + \cdots + C_N\, v^N = 0$$

Example

Consider a bond purchased at \$770.36 with a face value of \$1,000. This bond makes 10 coupon payments of \$50 each. The **IRR** for this bond can be calculated using Excel's **RATE** function:

$$\textbf{RATE}(10, 50, -770.36, 1000, 0) = 8.5\%$$

Important Caveat on Reinvestment

- YTM assumes coupon payments are reinvested at the same yield.
- If reinvestment occurs at different rates, the realized return will differ from YTM.
- Because interest rates fluctuate, exact reinvestment at YTM is rarely achieved in practice.

4 The Time Path of Bond Prices

As a bond approaches maturity, its price converges toward its face value. Whether the bond's price rises, falls, or remains constant depends on whether it initially trades at a discount, at a premium, or at par.

4.1 Bond Price vs. N : $\frac{\partial P}{\partial N}$

We are interested in examining $\frac{\partial P}{\partial N}$. For this, it is best to consider the following version of the bond price derived earlier in class:

$$P - 100 = 100\left((r-y) \times \frac{1-(1+y)^{-N}}{y}\right)$$

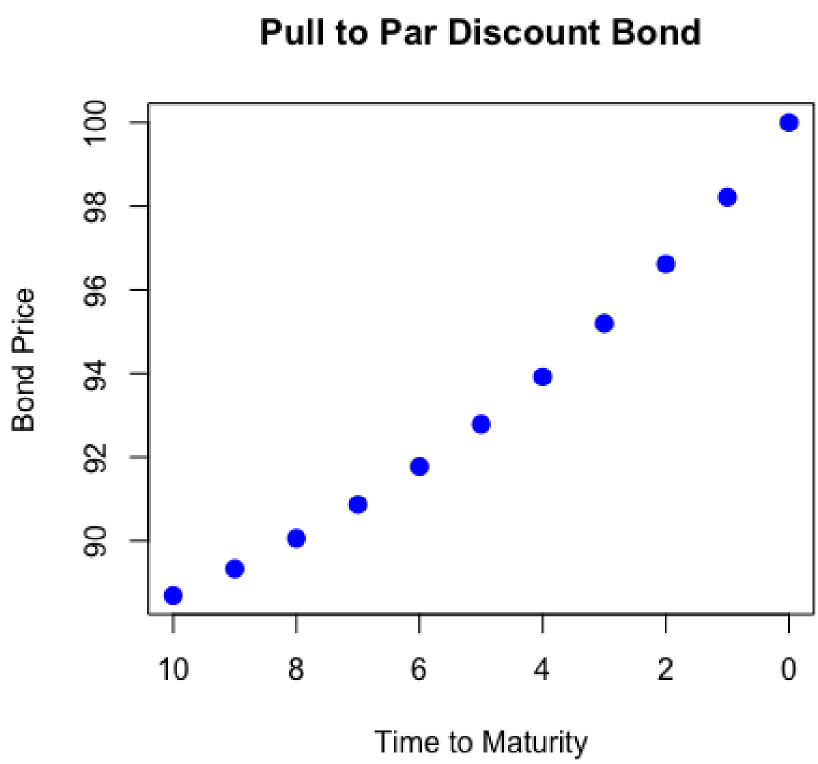
It is easy to see that

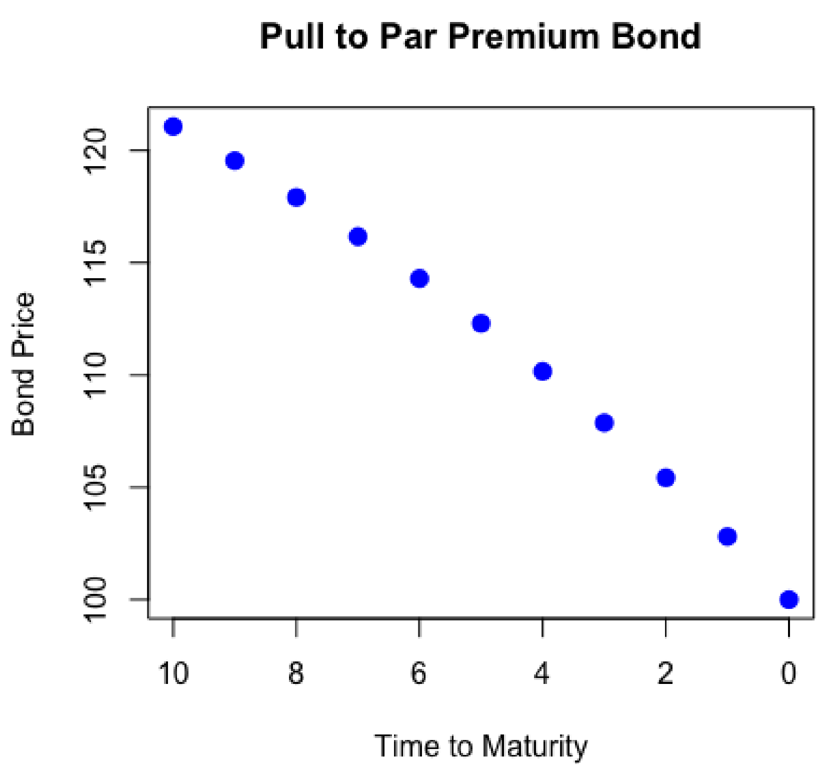
$$\text{sign}\frac{\partial P}{\partial N} = \text{sign}\,(y-r)$$

The above holds due to the fact that since $(1+y)^N > 1$. Therefore, $1-(1+y)^{-N} > 0$.

- For discount bonds ($r < y$), prices rise toward par.
- For premium bonds ($r > y$), prices fall toward par.
- Par bonds remain at face value throughout their life.
- This phenomenon is known as the **pull-to-par effect**.

Hence as a bond gets closer to maturity (e.g., as N gets smaller), the bond price will tend to increase for a discount bond ($r < y$) and decrease for a premium ($y < r$) bond. For a par bond, the bond price is not affected as it gets closer to maturity. That is, a par bond always trades at its face value.


Pull to Par Discount Bond
Bond Price
100
98
96
94
92
90
10
8
6
4
2
0
Time to Maturity


Pull to Par Premium Bond
Bond Price
120
115
110
105
100
10
8
6
4
2
0
Time to Maturity

4.2 Impact of Yield Changes: $\frac{\partial P}{\partial y}$

The price sensitivity of a bond to changes in the required yield can be measured in terms of the dollar price change or the percentage price change.

- For this, it is best to consider the following bond price formulation:

$$P = 100\, r \left(\frac{1}{(1+y)} + \cdots + \frac{1}{(1+y)^N} \right) + \frac{100}{(1+y)^N} \tag{10}$$

- Consider the change in bond price as the yield changes. It is easy to see that

$$\frac{\partial P}{\partial y} = -100\, r \sum_{k=1}^{N} k\ (1+y)^{-k-1} - 100\, N\, (1+y)^{-N-1} < 0 \tag{11}$$

- Next, compute the second partial derivative of the bond price with respect to yield. This relation will help us in deciding the shape of the relationship among others.

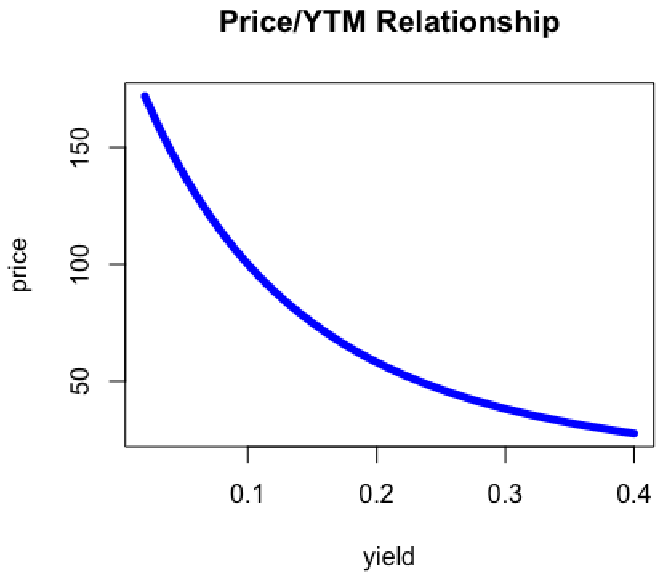
$$\frac{\partial^2 P}{\partial y^2} = 100\, r \sum_{k=1}^{N} k\ (k+1)\, (1+y)^{-k-1} + 100\, N\, (N+1)\, (1+y)^{-N-2} > 0$$

- Note that since $\frac{\partial^2 P}{\partial y^2} > 0$.
  - Bond prices are decreasing functions of yield:
  $$\frac{\partial P}{\partial y} < 0.$$
  - The relationship is nonlinear and convex:
  $$\frac{\partial^2 P}{\partial y^2} > 0.$$
  - As yields fall, prices rise more than they fall for an equivalent yield increase.
  - Price sensitivity declines as yield levels increase.

- The plot of bond price versus discount rate or yield is as follows:

- Clearly, $\frac{\partial P}{\partial y} < 0$. In other words, the bond price P is a decreasing function of y. That is, as the yield increases, the price falls, and as the yield decreases, the bond price rises.

  The above relation is depicted in the following non–linear graph:

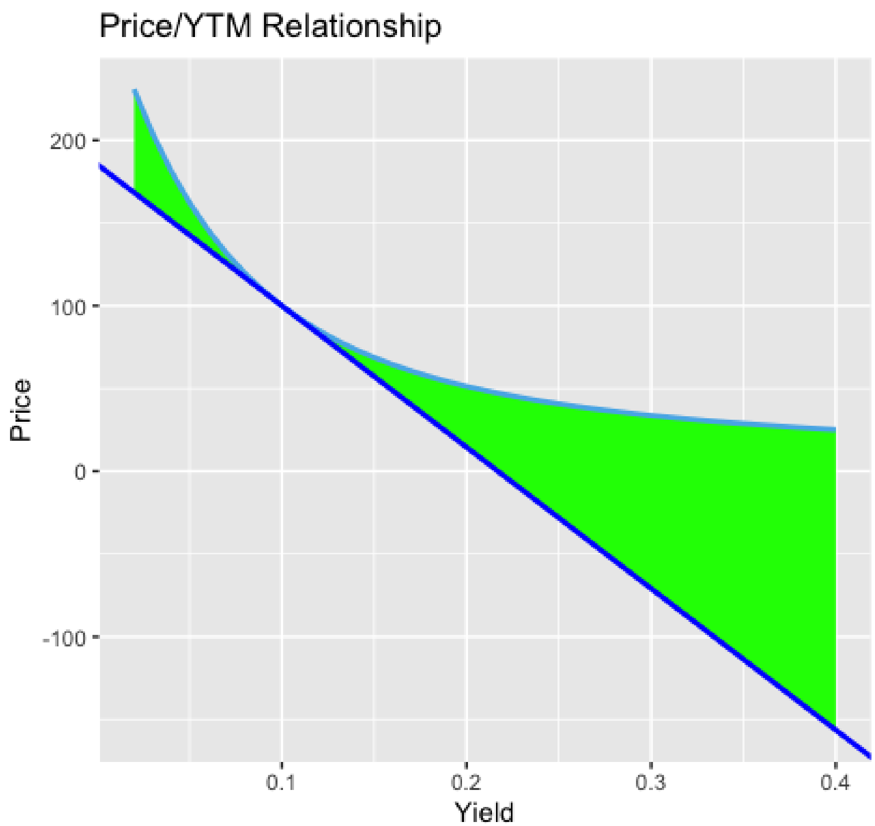


Therefore, the graph of P versus y is concave up.

- Therefore, we conclude that $P(y)$ is a decreasing, concave up function of y. Hence, the plot has the following shape:

4.3 Implications of Convexity

- As $y \uparrow$ bond prices will fall.
- As $y \downarrow$ bond prices will rise.
- The bond price sensitivity depends crucially on the level of the bond price.
- Bond price sensitivity is also known as the price volatility.
- The relation between the bond price as a function of yield is non–linear.
- Bond price changes are asymmetric for equal-sized yield increases and decreases.
- Convexity explains why price gains from falling yields exceed price losses from rising yields.
- Duration provides a linear approximation; convexity corrects that approximation.



- Since bond price as function of yield is convex, it implies that tangent line at any point is below the graph.

$$P(y) \geq P(\bar{y}) + P'(\bar{y})(y - \bar{y}))$$

- However, we do not know the magnitude of the approximate change in the bond price as yield changes.
- Moreover, the magnitude of change is non–symmetric.
- That is, for a given change in yield, Δy, the change in price, ΔP, is different for $\pm\Delta y$.
- Price Volatility depends on the current level of y.
- Please refer to the tabular data.

Example: Changes in y

Refer to the tabulated data:

- Suppose initially the yield is equal to $\bar{y} = 7\%$. From the table, the bond price at this yield is $P(\bar{y}) = \$121.07$.

- Consider a "small" change in the yield:

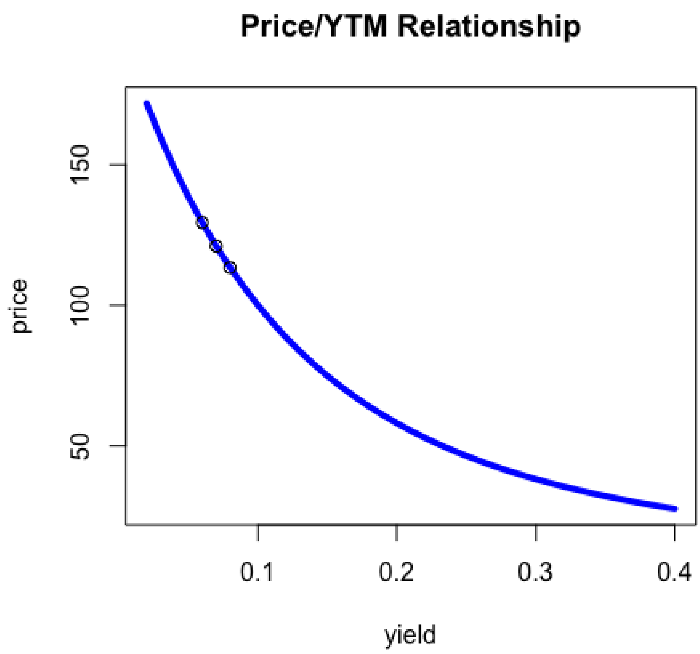
- Suppose the yield increases by 1%. That is, take $\Delta y = +.01$. In other words, we are assuming the yield increases by 100 basis points. The new bond price, that is the bond price when $y = 8\%$, is 113.42.
- Obviously, as y has increased the bond price decreased. That is as $y \uparrow$, $P \downarrow$. The change in the bond price is

$$\text{For} \quad \Delta y = +.01 \implies \Delta P = (113.42 - 121.07) = -7.65$$

- Next, suppose again we start at yield being $y = 7\%$. Now consider a yield decrease by 1%. That is, take $\Delta y = -.01$. Here, we are assuming the yield decreases by 100 basis points. The new bond price, that is the bond price when $y = 6\%$ is 129.44.
- Obviously, as y decreased the bond price increased. That is as $y \downarrow$, $P \uparrow$. The change in the bond price is

$$\text{For} \quad \Delta y = -.01 \implies \Delta P = (129.44 - 121.07) = +8.37$$

- We conclude that although $\frac{\partial P}{\partial y} < 0$, the response of bond price to changes in the yield is not symmetric in Δy.



Price/YTM Relationship
price
150
100
50
0.1
0.2
0.3
0.4
yield

Example: Change in y

Next, suppose an initial yield of $y = 20\%$. The point of this example is to highlight the impact of the yield change to the level of interest rate. Then consider what happens as the yield changes by $\pm\Delta y = .01$:

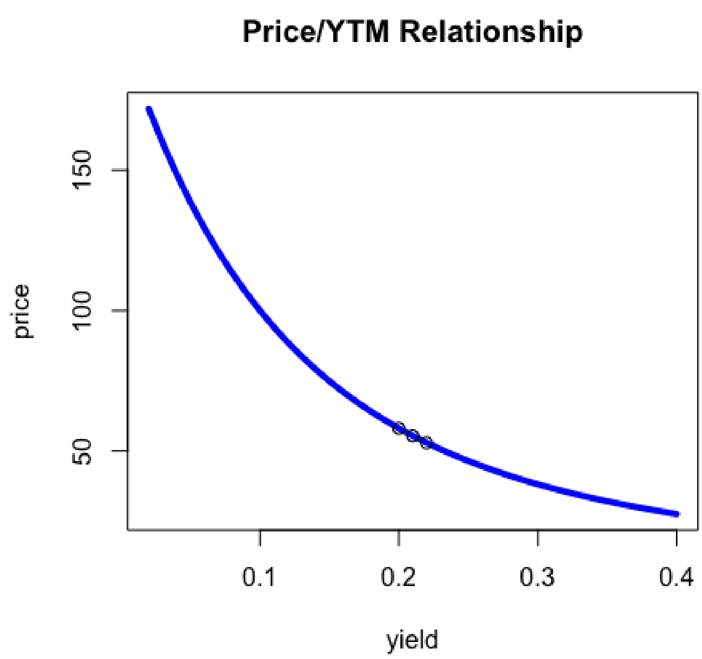
Yield	Bond Price
19%	\$60.94
20%	\$58.074
21%	\$55.40

Starting at $y = 20\%$ where $P(y) = \$58.074$, consider decreasing the yield by 100 basis points:

$$\text{For} \quad \Delta y = -.01 \implies \Delta P = +2.866$$

Similarly, starting at $y = 20\%$ where $P(y) = \$58.074$, consider increasing the yield by 100 basis points:

$$\text{For} \quad \Delta y = +.01 \implies \Delta P = -2.674$$



Remark–The impact of 100 basis point change in yield has an asymmetric impact on the bond price. In general, the bond price is less affected for a given change in yield when the yield is high.

In practice, we are interested in approximate changes in bond price to changes in the yield.

Yield %	**Bond Price $**
0.02	171.86068
0.03	159.71142
0.04	148.66537
0.05	138.60867
0.06	129.44035
0.07	121.07074
0.08	113.42016
0.09	106.41766
0.10	100.00000
0.11	94.11077
0.12	88.69955
0.13	83.72127
0.14	79.13554
0.15	74.90616
0.16	71.00064
0.17	67.38977
0.18	64.04731
0.19	60.94959
0.20	58.07528
0.21	55.40514
0.22	52.92179
0.23	50.60949
0.24	48.45401
0.25	46.44245
0.26	44.56310
0.27	42.80534
0.28	41.15950
0.29	39.61679
0.30	38.16921
0.31	36.80946
0.32	35.53088
0.33	34.32740
0.34	33.19344
0.35	32.12393
0.36	31.11421
0.37	30.16000
0.38	29.25739
0.39	28.40278

Table 1: Data: Bond Price versus Yield